The Effects of Sampling Frequency on Detrending Methods for Unit Root Tests

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Abstract

This paper analyses the effects of sampling frequency on detrending methods based on an underlying continuous time representation of the process of interest. Such an approach has the advantage of allowing for the explicit – and different – treatment of the ways in which stock and flow variables are actually observed. Some general results are provided before the focus turns to three particular detrending methods that have found widespread use in the conduct of tests for a unit root, these being GLS detrending, OLS detrending, and first differencing, and which correspond to particular values of the generic detrending parameter. In addition, three different scenarios concerning sampling frequency and data span, in each of which the number of observations increases, are considered for each detrending method. The limit properties of the detrending coefficient estimates, as well as an invariance principle for the detrended variable, are derived. An example of the application of the techniques to testing for a unit root, using GLS detrending on an intercept, is provided and the results of a simulation exercise to analyse the size and power properties of the test in the three different sampling scenarios are reported.

Keywords. Continuous time; detrending; sampling frequency.

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1. Introduction

It has become common in recent econometric practice to implement some form of detrending procedure prior to carrying out a test for a unit root in an observed time series. The three most widely used methods are detrending by first differencing, ordinary least squares (OLS) detrending, and generalised least squares (GLS) detrending, the latter having become popular since the work of Elliott, Rothenberg and Stock (1996). Once the data have been detrended it is then a matter of carrying out a unit root test using the detrended data, provided that the appropriate limit distribution is used to determine the critical value for the test. This is because the process of detrending affects the data used for constructing the test statistic and, hence, the form of the invariance principle that is used to describe the limit distribution.

In many situations researchers are increasingly faced with a choice of sampling frequencies with which to work, due to the rapidly expanding availability of time series data. Such choices are not necessarily innocuous, however, and a number of investigations have been carried out to assess the effects of sampling frequency and data span on the properties of estimators and test statistics. In the context of unit root testing, Perron (1991) demonstrated that, for the case of a stock variable observed at equispaced points in time, test power was influenced more by the data span than the number of observations *per se*, while Chambers (2004, 2008) carried out a similar analysis for the case of a flow variable observed as a sequence of integrals and also showed that test consistency requires an increasing data span.[1] Neither of these studies, however, considered deterministic components in the underlying processes and, hence, there was no need for any form of detrending. But given the prominent role that detrending now occupies in the field of unit root testing it would seem apposite to ascertain the effects of sampling frequency and data span on the different detrending methods available to researchers.

The main aim of this paper is, therefore, to derive the limit properties of the estimated coefficients in the detrending regressions and thereby to determine invariance principles for the resulting detrended data. The underlying model is formulated in continuous time which has two main advantages over a discrete time formulation for the analysis of the effects of sampling frequency. The first is that the underlying model is not tied to any particular (arbitrary) sampling frequency. The second advantage is that the form of model satisfied by the discrete time observations is invariant to their sampling frequency, a feature that is not necessarily true when aggregating a discrete time process; see, for example, the results in Brewer (1973), Weiss (1984) and Marcellino (1999). The continuous time framework is also ideally suited to handling the different ways in which measurements of stock and flow variables are made, the former being recorded at discrete (equispaced) points in time, the latter as integrals over the observation interval.

The general form of detrending regression itself extends the ideas of Chambers (2015b) and is also formulated in continuous time. Its discrete time equivalent, satisfied by the observations exactly, is presented in Theorem 1. The three particular detrending methods considered then correspond to particular values of the generic detrending parameter. The case of an observed stock variable is treated in detail in section 3. The asymptotic properties

[1] The importance of increasing span for estimator consistency has also been established in the context of cointegration by Chambers (2011).

are derived for three different sampling schemes corresponding to different scenarios for data span and sampling frequency. The invariance principle which drives subsequent results is presented in Lemma 2 and its implications for the (unobserved) detrended process in discrete time are outlined in Lemma 3. The main results for the three detrending methods then follow and are given in Theorems 2–4. Section 4 considers the case of detrending a flow variable which affects the form of the underlying invariance principle (Lemma 4), mainly through the effect of aggregation on the serial correlation properties. It is shown that the results for a stock variable continue to hold, with one minor exception, provided the appropriate long run variance is used. An application of the results using GLS detrending on an intercept with a stock variable is provided in section 5 in which the performance of a test statistic, based on the normalised autoregressive coefficient estimator, is assessed across the three sampling schemes in a simulation exercise. The simulation results are found to be in accordance with the predictions of the theory. Some concluding comments are provided in section 6. Three appendices are provided that contain proofs of theorems (Appendix A), proofs of lemmas (Appendix B), and statements and proofs of supplementary lemmas (Appendix C). Throughout, stock variables are represented by lower case characters (e.g. y) and flow variables by upper case (e.g. Y).

2. The model, detrending methods and some preliminary results

2.1. The model

The continuous time process of interest, $y_c(t)$, is assumed to consist of a (deterministic) trend component, $\psi' z(t)$, and a stochastic component, $u(t)$, the latter containing a potential unit root driven by a stationary process, $\eta(t)$, More formally the model is given by

$$y_c(t) = \psi' z(t) + u(t), \quad t > 0, \tag{1}$$
$$du(t) = \alpha u(t)dt + \eta(t)dt, \quad t > 0, \tag{2}$$

where α denotes the continuous time autoregressive parameter and (1) and (2) are initialised by $y_c(0)$ and $u(0)$, respectively. The parameter α is the object of interest in unit root testing scenarios and we shall be more precise about its specification in subsequent sub-sections. It is also assumed that $z(t) = [1, t, t^2, \ldots, z^m]'$ and ψ is an $(m+1) \times 1$ vector of parameters. More specifically we will focus on the cases $m = 0$ and $m = 1$ so that, respectively,

$$\psi' z(t) = \begin{cases} \psi_0, & m = 0, \\ \psi_0 + \psi_1 t, & m = 1; \end{cases} ; \tag{3}$$

higher-order polynomials could be considered but are rarely used in practice. Now suppose that $\hat{\psi}$ is an estimate of ψ. We can then define the detrended process in continuous time as

$$y(t) = y_c(t) - \hat{\psi}' z(t) = u(t) - (\hat{\psi} - \psi)' z(t),$$

which converges to the unobservable process $u(t)$ if $\hat{\psi}$ is a consistent estimator of ψ. We shall analyse such properties in what follows and derive the limit properties of the detrended discrete time equivalents of $y(t)$.

For the purposes of analysing the properties of various detrending methods under different sampling schemes it is not necessary to assume anything more specific about the process $\eta(t)$ other than it being stationary (and functionals of it satisfying an invariance principle; see below). For the conduct of unit root tests in the observed data, however, it may be necessary to make additional assumptions, depending on the type of test being conducted. For example, the tests of Phillips and Perron (1988) would not require further assumptions, while the tests of Said and Dickey (1984) would rely on a parameteric specification of the dynamics governing $\eta(t)$. In a continuous time framework Chambers (2015b) has considered unit root tests based on a discrete time series of skip-sampled (stock) data generated according to (1) and (2) with $\eta(t)$ satisfying the CARMA(p, q) specification[2]

$$\phi(D)\eta(t) = \theta(D)\epsilon(t),$$

where $\epsilon(t)$ is a continuous time white noise, D denotes the mean square differential operator, $\phi(z) = z^p + \phi_{p-1}z^{p-1} + \ldots + \phi_1 z + \phi_0$ and $\theta(z) = 1 + \theta_1 z + \ldots + \theta_{q-1}z^{q-1} + \theta_q z^q$. This specification leads to discrete time ARMA dynamics in the discrete time equivalent of (2) which is specified below.

We shall assume that a sequence of discrete time observations is available and that the *sampling interval* is h; this is the length of time between successive observations on stock variables and the interval of time over which the observations on flow variables are recorded. The corresponding *sampling frequency* is $1/h$. We also assume that the time *span* over which observations are recorded is denoted N, implying that the number of observations is $T = N/h$. We shall consider two types of variables, stocks and flows, whose discrete time observations are determined by:

$$\text{Stocks: } y_{th} = y_c(th), \quad t = 1, \ldots, T;$$

$$\text{Flows: } Y_{th} = \frac{1}{h}\int_{th-h}^{th} y_c(r)dr, \quad t = 1, \ldots, T.$$

For a stock variable[3] the observed sequence is therefore $y_h, y_{2h}, \ldots, y_{Th} = y_N$, while for a flow variable the observed sequence is given by $Y_h, Y_{2h}, \ldots, Y_{Th} = Y_N$. The properties of the discrete time observations generated by the underlying continuous time system (1) and (2) are given below.

Lemma 1. *Let $y_c(t)$ be generated by (1) with $\psi' z(t) = \psi_0 + \psi_1 t$. Then discrete time observations on stock variables satisfy*

$$y_{th} = \psi_0 + \psi_1 th + u_{th}, \quad t = 1, \ldots, T, \tag{4}$$

where $u_{th} = u(th)$ is determined by the stochastic difference equation

$$u_{th} = e^{\alpha h}u_{th-h} + v_{th}, \quad v_{th} = \int_{th-h}^{th} e^{\alpha(th-r)}\eta(r)dr, \quad t = 1, \ldots, T. \tag{5}$$

[2]It is also assumed that $q < p$; this condition ensures that the spectral density function of $\eta(t)$ is integrable and, therefore, that $\eta(t)$ has finite variance.

[3]It would also be possible to observe $y_0 = y(0)$ in principle for a stock variable, resulting in $T + 1$ observations, although for convenience we assume the observations start at $t = 1$.

For flow variables the discrete time observations satisfy

$$Y_{th} = \psi_0 + \psi_1 \left(th - \frac{h}{2} \right) + U_{th}, \quad t = 1, \ldots, T, \tag{6}$$

where the dynamics of U_{th} are governed by

$$U_h = \left(\frac{e^{\alpha h} - 1}{\alpha h} \right) u(0) + V_h, \quad V_h = \frac{1}{h} \int_0^h \int_0^s e^{\alpha(s-r)} \eta(r) dr ds, \tag{7}$$

$$U_{th} = e^{\alpha h} U_{th-h} + V_{th}, \quad V_{th} = \frac{1}{h} \int_{th-h}^{th} \int_{s-h}^{s} e^{\alpha(s-r)} \eta(r) dr ds, \quad t = 2, \ldots, T. \tag{8}$$

Alternatively, if $\psi' z(t) = \psi_0$, the discrete time representations above are still valid but with $\psi_1 = 0$ in (4) and (6).

Lemma 1 shows that the discrete time observations reflect the linear trend inherent in (1), although in the case of a flow variable the process of integration results in the trend being evaluated at the mid-point of the sampling interval i.e. $th - (h/2)$ rather than th itself. An implication of normalising the flow variables by $(1/h)$ is that the linear trend in continuous time is transformed into a linear trend (subject to the adjustment mentioned above) in discrete time. Without this normalisation the linear trend would be of the form $\psi_0 h + \psi_1 \left(th^2 - (h^2/2) \right)$.

The processes u_{th} and U_{th} appearing in Lemma 1 are driven by the stationary processes v_{th} and V_{th}, respectively, whose precise properties depend on the underlying process $\eta(t)$. In cases where $\eta(t)$ is CARMA(p, q) (with $q < p$) it can be shown (see Chambers and Thornton, 2012) that v_{th} is ARMA$(p, p-1)$ and V_{th} is ARMA(p, p). The additional order of moving average component in V_{th}, compared to v_{th}, becomes apparent by noting that V_{th} can be written in the form (assuming $\alpha \neq 0$)

$$V_h = \frac{1}{h} \int_0^h \left(\frac{e^{\alpha(h-r)} - 1}{\alpha} \right) \eta(r) dr,$$

$$V_{th} = \frac{1}{h} \int_{th-h}^{th} \left(\frac{e^{\alpha(th-r)} - 1}{\alpha} \right) \eta(r) dr + \frac{1}{h} \int_{th-2h}^{th-h} \left(\frac{e^{\alpha h} - e^{\alpha(th-h-r)}}{\alpha} \right) \eta(r) dr,$$

the second expression holding for $t = 2, \ldots, T$. This representation is obtained by changing the orders of integration in the double-integral representation in Lemma 1 as in, for example, McCrorie (2000). When $\alpha = 0$ the appropriate representation can be found by taking limits using the series expansion of e^x; we find that

$$\frac{e^{\alpha(th-r)} - 1}{\alpha} = (th - r) + O(\alpha), \quad \frac{e^{\alpha h} - e^{\alpha(th-h-r)}}{\alpha} = -(th - 2h - r) + O(\alpha)$$

and hence, in this case,

$$V_h = \frac{1}{h} \int_0^h (h - r) \eta(r) dr,$$

$$V_{th} = \frac{1}{h} \int_{th-h}^{th} (th - r) \eta(r) dr - \frac{1}{h} \int_{th-2h}^{th-h} (th - 2h - r) \eta(r) dr, \quad t = 2, \ldots, T,$$

which demonstrates the additional source of moving average in the limit when $\alpha = 0$.

2.2. Detrending methods

The objective is to detrend the observations so that the discrete time detrended series is consistent with detrending the underlying continuous time series under the given trend specification. Chambers (2015b) shows how GLS detrending can be achieved in continuous time for a stock variable and we extend the method below to a flow variable as well as considering other detrending methods. To motivate ideas, suppose α is known and use the substitution $u(t) = y_c(t) - \psi' z(t)$ in (2), yielding

$$dy_c(t) = \alpha y_c(t)dt + \psi' dz(t) - \alpha\psi' z(t)dt + \eta(t)dt.$$

Taking $m = 1$ we have $\psi = [\psi_0, \psi_1]'$, $z(t) = [1, t]'$ and $dz(t) = [0, dt]'$, in which case the above equation becomes

$$dy_c(t) = [\alpha y_c(t) - \psi_0\alpha + \psi_1(1 - \alpha t)]\, dt + \eta(t)dt. \tag{9}$$

Of course, α is unknown in practice and so the different detrending methods choose a particular value, say $\bar{\alpha}$, and proceed to estimate ψ_0 and ψ_1 based on the equation

$$dy_c(t) = [\bar{\alpha} y_c(t) - \psi_0\bar{\alpha} + \psi_1(1 - \bar{\alpha} t)]\, dt + \bar{\eta}(t)dt, \tag{10}$$

where $\bar{\eta}$ is a stationary continuous time process. From (10) it is possible to derive an exact representation for the discrete time observations which enables ψ_0 and ψ_1 to be estimated. We present the general result in Theorem 1 before looking at different possible choices for $\bar{\alpha}$.

Theorem 1. *Let $y_c(t)$ be generated according to (1) with $\psi' z(t) = \psi_0 + \psi_1 t$. Then detrending in continuous time is carried out by estimating the equation*

$$dy_c(t) = [\bar{\alpha} y_c(t) - \psi_0\bar{\alpha} + \psi_1\,(1 - \bar{\alpha} t)]\, dt + \bar{\eta}(t)dt, \quad t > 0, \tag{11}$$

where $\bar{\alpha}$ is the detrending parameter and $\bar{\eta}(t)$ is a stationary continuous time disturbance process. For a stock variable, estimation of (11) is equivalent to estimating

$$\tilde{y}_{th} = \psi' \tilde{z}_{th} + \tilde{w}_{th}, \quad t = 1, \ldots, T, \tag{12}$$

where $\tilde{y}_h = y_h$, $\tilde{z}_h = [1, h]'$,

$$\tilde{y}_{th} = y_{th} - e^{\bar{\alpha}h} y_{th-h}, \quad \tilde{z}_{th} = \begin{bmatrix} 1 - e^{\bar{\alpha}h} \\ th - e^{\bar{\alpha}h}(th - h) \end{bmatrix}, \quad t = 2, \ldots, T,$$

and $\tilde{w}_{th}$ denotes a stationary discrete time disturbance. For a flow variable the appropriate regression is

$$\tilde{Y}_{th} = \psi' \tilde{Z}_{th} + \tilde{W}_{th}, \quad t = 1, \ldots, T, \tag{13}$$

where $\tilde{Y}_h = Y_h$, $\tilde{Z}_h = [1, h/2]'$,

$$\tilde{Y}_{th} = Y_{th} - e^{\bar{\alpha}h} Y_{th-h}, \quad \tilde{Z}_{th} = \begin{bmatrix} 1 - e^{\bar{\alpha}h} \\ th - \frac{1}{2}h - e^{\bar{\alpha}h}\left(th - \frac{3}{2}h\right) \end{bmatrix}, \quad t = 2, \ldots, T,$$

and $\tilde{W}_{th}$ *denotes a stationary discrete time disturbance. Alternatively, if* $\psi' z(t) = \psi_0$*, the discrete time regressions above are still valid but with* $\psi_1 = 0$ *in (12) and (13).*

The continuous time detrending regressions in Theorem 1 mirror the detrending regressions that take place in a more familiar discrete time framework in which the variable of interest and deterministic terms are appropriately transformed. The first transformed observation is obtained directly from (1) (with an additional integration in the case of a flow variable) while the remaining transformed observations are consistent with the dynamics inherent in (11), subject to accounting for the temporal aggregation effects in an appropriate way. The presence of $h/2$ in $\tilde{Z}_h$ and $th - (h/2) - e^{\bar{\alpha}h}[th - (3h/2)]$ in $\tilde{Z}_{th}$ $(t = 2, \ldots, T)$ arise due to the integration associated with a flow variable. If the flow variable was not normalised by $1/h$ then both of these trend components (as well as the intercept) would need to be multiplied by h.

The regressions in Theorem 1 enable ψ to be estimated by OLS, leading to the estimators

$$\hat{\psi} = \left(\sum_{t=1}^{T} \tilde{z}_{th}\tilde{z}'_{th}\right)^{-1} \sum_{t=1}^{T} \tilde{z}_{th}\tilde{w}_{th} \quad \text{or} \quad \hat{\psi} = \left(\sum_{t=1}^{T} \tilde{Z}_{th}\tilde{Z}'_{th}\right)^{-1} \sum_{t=1}^{T} \tilde{Z}_{th}\tilde{W}_{th}$$

in the case of a stock variable or a flow variable, respectively. Although not stated explicitly above, the regression disturbances follow the same type of quasi-differencing as the observable variables, so that in the stock case, $\tilde{w}_h = u_h$ and $\tilde{w}_{th} = u_{th} - e^{\bar{\alpha}h} u_{th-h}$ $(t = 2, \ldots, T)$, while for a flow variable $\tilde{W}_1 = U_h$ and $\tilde{W}_{th} = U_{th} - e^{\bar{\alpha}h} U_{th-h}$ $(t = 2, \ldots, T)$. These definitions are required for the analysis of the limit properties of $\hat{\psi} - \psi$. The detrended series for stocks and flows are then given by

$$y^d_{th} = y_{th} - \hat{\psi}' z_{th} \quad \text{and} \quad Y^d_{th} = Y_{th} - \hat{\psi}' Z_{th},$$

respectively, where $z_{th} = [1, th]'$ and $Z_{th} = [1, th - (h/2)]'$.

We turn now to particular detrending methods and focus on three that have been employed in the literature, these being GLS detrending, OLS detrending, and first differencing. Each of these methods entails a particular choice of detrending parameter $\bar{\alpha}$ in place of the unknown α. In a purely discrete time framework, in which sampling frequency is ignored, the detrending regression (ignoring the initial observation for $t = 1$) for a generic variable y_t is of the form

$$y_t - \bar{\alpha} y_{t-1} = \psi'(z_t - \bar{\alpha} z_{t-1}) + w_t, \quad t = 2, \ldots, T,$$

where $z_t = [1, t]'$ and w_t is a stationary random disturbance. Under GLS detrending, $\bar{\alpha} = 1 + \bar{c}/T$ for some appropriate choice of constant $\bar{c}$, while OLS detrending sets $\bar{\alpha} = 0$ and first-differencing sets $\bar{\alpha} = 1$. The form of $\bar{\alpha}$ under GLS detrending mimics a popular approach to unit root testing in which the unknown parameter α is of the form $\alpha = 1 + c/T$ where $c < 0$ allows for a stationary near-unit root under the alternative hypothesis (the null of a unit root corresponding to $c = 0$). Elliot, Rothenberg and Stock (1996) suggest using $\bar{c} = -7$ when detrending on an intercept only and $\bar{c} = -13.5$ when using a linear trend. Note that, when $\bar{c}$ is fixed, $\bar{\alpha} \to 1$ as $T \to \infty$.

In analysing the effects of sampling frequency based on an underlying continuous time specification, the sample size, T, depends on both data span (N) and sampling interval (h): $T = N/h$. From Theorem 1 the discrete time detrending parameter is $e^{\bar{\alpha}h}$ and it is possible to choose $\bar{\alpha}$ so that the detrending reflects the usual discrete time approach but acknowledges the implicit temporal aggregation effects. For GLS detrending we therefore set $\bar{\alpha} = \bar{c}/N$ which results in $e^{\bar{\alpha}h} = e^{\bar{c}h/N} = e^{\bar{c}/T}$ and, noting that

$$e^{\bar{c}/T} = 1 + \frac{\bar{c}}{T} + O\left(\frac{\bar{c}^2}{T^2}\right),$$

we can see that the usual discrete time approach, using $1 + \bar{c}/T$ as the quasi-differencing parameter, truncates the series after the second term. For large sample sizes the usual GLS detrending approach may therefore provide a reasonably good approximation with temporally aggregated variables although for smaller samples the approximation will be less accurate. The remaining two detrending methods can be regarded as extreme cases of the GLS detrending parameter $\bar{c}$. The first differencing procedure is obtained by setting $\bar{c} = 0$, in which case we have $\bar{\alpha} = 0$ and, hence, $e^{\bar{\alpha}h} = 1$ for all sampling intervals h. It is easily seen from the results in Theorem 1 that the dependent variables become first differences (except for the initial observation) as do the deterministic terms. At the other extreme we can consider $\bar{c} \to -\infty$ in which case $\bar{\alpha} \to -\infty$ and $e^{\bar{\alpha}h} \to 0$, which corresponds to OLS detrending because, in effect, the underlying trend equation, (1), is being estimated directly by OLS, subject to accounting for the temporal aggregation. For convenience, the precise form of the regressors under each type of detrending is given for stock and flow variables in Table 1, in which $\tilde{z}_{th} = [\tilde{z}_{1,th}, \tilde{z}_{2,th}]'$ and $\tilde{Z}_{th} = [\tilde{Z}_{1,th}, \tilde{Z}_{2,th}]'$.

3. Asymptotic properties of detrending with stock variables

3.1. Some asymptotic results

Specifying an underlying continuous time model and working with its exact discrete representation enables alternative asymptotic regimes to be considered, as both the data span (N) and sampling interval (h) can be allowed to vary. We shall consider three different asymptotic sampling schemes, which were also considered by Zhou and Yu (2015) in the case of linear diffusion processes:

Scheme 1: h fixed, $N \to \infty$;
Scheme 2: $h \to 0$, $N \to \infty$;
Scheme 3: $h \to 0$, N fixed.

In all three cases the sample size $T = N/h \to \infty$. In order to derive the asymptotic properties of the detrended series an assumption needs to be made concerning the autoregressive parameter in the continuous time representation (2). From Lemma 1 we know that the autoregressive parameter for the discrete time stock variable u_{th} is $e^{\alpha h}$ and we therefore make the following assumption concerning α:

Assumption 1. The autoregressive parameter in (2) is $\alpha = c/N$ for some constant $c \leq 0$.

An immediate consequence of this assumption is that $e^{\alpha h} = e^{ch/N} = e^{c/T}$ which accords with the formulation in Phillips (1987) and allows for the treatment of a near-unit root as well as the analysis of the (power) properties of tests under stationary alternatives ($c < 0$). The properties of the process u_{th} are a key element in determining the properties of the detrended series and constitute the first main set of results in this section. The next assumption concerns the initial value $u(0)$.

Assumption 2. The initial value $u(0) = O_p(1)$.

This assumption ensures that $u(0)$ plays no role in the asymptotics in schemes 1 and 2 as $N \to \infty$. In fact, the weaker assumption that $u(0) = o_p(N^{1/2})$ would suffice under these two sampling schemes. Lemma 1 indicates that u_{th} is driven by the stationary process v_{th} and we also make the following assumption with a view to establishing an invariance principle (or functional central limit theorem) for the appropriately normalised disturbances v_{th}. The conditions concern the continuous time process $\eta(t)$ and partly involve the strong mixing coefficients defined for positive real values of s by $\alpha(s) = \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+s}^{\infty})$, where

$$\alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+s}^{\infty}) = \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+s}^{\infty}} \left| \Pr(G \cap H) - P(G)P(H) \right|,$$

and $\mathcal{F}_a^b$ denotes the sigma-field generated by $\eta(t)$ for $a \leq t \leq b$. A process is strong mixing if $\alpha(s) \to 0$ as $s \to \infty$, but the invariance principle requires these coefficients to satisfy a certain rate condition.

Assumption 3. $\eta(t)$ is a stationary continuous time process satisfying:
(a) $E\eta(t) = 0$.
(b) $E|\eta(t)|^{\beta} < \infty$ for some $\beta > 2$.
(c) $\eta(t)$ is strong mixing with mixing coefficients satisfying

$$\int_0^{\infty} \alpha(s)^{1-2/\beta} ds < \infty.$$

The conditions in Assumption 3 are satisfied if $\eta(t)$ is a Gaussian CARMA(p, q) process (with $q < p$) but they allow for much more general continuous time processes.[4] The mixing condition in part (c) is satisfied, for example, if the process is geometrically strong mixing i.e. if $\alpha(j) \leq e^{-j\theta}$ for some $\theta > 0$. Assumption 3 is used to establish the following result for partial sums of v_{th}.

Lemma 2. *Under Assumption 3, as $T \to \infty$ the functional*

$$x_T(r) = \frac{1}{T^{1/2}} \sum_{t=1}^{[Tr]} \left(\frac{v_{th}}{h^{1/2}} \right) \Rightarrow \sigma W(r), \quad r \in [0, 1],$$

[4] In the CARMA case, if $p = q$ then $\eta(t)$ has infinite variance but it may still be possible to obtain an invariance principle under stronger conditions and possibly a different rate of convergence. For example, Shao (1993) provides a set of conditions under which an invariance principle holds for stationary ρ-mixing processes with infinite variance.

where $W(r)$ is a Wiener process, σ^2 denotes the long run variance given by

$$\sigma^2 = \frac{1}{h}\left(Ev_{th}^2 + 2\sum_{k=1}^{\infty} Ev_{th}v_{th-kh}\right) = \begin{cases} 2\pi h f_\eta(0), & \alpha = 0, \\ 2\pi h\left(1-e^{\alpha h}\right)^2 \displaystyle\sum_{j=-\infty}^{\infty} \frac{f_\eta(2\pi j/h)}{h^2\alpha^2 + (2\pi j)^2}, & \alpha < 0, \end{cases}$$

and $f_\eta(\lambda)$ $(-\infty < \lambda < \infty)$ is the spectral density function of the continuous time process $\eta(t)$.

The normalisation of v_{th} by $h^{1/2}$ in Lemma 2 is due to the variance (and covariances) of v_{th} being $O(h)$. Also note that, as $Th = N$, an alternative form for the functional in Assumption 2 is

$$x_T(r) = \frac{1}{N^{1/2}}\sum_{t=1}^{[Tr]} v_{th}.$$

Although $N \to \infty$ in sampling schemes 1 and 2 it is fixed in scheme 3 which suggests that $\sum_{t=1}^{[Tr]} v_{th} \Rightarrow \sigma N^{1/2} W(r)$ in this case. The long run variance, σ^2, is presented in terms of the spectral density function of $\eta(t)$. When $\alpha = 0$ it is proportional to the spectrum at the origin but when $\alpha < 0$ the doubly-infinite summation arises from the process of moving from continuous time, where the spectral density is defined over $-\infty < \lambda < \infty$, to discrete time, where $-\pi < \lambda \leq \pi$; see, for example, Grenander and Rosenblatt (1957, p.57).

The building block for many of the results is contained in the following lemma, where $J_c(r)$ denotes the Ornstein-Uhlenbeck process which satisfies $dJ_c(r) = cJ_c(r)dr + dW(r)$ and has the solution $J_c(r) = \int_0^r e^{(r-s)c}dW(s)$. It is also convenient to define the constants

$$\delta_c = \left(\frac{e^c - 1}{c}\right), \quad \mu_c = \left(\frac{1-(1-c)e^c}{c^2}\right),$$

which appear in the limits in scheme 3.

Lemma 3. *Let $u(t)$ satisfy (2) and let $u_{th} = u(th)$. Then, under Assumptions 1–3:*
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N^{1/2}}u_{[Tr]h} \Rightarrow \sigma J_c(r), \quad \frac{1}{N^{3/2}}\sum_{t=1}^{T} u_{th} \Rightarrow \frac{\sigma}{h}\int_0^1 J_c(r)dr, \quad \frac{1}{N^{5/2}}\sum_{t=1}^{T} thu_{th} \Rightarrow \frac{\sigma}{h}\int_0^1 rJ_c(r)dr.$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\frac{1}{N^{1/2}}u_{[Tr]h} \Rightarrow \sigma J_c(r), \quad \frac{h}{N^{3/2}}\sum_{t=1}^{T} u_{th} \Rightarrow \sigma\int_0^1 J_c(r)dr, \quad \frac{h}{N^{5/2}}\sum_{t=1}^{T} thu_{th} \Rightarrow \sigma\int_0^1 rJ_c(r)dr.$$

Scheme 3. As $h \to 0$ with N fixed,

$$u_{[Tr]h} \Rightarrow e^{cr}u(0) + \sigma N^{1/2}J_c(r), \quad h\sum_{t=1}^{T} u_{th} \Rightarrow N\delta_c u(0) + \sigma N^{3/2}\int_0^1 J_c(r)dr,$$

$$h\sum_{t=1}^{T} thu_{th} \Rightarrow N^2\mu_c u(0) + \sigma N^{5/2}\int_0^1 rJ_c(r)dr.$$

It is interesting to note from Lemma 3 that the normalisation for the convergence of $u_{[Tr]h}$ is the same $(1/N^{1/2})$ in schemes 1 and 2 and is independent of h. The limits of sums of u_{th}, however, depend on the fixed value of h in scheme 1 but h is needed in the normalisations in scheme 2. The results for scheme 3 are not obtained from scheme 2 simply by fixing the value of N. This is because the initial value $u(0)$, which is asymptotically negligible in schemes 1 and 2, plays a non-negligible role when N is fixed and only h varies. Clearly, if $u(0) = 0$, this is not an issue.

All limit results in Lemma 3 depend on the parameter c via the process $J_c(r)$. This would not be the case if an alternative definition of α were to be assumed. For example, if, in place of Assumption 1, it was assumed that $\alpha = c/T$, so that the continuous time parameter α tends to zero in all three sampling schemes, then the discrete time autoregressive parameter becomes $e^{\alpha h} = e^{ch/T} = e^{ch^2/N}$. An implication of this[5] is that $J_c(r)$ is replaced by $J_{ch}(r)$ in scheme 1 while in schemes 2 and 3 $J_c(r)$ is replaced by $W(r)$ (along with a modification of the term multiplying $u(0)$ in scheme 3). This alternative specification does not appear to be realistic due to the dependence of the limit random process, $J_{ch}(r)$, on h in scheme 1 and the absence of any dependence on c in schemes 2 and 3.

The analysis of the properties of the estimators of the parameters under the three different detrending methods is facilitated by using a common notation for each. In the most general case of detrending on an intercept and trend this is given by

$$\hat{\psi} - \psi = Q^{-1}p = \frac{1}{|Q|}\begin{pmatrix} \Delta_0 \\ \Delta_1 \end{pmatrix}, \tag{14}$$

where $\Delta_0 = Q_{22}p_1 - Q_{12}p_2$, $\Delta_1 = Q_{11}p_2 - Q_{12}p_1$, $|Q| = Q_{11}Q_{22} - Q_{12}^2$, and Q and p are defined by

$$Q = \sum_{t=1}^{T} \tilde{z}_{th}\tilde{z}_{th}' = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}, \quad p = \sum_{t=1}^{T} \tilde{z}_{th}\tilde{w}_{th} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Clearly, the elements of Q and p depend on which detrending method is used, and their limit properties are defined for each method in some supplementary lemmas in Appendix C. These results are then incorporated into the proofs of the theorems that follow. As for the detrended series itself we shall be interested in functional central limit theorems (or invariance principles) for appropriately normalised versions of

$$y_{[Tr]h}^{d} = u_{[Tr]h} - \left(\hat{\psi}_0 - \psi_0\right) - \left(\hat{\psi}_1 - \psi_1\right)[Tr]h, \tag{15}$$

the relevant results following from Lemma 2 and the properties of $\hat{\psi} - \psi$. The results for each of the detrending methods are contained in Theorems 2–4 that follow, beginning with the OLS detrending results.

3.2. OLS detrending

Detrending by OLS involves estimation of (12) with $\bar{\alpha} = -\infty$ (implying $e^{\bar{\alpha}h} = 0$) and, hence, no (quasi-)differencing applied to the intercept or trend. It is convenient, for the

[5]We do not provide details but the results are readily obtained by following the steps in the proof of Lemma 3 but using this alternative definition of α.

presentation of the results, to define the following functionals of $J_c(r)$:

$$\begin{aligned}
\bar{J}_c(r) &= J_c(r) - \int_0^1 J_c(s)ds, \\
J_c^0 &= 4\int_0^1 J_c(s)ds - 6\int_0^1 sJ_c(s)ds, \\
J_c^1 &= 12\int_0^1 sJ_c(s)ds - 6\int_0^1 J_c(s)ds, \\
J_c^2(r) &= J_c(r) - (4-6r)\int_0^1 J_c(s)ds + (6-12r)\int_0^1 sJ_c(s)ds = J_c(r) - J_c^0 - rJ_c^1.
\end{aligned}$$

The results for the limit distribution of the estimator of ψ and the invariance principle for the detrended series are as follows.

Theorem 2. *Let $y_c(t)$ be generated according to (1) and (2), let y_{th}^d $(t = 1, \ldots, T)$ denote the detrended series and let Assumptions 1–3 hold.*

(a) If OLS detrending is carried out using only an intercept:
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N^{1/2}}(\hat{\psi}_0 - \psi_0) \Rightarrow \sigma\int_0^1 J_c(r)dr, \quad \frac{1}{N^{1/2}} y_{[Tr]h}^d \Rightarrow \sigma\bar{J}_c(r).$$

Scheme 2. As $N \to \infty$ and $h \to 0$, the results of scheme 1 continue to hold.
Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow \delta_c u(0) + \sigma N^{1/2}\int_0^1 rJ_c(r)dr, \quad y_{[Tr]h}^d \Rightarrow (e^{cr} - \delta_c)\, u(0) + \sigma N^{1/2}\bar{J}_c(r).$$

(b) If OLS detrending is carried out using an intercept and a trend:
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N^{1/2}}(\hat{\psi}_0 - \psi_0) \Rightarrow \sigma J_c^0, \quad N^{1/2}(\hat{\psi}_1 - \psi_1) \Rightarrow \sigma J_c^1, \quad \frac{1}{N^{1/2}} y_{[Tr]h}^d \Rightarrow \sigma J_c^2(r).$$

Scheme 2. As $N \to \infty$ and $h \to 0$, the results of scheme 1 continue to hold.
Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow 12k_1 u(0) + \sigma N^{1/2} J_c^0, \quad \hat{\psi}_1 - \psi_1 \Rightarrow \frac{12k_2}{N}u(0) + \frac{\sigma}{N^{1/2}}J_c^1,$$

$$y_{[Tr]h}^d \Rightarrow k_3(r)u(0) + \sigma N^{1/2} J_c^2(r),$$

where

$$k_1 = \frac{\delta_c}{3} - \frac{\mu_c}{2}, \quad k_2 = \mu_c - \frac{\delta_c}{2}, \quad k_3(r) = e^{cr} - 12(k_1 + k_2 r).$$

The large span asymptotics ($N \to \infty$) in Theorem 2 provide the same results regardless of whether the sampling interval (h) is fixed (scheme 1) or tends to zero (scheme 2). This is not true, however, for the underlying sums of u_{th} whose asymptotic properties are given in Lemma 3 and whose normalisations depend on h and N. When span is fixed the results are seen to depend quite explicitly on $u(0)$.

3.3. GLS detrending

We now turn to the case of GLS detrending, for which it is convenient to define the random variable

$$X(c,\bar{c}) = \frac{(1-\bar{c})J_c(1) + \bar{c}^2 \displaystyle\int_0^1 rJ_c(r)dr}{1-\bar{c}+\bar{c}^2/3} = \lambda J_c(1) + 3(1-\lambda)\int_0^1 rJ_c(r)dr,$$

where $\lambda = (1-\bar{c})/(1-\bar{c}+\bar{c}^2/3)$.

Theorem 3. *Let $y_c(t)$ be generated according to (1) and (2), let y^d_{th} $(t = 1, \ldots, T)$ denote the detrended series and let Assumptions 1–3 hold.*

(a) If GLS detrending is carried out using only an intercept:
Scheme 1: As $N \to \infty$ with h fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u_h, \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J_c(r).$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J_c(r).$$

Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad y^d_{[Tr]h} \Rightarrow \sigma N^{1/2} J_c(r) - (1-e^{cr})u(0).$$

(b) If GLS detrending is carried out using an intercept and a trend:
Scheme 1: As $N \to \infty$ with h fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u_h, \quad N^{1/2}(\hat{\psi}_1 - \psi_1) \Rightarrow \sigma X(c,\bar{c}), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma\left(J_c(r) - X(c,\bar{c})r\right).$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad N^{1/2}(\hat{\psi}_1 - \psi_1) \Rightarrow \sigma X(c,\bar{c}), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma\left(J_c(r) - X(c,\bar{c})r\right).$$

Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad \hat{\psi}_1 - \psi_1 \Rightarrow \frac{k_4}{N}u(0) + \frac{\sigma}{N^{1/2}}X(c,\bar{c}),$$

$$y^d_{[Tr]h} \Rightarrow \sigma N^{1/2}\left(J_c(r) - X(c,\bar{c})r\right) - (1-e^{cr}+k_4 r)u(0),$$

where

$$k_4 = \frac{\bar{c}^2\left(\mu_c - \dfrac{1}{2}\right) - (1-\bar{c})(1-e^c)}{1-\bar{c}+\dfrac{\bar{c}^2}{3}}.$$

The estimators of ψ_0 are inconsistent in all cases reported in Theorem 3 and there are also differences in the results between schemes 1 and 2. In the former the limit is determined by u_h while in the latter it is $u(0)$. Noting that $u_h = e^{\alpha h}u(0) + v_h$ and that $v_h \to 0$ in probability as $h \to 0$ explains the connection between these results. The limit properties of $y^d_{[Tr]h}$ are the same in schemes 1 and 2 and the influence of $u(0)$ is once again evident in scheme 3. The process $J_c(r) - X(c, \bar{c})r$ appearing in the limits in Theorem 3(b) is often denoted $V_c(r, \bar{c})$ or $V_{c,\bar{c}}(r)$ in the literature; see, for example, Elliott, Rothenberg and Stock (1996) in the former case and Chambers (2015a) in the latter.

3.4. Detrending by differencing

The final method of detrending we consider is differencing; the results are presented in Theorem 4, in which the process

$$J^3_c(r) = J_c(r) - rJ_c(1)$$

is defined for convenience.

Theorem 4. *Let $y_c(t)$ be generated according to (1) and (2), let y^d_{th} $(t = 1, \ldots, T)$ denote the detrended series and let Assumptions 1–3 hold.*

(a) If detrending is carried out by differencing using only an intercept:
Scheme 1: As $N \to \infty$ with h fixed,

$$\hat{\psi}_0 - \psi_0 = u_h, \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J_c(r).$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\hat{\psi}_0 - \psi_0 \xrightarrow{p} u(0), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J_c(r).$$

Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \xrightarrow{p} u(0), \quad y^d_{[Tr]h} \Rightarrow \sigma N^{1/2} J_c(r) - (1 - e^{cr})\, u(0).$$

(b) If detrending is carried out by differencing using an intercept and a trend:
Scheme 1: As $N \to \infty$ with h fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u_h, \quad N^{1/2}(\hat{\psi}_1 - \psi_1) \Rightarrow \sigma J_c(1), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J^3_c(r).$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad N^{1/2}(\hat{\psi}_1 - \psi_1) \Rightarrow \sigma J_c(1), \quad \frac{1}{N^{1/2}} y^d_{[Tr]h} \Rightarrow \sigma J^3_c(r).$$

Scheme 3. As $h \to 0$ with N fixed,

$$\hat{\psi}_0 - \psi_0 \Rightarrow u(0), \quad \hat{\psi}_1 - \psi_1 \Rightarrow \frac{\sigma N^{1/2} J_c(1) - (1 - e^c)u(0)}{N},$$

$$y^d_{[Tr]h} \Rightarrow \sigma N^{1/2} J^3_c(r) - [(1 - e^{cr}) - (1 - e^c)r]\, u(0).$$

The limit distributions of the detrended series coincide with those under GLS detrending when only an intercept is used in the regression. However, when a time trend is also included the limit properties are characterised by a Brownian bridge-type process (and which is a genuine Brownian bridge when $c = 0$).

4. Asymptotic properties of detrending with flow variables

When the variable of interest is a flow the observations are subject to a further integration over the sampling period as described in section 2. It is well known that this additional integration induces a further moving average component into the observed series. The process driving the results is now U_{th}, defined in Lemma 1, which in turn depends on V_{th}, whose limit properties are described below.

Lemma 4. *Under Assumption 3, as $T \to \infty$ the functional*

$$X_T(r) = \frac{1}{T^{1/2}} \sum_{t=1}^{[Tr]} \left(\frac{V_{th}}{h^{1/2}} \right) \Rightarrow \omega W(r), \quad r \in [0,1],$$

where $W(r)$ is a Wiener process and

$$\omega^2 = \frac{1}{h} \left(EV_{th}^2 + 2 \sum_{k=1}^{\infty} EV_{th} V_{th-kh} \right) = \begin{cases} 2\pi h f_\eta(0), & \alpha = 0, \\ \dfrac{2\pi \left(1 - e^{\alpha h}\right)^2 f_\eta(0)}{h\alpha^2}, & \alpha < 0, \end{cases}$$

denotes the long run variance.

In contrast to the stock case the long run variance in Lemma 4 depends only on the spectral density of $\eta(t)$ at the origin when $\alpha < 0$ and does not depend on the aliased frequencies. The normalisation of V_{th} by $h^{1/2}$ in the definition of the functional $X_T(r)$ is due to the variance (and covariances) of V_{th} being $O(h)$; recall that, although V_{th} is obtained by an additional process of integration over the interval $(th - h, th]$, it is also normalised by h.

The above invariance principle for V_{th} enables the properties of U_{th} to be determined. The main result is as follows.

Theorem 5. *Let $u(t)$ satisfy (2) and let U_{th} be defined as in Lemma 1. Then, under Assumptions 1–3, the results in Lemma 3 continue to hold with u_{th} replaced by U_{th} and σ replaced by ω.*

Theorem 5 shows that sampling a flow variable rather than a stock affects neither the form of the limits of the relevant functions of the variable nor the rates of convergence in any of the sampling schemes. It is not immediately clear whether these features necessarily translate across to the detrending regressions and detrended variable given that the form of the regressors is different under a flow variable. An investigation of these regressions with a flow variable yields the following result.

Theorem 6. *Let $y_c(t)$ be generated according to (1) and (2), let Y_{th}^d $(t = 1, \ldots, T)$ denote the detrended series and let Assumptions 1–3 hold. Then the conclusions of Theorems 2–4 remain valid with y_{th}^d replaced by Y_{th}^d and σ replaced by ω except for Theorem 3(b) where the* *constant k_4 in scheme 3 needs to be replaced by*

$$k_4' = \frac{\bar{c}^2\left(\mu_c - \dfrac{1}{2}\right) - (1-\bar{c})(1-e^c) - \dfrac{1}{2}}{1 - \bar{c} + \dfrac{\bar{c}^2}{3}}.$$

The same limit properties, including rates of convergence, therefore hold for flow data as for stock data. One of the reasons for this is that the flows are normalised by h, and without this normalisation different convergence rates would apply (at least in cases when $h \to 0$).

5. An application

As an illustration of the application of the preceding theoretical results we consider the task of testing for a zero root in continuous time with an observed stock variable. Perron (1991) found that the span of the data, rather than the number of observations *per se*, was the important determinant of the finite sample properties of tests for a random walk in equispaced data, but didn't consider the effects of detrending. We therefore consider a similar set-up but also allow for the effects of GLS detrending. The model is given by

$$y_c(t) = \psi_0 + u(t), \quad du(t) = \alpha u(t)dt + \sigma_u dW(t), \tag{16}$$

where $\alpha = c/N$. The discrete time observations, $y_{th} = y(th)$, satisfy

$$y_{th} = \psi_0 + u_{th}, \quad u_{th} = e^{\alpha h}u_{th-h} + v_{th}, \tag{17}$$

where v_{th} is Gaussian white noise with variance $\sigma_v^2 = \sigma_u^2(e^{2\alpha h} - 1)/2\alpha$. We consider the regression of the GLS-detrended variable, $y_{th}^d = y_{th} - \hat{\psi}_0$, on its lagged value, y_{th-h}^d, where ψ_0 is obtained in the manner outlined in Theorem 1 with $\psi_1 = 0$ and $\bar{\alpha} = -7/N$. This regression yields the OLS estimator of $\phi_h = e^{\alpha h}$, given by

$$\hat{\phi}_h = \frac{\sum_{t=1}^{T} y_{th-h}^d y_{th}^d}{\sum_{t=1}^{T}\left(y_{th-h}^d\right)^2}$$

from which an estimator of α can be obtained via $\hat{\alpha} = \log(\hat{\phi}_h)/h$. It is convenient, for the presentation of results, to define the functional

$$Z(X, Y) = \frac{\int_0^1 X(r)dY(r)}{\int_0^1 X(r)^2 dr},$$

where X and Y are random processes on $[0, 1]$, as well as the following two random functionals:

$$Z_1(c, N, u(0)) = N^{1/2}\sigma u(0)\left[c\int_0^1 J_c(r)dr - \int_0^1 (1-e^{cr})dW(r)\right] - c(1-\delta_c)u(0)^2,$$

$$Z_2(c, N, u(0)) = Nu(0)\left[\int_0^1 (1-e^{cr})^2 dr u(0) - 2N^{1/2}\sigma\int_0^1 (1-e^{cr})J_c(r)dr\right].$$

The limit results for $\hat{\phi}_h$ and $\hat{\alpha}$ for the three sampling schemes are presented in Theorem 7.

Theorem 7. *Let $y_c(t)$ be generated according to (16) and let y^d_{th} $(t = 1, \dots, T)$ denote the GLS-detrended series. Then:*
Scheme 1: As $N \to \infty$ with h fixed,

$$N(\hat{\phi}_h - \phi_h) \Rightarrow hZ(J_c, W), \quad N(\hat{\alpha} - \alpha) \Rightarrow Z(J_c, W).$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\frac{N}{h}(\hat{\phi}_h - \phi_h) \Rightarrow Z(J_c, W), \quad N(\hat{\alpha} - \alpha) \Rightarrow Z(J_c, W).$$

Scheme 3. As $h \to 0$ with N fixed,

$$\frac{1}{h}(\hat{\phi}_h - \phi_h) \Rightarrow \frac{N\sigma^2\int_0^1 J_c(r)dW(r) + Z_1(c, N, u(0))}{N^2\sigma^2\int_0^1 J_c(r)^2 dr + Z_2(c, N, u(0))},$$

$$(\hat{\alpha} - \alpha) \Rightarrow \frac{N\sigma^2\int_0^1 J_c(r)dW(r) + Z_1(c, N, u(0))}{N^2\sigma^2\int_0^1 J_c(r)^2 dr + Z_2(c, N, u(0))},$$

where $\sigma^2 = \sigma_v^2/h$.

The results in Theorem 7 rest on the invariance principles for the GLS-detrended variable $y^d_{[Tr]h}$ given in Theorem 3. However, it is not simply a case of using these results in the integral approximations of discrete sums as would usually be the case. This is principally because it is not possible to replace u_{th} with y^d_{th} in deriving the asymptotics due to the inconsistency of $\hat{\psi}_0$ reported in Theorem 3. In fact, the proof of Theorem 7 shows that y^d_{th} satisfies

$$y^d_{th} = \phi_h y^d_{th-h} + v_{th} - \lambda_{th},$$

where $\lambda_{th} = (\hat{\psi}_0 - \psi_0)(1-\phi_h)$, and it is the presence of λ_{th} that needs additional attention in deriving the results. Westerlund (2014), in particular, has recently highlighted this feature that arises in testing for a unit root using GLS-detrended data.

The results for schemes 1 and 2 in Theorem 7 are, essentially the same, because normalising by h in scheme 1 yields $(N/h)(\hat{\phi}_h - \phi_h) \Rightarrow Z(J_c, W)$, as in scheme 2. When $u(0) \neq 0$ the limit distributions in scheme 3 depend in a complex way on the value of $u(0)$. However, when $u(0) = 0$ we find that $(1/h)(\hat{\phi}_h - \phi_h) \Rightarrow (1/N)Z(J_c, W)$ or, equivalently, $(N/h)(\hat{\phi}_h - \phi_h) \Rightarrow Z(J_c, W)$, as in schemes 1 and 2.

The results in Theorem 7 highlight the importance of increasing span for the consistent estimation of α. Although $\phi_h = e^{\alpha h}$ can be consistently estimated in all three sampling schemes, in view of $\hat{\phi}_h - \phi_h = o_p(1)$, the same is not true of α in view of the result that $\hat{\alpha} - \alpha = O_p(1)$ in scheme 3.

The results presented in Theorem 7 provide a basis for a simulation study to assess the properties of a test for a zero root in continuous time i.e. a test of the null hypothesis $H_0 : \alpha = 0$ against the (stationary) alternative $H_1 : \alpha < 0$. Under the null there exists a unit root in discrete time as $\phi_h = 1$ when $\alpha = 0$ while $0 < \phi_h < 1$ when $\alpha < 0$. The test statistic under consideration is $N\hat{\alpha}$ and we take $u(0) = 0$ so that, under the null hypothesis, the limit distribution is

$$N\hat{\alpha} \Rightarrow Z(W, W) = \frac{\int_0^1 W(r)dW(r)}{\int_0^1 W(r)^2 dr}$$

in all three sampling schemes, noting that $J_0(r) = W(r)$. The 5% critical value for this distribution is -8.038 according to the numerical calculations of Perron (1989). Three data spans and sampling frequencies are considered, these being $N = \{25, 50, 100\}$ and $h = \{1, 1/4, 1/52\}$, respectively, and 10,000 replications were used for each combination of parameter values. If $h = 1$ is taken to correspond to a sampling interval of one year then $h = 1/4$ and $h = 1/52$ correspond to quarterly and weekly intervals, respectively, while the spans cover 25, 50 and 100 years. In order to assess the power properties of the test statistic, the parameter $\alpha = c/N$ was considered for $c = \{0, -2.5, -5.0, \ldots, -17.5, -20.0\}$. Table 2 contains the value of ϕ_h for each combination of N, h and c. For the smallest span ($N = 25$) and lowest frequency ($h = 1$) the value of ϕ_h falls rapidly as c becomes more negative, reaching 0.4493 for $c = -20$. As frequency increases ϕ_h remains much closer to unity, falling only to 0.9847 for $c = -20$. For larger spans the deviation of ϕ_h from unity lessens for a given value of h. This highlights an important trade-off – larger spans and/or higher sampling frequencies result in more observations but the coefficient being estimated becomes closer to unity so it is not entirely obvious which scenario is likely to yield highest test power.

An issue of practical relevance concerns the calculation of $\hat{\alpha}$ from $\hat{\phi}_h$, which involves $\log \hat{\phi}_h$. This is only possible provided that $\hat{\phi}_h > 0$,[6] and so the final panel of Table 2 reports the proportion of replications in which $\hat{\phi}_h < 0$; this only occurred for the smallest sample size when $N = 25$ and $h = 1$ (so that $T = 25$). As can be seen, this proportion was negligible under the null ($c = 0$), being equal to just 0.01%, rising monotonically to 3.81% when $c = -20$. This is due to the finite sample distribution of $\hat{\phi}_h$ shifting to the left as ϕ_h gets smaller and, hence, the probability that $\hat{\phi}_h$ is negative increases. In any case, obtaining a negative value of $\hat{\phi}_h$ in a reasonably-sized finite sample might suggest that the data were not consistent with a continuous time AR(1) process as it is known that $\phi_h > 0$ in this case for any value of the continuous time autoregressive parameter. The simulations, however, show that it is possible to obtain negative estimates in a small sample even with data generated by a continuous time model, although the proportions reported in Table 2 are smaller than many reported in Chambers (2005).

[6]Ignoring the possibility of complex values of the logarithm for negative arguments.

The simulated size and power (both raw and size-adjusted) of $N\hat{\alpha}$ are reported in Table 3. Under scheme 1, fixing h and allowing N to increase shows that the size of the test falls towards the nominal 5% level, while the size-adjusted power increases with N. The raw power is inflated at low frequency sampling owing to the over-sizing of the test in these situations. For scheme 2 the sampling frequency increases with span, and this is reflected in the sequence of (N, h) combinations $(25, 1)$, $(50, 1/4)$, $(100, 1/52)$. Moving through this sequence shows size falling towards the nominal 5% level and size-adjusted power increasing. Finally, for scheme 3, we need to fix N and consider the sequence of falling h values. The size of the test tends towards 5% while the size-adjusted power tends to increase in most cases. That the test in scheme 3 performs so well is perhaps surprising in view of $\hat{\alpha}$ being inconsistent for α. In effect, as $h \to 0$ it follows that $\phi_h \to 1$ for any value of α (or, equivalently, c), and the test is being asked to distinguish the effect of falling h from the true value of α that is being estimated inconsistently. The simulations suggest it performs remarkably well under such circumstances.

6. Concluding comments

This paper has analysed the effects of sampling frequency on detrending methods based on an underlying continuous time representation of the process of interest. Such an approach has the advantage of allowing for the explicit – and different – treatment of the ways in which stock and flow variables are actually observed. Some general results were provided before the focus turned to three particular detrending methods that have found widespread use in the conduct of tests for a unit root, these being GLS detrending, OLS detrending, and first differencing. In addition, three different scenarios concerning sampling frequency and data span, in each of which the number of observations increases, were considered for each detrending method. The limit properties of the detrending coefficient estimates, as well as an invariance principle for the detrended variable, were derived. An example of the application of the techniques to testing for a unit root, using GLS detrending on an intercept, was provided and a simulation exercise carried out to analyse the size and power properties of the test in the three different sampling scenarios.

The results presented here are likely to be of use in other situations where detrended data are used and the effects of sampling frequency and data span are of interest. One particular avenue currently being pursued is the analysis of testing for a unit root in continuous time ARMA processes of the type considered by Chambers and Thornton (2012). The detrending results obtained here feed in naturally to that investigation.

Appendix A. Proofs of theorems

Proof of Theorem 1. For a stock variable the equation for $t = 0$ is obtained by inserting this value into (1), which gives $y_0 = \psi_0 + \tilde{w}_0$ where $\tilde{w}_0 = u(0)$. For $t = 1, \ldots, T$ the discrete time representation is obtained from the solution to (11), given by

$$y_c(th) = e^{\bar{\alpha}th} y_c(0) - \psi_0 \bar{\alpha} \int_0^{th} e^{\bar{\alpha}(th-r)} dr + \psi_1 \int_0^{th} e^{\bar{\alpha}(th-r)} (1 - \bar{\alpha}r) dr + \int_0^{th} e^{\bar{\alpha}(th-r)} \bar{\eta}(r) dr. \quad (18)$$

This solution, which is unique in the mean square sense, can be used to derive the stochastic difference equation

$$y_c(th) = e^{\bar{\alpha}h} y_c(th-h) - \psi_0 \bar{\alpha} \int_{th-h}^{th} e^{\bar{\alpha}(th-r)} dr + \psi_1 \int_{th-h}^{th} e^{\bar{\alpha}(th-r)}(1-\bar{\alpha}r)dr + \tilde{w}_{th}, \quad (19)$$

where $\tilde{w}_{th} = \int_{th-h}^{th} e^{\bar{\alpha}(th-r)}\bar{\eta}(r)dr$. Evaluation of the deterministic integrals yields

$$y_c(th) = e^{\bar{\alpha}h} y_c(th-h) + \psi_0 \left(1 - e^{\bar{\alpha}h}\right) + \psi_1 \left(th - e^{\bar{\alpha}h}(th-h)\right) + \tilde{w}_{th}, \quad (20)$$

which yields the form for the regressors stated following (12).

For a flow variable the equation for $t = 1$ is obtained by integrating (1) over the interval $(0, h]$, with $\tau(t) = \psi_0 + \psi_1 t$, and noting that $\int_0^h r dr = h^2/2$; the disturbance term is given by $\tilde{W}_h = (1/h) \int_0^h u(r)dr$. For $t = 2, \ldots, T$ the equation is obtained by integrating (20) over $(th - h, th]$ which yields

$$\begin{aligned} \frac{1}{h}\int_{th-h}^{th} y_c(r)dr &= e^{\bar{\alpha}h}\frac{1}{h}\int_{th-2h}^{th-h} y_c(r)dr + \psi_0\left(1-e^{\bar{\alpha}h}\right)\frac{1}{h}\int_{th-h}^{th} dr \\ &\quad + \psi_1 \frac{1}{h}\int_{th-h}^{th}\left(r - e^{\bar{\alpha}h}(r-h)\right)dr + \tilde{W}_{th}, \end{aligned} \quad (21)$$

where $\tilde{W}_{th} = (1/h)\int_{th-h}^{th}\int_{r-h}^{r} e^{\bar{\alpha}(r-s)}\bar{\eta}(s)dsdr$. Evaluation of the integrals yields the form for the regressors stated following (13). □

Proof of Theorem 2. (a) Under OLS detrending on an intercept we find that

$$\hat{\psi}_0 = \frac{1}{T}\sum_{t=1}^{T} y_{th} = \psi_0 + \frac{h}{N}\sum_{t=1}^{T} u_{th}.$$

The results for $\hat{\psi}_0 - \psi_0 = (h/N)\sum_{t=1}^{T} u_{th}$ then follow from Lemma 3, and the properties for the detrended series are then obtained from $y^d_{[Tr]h} = u_{[Tr]h} - (\hat{\psi}_0 - \psi_0)$.
(b) Under OLS detrending on an intercept and trend note that $\tilde{z}_{th} = z_{th} = [1, th]'$. It follows that the elements of Q and p are given by

$$Q_{11} = T, \quad Q_{12} = \sum_{t=1}^{T} th, \quad Q_{22} = \sum_{t=1}^{T}(th)^2, \quad p_1 = \sum_{t=1}^{T} u_{th}, \quad p_2 = \sum_{t=1}^{T} thu_{th}.$$

The limit properties of the elements of q are provided in Lemma C2 while those of the elements of p appear in Lemma 3. Taking each sampling scheme in turn:
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N^4}|Q| = \frac{1}{N}Q_{11}\frac{1}{N^3}Q_{22} - \left(\frac{1}{N^2}Q_{12}\right)^2 \to \frac{1}{12h^2},$$

$$\frac{1}{N^{9/2}}\Delta_0 = \frac{1}{N^3}Q_{22}\frac{1}{N^{3/2}}p_1 - \frac{1}{N^2}Q_{12}\frac{1}{N^{5/2}}p_2 \Rightarrow \frac{\sigma}{h^2}\left(\frac{1}{3}\int_0^1 J_c(r)dr - \frac{1}{2}\int_0^1 rJ_c(r)dr\right),$$

$$\frac{1}{N^{7/2}}\Delta_1 = \frac{1}{N}Q_{11}\frac{1}{N^{5/2}}p_2 - \frac{1}{N^2}Q_{12}\frac{1}{N^{3/2}}p_1 \Rightarrow \frac{\sigma}{h^2}\left(\int_0^1 rJ_c(r)dr - \frac{1}{2}\int_0^1 J_c(r)dr\right).$$

The results for the elements of $\hat{\psi} - \psi$ follow straightforwardly, while for the detrended series we have

$$\frac{1}{N^{1/2}}y^d_{[Tr]h} = \frac{1}{N^{1/2}}u_{[Tr]h} - \frac{1}{N^{1/2}}(\hat{\psi}_0 - \psi_0) - N^{1/2}(\hat{\psi}_1 - \psi_1)\left[\frac{Nr}{h}\right]\frac{h}{N}$$

and the results follow noting that $[Nr/h](h/N) \to r$ as $N \to \infty$.
Scheme 2: As $N \to \infty$ and $h \to 0$,

$$\frac{h^2}{N^4}|Q| = \frac{h}{N}Q_{11}\frac{h}{N^3}Q_{22} - \left(\frac{h}{N^2}Q_{12}\right)^2 \to \frac{1}{12},$$

$$\frac{h^2}{N^{9/2}}\Delta_0 = \frac{h}{N^3}Q_{22}\frac{h}{N^{3/2}}p_1 - \frac{h}{N^2}Q_{12}\frac{h}{N^{5/2}}p_2 \Rightarrow \sigma\left(\frac{1}{3}\int_0^1 J_c(r)dr - \frac{1}{2}\int_0^1 rJ_c(r)dr\right),$$

$$\frac{h^2}{N^{7/2}}\Delta_1 = \frac{h}{N}Q_{11}\frac{h}{N^{5/2}}p_2 - \frac{h}{N^2}Q_{12}\frac{h}{N^{3/2}}p_1 \Rightarrow \sigma\left(\int_0^1 rJ_c(r)dr - \frac{1}{2}\int_0^1 J_c(r)dr\right).$$

These expressions yield the stated results for $\hat{\psi} - \psi$, while the result for the detrended series follows in the same way as in scheme 1 above.
Scheme 3: As $h \to 0$ with N fixed,

$$h^2|Q| = hQ_{11}hQ_{22} - (hQ_{12})^2 \to \frac{N^4}{12},$$

$$h^2\Delta_0 = hQ_{22}hp_1 - hQ_{12}hp_2 \Rightarrow N^4k_1u(0) + \sigma N^{9/2}\left(\frac{1}{3}\int_0^1 J_c(r)dr - \frac{1}{2}\int_0^1 rJ_c(r)dr\right),$$

$$h^2\Delta_1 = hQ_{11}hp_2 - hQ_{12}hp_1 \Rightarrow N^3k_2u(0) + \sigma N^{7/2}\left(\int_0^1 rJ_c(r)dr - \frac{1}{2}\int_0^1 J_c(r)dr\right),$$

where k_1 and k_2 are defined in the theorem. As for the detrended series we may write

$$y^d_{[Tr]h} = u_{[Tr]h} - (\hat{\psi}_0 - \psi_0) - N(\hat{\psi}_1 - \psi_1)\left[\frac{Nr}{h}\right]\frac{h}{N},$$

where pre-multiplication of $\hat{\psi}_1 - \psi_1$ by N (which is fixed) ensures that the limit can be expressed in a form that is easy to relate to those in schemes 1 and 2 while also ensuring that $[Nr/h](h/N) \to r$ as $h \to 0$. The result follows from Lemma 3 and the properties of $\hat{\psi} - \psi$. □

Proof of Theorem 3. (a) Under GLS detrending on an intercept we have

$$\hat{\psi}_0 = \frac{\sum_{t=1}^{T}\tilde{z}_{th}\tilde{y}_{th}}{\sum_{t=1}^{T}\tilde{z}_{th}^2} = \psi_0 + \frac{\sum_{t=1}^{T}\tilde{z}_{th}\tilde{u}_{th}}{\sum_{t=1}^{T}\tilde{z}_{th}^2},$$

where $\tilde{z}_{th}$ and $\tilde{y}_{th}$ are given in Table 1 and where $\tilde{u}_h = u_h$ and $\tilde{u}_{th} = u_{th} - e^{\bar{c}/T}u_{th-h}$

$(t = 2, \ldots, T)$. For the denominator,

$$\sum_{t=1}^{T} \tilde{z}_{th}^2 = 1 + \sum_{t=2}^{T} \left(1 - e^{\bar{c}/T}\right)^2 = 1 + (T-1)\left(1 - e^{\bar{c}/T}\right)^2 \to 1$$

in all three sampling schemes (see the results for the quantity Q_{11} in Lemma C3). Hence the limit properties of $\hat{\psi}_0 - \psi_0$ are determined by

$$\sum_{t=1}^{T} \tilde{z}_{th}\tilde{u}_{th} = \tilde{u}_h + \left(1 - e^{\bar{c}/T}\right)\sum_{t=2}^{T} \tilde{u}_{th} = u_h + \left(1 - e^{\bar{c}/T}\right)\sum_{t=2}^{T}\left(u_{th} - e^{\bar{c}/T}u_{th-h}\right),$$

which is equal to the quantity p_1 in Lemma C3 and which immediately yields the results for $\hat{\psi}_0 - \psi_0$. The limits for $y_{[Tr]h}^d = u_{[Tr]h} - (\hat{\psi}_0 - \psi_0)$ then follow from Lemma 3 and the properties of $\hat{\psi}_0 - \psi_0$.

(b) In the case of GLS detrending on an intercept and trend the elements of Q and p are defined in Lemma C3 which also gives their limit properties. Taking each sampling scheme in turn:

Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N}|Q| = Q_{11}\frac{1}{N}Q_{22} - \frac{1}{N}Q_{12}^2 \to h\left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right),$$

$$\frac{1}{N}\Delta_0 = \frac{1}{N}Q_{22}p_1 - \frac{1}{N^{1/2}}Q_{12}\frac{1}{N^{1/2}}p_2 \Rightarrow h\left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right)u_h,$$

$$\frac{1}{N^{1/2}}\Delta_1 = Q_{11}\frac{1}{N^{1/2}}p_2 - \frac{1}{N^{1/2}}Q_{12}p_1 \Rightarrow h\sigma\left((1-\bar{c})J_c(1) + \bar{c}^2\int_0^1 rJ_c(r)dr\right).$$

The results for the elements of $\hat{\psi} - \psi$ follow straightforwardly, while for $y_{[Tr]h}^d$ we obtain

$$\frac{1}{N^{1/2}}y_{[Tr]h}^d = \frac{1}{N^{1/2}}u_{[Tr]h} - N^{1/2}(\hat{\psi}_1 - \psi_1)\left[\frac{Nr}{h}\right]\frac{h}{N} + o_p(1),$$

from which the result follows.

Scheme 2: As $N \to \infty$ and $h \to 0$,

$$\frac{1}{hN}|Q| = Q_{11}\frac{1}{hN}Q_{22} - \frac{h}{N}\left(\frac{1}{h}Q_{12}\right)^2 \to \left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right),$$

$$\frac{1}{hN}\Delta_0 = \frac{1}{hN}Q_{22}p_1 - \frac{h}{N^{1/2}}\frac{1}{h}Q_{12}\frac{1}{hN^{1/2}}p_2 \Rightarrow \left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right)u(0),$$

$$\frac{1}{hN^{1/2}}\Delta_1 = Q_{11}\frac{1}{hN^{1/2}}p_2 - \frac{1}{N^{1/2}}\frac{1}{h}Q_{12}p_1 \Rightarrow \sigma\left((1-\bar{c})J_c(1) + \bar{c}^2\int_0^1 rJ_c(r)dr\right).$$

These expressions yield the stated results for $\hat{\psi} - \psi$, while a similar decomposition applies for $y_{[Tr]h}^d$ as in scheme 1.

Scheme 3: As $h \to 0$ with N fixed,

$$\frac{1}{h}|Q| = Q_{11}\frac{1}{h}Q_{22} - h\left(\frac{1}{h}Q_{12}\right)^2 \to N\left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right),$$

$$\frac{1}{h}\Delta_0 = \frac{1}{h}Q_{22}p_1 - h\frac{1}{h}Q_{12}\frac{1}{h}p_2 \Rightarrow N\left(1 - \bar{c} + \frac{\bar{c}^2}{3}\right)u(0),$$

$$\frac{1}{h}\Delta_1 = Q_{11}\frac{1}{h}p_2 - \frac{1}{h}Q_{12}p_1 \quad \Rightarrow \quad \left[(1-\bar{c})e^c + \bar{c}^2\mu_c - \left(1 - \bar{c} + \frac{\bar{c}^2}{2}\right)\right]u(0)$$
$$+\sigma N^{1/2}\left((1-\bar{c})J_c(1) + \bar{c}^2\int_0^1 rJ_c(r)dr\right).$$

The results for $\hat{\psi} - \psi$ now follow, while that for the detrended series is obtained from

$$y^d_{[Tr]h} = u_{[Tr]h} - \left(\hat{\psi}_0 - \psi_0\right) - \left(\hat{\psi}_1 - \psi_1\right)\left[\frac{Nr}{h}\right]h,$$

noting that $[Nr/h]h \to Nr$ as $h \to 0$. □

Proof of Theorem 4. (a) Under detrending by differencing using an intercept, we have

$$\hat{\psi}_0 = \psi_0 + \frac{\sum_{t=1}^{T}\tilde{z}^2_{th}}{\sum_{t=1}^{T}\tilde{z}_{th}\tilde{w}_{th}},$$

where $\tilde{z}_h = 1$, $\tilde{z}_{th} = 0$ $(t = 2, \ldots, T)$, $\tilde{w}_h = u_h$ and $\tilde{w}_{th} = u_h - u_{th-h}$ $(t = 2, \ldots, T)$. It is immediate that

$$\sum_{t=1}^{T}\tilde{z}^2_{th} = 1, \quad \sum_{t=1}^{T}\tilde{z}_{th}\tilde{w}_{th} = u_h,$$

and hence $\hat{\psi}_0 - \psi_0 = u_h$; this is the result for scheme 1. For schemes 2 and 3 recall that $u_h = e^{\alpha h}u(0) + v_h$ where $v_h = \int_0^h e^{\alpha(h-s)}\eta(s)ds$, so that $u_h \to u(0)$ in probability as $h \to 0$. For the detrended series we have, for schemes 1 and 2,

$$\frac{1}{N^{1/2}}y^d_{[Tr]h} = \frac{1}{N^{1/2}}u_{[Tr]h} + o_p(1) \Rightarrow \sigma J_c(r),$$

while for scheme 3 no normalisation is required and the stated limit applies.

(b) Under detrending by differencing using an intercept and a trend the elements of Q and p are

$$Q_{11} = 1, \quad Q_{12} = h, \quad Q_{22} = Th^2 = hN, \quad p_1 = u_h, \quad p_2 = hu_{Th}.$$

The appropriate normalisation for the elements of Q are immediate while those for p_1 and p_2 follow straightforwardly from Lemma 3. Considering each sampling scheme in turn:
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N}|Q| = Q_{11}\frac{1}{N}Q_{22} - \frac{1}{N}Q^2_{12} \to h,$$

$$\frac{1}{N}\Delta_0 = \frac{1}{N}Q_{22}p_1 - \frac{1}{N^{1/2}}Q_{12}\frac{1}{N^{1/2}}p_2 \Rightarrow hu_h,$$

$$\frac{1}{N^{1/2}}\Delta_1 = Q_{11}\frac{1}{N^{1/2}}p_2 - \frac{1}{N^{1/2}}Q_{12}p_1 \Rightarrow h\sigma J_c(1).$$

The results for the elements of $\hat{\psi} - \psi$ follow straightforwardly, while for $y^d_{[Tr]h}$ we obtain

$$\frac{1}{N^{1/2}} y^d_{[Tr]h} = \frac{1}{N^{1/2}} u_{[Tr]h} - N^{1/2}(\hat{\psi}_1 - \psi_1) \left[\frac{Nr}{h} \right] \frac{h}{N} + o_p(1),$$

from which the result follows.
Scheme 2: As $N \to \infty$ and $h \to 0$,

$$\frac{1}{hN}|Q| = Q_{11} \frac{1}{hN} Q_{22} - \frac{h}{N} \left(\frac{1}{h} Q_{12} \right)^2 \to 1,$$

$$\frac{1}{hN} \Delta_0 = \frac{1}{hN} Q_{22} p_1 - \frac{h}{N^{1/2}} \frac{1}{h} Q_{12} \frac{1}{hN^{1/2}} p_2 \Rightarrow u(0),$$

$$\frac{1}{hN^{1/2}} \Delta_1 = Q_{11} \frac{1}{hN^{1/2}} p_2 - \frac{1}{N^{1/2}} \frac{1}{h} Q_{12} p_1 \Rightarrow \sigma J_c(1).$$

These expressions yield the stated results for $\hat{\psi} - \psi$, while a similar decomposition applies for $y^d_{[Tr]h}$ as in scheme 1.
Scheme 3: As $h \to 0$ with N fixed,

$$\frac{1}{h}|Q| = Q_{11} \frac{1}{h} Q_{22} - h \left(\frac{1}{h} Q_{12} \right)^2 \to N,$$

$$\frac{1}{h} \Delta_0 = \frac{1}{h} Q_{22} p_1 - h \frac{1}{h} Q_{12} \frac{1}{h} p_2 \Rightarrow N u(0),$$

$$\frac{1}{h} \Delta_1 = Q_{11} \frac{1}{h} p_2 - \frac{1}{h} Q_{12} p_1 \Rightarrow (e^c - 1) u(0) + \sigma N^{1/2} J_c(1).$$

The results for $\hat{\psi} - \psi$ now follow, while that for the detrended series is obtained from

$$y^d_{[Tr]h} = u_{[Tr]h} - \left(\hat{\psi}_0 - \psi_0 \right) - \left(\hat{\psi}_1 - \psi_1 \right) \left[\frac{Nr}{h} \right] h,$$

again noting that $[Nr/h]h \to Nr$ as $h \to 0$. □

Proof of Theorem 5. Using the expressions for U_h and U_{th} $(t = 2, \ldots, T)$ in Lemma 1 we find, by backward substitution, that

$$\begin{aligned}
U_{th} &= e^{\alpha h} U_{th-h} + V_{th} \\
&= e^{\alpha(th-h)} U_h + \sum_{j=2}^{t} e^{\alpha(th-jh)} V_{jh} \\
&= e^{\alpha(th-h)} \left(\frac{e^{\alpha h} - 1}{\alpha h} \right) u(0) + e^{\alpha(th-h)} V_h + \sum_{j=2}^{t} e^{\alpha(th-jh)} V_{jh} \\
&= e^{\alpha th} \left(\frac{1 - e^{-\alpha h}}{\alpha h} \right) u(0) + \sum_{j=1}^{t} e^{\alpha(th-jh)} V_{jh} \\
&= e^{ct/T} \left(\frac{1 - e^{-c/T}}{c/T} \right) u(0) + \sum_{j=1}^{t} e^{c(t-j)/T} V_{jh},
\end{aligned}$$

noting that $\alpha h = c/T$. This expression only differs from the corresponding expression for u_{th}

in the proof of Lemma 3 through the extra term in parentheses involving c/T that multiplies $u(0)$ and the obvious replacement of v_{th} with V_{th}. It is then possible to write

$$U_{[Tr]h} = e^{c[Tr]/T}\delta_{-c/T}u(0) + N^{1/2}\int_0^r e^{(r-s)c}dX_T(s)$$

using the same arguments as in the proof of Lemma 3 and noting that

$$\left(\frac{1-e^{-c/T}}{c/T}\right) = \left(\frac{e^{-c/T}-1}{-c/T}\right) = \delta_{-c/T}$$

based on the definition of the constant δ_c introduced prior to the statement of Lemma 3. The results follow by noting that

$$\delta_{-c/T} = \frac{1-e^{-c/T}}{c/T} = 1 + O\left(\frac{1}{T}\right)$$

as $T \to \infty$. Similarly, when considering sums of U_{th}, the differences compared to sums of u_{th} occur through the presence of $\delta_{-c/T}$ and the replacement of v_{th} with V_{th} (and $x_T(r)$ with $X_T(r)$). We therefore find that

$$\begin{aligned}\sum_{t=1}^{T} u_{th} &= \frac{N}{h}\delta_c\delta_{-c/T}u(0) + \frac{N^{3/2}}{h}\int_0^1\int_0^r e^{(r-s)c}dX_T(s)dr, \\ \sum_{t=1}^{T} thu_{th-h} &= \frac{N^2}{h}\mu_c\delta_{-c/T}u(0) + \frac{N^{5/2}}{h}\int_0^1 r\int_0^{r-1/T} e^{(r-s)c}dX_T(s)dr,\end{aligned}$$

from which the results follow. □

Proof of Theorem 6. The cases with intercept only are straightforward to verify. In cases with an intercept and trend it is possible to express the elements of Q in terms of the same matrix used for stocks and to then analyse the properties of the difference in the different sampling schemes. Let Q^f be the matrix for the flows regression and Q^s the matrix for stocks. Then, for example, in the case of GLS detrending it can be shown that $Q^f = Q^s + \Gamma$ where

$$\Gamma = \sum_{t=1}^{T}\left(\gamma_{th}\gamma_{th}' - \gamma_{th}\tilde{z}_{th}' - \tilde{z}_{th}\gamma_{th}'\right),$$

$\gamma_h = (0, h/2)'$ and $\gamma_{th} = (0, (1-e^{\bar{c}/T})h/2)'$ $(t = 2, \ldots, T)$. It can be shown that $\Gamma_{11} = 0$ while

$$\begin{aligned}\Gamma_{12} &= -\frac{h}{2}\left[1 + (T-1)\left(1-e^{\bar{c}/T}\right)^2\right], \\ \Gamma_{22} &= \frac{h^2}{4}\left[1 + (T-1)\left(1-e^{\bar{c}/T}\right)^2\right] - h(1-e^{\bar{c}/T})^2\sum_{t=2}^{T} th - h^2(T-1)e^{\bar{c}/T}\left(1-e^{\bar{c}/T}\right).\end{aligned}$$

The following limits then follow with the normalisations used for the elements of Q^s:

$$\begin{aligned}
&\text{Scheme 1:} \quad && \Gamma_{12} = -\frac{h}{2} + o(1), \quad \frac{1}{N}\Gamma_{22} = o(1), \\
&\text{Scheme 2:} \quad && \frac{1}{h}\Gamma_{12} = -\frac{1}{2} + o(1), \quad \frac{1}{hN}\Gamma_{22} = o(1), \\
&\text{Scheme 3:} \quad && \frac{1}{h}\Gamma_{12} = -\frac{1}{2} + o(1), \quad \frac{1}{h}\Gamma_{22} = o(1).
\end{aligned}$$

The non-zero limits of the normalised Γ_{12} elements, however, ultimately have no effect on the properties of $\hat{\psi} - \psi$ because they are subject to additional normalisation through which they are eliminated asymptotically. The one instance where such effects are not eliminated concerns the element p_2 under GLS detrending in scheme 3. Here it is possible to show that $p_2 = p_{21} + p_{22}$ where

$$\begin{aligned}
p_{21} &= \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} thU_{th} + \left[Ne^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) + he^{\bar{c}/T}\right] U_{Th}, \\
p_{22} &= -\frac{h}{2}\left[e^{\bar{c}/T}U_h + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} U_{th} + e^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) U_{Th}\right].
\end{aligned}$$

The first term is of the same form as p_2 for stocks (see the proof of Theorem 3(b)) so the limits are given in Lemma C3 (with σ replaced by ω). For the second term we have:

$$\begin{aligned}
&\text{Scheme 1:} \quad && \frac{1}{N^{1/2}} p_{22} \Rightarrow 0, \\
&\text{Scheme 2:} \quad && \frac{1}{hN^{1/2}} p_{22} \Rightarrow 0, \\
&\text{Scheme 3:} \quad && \frac{1}{h} p_{22} \Rightarrow -\frac{1}{2}u(0).
\end{aligned}$$

Hence the limits for p_2 are of the same form as stocks in schemes 1 and 2 but it is the non-zero limit of $(1/h)p_{22}$ in scheme 3 that ultimately feeds through into the constant k_4' defined in the Theorem. □

Proof of Theorem 7. We begin by noting that the detrended series is given by

$$y_{th}^d = y_{th} - \hat{\psi}_0 = u_{th} - (\hat{\psi}_0 - \psi_0),$$

using the fact that $y_{th} = \psi_0 + u_{th}$. In order to derive the law of motion for y_{th}^d we substitute $u_{th} = y_{th}^d + (\hat{\psi}_0 - \psi_0)$ into the equation for u_{th} in (17), yielding

$$y_{th}^d = \phi_h y_{th-h}^d + v_{th} - \lambda_{th},$$

where $\lambda_{th} = (\hat{\psi}_0 - \psi_0)(1 - \phi_h)$. It therefore follows that

$$\hat{\phi}_h = \frac{\sum_{t=1}^{T} y_{th-h}^d y_{th}^d}{\sum_{t=1}^{T} (y_{th-h}^d)^2} = \phi_h + \frac{\sum_{t=1}^{T} y_{th-h}^d (v_{th} - \lambda_{th})}{\sum_{t=1}^{T} (y_{th-h}^d)^2}.$$

Given that $\sum_{t=1}^{T} y_{th-h}^d \lambda_{th} = (1-\phi_h)(\hat{\psi}_0-\psi_0) \sum_{t=1}^{T} y_{th-h}^d$ it follows that we need to determine the limit properties of the following three quantities: (i) $\sum_{t=1}^{T} y_{th-h}^d$; (ii) $\sum_{t=1}^{T}(y_{th-h}^d)^2$; and (iii) $\sum_{t=1}^{T} y_{th-h}^d v_{th}$. The aim is to express these quantities in terms of $y_{[Tr]h}^d$ and then to use the results in Theorem 3 in conjunction with the continuous mapping theorem. It follows straightforwardly that

$$\frac{h}{N}\sum_{t=1}^{T} y_{th-h}^d = \int_0^1 y_{[Tr]h}^d dr, \quad \frac{h}{N}\sum_{t=1}^{T}(y_{th-h}^d)^2 = \int_0^1 (y_{[Tr]h}^d)^2 dr,$$

noting that $h/N = 1/T$, while use of the expression $v_{th} = h^{1/2}T^{1/2}\int_{(t-1)/T}^{t/T} dx_T(r)$ results in (using $h^{1/2}T^{1/2} = N^{1/2}$)

$$\sum_{t=1}^{T} y_{th-h}^d v_{th} = N^{1/2}\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T} y_{[Tr]h}^d dx_T(r) = N^{1/2}\int_0^1 y_{[Tr]h}^d dx_T(r),$$

where $x_T(r)$ is defined in Lemma 2. Taking each sampling scheme in turn:
Scheme 1: From Theorem 3 we know that $N^{-1/2}y_{[Tr]h}^d \Rightarrow \sigma J_c(r)$ and so

$$\frac{1}{N^{3/2}}\sum_{t=1}^{T} y_{th-h}^d = \frac{1}{h}\int_0^1 \frac{1}{N^{1/2}} y_{[Tr]h}^d dr \Rightarrow \frac{\sigma}{h}\int_0^1 J_c(r)dr,$$

$$\frac{1}{N^2}\sum_{t=1}^{T}(y_{th-h}^d)^2 = \frac{1}{h}\int_0^1 \left(\frac{1}{N^{1/2}} y_{[Tr]h}^d\right)^2 dr \Rightarrow \frac{\sigma^2}{h}\int_0^1 J_c(r)^2 dr,$$

$$\frac{1}{N}\sum_{t=1}^{T} y_{th-h}^d v_{th} = \int_0^1 \frac{1}{N^{1/2}} y_{[Tr]h}^d dx_T(r) \Rightarrow \sigma^2\int_0^1 J_c(r)dW(r),$$

the last result using the convergence of $x_T(r)$ in Lemma 2. Theorem 3 also shows that $\hat{\psi}_0 - \psi_0 \Rightarrow u_h$ and so

$$\frac{1}{N^{1/2}}\sum_{t=1}^{T} y_{th-h}^d \lambda_{th} = N(1-e^{\alpha h})(\hat{\psi}_0 - \psi_0)\frac{1}{N^{3/2}}\sum_{t=1}^{T} y_{th-h}^d \Rightarrow -c\sigma u_h \int_0^1 J_c(r)dr,$$

which also uses Lemma C1 for $1 - e^{\alpha h} = 1 - e^{c/T}$. It follows that

$$N(\hat{\phi}_h - \phi_h) = \frac{\frac{1}{N}\sum_{t=1}^{T} y_{th-h}^d v_{th}}{\frac{1}{N^2}\sum_{t=1}^{T}(y_{th-h}^d)^2} + o_p(1)$$

which leads to the stated limit distribution.
Scheme 2: Proceeding in a similar fashion we find that

$$\frac{h}{N^{3/2}}\sum_{t=1}^{T} y_{th-h}^d = \int_0^1 \frac{1}{N^{1/2}} y_{[Tr]h}^d dr \Rightarrow \sigma\int_0^1 J_c(r)dr,$$

$$\frac{h}{N^2}\sum_{t=1}^{T}(y_{th-h}^d)^2 = \int_0^1 \left(\frac{1}{N^{1/2}} y_{[Tr]h}^d\right)^2 dr \Rightarrow \sigma^2 \int_0^1 J_c(r)^2 dr,$$

$$\frac{1}{N}\sum_{t=1}^{T} y_{th-h}^d v_{th} = \int_0^1 \frac{1}{N^{1/2}} y_{[Tr]h}^d dx_T(r) \Rightarrow \sigma^2 \int_0^1 J_c(r) dW(r),$$

$$\frac{1}{N^{1/2}}\sum_{t=1}^{T} y_{th-h}^d \lambda_{th} = \frac{N}{h}(1-e^{\alpha h})(\hat{\psi}_0 - \psi_0)\frac{h}{N^{3/2}}\sum_{t=1}^{T} y_{th-h}^d \Rightarrow -c\sigma u(0)\int_0^1 J_c(r) dr,$$

resulting in

$$\frac{N}{h}(\hat{\phi}_h - \phi_h) = \frac{\frac{1}{N}\sum_{t=1}^{T} y_{th-h}^d v_{th}}{\frac{h}{N^2}\sum_{t=1}^{T}(y_{th-h}^d)^2} + o_p(1)$$

and the limit distribution follows.
Scheme 3: In this case $y_{[Tr]h}^d \Rightarrow \sigma N^{1/2} J_c(r) - (1-e^{cr})u(0)$ and so:

$$h\sum_{t=1}^{T} y_{th-h}^d = N\int_0^1 y_{[Tr]h}^d dr \Rightarrow N\int_0^1 \left(\sigma N^{1/2} J_c(r) - (1-e^{cr})u(0)\right) dr,$$

$$h\sum_{t=1}^{T}(y_{th-h}^d)^2 = N\int_0^1 \left(y_{[Tr]h}^d\right)^2 dr \Rightarrow N\int_0^1 \left(\sigma N^{1/2} J_c(r) - (1-e^{cr})u(0)\right)^2 dr,$$

$$\sum_{t=1}^{T} y_{th-h}^d v_{th} = N^{1/2}\int_0^1 y_{[Tr]h}^d dx_T(r) \Rightarrow N^{1/2}\int_0^1 \left(\sigma N^{1/2} J_c(r) - (1-e^{cr})u(0)\right)\sigma dW(r),$$

$$\begin{aligned}\sum_{t=1}^{T} y_{th-h}^d \lambda_{th} &= \frac{1}{h}(1-e^{\alpha h})(\hat{\psi}_0 - \psi_0) h\sum_{t=1}^{T} y_{th-h}^d \\ &\Rightarrow -cu(0)\int_0^1 \left(\sigma N^{1/2} J_c(r) - (1-e^{cr})u(0)\right) dr.\end{aligned}$$

Combining these results and rearranging yields the required limit distribution.

Finally, turning to $\hat{\alpha} = (1/h)\log\hat{\phi}_h$, we begin with a mean value expansion of $\log\hat{\phi}_h$ around ϕ_h, yielding

$$\log\hat{\phi}_h = \log\phi_h + \frac{1}{\phi_h^*}(\hat{\phi}_h - \phi_h),$$

where $|\phi_h^* - \phi_h| \leq |\hat{\phi}_h - \phi_h|$. As in the proof of Theorem 3.3 of Wang and Yu (2014) it can be shown that $\lim_{T\to\infty}\Pr(|\phi_h^* - 1| > \epsilon) = 0$ for some $\epsilon > 0$; this applies to all three sampling schemes as $T \to \infty$ in each of them. Hence

$$N(\hat{\alpha} - \alpha) = \frac{N}{h}(\log\hat{\phi}_h - \log\phi_h) = \frac{N}{h}(\hat{\phi}_h - \phi_h) + o_p(1).$$

The results follow immediately. □

Appendix B. Proofs of lemmas

Proof of Lemma 1. For a stock variable, $y_{th} = y_c(th)$ and (4) is immediate. The dynamics for u_{th} are obtained from the unique mean square solution to (2), given by

$$u(th) = e^{\alpha th}u(0) + \int_0^{th} e^{\alpha(th-r)}\eta(r)dr. \tag{22}$$

This can be used to relate $u_{th} = u(th)$ to u_{th-h}:

$$\begin{aligned} u(th) &= e^{\alpha h}\left(e^{\alpha(th-h)}u(0) + \int_0^{th-h} e^{\alpha(th-h-r)}\eta(r)dr\right) + \int_{th-h}^{th} e^{\alpha(th-r)}\eta(r)dr \\ &= e^{\alpha h}u(th-h) + \int_{th-h}^{th} e^{\alpha(th-r)}\eta(r)dr, \end{aligned} \tag{23}$$

which yields (5). The results for a flow variable are obtained by integrating (1):

$$Y_{th} = \frac{1}{h}\int_{th-h}^{th} y_c(r)dr = \frac{1}{h}\int_{th-h}^{th} (\psi_0 + \psi_1 r + u(r))\, dr.$$

Noting that $\int_{th-h}^{th} rdr = th^2 - (h^2/2)$ yields (6). The equation relating U_h to $u(0)$ is obtained by integrating (22) over the interval $(0, h]$, yielding

$$U_h = \frac{1}{h}\int_0^h e^{\alpha s}dsu(0) + \frac{1}{h}\int_0^h \int_0^s e^{\alpha(s-r)}\eta(r)drds.$$

The stated equation results from evaluating the integral multiplying $u(0)$. Finally, the equation for the law of motion of $U_{th} = (1/h)\int_{th-h}^{th} u(r)dr$ for $t = 2, \ldots, T$ is obtained via a further integration of (23). □

Proof of Lemma 2. We verify that the conditions of Corollary 2.2 of Phillips and Durlauf (1986) are satisfied; these are essentially the same as those in Assumption 3 but applied to v_{th} instead of $\eta(t)$. Assumption 3(a) implies immediately that $Ev_{th} = 0$ while from Assumption 3(b) we obtain

$$E|v_{th}|^\beta = E\left|\int_0^h e^{\alpha s}\eta(th-s)ds\right|^\beta \leq E|\eta(th)|^\beta \left[\int_0^h e^{\alpha s}ds\right]^\beta < \infty,$$

the inequality arising from Lemma A3 of Chambers (2003). To verify the mixing condition we note that v_{th} is a measurable function of $\eta(t)$ over a finite interval and so inherits the same mixing properties; see, for example Theorem 14.1 of Davidson (1994). Thus if α_j^v denotes the strong mixing coefficient for v_{th} then $\sum_{j=1}^\infty (\alpha_j^v)^{1-2/\beta} < \infty$ under Assumption 3(c). The validity of the invariance principle is thereby established.

To derive the long run variance σ^2, we use the representation $\sigma^2 = 2\pi f_v^d(0)$, where $f_v^d(\lambda)$ $(-\pi < \lambda \leq \pi)$ denotes the spectral density function of the discrete time process v_{th}. In order to derive $f_v^d(\lambda)$ we first derive the spectral density of v_{th} *as a continuous time* *process*, denoted $f_v^c(\lambda)$ $(-\infty < \lambda < \infty)$, and then apply the folding formula

$$f_v^d(\lambda) = \frac{1}{h} \sum_{j=-\infty}^{\infty} f_v^c\left(\frac{\lambda + 2\pi j}{h}\right), \quad -\pi < \lambda \leq \pi;$$

see, for example, Grenander and Rosenblatt (1957, p.57). It is convenient to use filter notation to link $\eta(t)$ and v_{th}:

$$v_{th} = \int_0^h e^{\alpha s}\eta(th - s)ds = \int_0^h e^{\alpha s}e^{-sD}\eta(th)ds = g_h^\alpha(D)\eta(th),$$

noting that $\eta(th - s) = e^{-sD}\eta(th)$ and where

$$g_h^\alpha(z) = \int_0^h e^{(\alpha - z)s}ds = \frac{e^{(\alpha - z)h} - 1}{\alpha - z}.$$

As a continuous time process the spectral density of v_{th} is

$$f_v^c(\lambda) = |g_h^\alpha(-i\lambda)|^2 f_\eta(\lambda) = \left(\frac{1 + e^{2\alpha h} - 2e^{\alpha h}\cos h\lambda}{\alpha^2 + \lambda^2}\right) f_\eta(\lambda), \quad -\infty < \lambda < \infty.$$

Applying the folding formula yields

$$f_v^d(\lambda) = h\left(1 + e^{2\alpha h} - 2e^{\alpha h}\cos\lambda\right) \sum_{j=-\infty}^{\infty} \frac{f_\eta(\lambda + 2\pi j/h)}{h^2\alpha^2 + (\lambda + 2\pi j)^2}, \quad -\pi < \lambda \leq \pi.$$

When $\alpha < 0$ setting $\lambda = 0$ yields the stated result straightforwardly. However, when $\alpha = 0$, note that

$$(1 - e^{\alpha h})^2 = \alpha^2 h^2 + O(\alpha^3)$$

as $\alpha \to 0$, thereby nullifying the contributions for $j \neq 0$, which leaves the term for $j = 0$ and the result follows. □

Proof of Lemma 3. First note that we can write, by backward substitution,

$$u_{th} = e^{\alpha h}u_{th-h} + v_{th} = e^{\alpha th}u(0) + \sum_{j=1}^{t} e^{\alpha(t-j)h}v_{jh} = e^{ct/T}u(0) + \sum_{j=1}^{t} e^{c(t-j)/T}v_{jh},$$

using the fact that $\alpha h = ch/N = c/T$. Consider, first,

$$\begin{aligned} u_{[Tr]h} &= e^{c[Tr]/T}u(0) + \sum_{j=1}^{[Tr]} e^{c([Tr]-j)/T}v_{jh} \\ &= e^{c[Tr]/T}u(0) + \sum_{j=1}^{[Tr]} e^{c([Tr]-j)/T}h^{1/2}T^{1/2}\int_{(j-1)/T}^{j/T} dx_T(s) \\ &= e^{c[Tr]/T}u(0) + N^{1/2}\int_0^r e^{(r-s)c}dx_T(s). \end{aligned}$$

The limits for $u_{[Tr]h}$ follow using the invariance principle for $x_T(r)$ in Lemma 2.

Turning to sums of u_{th} we obtain

$$\begin{aligned}
\sum_{t=1}^{T} u_{th} &= \sum_{t=1}^{T} e^{ct/T} u(0) + \sum_{t=1}^{T}\sum_{j=1}^{t} e^{c(t-j)/T} v_{jh} \\
&= T\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T} e^{ct/T} dr u(0) \\
&\quad + h^{1/2}T^{3/2}\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T} dr \sum_{j=1}^{t} e^{c(t-j)/T}\int_{(j-1)/T}^{j/T} dx_T(s) \\
&= \frac{N}{h}\int_0^1 e^{cr} dr u(0) + \frac{N^{3/2}}{h}\int_0^1\int_0^r e^{(r-s)c} dx_T(s) dr.
\end{aligned}$$

The results follow by noting that $\int_0^1 e^{cr}dr = (e^c - 1)/c = \delta_c$.

Finally, sums of thu_{th} are handled by noting that

$$\sum_{t=1}^{T} thu_{th} = \sum_{t=1}^{T} the^{ct/T} u(0) + \sum_{t=1}^{T} th \sum_{j=1}^{t} e^{c(t-j)/T} v_{jh} = Ahu(0) + hB,$$

where

$$\begin{aligned}
A &= \sum_{t=1}^{T} te^{ct/T} = T^2 \sum_{t=1}^{T}\left(\frac{t}{T}\right)\int_{(t-1)/T}^{t/T} e^{ct/T} dr = T^2\int_0^1 re^{cr} dr = T^2\mu_c, \\
B &= \sum_{t=1}^{T} t \sum_{j=1}^{t} e^{c(t-j)/T} v_{jh} \\
&= h^{1/2}T^{5/2}\sum_{t=1}^{T}\left(\frac{t}{T}\right)\int_{(t-1)/T}^{t/T} dr \sum_{j=1}^{t} e^{c(t-j)/T}\int_{(j-1)/T}^{j/T} dx_T(s) \\
&= h^{1/2}T^{5/2}\int_0^1 r\int_0^{r-1/T} e^{(r-s)c} dx_T(s) dr,
\end{aligned}$$

and where $\mu_c = \int_0^1 re^{cr}dr = [1 - (1-c)e^c]/c^2$. It follows that

$$\sum_{t=1}^{T} thu_{th-h} = \frac{N^2}{h}\mu_c u(0) + \frac{N^{5/2}}{h}\int_0^1 r\int_0^{r-1/T} e^{(r-s)c} dx_T(s) dr,$$

and the stated results are a consequence of this expression. □

Proof of Lemma 4. It is convenient to define the random process

$$\xi_{th} = \int_{th-h}^{th} \eta(s) ds, \quad t = 1, \ldots, T,$$

which is simply v_{th} when $\alpha = 0$. This enables us to express V_{th} in terms of v_{th} and ξ_{th}, using the expressions in Remark 3, in the form

$$\begin{aligned} V_h &= \frac{1}{\alpha h} v_h - \frac{1}{\alpha h} \xi_h, \\ V_{th} &= \frac{1}{\alpha h} v_{th} - \frac{1}{\alpha h} \xi_{th} + \frac{e^{\alpha h}}{\alpha h} \xi_{th-h} - \frac{1}{\alpha h} v_{th-h}, \\ &= \frac{1}{\alpha h} (v_{th} - v_{th-h}) - \frac{1}{\alpha h} (\xi_{th} - e^{\alpha h} \xi_{th-h}), \quad t = 2, \ldots, T. \end{aligned}$$

Now consider

$$\begin{aligned} \sum_{t=1}^{T} V_{th} &= \frac{1}{\alpha h} v_h - \frac{1}{\alpha h} \xi_h + \frac{1}{\alpha h} \sum_{t=2}^{T} (v_{th} - v_{th-h}) - \frac{1}{\alpha h} \sum_{t=2}^{T} (\xi_{th} - e^{\alpha h} \xi_{th-h}) \\ &= \frac{1}{\alpha h} v_{Th} - \frac{1}{\alpha h} \left(\xi_{Th} + \sum_{t=2}^{T-1} (1 - e^{\alpha h}) \xi_{th} - e^{\alpha h} \xi_h + \xi_h \right) \\ &= \frac{1}{\alpha h} v_{Th} - \frac{1}{\alpha h} \left((1 - e^{\alpha h}) \sum_{t=1}^{T} \xi_{th} + e^{\alpha h} \xi_{Th} \right). \end{aligned}$$

We know, from Lemma 2, that (setting $\alpha = 0$),

$$\frac{1}{T^{1/2}} \sum_{t=1}^{[Tr]} \left(\frac{\xi_{th}}{h^{1/2}} \right) \Rightarrow \sigma W(r)$$

as $T \to \infty$, where $\sigma^2 = 2\pi h f_\eta(0)$. It follows that

$$\frac{1}{T^{1/2}} \sum_{t=1}^{[Tr]} \left(\frac{V_{th}}{h^{1/2}} \right) = -\frac{(1 - e^{\alpha h})}{\alpha h} \frac{1}{T^{1/2}} \sum_{t=1}^{[Tr]} \left(\frac{\xi_{th}}{h^{1/2}} \right) + o_p(1) \Rightarrow \omega W(r),$$

as $T \to \infty$, where the long run variance is given by

$$\omega^2 = \sigma^2 \left(\frac{(1 - e^{\alpha h})}{\alpha h} \right)^2 = \frac{2\pi (1 - e^{\alpha h})^2 f_\eta(0)}{h \alpha^2}$$

when $\alpha \neq 0$. The result for the case when $\alpha = 0$ is obtained from above by noting that

$$\frac{(1 - e^{\alpha h})}{\alpha h} = 1 + O(\alpha)$$

as $\alpha \to 0$, yielding the stated expression for ω^2.[7] □

Appendix C. Supplementary lemmas

Lemma C1. *Let $\bar{c}$ be a fixed constant and let $T = N/h$. Then:*
Scheme 1: As $N \to \infty$ with h fixed,

$$N \left(1 - e^{\bar{c}/T} \right) \to -\bar{c} h, \quad N^2 \left(1 - e^{\bar{c}/T} \right)^2 \to \bar{c}^2 h^2, \quad N e^{\bar{c}/T} \left(1 - e^{\bar{c}/T} \right) \to -\bar{c} h.$$

[7] The same result can also be obtained using filters in the same spirit as the proof of Lemma 2.

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\frac{N}{h}\left(1-e^{\bar{c}/T}\right) \to -\bar{c}, \quad \frac{N^2}{h^2}\left(1-e^{\bar{c}/T}\right)^2 \to \bar{c}^2, \quad \frac{N}{h}e^{\bar{c}/T}\left(1-e^{\bar{c}/T}\right) \to -\bar{c}.$$

Scheme 3. As $h \to 0$ with N fixed,

$$\frac{1}{h}\left(1-e^{\bar{c}/T}\right) \to -\frac{\bar{c}}{N}, \quad \frac{1}{h^2}\left(1-e^{\bar{c}/T}\right)^2 \to \frac{\bar{c}^2}{N^2}, \quad \frac{1}{h}e^{\bar{c}/T}\left(1-e^{\bar{c}/T}\right) \to -\frac{\bar{c}}{N}.$$

Proof of Lemma C1. All proofs follow from the expansion

$$e^{\bar{c}/T} = 1 + \frac{\bar{c}}{T} + O\left(\frac{1}{T^2}\right) = 1 + \frac{\bar{c}h}{N} + O\left(\frac{h^2}{N^2}\right)$$

which can be used to establish the limits of the stated quantities under the different sampling schemes. □

Lemma C2. *Let*

$$Q_{11} = T, \quad Q_{12} = \sum_{t=1}^{T} th, \quad Q_{22} = \sum_{t=1}^{T} (th)^2.$$

Then:
Scheme 1: As $N \to \infty$ with h fixed,

$$\frac{1}{N}Q_{11} = \frac{1}{h}, \quad \frac{1}{N^2}Q_{12} \to \frac{1}{2h}, \quad \frac{1}{N^3}Q_{22} \to \frac{1}{3h}.$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$\frac{h}{N}Q_{11} = 1, \quad \frac{h}{N^2}Q_{12} \to \frac{1}{2}, \quad \frac{h}{N^3}Q_{22} \to \frac{1}{3}.$$

Scheme 3. As $h \to 0$ with N fixed,

$$hQ_{11} = N, \quad hQ_{12} \to \frac{N^2}{2}, \quad hQ_{22} \to \frac{N^3}{3}.$$

Proof of Lemma C2. The results for $Q_{11} = T = N/h$ are immediate. For Q_{12} we have

$$Q_{12} = h\sum_{t=1}^{T} t = \frac{h}{2}T(T+1) = \frac{1}{2}\left(\frac{N^2}{h} + N\right),$$

while for Q_{22} a similar procedure establishes that

$$Q_{22} = h^2\sum_{t=1}^{T} t^2 = \frac{h^2}{6}T(T+1)(2T+1) = \frac{1}{6}\left(\frac{2N^3}{h} + 3N^2 + hN\right).$$

The results follow straightforwardly. □

Lemma C3. *Let*

$$\begin{aligned}
Q_{11} &= 1+(T-1)\left(1-e^{\bar{c}/T}\right)^2, \\
Q_{12} &= h+\left(1-e^{\bar{c}/T}\right)\sum_{t=2}^{T}\left(th-e^{\bar{c}/T}(th-h)\right), \\
Q_{22} &= h^2+\sum_{t=2}^{T}\left(th-e^{\bar{c}/T}(th-h)\right)^2, \\
p_1 &= u_h+\left(1-e^{\bar{c}/T}\right)\sum_{t=2}^{T}\left(u_{th}-e^{\bar{c}/T}u_{th-h}\right), \\
p_2 &= hu_h+\sum_{t=2}^{T}\left(th-e^{\bar{c}/T}(th-h)\right)\left(u_{th}-e^{\bar{c}/T}u_{th-h}\right).
\end{aligned}$$

Then:
Scheme 1: As $N \to \infty$ with h fixed,

$$Q_{11} \to 1, \quad Q_{12} \to h\left(1-\bar{c}+\frac{\bar{c}^2}{2}\right), \quad \frac{1}{N}Q_{22} \to h\left(1-\bar{c}+\frac{\bar{c}^2}{3}\right),$$

$$p_1 \Rightarrow u_h, \quad \frac{1}{N^{1/2}}p_2 \Rightarrow \sigma h\left[(1-\bar{c})J_c(1)+\bar{c}^2\int_0^1 rJ_c(r)dr\right].$$

Scheme 2. As $N \to \infty$ and $h \to 0$,

$$Q_{11} \to 1, \quad \frac{1}{h}Q_{12} \to \left(1-\bar{c}+\frac{\bar{c}^2}{2}\right), \quad \frac{1}{hN}Q_{22} \to \left(1-\bar{c}+\frac{\bar{c}^2}{3}\right),$$

$$p_1 \Rightarrow u(0), \quad \frac{1}{hN^{1/2}}p_2 \Rightarrow \sigma\left[(1-\bar{c})J_c(1)+\bar{c}^2\int_0^1 rJ_c(r)dr\right].$$

Scheme 3. As $h \to 0$ with N fixed,

$$Q_{11} \to 1, \quad \frac{1}{h}Q_{12} \to \left(1-\bar{c}+\frac{\bar{c}^2}{2}\right), \quad \frac{1}{h}Q_{22} \to N\left(1-\bar{c}+\frac{\bar{c}^2}{2}\right).$$

$$p_1 \Rightarrow u(0), \quad \frac{1}{h}p_2 \Rightarrow \left[(1-\bar{c})e^c+\bar{c}^2\mu_c\right]u(0)+\sigma N^{1/2}\left[(1-\bar{c})J_c(1)+\bar{c}^2\int_0^1 rJ_c(r)dr\right].$$

Proof of Lemma C3. From the series expansion of $e^{\bar{c}/T}$ it can be shown that

$$\left(1-e^{\bar{c}/T}\right)^2 = \frac{\bar{c}^2}{T^2}+O\left(\frac{1}{T^3}\right) = \frac{\bar{c}^2h^2}{N^2}+O\left(\frac{h^3}{N^3}\right)$$

and hence

$$Q_{11} = 1+\left(\frac{N}{h}-1\right)\left[\frac{\bar{c}^2h^2}{N^2}+O\left(\frac{h^3}{N^3}\right)\right] = 1+\frac{\bar{c}^2h}{N}+O\left(\frac{h^2}{N^2}\right) \to 1$$

in all three sampling schemes. For Q_{12} we can write

$$Q_{12} = h + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=2}^{T} th + (T-1)he^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right).$$

Now $(T-1)h = N - h$ and $\sum_{t=2}^{T} th = \sum_{t=1}^{T} th - h$ which gives

$$\begin{aligned} Q_{12} &= h + \left(1 - e^{\bar{c}/T}\right)^2 \left(\sum_{t=1}^{T} th - h\right) + (N-h)e^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) \\ &= h\left[1 - \left(1 - e^{\bar{c}/T}\right)^2 - e^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right)\right] + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} th + Ne^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right). \end{aligned}$$

But the coefficient multiplying h is simply $e^{\bar{c}/T}$ while $\sum_{t=1}^{T} th$ can be simplified (see the entry for Q_{12} in Lemma C2), resulting in

$$Q_{12} = he^{\bar{c}/T} + Ne^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) + \frac{1}{2}\left(\frac{N^2}{h} + N\right)\left(1 - e^{\bar{c}/T}\right)^2.$$

Turning to Q_{22} note that

$$\begin{aligned} \left(th - e^{\bar{c}/T}(th - h)\right)^2 &= \left[\left(1 - e^{\bar{c}/T}\right) th + he^{\bar{c}/T}\right]^2 \\ &= \left(1 - e^{\bar{c}/T}\right)^2 (th)^2 + 2he^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) th + h^2 e^{2\bar{c}/T}. \end{aligned}$$

It follows that

$$\begin{aligned} Q_{22} &= h^2 + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=2}^{T} (th)^2 + 2he^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} th + h^2(T-1)e^{2\bar{c}/T} \\ &= h^2 + \left(1 - e^{\bar{c}/T}\right)^2 \left(\sum_{t=1}^{T} (th)^2 - h^2\right) + 2he^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right)\left(\sum_{t=1}^{T} th - h\right) \\ &\quad + h^2(T-1)e^{2\bar{c}/T} \\ &= h^2\left[1 - \left(1 - e^{\bar{c}/T}\right)^2 - 2e^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) - e^{2\bar{c}/T}\right] + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} (th)^2 \\ &\quad + 2he^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) \sum_{t=1}^{T} th + h^2 T e^{2\bar{c}/T} \end{aligned}$$

But the coefficient on h^2 is zero while $h^2 T = hN$ and Lemma C2 provides expressions for the sums involving th and $(th)^2$. Utilising these simplifications results in

$$Q_{22} = \frac{1}{6}\left(\frac{2N^3}{h} + 2N^2 + hN\right)\left(1 - e^{\bar{c}/T}\right)^2 + (N^2 + hN)e^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) + hNe^{2\bar{c}/T}.$$

The limits for these quantities under each sampling scheme now follow.

Turning to p_1 we have

$$\begin{aligned}
p_1 &= u_h + \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} \left(u_{th} - e^{\bar{c}/T} u_{th-h}\right) \\
&= u_h + \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} u_{th} - e^{\bar{c}/T} \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} u_{th-h} \\
&= u_h + \left(1 - e^{\bar{c}/T}\right) \left(\sum_{t=1}^{T} u_{th} - u_h\right) - e^{\bar{c}/T} \left(1 - e^{\bar{c}/T}\right) \left(\sum_{t=1}^{T} u_{th} - u_{Th}\right) \\
&= e^{\bar{c}/T} u_h + \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} u_{th} + e^{\bar{c}/T} \left(1 - e^{\bar{c}/T}\right) u_{Th}.
\end{aligned}$$

The limit results for u_{th} in Lemma 3 allied with the limits in Lemma C1 for expressions involving $e^{\bar{c}/T}$ can be used to establish that

$$p_1 \Rightarrow \begin{cases} u_h, & \text{scheme 1,} \\ u(0), & \text{schemes 2 and 3.} \end{cases}$$

Finally, turning to p_2 we find that

$$\begin{aligned}
p_2 &= hu_h + \sum_{t=2}^{T} \left(th - e^{\bar{c}/T}(th - h)\right) \left(u_{th} - e^{\bar{c}/T} u_{th-h}\right) \\
&= hu_h + \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} th \left(u_{th} - e^{\bar{c}/T} u_{th-h}\right) + he^{\bar{c}/T} \sum_{t=2}^{T} \left(u_{th} - e^{\bar{c}/T} u_{th-h}\right) \\
&= hu_h + \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} thu_{th} - e^{\bar{c}/T} \left(1 - e^{\bar{c}/T}\right) \sum_{t=2}^{T} thu_{th-h} \\
&\quad + he^{\bar{c}/T} \sum_{t=2}^{T} u_{th} - he^{2\bar{c}/T} \sum_{t=2}^{T} u_{th-h}.
\end{aligned}$$

We can use the substitutions

$$\sum_{t=2}^{T} thu_{th} = \sum_{t=1}^{T} thu_{th} - hu_h,$$

$$\begin{aligned}
\sum_{t=2}^{T} thu_{th-h} = \sum_{t=1}^{T-1} (th + h) u_{th} &= \sum_{t=1}^{T} (th + h) u_{th} - (Th + h) u_{Th} \\
&= \sum_{t=1}^{T} thu_{th} + h \sum_{t=1}^{T} u_{th} - (Th + h) u_{Th},
\end{aligned}$$

$$\sum_{t=2}^{T} u_{th} = \sum_{t=1}^{T} u_{th} - u_h, \quad \sum_{t=2}^{T} u_{th-h} = \sum_{t=1}^{T-1} u_{th} = \sum_{t=1}^{T} u_{th} - u_{Th},$$

which enable the results in Lemma 3 to be applied directly. Making these substitutions and rearranging it can be shown that

$$p_2 = \left(1 - e^{\bar{c}/T}\right)^2 \sum_{t=1}^{T} thu_{th} + \left[Ne^{\bar{c}/T}\left(1 - e^{\bar{c}/T}\right) + he^{\bar{c}/T}\right] u_{Th}.$$

Lemmas 3 and C1 then yield the limit properties of p_2 under the three sampling schemes. □

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Table 1

Transformed variables for detrending regressions

Stock variable							
$\bar{\alpha}$	$e^{\bar{\alpha}h}$	$\tilde{y}_h$	$\tilde{z}_{1,h}$	$\tilde{z}_{2,h}$	$\tilde{y}_{th}$	$\tilde{z}_{1,th}$	$\tilde{z}_{2,th}$
GLS detrending							
$\frac{\bar{c}}{N}$	$e^{\bar{c}/T}$	y_h	1	h	$y_{th} - e^{\bar{c}/T} y_{th-h}$	$1 - e^{\bar{c}/T}$	$th - e^{\bar{c}/T}(th - h)$
Differencing							
0	1	y_h	1	h	$y_{th} - y_{th-h}$	0	h
OLS detrending							
$-\infty$	0	y_h	1	h	y_{th}	1	th

Flow variable							
$\bar{\alpha}$	$e^{\bar{\alpha}h}$	$\tilde{Y}_h$	$\tilde{Z}_{1,h}$	$\tilde{Z}_{2,h}$	$\tilde{Y}_{th}$	$\tilde{Z}_{1,th}$	$\tilde{Z}_{2,th}$
GLS detrending							
$\frac{\bar{c}}{N}$	$e^{\bar{c}/T}$	Y_h	1	$\frac{h}{2}$	$Y_{th} - e^{\bar{c}/T} Y_{th-h}$	$1 - e^{\bar{c}/T}$	$th - \frac{h}{2} - e^{\bar{c}/T}\left(th - \frac{3h}{2}\right)$
Differencing							
0	1	Y_h	1	$\frac{h}{2}$	$Y_{th} - Y_{th-h}$	0	h
OLS detrending							
$-\infty$	0	Y_h	1	$\frac{h}{2}$	Y_{th}	1	$th - \frac{h}{2}$

Table 2

Discrete time autoregressive parameter ($\phi_h = e^{\alpha h} = e^{c/T}$)

	c								
h	0.0	−2.5	−5.0	−7.5	−10.0	−12.5	−15.0	−17.5	−20.0
$N = 25$									
1	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493
1/4	1.0000	0.9753	0.9512	0.9277	0.9048	0.8825	0.8607	0.8395	0.8187
1/52	1.0000	0.9981	0.9962	0.9942	0.9923	0.9904	0.9885	0.9866	0.9847
$N = 50$									
1	1.0000	0.9512	0.9048	0.8607	0.8187	0.7788	0.7408	0.7047	0.6703
1/4	1.0000	0.9876	0.9753	0.9632	0.9512	0.9394	0.9277	0.9162	0.9048
1/52	1.0000	0.9990	0.9981	0.9971	0.9962	0.9952	0.9942	0.9933	0.9923
$N = 100$									
1	1.0000	0.9753	0.9512	0.9277	0.9048	0.8825	0.8607	0.8395	0.8187
1/4	1.0000	0.9938	0.9876	0.9814	0.9753	0.9692	0.9632	0.9572	0.9512
1/52	1.0000	0.9995	0.9990	0.9986	0.9981	0.9976	0.9971	0.9966	0.9962
Proportion of negative estimates of ϕ_h when $N = 25$ and $h = 1$									
	0.0001	0.0005	0.0011	0.0022	0.0045	0.0085	0.0169	0.0252	0.0381

Table 3

Simulated size and power of unit root test statistic $N\hat{\alpha}$

		c								
N	h	0.0	−2.5	−5.0	−7.5	−10.0	−12.5	−15.0	−17.5	−20.0
Size ($c = 0$) and raw power										
25	1	14.67	34.39	54.85	72.54	84.75	92.16	95.83	97.83	98.90
	1/4	7.32	20.71	42.50	65.36	83.39	93.01	97.65	99.38	99.88
	1/52	5.08	14.43	32.35	55.53	76.58	90.59	97.12	99.31	99.90
50	1	10.56	26.57	47.83	68.71	84.17	92.92	97.10	98.91	99.66
	1/4	6.53	18.53	37.58	61.07	80.64	92.14	97.51	99.49	99.87
	1/52	5.13	14.98	32.18	55.11	76.36	90.40	96.97	99.35	99.90
100	1	7.60	21.05	41.28	64.62	82.36	92.89	97.86	99.51	99.90
	1/4	5.47	15.89	34.18	57.21	78.34	91.20	97.43	99.42	99.91
	1/52	4.94	14.32	31.43	54.45	75.16	89.90	96.91	99.35	99.92
Size ($c = 0$) and size-adjusted power										
25	1	14.67	13.46	25.70	40.62	55.17	68.64	79.06	85.92	90.49
	1/4	7.32	14.71	31.65	53.31	72.97	86.90	94.28	97.95	99.41
	1/52	5.08	14.20	32.04	54.95	76.08	90.37	96.97	99.28	99.89
50	1	10.56	13.26	26.65	44.03	61.41	76.92	87.20	93.50	96.87
	1/4	6.53	14.29	30.67	52.15	72.95	87.37	95.04	98.60	99.73
	1/52	5.13	14.66	31.70	54.46	75.96	90.17	96.86	99.31	99.88
100	1	7.60	14.75	31.42	52.31	71.86	86.52	94.27	98.15	99.50
	1/4	5.47	15.07	32.34	55.35	76.21	90.06	96.94	99.24	99.88
	1/52	4.94	14.41	31.63	54.73	75.40	90.04	96.95	99.36	99.92