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EQUILIBRIUM LABOR TURNOVER, FIRM GROWTH, AND UNEMPLOYMENT

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EQUILIBRIUM LABOR TURNOVER, FIRM GROWTH, AND UNEMPLOYMENT

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This paper considers equilibrium quit turnover in a frictional labor market with costly hiring by firms, where large firms employ many workers and face both aggregate and firm specific productivity shocks. There is exogenous firm turnover as new (small) startups enter the market over time, while some existing firms fail and exit. Individual firm growth rates are disperse and evolve stochastically. The paper highlights how dynamic monopsony, where firms trade off lower wages against higher (endogenous) employee quit rates, yields excessive job-to-job quits. Such quits directly crowd out the reemployment prospects of the unemployed. With finite firm productivity states, stochastic equilibrium is fully tractable and can be computed using standard numerical techniques.

KEYWORDS: Labor turnover, stochastic equilibrium, unemployment.

1. INTRODUCTION

QUIT TURNOVER IS A QUANTITATIVELY IMPORTANT PROCESS: Fallick and Fleishman (2004) report for the United States that around 40% of new jobs created are filled by job-to-job transitions. Quit turnover is also highly procyclical (e.g., Menzio and Shi (2011)). This paper considers equilibrium quit turnover in a frictional labor market where large firms employ many workers and there are aggregate and firm specific productivity shocks. It highlights how dynamic monopsony, where firms trade off lower wages against higher (endogenous) employee quit rates, yields excessive job-to-job quits. Numerical examples in our companion paper (Coles and Mortensen (2015)) demonstrate how such turnover generates too high unemployment relative to the competitive allocation. This occurs as quit turnover here directly crowds out the reemployment prospects of the unemployed. Furthermore when a new job is filled by an already employed worker, that job created is matched by a job destroyed at the quitter's previous employer (which has to pay recruitment costs to hire a replacement). Such quit turnover, being strongly procyclical, also has important implications for equilibrium unemployment dynamics. This paper thus identifies a new, tractable framework for understanding equilibrium wage dispersion, quit turnover, and unemployment, one which has genuine policy implications.

Even though quit turnover is large and pervasive, the literature on stochastic equilibrium unemployment with on-the-job search is small. The reason is

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that on-the-job search creates a major theoretical hurdle: that, in general, the level of unemployment U and distribution of employment across firms $G(\cdot)$ are payoff relevant state variables. Of course in steady state these objects can be solved for directly (e.g., Burdett and Mortensen (1998)). But as a stochastic environment implies $G(\cdot)$ evolves endogenously and is infinitely dimensional, the stochastic extension is deeply problematic.

One successful approach is to adopt a one firm/one job technology with free entry of vacancies and a constant returns to scale matching function. Menzio and Shi (2011) assume directed on-the-job search and, as job-to-job transitions do not crowd out the unemployed, find equilibrium unemployment and directed quit turnover are efficient. Lise and Robin (2015) instead adopt the sequential auction approach of Postel-Vinay and Robin (2002).[2] Unemployment there is instead inefficiently low, because firms extract full surplus and a congestion externality in the matching function implies firms invest in too many vacancies. Both frameworks are tractable, however, because they exhibit a very special property: that the equilibrium value functions (or match surplus function in Lise and Robin (2015)) are independent of unemployment U and the distribution of employment $G(\cdot)$. Away from the constant returns matching environment with free entry, however, equilibrium with on-the-job search finds the value functions generally depend on U and G.

A different approach instead assumes hiring is costless and that exogenous contact rates evolve stochastically (e.g., Robin (2011) and the large part of Moscarini and Postel Vinay (2013)). This approach is tractable, because once it is established that equilibrium wages induce workers to only quit to more productive firms, exogenous contact rates imply U and G can be computed independently of actual wage outcomes and so yields a theory of firm-size dynamics.[3]

The more interesting case of costly hiring by firms creates a fundamental problem: the evolution of $G(\cdot)$ depends on endogenous hiring outcomes. This paper identifies a tractable stochastic framework in which firms face recruitment costs to screen prospective hires and train new employees into firm practices. We also allow firm turnover (via new startup companies and stochastic firm death rates) and introduce a dynamically consistent wage determination process wherein each firm's optimal wage strategy depends on its productivity, which is both private information and subject to shocks. Most importantly, on the assumption that firm recruitment costs exhibit constant returns to scale (that is, a firm that is twice as large and wishes to hire twice as many new recruits has twice the recruitment costs), we show stochastic equilibria have

[2]Formally, this paper assumes a competitive vacancy market where firms take the price of a vacancy ticket as parametric.

[3]There has been much recent work on understanding how employment growth varies across firms over the cycle; see for example Moscarini and Postel-Vinay (2012, 2013), Haltiwanger, Jarmin, and Miranda (2013), Haltiwanger, Hyatt, and McEntarfar (2014).

a particular simplifying property. Given a finite number of firm productivity states, Theorem 1 identifies equilibria in which the *finite* employment vector $\underline{N}$, where N_i is the measure of workers employed by firms in productivity state $i = 1, \dots, I$, is a sufficient statistic for the *infinitely dimensional* $G(\cdot)$ in the equilibrium value functions. The identical firms case is particularly tractable, for then unemployment U, a scalar, is sufficient for $G(\cdot)$. Indeed we fully characterize the unique equilibrium for the identical firms case with no shocks, describe analytically the impulse response of the economy to an aggregate productivity shock, and so identify the market failures implied by dynamic monopsony. Theorem 2 identifies how equilibrium can be directly computed using value function iteration with multiple firm productivity states and both aggregate and firm productivity shocks. Coles and Mortensen (2015) provide numerical examples and further insights.

Following the description of the model, Sections 3 and 4 describe the set of equilibrium wage and hiring strategies of (heterogeneous) firms and equilibrium quit strategies of workers. Section 5 describes stochastic equilibria with finite firm productivity states and fully characterizes equilibrium with identical firms and no shocks. All proofs are given in the Supplemental Material (Coles and Mortensen (2016)).

2. THE MODEL

Time is continuous and is denoted $t \in [0, \infty)$. There is a unit measure of equally productive workers who are risk neutral, infinitely lived, and discount the future at rate $r > 0$. Each is either (i) employed earning some wage w, (ii) unemployed with home production $b \geq 0$, or (iii) an entrepreneur trying to start up a new company.

Each firm $x \in [0, 1]$ is risk neutral and has a constant returns to scale technology. When firm x employs $n \in \mathbb{N}^+$ workers, its flow revenue is $np(x, \theta)$, where $\theta \in [\underline{\theta}, \overline{\theta}]$ is an aggregate productivity parameter that evolves according to a Poisson process with parameter $\alpha \geq 0$ and transition probabilities $H(\theta'|\theta)$. Output per employee $p(\cdot)$ is increasing in both arguments and $p(0, \underline{\theta}) > b$. For the moment, we assume $p(\cdot)$ is strictly increasing in x, but we relax this assumption later to consider finite productivity states. The variable x is private information to the firm.

Firms post wages and cannot precommit to future wages. As workers are equally productive, antidiscrimination legislation requires the firm to pay the same wage to employees doing the same job regardless of race, gender, sexuality, and so forth. Should an employee receive an outside job offer, the firm does not match offers. Instead the employee forms rational expectations on future wages at the two firms and takes employment at the firm that offers greater expected value. We adopt the tie-breaking convention that the worker accepts the outside offer when indifferent. There is no recall once a worker rejects a job.

There are hiring costs: if firm x with n employees hires at flow rate H, its flow hiring cost is $p(x,\theta)C(H,n)$, where $C(\cdot)$ describes foregone production due to the recruitment process. Constant returns to scale requires $C(\cdot)$ to be homogenous of degree 1. Hiring costs can thus be written as $np(x,\theta)c(h)$, where $h = H/n$ is the firm's hire rate per employee and $c(h) \equiv C(h,1)$, which is assumed to be continuously differentiable and strictly convex with $c(0) = c'(0) = 0$.

There is firm turnover: new firms are created at an exogenous rate $\mu > 0$ while existing firms die at exogenous rate $\delta(\theta) \geq 0$. At startup, a firm has one employee and productivity x is considered a random draw from the cumulative distribution function (c.d.f.) $\Gamma_0(\cdot)$. As $p(\cdot)$ is arbitrary, we normalize Γ_0 to the uniform distribution. There are independent firm specific productivity shocks that occur at rate $\gamma \geq 0$, whereupon new firm productivity $x' \in [0,1]$ is considered a random draw from c.d.f. $\Gamma_1(\cdot|x)$. Throughout we assume first order stochastic dominance in Γ_1 and $\Gamma_1(0|0) = 1$; that is, the lowest productivity state is absorbing.

Let $G(x,n)$ denote the measure of workers who are not employed plus those who are employed at firms with productivity parameter strictly less than $x \in [0,1]$ and firm size no greater than n. Thus $U = G(0,0)$ is the measure of workers who are not employed. The employment distribution $G(\cdot)$ is payoff relevant since it determines the distribution of employee quit rates. There is no explicit matching function. Instead job offers are randomly allocated across all workers, regardless of employment status.

Agents who are not employed choose either to be home producers (with flow output b) or entrepreneurs. Let $E_t \leq U_t$ denote the measure of nonemployed agents who choose to be entrepreneurs at date t. There is perfect crowding out: each entrepreneur successfully starts up a new firm at rate μ/E_t. Should an entrepreneur successfully create a new startup, he/she sells the startup company for its value and becomes the firm's first employee. In this way, each startup begins life with $n = 1$ and $x \sim \Gamma_0$. Throughout we assume μ/b sufficiently small that some unemployed workers always choose to be home producers and so $E_t < U_t$. This simplifies since it ensures no employed worker wishes to quit into unemployment to become an entrepreneur.

3. STATIONARY BAYESIAN EQUILIBRIA

In what follows, (θ, G) is the aggregate state. We consider pure strategy, Bayesian equilibria where employee beliefs on firm state x are consistent with the set of equilibrium wage strategies and Bayes rule. Suppose firm x with n' employees posts wage w'. A stationary Bayesian equilibrium requires that the firm's job offer, defined as (w', n', θ, G), is sufficient information for workers to predict future wages at the firm; that is, the firm's history of past wages provides no additional useful information.[4] Let $W' = W(w', n', \theta, G)$ denote the worker's expected value of employment given this job offer. If the worker receives an outside job offer (w'', n'', θ, G), the worker computes its value $W'' = W(w'', n'', \theta, G)$ and quits if $W'' \geq W'$. As offers are randomly allocated, $\lambda = \lambda(\theta, G)$ denotes the rate at which any individual worker receives a job offer in aggregate state (θ, G), while $F(W, \theta, G)$ denotes the probability the job offer received has value strictly less than W. These objects will be endogenously determined by aggregating across the equilibrium wage posting and recruitment strategies of firms. Given job offer (w', n', θ, G), the firm's quit rate is then

$$q(w', n', \theta, G) = \lambda(\cdot)\left[1 - F(W', \cdot)\right], \tag{1}$$

where $W' = W(w', n', \theta, G)$.

As, conditional on job offer (w', n, θ, G), the firm's quit rate $q(\cdot)$ is history independent, a firm $x \in [0, 1]$ with n employees adopts an optimal wage strategy $w = w(x, n, \theta, G)$ and an optimal hire strategy $h = h(x, n, \theta, G)$, and so generates expected discounted profit $\Pi(x, n, \theta, G)$. If the firm posts wage $w' = b$, an employee does not quit into unemployment (remaining employed at his/her current employer has a positive option value (wages may be higher tomorrow) and the worker can always quit tomorrow). As paying wage b yields strictly positive profit, equilibrium implies firms must make strictly positive profit. Denoting the value of being unemployed as $V_u(\theta, G)$, strictly positive profit further implies that any equilibrium job offer (w, n, θ, G) must yield employment value $W \geq V_u(\theta, G)$ (otherwise all workers quit into unemployment, which yields zero profit). Standard arguments thus imply the Bellman equation

$$\begin{aligned}
(r+\delta)\Pi(x, n, \theta, G) \\
= \max_{w', h' \geq 0} \Big\langle & n\left[p(x, \theta) - w'\right] - np(x, \theta)c(h') \\
& + nh'\left[\Pi(x, n+1, \cdot) - \Pi(x, n, \cdot)\right] \\
& + nq(w', \cdot)\left[\Pi(x, n-1, \cdot) - \Pi(x, n, \cdot)\right] \\
& + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left[\Pi(x, n, \theta', \cdot) - \Pi(x, n, \theta, \cdot)\right] dH(\theta' | \theta) \\
& + \gamma \int_0^1 \left[\Pi(z, n, \cdot) - \Pi(x, n, \cdot)\right] d\Gamma_1(z|x) + \frac{\partial \Pi}{\partial t} \Big\rangle,
\end{aligned} \tag{2}$$

subject to $W(w', n, \theta, G) \geq V_u(\theta, G)$, where $\partial\Pi/\partial t$ describes the total effect on $\Pi(\cdot)$ through the dynamic evolution of G.

[4] Coles (2001) considers symmetric information, but allows reputation effects. The asymmetric information approach adopted here is more natural.

3.1. *Firm-Size-Independent Strategies*

In principle, worker quit rates need not be independent of firm size n. For example with a nonconstant returns hiring technology, firm size is relevant information for employees when trying to forecast future wages.[5] Constant returns, however, implies a critical property: we can restrict attention to equilibria in which optimal firm strategies are firm-size-independent.[6] In that case, and with a slight abuse of notation, the relevant state variable $G(\cdot)$ reduces to $G(x)$, where $1 - G(x)$ is the measure of workers employed at firms with productivity no less than x.

Thus consider stationary Bayesian equilibria in which firm strategies are size-independent. As wages paid do not depend on firm size, the value of being employed $W(w', n', \theta, G)$ simplifies to $W(w', \theta, G)$. As this now implies equilibrium quit rate (1) is firm-size-independent, (2) has linear solution $\Pi(x, n, \theta, G) = nv(x, \theta, G)$, where $v(x, \theta, G)$ is the firm's profit per employee and is given by

$$\begin{aligned}
(3) \qquad & (r + \delta + \gamma + \alpha)v(x, \theta, G) \\
&= \max_{w', h' \geq 0} \Big\langle p(x, \theta) + \big[h'v(x, \cdot) - p(x, \theta)c\big(h'\big)\big] \\
&\quad - \big[w' + q\big(w', \cdot\big)v(x, \cdot)\big] + \gamma \int_0^1 v(z, \cdot)\, d\Gamma_1(z|x) \\
&\quad + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \big[v\big(x, \theta', \cdot\big)\big]\, dH\big(\theta'|\theta\big) + \frac{\partial v}{\partial t} \Big\rangle, \\
& \text{subject to} \quad W\big(w', \theta, G\big) \geq V_u(\theta, G).
\end{aligned}$$

The transversality condition necessary for a solution to this dynamic programming problem is

$$(4) \qquad \lim_{t \to \infty} E_0\big[e^{-rt} v(x_t, \theta_t, G_t) | x_0, \theta_0, G_0\big] = 0.$$

As productivity $p(\cdot)$ is strictly increasing in x and there is first order stochastic dominance in Γ_1, it follows that $v(\cdot)$ is strictly increasing in x.[7] The

[5]Section 6 in Moscarini and Postel-Vinay (2013) considers a costly vacancy posting structure where $C(v, n) = c(v)$ with $c(\cdot)$ strictly convex. Should large firms have greater worker turnover, they on average hire more and so face higher marginal vacancy costs. They are thus more willing to pay higher wages to reduce turnover.

[6]Although wages paid do not depend on firm size, Coles and Mortensen (2015) find that steady state implies wages and firm size are positively correlated.

[7]See the proof of Lemma 2 in Coles and Mortensen (2012).

Bellman equation (3) implies optimal $h = h(x, \theta, G)$ is firm-size-independent where

$$c'(h) = \frac{v(x, \theta, G)}{p(x, \theta)}. \tag{5}$$

The equilibrium dynamics are thus consistent with Gibrat's law: firm size at firm (x, n, θ, G) grows at rate $g(x, \theta, G) = h(\cdot) - q(\cdot)$, which depends on productivities (x, θ) and distribution G, but is otherwise independent of firm size.

3.2. *Signalling and Beliefs*

The quit rate (1) and Bellman equation (3) imply each firm's wage strategy solves

$$\begin{aligned} w(x, \cdot) = \arg\min_{w'} \{ w' + \lambda(\cdot)[1 - F(W(w', \cdot), \cdot)] v(x, \cdot) \} \\ \text{subject to} \quad W(w', \theta, G) \geq V_u(\theta, G). \end{aligned} \tag{6}$$

The firm's optimal wage strategy targets the quit margin: each firm trades off paying a lower wage against inducing a higher quit rate (where replacing workers who quit is costly). Lemma 1 now establishes that more productive firms post higher wages.

LEMMA 1: *In any stationary Bayesian equilibrium with firm-size independence*, $w(\cdot)$ *is nondecreasing in* x.

All proofs are given in the Supplemental Material.

Lemma 1 does not imply wages are strictly increasing in x and so does not rule out the possibility of mass points in the distribution of posted wages. The usual argument to rule out mass points is that a firm in the supposed mass point can strictly increase profit by paying a marginally higher wage. That argument, however, does not apply when there is no precommitment on future wages. The Supplemental Material describes an equilibrium with partial pooling where a positive mass of firms post the same highest wage $\overline{w}(\theta, G)$. In that example, a wage deviation $w' > \overline{w}(\theta, G)$ lies outside the support of equilibrium posted wages and so Bayes rule does not then apply. By specifying workers believe $x = 0$ when $w' > \overline{w}(\theta, G)$, Lemma 1 implies employees expect the deviating firm will thereafter announce the lowest market wage $\underline{w}(\theta, G)$. As the deviating higher wage *reduces* expected $W(\cdot)$, (6) implies any such deviation is never optimal. Thus equilibria with mass points exist.

Equilibria with mass points require that employees may perceive themselves as worse off (in expectation) when offered a pay raise and so such pay raises are not offered in equilibrium. To rule out such perverse expectation formation, we restrict out-of-equilibrium beliefs as follows. Let $F^p(x|w', \theta, G)$ denote the worker's posterior belief that the firm's productivity is no greater than $x \in [0, 1]$ given job offer (w', θ, G). For any two wages $w' > w''$ in the support of equilibrium posted wages, Lemma 1 and Bayes rule imply $F^p(x|w', \theta, G)$ first order stochastically dominates $F^p(x|w'', \theta, G)$. The rest of the paper restricts beliefs to have this property for all wages.

- *Monotone Beliefs*: Given any two wages w', w'' with $w' > w''$, then $F^p(x|w', \theta, G)$ first order stochastically dominates $F^p(x|w'', \theta, G)$.

Given any two job offers (w', θ, G) and (w'', θ, G) with $w' > w''$, monotone beliefs imply firm w' is not only believed to have a current productivity that first order stochastically dominates the lower wage firm, first order stochastic dominance in Γ_1 and Lemma 1 imply the predicted distribution of wages paid at the high wage firm, at any future date, first order stochastically dominates those of the lower wage firm. Thus given current wage $w' > w''$, the worker strictly prefers employment at the higher wage firm. As this implies $W(\cdot)$ is increasing in w, the no-recall assumption further implies unemployed workers adopt a reservation wage strategy: each accepts any job offer (w', θ, G) satisfying $w' \geq R(\theta, G)$, where $W(R, \theta, G) = V_u(\theta, G)$. Lemma 2 now shows that a stationary Bayesian equilibrium with monotone beliefs must be fully separating: that higher productivity firms post strictly higher wages (no mass points) and enjoy strictly lower quit rates.

LEMMA 2—Equilibrium Beliefs and the Reservation Wage: *In any state* (θ, G), *a stationary Bayesian equilibrium with firm-size independence and monotone beliefs implies the following statements*:

(i) *The distribution of posted wages is continuous and has connected support*.

(ii) *Equilibrium wage strategies* $w(x, \theta, G)$ *are strictly increasing in* $x \in [0, 1]$, *where the lowest wage paid is* $w(0, \theta, G) = R(\theta, G) = b$.

(iii) *Given any job offer* (w', θ, G), *employees believe* $x = \widehat{x}(w', \theta, G)$, *where* $\widehat{x} \in [0, 1]$ *solves*

$$w(\widehat{x}, \theta, G) = w' \quad \text{when} \quad w' \in \big[b, w(1, \theta, G)\big],$$
$$\widehat{x} = 0 \quad \text{when} \quad w' < b,$$
$$\widehat{x} = 1 \quad \text{when} \quad w' > w(1, \theta, G).$$

(iv) *An employee on wage* $w' \geq b$ *quits to any outside offer* $w'' \geq w'$.

(v) *An employee on wage* $w' < b$ *quits into unemployment*.

Lemma 2(ii) establishes the lowest productivity ($x = 0$) firm posts a wage equal to the reservation wage $R(\theta, G)$, which equals b. This latter result requires workers to have equal access to the same job offer technology. A different approach might instead assume employed workers receive job offers at a lower exogenous rate $\lambda_1 < \lambda$. In that case, the reservation wage R solves

$$R(\theta, G) = b + (\lambda - \lambda_1) \int_R^{\overline{w}} \big[W(w, \cdot) - V_u(\cdot)\big]\, dF(w, \cdot). \tag{7}$$

Assuming $\lambda_1 < \lambda$ helps in numerical work because (empirically) worker quit rates are low relative to unemployed worker job finding rates. But this assumption distorts the wage equation since it implies that the unemployed enjoy more efficient search and thus $R > b$. This latter property is unreasonable because, if anything, network effects suggest employed workers have better access to the job search technology. The tension arises due to the assumption of exogenous job search effort.[8] Unfortunately allowing endogenous job search effort within an equilibrium stochastic framework is too complex. By assuming all workers have equal access to the same job offer technology, this not only makes the analysis tractable, it yields "sticky" wages over the cycle since the infimum of the wage distribution is tied down to b. The model partly captures the fact that unemployed workers enjoy high reemployment rates relative to employed worker quit turnover as, here, those rates are $\lambda + \mu/E$.[9]

4. DEFINITION AND CHARACTERIZATION OF EQUILIBRIUM

As firm (x, n, θ, G) hires at rate $H = nh(x, \theta, G)$, it must make job offers at rate $H/G(x)$ since, with random offers, its job offer is only accepted with probability $G(x)$. Aggregating job offer rates across all firms implies each worker receives a job offer at rate

$$\lambda(\theta, G) = \int_0^1 \frac{h(x, \theta, G)}{G(x)}\, dG(x), \tag{8}$$

where $dG(x)$ describes the measure of workers employed at type x firms. Let $\widehat{F}(x, \theta, G)$ denote the fraction of job offers made by firms with productivity no greater than x in aggregate state (θ, G). The same aggregation argument implies

$$\lambda\big[1 - \widehat{F}(x, \cdot)\big] = \int_x^1 \frac{h(z, \theta, G)}{G(z)}\, dG(z). \tag{9}$$

Equation (9) thus describes the equilibrium quit rate at firm (believed to be) x in state (θ, G) and so implies (10) below. We can now formally define equilibrium.

[8]Endogenous job search effort with equal access to the job search technology not only implies the unemployed have a higher arrival rate of job offers (via greater search effort), but also $R = b$ (e.g., Lise (2013)).

[9]Equation (S4) in the Supplemental Material establishes equilibrium

$$E(\theta, G) = \frac{\mu}{b} \int_0^1 \big[v(x, \cdot) + W\big(w(x), \cdot\big) - V_u(\cdot)\big]\, d\Gamma_0(x),$$

where we assume μ/b sufficiently small that $E < U$ along the equilibrium path (i.e., not all unemployed workers try to start up new businesses).

DEFINITION OF EQUILIBRIUM: A stationary Bayesian equilibrium with firm-size independence and monotone beliefs is the set $\langle v, h, q, w, \widehat{x}\rangle$ such that for all $x \in [0, 1]$ and (θ, G), the following statements hold:

(Di) Employee value $v(x, \theta, G)$ satisfies (3).

(Dii) Hire strategy $h(x, \theta, G)$ satisfies (5).

(Diii) Quit function $q(w', \theta, G)$ satisfies

$$q\big(w', \theta, G\big) = \int_{\widehat{x}(w',\cdot)}^{1} \frac{h(z, \theta, G)}{G(z)}\, dG(z). \tag{10}$$

(Div) Wage strategy $w(x, \theta, G)$ satisfies

$$w(x, \theta, G) = \arg\min_{w' \geq b}\big[w' + q\big(w', \cdot\big)v(x, \cdot)\big]. \tag{11}$$

(Dv) Beliefs $\widehat{x}(w', \theta, G)$ are given by Lemma 2(iii).

(Dvi) State (θ, G) is a Markov process that evolves consistently with the equilibrium hire and quit strategies.

For the case that the density of $G(\cdot)$ exists,[10] Proposition 1 describes the equilibrium wage equation.

PROPOSITION 1: *If G is differentiable*, *equilibrium $w(\cdot)$ is the solution to the initial value problem*

$$\frac{\partial w}{\partial x} = \frac{h(x, \cdot)G'(x)}{G(x)} v(x, \cdot) \quad \textit{subject to} \quad w(0, \cdot) = b. \tag{12}$$

As firms make strictly positive profit, the solution to this initial value problem implies equilibrium wage strategies $w(x, \cdot)$ are strictly increasing in x and so are indeed fully revealing. At its equilibrium wage $w = w(x, \cdot)$, firm x has quit rate $\int_x \frac{h(z,\cdot)G'(z)}{G(z)}\, dz$ and so $[-\frac{h(x,\cdot)G'(x)}{G(x)}]$ describes its marginal quit rate. Equation (12) describes the optimal monopsony wage of firm x: the marginal cost to paying a higher wage to an incumbent employee equals its marginal return, which is the marginal fall in the employee's quit rate times the firm's value $v(\cdot)$ to the continued relationship.

As a worker's outside offer is a random draw from the set of hiring firms, the equilibrium wage equation $w(\cdot)$ has the structure of a first price auction with private independent values. Less productive firms, those with $x_L < x$ and thus

[10] If G is not differentiable, then

$$dw = h(x, \cdot)v(x, \cdot)d\big[\log G(x)\big].$$

employee value $v_L < v(x,\cdot)$, bid a low wage $w_L = w(x_L,\cdot)$. At that wage point w_L, firm $x > x_L$ finds the marginal return to a lower quit rate,

$$\frac{h(x_L,\cdot)G'(x_L)}{G(x_L)}v(x,\cdot) > \frac{h(x_L,\cdot)G'(x_L)}{G(x_L)}v_L = \frac{\partial w(x_L)}{\partial x},$$

exceeds the marginal cost to raising its wage: it thus bids a higher wage $w > w_L$. Of course it does not raise wage beyond its equilibrium wage $w(x,\cdot)$ because beyond that point, the increased retention rate no longer compensates for the higher wage paid. In a static one-shot first price auction, the information revealed by an optimal bid plays no role. Here with repeated trade and in a stationary Bayesian equilibrium, the equilibrium wage w reveals the firm's current productivity $x = \widehat{x}(w,\cdot)$, which, in turn, yields sufficient information for workers to predict future wages.

5. CHARACTERIZATION OF STOCHASTIC EQUILIBRIA

Our companion paper (Coles and Mortensen (2015)) fully characterizes steady state equilibria with a continuum of firm states and compares the market outcome to the competitive allocation. Here the focus is on characterizing stochastic equilibria. The key difficulty is that the state space, in general, includes $G(\cdot)$.

Suppose from now on finite $I \geq 1$ firm productivities. As Γ_0 is normalized to the uniform distribution, we partition the interval $[0,1]$ into a grid $[x_{i-1}, x_i)$ with $x_0 = 0$, $x_i = x_{i-1} + \gamma_{0i}$, and $x_I = 1$, where $\gamma_{0i} > 0$ describes the probability a new startup draws productivity $x \in [x_{i-1}, x_i)$. Firms with $x \in [x_{i-1}, x_i)$ are referred to as state i firms where each has the same productivity $p(x,\theta) = p_i(\theta)$ that is strictly increasing in $i = 1,\dots,I$ and face independent firm productivity shocks that occur at rate γ with transition probabilities $\pi_{ii'}$ that satisfy first order stochastic dominance and $\pi_{11} = 1$.

A stationary Bayesian equilibrium implies all state i firms enjoy the same value $v = v_i(\theta, G)$. There is, however, wage dispersion between state i firms because state i firms that pay higher wages enjoy lower quit rates. As we do not consider mixed strategy equilibria, we must describe how firms select those wage strategies. To that end, define rank $\chi \in [0,1]$ of firm $x \in [x_{i-1}, x_i)$ as $\chi = [x - x_{i-1}]/[x_i - x_{i-1}]$. As Γ_0 is uniformly distributed, each firm's initial draw x yields a state i and a rank $\chi \sim U[0,1]$. We assume that should a state i firm with $x \in [x_{i-1}, x_i)$ receive a productivity shock and so enter state i', realized $x' = x_{i'-1} + \chi[x_{i'} - x_{i'-1}]$ is rank-χ-preserving.[11] Such a shock process identifies a $\Gamma_1(\cdot)$ consistent with the previous section. We now characterize the

[11]Although allowing rank shocks (consistent with first order stochastic dominance in Γ_1) is possible, and so firms might select different pure wage strategies over time, such a variation does not affect the characterization of equilibrium below and so is omitted.

corresponding stationary Bayesian equilibrium where $w(\cdot)$ is a strictly increasing function of x; that is, state i firms select wage strategies that increase by rank χ. We first describe equilibrium off the equilibrium path.

Suppose state i firm $x \in [x_{i-1}, x_i)$ deviates to any wage $w' \neq w(x, \theta, G)$. The definition of equilibrium implies employees update to a unique (history independent) belief $\widehat{x}(\cdot) \in [0, 1]$. Corresponding to that belief $\widehat{x}$ is a unique state $\widehat{i}$ and corresponding rank $\widehat{\chi} \in [0, 1]$ such that $x_{\widehat{i}-1} + \widehat{\chi}[x_{\widehat{i}} - x_{\widehat{i}-1}] = \widehat{x}$. As a stationary Bayesian equilibrium implies the current wage is sufficient information to predict future wages, workers therefore predict future wages based on this updated rank $\widehat{\chi}$. Given its realized state i', the firm's equilibrium wage strategy, in the continuation, must therefore be to post wage $w = w(x', \cdot)$ with $x' = x_{i'-1} + \widehat{\chi}[x_{i'} - x_{i'-1}]$; that is, each firm posts wage $w(\cdot)$ consistent with its current state i' and rank $\widehat{\chi}$. The equilibrium is dynamically consistent for:

(i) should a firm change its wage, say, to a lower rank $\widehat{\chi} < \chi$, monotone beliefs imply that its employees expect the firm to announce wages consistent with that lower rank in the entire future (and amend their quit strategies accordingly). As the firm is indifferent to adopting this wage strategy, it is an equilibrium that it subsequently posts wage w consistent with rank $\widehat{\chi}$, and

(ii) the proof of Proposition 1 establishes that posting wage $w \notin [w(x_{i-1}, \cdot), w(x_i, \cdot)]$ is strictly payoff decreasing.

Of course along the equilibrium path, the firm announces wages consistent with its rank χ allocated at startup.

Given such behavior off the equilibrium path, we now describe the equilibrium path. Let $\underline{v} = (v_1, v_2, \dots, v_I)$ denote the vector of type i firm values in a stationary Bayesian equilibrium. Define the generic type i hire function

$$h_i^*(v, \theta) = \arg\max_{h'}\big[h'v - p_i(\theta)c\big(h'\big)\big],$$

and note that $h_i^*(v, \theta)$ with $v = v_i$ describes the optimal hiring rate of type i firms. Let $N_i = G(x_i) - G(x_{i-1})$ denote total employment in state i firms, let $\underline{N}$ denote the corresponding employment vector where $U = 1 - \sum_i N_i$.

We solve for the equilibrium recursively, starting with the lowest rank $\chi = 0$, state $i = 1$ firm. Equation (10) implies its quit rate is

$$q_1 = \int_0^1 \frac{h(z, \cdot)\, dG(z)}{G(z)} = \sum_{i=1}^{I}\left[\int_{x_{i-1}}^{x_i} \frac{h_i^*(v_i, \theta)\, dG(z)}{G(z)}\right] = \sum_{i=1}^{I} h_i^*(v_i, \theta) \ln\left[\frac{U + \sum_{j=1}^{i} N_j}{U + \sum_{j=1}^{i-1} N_j}\right]. \tag{13}$$

Note the key simplification: the equilibrium quit function $q_1(\cdot)$ depends on $\underline{N}$ but is otherwise independent of $G(\cdot)$. As this firm pays wage $w_1 = b$, the Bellman equation (3) implies v_1 evolves according to (14) in Theorem 1 below with $i = 1$. The wage equation (Proposition 1) implies the highest rank $\chi = 1$, state 1 firm pays wage

$$w_2 = b + v_1 h_1^*(v_1, \theta)\left[\ln \frac{U + N_1}{U}\right].$$

Now consider the lowest rank state 2 firm. The lowest rank state 2 firm does not pay wage $w' < w_2$, as otherwise its employees believe it is a state 1 firm, which then triggers a too high quit rate. Thus the lowest rank state 2 firm posts wage w_2.[12] Forward induction now establishes the result.

THEOREM 1: *A stationary Bayesian equilibrium with firm-size independence, monotone beliefs, and finite firm productivity states implies $\underline{N}$ is a sufficient statistic for $G(\cdot)$, where equilibrium vector values $\underline{v} = \underline{v}(\underline{N}, \theta)$ satisfy*

$$(14) \qquad \big(r + \delta(\theta) + q_i\big)v_i = p_i(\theta) - w_i + \max_{h\geq 0}\big\{hv_i - p_i(\theta)c(h)\big\} + \alpha \int \big[v_i\big(\theta', \cdot\big) - v_i(\theta, \cdot)\big]\, dH\big(\theta'|\theta\big) + \gamma \sum_{i'=1}^{I} \pi_{ii'}(v_{i'} - v_i) + \sum_j \frac{\partial v_i}{\partial N_j}\dot{N}_j$$

for $i = 1, \dots, I$, with

$$(15) \qquad w_i = b + \sum_{j=1}^{i-1} v_j h_j^*(v_j, \theta)\left[\ln \frac{U + N_1 + \cdots + N_j}{U + N_1 + \cdots + N_{j-1}}\right],$$

$$(16) \qquad q_i = \sum_{j=i}^{I} h_j^*(v_j, \theta) \ln\left[\frac{U + N_1 + \cdots + N_j}{U + N_1 + \cdots + N_{j-1}}\right],$$

and equilibrium turnover

$$(17) \qquad \dot{N}_i = \mu\gamma_{0i} + [U + N_1 + \cdots + N_{i-1}]h_i^* \ln\left[\frac{U + N_1 + \cdots + N_i}{U + N_1 + \cdots + N_{i-1}}\right] - \delta N_i - q_{i+1}N_i + \sum_{j\neq i}\gamma_{ji}N_j - \sum_{j\neq i}\gamma_{ij}N_i.$$

[12]If, in equilibrium, the wage paid by the lowest rank type 2 firms is strictly greater than w_2, then monotone beliefs imply posting wage w_2 is a profitable deviation (its quit rate is unchanged and it makes greater flow profit).

A general existence proof of stationary Bayesian equilibria is not currently available. Below we fully describe the unique equilibrium when $I = 1$ and $\alpha = 0$. In related work, Robin (2011) and Moscarini and Postel-Vinay (2013) provide an existence proof when contact rates evolve exogenously, described by a pair $(\lambda_0(\theta), \lambda_1(\theta))$. We can prove existence for the analogous case that hiring rates are exogenous (see Coles and Mortensen (2012)). Exogenous contact rates, however, are not the interesting case.

Theorem 2 identifies a more useful characterization of equilibrium. Specifically integrating (14) forward over an (arbitrarily small) time period $\Delta > 0$ yields (18).

THEOREM 2: *Over* (*arbitrarily small*) *time period* $\Delta > 0$, *a stationary Bayesian equilibrium with firm-size independence*, *monotone beliefs*, *and finite firm productivity states implies*

$$v_i(\underline{N}, \theta) = \max_{h_i \geq 0} \Bigg\{ \Bigg[p_i(\theta) - w_i + h_i v_i(\underline{N}, \theta) - p_i(\theta) c(h_i) + \alpha \int v_i(\underline{N}, \theta')\, dH(\theta'|\theta) + \gamma \sum_{j=1}^{I} \pi_{ij} v_j(\underline{N}, \theta) \Bigg] \Delta + e^{-(r+\delta+q_i+\alpha+\gamma)\Delta} v_i(\underline{N}^{\Delta}, \theta) \Bigg\} \tag{18}$$

for $i = 1, \ldots, I$, *with* $\underline{N}^{\Delta} = \underline{N} + \dot{\underline{N}}\Delta$, *and* w_i, q_i, *and* $\dot{\underline{N}}$ *given by* (15)–(17) *in Theorem* 1.

Equation (18) describes a system of recursive equations for $v_i(\underline{N}, \theta)$ that can be directly computed using value function iteration. Specifically by using (15) and (16) to substitute out w_i and q_i, the map is directly analogous to that of a stochastic capital accumulation model with vector of "assets" $\underline{N}$ but equilibrium investment strategies $\underline{h}$ are determined non-cooperatively. Coles and Mortensen (2015) solve this system numerically with $I = 3$ and identify the impulse response of the economy to an unforseen aggregate productivity shock. The following discussion illustrates those insights analytically for the identical firms case.

5.1. *The Homogenous Firms Case*

Suppose $I = 1$ and $\alpha = 0$ (no aggregate shocks), and so let $p(\cdot) = \overline{p}$ for all $x \in [0, 1]$. Theorem 1 implies unemployment U is a sufficient statistic for $G(\cdot)$ and so let equilibrium $v = v(U)$. As each firm recruits at the same (generic) rate $h^*(v)$, (8) implies aggregate job offer rate

$$\lambda = \int_0^1 \frac{h^*(v)\, dG(z)}{G(z)} = -h^*(v) \ln U,$$

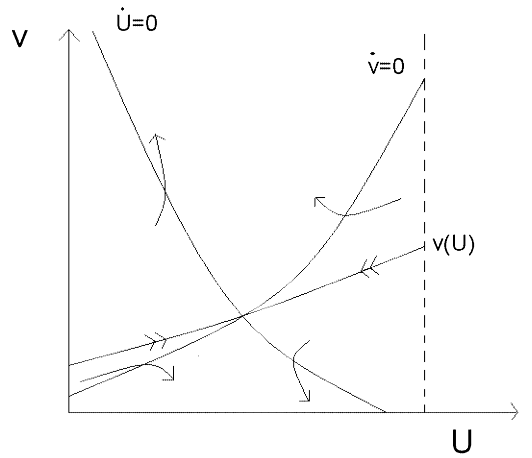

FIGURE 1.—Phase Diagram (v, U).

which depends on U but is otherwise independent of G. Putting $x = 0$ in (3) and letting $\dot{v} = [\partial v/\partial U]\dot{U}$ yields

$$\dot{v} = \big[r + \delta - h^*(v)\ln U\big]v - \Big(\overline{p} - b + \max_{h\geq 0}\big[hv - c(h)\big]\Big), \tag{19}$$

where unemployment evolves according to

$$\dot{U} = \delta(1 - U) + \big[h^*(v)\ln U\big]U - \mu. \tag{20}$$

Equations (19) and (20) describe an autonomous differential equation system for (v, U). Coles and Mortensen (2012) establish that this system has a unique steady state that is a saddle. Figure 1 depicts its phase diagram. Because v on a solution trajectory above the stable saddle path fails the transversality condition (4), while any one below the steady state ultimately yields zero v (which contradicts strictly positive profit), the stable saddle path describes the unique stationary equilibrium.

Equations (19) and (20) reveal the underlying market structure: equilibrium requires that each firm's optimal hiring rate h^* is a best response to the quit rates induced by current (and expectations of future) competitor hiring rates h^*. As firms are equally productive and recruitment is costly, all quits are socially inefficient. Indeed, even with heterogeneous firms, Coles and Mortensen (2015) show the competitive (steady state) equilibrium implies all firms pay the same market clearing wage and there is no job-to-job turnover. Dynamic monopsony instead generates excess quit turnover because firms trade off paying lower wages against higher quit rates and equilibrium wage dispersion implies workers in low paid employment quit for better wages.

The phase diagram demonstrates the economy is stable but the adjustment dynamics are potentially slow. Suppose the economy is in steady state and consider the impulse response of the economy to a permanent (unforseen) increase in productivity, say from $\overline{p}$ to $\theta\overline{p}$ with $\theta > 1$. On the phase diagram, this shifts up the $\dot{v} = 0$ locus, but does not affect the $\dot{U} = 0$ locus. At the initial level of unemployment, the positive productivity shock implies the economy jumps to a higher saddle path, which implies an initial increase in both $v = v(U, \theta)$ and firm recruitment rates $h^*(v, \theta)$. Computing gross flows (when $I = 1$) yields

$$\text{gross hire flows} = \int_0^1 h^*(v, \theta)\, dG(x) = [1 - U]h^*(v, \theta),$$

$$\text{gross quit flows} = \int_0^1 q(x, \theta)\, dG(x) = [1 - U + U \log U]h^*(v, \theta).$$

The increase in aggregate productivity θ not only increases gross hires, but the increased hiring rate h^* also triggers an increase in gross quits because already employed workers seek better wages. As a job-to-job transition implies a job is destroyed at the quitter's previous employer, such turnover implies a more modest increase in net job creation. Endogenous quit turnover thus makes unemployment more persistent along the equilibrium adjustment path.[13] As unemployment declines along that path, the tightening labor market implies both v and $h^*(v, \cdot)$ fall over time.

6. CONCLUSION

This paper has identified a tractable stochastic equilibrium model of job and labor flows that seems ideal for both macropolicy applications and microempirical analysis. We have formally shown how dynamic monopsony generates excessive quit turnover. As this crowds out the reemployment prospects of the unemployed, equilibrium unemployment levels are not efficient. Tractability is obtained via three key assumptions: that firm recruitment costs exhibit constant returns to scale (so that firm policies are firm-size-independent), that all workers have equal access to the job offer technology (so that $R = b$), and a finite number of firm productivity states.

As there is no small surplus assumption, small productivity shocks here do not generate large employment fluctuations. Robin (2011) and Menzio and Shi (2011) both find the main channel for generating large unemployment volatility in this class of model is through the destruction of marginal jobs in recessions. Even so, Robin (2011) points out that endogenous job destruction (driven by a single aggregate productivity variable) is too lumpy to fit the data well (excess kurtosis). Allowing a more flexible exogenous job (or firm) destruction process

[13]The numerical examples in Coles and Mortensen (2015) suggest a half-life of 20 months.

$\delta = \delta(x, \theta)$, where low productivity jobs are more likely to be destroyed in low aggregate states, would seem an important direction for future research.

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