SEMIGROUPS WITH OPERATION-COMPATIBLE GREEN'S QUASIORDERS

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Abstract. We call a semigroup on which the Green's quasiorder $\leq_{\mathcal{J}}$ ($\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$) is operation-compatible, a $\leq_{\mathcal{J}}$-compatible ($\leq_{\mathcal{L}}$-compatible, $\leq_{\mathcal{R}}$-compatible) semigroup. We study the classes of $\leq_{\mathcal{J}}$-compatible, $\leq_{\mathcal{L}}$-compatible and $\leq_{\mathcal{R}}$-compatible semigroups, using the smallest operation-compatible quasiorders containing Green's quasiorders as a tool. We prove a number of results, including the following. The class of $\leq_{\mathcal{L}}$-compatible ($\leq_{\mathcal{R}}$-compatible) semigroups is closed under taking homomorphic images. A regular periodic semigroup is $\leq_{\mathcal{J}}$-compatible if and only if it is a semilattice of simple semigroups. Every negatively orderable semigroup can be embedded into a negatively orderable $\leq_{\mathcal{J}}$-compatible semigroup.

1. Introduction

Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are one of the most important tools in studying the structure of semigroups. They can also be viewed from a less common angle: as being defined via quasiorders (or preorders), which we shall refer to as *Green's quasiorders* and denote by $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$, respectively. Studying the properties of these quasiorders is of interest, because of the importance of Green's relations and due to the fact that in a certain sense these associated quasiorders contain 'more information' about a semigroup than Green's relations: given only a Green's quasiorder on a semigroup we can reconstruct the corresponding Green's relation, whereas the converse is not true. We shall call a semigroup S *$\leq_{\mathcal{L}}$-compatible*, *$\leq_{\mathcal{R}}$-compatible* and *$\leq_{\mathcal{J}}$-compatible*, respectively, if $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ is operation-compatible on S. The aim of this paper is to explore some properties of the classes of *$\leq_{\mathcal{L}}$-compatible*, *$\leq_{\mathcal{R}}$-compatible* and *$\leq_{\mathcal{J}}$-compatible* semigroups. These classes are natural to consider; operation-compatible quasiorders have the convenient property that the equivalences induced by them are congruences, hence yield factor semigroups. We shall denote the smallest operation-compatible quasiorders containing $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ by $\leq_{\mathring{\mathcal{J}}}$,

$\leq_{\mathring{\mathcal{R}}}$ and $\leq_{\mathring{\mathcal{J}}}$, respectively. In [9] it was shown that there is a close connection between $\leq_{\mathring{\mathcal{J}}}$ and the filters of a semigroup, and thus $\leq_{\mathring{\mathcal{J}}}$ can be used to determine the lattice of filters and the largest semilattice image of a semigroup.

2. Definitions and observations

2.1. **Main concepts.** A *quasiorder* (or *preorder*) on a set is a reflexive and transitive relation. If S is a semigroup, by S^1 one denotes S if it has an identity element or, otherwise, S with an added identity element. We shall call *Green's quasiorders* the relations defined on every semigroup as follows:

Definition 2.1. For any elements s, t of a semigroup S let

- $s \leq_{\mathcal{L}} t$ if and only if $s = xt$ for some $x \in S^1$,
- $s \leq_{\mathcal{R}} t$ if and only if $s = ty$ for some $y \in S^1$,
- $s \leq_{\mathcal{J}} t$ if and only if $s = xty$ for some $x, y \in S^1$.

It is easy to show that the relations $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ are quasiorders and that $\mathcal{L} = \leq_{\mathcal{L}} \cap \leq_{\mathcal{L}}^{-1}$, $\mathcal{R} = \leq_{\mathcal{R}} \cap \leq_{\mathcal{R}}^{-1}$ and $\mathcal{J} = \leq_{\mathcal{J}} \cap \leq_{\mathcal{J}}^{-1}$.

A quasiorder $\leq$ on a semigroup S is *left (right) operation-compatible* if for all $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ ($ac \leq bc$). A quasiorder is *operation-compatible* if it is both left and right operation-compatible. Clearly, $\leq_{\mathcal{L}}$ ($\leq_{\mathcal{R}}$) is right (left) operation-compatible on every semigroup. However, Green's quasiorders are not operation-compatible in general. As operation-compatible quasiorders on any semigroup form a complete lattice, for any quasiorder on a semigroup there exists a smallest operation-compatible quasiorder containing it.

Definition 2.2. We call a semigroup $\leq_{\mathcal{J}}$*-compatible* ($\leq_{\mathcal{L}}$*-compatible*, $\leq_{\mathcal{R}}$*-compatible*) if $\leq_{\mathcal{J}}$ ($\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$) is operation-compatible on S.

Definition 2.3. Denote by $\leq_{\mathring{\mathcal{J}}}$ ($\leq_{\mathring{\mathcal{L}}}$, $\leq_{\mathring{\mathcal{R}}}$) the smallest operation-compatible quasiorder containing $\leq_{\mathcal{J}}$ ($\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$).

Relations $\leq_{\mathring{\mathcal{J}}}$, $\leq_{\mathring{\mathcal{L}}}$ and $\leq_{\mathring{\mathcal{R}}}$ will be a useful instrument for us in this paper because, obviously, a semigroup is $\leq_{\mathcal{J}}$*-compatible* ($\leq_{\mathcal{L}}$*-compatible*, $\leq_{\mathcal{R}}$*-compatible*) if and only if $\leq_{\mathcal{J}} = \leq_{\mathring{\mathcal{J}}}$ ($\leq_{\mathcal{L}} = \leq_{\mathring{\mathcal{L}}}$, $\leq_{\mathcal{R}} = \leq_{\mathring{\mathcal{R}}}$).

In [10] a description of $\leq_{\mathring{\mathcal{J}}}$, $\leq_{\mathring{\mathcal{J}}}$ and $\leq_{\mathring{\mathcal{R}}}$ was given. In Lemma 2.1 below we give another description, which will be convenient later. In this lemma, for any relation θ, $\overline{\theta}$ denotes the transitive closure of θ.

Let S be a semigroup. Define the relation $\prec_{\mathring{\mathcal{J}}}$ as follows: for any $s, t \in S$ let $s \prec_{\mathring{\mathcal{J}}} t$ if and only if $s = t_1 s_1 t_2$ and $t = t_1 t_2$ for some $t_1, t_2, s_1 \in S^1$. Define the relation $\prec_{\mathring{\mathcal{L}}}$ ($\prec_{\mathring{\mathcal{R}}}$) as follows: for any $s, t \in S$ let $s \prec_{\mathring{\mathcal{L}}} t$ ($s \prec_{\mathring{\mathcal{R}}} t$) if and only if $t_2 \in S$, $t_1, s_1 \in S^1$ ($t_1 \in S$, $t_2, s_1 \in S^1$).

Lemma 2.1. *In every semigroup*
- *(1)* $\leq_{\mathring{\mathcal{J}}} = \overline{\prec_{\mathring{\mathcal{J}}}}$
- *(2)* $\leq_{\mathring{\mathcal{L}}} = \overline{\prec_{\mathring{\mathcal{L}}}}$
- *(3)* $\leq_{\mathring{\mathcal{R}}} = \overline{\prec_{\mathring{\mathcal{R}}}}$.

Proof. We only prove Statement 1, since Statements 2 and 3 can be verified similarly.

If $a \leq_{\mathcal{J}} b$ then $a = sbt$ for some $s, t \in S^1$; since $a = sbt \prec_{\mathring{\mathcal{J}}} sb \prec_{\mathring{\mathcal{J}}} b$, we have $a \overline{\prec_{\mathring{\mathcal{J}}}} b$. Therefore, $\leq_{\mathcal{J}} \subseteq \overline{\prec_{\mathring{\mathcal{J}}}}$. It is obvious that if $a \prec_{\mathring{\mathcal{J}}} b$ then for any $s, t \in S^1$, $sat \prec_{\mathring{\mathcal{J}}} sbt$. Hence, if $a \overline{\prec_{\mathring{\mathcal{J}}}} b$ then for any $s, t \in S^1$ $sat \overline{\prec_{\mathring{\mathcal{J}}}} sbt$. Therefore, $\overline{\prec_{\mathring{\mathcal{J}}}}$ is operation-compatible. Obviously, $\overline{\prec_{\mathring{\mathcal{J}}}}$ is transitive. Therefore, $\leq_{\mathring{\mathcal{J}}} \subseteq \overline{\prec_{\mathring{\mathcal{J}}}}$.

It is obvious that $\prec_{\mathring{\mathcal{J}}}$ is contained in the operation-compatible closure of $\leq_{\mathcal{J}}$. Hence, $\overline{\prec_{\mathring{\mathcal{J}}}}$ is contained in the transitive operation-compatible closure of $\leq_{\mathcal{J}}$, which is exactly $\leq_{\mathring{\mathcal{J}}}$. Therefore, $\leq_{\mathring{\mathcal{J}}} \supseteq \overline{\prec_{\mathring{\mathcal{J}}}}$. □

2.2. Examples of classes of $\leq_{\mathcal{J}}$-compatible semigroups.

Proposition 2.1. Every group and every commutative semigroup is $\leq_{\mathcal{J}}$-compatible, $\leq_{\mathcal{L}}$-compatible and $\leq_{\mathcal{R}}$-compatible.

Proof. The result follows from the fact that in a group or in a commutative semigroup $\prec_{\mathring{\mathcal{J}}} \subseteq \leq_{\mathcal{J}}$, $\prec_{\mathring{\mathcal{L}}} \subseteq \leq_{\mathcal{L}}$ and $\prec_{\mathring{\mathcal{R}}} \subseteq \leq_{\mathcal{R}}$ and from Lemma 2.1. □

As we shall see in Sections 4 and 5, every band is $\leq_{\mathcal{J}}$-compatible, but not necessarily $\leq_{\mathcal{L}}$-compatible and $\leq_{\mathcal{R}}$-compatible.

2.3. **Monoids.** As the following statements demonstrate, results concerning $\leq_{\mathring{\mathcal{J}}}$ are not affected by a semigroup being a monoid; however, results concerning $\leq_{\mathring{\mathcal{L}}}$ and $\leq_{\mathring{\mathcal{R}}}$ are affected by this fact.

Proposition 2.2. Consider a semigroup S and a monoid $M = S \cup 1$ with the neutral element 1, where $1 \notin S$. Then the relation $\leq_{\mathring{\mathcal{J}}}$ on S is equal to the restriction of $\leq_{\mathring{\mathcal{J}}}$ on S.

Proof. This follows from the description of $\leq_{\mathring{\mathcal{J}}}$ in Lemma 2.1. □

Proposition 2.3. In every monoid $\leq_{\mathring{\mathcal{J}}} = \leq_{\mathring{\mathcal{L}}} = \leq_{\mathring{\mathcal{R}}}$.

Proof. From the definition it follows that in any monoid M we have $\prec_{\mathring{\mathcal{J}}} = \prec_{\mathring{\mathcal{L}}} = \prec_{\mathring{\mathcal{R}}}$. Therefore, by Lemma 2.1, $\leq_{\mathring{\mathcal{J}}} = \leq_{\mathring{\mathcal{L}}} = \leq_{\mathring{\mathcal{R}}}$. □

3. Congruences

3.1. **Induced equivalence relations.** For any element s in a semigroup S and any congruence θ on S, s^θ shall denote the image of s under the natural homomorphism $S \to S/\theta$.

Lemma 3.1. *Let S be a semigroup and let $s, t \in S$ be such that $s \leq_{\mathring{\mathcal{J}}} t$ ($s \leq_{\mathring{\mathcal{L}}} t$, $s \leq_{\mathring{\mathcal{R}}} t$). Then for any congruence θ on S, $s^\theta \leq_{\mathring{\mathcal{J}}} t^\theta$ ($s^\theta \leq_{\mathring{\mathcal{L}}} t^\theta$, $s^\theta \leq_{\mathring{\mathcal{R}}} t^\theta$) in S/θ.*

Proof. If $s \leq_{\mathring{\mathcal{J}}} t$ then by Lemma 2.1 there exist $s = s_0, s_1, \ldots, s_n = t \in S$ such that $s_i \prec_{\mathring{\mathcal{J}}} s_{i+1}$ for every $0 \leq i \leq n-1$. Fix an arbitrary $0 \leq i \leq n-1$. Then $s_i = abc$ and $s_{i+1} = ac$ for some $a, b, c \in S^1$. Hence $s_i^\theta = a^\theta b^\theta c^\theta$ and $s_{i+1}^\theta = a^\theta c^\theta$ (where for $1_S \in S^1$ we have $1_S^\theta = 1_T \in T^1$), and so $s_i^\theta \prec_{\mathring{\mathcal{J}}} s_{i+1}^\theta$. Therefore by Lemma 2.1 $s^\theta \leq_{\mathring{\mathcal{J}}} t^\theta$. (The proof is similar for $\leq_{\mathring{\mathcal{L}}}$ and $\leq_{\mathring{\mathcal{R}}}$.) □

Definition 3.1. Denote by $\mathring{\mathcal{J}}$, $\mathring{\mathcal{L}}$ and $\mathring{\mathcal{R}}$ the equivalences $\leq_{\mathring{\mathcal{J}}} \cap \leq_{\mathring{\mathcal{J}}}^{-1}$, $\leq_{\mathring{\mathcal{L}}} \cap \leq_{\mathring{\mathcal{L}}}^{-1}$ and $\leq_{\mathring{\mathcal{R}}} \cap \leq_{\mathring{\mathcal{R}}}^{-1}$, respectively.

For any operation-compatible quasiorder $\leq$, $\leq \cap \leq^{-1}$ is a congruence (see [13] for instance), hence $\mathring{\mathcal{J}}$, $\mathring{\mathcal{L}}$ and $\mathring{\mathcal{R}}$ are congruences.

Definition 3.2. Let us say that a semigroup is $\mathring{\mathcal{J}}$*-trivial* ($\mathring{\mathcal{L}}$*-trivial*, $\mathring{\mathcal{R}}$*-trivial*) if $\mathring{\mathcal{J}}$ ($\mathring{\mathcal{L}}$, $\mathring{\mathcal{R}}$) is the identity relation on S.

We call a quasiorder on a semigroup S a *negative quasiorder* if $st \leq s$ and $st \leq t$ for every s, t in S; S is called *negatively orderable* if there exists an operation-compatible negative partial order on S.

Proposition 3.1. A semigroup is $\mathring{\mathcal{J}}$-trivial if and only if it is negatively orderable.

Proof. If a semigroup S is $\mathring{\mathcal{J}}$-trivial then, obviously, $\leq_{\mathcal{J}}$ is an operation-compatible negative partial order on S. If there is an operation-compatible negative partial order $\leq$ on S then $\prec_{\mathring{\mathcal{J}}} \subseteq \leq$, by the definition of $\prec_{\mathring{\mathcal{J}}}$, hence, $\leq_{\mathring{\mathcal{J}}} \subseteq \leq$, therefore, $\leq_{\mathring{\mathcal{J}}}$ is an order and, hence, $\mathring{\mathcal{J}}$ is the identity relation. □

According to the usual convention, let us call a congruence θ on a semigroup S a *$\mathring{\mathcal{J}}$-trivial congruence* (*$\mathring{\mathcal{L}}$-trivial congruence*, *$\mathring{\mathcal{R}}$-trivial congruence*) if S/θ is a $\mathring{\mathcal{J}}$-trivial semigroup ($\mathring{\mathcal{L}}$-trivial semigroup, $\mathring{\mathcal{R}}$-trivial semigroup).

Proposition 3.2. In any semigroup S, the congruence $\mathring{\mathcal{J}}$ ($\mathring{\mathcal{L}}$, $\mathring{\mathcal{R}}$) is the smallest $\mathring{\mathcal{J}}$-trivial ($\mathring{\mathcal{L}}$-trivial, $\mathring{\mathcal{R}}$-trivial) congruence.

Proof. Let S be a semigroup. First we prove that $\mathring{\mathcal{J}}$ is contained in every $\mathring{\mathcal{J}}$-trivial congruence on S. Let θ be a $\mathring{\mathcal{J}}$-trivial congruence on S and let $s, t \in S$ be such that $s \; \mathring{\mathcal{J}} \; t$. Then we have $s \leq_{\mathring{\mathcal{J}}} t$ and $t \leq_{\mathring{\mathcal{J}}} s$. By Lemma 3.1 in the factor semigroup S/θ we have $s^\theta \leq_{\mathring{\mathcal{J}}} t^\theta$ and $t^\theta \leq_{\mathring{\mathcal{J}}} s^\theta$. Then $t^\theta \; \mathring{\mathcal{J}} \; s^\theta$ and since θ is a $\mathring{\mathcal{J}}$-trivial congruence, we have $t^\theta = s^\theta$. Therefore $\mathring{\mathcal{J}} \subseteq \theta$.

We show that $\mathring{\mathcal{J}}$ is a $\mathring{\mathcal{J}}$-trivial congruence on S. Suppose that $s^{\mathring{\mathcal{J}}} \leq_{\mathring{\mathcal{J}}} t^{\mathring{\mathcal{J}}}$ and $t^{\mathring{\mathcal{J}}} \leq_{\mathring{\mathcal{J}}} s^{\mathring{\mathcal{J}}}$ for some s and t in S. Then – by Lemma 2.1 – there exists a sequence $s = s_0, \ldots, s_n = t$ in S such that $s_i^{\mathring{\mathcal{J}}} \prec_{\mathring{\mathcal{J}}} s_{i+1}^{\mathring{\mathcal{J}}}$ for every $0 \leq i \leq n-1$. By definition of $\prec_{\mathring{\mathcal{J}}}$ for every $0 \leq i \leq n-1$ there exist $a_i, b_i, c_i \in S^1$ such that $s_i^{\mathring{\mathcal{J}}} = a_i^{\mathring{\mathcal{J}}} b_i^{\mathring{\mathcal{J}}} c_i^{\mathring{\mathcal{J}}}$ and $s_{i+1}^{\mathring{\mathcal{J}}} = a_i^{\mathring{\mathcal{J}}} c_i^{\mathring{\mathcal{J}}}$ (where for $1_S \in S^1$, $1_S^{\mathring{\mathcal{J}}}$ is defined as $1_S^{\mathring{\mathcal{J}}} = 1_{S/\mathring{\mathcal{J}}} \in (S/\mathring{\mathcal{J}})^1$). Then $s_i^{\mathring{\mathcal{J}}} = a_i^{\mathring{\mathcal{J}}} b_i^{\mathring{\mathcal{J}}} c_i^{\mathring{\mathcal{J}}} = (a_i b_i c_i)^{\mathring{\mathcal{J}}}$ and $s_{i+1}^{\mathring{\mathcal{J}}} = a_i^{\mathring{\mathcal{J}}} c_i^{\mathring{\mathcal{J}}} = (a_i c_i)^{\mathring{\mathcal{J}}}$, hence $s_i \; \mathring{\mathcal{J}} \; a_i b_i c_i \leq_{\mathring{\mathcal{J}}} a_i c_i \; \mathring{\mathcal{J}} \; s_{i+1}$, thus $s_i \leq_{\mathring{\mathcal{J}}} s_{i+1}$ for every $0 \leq i \leq n-1$. By transitivity $s \leq_{\mathring{\mathcal{J}}} t$ follows. Similarly we can show that $t \leq_{\mathring{\mathcal{J}}} s$, thus $s \; \mathring{\mathcal{J}} \; t$ and so $s^{\mathring{\mathcal{J}}} = t^{\mathring{\mathcal{J}}}$ holds. Therefore $S/\mathring{\mathcal{J}}$ is a $\mathring{\mathcal{J}}$-trivial semigroup and $\mathring{\mathcal{J}}$ is a $\mathring{\mathcal{J}}$-trivial congruence.

The statement regarding the congruences $\mathring{\mathcal{L}}$ and $\mathring{\mathcal{R}}$ can be proved similarly. □

As a comment to the previous result, we would like to emphasize that we do not say that every congruence containing $\mathring{\mathcal{J}}$ is $\mathring{\mathcal{J}}$-trivial. For instance, a free semigroup obviously has non-$\mathring{\mathcal{J}}$-trivial factor semigroups, and it is $\mathring{\mathcal{J}}$-trivial. Indeed, let A be an alphabet. Then – by Lemma 2.1 – it is easy to show that for any u, v in the free semigroup A^+ we have $u \leq_{\mathring{\mathcal{J}}} v$ if and only if v is a subword of u. Hence $u \leq_{\mathring{\mathcal{J}}} v$ and $v \leq_{\mathring{\mathcal{J}}} u$ imply $u = v$, and so A^+ is $\mathring{\mathcal{J}}$-trivial.

One might think incorrectly that if in a semigroup $\mathcal{J} = \mathring{\mathcal{J}}$ ($\mathcal{L} = \mathring{\mathcal{L}}$, $\mathcal{R} = \mathring{\mathcal{R}}$) then it is a $\leq_{\mathcal{J}}$-compatible ($\leq_{\mathcal{L}}$-compatible, $\leq_{\mathcal{R}}$-compatible) semigroup. However, this is wrong even in semigroups which are $\mathring{\mathcal{J}}$-trivial; now we present an example of a $\mathring{\mathcal{J}}$-trivial semigroup which is not $\leq_{\mathcal{J}}$-compatible.

Example 3.1. For any positive integer n, the semigroup OE_n of all order-preserving decreasing mappings on an n-element set is well known to be negatively orderable (we cannot find this observation in the literature formulated explicitly, although it is implicit in, for instance, [6]). Hence, OE_n is $\mathring{\mathcal{J}}$-trivial. Consider the mappings $\alpha, \beta \in OE_4$ defined as follows. Let $\alpha : 4 \mapsto 3, 3 \mapsto 2, 2 \mapsto 1$ and $\beta : 4 \mapsto 3, 3 \mapsto 3, 2 \mapsto 1$ (and $1 \mapsto 1$, as in every element of OE_n). Then $\alpha \not\leq_{\mathcal{J}} \beta$, since $rank(\alpha) \not\leq rank(\beta)$ (where the rank of a mapping is the size of its image) . Let us demonstrate that $\alpha \leq_{\mathring{\mathcal{J}}} \beta$ ($\alpha \leq_{\mathring{\mathcal{J}}} \beta$, $\alpha \leq_{\mathring{\mathcal{R}}} \beta$). Indeed, let $\alpha_1 : 4 \mapsto 4, 3 \mapsto 2, 2 \mapsto 2$, $\beta_1 : 4 \mapsto 4, 3 \mapsto 3, 2 \mapsto 1$ and $\beta_2 : 4 \mapsto 3, 3 \mapsto 3, 2 \mapsto 2$. It is easy to see that $\beta = \beta_1\beta_2$ and $\alpha = \beta_1\alpha_1\beta_2$, hence by Lemma 2.1 $\alpha \leq_{\mathring{\mathcal{J}}} \beta$ ($\alpha \leq_{\mathring{\mathcal{J}}} \beta$, $\alpha \leq_{\mathring{\mathcal{R}}} \beta$).

As to an example of a completely different kind, any free semigroup with at least two generators is also a $\mathring{\mathcal{J}}$-trivial semigroup with $\leq_{\mathring{\mathcal{J}}} \neq \leq_{\mathcal{J}}$.

3.2. **Homomorphic images of $\leq_{\mathcal{L}}$-compatible, $\leq_{\mathcal{R}}$-compatible and $\leq_{\mathcal{J}}$-compatible semigroups.** The class of $\leq_{\mathcal{J}}$-compatible semigroups is not closed with respect to subsemigroups (for example, a counterexample can be produced on the basis of Corollary 6.1 below). However, the following is true:

Theorem 3.1. *The class of $\leq_{\mathcal{J}}$-compatible ($\leq_{\mathcal{L}}$-compatible, $\leq_{\mathcal{R}}$-compatible) semigroups is closed under taking homomorphic images.*

Proof. Let S be a $\leq_{\mathcal{J}}$-compatible semigroup and let T be a homomorphic image of S under a homomorphism $\alpha : S \to T$. Let $s, t \in S$ be such that $\alpha(s) \leq_{\mathring{\mathcal{J}}} \alpha(t)$ in T. Then – by Lemma 2.1 – there is a sequence $s = s_0, s_1, \ldots, s_n = t \in S$ such that for every $0 \leq i \leq n-1$, $\alpha(s_i) = \alpha(a_i)\alpha(b_i)\alpha(c_i)$ and $\alpha(s_{i+1}) = \alpha(a_i)\alpha(c_i)$ for some $a_i, b_i, c_i \in S^1$ (where for $1_S \in S^1$, $\alpha(1_S)$ is defined as $\alpha(1_S) = 1_T \in T^1$). Let us fix an index $0 \leq i \leq n-1$. Then $a_ib_ic_i \leq_{\mathring{\mathcal{J}}} a_ic_i$ in S and as $\leq_{\mathring{\mathcal{J}}} = \leq_{\mathcal{J}}$ in S, there exist $u_i, v_i \in S^1$ such that $a_ib_ic_i = u_ia_ic_iv_i$. Then $\alpha(s_i) = \alpha(a_ib_ic_i) = \alpha(u_ia_ic_iv_i) = \alpha(u_i)\alpha(a_ic_i)\alpha(v_i) \leq_{\mathcal{J}} \alpha(a_ic_i) = \alpha(s_{i+1})$. Hence, $\alpha(s_i) \leq_{\mathcal{J}} \alpha(s_{i+1})$ and by transitivity, $\alpha(s) \leq_{\mathcal{J}} \alpha(t)$. For $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ the statement can be proved similarly. □

4. Regular periodic $\leq_{\mathcal{J}}$-compatible ($\leq_{\mathcal{L}}$-compatible, $\leq_{\mathcal{R}}$-compatible) semigroups

In this section we shall provide a description of regular periodic $\leq_{\mathcal{J}}$-compatible, $\leq_{\mathcal{L}}$-compatible and $\leq_{\mathcal{R}}$-compatible semigroups.

By $\mathcal{J}^\sharp$ one denotes the smallest congruence containing $\mathcal{J}$. It is well known that in regular semigroups the congruence $\mathcal{J}^\sharp$ plays a special role: it is the smallest semilattice congruence; see, for instance, Proposition 3.2.3 in [5].

Lemma 4.1. *In a regular semigroup, $\mathcal{J}^\sharp = \mathring{\mathcal{J}}$.*

Proof. From Proposition 3.2 and from $\mathcal{J} \subseteq \mathring{\mathcal{J}}$ it follows that $\mathcal{J}^\sharp \subseteq \mathring{\mathcal{J}}$. Let us prove that $\mathring{\mathcal{J}} \subseteq \mathcal{J}^\sharp$. Indeed, by Proposition 3.2, $\mathring{\mathcal{J}}$ is the smallest $\mathring{\mathcal{J}}$-trivial congruence. At the same time, $\mathcal{J}^\sharp$ is the smallest semilattice congruence. Since every semilattice is $\mathring{\mathcal{J}}$-trivial, $\mathcal{J}^\sharp$ is a $\mathring{\mathcal{J}}$-trivial congruence, hence by Proposition 3.2 $\mathring{\mathcal{J}} \subseteq \mathcal{J}^\sharp$. □

Example 4.1. As the following example shows, in a regular semigroup $\mathcal{R}^\sharp \neq \mathring{\mathcal{R}}$ in general. Consider the variety $\mathbf{MK_1}$ of semigroups defined by the identities $x = x^2$ and $xy = xyx$ within the variety of all semigroups (the notation was first introduced in [11]). Let B denote the band which is free in $\mathbf{MK_1}$ with generators $A = \{a_1, \ldots, a_n\}$ for some $n \geq 3$. Since B is a band, it is a regular semigroup. It is easy to see that $\mathcal{R}$ is the identity relation on B, hence $\mathcal{R}^\sharp = \mathcal{R}$ is also the identity. We show that $\mathring{\mathcal{R}}$ is not the identity on B. For a_1, a_2, a_3 in B we have $a_1a_2a_3 \geq_{\mathring{\mathcal{R}}} a_1a_3a_2a_3 = a_1a_3a_2$ and $a_1a_3a_2 \geq_{\mathring{\mathcal{R}}} a_1a_2a_3a_2 = a_1a_2a_3$, hence $a_1a_2a_3 \ \mathring{\mathcal{R}} \ a_1a_3a_2$. It is easy to see – and it also follows from Lemma 5.1 which will be proved in Subsection 5.2 – that $a_1a_2a_3 \neq a_1a_3a_2$ in B. Therefore $\mathring{\mathcal{R}}$ is not the identity relation on B and thus $\mathcal{R}^\sharp \neq \mathring{\mathcal{R}}$. Similarly we can show that in a regular semigroup $\mathcal{L}^\sharp \neq \mathring{\mathcal{L}}$ in general.

Lemma 4.2. *If S is a $\leq_{\mathcal{J}}$-compatible band of simple semigroups then S is a $\leq_{\mathcal{J}}$-compatible semigroup.*

Proof. Let S be a $\leq_{\mathcal{J}}$-compatible band of simple semigroups and let θ be a $\leq_{\mathcal{J}}$-compatible band congruence on S such that each θ-class is a simple semigroup and let $R = S/\theta$. For every $s \in S$ let θ_s denote the θ-class of s. Let $s, t \in S$ be such that $s \leq_{\mathring{\mathcal{J}}} t$. Then by Lemma 3.1 we have $s^\theta \leq^R_{\mathring{\mathcal{J}}} t^\theta$. Since R is a $\leq_{\mathcal{J}}$-compatible band, it implies $s^\theta \leq^R_{\mathcal{J}} t^\theta$ and thus $s^\theta = x^\theta t^\theta y^\theta$ for some $x, y \in S^1$ (where for $1_S \in S^1$, 1_S^θ is defined as $1_S^\theta = 1_R \in R^1$). Hence $s^\theta = x^\theta t^\theta y^\theta = (xty)^\theta$ and thus $s \ \theta \ xty$ and – since θ_s is a simple semigroup – $s \leq_{\mathcal{J}} xty \leq_{\mathcal{J}} t$.

□

The following statement is a classical result, see, for instance, Theorem 1.3.10 in [7] or Theorem 4.1.3 in [8]:

Theorem 4.1. *(Clifford's Theorem) Every completely regular semigroup is a semilattice of completely simple semigroups.*

Corollary 4.3. *Every completely regular semigroup is a $\leq_{\mathcal{J}}$-compatible semigroup.*

Since every band is completely regular, by Corollary 4.3:

Corollary 4.4. *Every band is a $\leq_{\mathcal{J}}$-compatible semigroup.*

Theorem 4.2. *For a regular periodic semigroup S the following are equivalent:*

(1) S is a $\leq_{\mathcal{J}}$-compatible semigroup
(2) S is a band of simple semigroups
(3) S is a semilattice of simple semigroups.

Proof. 1 ⇒ 2 Let S be a regular periodic $\leq_{\mathcal{J}}$-compatible semigroup. Then $\mathcal{J} = \mathring{\mathcal{J}}$ in S, hence by Proposition 3.2, $\mathcal{J}$ is a $\mathring{\mathcal{J}}$-trivial congruence on S. Therefore $B = S/\mathcal{J}$ is a $\mathring{\mathcal{J}}$-trivial semigroup. Since S

is regular, every $\mathcal{J}$-class of S contains an idempotent. It follows that each $\mathcal{J}$-congruence class of S is a semigroup, hence B is a band. We show that every $\mathcal{J}$-congruence class is a simple semigroup. For any element $s \in S$ let J_s^S, L_s^S, R_s^S denote the $\mathcal{J}, \mathcal{L}$ and $\mathcal{R}$-class, respectively of s in S. Let T be an arbitrary $\mathcal{J}$-class of S. We show that $\mathcal{L}^T = \mathcal{L}^S|_T$ and $\mathcal{R}^T = \mathcal{R}^S|_T$. Let $s, t \in T$ be such that $s\ \mathcal{L}^S\ t$. Let $e \in L_s^S$ be an idempotent (as S is regular, such an idempotent exists, see Proposition 2.3.2 in [8]) and let $s' \in J_s^S$ be an inverse of s such that $s's = e$. (Such an inverse exists, see [8]). Then e is a right identity in L_s^S (see Proposition 2.3.3 in [8]), therefore $t = te = tss's$ and thus $t \leq_{\mathcal{L}}^T s$. Similarly we can show $s \leq_{\mathcal{L}}^T t$, hence $s\mathcal{L}^T t$ follows. Therefore $\mathcal{L}^T = \mathcal{L}^S|_T$ and $\mathcal{R}^T = \mathcal{R}^S|_T$ can be verified similarly. Since S, T are periodic, we have $\mathcal{J}^T = \mathcal{L}^T \circ \mathcal{R}^T = \mathcal{L}^S|_T \circ \mathcal{R}^S|_T = \mathcal{J}^S|_T = T \times T$ and thus T is a simple semigroup.

2 ⇒ 1 It follows from Lemma 4.2 and Corollary 4.4.

3 ⇒ 2 This implication is trivial.

1 ⇒ 3 Let S be a regular periodic $\leq_{\mathcal{J}}$-compatible semigroup. Then by Proposition 3.2 and Lemma 4.1, $\mathcal{J} = \mathring{\mathcal{J}} = \mathcal{J}^\sharp$ is a semilattice congruence on S. Above we proved that each $\mathcal{J}$-class in a regular periodic semigroup is a simple semigroup, thus S is a semilattice of simple semigroups. □

Definition 4.1. A band is called a *left (right) normal band* if it satisfies the identity $xyz = xzy$ $(xyz = yxz)$.

Lemma 4.5. *In any left normal band* $\leq_{\mathring{\mathcal{R}}} = \leq_{\mathcal{R}} \subseteq \leq_{\mathcal{L}}$*; in any right normal band* $\leq_{\mathring{\mathcal{J}}} = \leq_{\mathcal{L}} \subseteq \leq_{\mathcal{R}}$.

Proof. Let B be a left normal band. The containment $\leq_{\mathcal{R}} \subseteq \leq_{\mathring{\mathcal{R}}}$ trivially holds. To verify $\leq_{\mathring{\mathcal{R}}} \subseteq \leq_{\mathcal{R}}$ it is sufficient to show that $\leq_{\mathcal{R}}$ is operation-compatible. Clearly, $\leq_{\mathcal{R}}$ is left operation-compatible. We show that $\leq_{\mathcal{R}}$ is also right operation-compatible. Let $s, t \in B$ be such that $s \leq_{\mathcal{R}} t$, namely, $s = tr$ for some $r \in B$. Then for any $u \in B$, $su = tru = tur \leq_{\mathcal{R}} tu$, thus $\leq_{\mathcal{R}}$ is right operation-compatible, and hence $\leq_{\mathring{\mathcal{R}}} = \leq_{\mathcal{R}}$.

As to the second part of the statement, let $s, t \in B$ be such that $s \leq_{\mathcal{R}} t$, namely, $s = tr$ for some $r \in B$. Then $s = tr = ttr = trt \leq_{\mathcal{L}} t$ and thus, $\leq_{\mathcal{R}} \subseteq \leq_{\mathcal{L}}$.

The dual statement can be proved similarly. □

Lemma 4.6. *Every left normal band is $\mathring{\mathcal{R}}$-trivial, and every right normal band is $\mathring{\mathcal{L}}$-trivial.*

Proof. Let B be a left normal band. Let $e, f \in B$ be such that $e \leq_{\mathring{\mathcal{R}}} f$ and $f \leq_{\mathring{\mathcal{R}}} e$. By Lemma 4.5 it implies $e \leq_{\mathcal{R}} f$ and $f \leq_{\mathcal{R}} e$, hence there exist $x, y \in B$ such that $e = fx$, $f = ey$. Then $f = ey = fxy = eyxy = ey^2x = eyx = fx = e$ holds.

The dual statement can be proved similarly. □

Lemma 4.7. *Let S be a band. The following conditions are equivalent:*

(1) S is $\mathring{\mathcal{L}}$-trivial ($\mathring{\mathcal{R}}$-trivial);

(2) S is $\mathring{\mathcal{L}}$-trivial ($\mathring{\mathcal{R}}$-trivial) and $\leq_{\mathcal{L}}$-compatible($\leq_{\mathcal{R}}$-compatibl9);

(3) S is a right (left) normal band.

Proof. *3 ⇒ 2* By Lemma 4.6 every right (left) normal band is $\mathring{\mathcal{L}}$-trivial ($\mathring{\mathcal{R}}$-trivial). By Lemma 4.5 every right (left) normal band is an $\leq_{\mathcal{L}}$-compatible semigroup ($\leq_{\mathcal{R}}$-compatible semigroup).

2 ⇒ 1 Obvious.

1 ⇒ 3 Indeed, in a band we have $xyz \leq_{\mathring{\mathcal{R}}} xzyzy = xzy$. In the same way, $xzy \leq_{\mathring{\mathcal{R}}} xyz$. If the band is $\mathring{\mathcal{R}}$-trivial then $xyz = xzy$, hence, the band is left normal. The result for right normal bands can be proved in the same way. □

Theorem 4.3. *A regular periodic semigroup is an $\leq_{\mathcal{L}}$-compatible semigroup ($\leq_{\mathcal{R}}$-compatible semigroup) if and only if it is a right normal band (left normal band) of $\mathcal{L}$-simple ($\mathcal{R}$-simple) semigroups.*

Proof. Let S be a regular periodic semigroup which is a right normal band of $\mathcal{L}$-simple semigroups; thus, there is a congruence θ on S such that θ is a right normal band congruence and every θ-class is $\mathcal{L}$-simple. We show that $\leq_{\mathcal{L}} = \leq_{\mathring{\mathcal{J}}}$ in S. Clearly, $\leq_{\mathcal{L}} \subseteq \leq_{\mathring{\mathcal{J}}}$. Let $s, t \in S$ be such that $s \leq_{\mathring{\mathcal{J}}} t$. Let $B = S/\theta$. For any $s \in S$ let s^θ denote the image of s under the natural homomorphism $S \to S/\theta$. By Lemma 3.1 $s^\theta \leq_{\mathring{\mathcal{J}}} t^\theta$ follows. By Lemma 4.7 B is $\leq_{\mathcal{L}}$-compatible semigroup, hence $s^\theta \leq_{\mathring{\mathcal{J}}} t^\theta$ implies $s^\theta \leq_{\mathcal{L}} t^\theta$ and therefore $s^\theta = x^\theta t^\theta$ for some $x \in S^1$ (where $1^\theta = 1_B \in B^1$). Then $s^\theta = x^\theta t^\theta = (xt)^\theta$, hence $s\ \theta\ (xt)$ and since each θ-class of S is $\mathcal{L}$-simple, it implies $s \leq_{\mathcal{L}} xt \leq_{\mathcal{L}} t$. Thus $\leq_{\mathring{\mathcal{J}}} \subseteq \leq_{\mathcal{L}}$ and hence $\leq_{\mathring{\mathcal{J}}} = \leq_{\mathcal{L}}$.

For the other direction, let S be a regular periodic $\leq_{\mathcal{L}}$-compatible semigroup. Then clearly, $\mathcal{L} = \mathring{\mathcal{L}}$ on S and hence by Proposition 3.2, $\mathcal{L}$ is the smallest $\mathring{\mathcal{L}}$-trivial congruence on S. In a regular semigroup every $\mathcal{L}$-class of S contains an idempotent (see [8]), hence $B = S/\mathcal{L} = S/\mathring{\mathcal{L}}$ is a band. By Proposition 3.2 B is $\mathring{\mathcal{L}}$-trivial, therefore – by Lemma 4.7 – B is a right normal band. Let L be an arbitrary $\mathcal{L}$-class of S. Since $\mathcal{L}$ is a band congruence on S, L is a subsemigroup of S. Let $s, t \in L$ be arbitrary elements and m be a positive integer such that $s^\omega = s^m$. Then $s^\omega \in L$ is a right identity in L (see [8]), hence $ts^m = ts^\omega = t$ and thus $t \leq_{\mathcal{L}}^L s$ in L. Similarly, $s \leq_{\mathcal{L}}^L t$. Therefore, L is an $\mathcal{L}$-simple semigroup. □

5. Counterexamples

Every completely regular semigroup is $\leq_{\mathcal{J}}$-compatible; an example below shows that not every inverse semigroup is $\leq_{\mathcal{J}}$-compatible.

Every band is $\leq_{\mathcal{J}}$-compatible; an example below shows that not every band is $\leq_{\mathcal{L}}$-compatible ($\leq_{\mathcal{R}}$-compatible).

5.1. **An inverse semigroup which is not $\leq_{\mathcal{J}}$-compatible.** We shall demonstrate through an example that not every inverse semigroup is a $\leq_{\mathcal{J}}$-compatible semigroup. Consider the set $X = \{a, b, c\}$ and define the partial transformations α, β, γ on X as follows: $\alpha = \{(a, b), (b, c)\}$, $\beta = \{(b, c), (c, a)\}$, $\gamma = \{(c, a), (a, b)\}$. The inverses (in the relation sense) α^{-1}, β^{-1} and γ^{-1} of these partial transformations are also partial transformations on X. Let S denote the semigroup of partial transformations generated by $\{\alpha, \beta, \gamma, \alpha^{-1}, \beta^{-1}, \gamma^{-1}\}$. It is known that this semigroup is an inverse semigroup. Then $\alpha\gamma = \{(b, a)\}$ and $\alpha\beta\gamma = \{(a, a), (b, b)\}$. Since $|Im(\alpha\beta\gamma)| = 2 > 1 = |Im(\alpha\gamma)|$, we have $\alpha\beta\gamma \nleq_{\mathcal{J}} \alpha\gamma$, but clearly $\alpha\beta\gamma \leq_{\mathring{\mathcal{J}}} \alpha\gamma$. Therefore $\leq_{\mathring{\mathcal{J}}} \neq \leq_{\mathcal{J}}$ in S.

5.2. **A band which is not $\leq_{\mathcal{R}}$-compatible.** As we have seen in Corollary 4.4 every band is a $\leq_{\mathcal{J}}$-compatible semigroup. Here we shall show through two examples that the analogous statements involving $\leq_{\mathcal{L}}$-compatible and $\leq_{\mathcal{R}}$-compatible semigroups, respectively, do not hold.

Like in Example 4.1, consider the variety $\mathbf{MK_1}$ of bands defined by the identities $x = x^2$ and $xy = xyx$ within the variety of all semigroups. Let B denote the band which is free in $\mathbf{MK_1}$ with generators $A =$

$\{a_1, \ldots, a_n\}$ for some $n \geq 3$. Obviously, $B \cong A^+/\theta$ where θ is the smallest $\mathbf{MK_1}$-congruence on A^+.

For any word $w \in A^+$ the *content of* w, denoted by $c(w)$, is the set of all letters of w; let $f(w)$ denote the first letter of w and let $i(w)$ denote the subword of w obtained by keeping only the first occurrence of each letter of w and deleting all other letters of w. Let $\overline{w}$ denote the image of w under the natural homomorphism $A^+ \to A^+/\theta$.

The following two facts can be found in literature [12, 3, 4] and are not difficult to prove.

Lemma 5.1. *Let $v, w \in A^+$ be arbitrary words. Then*
(1) we have $w \,\theta\, i(w)$;
(2) we have $v \,\theta\, w$ if and only if $i(v) = i(w)$;
(3) if $v \,\theta\, w$ then $c(v) = c(w)$ and $f(v) = f(w)$.

Lemma 5.2. *Let $a, b \in B$ and $i(a) = x_1x_2 \ldots x_p$, $i(b) = y_1y_2 \ldots y_q$. We have $a \leq_{\mathcal{R}} b$ if and only if $q \leq p$ and $x_i = y_i$ for every $1 \leq i \leq q$.*

Theorem 5.1. *Let $a, b \in B$. We have $a \leq_{\overset{\circ}{\mathcal{R}}} b$ if and only if $c(b) \subseteq c(a)$ and $f(a) = f(b)$.*

Proof. ($\Rightarrow$) Suppose $a \leq_{\overset{\circ}{\mathcal{R}}} b$. Then by Lemma 2.1 for some positive integer m there exist elements $a = a_0, a_1, \ldots, a_m = b \in B$ such that $a_i \prec_{\overset{\circ}{\mathcal{R}}} a_{i+1}$ for every $0 \leq i \leq m-1$. Let us fix an arbitrary index $0 \leq i \leq m-1$. By definition there exist $d \in B$ and $e, f \in B^1$ such that $a_i = def$ and $a_{i+1} = df$. Let $w_{a_i}, w_{a_{i+1}}, w_d \in A^+$ and $w_e, w_f \in A^*$ be such that $a_i = \overline{w_{a_i}}, a_{i+1} = \overline{w_{a_{i+1}}}, d = \overline{w_d}, e = \overline{w_e}$ and $f = \overline{w_f}$ (where for the empty word $\lambda \in A^*$, $\overline{\lambda}$ is defined as $\overline{\lambda} = 1 \in B^1$). Then clearly $a_i = \overline{w_d}\ \overline{w_e}\ \overline{w_f} = \overline{w_dw_ew_f}$ and $a_{i+1} = \overline{w_dw_f}$. Therefore $c(a_{i+1}) = c(w_dw_f) \subseteq c(w_dw_ew_f) = c(a_i)$ and $f(a_{i+1}) = f(w_dw_f) = f(w_dw_ew_f) = f(a_i)$, hence $c(b) \subseteq c(a)$ and $f(a) = f(b)$ follows.

($\Leftarrow$) Suppose $c(b) \subseteq c(a)$ and $f(a) = f(b)$ and let $a = \overline{x_1x_2 \ldots x_p}$, $b = \overline{y_1y_2 \ldots y_q}$ for some $x_1x_2 \ldots x_p, y_1y_2 \ldots y_q \in A^+$. Then by definition $\{y_1, y_2, \ldots, y_q\} = c(b) \subseteq c(a) = \{x_1, x_2, \ldots, x_p\}$ and $x_1 = f(a) = f(b) = y_1$. Also $i(y_1x_1x_2 \ldots x_py_2 \ldots y_q) = i(x_1x_1x_2 \ldots x_py_2 \ldots y_q) = i(x_1x_2 \ldots x_p)$. Therefore by Lemma 5.1 we have $a = \overline{x_1x_2 \ldots x_p} = \overline{y_1x_1x_2 \ldots x_py_2 \ldots y_q} = \overline{y_1}\ \overline{x_1x_2 \ldots x_p}\ \overline{y_2 \ldots y_q} \leq_{\overset{\circ}{\mathcal{R}}} \overline{y_1}\ \overline{y_2 \ldots y_q} = \overline{y_1y_2 \ldots y_q} = b$, hence $a \leq_{\overset{\circ}{\mathcal{R}}} b$. □

Corollary 5.3. *Let $a, b \in B$ and $i(a) = x_1x_2 \ldots x_p$, $i(b) = y_1y_2 \ldots y_q$. We have $a \leq_{\overset{\circ}{\mathcal{R}}} b$ if and only if $\{y_1, y_2, \ldots, y_q\} \subseteq \{x_1, x_2, \ldots, x_p\}$ and $x_1 = y_1$.*

Proof. The statement follows from Lemma 5.1 and Theorem 5.1. □

Example 5.1. Let $n \geq 3$ be an integer. Then the free semigroup in $\mathbf{MK_1}$ over an n-element set is not an $\leq_{\mathcal{R}}$-compatible semigroup. This follows from the fact that the conditions describing $\leq_{\mathcal{R}}$ in Corollary 5.3 and $\leq_{\mathcal{R}}$ in Lemma 5.2 are clearly not equivalent.

The dual variety $\mathbf{MK_2}$ of bands is defined by the identities $x = x^2$ and $yx = xyx$ within the variety of all semigroups. Similarly to the above proof we can show that for any integer $n \geq 3$ the semigroup which is free in $\mathbf{MK_2}$ over an n-element set is not an $\leq_{\mathcal{L}}$-compatible semigroup.

6. Embedding into a $\leq_{\mathcal{J}}$-compatible semigroup

Every semigroup can be embedded into a simple semigroup as was proved by R. H. Bruck (see [1] or [2]). Since every simple semigroup is clearly a $\leq_{\mathcal{J}}$-compatible semigroup, we have the following statement:

Corollary 6.1. *Every semigroup can be embedded into a $\leq_{\mathcal{J}}$-compatible semigroup.*

In the rest of this section we shall show that if a semigroup S is $\mathring{\mathcal{J}}$-trivial then S can be embedded into a $\leq_{\mathcal{J}}$-compatible semigroup which is also $\mathring{\mathcal{J}}$-trivial.

Let S be an arbitrary semigroup. For each triple $(a, b, c) \in S^3$ let us introduce new elements $\overrightarrow{abc}$ and $\overleftarrow{abc}$ (not contained by S), and let $A = \{\overrightarrow{abc}, \overleftarrow{abc} \mid a, b, c \in S\}$. Consider the free semigroup $(S \cup A)^+$. For any word $w \in S^+$ let $\overline{w}$ denote the element of S represented by w. Let $\approx$ denote the congruence on $(S \cup A)^+$ generated by the set of all relations of the form $st = \overline{st}$ where $s, t \in S$.

Let $\sim$ denote the congruence on $(S \cup A)^+$ generated by the set of relations of the form $\overrightarrow{abc}\, ac\, \overleftarrow{abc} = abc$ where $a, b, c \in S$. Let θ denote the smallest congruence on $(S \cup A)^+$ containing $\approx$ and $\sim$, and let $\overleftrightarrow{S} = (S \cup A)^+/\theta$. For any $w \in (S \cup A)^+$ let $\theta(w)$ denote the image of w under the natural homomorphism $(S \cup A)^+ \to (S \cup A)^+/\theta = \overleftrightarrow{S}$.

By a $\approx$-*step* we shall understand replacing, in a word $w \in (S \cup A)^+$, a two-letter factorword of the form st by a one-letter factorword $\overline{st}$ or vice versa, a factorword of the form $\overline{st}$ by st, for some $s, t \in S$. By a $\sim$-*step* we shall understand replacing, in a word $w \in (S \cup A)^+$, a factorword of the form $\overrightarrow{abc}\,ac\,\overleftarrow{abc}$ by abc or vice versa, a factorword of the form abc by $\overrightarrow{abc}\,ac\,\overleftarrow{abc}$, for some $a, b, c \in S$. By an *inserting step* we shall understand inserting a letter from $S \cup A$ somewhere between two letters of a word $w \in (S \cup A)^+$, or before the first or after the last letter of w.

The following statement is straightforward, as it follows immediately from the definition of θ:

Lemma 6.2. *For any words $v, w \in (S \cup A)^+$, $\theta(v) = \theta(w)$ holds if and only if there is a finite sequence $v = v_0, v_1, \ldots, v_n = w \in (S \cup A)^+$ such that for every $0 \le i \le n-1$, v_{i+1} can be obtained by applying one $\approx$-step or one $\sim$-step to v_i.*

Lemma 6.3. *If $\theta(v) \le_{\mathring{\mathcal{J}}} \theta(w)$ for some $v, w \in (S \cup A)^+$ then there exists a sequence $w = w_0, w_1, \ldots, w_n = v \in (S \cup A)^+$ such that for every $0 \le i \le n-1$ w_{i+1} can be obtained from w_i by one $\approx$-step or one $\sim$-step or one inserting step.*

Proof. Since $\le_{\mathring{\mathcal{J}}}$ is the transitive closure of $\le'_{\mathring{\mathcal{J}}}$, hence, it is sufficient to prove the statement for words $v, w \in (S \cup A)^+$ such that $\theta(v) \le'_{\mathring{\mathcal{J}}} \theta(w)$. Let $v, w \in (S \cup A)^+$ be such that $\theta(v) \le'_{\mathring{\mathcal{J}}} \theta(w)$. Then by definition there exist $w_1, w_2, u \in (S \cup A)^*$ such that $\theta(w) = \theta(w_1)\theta(w_2)$ and $\theta(v) = \theta(w_1)\theta(u)\theta(w_2)$ (where for the empty word λ we put $\theta(\lambda) = 1 \in \overleftrightarrow{S}^1$). Then $\theta(w) = \theta(w_1 w_2)$ and $\theta(v) = \theta(w_1 u w_2)$. By Lemma 6.2, $w_1 w_2$ can be obtained from w by $\approx$- and $\sim$- steps and similarly, v can be obtained from $w_1 u w_2$ by $\approx$- and $\sim$-steps. Clearly, $w_1 u w_2$ can be obtained from $w_1 w_2$ by inserting steps, hence the statement follows. □

Starting from now, when we speak about factorwords or subwords of a word w, we shall normally mean factorwords or subwords whose position within w is fixed. This should not lead to confusion.

Now we are going to extend the $\overline{w}$ notation to certain 'good words' over $(S \cup A)^+$. Let us call a word $w \in (S \cup A)^+$ a *bracketed word* if the first and last letters of w are $\overrightarrow{abc}$ and $\overleftarrow{abc}$, respectively, for some $a, b, c \in S$. For a bracketed word $w \in (S \cup A)^+$ with first letter $\overrightarrow{abc}$ let $\overline{w}$ be defined as $\overline{w} = \overline{abc}$. Let us call a sequence $w_1, w_2, \ldots, w_k \in (S \cup A)^+$ of words a *good sequence* if for every $1 \le i \le k$ either $w_i \in S^+$ or w_i is a bracketed word. For a good sequence $w_1, w_2, \ldots, w_k \in (S \cup A)^+$ define $\pi(w_1, w_2, \ldots, w_k)$ as $\pi(w_1, w_2, \ldots, w_k) = \prod_{i=1}^{k} \overline{w_i}$.

Let $w \in (S \cup A)^+$ be an arbitrary word. Let us call a good sequence $w_1, w_2, \ldots, w_k$ a *good factor-sequence of* w, if w can be written in the form $w = u_0 w_1 u_1 w_2 \ldots w_k u_k$, for some $u_i \in (S \cup A)^*$, $0 \leq i \leq k$. For any word $w \in (S \cup A)^+$, define the *trace* $Tr(w)$ of w as the set $Tr(w)$ consisting of elements $\pi(w_1, \ldots, w_k)$ for all good factor-sequences $w_1, \ldots, w_k$ of w. Let us call a good factor-sequence $w_1, \ldots, w_k$ of w *S-merged* if it contains all letters from S occurring in w and such that any two words $w_1, \ldots, w_k$ from S^+ do not neighbor one another within w; in other words, if $w = u_0 w_1 u_1 w_2 u_2 \ldots w_k u_k$ for some $u_i \in (S \cup A)^*$, $0 \leq i \leq k$, then we have $u_i \in A^*$ for each u_i, and if $w_j, w_{j+1} \in S^+$ then u_j is not empty.

The following statement is easy to prove:

Lemma 6.4. *Let $w \in (S \cup A)^+$. For any good factor-sequence $w_1, w_2, \ldots, w_k$ of w there is an S-merged good factor-sequence $y_1, \ldots, y_m$ of w such that for every $1 \leq i \leq k$ there is a $1 \leq j \leq m$ such that w_i is a factorword of y_j and $\pi(y_1, \ldots, y_m) \leq_{\mathring{\mathcal{J}}} \pi(w_1, \ldots, w_k)$.*

Proof. Let $s_1 s_2 \ldots s_l$ be the subword of w which we obtain by deleting the factorwords $w_1, w_2, \ldots, w_k$ from w and also deleting all letters of w from A. If $s_1 s_2 \ldots s_l$ is not the empty word then for every $1 \leq i \leq l$ we have $s_i \in S$, hence s_i is a (one-letter) good factorword of w. Consider the factor-sequence $v_1, v_2, \ldots, v_{k+l}$ of w which consists of all the factors w_i, $1 \leq i \leq k$ and s_j, $1 \leq j \leq l$. Then $v_1, v_2, \ldots, v_{k+l}$ is a good factor-sequence which contains all letters of w from S. If $s_1 s_2 \ldots s_l$ is the empty word then let $l = 0$ and let $v_1, v_2, \ldots, v_k$ be identical to $w_1, w_2, \ldots, w_k$. In both cases – as $w_1, \ldots, w_k$ is a subsequence of $v_1, v_2, \ldots, v_{k+l}$ – we have $\pi(v_1, v_2, \ldots, v_{k+l}) = \Pi_{i=1}^{k+l} \overline{v_i} \leq_{\mathring{\mathcal{J}}} \Pi_{i=1}^{k} \overline{w_i} = \pi(w_1, w_2, \ldots, w_k)$.

If $v_1, \ldots, v_{k+l}$ is S-merged then the proof is complete. Otherwise there exists an index $1 \leq i \leq k + l - 1$ such that $v_i, v_{i+1} \in S^+$ and v_i and v_{i+1} are neighboring factorwords in w. Let $v_i' = v_i v_{i+1} \in S^+$ be the word obtained by the concatenation of the words v_i and v_{i+1}. Then $v_1, \ldots, v_{i-1}, v_i', v_{i+2}, v_{i+3}, \ldots, v_{k+l}$ is a good factor-sequence of w. Since $\overline{v_i'} = \overline{v_i v_{i+1}}$, we have $\pi(v_1, \ldots, v_{i-1}, v_i', v_{i+2}, v_{i+3}, \ldots, v_{k+l}) = \pi(v_1, v_2, \ldots, v_{k+l})$. By the repeated use of such concatenations of factorwords eventually we shall obtain an S-merged good factor-sequence $y_1, y_2, \ldots, y_m$ of w such that $\pi(y_1, y_2, \ldots, y_m) = \pi(v_1, \ldots, v_{k+l}) \leq_{\mathring{\mathcal{J}}} \pi(w_1, \ldots, w_k)$. (The process will terminate after finitely many steps, since by each concatenation we decrease the number of factorwords in our good factor-sequence by one.) It is easy to see that for every $1 \leq i \leq n$, w_i is a factorword of y_j for some $1 \leq j \leq m$. □

Lemma 6.5. *If $v, w \in (S \cup A)^+$ are such that $\theta(v) \leq_{\mathring{\mathcal{J}}} \theta(w)$ then for any $r \in Tr(w)$ there exists $r' \in Tr(v)$ such that $r' \leq_{\mathring{\mathcal{J}}} r$.*

Proof. By Lemma 6.3, it is sufficient to prove the statement for the cases when v can be obtained from w by one $\approx$- , one $\sim$- or one inserting step. Let $r \in Tr(w)$ be arbitrary and let $w_1, \ldots, w_k$ be a good factor-sequence of w such that $r = \pi(w_1, \ldots, w_k)$. By Lemma 6.4 it is sufficient to prove the statement for the case when $w_1, \ldots, w_k$ is S-merged.

Case 1: v can be obtained from w by one $\approx$-step. Let $s, t \in S$ be such that by changing the factorword st in w to $\overline{st}$ or changing the factorword $\overline{st}$ to st, we can obtain v. Let z and z' denote the factorword which is changed before and after the change, respectively. Then z is a factorword of w_i for some $1 \leq i \leq k$ (as $w_1, \ldots, w_k$ is S-merged). Let w_i' denote the factorword obtained from w_i by changing the factorword z of w_i to z'. Then clearly, $w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v and $r' = \pi(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k) = \pi(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_k) = r$, hence the statement follows.

Case 2: v can be obtained from w by one $\sim$-step. Let z and z' denote the factorwords of w and v, respectively, such that z is changed to z' in the $\sim$-step. If $z = \overrightarrow{abc}ac\overleftarrow{abc}$ for some $a, b, c \in S$ then from the definition of a $\sim$-step it follows that we have one of two situations: (1) z is a factorword of some w_i where w_i is a bracketed word; or (2) $z = \overrightarrow{abc}w_i\overleftarrow{abc}$ where $w_i = ac$, for some $1 \leq i \leq k$.

In case (1), z is a factorword of w_i for some $1 \leq i \leq k$ where w_i is a bracketed word. Let w_i' denote the factorword obtained from w_i by changing z to z'. If the first letters of z and w_i are identical then $\overline{z'} = \overline{w_i}$ and since $w_1, \ldots, w_{i-1}, z', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v, the statement follows. If the first letters of z and w_i are different then $\overline{w_i'} = \overline{w_i}$ and as $w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v, the statement follows. In case (2), when $z = \overrightarrow{abc}w_i\overleftarrow{abc}$ then $\overline{z'} = \overline{abc} \leq_{\mathring{\mathcal{J}}} \overline{ac} = \overline{w_i}$, hence $\pi(w_1, \ldots, w_{i-1}, z', w_{i+1}, \ldots, w_k) \leq_{\mathring{\mathcal{J}}} \pi(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_k)$ and since $w_1, \ldots, w_{i-1}, z', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v, the statement follows.

Now consider the opposite direction. If $z = abc$ for some $a, b, c \in S$ then z is a factorword of w_i for some $1 \leq i \leq k$. If $w_i \in S^+$ then $w_i = y_1abcy_2$ for some $y_1, y_2 \in S^*$ and $w_1, \ldots, w_{i-1}, y_1, \overrightarrow{abc}ac\overleftarrow{abc}, y_2, w_{i+1}, \ldots, w_k$ is a good factor-sequence of v and as $\overline{w_i} = \overline{y_1}\ \overline{abc}\ \overline{y_2} = \overline{y_1}\overline{\overrightarrow{abc}\ ac\overleftarrow{abc}}\ \overline{y_2}$ thus

$$\pi(w_1, \ldots, w_k) = \pi(w_1, \ldots, w_{i-1}, y_1, \overrightarrow{abc}ac\overleftarrow{abc}, y_2, w_{i+1}, \ldots, w_k),$$

hence the statement follows. If w_i is a bracketed word then let w_i' be the word obtained from w_i by changing z to z'. Then clearly $\overline{w_i'} = \overline{w_i}$, thus $\pi(w_1, \ldots, w_k) = \pi(w_1, \ldots w_{i-1}, w_i', w_{i+1}, \ldots w_k)$ and since $w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v, the statement follows.

Case 3: v can be obtained from w by one inserting step. Let $x \in S \cup A$ denote the letter inserted into w in the inserting step. For any factorword w_i of w let us say that x splits w_i if x is inserted into w between two consecutive letters of w_i. If z does not split w_i for any $1 \leq i \leq k$ then $w_1, \ldots, w_k$ is a good factor-sequence of v. If x splits w_i for some $1 \leq i \leq k$ then let $y_1, y_2 \in S^+$ be such that $w_i = y_1y_2$ and x is inserted between y_1 and y_2 in the inserting step. If $w_i \in S^+$ then $w_1, \ldots, w_{i-1}, y_1, y_2, w_{i+1}, \ldots, w_k$ is a good factor-sequence of v and $\pi(w_1, \ldots, w_{i-1}, y_1, y_2, w_{i+1}, \ldots, w_k) = \pi(w_1, \ldots, w_k)$. If w_i is a bracketed word then let $w_i' = y_1xy_2$. Then $w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k$ is a good factor-sequence of v, and as $\overline{w_i'} = \overline{w_i}$, therefore $\pi(w_1, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k) = \pi(w_1, \ldots, w_k)$. Hence, in both cases the statement follows. □

The following statement is easy to prove:

Lemma 6.6. *If $w \in S^+$ then for every $t \in Tr(w)$, $\overline{w} \leq_{\mathring{\mathcal{J}}} t$.*

Lemma 6.7. *Let S be $\mathring{\mathcal{J}}$-trivial. Then if $v, w \in S^+$ are such that $\theta(v)\ \mathring{\mathcal{J}}\ \theta(w)$ then $\overline{v} = \overline{w}$.*

Proof. Since $w_1 = w$ is a good factor-sequence of w and $\theta(v) \leq_{\mathring{\mathcal{J}}} \theta(w)$, by Lemma 6.5 there exists $r \in Tr(v)$ such that $r \leq_{\mathring{\mathcal{J}}} \overline{w}$. By Lemma 6.6 $\overline{v} \leq_{\mathring{\mathcal{J}}} r$, hence $\overline{v} \leq_{\mathring{\mathcal{J}}} \overline{w}$. Similarly, $\overline{w} \leq_{\mathring{\mathcal{J}}} \overline{v}$ holds and by $\mathring{\mathcal{J}}$-triviality of S, $\overline{v} = \overline{w}$ follows. □

Let $\widehat{S} = \overleftrightarrow{S}/\mathring{\mathcal{J}}$ and let τ denote the natural homomorphism $(S \cup A)^+ \to \overleftrightarrow{S}/\mathring{\mathcal{J}} = \widehat{S}$.

Lemma 6.8. *S can be embedded into the semigroup $\widehat{S}$ and $\widehat{S}$ is a $\mathring{\mathcal{J}}$-trivial semigroup.*

Proof. Define the map $\alpha : S \to \widehat{S}$ in the following way: for any $s \in S$ let $\alpha(s) = \tau(w)$ where $w \in (S \cup A)^+$ is such that $s = \overline{w}$. Then τ is clearly well-defined and a homomorphism. By Lemma 6.7, τ is injective, hence is an embedding of S into $\widehat{S}$. By Proposition 3.2 $\widehat{S}$ is a $\mathring{\mathcal{J}}$-trivial semigroup. □

Consider the infinite sequence $S = T_0, T_1, \ldots$ of semigroups such that $T_{i+1} = \widehat{T_i}$ for every $i \geq 0$ and define the semigroup T as the projective limit of $S = T_0, T_1, \ldots$ that is: let the set of elements of T be equal to $\bigcup_{i=0}^{\infty} T_i$; if $s, t \in T$ then let k be the smallest index such that $s, t \in T_k$ and define the product of s and t in T as the product of s and t in T_k.

Lemma 6.9. *If S is a $\mathring{\mathcal{J}}$-trivial semigroup and T is defined as above then:*

(1) S can be embedded into T
(2) T is $\mathring{\mathcal{J}}$-trivial
(3) T is a $\leq_{\mathcal{J}}$-compatible semigroup.

Proof. 1. This is obvious from the definition of T.

2. Suppose $s, t \in T$ and $s \mathrel{\mathring{\mathcal{J}}} t$. Then, thanks to our description of $\mathring{\mathcal{J}}$ in Lemma 2.1, we also have $s \mathrel{\mathring{\mathcal{J}}} t$ within one of the semigroups T_i. The semigroup T_i is $\mathring{\mathcal{J}}$-trivial by Proposition 3.2; therefore, $s = t$. Hence, T is $\mathring{\mathcal{J}}$-trivial.

3. We only need to prove that for any $s, t \in T$ if $s \leq_{\mathring{\mathcal{J}}} t$ then $s \leq_{\mathcal{J}} t$. Indeed, suppose that $s \leq_{\mathring{\mathcal{J}}} t$. By Lemma 2.1, it is sufficient to consider the case $s \leq'_{\mathring{\mathcal{J}}} t$. By the definition of $\leq'_{\mathring{\mathcal{J}}}$, there exist elements $s_1, t_1, t_2 \in T^1$ such that $s = t_1 s_1 t_2$, $t = t_1 t_2$. Assume that $s_1, t_1, t_2 \in T$; if some of these elements are equal to 1, the proof can be easily modified accordingly. Consider a semigroup T_i containing all these elements $s_1, t_1, t_2 \in T$. In the semigroup $\overleftrightarrow{T_i}$ we have $\overrightarrow{t_1 s_1 t_2}\, t\, \overleftarrow{t_1 s_1 t_2} = \overrightarrow{t_1 s_1 t_2}\, t_1 t_2\, \overleftarrow{t_1 s_1 t_2} = t_1 s_1 t_2 = s$ and thus $s \leq_{\mathcal{J}} t$ in $\overleftrightarrow{T_i}$. Clearly this inequality is preserved when we factorise by $\mathring{\mathcal{J}}$ to produce T_{i+1}. Since T_{i+1} is a subsemigroup of T, we have $s \leq_{\mathcal{J}} t$ in T. □

From the results of this section, the theorem below follows.

Theorem 6.1. *Every $\mathring{\mathcal{J}}$-trivial semigroup can be embedded into a $\mathring{\mathcal{J}}$-trivial $\leq_{\mathcal{J}}$-compatible semigroup.*

In the beginning of this section we have recalled that every semigroup can be embedded into a simple (that is, $\mathcal{J}$-simple) semigroup. However, it is easy to show that not every semigroup can be embedded into an $\mathcal{L}$-simple (or $\mathcal{R}$-simple) semigroup. There is a certain analogy between this and what happens with $\mathring{\mathcal{J}}$-trivial semigroups (as described in Theorem 6.1) versus $\mathring{\mathcal{L}}$-trivial (or $\mathring{\mathcal{R}}$-trivial) semigroups, see the example below.

Lemma 6.10. *Let S be a semigroup and let $s, t, a, b \in S$. If $s \leq_{\mathcal{L}} t$ and $ta = tb$ then $sa = sb$.*

Proof. Since $s \leq_{\mathcal{L}} t$, one has $s = ct$ for some $c \in S^1$ and thus $sa = cta = ctb = sb$. □

Example 6.1. We give an example of an $\mathring{\mathcal{L}}$-trivial semigroup which cannot be embedded into any $\leq_{\mathcal{L}}$-compatible semigroup (not only into an $\mathring{\mathcal{L}}$-trivial $\leq_{\mathcal{L}}$-compatible semigroup). As we stated in Example 3.1 OE_4 is an $\mathring{\mathcal{L}}$-trivial semigroup. Consider the following mappings in OE_4. Let $\alpha_1 : 4 \mapsto 4$, $3 \mapsto 3$, $2 \mapsto 1$ (and $1 \mapsto 1$, as in every element of OE_n). Let $\alpha_2 : 4 \mapsto 4$, $3 \mapsto 2$, $2 \mapsto 2$. Let $\alpha_3 : 4 \mapsto 3, 3 \mapsto 3, 2 \mapsto 2$. Let $\alpha = \alpha_1\alpha_3$ and let $\beta = \alpha_1\alpha_2\alpha_3$. By definition, $\beta \leq_{\mathring{\mathcal{J}}} \alpha$. It is easy to see that $\alpha\alpha_1 = \alpha\alpha_3$. However, $\beta\alpha_1 \neq \beta\alpha_3$. Therefore, by Lemma 6.10, in no semigroup containing OE_4 as a subsemigroup, we can have $\beta \leq_{\mathcal{L}} \alpha$.

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