Integrability, Recursion Operators and Soliton Interactions

$$L(\lambda)\psi \equiv i\sigma_3 \frac{d\psi}{dx} + \begin{pmatrix} -\frac{1}{2}|q|^2 - \lambda^2 & (\lambda+\alpha)q(x,t) \\ (\lambda-\alpha)q^*(x,t) & -\frac{1}{2}|q|^2 - \lambda^2 \end{pmatrix} \psi(x,t,\lambda) = 0,$$

$$q(x,t) = u(x,t)\exp\left(2i\int_x^\infty dy\,|u|^2\right),$$

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - i\frac{\partial |u|^2 u}{\partial x} + 2\alpha|u|^2 u(x,t) = 0.$$

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Editors

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Proceedings of the International Conference on

INTEGRABILITY, RECURSION OPERATORS AND SOLITON INTERACTIONS

CONTENTS

Preface

This volume contains selected papers based on the talks, presented at the Conference 'Integrability, Recursion Operators and Soliton Interactions', held in Sofia, Bulgaria (29–31 August 2012) at the Institute for Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences. Included are also invited papers presenting new research developments in the thematic area. The Conference was dedicated to the 65-th birthday of our esteemed colleague and friend Vladimir Gerdjikov. The event brought together more than 30 scientists, from 6 European countries to celebrate Vladimir's scientific achievements. All participants enjoyed a variety of excellent talks in a friendly and stimulating atmosphere. The main topics of the conference were those where Vladimir has contributed enormously during his career: integrable nonlinear partial differential equations, underlying algebraic and geometric structures of the integrable systems, soliton solutions, soliton interactions, quantum integrable systems, discrete integrable systems and applications of the nonlinear models. The papers, included in this volume will be useful to researchers with interests in these areas.

The organizers would like to express their gratitude to all participants and authors for their contribution to the success of the conference, and to Prof. Dimitar Tonev, director of the INRNE and the deputy director Prof. Lachezar Georgiev for their support.

Last, but nor least important, the organizers are grateful to Prof. Ivaïlo Mladenov and Dr. Mariana Hadjilazova for their help in the preparation of this volume.

Boyka Aneva
Georgi Grahovski
Rossen Ivanov
Dimitar Mladenov

Dear participants in the IROSI conference,
Dear friends and colleagues,

It is a great pleasure to welcome you here in Sofia at our Institute. With this conference we are celebrating the 65-th birthday of our esteemed colleague Vladimir Gerdjikov and the achievements of his active, fruitful and interesting career.

Vladimir was born on 7-th January 1947 in Stara Zagora, Bulgaria. He has studied in the Physics Department of the University of Sofia during the period 1964-1969. He did his Master Thesis at Göteborg Techniska Högskolan, Sweden, under the supervision of Professor Karl-Erik Eriksson. He defended it at the end of 1969 at the Faculty of Physics of Sofia University. In the 1970–71 Vladimir has been appointed as a physicist at the Institute of Physics of the Bulgarian Academy of Sciences, Sofia. His career path after that is

1971–1974
    Graduate student at Leningrad State University, USSR, where in 1974 he defended his PhD thesis "On the infrared singularities of the Quantum Electrodynamics" supervised by L. D. Faddeev and P. P. Kulish from the Leningrad Division of Steklov Institute of Mathematics;

1975–1976
    Research associate at INRNE, Sofia;

1977–1983
    Research associate and senior research associate at the Laboratory of Theoretical Physics at the Joint Institute for Nuclear Research, Dubna, USSR;

1984–1996
    Research associate and senior research associate at INRNE;

1987
    Doctor of Science with a Thesis "Generating Operators of Soliton–Type Nonlinear Evolution Equations, Related to the Semisimple Lie Algebras" defended at the Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research, Dubna, USSR;

1996
    Full Professor at INRNE, Sofia;

2002
    Founder and Head of the Laboratory of solitons, coherence and geometry at the Institute for Nuclear Research and Nuclear Energy.

Professor Gerdjikov has been very successful in his administrative work as well. He worked very actively as a member of the General Assembly of the Bulgarian Academy of Sciences (2004–2008) as a member of the Scientific council of the

Bulgarian Academy of Sciences (2004-2008) and as a Deputy Director of the Institute (2005–2008).

Vladimir Gerdjikov is an author of more than 190 scientific publications, including: more than 100 papers in scientific journals; more than 90 published reports in proceedings of international conferences; one monograph and co-editor of 5 proceedings volumes; more than 1000 independent citations, organizer and co-organizer of 13 conferences, workshops and symposia.

His main fields of research and achievements will be presented and celebrated at this conference, which I'm sure will be very fruitful and interesting.

I like very much his talent and enthusiasm to work with the young people. Indeed, he has supervised to completion 6 PhD students. This is one of the best results achieved in the Institute by a research supervisor! He has supervised also several MSc and BSc diploma theses.

All these results have been obtained by a tremendous amount of work. For his achievements Gerdjikov as a team leader has been awarded the INRNE Prizes for best work in theoretical physics two times: in 1998 and 2007.

Gerdjikov is a person who is addicted to work in a good sense, in a sense that he enjoys his work. Being always busy is something typical for his style, this is a feature which makes him unique. However this is possibly the only way of achieving such profound results - both in terms of quantity and quality.

I am very glad that many colleagues from abroad have gathered here for Vladimir's celebration. I am glad that here are also several young colleagues, some of whom as organizers of this nice event.

I would like to give Vladimir an official address as a memory of this special occasion.

I wish him many more successful years ahead in his career, more students, more conferences and symposia and to continue to be as energetic as ever!

Prof. D. Tonev, PhD
Director of INRNE
Sofia, August 29, 2012

Professor Vladimir S. Gerdjikov. Short Biography

Vladimir Gerdjikov was born on 7th January 1947 in Stara Zagora, Bulgaria. He obtained his higher education from the Physics Faculty of the University of Sofia during 1964-1969. Among his lecturers were several distinguished Bulgarian scientists such as Professors Christo Christov, Ivan Zlatev, Angel Nikolov, Alipi Mateev, Assen Datsev. Vladimir then prepared his Master Thesis in 1969 at Göteborg Techniska Hogskolan, Sweden, under the supervision of K.E. Eriksson, and defended it the same year at the Theoretical Physics Department of the Physics Faculty of the University of Sofia.

His first appointment was during the years 1970-71 as a physicist at the Institute of Physics of the Bulgarian Academy of Sciences, Sofia, Bulgaria.

During 1971-74, Vladimir Gerdjikov was a graduate student at Leningrad State University and in 1974 he completed and successfully defended his PhD thesis *"On the Infrared Singularities of the Quantum Electrodynamics"* at the Physics Department of Leningrad State University, Leningrad, USSR. His research was supervised by Ludvig D. Faddeev and Peter P. Kulish from the Leningrad Division of Steklov Institute of Mathematics.

Following his return to Sofia, during 1975-76 Vladimir Gerdjikov worked as a research associate at INRNE. Later on (1977-83), he moved to the Joint Institute for Nuclear Research, Dubna, USSR where he was a research associate and senior research associate at the Laboratory of Theoretical Physics. In 1987 he visited Dubna for three months and defended his Doctor of science thesis *"Generating Operators of Soliton Type Nonlinear Evolution Equations, Related to the Semisimple Lie Algebras"*.

These two long term visits (Leningrad 1971–1974 and Dubna 1977–1983) formed to a greater extent the scientific tastes and abilities of Vladimir. Being in Dubna he used the good chance to renew his collaboration with the Leningrad Division of Steklov Institute and especially with P. Kulish, but now on a new topic - integrable systems, both classical and quantum, and inverse scattering method. This topic was being developed also in Dubna by V. Makhan'kov, V. Gerdt and others. Besides, at the end of 1970's Vladimir Zakharov started a seminar on soliton theory in Moscow. This gave Vladimir an excellent opportunity not only to follow the

latest achievements in the field, but also to start personal contacts with many of the members of Zakharov's school, one of the leading groups in the world in the area of nonlinear sciences, including Sergey Manakov, Alexander Mikhailov, Evgeny Kuznetsov and others.

Also in Dubna he had the chance to meet Francesco Calogero and Marco Boiti. Although brief, these two meetings demonstrated common scientific interests and in fact led to a long term informal collaboration. Indeed, in 1985 Boiti invited him for three months to Lecce University, which started the Italian 'period' which still goes on. Another very important factor was the series of NEEDS biannual conferences which were held in Gallipoli, near Lecce, and where Vladimir was a regular participant.

The series of informal, but regular visits of Vladimir to Italy, besides Lecce University, included also Salerno University (G. Vilasi, G. Sparanno, M. Salerno), University of Naples (G. Marmo), University of Rome (F. Calogero, A. Degasperis, O. Ragnisco) and others.

Let me now briefly outline the topics to which Vladimir has contributed.

His PhD thesis deals with the infrared singularities in quantum electrodynamics. He proved that the quasiclassical results of Faddeev and Kulish hold true also for the quantum case. Thus, the infrared divergencies and the infrared singularities can be removed if instead of the naked charged particles, one considers their coherent states.

His next field became the soliton theory and the integrable models.

The idea that the inverse scattering method is a generalized Fourier transform started from the famous AKNS paper. Kaup was the first who realized that this fact is based on the completeness relation of the 'squared solutions' of the 2×2 Lax operator L and derived it. Soon after that, Gerdjikov and Khristov proposed an alternative rigorous proof of Kaup's result. Further development of this field initiated by Gerdjikov and collaborators included explicitly the gauge covariant formulation of the 'squared solutions' and the expansions over them. Thus, it became possible to derive all fundamental properties of the relevant soliton equations on the same footing. These expansions are actually spectral decompositions of the recursion operators $\Lambda_\pm$ which generate both the class of soliton equations and their Hamiltonian hierarchy. In a number of cases this approach allows one to derive the action-angle variables. Most of these results are summarized in the monograph with Gaetano Vilasi and Alexandar Yanovsky *"Integrable Hamiltonian Hierarchies"* published by Springer 'Lecture Notes in Physics' in 2008.

The expansions over the 'squared solutions' and their completeness relations were generalized in several directions. The first one concerned the Lax operators, which are quadratic 2×2 bundles. They allow one to solve the hierarchy containing

the derivative nonlinear Schroödinger equation and several of its generalizations, including the well known Gerdjikov-Ivanov (GI) equation.

The second one considered the generalized Zakharov-Shabat (GZS) systems of $n \times n$ Lax operators. Using such Lax operators one is able to solve the N-wave equations, the multicomponent nonlinear Schroödinger equation (MNLS) equations and their gauge equivalent. It is important here to introduce the 'squared solutions' using the fundamental analytic solutions (FAS) of L.

The third one concerned the Caudrey-Beals-Coifman (CBC) system whose FAS and 'squared solutions' are constructed in a substantially more elaborated way than for GZS. In the last years CBC systems with $\mathbb{Z}_h$ and $\mathbb{D}_h$-symmetries were treated and the completeness of their 'squared solutions' was also proved. Using it one can derive the action-angle variables for the 2-dimensional Toda field hierarchy.

In the early 1990's Vladimir started working, together with Ivan Uzunov, on another topic – N-soliton interactions in adiabatic approximation. In 1996 they derived a dynamical system for the $4N$ soliton parameters. Soon after that, a simplification of this system was proposed, which is known now as the complex Toda chain. An important consequence of this is the possibility to predict the asymptotic behavior of N-soliton trains. Since then this model has been tested a number of times and has been shown to describe adequately the soliton interactions not only for the nonlinear Schrödinger equation, but also of its vector generalization – the Manakov model. Both these equations find numerous applications in nonlinear optics, plasma physics, hydrodynamics etc.

His ability to organize research activities and collaborations culminated in 2002 when, together with several other colleagues, Vladimir founded the Laboratory of Solitons, Coherence and Geometry at the Institute for Nuclear Research and Nuclear Energy, and he has acted as the Head of the Laboratory ever since then. The scientific activities of the Laboratory, besides the soliton theory and soliton interactions, was extended to problems in quantum mechanics, differential geometry and the theory of coherent states. Since 2006 the Laboratory was joined by DSc Nikolay Kostov (1956-2011), who contributed very much to the study of multicomponent soliton equations and their reductions. Several classes of special reductions of the N-wave equations and the MNLS equations related to simple Lie algebras were analyzed. These include several special MNLS equations describing Bose-Einstein condensates.

During the last years Vladimir started another trend for constructing new types of integrable interactions based on the Riemann-Hilbert Problem with canonical normalization combined with Mikhailov's reduction group.

Vladimir was an organizer and co-organizer of several conferences, workshops and symposia. Most recently he was a co-organizer of the two international workshops on Complex structures, Integrability and Vector Fields together with Stancho Dimiev and Kouei Sekigawa (2008, 2010) held in Sofia. Together with Boyka Aneva and Georgi Grahovski he organized the International conference "Symmetries and Integrability of Difference Equations", SIDE-9 held in 2010 in Varna, Bulgaria.

He achieved remarkable results as a teacher and supervisor. He supervised 6 successful PhD dissertations, defended at various institutions: JINR, Dubna, USSR, INRNE and Cergy Pontoise University, Paris, France. He was a supervisor of several MSc and BSc theses as well. It is not an exaggeration to say that he has created a school of scientists, raised under his guidance, advice and mentorship.

Vladimir has designed and delivered two advanced lecture courses - on Soliton Theory and on the Theory of Lie Groups and Lie algebras for final year students and young researchers. He still works very actively in the area of integrability, shares his ideas and experience with younger colleagues, travels widely and collaborates with many scientists around the world.

In conclusion, I would like to wish our teacher, mentor, colleague and friend Vladimir Gerdjikov a long life, good health, a lot of energy and enthusiasm, and many more wonderful scientific ideas and results. I am sure that Vladimir will continue to be a source of inspiration, optimism and creativity to everyone who knows him.

Rossen Ivanov
Dublin Institute of Technology
Dublin Ireland

Opening speech
Dear Colleagues and Friends,

During the opening I heard a lot of nice words about myself, and I could not believe they were all true. It is impossible for one person to do so much work all by himself. And if I succeeded, that was because there were a lot of colleagues and friends helping me.

First of all I would like to thank Rossen Ivanov and Georgi Grahovski who are former graduate students of mine, for the idea to organize this conference. To say frankly, in the beginning I was reluctant, because it is not such a big deal to get that old. After all I agreed, not because of myself, but because such an event would be beneficial to my colleagues, friends and future students who may come around.

I am grateful to the Directorate of the Institute: Professor Dimitar Tonev and his deputy Professor Lachezar Georgiev for the support that made the conference possible.

Next I am grateful to my teachers at the University and mostly to academician Khristo Khristov. His lectures introduced not only me, but most of my colleagues in the field of theoretical and mathematical physics. I learned a lot from Peter Kulish and Ludvig Faddeev during the three years in Leningrad, and from Evgeny Khristov during the years I spent in Dubna.

The results in my papers wouldn't have been possible without co-authors, and I was lucky to work with a number of young people eager to understand and learn. So it is natural that I start with my former graduate students, noting briefly the results they are responsible for:

- Michail Ivanov – the GI equation;
- Alexander Yanovski – the gauge covariant approach to the recursion operators;
- Yordan Vaklev (1942 – 1999) and Michail Ivanov – the gauge covariant difference evolution equations;
- Evstati Evstatiev – his timely and precise observation lead us to the discovery of the complex Toda chain;
- Rossen Ivanov – reductions of N-waves, complete integrability of Camassa-Holm equation and CTC related to semisimple Lie algebras;
- Georgi Grahovski – reductions of N-waves, real Hamiltonian forms of 2-dim Toda's and many others;
- Tihomir Valchev, Victor Atanasov – soliton solutions to MNLS and their reductions, applications to Bose-Einstein condensates.

Thanks are due also to my co-authors who were able to put up with my character and still produce good results:

- Marco Boiti, Flora Pempinelli – Wadati-Konno-Ichikawa equation;
- Gaetano Vilasi and Alexander Yanovsky – for long years of collaboration during which we completed the monograph;
- Ivan Uzunov for proposing the study of N-soliton interactions and David Kaup, Evstati Evstatiev, Georgi Diankov for analyzing and testing the CTC;
- Nikolay Kostov (1956 – 2011) came up with most of the ideas for the papers on Bose-Einstein condensates (BEC);
- Giuseppe Marmo for patiently teaching me differential geometry, Hamiltonian dynamics that lead to the construction of real Hamiltonian forms jointly with Gaetano Vilasi and Assen Kyuldjiev;
- Mario Salerno, Vladimir Konotop, Bakhtiyor Baizakov, Victor Enol'skii – modulational instability of periodic solutions to MNLS and BEC;
- Evgeny Doktorov (1947 – 2011), Natalia Matsuka, Jianke Yang – CTC for derivative NLS;
- Rossen Dandoloff, Nikolay Kostov (1956 – 2011), Georgi Grahovski – differential geometry and Manakov model;
- Adrian Constantin, Rossen Ivanov – Camassa-Holm equation;
- Alexander Mikhailov, Tihomir Valchev, Georgi Grahovski – for deriving new types of soliton equations – the GMV equation;
- Radha Balakrishnan, Rossen Dandoloff, Dimiter Pushkarov, Avadh; Saxena – for organizing and editing the proceedings of the conference on "Topical Issue on Geometry, Integrability and Nonlinearity in Condensed Matter Physics", Bansko (2001);
- Milcho Tsvetkov and Plamen Fiziev – for organizing and editing the proceedings of the conference dedicated to Georgi Manev (1984 – 1965);
- Ivaïlo Mladenov, Stancho Dimiev, Kouei Sekigawa, Yasuo Matsushita – for organizing and editing the proceedings of two conferences on CSIVF in 2008 and 2010;
- Boyka Aneva, Plamen Iliev, Vassilis Papageorgiou. for organizing and editing the proceedings of SIDE-9 conference (2010) as special issue of SIGMA.

Special thanks to

- Francesco Calogero, Marco Boiti, Flora Pempinelli, Giusseppe Marmo, Gaetano Vilasi, Andrey Pogrebkov, Boris Konopelchenko, Barbara Prinari for giving me the chance to participate in a number of conferences on solitons;

- Ivaïlo Mladenov, Mariana Hadzhilazova, Akira Yoshioka for organizing the traditional conferences on Geometry, Integrability and Quantization in Varna.

Thus, I was able to enjoy south Italy and the Black sea and to have fruitful discussions with many famous scientists!

Finally and most of all I thank my parents and my family!!

In 1976 I married my wife Svetla, who at that time was working at Dubna. In 1977 I arrived to Dubna as a husband of my wife. That was crucial, because:

- I lost my freedom but gained a beautiful daughter and a son;
- I lost a bet to Evgeni Khristov but gained a teacher and a co-author;
- I had the chance to meet Vladimir Zakharov and his team in Moscow and Chernogolovka;
- I completed most of my DSc thesis and later defended it.

Best wishes to all of you.

Vladimir Gerdjikov
INRNE, Bulgarian Academy of Sciences
Sofia, Bulgaria
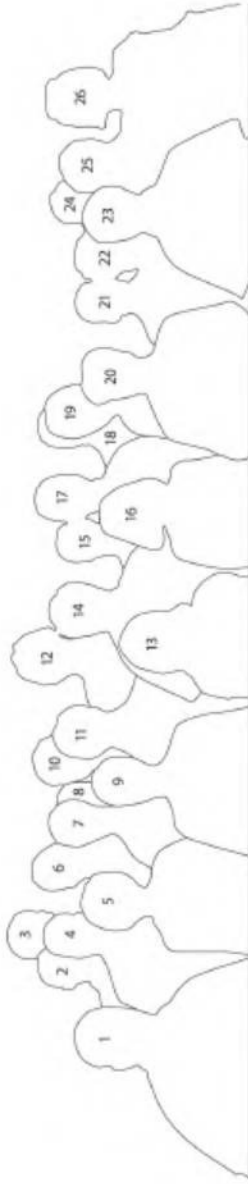
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24. Assen Kyuldjiev
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DEFORMED SQUEEZED STATE SOLUTION TO THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

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Abstract. Deformed squeezed states are generalized intelligent Schrödinger sates. They are implemented for an exact solution of the stationary n-species stochastic diffusion boundary problem.

1. Introduction

Coherent and squeezed states have a wide range of applications to various problems in many different areas of physics [1–4].

By origin the coherent states are quantum states, but at the same time they are parametrized by points in the phase space of a classical system. This makes them very suitable for the study of systems where one encounters a relationship between classical and quantum descriptions. From this point of view, interacting many-particle systems with stochastic dynamics provide an appropriate playground to enhance the utility of generalized coherent states.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, ..., n-1$ at a site $i = 1, 2, ..., L$ of a linear chain. A configuration on the lattice at a time t is determined by the set of occupation numbers $s_1, s_2, ..., s_L$ and a transition to another configuration s' during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s, s')\mathrm{d}t$. The time evolution of the stochastic system is governed by the master equation

$$\frac{\mathrm{d}P(s, t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s, s') P(s', t) \tag{1}$$

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for the probability $P(s, t)$ of finding the configuration s at a time t. With the restriction of dynamics that changes of configuration can only occur at two adjacent sites, the rates for such changes depend only on these sites. The two-site rates $\Gamma \equiv \Gamma_{jl}^{ik}$, $i, j, k, l = 0, 1, 2, ..., n - 1$ are assumed to be independent from the position in the bulk. At the boundaries, i.e., sites 1 and L, additional processes can take place with single-site rates L_k^i and R_k^i, $i, k = 0, 1, ..., n - 1$.

For processes where each lattice site can be occupied by a finite number of different-type particles, the master equation can be mapped to a Schrödinger equation in imaginary time of an n-state quantum spin-S Hamiltonian ($n = 2S + 1$ distinct states) with nearest-neighbour interaction in the bulk and single-site boundary terms

$$\frac{dP(t)}{dt} = -HP(t), \qquad H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}. \qquad (2)$$

The probability distribution thus becomes a state vector in the configuration space of the quantum spin chain and the ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary. The mapping provides a connection with integrable quantum spin chains and allows for exact results of the stochastic dynamics with the formalism of quantum mechanics.

A different description, which is also based on the relationship of a Markov process probability distribution with the quantum Hamiltonian picture, is the matrix-product states approach to stochastic dynamics [8,9]. The idea is that the stationary probability distribution, i.e., the ground state of a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms can be expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra

$$\Gamma_{jl}^{ik} D_i D_k = x_l D_j - x_j D_l, \qquad i, j, k, l = 0, 1, ..., n - 1 \qquad (3)$$

determined by the dynamics of the process. For diffusion processes that will be considered in this paper, $\Gamma_{ki}^{ik} = g_{ik}$ and the n-species diffusion quadratic algebra has the form

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \qquad (4)$$

where g_{ik} and g_{ki} are positive (or zero) probability rates, x_i are c-numbers and $i, k = 0, 1, ..., n - 1$. (No summation over repeated indices in equation 4.) The algebra has a Fock representation in an auxiliary Hilbert space where the n generators D act as operators. For systems with periodic boundary conditions, the stationary probability distribution is related to the expression

$$P(s_1, ..., s_L) = \text{Tr}(D_{s_1} D_{s_2} ... D_{s_L}). \qquad (5)$$

When boundary processes are considered the stationary probability distribution is related to a matrix element in the auxiliary vector space

$$P(s_1, ..., s_L) = \langle w | D_{s_1} D_{s_2} ... D_{s_L} | v \rangle \tag{6}$$

with respect to the vectors $|v\rangle$ and $\langle w|$, determined by the boundary conditions

$$\langle w | (L_i^k D_k + x_i) = 0, \qquad (R_i^k D_k - x_i) | v \rangle = 0 \tag{7}$$

where the x-numbers sum up to zero, because of the form of the boundary rate matrices

$$L_i^i = -\sum_{j=0}^{n-1} L_j^i, \qquad R_i^i = -\sum_{j=0}^{n-1} R_j^i, \qquad \sum_{i=0}^{n-1} x_i = 0. \tag{8}$$

These relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_k$, $i = 1, 2, ..., L$, $k = 0, 1, ..., n-1$ if a site i is occupied by a k-type particle. The number of all possible configurations of an n-species stochastic system on a chain of L sites is n^L and this is the dimension in the configuration space of the stationary probability distribution as a state vector. Each component of this vector, i.e., the (unnormalized) steady-state weight of a given configuration, is a trace or an expectation value in the auxiliary space given by equation (5) or equation (6). The quadratic algebra reduces the number of independent components to only monomials symmetrized upon using the relations (4).

The boundary vectors with respect to which one determines the stationary probability distribution of the n-species diffusion process are generalized, coherent or squeezed states of the deformed Heisenberg algebra underlying the algebraic solution of the corresponding quadratic algebra.

In this paper we consider a deformed squeezed state solution to the asymmetric simple exclusion process (ASEP). We present the stationary solution to the 2-species boundary ASEP and then generalize the boundary problem solution of the n-species stochastic diffusion process.

2. Deformed Coherent and Squeezed States

For completeness we first recall the definitions of a deformed coherent and squeezed state of the deformed Heisenberg algebra. It is generated by the operators a, a^+ and $q^{\pm N}$ with defining relations

$$aa^+ - qa^+a = 1, \qquad q^N a^+ = qa^+ q^N, \qquad q^N a = q^{-1}aq^N \tag{9}$$

where $0 < q < 1$ is a real parameter and

$$a^+ a = \frac{1 - q^N}{1 - q} \equiv [N]. \tag{10}$$

A Fock representation is obtained in a Hilbert space spanned by the orthonormal basis $\frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle = |n\rangle$, $n = 0, 1, 2, ...$ and $\langle n|n'\rangle = \delta_{nn'}$

$$a|0\rangle = 0, \qquad a|n\rangle = [n]^{1/2}|n-1\rangle, \qquad a^+|n\rangle = [n+1]^{1/2}|n+1\rangle. \qquad (11)$$

The Hilbert space consists of all elements $|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle$ with complex f_n and finite norm with respect to the scalar product $\langle f|f\rangle = \sum_{n=0}^{\infty} |f_n|^2$. The q-deformed oscillator algebra has a Bargmann–Fock representation on the Hilbert space of entire analytic functions.

Generalized or q-deformed coherent states are defined as the eigenstates of the deformed annihilation operator a and are labelled by a continuous (in general complex) variable z

$$a|z\rangle = z|z\rangle, \qquad |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle. \qquad (12)$$

These vectors belong to the Hilbert space for $|z|^2 < [\infty] = \frac{1}{1-q}$.

The scalar product of two coherent states for different values of the parameter z is non-vanishing

$$\langle z|z'\rangle = \sum_{0}^{\infty} \frac{(\bar{z}z')^n}{[n]!} = e_q^{\bar{z}z'} \qquad (13)$$

and they can be properly normalized with the help of the q-exponent on the RHS of (13)

$$|z\rangle = \exp_q\left(-\frac{|z|^2}{2}\right)\exp_q(za^+)|0\rangle. \qquad (14)$$

The q-deformed coherent states reduce to the conventional coherent states of a one-dimensional Heisenberg algebra in the limit $q \to 1^-$. These generalized coherent states carry the basic characteristics of the conventional ones, namely continuity and completeness (resolution of unity).

To recall the definition of a squeezed state [3], [14–16] we write the deformed Heizenberg algebra in an equivalent commutator form

$$[a, a^+] = q^N \qquad (15)$$

Then there is a two-parameter-dependent linear map to a pair of "quasi"-oscillators with (the symbol of) a "quasiparticle" number operator $\mathcal{N}$

$$A = \mu a + \nu a^+, \qquad A^+ = \bar{\mu}a^+ + \bar{\nu}a. \qquad (16)$$

These operators generate a deformed Heisenberg algebra

$$[A, A^+] = q^{\mathcal{N}} \qquad (17)$$

provided that

$$q^{\mathcal{N}} = (|\mu|^2 - |\nu|^2)q^N. \qquad (18)$$

In the limit $q \to 1^-$ the relation between the parameters of the conventional squeezed state is recovered [17]. In the deformed "quasi"-oscillator algebra Fock representation space with a vacuum $|0\rangle_s$ one can define a normalizable coherent state $|\zeta\rangle_s$ as the eigenvector of the annihilation operator A

$$|\zeta\rangle_s = e_q^{-\frac{1}{2}|\zeta|^2} e_q^{\zeta A^+} |0\rangle_s. \tag{19}$$

A squeezed state of the deformed creation and annihilation operators is a normalized solution of the eigenvalue equation

$$(\mu a + \nu a^+)|\zeta, \mu, \nu\rangle_s = \zeta|\zeta, \mu, \nu\rangle_s = A|\zeta\rangle_s. \tag{20}$$

This proposition is motivated by the analogy with the non-deformed case [18–20] and by the fact that such normalized eigenstate vectors of the written above linear combination of q-deformed oscillators appear in the solution of the boundary problem of a many-particle non-equilibrium system.

3. Physical Applications

As a physical we consider a diffusion process with n species on a chain of L sites with nearest-neighbour interaction with exclusion, i.e., a site can be either empty or occupied by a particle of a given type. In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a first-type particle at a site $i, ..., s_i = n - 1$ if there is an $(n-1)$th-type particle at a site i. On successive sites the species i and k exchange places with probability $g_{ik}\mathrm{d}t$, where $i, k = 0, 1, 2, ..., n - 1$. With $i < k$, g_{ik} are the probability rates of hopping to the left, and g_{ki} to the right. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle, or each of the sites is occupied by a particle of a different type. The n-species symmetric simple exclusion process is known as the lattice gas model of particle hopping between nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$. The n-species asymmetric simple exclusion process with hopping in a preferred direction is the diffusion-driven lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles n_i of each species in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. In most studied examples one considers phase transitions inducing boundary processes when a particle of type k, $k = 1, 2, ..., n - 1$ is added with a rate L_k^0 and/or removed with a rate L_0^k at the left end of the chain, and it is removed with a rate R_0^k and/or added with a rate R_k^0 at the right end of the chain.

In the matrix-product states approach the boundary rate matrices define the boundary vectors with respect to which the stationary probability distribution is related to an expectation value of product of matrices obeying the quadratic algebra (4). The problem to be solved is to find matrix representations of the quadratic algebra consistent with the boundary conditions (7), namely that the combinations $L_i^k D_k + x_i$ and $R_i^k D_k - x_i$ have common vectors with eigenvalue zero, where the only nonvanishing boundary rates are $L_0^k, L_k^0, R_0^k, R_k^0$, $k = 1, 2, ..., n - 1$. Once this problem is solved important physical quantities like correlation functions, currents, density profiles can be obtained which is the advantage of the matrix-product states approach. Despite the extensive study of simple generalizations of the exclusion process solutions of systems of n- species is lacking.

We are implementing here the deformed squeezed states and the deformed coherent states, introduced in the previous section, to obtain a solution to the general n boundary value problem.

3.1. The Two-Species Model with Incoming and Outgoing Particles at Both Boundaries

As an example we consider the two-species partially asymmetric simple exclusion process. We simplify the notations, namely at the left boundary a particle can be added with probability αdt and removed with probability γdt, and at the right boundary it can be removed with probability βdt and added with probability δdt. The system is described by the configuration set $s_1, s_2, ..., s_L$ where $s_i = 0$ if a site $i = 1, 2, ..., L$ is empty and $s_i = 1$ if a site i is occupied by a particle. The particles hop with a probability $g_{01} dt$ to the left and with a probability $g_{10} dt$ to the right, where without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The quadratic algebra $D_1 D_0 - q D_0 D_1 = D_0 + D_1$ is solved by a pair of deformed oscillators a, a^+ (see [23]). The boundary conditions have the form

$$(\beta D_1 - \delta D_0)|v\rangle = |v\rangle, \qquad \langle w|(\alpha D_0 - \gamma D_1) = \langle w|. \qquad (21)$$

For a given configuration $(s_1, s_2, ..., s_L)$ the stationary probability is given by the expectation value

$$P(s) = \frac{\langle w|D_{s_1} D_{s_2} ... D_{s_L}|v\rangle}{Z_L} \qquad (22)$$

where $D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. Within the matrix-product ansatz, one can also evaluate physical quantities such as the current J through a bond between site i and site

$i + 1$, the mean density $\langle s_i \rangle$ at a site i, the two-point correlation function $\langle s_i s_j \rangle$

$$J = \frac{Z_{L-1}}{Z_L}$$

$$\langle s_i \rangle = \frac{\langle w | (D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{L-i} | v \rangle}{Z_L} \tag{23}$$

$$\langle s_i s_j \rangle = \frac{\langle w | (D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{j-i-1} D_1 (D_0 + D_1)^{L-j} | v \rangle}{Z_L}$$

and higher correlation functions. In terms of the deformed boson operators the boundary conditions read

$$(\beta a - \delta a^+) | v \rangle = \sqrt{1 - q} \left(1 - \frac{\beta - \delta}{1 - q} \right) | v \rangle$$

$$\langle w | (\alpha a^+ - \gamma a) = \langle w | \left(1 - \frac{\alpha - \gamma}{1 - q} \right) \sqrt{1 - q}. \tag{24}$$

Hence, according to equation (20), the boundary vectors $| v \rangle$ and $\langle w |$ are squeezed coherent states, eigenstates of an annihilation and a creation operator A, A^+

$$(\beta a - \delta a^+) | v \rangle = A | v \rangle = v | v \rangle, \qquad \langle w | (\alpha a^+ - \gamma a) = \langle w | A^+ = \langle w | w \tag{25}$$

corresponding to the eigenvalues

$$v(\beta, \delta) = \sqrt{1 - q} \left(1 - \frac{\beta - \delta}{1 - q} \right), \quad w(\alpha, \gamma) = \sqrt{1 - q} \left(1 - \frac{\alpha - \gamma}{1 - q} \right). \tag{26}$$

The explicit form of the (unnormalized) vectors in the oscillator Fock space representation is given by

$$\langle w | = \sum_{n=0}^{\infty} \frac{w^n(\alpha, \gamma)}{\sqrt{[n]!}} | n \rangle, \qquad | v \rangle = \sum_{n=0}^{\infty} \frac{v^n(\beta, \delta)}{\sqrt{[n]!}} | n \rangle.$$

As already noted the operators A and A^+ are not each other's Hermitian conjugate. To find the expectation values of normally ordered monomials in D_0 and D_1, we make use of the inverse transformation

$$a = \frac{\alpha}{\alpha \beta - \gamma \delta} A + \frac{\delta}{\alpha \beta - \gamma \delta} A^+, \qquad a^+ = \frac{\beta}{\alpha \beta - \gamma \delta} A^+ + \frac{\gamma}{\alpha \beta - \gamma \delta} A. \tag{27}$$

Hence with $\Delta = \alpha \beta - \gamma \delta \neq 0$

$$D_0 + D_1 = \frac{2}{1 - q} + \frac{\alpha + \gamma}{\Delta \sqrt{1 - q}} A + \frac{\beta + \delta}{\Delta \sqrt{1 - q}} A^+ \tag{28}$$

and the normalization factor $\langle w | (D_0 + D_1)^L | v \rangle$ to the stationary probability distribution can be easily calculated in terms of the operators A and A^+. One has

$$(D_0 + D_1)^L = \left(\frac{2}{1-q} + \frac{\alpha+\gamma}{\Delta\sqrt{1-q}}A + \frac{\beta+\delta}{\Delta\sqrt{1-q}}A^+ \right)^L$$

$$= \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}\Delta^{-m}}{(1-q)^{L-\frac{m}{2}}} ((\alpha+\gamma)A + (\beta+\delta)A^+)^m. \quad (29)$$

To evaluate the above expression one makes use of the eigenvalue properties of the squeezed states $\langle w|$ and $|v\rangle$. For the purpose, one first applies the procedure for normal ordering of polynomials in A, A^+. To evaluate the normalization factor for boundary processes with incoming particles and outgoing particles at both ends of the chain one has to normally order boson operators whose deformed commutator is not normalized to unity. We spare the technical details which result in the formula

$$((\alpha+\gamma)A + (\beta+\delta)A^+)^m = \sum_{k=0}^{[m/2]} S_k^m (\alpha+\gamma)^k (\beta+\delta)^k$$

$$\times \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!} ((\beta+\delta)A^+)^l((\alpha+\gamma)A)^{m-2k-l}. \quad (30)$$

One explores next the eigenvalue properties of the operators A, A^+ with respect to the vectors $|v\rangle$ and $\langle w|$ and finds the normalization factor $\langle w|(D_0+D_1)^L|v\rangle = Z_L$ to the stationary probability distribution

$$Z_L = \sum_{m=0}^{L} \left(\binom{L}{m} \right) \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{(\alpha+\gamma)^k (\beta+\delta)^k}{(\alpha\beta-\gamma\delta)^m}$$

$$\times \left(\binom{m-2k}{l} \right)_q ((\beta+\delta)w)^l((\alpha+\gamma)v)^{m-2k-l}. \quad (31)$$

Consequently one directly obtains an expression for the current J. We note that an explicit formula for the normalization factor to the stationary probability distribution (and hence for the current) of the two-species diffusion system with incoming and outgoing particles at both boundaries has not been written elsewhere. Using the prescription of normal ordering, one can readily calculate the correlation functions and any other quantity of interest like density profiles, etc. Since none of the physical quantities of interest for this process have been presented elswhere, this strongly supports the squeezed coherent state solution as a powerful method for the study of stochastic systems.

3.2. Deformed Squeezed State Solution of the Boundary Problem for the n-Species Process

The algebra for the n-species open asymmetric exclusion process of a diffusion system coupled at both boundaries to external reservoirs of particles of fixed density has the form

$$D_{n-1}D_0 - qD_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0$$

$$D_0D_k - q_kD_kD_0 = -\frac{x_0}{g_k}D_k$$

$$D_kD_{n-1} - q_kD_{n-1}D_k = \frac{x_{n-1}}{g_k} \tag{32}$$

$$D_kD_l - q_{kl}^{-1}D_lD_k = 0$$

where $k, l = 1, 2, ..., n - 2$, $x_0 + x_{n-1} = 0$ and

$$q = \frac{g_{0,n-1}}{g_{n-1,0}}, \qquad q_{kl} = \frac{g_{kl}}{g_{lk}}, \qquad q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}}. \tag{33}$$

The equalities in the last formula, together with the relations

$$g_k = g_{0k} = g_{k,n-1}, \qquad g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0} \tag{34}$$

yield a mapping to the commutation relations of a q-deformed Heisenberg algebra (see [13]) of $n - 1$ oscillators $a_k, a_k^+, k = 0, 1, 2, ..., n - 2$. A solution is obtained by a shift of the oscillators a_0, a_0^+

$$D_0 = \frac{x_0}{g_{n-1,0}}\left(\frac{1}{1-q} + \frac{a_0^+}{\sqrt{1-q}}\right)$$

$$D_{n-1} = \frac{-x_{n-1}}{g_{n-1,0}}\left(\frac{1}{1-q} + \frac{a}{\sqrt{1-q}}\right) \tag{35}$$

and by the identification of the rest of the generators D_k, $k = 1, 2, ..., n - 2$ with the remaining $n - 2$ creation operators a_k^+

$$D_k = a_k^+, \qquad k \neq 0. \tag{36}$$

For the phase transition inducing boundary processes, when a particle of type k is added with a rate L_k^0 and removed with a rate L_0^k at the left end of the chain and when it is removed with a rate R_0^k and added with a rate R_k^0 at the right end of the

chain, the boundary vectors are defined by the systems of equations

$$
\begin{aligned}
\langle w| \left((-L_1^0 - L_2^0 - \ldots - L_{n-1}^0)D_0 + L_0^1 D_1 \right) & \\
+ \langle w| \left(L_0^2 D_2 + \ldots + L_0^{n-1} D_{n-1} + x_0 \right) &= 0 \\
\langle w|(L_1^0 D_0 - L_0^1 D_1) &= 0 \\
\langle w|(L_2^0 D_0 - L_0^2 D_2) &= 0 \\
&\vdots \\
\langle w|(L_{n-2}^0 D_0 - L_0^{n-2} D_{n-2}) &= 0 \\
\langle w|(L_{n-1}^0 D_0 - L_0^{n-1} D_{n-1} + x_{n-1}) &= 0
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
\left((-R_1^0 - R_2^0 - \ldots - R_{n-1}^0)D_0 + R_0^1 D_1 \right) |v\rangle & \\
+ \left(R_0^2 D_2 + \ldots + R_0^{n-1} D_{n-1} - x_0 \right) |v\rangle &= 0 \\
(R_1^0 D_0 - R_0^1 D_1)|v\rangle &= 0 \\
(R_2^0 D_0 - R_0^2 D_2)|v\rangle &= 0 \\
&\vdots \\
(R_{n-2}^0 D_0 - R_0^{n-2} D_{n-2})|v\rangle &= 0 \\
(R_{n-1}^0 D_0 - R_0^{n-1} D_{n-1} - x_{n-1})|v\rangle &= 0.
\end{aligned}
\tag{38}
$$

The two systems are similar and can be solved by the same procedure. From the second to the last but one equation in (37) and (38), one has

$$
\begin{aligned}
\langle w|L_0^k D_k &= \langle w|L_k^0 D_0 \tag{39} \\
R_0^k D_k |v\rangle &= R_k^0 D_0 |v\rangle \tag{40}
\end{aligned}
$$

for $k = 1, 2, \ldots n - 2$. Hence one inserts equations (39) in the first equation of the system (37) and equations (40) in the first equation of the system (38) to obtain in both cases an equation that coincides with the last equation of the corresponding systems. Thus the system for the left and right boundary vectors are reduced to the pair of equations

$$
\langle w|(L_{n-1}^0 D_0 - L_0^{n-1} D_{n-1}) = \langle w|(R_0^{n-1} D_{n-1} - R_{n-1}^0 D_0)|v\rangle = |v\rangle. \tag{41}
$$

Making use of the explicit solution for D_{n-1} and D_0 as shifted deformed oscillators (with $x_0 = -x_1 = 1$), we rewrite equations (41) as

$$
(R_0^{n-1}a_0 - R_{n-1}^0 a_0^+)|v\rangle = \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right)|v\rangle
$$

$$
\langle w|(L_{n-1}^0 a_0^+ - L_0^{n-1}a_0) = \langle w|\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right)\sqrt{1-q}.
$$

(42)

The latter equations, in accordance with equation (20), determine the boundary vectors as squeezed coherent states of the deformed boson operators a_0, a_0^+ corresponding to the eigenvalues

$$
v = \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right)
$$

$$
w = \sqrt{1-q}\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right).
$$

(43)

The explicit form of these vectors is readily written, namely

$$
\langle w| = \langle n|\sum_{n=0}^{\infty}\frac{w^n}{\sqrt{[n]!}}\, e_q^{-\frac{1}{2}vw} \qquad \text{and} \qquad |v\rangle = e_q^{-\frac{1}{2}vw}\sum_{n=0}^{\infty}\frac{v^n}{\sqrt{[n]!}}|n\rangle.
$$

We thus conclude that the left and right boundary vectors are squeezed coherent states of the shifted deformed annihilation and creation operators D_{n-1} and D_0, associated with the non-zero boundary parameters x_{n-1} and x_0, and with eigenvalues depending on the right and left boundary rates

$$
(R_0^{n-1}a_0 - R_{n-1}^0 a_0^+)|v\rangle = A|v\rangle = v|v\rangle
$$

$$
\langle w|(L_{n-1}^0 a_0^+ - L_0^{n-1}a) = \langle w|A^+ = \langle w|w
$$

(44)

where the eigenvalues v and w are given by the formulas in (43). The operators A and A^+ satisfy the same deformed commutation relations as a and a^+, as was outlined in section 3, with the only difference that they are not Hermitian-conjugate. However their conjugation property is consistent with the involution of the quadratic algebra (4) which reflects the left-right symmetry of the model. From the inverse linear maps, with $R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0 \neq 0$, we obtain

$$
a_0 = \frac{L_{n-1}^0}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A + \frac{R_{n-1}^0}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A^+
$$

$$
a_0^+ = \frac{R_0^{n-1}}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A^+ + \frac{L_0^{n-1}}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A
$$

(45)

with the help of which the mean values of the generators D_0, D_{n-1} and the rest D_k for $k = 1, 2, \ldots, n-2$ are readily found

$$\langle w|D_0|v\rangle = \frac{1}{g_{n-1,0}(R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0)}\left(\frac{1}{1-q} + \frac{R_0^{n-1}w + L_0^{n-1}v}{\sqrt{1-q}}\right)$$

$$\langle w|D_{n-1}|v\rangle = \frac{1}{g_{n-1,0}(R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0)}\left(\frac{1}{1-q} + \frac{R_{n-1}^0 w + L_{n-1}^0 v}{\sqrt{1-q}}\right)$$

$$\langle w|D_k|v\rangle = \frac{L_k^0}{L_0^k}\langle w|D_0|v\rangle = \frac{R_k^0}{R_0^k}\langle w|D_0|v\rangle. \tag{46}$$

With these expressions at hand, it is easy to calculate the expectation value of any monomial of the form $\langle w|D_{s_1}D_{s_2}\cdots Ds_L|v\rangle$ (where $D_{s_i} = D_j$ for $i = 1, 2, \ldots, L$, $j = 0, 1, 2, \ldots, n-1$), which enters the stationary probability distribution, the current, the correlation functions. One first makes use of the algebra to bring all generators D_k for $k = 1, 2, \ldots, n-2$ to the very right or to the very left, which results in an expression of the expectation value as a power in D_0 and D_{n-1}. Then one writes the arbitrary power of D_0, D_{n-1} as a normally ordered product of A and A^+ to obtain, upon using the eigenvalue properties of the latter, an expression for the relevant physical quantity in terms of the probability-rate-dependent boundary eigenvalues v and w.

We note that if the boundary processes are such that there are only incoming particles of $(n-1)$th-type at the left boundary and only outgoing $(n-1)$th-type particles at the right boubdary, i.e., $L_0^{n-1} = R_{n-1}^0 = 0$ in (44), then the eigenstate equations define the boundary vectors $|v\rangle$ and $\langle w|$ as q-deformed coherent states. Using the eigenvalue properties of the latter one can likewise obtain the physical quantities of interest for the system. The value $q \neq 0$ corresponds to a partially asymmetric while $q = 0$ to a totally asymmetric diffusion in the bulk of the $n-1$-type particle. The deformed oscillator coherent states defined for $0 < q < 1$ and for $q = 0$ provide a unified description of both the partially and the totally asymmetric hopping of a given type of particle.

To summarize, we have applied the q-deformed squeezed states to obtain within the matrix-product states approach a boundary problem solution to a multiparticle (general n) open stochastic system of lattice Brownian motion.

References

[1] Schrödinger E., *Der stetige Übergang von der Mikro- zur Makromechanik*, Naturwissenschaften **14** (1926) 664–666.

[2] Glauber R., *The Quantum Theory of Optical Coherence*, Phys. Rev. **130** (1963) 2529–2539.

[3] Perelomov A., *Generalized Coherent States and Their Applications*, Springer, New York 1986.

[4] Gilmore R., *Geometry of Symmetrized States*, Ann. Phys. (NY) **74** (1972) 391–463; Zhang W., Feng D. and Gilmore R., *Coherent States: Theory and Some Applications*, Rev. Mod. Phys. **62** (1990) 867–927.

[5] Aric M. and Coon D., *Hilbert Spaces of Analytic Functions and Generalized Coherent States*, J. Math. Phys. **17** (1976) 524–527.

[6] Jurco B., *On Coherent States for the Simplest Quantum Groups*, Lett. Math. Phys.**21** (1991) 51–58; Jurco B. and Stovicek P., *Coherent States for Quantum Compact Groups*, Comm. Math. Phys. **182** (1996) 221–251.

[7] Klauder J. and Skagerstam B., *Coherent States*, World Scientific, Singapore 1985.

[8] Derrida B., Evans M., Hakim V. and Pasquier V., *Exact Solution of a 1D Asymmetric Exclusion Model Using a Matrix Formulation*, J. Phys. A: Math. & Gen. **26** (1993) 1493–1517.

[9] Krebs K. and Sandow S., *Matrix Product Eigenstates for One-Dimensional Stochastic Models and Quantum Spin Chains*, J. Phys. A: Math. & Gen. **30** (1997) 3165–3173.

[10] Isaev A., Pyatov P. and Rittenberg V., *Diffusion Algebras*, Eprint ArXiv: cond-matt/0103603, (2001)

[11] B.Aneva, *The Noncommutative Space of Stochastic Diffusion Systems*, J. Phys. A: Math. & Gen. **35** (2002) 859–878.

[12] Wess J. and Zumino B., *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. (Proc. Suppl.) B **18** (1991) 302–312.

[13] Fairlie D. and Zachos C., *Multiparameter Associative Generalizations of Canonical Commutation Relations and Quantized Planes*, Phys. Lett. B **256** (1991) 43–49.

[14] Stoler, D. *Equivalence Classes of Minimum Uncertainty Packets*, Phys. Rev. D**1** (1970) 3217–3219; *Generalized Coherent States*, Phys. Rev. D **4** (1971) 2309–2312.

[15] McDermott R. and Solomon A. *An Analogue of the Unitary Displacement Operator for the q-Oscillator*, J. Phys. A: Math. & Gen. **27** (1994) 2037–2044.

[16] Penson K. and Solomon A., *New Generalized Coherent States*, J. Math. Phys. **40** (1999) 2354–2363.

[17] Yuen H., *Two-Photon Coherent States of the Radiation Field*, Phys. Rev. A **13** (1976) 2226–2243.

[18] Wodkiewicz K. and Eberly J., *Coherent States, Squeezed Fluctuations and the SU(2) and SU(1,1) Groups in Quantum Optics Applications*, J. Opt. Soc. Am. B **2** (1985) 458–466.

[19] Nieto M. and Truax D., *Squeezed States for General Systems*, Phys. Rev. Lett. **71** (1993) 2843–2846.

[20] Trifonov D., *Generalized Intelligent States and Squeezing*, J. Math. Phys. **35** (1994) 2297–2308.

[21] Katriel J. and Solomon A., *Generalized q-Bosons and Their Squeezed States*, J. Phys. A: Math. & Gen. **24** (1991) 2093–2106.

[22] Celeghini E., Raseti M. and Vitielo G., *Squeezing and Quantum Groups*, Phys. Rev. Lett. **66** (1991) 2056–2059.

[23] Blythe R., Evans M., Colaiori F. and Essler F., *Exact Solution of a Partially Asymmetric Exclusion Model Using a Deformed Oscillator Algebra*, J. Phys. A: Math. & Gen. **33** (2000) 2313–2332.

SCHRÖDINGER MINIMUM UNCERTAINTY STATES OF EM-FIELD IN NONSTATIONARY MEDIA WITH NEGATIVE DIFFERENTIAL CONDUCTIVITY

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Abstract. Quantization of the electromagnetic field in non-stationary media (linear with respect to E, with negative differential conductivity) is investigated. The dynamical invariants and statistical properties of the field are found in such media. It is shown that in the eigenstates of linear dynamical invariant, the Schrodinger uncertainty relation is minimized. The time evolution of the tree independent second-order statistical moments (quantum fluctuations: covariance cov(q,p), var(q) and var(p)) are found out.

1. Introduction

The increasing use of energy as a result of the Industrial Revolution has brought a number of serious problems. Understanding the process of photosynthesis will play key role to solve these problems and to develop alternative energy sources in two aspects

- producing alternative fuels (such as H_2, biofuel etc.)
- producing directly Electricity using artificial photosynthesis in Dye-sensitized solar cells.

In the first processes (photosynthesis) the solar light is transformed into chemical energy, saved in molecule **adenosine triphosphate** (ATP). Thus a universal accumulator of energy is formed for widdely distributed biological processes [22].

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During the process of photosynthesis [23], the ATP is formed from **adenosine diphosphate** (ADP) and inorganic phosphate. Besides light this conversion requires a donor of electrons as well as protons received from water.

The chemical energy stored in this "biological batery" (ATP) is used by plants to synthesize carbohydrates from CO_2 and H_2O.

Similar and more simplified processes are observed at artificial photosynthesis in **Dye-Sensitized Solar Cells** (DSSC), where the sun light energy absorbed by Ruthenium-polypyridine dye and is transform in the electrical energy, like the chlorophyll in green leaves. Absorbed photons create an excited state of the dye, from which an electron is transported directly into the conduction band of the porous TiO2 in the artificial cell (see also [7]).

2. Motivation for Our Approach

It is well known that electron transport in photosynthesis has Quantum Nature [15]. It is due to the tunneling effect of electron through the barrier, with the action of light quanta. The electron is being tunneled from one carrier to another (starting from chlorophyll molecule to acceptor molecule) with probability depending on the width and height of the barrier. The probability decreases exponentially with increasing the barrier's size.

We are going to aproach only part of the photosynthesis problem, concerning the creation of electrons and charge transport in photosensitive dyes (e.g., in chlorophyll and ruthenium-polypyridine dyes, at photosynthesis and artificial photosynthesis respectively), using tools from quantum mechanics.

For this purpose we consider electron transport chain in thylakoid membrane (or dye solution in DSSC) as a linear media with **Negative Differential Conductivity** (NDC). The light interacts with dye molecule, excites electrons, which overcomes the subsequent quantum potential barrier. This transfers electron from one molecule carrier to an other carrier in the whole chain of electron transport.

3. Schrödinger Uncertainty Relation and Dynamical Invariants in QM

The description of quantum systems is fundamental for understanding many problems in physics and particulary in chemistry. One of the most revolutionary consequences that quantum mechanics bequeathed as a fundamental principle in physics is the refusal of strong determinism. That is why the uncertainty relation plays fundamental role in this science. In 1930, a few years after Heisenberg, Schrödinger

had generalized the famous **Uncertainty Relation** (UR) in quantum mechanics (QM) [2, 16, 18]

$$(\Delta q)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4} + \text{Cov}^2(q, p). \tag{1}$$

The above inequality shows the general connection between all three independent statistical moments of second order of two quantum variables q and p - the covariance $\text{Cov}(q, p)$

$$\text{Cov}(q, p) = \frac{1}{2}\langle qp + pq \rangle - \langle q \rangle \langle p \rangle \tag{2}$$

and the variances $(\Delta q)^2$ and $(\Delta p)^2$ defined as particular case of covariance

$$(\Delta q)^2 = \text{Cov}(q, q) \quad \text{respectively} \quad (\Delta p)^2 = \text{Cov}(p, p). \tag{3}$$

Canonical variables $\hat{q}$ and $\hat{p}$ satisfy the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hbar\,\hat{1}. \tag{4}$$

In terms of the covariance matrix $\sigma_M(q, p)$ [6], [10] the uncertainty relation (1) takes the form

$$\det[\sigma_M(q, p)] \geq \frac{\hbar^2}{4}. \tag{5}$$

Other important notions of QM are the dynamical invariants (integrals of motion) $\hat{I}$. These are operators which do no depend on the time t. Using the definition of total derivative in QM of certain quantum system with Hamiltonian $\hat{H}$, the dynamical invariants $\hat{I}$ are defined as solutions to the equation [12]

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} - \frac{i}{\hbar}[\hat{I}, \hat{H}] = 0. \tag{6}$$

The canonical commutation relations (4) show that quadratic in $\hat{q}$ and $\hat{p}$ Hamiltonians admit linear in $\hat{q}$ and $\hat{p}$ dynamical invariants. In [14] a family of (non-Hermitian) invariants $\hat{A}$ for the general nonstationary quadratic Hamiltonian

$$\hat{H} = a(t)\hat{p}^2 + b(t)(\hat{p}\hat{q} + \hat{q}\hat{p}) + c(t)\hat{q}^2 + d(t)\hat{p} + e(t)\hat{q} + f(t) \tag{7}$$

have been constructed in the form

$$\hat{A}(t) = \sqrt{\frac{a}{\hbar}}\left[\epsilon\hat{p} + \frac{1}{a}\left(\epsilon b - \dot{\epsilon} - \frac{\dot{a}}{2a}\epsilon\right)\hat{q}\right] \tag{8}$$

where ϵ is any solution of the second order equation (classical oscillator equation)

$$\ddot{\epsilon} + \Omega^2(t)\epsilon = 0. \tag{9}$$

Actually $\hat{A}^\dagger(t)$ and $\hat{A}(t)$ are generalization of boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ of the stationary oscillator (with $\Omega = $ constant). The time-dependent

coefficients $a(t), b(t)$ and $c(t)$ in (7) establish the connection between the Hamiltonian $\hat{H}$ and the frequency $\Omega(t)$ of classical non-stationary harmonic oscillator

$$\Omega^2 = 4ac + 2b\frac{\dot{a}}{a} + \frac{\ddot{a}}{2a} - \frac{3\dot{a}^2}{4a^2} - 4b^2 - 2\dot{b}. \tag{10}$$

The linear part in the Hamiltonian (7) is not essential for the classical non-stationary harmonic oscillator, so it is assumed that $d(t) = e(t) = f(t) = 0$.

The commutator $[\hat{A}, \hat{A}^\dagger]$ is presented by Wronsky determinant W

$$[\hat{A}, \hat{A}^\dagger] = \frac{i}{2}(\epsilon\dot{\epsilon}^* - \epsilon^*\dot{\epsilon}) \equiv \frac{i}{2}W \tag{11}$$

so that $[\hat{A}, \hat{A}^\dagger] = \hat{1}$ iff

$$\epsilon = |\epsilon|\, e^{i\int_0^t \frac{dt'}{|\epsilon(t')|^2}}. \tag{12}$$

4. Quantization of EM Field in Linear Media with Negative Differential Conductivity

The Maxwell equations in non-stationary linear media have the form

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mu(t)\boldsymbol{H}(\boldsymbol{r}, t), \qquad \boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon(t)\boldsymbol{E}(\boldsymbol{r}, t), \qquad \boldsymbol{j} = \sigma(t)\boldsymbol{E} \tag{13}$$

$$\mathrm{div}\,\boldsymbol{D} = 0, \qquad\qquad \mathrm{rot}\,\boldsymbol{H} = \frac{\partial}{\partial t}\boldsymbol{D} + \sigma(t)\boldsymbol{E}$$

$$\tag{14}$$

$$\mathrm{div}\,\boldsymbol{B} = 0, \qquad\qquad \mathrm{rot}\,\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}.$$

Note that $\varepsilon(t) = \varepsilon_r(t)\varepsilon_0$ is the dielectric permittivity, and differs from the solution $\epsilon(t)$ of classical oscillator equation (9).

A scheme was proposed for quantizing the damped light in conducting media [3] (see references therein). We are going to apply the quantization not only for non-stationary media ($\epsilon(t), \mu(t)$ and $\sigma(t)$), but for a case of negative differential conductivity. For convenience we will consider one dimension case (in x direction). A linear and homogenous media could have some resistivity R (which is a positive constant). There are some special cases, when the resistivity (respectively - the conductivity) vary with the applied voltage. For example, this is the case with tunnel diodes, which are represented as over-dopped semiconductors with very narrow $p - n$-junction, playing a role of quantum mechanical potential barrier [4, 5] (see Fig. 1). One analytical expression [5] for such I-V characteristic is shown here

$$I = \frac{U}{R_0}\exp\left[-(\frac{U}{U_0})^m\right] + I_s\exp\left[\frac{U}{\eta U_{th}} - 1\right] \tag{15}$$

where R_0, U_0, I_s, η and U_{th} are appropriate constants.
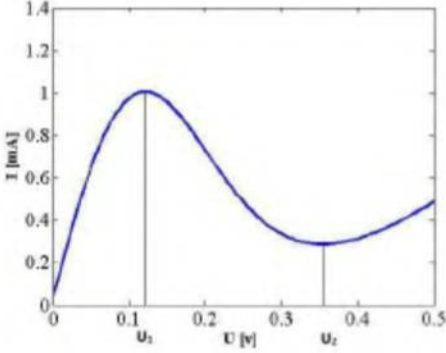
Fig. 1. I-U characteristic of device with negative differential conductivity.
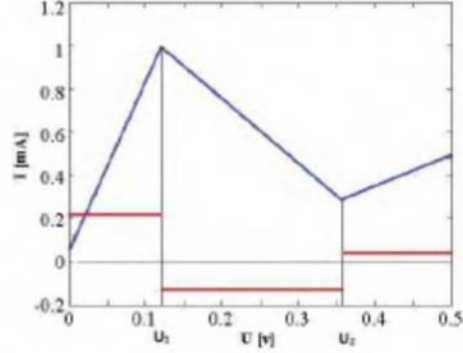
Fig. 2. Idealized I-U characteristic and the first derivative of σ, $\sigma_{\mathrm{diff}} < 0$ for $U_1 < U < U_2$.

To escape the problems with quantization in such media, we consider a hypothetic one, consisted of three different voltage domains, presenting the $I = I(U)$ with strait lines, as is shown on Fig. 2. This point of view is reasonable, because as is seen from Fig. 1, in all three domains there exists smaller sections, where the currents $I = I(U)$ could be presented approximately as strait lines. It is obvious that in the first and in the third domains, the secondary quantization could be solved in standard way (see for example [8,9,21]). Here we focus our attention mainly on the interesting second domain, which is called the regime with negative differential resistance (we do not take into account the transitions between domains, and leave this problem for future investigations). For the domain with negative differential conductivity around inflex point we always could apply linear approach. So, for our simplified model we have

$$\sigma = \text{const} > 0 \quad \text{but} \quad \sigma_{\mathrm{diff}} = \frac{\mathrm{d}\sigma(U)}{\mathrm{d}U} < 0 \qquad \text{for} \quad U_1 < U < U_2. \qquad (16)$$

For this domain, where the differential conductivity is negative, we are going to find analytical solution for this quantum problem, when the condition (16) is satisfied also. Applying the Coulomb gauge, one can define vectors fields as

$$\boldsymbol{B} = \text{rot}\,\boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \qquad (17)$$

and from Maxwell equations (14) we obtain the equation for $\boldsymbol{A}(\boldsymbol{r}, t)$

$$\nabla^2 \boldsymbol{A} - \mu(\sigma + \dot{\varepsilon})\frac{\partial \boldsymbol{A}}{\partial t} - \varepsilon\mu\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0 \qquad (18)$$

As usual (see e.g. the books [11, 17, 24]) we expand vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ in terms of mode functions $\boldsymbol{u}_l(\boldsymbol{r}) = \boldsymbol{e}_{l,\xi} u_{l,\xi}(\boldsymbol{r})$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{l,\xi} \boldsymbol{e}_{l,\xi} u_{l,\xi}(\boldsymbol{r}) q_{l,\xi}(t) \tag{19}$$

which satisfy the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_{0,l}^2}{c^2} \right) u_{l,\xi}(\boldsymbol{r}) = 0. \tag{20}$$

From Maxwell equations (14) it follows that (in case of linear media (13)) the time-dependent factors q_l are to obey the following linear equation (furthermore, unless otherwise stated, we suppress the polarization index ξ)

$$\frac{\partial^2 q_l}{\partial t^2} + \frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} \frac{\partial q_l}{\partial t} + \omega_l^2(t) q_l = 0, \qquad \omega_l^2(t) = \frac{\omega_{0,l}^2}{c^2 \varepsilon(t) \mu(t)}. \tag{21}$$

One can see that the equation (21) could be obtained from the classical Hamilton equation with Hamilton function

$$H_l = \frac{1}{2} \left[\frac{1}{\varepsilon_0} e^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'} p_l^2 + \varepsilon_0 \omega_l^2(t) e^{\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'} q_l^2 \right] \tag{22}$$

Introducing the canonical operators $q->\hat{q}_l$ and $p->\hat{p}_l$, which obey the commutation relation (4) we receive for the total Hamiltonian of the EMF as a sum over all modes [3]

$$\hat{H} = \sum_l \left[\frac{1}{2\varepsilon_0} e^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'} \hat{p}_l^2 + \frac{\varepsilon_0 \omega_l^2(t)}{2} e^{\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'} \hat{q}_l^2 \right] \equiv \sum_l \hat{H}_l. \tag{23}$$

It can be seeing that time-dependent coefficients in equation (23) are

$$a(t) = \frac{1}{2\varepsilon_0} e^{-\int_0^t \frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} dt}, \qquad b(t) = 0, \qquad c(t) = \frac{\varepsilon_0 \omega_l^2(t)}{2} e^{\int_0^t \frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} dt} \tag{24}$$

and for the frequency $\Omega_l(t)$ of the non-stationary harmonic oscillator (10) we get

$$\Omega_l^2(t) = \omega_l^2(t) - \frac{1}{2} \frac{d}{dt} \left(\frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} \right) - \frac{1}{4} \left(\frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} \right)^2. \tag{25}$$

In this case the invariants that satisfy the canonical boson relation $[\hat{A}, \hat{A}^\dagger] = \hat{1}$ are

$$\hat{A}_l = \frac{1}{\sqrt{2\hbar\varepsilon_0}} e^{-\frac{1}{2} \int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'}$$

$$\times \left[\epsilon_l \hat{p}_l - \varepsilon_0 e^{\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} dt'} \left(\dot{\epsilon}_l - \frac{1}{2} \frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)} \epsilon_l \right) \hat{q}_l \right]. \tag{26}$$

The linear invariants $\hat{A}_l$ and the quadratic ones $\hat{A}_l^\dagger \hat{A}_l$ have the following eigen-functions [14] ($\psi_{\alpha_l}(q_l,t) = \langle q_l|\alpha_l;t\rangle$, $\psi_{n_l}(q_l,t) = \langle q_l|n_l;t\rangle$

$$\psi_{\alpha_l}(q_l,t) = \psi_0(q_l,t)\exp\left[\sqrt{\frac{2}{a\hbar}}\frac{\alpha_l}{\epsilon_l}q_l - \frac{\epsilon_l^*}{2\epsilon_l}\alpha_l^2 - \frac{1}{2}|\alpha_l|^2\right] \qquad (27)$$

$$\psi_{n_l}(q_l,t) = \psi_0(q_l,t)\frac{(\epsilon_l^*/2\epsilon_l)^{n_l/2}}{\sqrt{n_l!}}H_{n_l}(x_l), \qquad x_l = \frac{q_l}{|\epsilon_l|\sqrt{a}} \qquad (28)$$

with eigenvalues α_l and n_l respectively. Here $H_n(x)$ are Hermite polynomials and $\psi_0(q_l,t)$ are the ground state wave functions ($\hat{A}_l\psi_0 = 0$)

$$\psi_0(q_l,t) = \left(\epsilon_l(\pi a\hbar)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\exp\left[\frac{i}{2a\hbar}\left(\frac{\dot{\epsilon}_l}{\epsilon_l} + \frac{\dot{a}}{2a}\right)q_l^2\right]. \qquad (29)$$

These time-dependent wave functions are normalized solutions to the Schrödinger equation with Hamiltonian $\hat{H}_l$, equation (23). Since $\hat{A}(t)$ and $\hat{A}_l^\dagger(t)\hat{A}_l(t)$ are dynamical invariant, the eigenvalues α_l and n_l are constant in time.

The system of $|\alpha_l;t\rangle$ is overcomplete in the one mode Hilbert space H_l (the set of $|n_l;t\rangle$ being complete)

$$\frac{1}{\pi}\int |\alpha_l;t\rangle\langle t;\alpha_l|d^2\alpha_l = \sum_{n_l}|n_l;t\rangle\langle t;n_l| = \hat{1}_l. \qquad (30)$$

These states $|\alpha_l;t\rangle$ minimize the general uncertainty relation of Schrödinger

$$(\Delta q)^2(\Delta p)^2 = \frac{\hbar^2}{4} + \text{Cov}^2(q,p). \qquad (31)$$

According to the terminology of references [13,14] the states $|\alpha_l;t\rangle$ may be called generalized **Coherent States** (CS) of nonstationary system with Hamiltonian $\hat{H}_l$, equation (23). For the purpose of this paper and to make it more readable for physicist-experimentalists, biologist etc. we will call the states minimizing relation (31) **Schrödinger Minimum Uncertainty States** (SMUS) as it is done in [19]. Because the Hamiltonian $\hat{H}$, equation (23), is a sum over l, the SMUS for EM field with finite number of modes are product over l of one mode SMUS $|\alpha_l;t\rangle$.

The vector potential operator takes the form

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sum_l \boldsymbol{u}_l(\boldsymbol{r})\hat{q}_l. \qquad (32)$$

Note that it differs from linear invariants $\hat{A}(t)$, and it is denoted here in bold face. Replacing it in (17) one obtains the quantized electric and magnetic fields

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = -\frac{1}{\epsilon_0} \mathrm{e}^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} \mathrm{d}t'} \sum_l \boldsymbol{u}_l(\boldsymbol{r}) \hat{p}_l \tag{33}$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r}, t) = \mathrm{e}^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')} \mathrm{d}t'} \sum_l \nabla \times \boldsymbol{u}_l(\boldsymbol{r}) \hat{q}_l. \tag{34}$$

Using the time derivatives of operators $\hat{q}_l$, $\hat{p}_l$ of the form

$$\frac{\mathrm{d}\hat{o}_l}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{o}_l, \hat{H}] \tag{35}$$

we check that all Maxwell equations (14) are satisfied by operator fields $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{D}} = \varepsilon\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{H}}$ and $\hat{\boldsymbol{B}} = \mu\hat{\boldsymbol{H}}$.

Evolution of second order statistical moments in SMUS. All three quantum-mechanical statistical moments for canonical operators $\hat{q}_l$ and $\hat{p}_l$ are defined in the evolved SMUS $|\alpha_l; t\rangle$. Using the general formulae [20] we find the variances

$$(\Delta q_l)^2_{\alpha_l} = \frac{\hbar\varepsilon_0^{-1}\mathrm{e}^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}}{2}\rho_l^2, \qquad \rho_l = |\epsilon_l(t)|$$

$$(\Delta p_l)^2_{\alpha_l} = \frac{\hbar}{\varepsilon_0^{-1}\mathrm{e}^{-\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}}\left[\frac{1}{2\rho_l^2} + \left(\dot{\rho}_l(t) - \frac{1}{2}\frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)}\rho_l\right)^2\right]. \tag{36}$$

From the general formula derived in [1], we obtain the covariance $\mathrm{Cov}(q, p)$ in terms of the negative differential conductivity

$$\mathrm{Cov}(q_l, p_l)_{\alpha_l} = -\frac{\hbar}{2}\rho_l\left(\dot{\rho}_l - \frac{1}{2}\frac{\sigma(t) + \dot{\varepsilon}(t)}{\varepsilon(t)}\rho_l\right). \tag{37}$$

Thus, we find the three statistical moments $((\Delta q_l)^2, (\Delta p_l)^2$ and $\mathrm{Cov}(q, p))$ in the case of media with negative differential conductivity.

5. Vector Operators for EM Field and Their Statistical Properties

To express the statistical properties of vector operators $\hat{\boldsymbol{E}}(\boldsymbol{r}, t)$ and $\hat{\boldsymbol{B}}(\boldsymbol{r}, t)$ for EM field, it is convenient to present the Hermitian operators $\hat{q}$, $\hat{p}$ in terms of the invariants $\hat{A}$, $\hat{A}\dagger$. Taking into account (26) we get the following expressions

$$\hat{q}_l = \left(\frac{\hbar}{2\varepsilon_0}\right)^{1/2} \mathrm{e}^{-\frac{1}{2}\int_0^t \frac{\sigma(t') + \dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'} \left(-\mathrm{i}\epsilon_l(t)\hat{A}_l^\dagger(t) + \mathrm{i}\epsilon_l^*(t)\hat{A}_l(t)\right) \tag{38}$$

$$\hat{p}_l = \left(\frac{\hbar\varepsilon_0}{2}\right)^{1/2} \mathrm{e}^{\frac{1}{2}\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}$$

$$\times \left(-\mathrm{i}(\dot{\epsilon}_l - \frac{1}{2}\frac{\sigma(t)+\dot{\varepsilon}(t)}{\varepsilon(t)}\epsilon_l)\hat{A}_l^\dagger + \mathrm{i}(\dot{\epsilon}_l^* - \frac{1}{2}\frac{\sigma(t)+\dot{\varepsilon}(t)}{\varepsilon(t)}\epsilon_l^*)\hat{A}_l\right). \quad (39)$$

We shall consider the case of periodic boundary conditions with *complex mode functions* $\boldsymbol{u}_{l,\xi}^{(\pm)}(\boldsymbol{r}) = V^{-1/2}\boldsymbol{e}_{l,\xi}\exp(\pm\mathrm{i}\boldsymbol{k}_l\cdot\boldsymbol{r})$ [24], where $\boldsymbol{e}_{l,\xi}$ is the polarization vector of mode l, with wavevector $\boldsymbol{k}_l$. With these modes the vector potential operator, which obeys the equation (18) takes the following form

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sqrt{\frac{\hbar}{2\varepsilon_0}}\mathrm{e}^{-\frac{1}{2}\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}\sum_{l,\xi}\boldsymbol{e}_{l,\xi}\left[u_{l,\xi}^*(\boldsymbol{r})\epsilon_l\hat{A}_{l,\xi}(t) + h.c.\right]. \quad (40)$$

Replacing the vector potential operator $\hat{\boldsymbol{A}}(\boldsymbol{r},t)$ in the relations (17) we receive vector operator for EM field in the form

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = \sqrt{\frac{\hbar}{2\varepsilon_0}}\mathrm{e}^{-\frac{1}{2}\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}$$

$$\times \sum_{l,\xi}\boldsymbol{e}_{l,\xi}\left[\left(\frac{1}{2}\frac{\sigma(t)+\dot{\varepsilon}(t)}{\varepsilon(t)}\epsilon_l - \dot{\epsilon}_l\right)u_{l,\xi}^*(\boldsymbol{r})\hat{A}_{l,\xi}(t) + h.c.\right] \quad (41)$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \mathrm{i}\sqrt{\frac{\hbar}{2\varepsilon_0}}\mathrm{e}^{-\frac{1}{2}\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}\sum_{l,\xi}\boldsymbol{k}_l\times\boldsymbol{e}_{l,\xi}\left[u_{l,\xi}^*(\boldsymbol{r})\epsilon_l\hat{A}_{l,\xi}(t) - h.c.\right]. \quad (42)$$

The commutators between the j and m components of $\hat{\boldsymbol{E}}_l(\boldsymbol{r},t)$ and $\hat{\boldsymbol{B}}_l(\boldsymbol{r},t)$ are C-numbers, vanishing for $j = m$

$$[\hat{E}_{l,j}(\boldsymbol{r},t), \hat{B}_{l,m}(\boldsymbol{r},t)] = \mathrm{i}\frac{\hbar}{\varepsilon_0 V}\mathrm{e}^{-\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'} \quad (43)$$

$$\times \sum_{\xi}\boldsymbol{e}_{l,\xi}(\boldsymbol{k}_l\times\boldsymbol{e}_{l,\xi})_j\,\mathrm{Re}\left(\dot{\epsilon}_l\epsilon_l^* - |\epsilon_l|^2\frac{1}{2}\frac{\sigma(t)+\dot{\varepsilon}(t)}{\varepsilon(t)}\right)\delta_{jm}.$$

The three second moments are found as

$$(\Delta E_l)_\alpha^2 = \frac{\hbar}{2\varepsilon_0 V}\mathrm{e}^{-\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}\left|\frac{1}{2}\frac{\sigma(t)+\dot{\varepsilon}(t)}{\varepsilon(t)}|\epsilon_l|^2 - \dot{\epsilon}_l^*\right|^2$$

$$(\Delta B_l)_\alpha^2 = k_l^2\frac{\hbar}{2\varepsilon_0 V}\mathrm{e}^{-\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}|\epsilon_l|^2 \quad (44)$$

$$\mathrm{Cov}(E_l, B_l)_\alpha = -k_l\frac{\hbar}{2\varepsilon_0 V}\mathrm{e}^{-\int_0^t \frac{\sigma(t')+\dot{\varepsilon}(t')}{\varepsilon(t')}\mathrm{d}t'}\mathrm{Im}\left(\frac{\sigma(t)+\dot{\varepsilon}(t)}{2\varepsilon(t)}|\epsilon_l|^2 - \dot{\epsilon}_l\epsilon_l^*\right)$$

(note also the presence of conductivity $\sigma(t)$ (with $\sigma_{\text{diff}} < 0$, which may vary in time also) in the expressions of all the above averages).

Schrödinger Uncertainty Relation for the j and m components of $\hat{\boldsymbol{E}}_l(\boldsymbol{r}, t)$ and $\hat{\boldsymbol{B}}_l(\boldsymbol{r}, t)$ in SMUS take the form

$$(\Delta E_l)_\alpha^2 \, (\Delta B_l)_\alpha^2 - \text{Cov}^2(E_l, B_l)_\alpha = \frac{\hbar^2}{4} \, |\langle [E_l, B_l] \rangle_\alpha|^2 \,. \tag{45}$$

Thus the time-evolved SMUS $|\alpha_l; t\rangle$ in nonstationary and/or conductive media are minimizing uncertainty states with respect to the photon ladder operator quadratures $\hat{q}_l$, $\hat{p}_l$, and with respect to the electric and magnetic field components as well. The time evolution of these states can exhibit q_l-p_l and E_l-B_l covariance and squeezing.

Conclusion

Quantization of the electromagnetic field in non-stationary media (linear with respect to $\boldsymbol{E}$ and arbitrary with respect to time t) is investigated. The model presented here allow to be investigated and to perform secondary quantization in media with negative differential conductivity also. The dynamical invariants and statistical properties of the field are found in such media. It is shown that in the eigenstates of linear dynamical invariant, the Schrodinger uncertainty relation is minimized. The time evolution of the tree independent second-order statistical moments (quantum fluctuations: covariance cov(q,p), var(q) and var(p)) are found out. The model developed here, could be involved in quantum-mechanical explanation of electrons transport, when the electron jumps from one dye molecule to an other, overcoming the potential barrier between them. Thus, the tunnel effect, leading to negative differential conductivity, play essential role not only in chlorophyll, but in electron transport in ruthenium-polypyridine dyes at artificial photosynthesis, also.

References

[1] Angelow A., *Light Propagation in Nonlinear Waveguide and Classical Two-dimensional Oscillator*, Physica A **256** (1998) 485–498.

[2] Angelow A. and Batoni M., *Translation with Annotation of the Original Paper of E. Schrödinger (1930) in English*, Bulg. J. Phys. **26** (1999) 193–203
http://arxiv.org/abs/quant-ph/9903100.

[3] Choi J., *Coherent and Squeezed States of Light in Linear Media with Time-dependent Parameters by Lewis-Riesenfeld Invariant Operator Method*, J. Phys. A **39** (2006) 669–684.

[4] Esaki L., *New Phenomenon in Narrow Germanium p-n Junction*, Phys. Rev. **109** (1958) 603-604.

[5] Simin G., *Tunnel Diodes*, Elctronics & Comp. Tech. Course ELCT, **563** (2009) 1-30, http://www.ee.sc.edu/personal/faculty/simin/ELCT563/08 Tunnel Diodes.pdf.

[6] Gardiner C., *Handbook on Stochastic Methods*, Springer, Heidelberg 1983.

[7] Lubner C., Applegate A., Knorzer P., Ganago A., Donald A., Bryant D., Happe T. and Golbeck J., *Solar Hydrogen-producing Bionanodevice Outperforms Natural Photosynthesis*, PNAS (2011) **108** 20988–20991.

[8] Gosson M., *The Principles of Newtonian and Quantum Mechanics: The Need for Planck's Constant h*, Imperial College Press, London 2001.

[9] Gosson M., *Symplectic Geometry and Quantum Mechanics*, Operator Theory: Advances and Applications **166**, I. Gohberg (Ed), Birkhäuser, Basel 2006.

[10] Korn G. and Korn T., *Mathematical Handbook*, McGraw-Hill, New York 1961.

[11] Lousell W., *Quantum Statistical Properties of Radiation*, Wiley, New York 1973.

[12] Malkin I. and Man'ko V., *Dynamicheskie Sistemyi i Kogerentnyie Sostoianija Kvantovyih Sistem* (in Russian), Nauka, Moscow 1979.

[13] Malkin I., Man'ko V. and Trifonov D., *Coherent States and Transition Probabilities in a Time-Dependent Electromagnetic Field*, Phys. Rev. D **2** (1970) 1371–1385.

[14] Malkin I., Man'ko V. and Trifonov D., *Linear Adiabatic Invariants and Coherent States*, J. Math. Phys. **14** (1973) 576–582.

[15] Rubin A., *Primary Processes of Photosynthesis* (in Russian), Sorosovskij Obrazovatelnyj Journal **10** (1997) 79–84.

[16] Schrödinger E., *Zum Heisenbergschen Unschärfeprinzip*, Ber. Kgl. Akad. Wiss. Berlin **19** (1930) 296–303.

[17] Scully M. and Zubairy M., *Quantum Optics*, Cambridge University Press, Cambridge 1997.

[18] Trifonov D., Nikolov B. and Mladenov I., *On the Uncertainty Relations in Stochastic Mechanics*, J. Geom. Symmetry Phys. **16** (2009) 57-75.

[19] Trifonov D., *Completeness and Geometry of Schrödinger Minimum Uncertainty States*, J. Math. Phys. **34** (1993) 100–110.

[20] Trifonov D., *Coherent States and Uncertainty Relations*, Phys. Lett. A **48** (1974), 165; Trifonov D., *Coherent States and Evolution of Uncertainty Products*, ICTP Internal Report **IC/75/2** (1975) 1-6; Bulg. J. Phys. **2** (1975) 303–311 (Here expressions for the variances of $\hat{q}$ and $\hat{p}$ should be interchanged).

[21] Trifonov D. and Angelow A., *Dynamical Invariants and Robertson-Schrödinger Correlated States of Electromagnetic Field in Nonstationary Linear Media*, AIP Conf. Proc. **1340** (2011) 221–233.

[22] Volkeinstein M., *Molekulqrnaq Biofizika* (in Russian), Nauka, Moscow 1975.

[23] Volkeinstein M., *Biofizika* (in Russian), Nauka, Moscow 1981.

[24] Walls D. and Milburn G., *Quantum Optics*, 2nd Edn, Springer, Heidelberg 2008.

SOLVABLE AND/OR INTEGRABLE MANY-BODY MODELS ON A CIRCLE[*]

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Abstract. Various many-body models are treated, which describe N points confined to move on a plane circle. Their Newtonian equations of motion (accelerations equal forces) are *integrable*, i.e., they allow the *explicit* exhibition of N *constants of motion* in terms of the dependent variables and their time-derivatives. Some of these models are moreover *solvable* by purely algebraic operations, by (explicitly performable) quadratures and, finally, by functional inversions. The techniques to manufacture these models are not new, but some of these models are themselves new and others are reinterpretations of known models.

1. Introduction

The investigation of the time evolution of an arbitrary number N of point-particles the dynamics of which is determined by Newtonian equations of motion (accelerations equal forces) is of course a fundamental topic in physics and mathematics. The identification in this context of models *amenable to exact treatments* is a major area of research in mathematical physics and applied mathematics, having a centuries-old history and having been boosted by developments in the last few decades, which also impacted several areas of physics beyond mechanics and many fields of pure mathematics. An interesting related development which is now becoming of interest is the study of such models in which the motion is restricted to lie on an *a priori* prescribed manifold: see for instance [1, 5, 6, 8]. In this paper we make some initial, simple steps in this direction by focussing on

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various many-body models describing the evolution of N points whose positions on a plane are characterized by N *unit* two-vectors, thereby forcing their motion to be confined to *a circle of unit radius centered at the origin*. All these models are characterized by *Newtonian* equations of motion: accelerations equal forces, which in these models are of *one-body*, *two-body* or, in some cases, *many-body* type, and might depend on the velocities of the moving particles in addition to their positions. All these models are *autonomous*: their equations of motion are time-independent. They are *all amenable to exact treatments*: in particular they *all* allow the explicit identification of N *constants of motion* in terms of the N dependent variables and their N time-derivatives (for terminological simplicity we hereafter call such models *integrable*). In some cases their *initial-value problems* can be moreover *solved* by (explicitly performable) quadratures and subsequent functional inversions, preceded by purely algebraic operations, such as solving systems of *linear* constant-coefficients ODEs, or (equivalently) evaluating the N *eigenvalues* of known (time-dependent) $N \times N$ matrices or (equivalently) the N *zeros* of known (time-dependent) polynomials of degree N (for terminological simplicity we hereafter call such models *solvable*). The techniques to manufacture these models are *not new*. Some of these models are themselves *new*, others are essentially *reinterpretations of known models*. The dynamics of these models are not analyzed in detail; but in some cases the main features of their behavior are ascertained, for instance for *isochronous* models the time evolution of which is *isochronous* (i.e., *completely periodic* with a *fixed* period independent of the initial data), or for models *all* motions of which are *multiply periodic*.

The equations of motion of the N-body problems treated below are listed with minimal comments in the following Section 2, to facilitate the hasty reader wishing to get an immediate idea of the findings reported in this paper. These results are then proven in the subsequent Section 3. The titles of its subsections indicate case-by-case the techniques employed to arrive at the relevant results. Finally, a terse Section 4 entitled "Outlook" outlines possible developments, to be eventually reported in other papers. Some mathematical details are confined to two Appendices.

2. Many-Body Models on a Circle Amenable to Exact Treatments

In the following subsections we display, with minimal comments, various N-body problems of Newtonian type (accelerations equal forces) describing motions on a circle and amenable to exact treatments (detailed in the following Section 3). But we provide firstly a terse subsection devoted to notation.

2.1. Notations

The models under consideration generally feature N points moving in a plane. We identify these N points by three-vectors $\vec{r}_n$, $n = 1, 2, ..., N$ for which we use the following three-dimensional notation:

$$\vec{r}_n \equiv (\cos\theta_n,\ \sin\theta_n,\ 0) \equiv (x_n,\ y_n,\ 0)\,. \tag{1}$$

Hereafter N is an *arbitrary positive integer* (generally $N \geq 2$) and indices such as n, m, ℓ run over the *positive integers* from 1 to N (unless otherwise explicitly indicated).

Clearly these vectors $\vec{r}_n$ have *unit* length

$$\vec{r}_n \cdot \vec{r}_n = 1. \tag{2a}$$

Throughout this paper the dot sandwiched among two vectors denotes the standard *scalar* product, so that for instance

$$\vec{r}_n \cdot \vec{r}_m = \cos(\theta_n - \theta_m)\,. \tag{2b}$$

It is moreover convenient to introduce the *unit* vector $\hat{z}$ orthogonal to the xy-plane,

$$\hat{z} \equiv (0,\ 0,\ 1) \tag{3}$$

and to denote by the "wedge" symbol $\wedge$ the standard (three-dimensional) *vector* product, so that

$$\hat{z} \wedge \vec{r}_n = -\vec{r}_n \wedge \hat{z} = (-\sin\theta_n,\ \cos\theta_n,\ 0) \tag{4a}$$

$$(\hat{z} \wedge \vec{r}_m) \cdot \vec{r}_n = (\vec{r}_m \wedge \vec{r}_n) \cdot \hat{z} = \sin(\theta_n - \theta_m)\,. \tag{4b}$$

Hereafter we deal with time-dependent vectors

$$\vec{r}_n(t) \equiv (\cos\theta_n(t),\ \sin\theta_n(t),\ 0) \tag{5}$$

and superimposed dots indicate derivatives with respect to the time variable t so that, for instance

$$\dot{\vec{r}}_n = \dot{\theta}_n\ (-\sin\theta_n,\ \cos\theta_n,\ 0) = \dot{\theta}_n\ \hat{z} \wedge \vec{r}_n \tag{6a}$$

$$\ddot{\vec{r}}_n = \ddot{\theta}_n(-\sin\theta_n,\ \cos\theta_n,\ 0) - \dot{\theta}_n^2(\cos\theta_n,\ \sin\theta_n,\ 0) = \ddot{\theta}_n\ \hat{z} \wedge \vec{r}_n - \dot{\theta}_n^2\ \vec{r}_n. \tag{6b}$$

Note that here we omitted, for notational simplicity, to indicate *explicitly* the time-dependence of the quantities appearing in these N equations; we will often do this below without repeating this warning.

Several other identities are reported in Appendix A: they are useful to obtain the results reported below, but are not necessary to understand the findings reported in the following subsections.

2.2. Two Models Obtained via Techniques of Generalized Lagrangian Interpolation

First model

$$\mu_n \, \ddot{\vec{r}}_n \; = \; -\mu_n \, \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \, \vec{r}_n$$

$$+ \hat{z} \wedge \vec{r}_n \, \left\{ \left[\mu_n \! \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) + \eta_n \! \left(\vec{r}_n \wedge \dot{\vec{r}}_n \right) \cdot \hat{z} \right] \, \sum_{\ell=1, \, \ell \neq n}^{N} \, \left[\frac{\left(\vec{r}_\ell \cdot \vec{r}_n \right)}{\left(\vec{r}_\ell \wedge \vec{r}_n \right) \cdot \hat{z}} \right] \right.$$

$$\left. + \left[\left(\vec{r}_n \wedge \dot{\vec{r}}_n \right) \cdot \hat{z} \right] \, \sum_{\ell=1, \, \ell \neq n}^{N} \, \left[\frac{\sigma_n \left(\underline{\vec{r}} \right)}{\sigma_\ell \left(\underline{\vec{r}} \right)} \, \frac{\mu_\ell \left(\vec{r}_\ell \wedge \dot{\vec{r}}_\ell \right) \cdot \hat{z} + \eta_\ell}{\left(\vec{r}_\ell \wedge \vec{r}_n \right) \cdot \hat{z}} \right] \right\} \qquad (7a)$$

$$\sigma_n \left(\underline{\vec{r}} \right) = \prod_{\ell=1, \, \ell \neq n}^{N} \, \left[\left(\vec{r}_\ell \wedge \vec{r}_n \right) \cdot \hat{z} \right] . \qquad (7b)$$

Second model

$$\mu_n \, \ddot{\vec{r}}_n \; = \; -\mu_n \, \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \, \vec{r}_n$$

$$+ \sum_{\ell=1, \, \ell \neq n}^{N} \, \left\{ [\left(\vec{r}_\ell \wedge \vec{r}_n \right) \cdot \hat{z}]^{-1} \left\{ \left[\left(\vec{r}_n \wedge \dot{\vec{r}}_n \right) \cdot \hat{z} \right] \left[\mu_\ell \left(\vec{r}_\ell \wedge \dot{\vec{r}}_\ell \right) \cdot \hat{z} + \eta_\ell \right] \right. \right.$$

$$\left. \left. + \left[\mu_n \left(\vec{r}_n \wedge \dot{\vec{r}}_n \right) \cdot \hat{z} + \eta_n \right] \left[\left(\vec{r}_\ell \wedge \dot{\vec{r}}_\ell \right) \cdot \hat{z} \right] \right\} \, \left(\vec{r}_\ell \wedge \vec{r}_n \right) \right\} . \qquad (8)$$

In these Newtonian equations μ_n and η_n are $2N$ arbitrary constants, and for the rest of the notation see Subsection 2.1. Note in particular the property (2a), implying that the N vectors $\vec{r}_n$ have *unit* modulus, hence that the N points whose time evolution is determined by these equations of motion are constrained to move on the circle of *unit* radius centered at the origin of the Cartesian plane.

These equations of motion are *covariant*, implying that the corresponding N-body problems are *rotation-invariant*.

These two N-body problems are both *integrable*: they possess N *constants of motion*, the explicit expressions of which in terms of the vectors $\vec{r}_n$ and their time-derivatives $\dot{\vec{r}}_n$ are displayed in the following Subsection 3.1. The equations of motion of the first, (7a), of these two models feature *many-body* forces due to the presence in their right-hand ("forces") sides of the quantities $\sigma_n \left(\underline{\vec{r}} \right)$, see (7b), but *their initial-value problem is solvable by purely algebraic operations*. Nevertheless their time evolution can be quite complicated (detailed analyses are not performed in this paper. The fact that *solvable* models can exhibit quite complicated dynamics is of course well known, see for instance the papers where a three-body model is studied the time evolution of which is highly nontrivial in spite of the fact that

its Aristotelian equations of motion–*velocity equal forces*–are quite neat and that its initial-value problem can be reduced to solving a single algebraic equation [3], [4],[7]).

2.3. Two *Solvable* Models Obtained via a Reinterpretation of Known Models

The *first model* is merely a transcription of the *solvableSutherland model*, see Subsection 3.2. It reads as follows

$$\ddot{\vec{r}}_n = - \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \vec{r}_n + g^2 \, \hat{z} \wedge \vec{r}_n \sum_{\ell=1, \, \ell \neq n}^{N} \left\{ \frac{\vec{r}_n \cdot \vec{r}_\ell}{[(\vec{r}_\ell \wedge \vec{r}_n) \cdot \hat{z}]^3} \right\} . \tag{9}$$

Here g is an *arbitrary* "coupling constant", and the rest of the notation is, we trust, clear (see Subsection 2.1).

The *second model* is also merely a transcription of a well-known *solvable* model (of goldfish type), see Subsection 3.2. It reads as follows

$$\ddot{\vec{r}}_n = - \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \vec{r}_n + g_0 \, \hat{z} \wedge \vec{r}_n + g_1 \, \dot{\vec{r}}_n$$

$$+ \hat{z} \wedge \vec{r}_n \sum_{\ell=1, \, \ell \neq n}^{N} \left\{ \frac{2 \dot{\vec{r}}_n \cdot \dot{\vec{r}}_\ell + g_2 \left[\left(\dot{\vec{r}}_n \wedge \vec{r}_\ell + \dot{\vec{r}}_\ell \wedge \vec{r}_n \right) \cdot \hat{z} \right] + g_3 \, \vec{r}_n \cdot \vec{r}_\ell}{(\vec{r}_\ell \wedge \vec{r}_n) \cdot \hat{z}} \right\} . \tag{10}$$

Here the four constants g_0, g_1, g_2 and g_3 are *arbitrary* constants, and the rest of the notation is, we trust, clear (see Subsection 2.1).

These equations of motion are *covariant*, implying that the corresponding N-body problems are *rotation-invariant*.

2.4. Two N-body Problems on a Circle Obtained by Changes of Dependent Variables

These two *solvable* models are merely transcriptions of two well-known one-dimensional *solvable* models, see Subsection 3.3. The *first model* reads as follows

$$\ddot{\vec{r}}_n = - \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \vec{r}_n - \hat{z} \wedge \vec{r}_n \left\{ 2 \left[\left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \frac{y_n}{x_n} \right] \right.$$

$$\left. + 4 \, x_n \, y_n - x_n^5 \sum_{\ell=1, \, \ell \neq n}^{N} \left[\frac{y_\ell}{(\vec{r}_\ell \wedge \vec{r}_n) \cdot \hat{z}} \right]^3 \right\} . \tag{11a}$$

Here $x_n \equiv \cos \theta_n$ and $y_n \equiv \sin \theta_n$ are the two Cartesian components in the plane of the vector $\vec{r}_n$, see (1).

This model is *isochronous* with period π

$$\vec{r}_n \left(t \pm \pi \right) = \vec{r}_n \left(t \right). \tag{11b}$$

The *second model* reads as follows

$$\ddot{\vec{r}}_n = - \left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \vec{r}_n - \hat{z} \wedge \vec{r}_n \left\{ 2 \left[\left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n \right) \frac{y_n}{x_n} \right] \right.$$
$$\left. + x_n \, y_n - x_n \sum_{\ell=1, \, \ell \neq n}^{N} \left\{ \frac{2 + x_n^2 \, x_\ell^2}{x_\ell \, \left[(\vec{r}_\ell \wedge \vec{r}_n) \cdot \hat{z} \right]} \right\} \right\}. \tag{12}$$

Here $x_n \equiv \cos \theta_n$ and $y_n \equiv \sin \theta_n$ are again the two Cartesian components in the plane of the vector $\vec{r}_n$, see (1).

All solutions of this model are *multiply periodic*, see Subsection 3.3.

Note that–in contrast to the equations of motions reported in the two preceding subsections–those displayed herein, (11a) and (12), are *not* written in *covariant* fashion, i.e., without any explicit appearance of the Cartesian components $x_n \equiv \cos \theta_n$ and $y_n \equiv \sin \theta_n$ of the vector $\vec{r}_n$. Indeed these equations of motion are *not* rotation-invariant, or equivalently, they are *not* invariant for translations along the circle (on which the motions take place due to the constraint (2a)).

3. Proofs

In the following subsections we substantiate the findings reported in the preceding Section 2.

3.1. Solvable and Integrable Models on the Circle Manufactured via Techniques of Generalized Lagrangian Interpolation

In this subsection we employ the technique to manufacture many-body models amenable to exact treatments introduced in [2] (see in particular Chapter 3 of this book, entitled "N-body problems treatable via techniques of exact Lagrangian interpolation in spaces of one or more dimensions"). We begin with a terse review of this method, in the specific case of one-dimensional space with an appropriate choice of the set of "seeds" (namely, of the N functions providing the point of departure for the generalized Lagrangian interpolation approach).

The set of seeds we conveniently take as basis for our treatment are the N functions

$$\{ s_n \left(\theta \right) \}_{n=1}^{N} = \{ \exp \left[\mathrm{i} \, \left(2 \, n - N - 1 \right) \, \theta \right] \}_{n=1}^{N}$$
$$= \{ \exp \left[\mathrm{i} \, \left(1 - N \right) \, \theta \right], \, \exp \left[\mathrm{i} \, \left(3 - N \right) \, \theta \right], \ldots$$
$$\ldots \exp \left[\mathrm{i} \, \left(N - 3 \right) \, \theta \right], \, \exp \left[\mathrm{i} \, \left(N - 1 \right) \, \theta \right] \}. \tag{13}$$

Remark 1. *These exponential functions with imaginary argument are complex, but clearly this set of seeds could be replaced without significant changes by an equivalent set featuring instead sines and cosines of real arguments. The use of exponentials merely facilitates some of the following developments. Likewise the factor two in the argument of these functions has been introduced merely to yield neater versions of the equations of motions that will be obtained, see below. The fact that these seeds are invariant under the transformation $\theta \Rightarrow \theta + 2\pi$ suggests to interpret the variable θ as an angle in the plane.*

We then consider a function $f(\theta)$ representable as a *linear* superposition of these N seeds

$$f(\theta) = \sum_{n=1}^{N} [h_n \, s_n(\theta)] \tag{14a}$$

where the N coefficients h_n are *a priori* arbitrary numbers. And we denote with f_n the N values that this function takes at the N (*arbitrarily assigned*) "nodes" $\theta = \theta_n$

$$f_n = f(\theta_n) \tag{14b}$$

and we display the representation of this function in terms of these N values, via the ("generalized Lagrangian interpolation") formula

$$f(\theta) = \sum_{n=1}^{N} \left[f_n \, q^{(n)}(\theta \, | \underline{\theta}) \right]. \tag{14c}$$

The N "interpolational functions" $q^{(n)}(\theta \, | \underline{\theta})$ depend on the variable θ and on the N nodes θ_n (hence on the N-vector having these nodes as its components, hereafter denoted as $\underline{\theta} \equiv (\theta_1, \theta_2, ..., \theta_N)$). They are themselves *linear* superpositions of the seeds $s_n(\theta)$, to insure consistency among (14c) and (14a); and they feature the property

$$q^{(n)}(\theta_m \, | \underline{\theta}) = \delta_{nm} \tag{15}$$

to insure consistency among (14c) and (14b) (here and hereafter δ_{nm} is the Kronecker symbol: $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$).

The explicit representation of these interpolational functions $q^{(n)}(\theta \, | \underline{\theta})$ in terms of the N seeds $s_n(\theta)$ and the N nodes θ_n reads [2]

$$q^{(n)}(\theta \, | \underline{\theta}) = \frac{\Delta(\theta_1, \ldots, \theta_{n-1}, \theta, \theta_{n+1}, \ldots, \theta_N)}{\Delta(\theta_1, \ldots, \theta_N)} \tag{16a}$$

where

$$\Delta(\underline{\theta}) = \begin{vmatrix} s_1(\theta_1) & s_2(\theta_1) & \ldots & s_N(\theta_1) \\ s_1(\theta_2) & s_2(\theta_2) & \ldots & s_N(\theta_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(\theta_N) & s_2(\theta_N) & \ldots & s_N(\theta_N) \end{vmatrix}. \tag{16b}$$

This determinant–with the set of seeds (13)–is of Vandermonde type hence it can be explicitly evaluated, yielding for the interpolational functions the expression

$$q^{(n)}\left(\theta \mid \underline{\theta}\right) = s_1\left(\theta - \theta_n\right) \prod_{\ell=1,\ \ell\neq n}^{N} \left[\frac{\exp\left(2\,\mathrm{i}\,\theta\right) - \exp\left(2\,\mathrm{i}\,\theta_\ell\right)}{\exp\left(2\,\mathrm{i}\,\theta_n\right) - \exp\left(2\,\mathrm{i}\,\theta_\ell\right)}\right]. \qquad (17)$$

The next step is to introduce the time variable t. As in [2], we assume hereafter that the N seeds $s_n\left(\theta\right)$ are time-independent. We moreover assume the function $f\left(\theta\right)$ to be also time-independent (thereby simplifying the more general treatment of [2]). A time-dependence is only introduced for the nodes $\theta_n \equiv \theta_n\left(t\right)$. Indeed they shall be the dependent variables of the dynamical systems we manufacture. Of course the fact that the nodes $\theta_n\left(t\right)$ evolve over time entails that the values f_n taken by the function $f\left(\theta\right)$ at these nodes (see (14b)) also evolve over time

$$f_n \equiv f_n\left(t\right) = f\left[\theta_n\left(t\right)\right]. \qquad (18)$$

We then posit a convenient relation among the time evolution of the N nodes $\theta_n\left(t\right)$ and the time evolution of the N quantities $f_n\left(t\right)$, by setting

$$f_n\left(t\right) = \rho_n\left[\underline{\theta}\left(t\right)\right]\dot{\theta}_n\left(t\right) + \gamma_n\left[\underline{\theta}\left(t\right)\right]. \qquad (19)$$

Here we introduced the $2N$ functions $\rho_n\left(\underline{\theta}\right)$ and $\gamma_n\left(\underline{\theta}\right)$ of the N nodes θ_n, that will be assigned later at our convenience (but note that we forsake–again, for simplicity–the possibility to assign an *explicit* time-dependence to these functions, in addition to their dependence on the N nodes).

The next step is to ascertain the time dependence of the N nodes $\theta_n \equiv \theta_n\left(t\right)$ implied by these assignments. To this end we time-differentiate the relation (19), getting the following expressions for the second time-derivatives of the N nodes $\theta_n \equiv \theta_n\left(t\right)$

$$\rho_n\left(\underline{\theta}\right)\ddot{\theta}_n = \dot{f}_n - \sum_{m=1}^{N}\left\{\left[\frac{\partial\,\gamma_n\left(\underline{\theta}\right)}{\partial\,\theta_m} + \frac{\partial\,\rho_n\left(\underline{\theta}\right)}{\partial\,\theta_m}\dot{\theta}_n\right]\dot{\theta}_m\right\}. \qquad (20)$$

Our next step is to evaluate the quantity $\dot{f}_n$, which (see (18)) reads

$$\dot{f}_n = \frac{\partial\,f\left(\theta_n\right)}{\partial\,\theta_n}\dot{\theta}_n. \qquad (21)$$

To evaluate this quantity we can use the finite-dimensional representation of the differential operator, yielding (for functions which are *linear* superpositions of the seeds $s_n\left(\theta\right)$, see (14)), the *exact* formula [2]

$$\frac{\partial\,f\left(\theta_n\right)}{\partial\,\theta_n} = \sum_{m=1}^{N}\left[D_{nm}\left(\underline{\theta}\right)f_m\right] \qquad (22a)$$

with the $N \times N$ matrix D defined componentwise as follows [2]

$$D_{nm}(\underline{\theta}) = \frac{\partial \, q^{(m)}(\theta \,|\underline{\theta})}{\partial \, \theta} \quad \text{evaluated at } \theta = \theta_n \tag{22b}$$

hence in our case (see (13) and (16)) reading

$$D_{nm}(\underline{\theta}) = \delta_{nm} \sum_{\ell=1, \, \ell \neq n}^{N} \cot(\theta_n - \theta_\ell) + \frac{\sigma_n(\underline{\theta})}{\sigma_m(\underline{\theta})} \frac{1 - \delta_{nm}}{\sin(\theta_n - \theta_m)} \tag{23a}$$

$$\sigma_n(\underline{\theta}) = \prod_{\ell=1, \, \ell \neq n}^{N} [\sin(\theta_n - \theta_\ell)] . \tag{23b}$$

Note that this definition coincides, via (4b), with (7b).

We therefore conclude that system (20) yields the following set of N Newtonian equations of motion for the dependent variables $\theta_n \equiv \theta_n(t)$

$$\begin{aligned}
\rho_n(\underline{\theta}) \, \ddot{\theta}_n &= \dot{\theta}_n \left[\rho_n(\underline{\theta}) \, \dot{\theta}_n + \gamma_n(\underline{\theta}) \right] \sum_{\ell=1, \, \ell \neq n}^{N} [\cot(\theta_n - \theta_\ell)] \\
&+ \dot{\theta}_n \sum_{\ell=1, \, \ell \neq n}^{N} \left\{ \frac{\sigma_n(\underline{\theta})}{\sigma_\ell(\underline{\theta})} \frac{\left[\rho_\ell(\underline{\theta}) \, \dot{\theta}_\ell + \gamma_\ell(\underline{\theta}) \right]}{\sin(\theta_n - \theta_\ell)} \right\} \\
&- \sum_{m=1}^{N} \left\{ \left[\frac{\partial \, \rho_n(\underline{\theta})}{\partial \, \theta_m} \dot{\theta}_n + \frac{\partial \, \gamma_n(\underline{\theta})}{\partial \, \theta_m} \right] \dot{\theta}_m \right\} .
\end{aligned} \tag{24}$$

Of course to obtain this system of N second-order ODEs we also used (19).

Let us now emphasize that, as a consequence of the way these N-body problems have been manufactured, they are *integrable*. It is indeed plain that the time independence of the function $f(\theta)$ entails (via (14a), (14b) and (19)) the relations

$$\sum_{m=1}^{N} \{ h_m \, s_m [\theta_n(t)] \} = \rho_n [\underline{\theta}(t)] \, \dot{\theta}_n(t) + \gamma_n [\underline{\theta}(t)] . \tag{25a}$$

Here we have displayed the *time-dependence* of the various quantities, in order to emphasize the *time-independence* of the N coefficients h_m, which can actually be evaluated by solving this system of N *linear* equations, thereby obtaining (via (16)) the following formulas

$$h_m = q^{(m)}(\vartheta_m \,|\underline{\theta}), \qquad \vartheta_m \equiv \frac{\mathrm{i} \log \left[\rho_m(\underline{\theta}) \, \dot{\theta}_m + \gamma_m(\underline{\theta}) \right]}{2 \, m - N - 1} \tag{25b}$$

where of course the N nodes $\theta_m \equiv \theta_m(t)$ and their N time derivatives $\dot{\theta}_m \equiv \dot{\theta}_m(t)$ can be evaluated at any arbitrary time t. It is thus plain that the N-body systems (25) are *integrable* for any *arbitrary* assignment of the $2N$ functions $\rho_m(\underline{\theta})$

and $\gamma_m (\underline{\theta})$ of the N dependent variables θ_n, with these N quantities h_m providing N *constants of motion* given by explicit (generally nontrivial) expressions in terms of the N nodes θ_n and their N time-derivatives $\dot{\theta}_n$.

We are still free to assign the $2N$ functions $\rho_n (\underline{\theta})$ and $\gamma_n (\underline{\theta})$. There are two natural choices.

The first one reads simply

$$\rho_n (\underline{\theta}) = \mu_n, \qquad \gamma_n (\underline{\theta}) = \eta_n \tag{26}$$

with μ_n and η_n arbitrary *constant* parameters. It clearly yields (see (25)) an N-body system characterized by the following set of Newtonian equations of motion

$$
\mu_n \ddot{\theta}_n = \dot{\theta}_n \left(\mu_n \dot{\theta}_n + \eta_n \right) \sum_{\ell=1,\ \ell\neq n}^{N} [\cot (\theta_n - \theta_\ell)]
$$
$$
+ \dot{\theta}_n \sum_{\ell=1,\ \ell\neq n}^{N} \left[\frac{\sigma_n (\underline{\theta})}{\sigma_\ell (\underline{\theta})} \frac{\left(\mu_\ell \dot{\theta}_\ell + \eta_\ell \right)}{\sin (\theta_n - \theta_\ell)} \right]. \tag{27}
$$

Here the functions $\sigma_n (\underline{\theta})$ of the N nodes θ_m are of course defined by (23b).

The second assignment of the $2N$ functions $\rho_n (\underline{\theta})$ and $\gamma_n (\underline{\theta})$ is suggested by the structure of system (25). It reads

$$\rho_n (\underline{\theta}) = \mu_n \sigma_n (\underline{\theta}), \qquad \gamma_n (\underline{\theta}) = \eta_n \sigma_n (\underline{\theta}) \tag{28}$$

where again μ_n and η_n are arbitrary *constant* parameters and the functions $\sigma_n (\underline{\theta})$ are defined as above, see (23b), implying (by logarithmic differentiation)

$$
\frac{\partial \gamma_n (\underline{\theta})}{\partial \theta_m} = \gamma_n (\underline{\theta}) \left\{ \delta_{nm} \sum_{\ell=1, \ell\neq n}^{N} [\cot (\theta_n - \theta_\ell)] - (1 - \delta_{nm}) \cot (\theta_n - \theta_m) \right\}
$$
$$\tag{29a}$$

and likewise

$$
\frac{\partial \rho_n (\underline{\theta})}{\partial \theta_m} = \rho_n (\underline{\theta}) \left\{ \delta_{nm} \sum_{\ell=1, \ell\neq n}^{N} [\cot (\theta_n - \theta_\ell)] - (1 - \delta_{nm}) \cot (\theta_n - \theta_m) \right\}.
$$
$$\tag{29b}$$

Thereby the N-body system gets characterized by the following, simpler set of Newtonian equations of motion

$$
\mu_n \ddot{\theta}_n = \sum_{\ell=1, \ell\neq n}^{N} \left[\frac{\dot{\theta}_n \left(\mu_\ell \dot{\theta}_\ell + \eta_\ell \right) + \left(\mu_n \dot{\theta}_n + \eta_n \right) \dot{\theta}_\ell \cos (\theta_n - \theta_\ell)}{\sin (\theta_n - \theta_\ell)} \right]. \tag{30}
$$

The differences among these two N-body systems, (28) and (30), deserve to be emphasized: the N-body model (28) involves *many-body* forces, due to the presence

of the functions $\sigma_n(\underline{\theta})$ and $\sigma_\ell(\underline{\theta})$ in its right-hand (*forces*) side; while the N-body model (30) only involves *two-body* forces. Both systems can be integrated once, corresponding to the transition from their N *second-order* Newtonian equations of motion to the corresponding N *first-order* ODEs (25a). On the other hand, as we show below, only the first of these two *integrable* systems is *solvable*.

Indeed, for the first system (but not for the second!), the N first-order ODEs (25a) are *uncoupled*, reading simply, via (26)

$$\mu_n\,\dot{\theta}_n = -\eta_n + \sum_{m=1}^{N}\left[h_m\,s_m\left(\theta_n\right)\right] \tag{31a}$$

or, equivalently (see (13))

$$\mu_n\,\exp\left[(N+1)\,\mathrm{i}\,\theta_n\right]\,\dot{\theta}_n = -\eta_n\,\exp\left[(N+1)\,\mathrm{i}\,\theta_n\right] + \sum_{m=1}^{N}\left[h_m\,\exp\left(2\,m\,\mathrm{i}\,\theta_n\right)\right] \tag{31b}$$

where the N quantities h_n are explicitly known in terms of the $2N$ initial data $\theta_n(0)$, $\dot{\theta}_n(0)$ (via (25b), (26) and (17), see Appendix B).

These first-order ODEs can be integrated, we confine the relevant developments to Appendix B.

Although the technique to manufacture these two *solvable* and *integrable* N-body problems, (28) and (30), is *not* new [2], these models are, to the best of our knowledge, themselves *new*. And therefore a detailed discussion of the actual behavior of these systems has not yet been done. In the present paper we limit our consideration to pointing out how these models can be reformulated to describe the evolution of N points whose positions on a plane are characterized by N *unit* two-vectors $\vec{r}_n(t)$, see the notation introduced in Subsection 2.1. To this end one utilizes the formulas (6b), (2b), (4b) and the relevant ones among those conveniently collected in Appendix A. And it is plain that one thereby obtains the two models (7) and (8).

3.2. Solvable Models on the Circle Manufactured by Reinterpreting Known Solvable Models

In this section we tersely indicate how to obtain the two models (9) and (11).

The *first model* obtains from the N-body system characterized by the following Newtonian equations of motion (with velocity-independent two-body forces)

$$\ddot{\theta}_n = g^2\,\sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{\cos\left(\theta_n-\theta_\ell\right)}{\sin^3\left(\theta_n-\theta_\ell\right)}\right]. \tag{32}$$

Here g is an arbitrary "coupling constant", and the rest of the notation is, we trust, clear.

This is a well-known *solvable* many-body problem, generally associated with the name of Bill Sutherland, who was the first to show the possibility to treat this N-body problem by exact methods (originally in a quantal context [10], [11]). Its treatment in a classical (Hamiltonian) context is provided in several textbooks, see for instance [9] [2] [12].

It is plain that the model (9) is merely the transcription of this model via the notation of Subsection 2.1.

The *second model* obtains from the N-body system characterized by the following Newtonian equations of motion (with velocity-dependent one-body and two-body forces)

$$\ddot{\theta}_n = g_0 + g_1 \dot{\theta}_n + \sum_{\ell=1,\ell\neq n}^{N} \left\{ \left[2\dot{\theta}_n \dot{\theta}_\ell + g_2 \left(\dot{\theta}_n + \dot{\theta}_\ell \right) + g_3 \right] \cot \left(\theta_n - \theta_\ell \right) \right\}. \quad (33)$$

Here g_0, g_1, g_2 and g_3 are 4 arbitrary coupling constants, and we again trust the rest of the notation to be clear.

This is also a well known *solvable* model, see for instance equation (2.3.5-12) on page 199 of [2].

And it is again plain that the model (11) is merely the transcription of this model via the notation of Subsection 2.1 and Appendix A.

3.3. How to Manufacture N-Body Problems with *Angles* as Dependent Variables

In the preceding subsection we have shown how certain N-body models with dependent variables naturally interpretable as *angles* can be reformulated as N-body models describing the time evolution on a plane of particles *constrained to move on a circle*. In this subsection we indicate how, via a simple change of dependent variables, essentially *any* N-body model can be reformulated so that its dependent variables can be interpreted as *angles*, hence subsequently it can also be reformulated (in fact in many ways) so that it describes the time evolution of particles *constrained to move on a plane circle*.

The trick to achieve this goal is quite elementary and general; we illustrate it below via two examples.

Consider an N-body model in which the positions of the N point-particles–moving in one-dimensional space–are identified by N coordinates $z_n \equiv z_n(t)$, and perform the change of dependent variables by positing, say

$$z_n(t) = \tan[\theta_n(t)]. \quad (34)$$

Remark 2. *Of course this assignment defines $\theta_n(t)$ only $\mathrm{mod}\,(\pi)$ and clearly many other assignments could be instead made–different but having an analogous effect, such as $z_n = 1/\sin(2\theta_n)$, or $z_n = \tan^3\theta_n$, etc.*

In the *first example* we take as point of departure the N-body problem characterized by the Newtonian equations of motion

$$\ddot{z}_n = -4\,z_n + g^2 \sum_{\ell=1,\,\ell\neq n}^{N} \left[(z_n - z_\ell)^{-3}\right]. \tag{35a}$$

Here g is an arbitrary (real) coupling constant. This is a well-known *solvable* model (see for instance [2]); it is *isochronous*, all its solutions being *completely periodic with period* π

$$z_n(t \pm \pi) = z_n(t). \tag{35b}$$

Via the change of dependent variables (34) the equations of motion (35a) become (as the diligent reader will easily verify, utilizing if need be the identities reported in the last part of Appendix A)

$$\ddot{\theta}_n = -2\dot{\theta}_n^2\,\tan\theta_n - 4\sin\theta_n\,\cos\theta_n + g^2 \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{\cos^5\theta_n\,\sin^3\theta_\ell}{\sin^3(\theta_n - \theta_\ell)}\right]. \tag{36a}$$

Remark 3. *This model of course hereditates the property of isochrony of the model (35a) it has been obtained from*

$$\theta_n(t \pm \pi) = \theta_n(t)\,\mathrm{mod}\,(\pi). \tag{36b}$$

The next task is to transform these equations of motion, (36a), into equations of motion for points moving in the plane but constrained to stay on a *circle* of *unit* radius centered at the origin. To realize this goal one may now use the change of dependent variables from the *angles* θ_n to the *vectors* $\vec{r}_n$ described in Subsection 2.1, using if need be the identities reported in the first part of Appendix A. And it is plain that in this manner one arrives at the equations of motion (11a).

In the *second example* we take as point of departure the well-known *solvable* N-body problem characterized by the following Newtonian equations of motion (see equation (2.3.4.2-1) on page 188 of [2])

$$\ddot{z}_n = -z_n + \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell + 1}{z_n - z_\ell}\right). \tag{37}$$

All solutions of this model are *multiply periodic*, being (generally nonlinear) superpositions of the N functions $b_m(t) = \cos(\sqrt{m}\,t + \beta_m)$, $m = 1, ..., N$ (with the N phases β_m depending on the initial data). For special initial data only functions $b_m(t)$ with m a *squared-integer* contribute [2], yielding solutions *completely periodic* with period 2π.

Via the change of dependent variables (34) equations of motion (37) become (as the diligent reader will easily verify, utilizing again, if need be, the identities reported in the last part of Appendix A)

$$\ddot{\theta}_n = -2\,\dot{\theta}_n^2\,\tan\theta_n - \sin\theta_n\,\cos\theta_n$$

$$+\cos\theta_n \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{2\,\dot{\theta}_n\,\dot{\theta}_\ell + \cos^2\theta_n\,\cos^2\theta_\ell}{\cos\theta_n\,\sin\left(\theta_n - \theta_\ell\right)}\right].$$

(38)

Then we transform these equations of motion into equations of motion for points moving in the plane but constrained to stay on a *circle* of *unit* radius centered at the origin, by using again the change of dependent variables from the *angles* θ_n to the *vectors* $\vec{r}_n$ described in Subsection 2.1 via–if need be–the identities reported in the first part of Appendix A. And it is plain that in this manner one arrives at the equations of motion (12).

4. Outlook

Our original motivation to undertake this line of research was the intention to manufacture N-body problems amenable to exact treatments describing motions on a sphere, or more generally on manifolds. We consider the results reported in this paper as a modest first step in that direction. We also believe that the actual behavior of the *new* models reported in this paper–see (7) and (8)–shall eventually deserve a more detailed scrutiny than that provided in Subsection 3.1.

Appendix A. Identities

It is plain that the notation introduced in Subsection 2.1 entails the following additional identities

$$\dot{\vec{r}}_n \cdot \vec{r}_n = 0, \qquad \dot{\vec{r}}_n \cdot \dot{\vec{r}}_n = \dot{\theta}_n^2, \qquad \left(\vec{r}_n \wedge \dot{\vec{r}}_n\right)\cdot\hat{z} = \dot{\theta}_n \tag{39a}$$

$$\ddot{\vec{r}}_n \cdot \vec{r}_n = -\dot{\theta}_n^2, \qquad \ddot{\vec{r}}_n \cdot (\hat{z} \wedge \vec{r}_n) = \ddot{\theta}_n \tag{39b}$$

$$\dot{\vec{r}}_n \cdot \vec{r}_m = -\dot{\theta}_n\,\sin\left(\theta_n - \theta_m\right) \tag{39c}$$

$$\dot{\vec{r}}_n \cdot \dot{\vec{r}}_m = \dot{\theta}_n\,\dot{\theta}_m\,\cos\left(\theta_n - \theta_m\right) \tag{39d}$$

$$\hat{z} \wedge \dot{\vec{r}}_n = -\dot{\theta}_n\,\vec{r}_n, \qquad \hat{z} \wedge \ddot{\vec{r}}_n = -\ddot{\theta}_n\,\vec{r}_n - \dot{\theta}_n^2\,\hat{z} \wedge \vec{r}_n \tag{40}$$

$$\left(\dot{\vec{r}}_n \wedge \vec{r}_m\right)\cdot\hat{z} = -\dot{\theta}_n\,\cos\left(\theta_n - \theta_m\right), \qquad \left(\dot{\vec{r}}_n \wedge \dot{\vec{r}}_m\right)\cdot\hat{z} = -\dot{\theta}_n\,\dot{\theta}_m\,\sin\left(\theta_n - \theta_m\right).$$

(41)

We also display here some relations among the time-dependent "coordinates"

$$z_n \equiv z_n(t) = \tan \theta_n(t) \tag{42a}$$

and the "angles" $\theta_n \equiv \theta_n(t)$

$$z_n - z_m = \frac{\sin(\theta_n - \theta_m)}{\cos\theta_n \, \cos\theta_m}, \qquad \frac{1}{z_n - z_m} = \frac{\cos\theta_n \, \cos\theta_m}{\sin(\theta_n - \theta_m)} \tag{42b}$$

$$\dot{z}_n = \frac{\dot{\theta}_n}{\cos^2\theta_n}, \qquad \dot{z}_n \, z_m = \frac{\dot{\theta}_n \, \sin\theta_m}{\cos^2\theta_n \, \cos\theta_m}, \qquad \dot{z}_n \, \dot{z}_m = \frac{\dot{\theta}_n \, \dot{\theta}_m}{\cos^2\theta_n \, \cos^2\theta_m} \tag{43}$$

$$
\begin{aligned}
\frac{\dot{z}_n + \dot{z}_m}{z_n - z_m} &= \frac{\dot{\theta}_n \, \cos^2\theta_m + \dot{\theta}_m \, \cos^2\theta_n}{\cos\theta_n \, \cos\theta_m \, \sin(\theta_n - \theta_m)} \\
\frac{\dot{z}_n z_m + \dot{z}_m \, z_n}{z_n - z_m} &= \frac{\dot{\theta}_n \, \sin\theta_m \, \cos\theta_m + \dot{\theta}_m \, \sin\theta_n \, \cos\theta_n}{\cos\theta_n \, \cos\theta_m \, \sin(\theta_n - \theta_m)} \\
\frac{\dot{z}_n \, \dot{z}_m}{z_n - z_m} &= \frac{\dot{\theta}_n \, \dot{\theta}_m}{\cos\theta_n \, \cos\theta_m \, \sin(\theta_n - \theta_m)}
\end{aligned}
\tag{44}
$$

$$\ddot{z}_n = \frac{\ddot{\theta}_n}{\cos^2\theta_n} + \frac{2\,\dot{\theta}_n^2 \, \sin\theta_n}{\cos^3\theta_n} = \frac{\ddot{\theta}_n + 2\,\dot{\theta}_n^2 \, \tan\theta_n}{\cos^2\theta_n}. \tag{45}$$

Appendix B. Solution of the System (31)

In this Appendix we indicate how the initial-value problem of the system of N (decoupled) first-order ODEs (31) is solved.

Let us, for notational convenience, make here the following change of variables

$$\zeta_n(t) = \exp[i\,\theta_n(t)] \tag{46a}$$

entailing

$$\dot{\zeta}_n(t) = i\,\dot{\theta}_n(t)\,\exp[i\,\theta_n(t)]. \tag{46b}$$

We then use the relation (46a) to rewrite the equations of motion (31) as follows:

$$\mu\,\zeta^N\,\dot{\zeta} = i\left[-\eta\,\zeta^{N+1} + \sum_{m=1}^{N}\left(h_m\,\zeta^{2m}\right)\right]. \tag{47}$$

Remark 4. *Let us emphasize that, in the last formula and below (in this Appendix), as a notational simplification, we omit to indicate explicitly the time-dependence of the dependent variable $\zeta_n \equiv \zeta_n(t)$, as well as its dependence on the index n; and likewise the dependence on this index n of the parameters μ_n and η_n.*

The ODE (47) can clearly be solved by the following quadrature

$$\int\limits_{\zeta(0)}^{\zeta(t)} d\xi \, \xi^{N-2} \left\{ -\eta \, \xi^{N-1} + \sum_{m=1}^{N} \left[h_m \, \xi^{2(m-1)} \right] \right\}^{-1} = \frac{i \, t}{\mu}. \qquad (48)$$

To perform the integration it is convenient to introduce the $2(N-1)$ zeros ξ_j of the polynomial of degree $2(N-1)$ appearing in the denominator of the integrand,

$$-\eta \, \xi^{N-1} + \sum_{m=1}^{N} \left[h_m \, \xi^{2(m-1)} \right] = h_N \prod_{j=1}^{2(N-1)} (\xi - \xi_j) \qquad (49a)$$

and then the $2(N-1)$ "residues" ϕ_j defined by setting

$$\left\{ -\eta \, \xi^{N-1} + \sum_{m=1}^{N} \left[h_m \, \xi^{2(m-1)} \right] \right\}^{-1} = h_N^{-1} \sum_{j=1}^{2(N-1)} \left(\frac{\phi_j}{\xi - \xi_j} \right). \qquad (49b)$$

Note that these formulas imply that the computation of, firstly, the $2(N-1)$ zeros ξ_j, and, secondly, the $2(N-1)$ residues ϕ_j, is a purely *algebraic* task (although not one that can be analytically performed for $N \geq 3$) and hence these quantities can in principle be considered known functions of the parameter η (from which they inherit a dependence on the index n, see Remark 4 and of the N constants of motion h_m. As for these N quantities h_m (which are of course independent of the index n) they are–in the context of the *initial-value* problem for the dynamical system (28)–explicitly given by the formulas (25b) at $t = 0$ (let us reiterate that these expressions of the N constants of motion h_m are valid throughout the time evolution, and of course, in particular, at the *initial* time $t = 0$).

The final step is to perform the integration in the left-hand side of (48). Via (49b) the key ingredient to do so is the formula

$$\begin{aligned}
\int\limits_{\zeta_0}^{\zeta} d\xi \, \frac{\xi^{N-2}}{\xi - \xi_0} &= \int\limits_{\zeta_0 - \xi_0}^{\zeta - \xi_0} d\xi \, \frac{(\xi + \xi_0)^{N-2}}{\xi} \\
&= \int\limits_{\zeta_0 - \xi_0}^{\zeta - \xi_0} d\xi \, \sum_{k=0}^{N-2} \left[\binom{N-2}{k} \xi^{k-1} \, \xi_0^{N-2-k} \right] \\
&= \xi_0^{N-2} \log \left(\frac{\zeta - \xi_0}{\zeta_0 - \xi_0} \right) \\
&\quad + \sum_{k=1}^{N-2} \left\{ \binom{N-2}{k} \frac{\xi_0^{N-2-k}}{k} \left[(\zeta - \xi_0)^k - (\zeta_0 - \xi_0)^k \right] \right\}.
\end{aligned} \qquad (50)$$

Acknowledgements

References

[1] Auslander L. and Markus L., *Classical Differential Equations on Manifolds*, Trans. Amer. Math. Soc. **91** (1959) 113-128.

[2] Calogero F., *Classical Many-Body Problems Amenable to Exact Treatments*, Lecture Notes in Physics Monographs **m66**, Springer, Berlin 2001.

[3] Calogero F., Gómez-Ullate D., Santini P. and Sommacal M., *The Transition From Regular to Irregular Motions, Explained as Travel on Riemann Surfaces*, J. Phys. A: Math. Gen. **38** (2005) 8873-8896.

[4] Calogero F., Gómez-Ullate D., Santini P. and Sommacal M., *Towards a Theory of Chaos Explained as Travel on Riemann Surfaces*, J. Phys. A.: Math. & Theor. **42** (2009) 15205-15231.

[5] Crouch P. and Grossman R., *Numerical Integration of Ordinary Differential Equations on Manifolds*, J. Nonlinear Sci. **3** (1993) 1-33.

[6] Erman F. and Turgut O., *A Many-Body Problem with Point-Interactions on Two-Dimensional Manifolds*, arXiv:1204.2171v2[math-ph].

[7] Gómez-Ullate D., Santini P., Sommacal M. and Calogero F., *Understanding Complex Dynamics by Means of an Associated Riemann Surface*, Physica D **241** (2012) 1291-1305.

[8] Hairer E., *Solving Differential Equations on Manifolds*, Lecture Notes, Université de Genève 2011.

[9] Perelomov A., *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser, Basel 1990.

[10] Sutherland B., *Exact Results for a Quantum Many-Body Problem in One Dimension*, Phys. Rev. A **4** (1971) 2019-2021.

[11] Sutherland B., *Exact Results for a Quantum Many-Body Problem in One Dimension II*, Phys. Rev. A **5** (1972) 1372-1376.

[12] Sutherland B., *Beautiful Models - 70 Years of Exactly Solved Quantum Many-Body Problems*, World Scientific, Singapore 2004.

EULER DECOMPOSITION IN NON-ORTHOGONAL BASES

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Abstract. Here we obtain various covariant expressions for the generalized
Euler decompositions of three-dimensional rotations and pseudo-rotations
based on the vector parameterization developed by *Rodrigues*, *Gibbs* and
Fedorov [3, 8]. When the chosen rotational axes form a (generally non-
orthogonal) basis, the solutions may be written explicitly in terms of the
coordinates of the compound vector parameter in this basis. An alternative
version of these results is based on considering the entries of the (pseudo-
)rotational matrix given by *Rodrigues*' formula. Apart from pure geometry
and rigid body mechanics [1, 9, 10], they find applications in areas that vary
from robotics and image processing [7], through crystallography and diffrac-
tometry [4] to relativity, quantum mechanics and gauge field theories [2,5,6].

1. Vector Algebra in Non-Orthogonal Bases

We remind that if $\{\hat{\mathbf{c}}_k\}$ is a basis in $\mathbb{R}^n$, each vector in this space can be expanded as
$\mathbf{x} = x^k \hat{\mathbf{c}}_k$ (summation over repeated upper and lower indices is assumed through-
out the text), where the coefficients, or the parallel projections, happen to coincide
with the orthogonal projections $x^k = x_k = \hat{\mathbf{c}}_k(\mathbf{x})$ as long as the basis is orthonor-
mal, i.e., $(\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j) = \delta_{ij}$. In the generic case of non-orthogonal bases, however,
there is a crucial difference between upper and lower indices and although the
above expansion is still valid, the vector components are equal to the orthogonal

projections in the *dual basis* $\{\hat{\mathbf{c}}^k\}$

$$x^k = (\hat{\mathbf{c}}^k, \mathbf{x}), \qquad \hat{\mathbf{c}}^i(\hat{\mathbf{c}}_j) = \delta^i_j.$$

This dual basis can easily be constructed with the aid of the skew-symmetric *Levi-Civita* symbol ε_{ijk}, which in $\mathbb{R}^3$ gives the usual cross product

$$\varepsilon_{ijk}\hat{\mathbf{c}}^k = \frac{\hat{\mathbf{c}}_i \times \hat{\mathbf{c}}_j}{(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)}, \qquad (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3) = \omega \qquad (1)$$

and one may consider the bases $\{\hat{\mathbf{c}}_k\}$ and $\{\hat{\mathbf{c}}^k\}$ as mutually dual, which generates duality between contravariant (upper) and covariant (lower) indices. The latter can be made even more explicit by introducing the metric tensor

$$g: \quad g_{ik} = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_k), \qquad g^{-1}: \quad g^{ik} = (\hat{\mathbf{c}}^i, \hat{\mathbf{c}}^k) \qquad (2)$$

that allows for transforming one type of coordinates into the other

$$x^i = g^{ik}x_k, \qquad\qquad x_i = g_{ik}x^k. \qquad (3)$$

Moreover, the volumes in the two bases are easily seen to be mutually reciprocal

$$(\hat{\mathbf{c}}^1, \hat{\mathbf{c}}^2, \hat{\mathbf{c}}^3) = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)^{-1} = \omega^{-1}$$

and since g_{ik} is a *Gram* matrix, its determinant is related to ω by

$$|g| = \det g_{ik} = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)^2 = \omega^2, \qquad |g|^{-1} = \det g^{ik} = (\hat{\mathbf{c}}^1, \hat{\mathbf{c}}^2, \hat{\mathbf{c}}^3)^2 = \omega^{-2}.$$

2. Quaternions and Vector Parameters

We may take advantage of the local isomorphism between the *Lie* groups $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ to construct our main tool, the *vector parameter*, (also known as *Rodrigues'* or *Gibbs'* vector). Identifying $\mathsf{SU}(2) \cong \mathbb{S}^3$ with the set of unit quaternions, we obtain a convenient representation for the rotation group via projection

$$0 \to \mathbb{Z}_2 \to \mathsf{SU}(2) \to \mathsf{SO}(3) \to 0.$$

More precisely, we may choose a basis of $\mathfrak{su}(2)$ in the form

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad (4)$$

and introduce the set of unit quaternions as

$$\zeta = \zeta_0 + \zeta_1\mathbf{i} + \zeta_2\mathbf{j} + \zeta_3\mathbf{k}, \qquad |\zeta|^2 = 1, \qquad \zeta_\mu \in \mathbb{R}$$

with norm given by

$$|\zeta|^2 = \frac{1}{2}\mathrm{Tr}(\zeta\bar{\zeta}) = \det(\zeta) = \sum_{\mu=0}^{3} \zeta_\mu^2$$

where $\bar{\zeta} = \zeta_0 - \zeta_1 \mathbf{i} - \zeta_2 \mathbf{j} - \zeta_3 \mathbf{k}$ stands for the *conjugate* quaternion.

Next, we associate with each vector $\mathbf{x} \in \mathbb{R}^3$ a skew-hermitian matrix given by

$$\mathbf{x} \to \mathbf{X} = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$$

where x_i are the *Cartesian* coordinates of $\mathbf{x}$ in the default basis.

Now we let $\mathsf{SU}(1,1)$ act in its *Lie* algebra via the adjoint representation Ad_ζ : $\mathbf{X} \to \zeta \mathbf{X} \bar{\zeta}$, which can be viewed as a norm-preserving automorphism of $\mathbb{R}^3$. It is not difficult to see that the orthogonal matrix transforming the corresponding *Cartesian* coordinates of three-dimensional vectors is given by

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^T + 2\zeta_0 \boldsymbol{\zeta}^\times \tag{5}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^3$ stands for the *imaginary*, or *vector* part of the quaternion $\zeta = (\zeta_0, \boldsymbol{\zeta})$ and we refer to ζ_0 as its *real*, or *scalar* part. Then, using the substitution

$$\zeta_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\mathbf{n}$$

one easily recovers the famous *Rodrigues*' formula

$$\mathcal{R}(\mathbf{n}, \varphi) = \cos\varphi\,\mathcal{I} + (1 - \cos\varphi)\,\mathbf{n} \otimes \mathbf{n}^T + \sin\varphi\,\mathbf{n}^\times \tag{6}$$

which shows that ζ acts as a rotation by an angle φ about the axis determined by the unit vector $\mathbf{n}$. In particular $\mathcal{R}(\zeta)$ is a half-turn for a purely imaginary quaternion ($\zeta_0 = 0 \Rightarrow \boldsymbol{\zeta} = \mathbf{n}$) and the identity transformation in the scalar case ($\zeta_0 = 1 \Rightarrow \boldsymbol{\zeta} = \mathbf{0}$). The correspondence between quaternions ζ and vector parameters $\mathbf{c}$ is then naturally given by the stereographic projection

$$\mathbb{S}^3 \longrightarrow \mathbb{RP}^3 : \qquad \zeta \longrightarrow \mathbf{c} = \frac{\boldsymbol{\zeta}}{\zeta_0} \tag{7}$$

which can be lifted back to the two-sheeted cover as

$$\zeta_0^\pm = \pm(1 + \mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_0^\pm \mathbf{c}. \tag{8}$$

In that sense, vector parametrization appears to be a very natural choice for the description of the projective group $\mathsf{SO}(3)$. Note that with the help of (7) and (8), *Rodrigues*' formula easily takes the simple rational form

$$\mathcal{R}(\mathbf{c}) = \frac{(1 - \mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c} \otimes \mathbf{c}^T + 2\,\mathbf{c}^\times}{1 + \mathbf{c}^2}. \tag{9}$$

The above correspondence (7) also allows for deriving a vector parameter composition law from quaternion multiplication

$$\zeta = \xi\,\eta = (\xi_0\eta_0 - (\boldsymbol{\xi}, \boldsymbol{\eta}),\ \eta_0\boldsymbol{\xi} + \xi_0\boldsymbol{\eta} + \boldsymbol{\xi} \times \boldsymbol{\eta})\,.$$

After stereographic projection $\eta \to \mathbf{c}_1,\ \xi \to \mathbf{c}_2$ and $\zeta \to \mathbf{c}$ we easily obtain

$$\mathbf{c} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)}. \tag{10}$$

It is also not hard to see that for a composition of three rotations

$$\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$$

one has respectively

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_3 \times \mathbf{c}_2 + \mathbf{c}_3 \times \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1 + (\mathbf{c}_3 \times \mathbf{c}_2) \times \mathbf{c}_1 - (\mathbf{c}_3, \mathbf{c}_2)\,\mathbf{c}_1}{1 - (\mathbf{c}_3, \mathbf{c}_2) - (\mathbf{c}_3, \mathbf{c}_1) - (\mathbf{c}_2, \mathbf{c}_1) - (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}.$$

Moreover, we note that the composition is associative but non-commutative and can be used as a substitute of the standard matrix representation since

$$\langle\, \mathbf{c},\, \mathbf{0}\, \rangle = \langle\, \mathbf{0},\, \mathbf{c}\, \rangle = \mathbf{c}, \qquad \langle\, \mathbf{c},\, -\mathbf{c}\, \rangle = \mathbf{0}, \qquad \mathcal{R}(\mathbf{0}) = \mathcal{I}, \qquad \mathcal{R}(-\mathbf{c}) = \mathcal{R}^{-1}(\mathbf{c}).$$

Compared to the latter, it has several major advantages: first, it is much simpler for computations, second, *Rodrigues*' formula and many other useful expressions appear as rational functions, rewritten in terms of $\mathbf{c}$, and finally, as we already have mentioned, the vector-parameter gives a topologically correct description of $\mathsf{SO}(3) \cong \mathbb{RP}^3$ as it is a projective quantity by construction.

3. Euler Decomposition

Our task consists of finding the angles (or scalar parameters) of rotation about initially given oriented axes with unit vectors $\hat{\mathbf{c}}_k$ (possibly coplanar, but such that $\hat{\mathbf{c}}_2$ is not parallel to any of the other two) in the decomposition

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_3)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1) \tag{11}$$

for an initially given compound rotation with vector parameter $\mathbf{c} = \tau\mathbf{n}$, where $\mathbf{n}^2 = 1$ and $\tau = \tan\dfrac{\varphi}{2}$ (rotation by an angle φ about the oriented axis $\mathbf{n}$). Instead of the angles φ_k, we consider the *scalar parameters* τ_k determined by $\mathbf{c}_k = \tau_k\,\hat{\mathbf{c}}_k$. The first step is to insert (11) in a suitably chosen scalar product

$$(\hat{\mathbf{c}}_3, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_3, \mathcal{R}(\mathbf{c}_2)\,\hat{\mathbf{c}}_1) \tag{12}$$

where we have made use of the fact that $\hat{\mathbf{c}}_k$ is an eigenvector of $\mathcal{R}(\mathbf{c}_k)$ with unit eigenvalue. In combination with (9), this leads to a quadratic equation for τ_2 in the form

$$(r_{31} + g_{31} - 2g_{12}g_{23})\,\tau_2^2 + 2\omega\,\tau_2 + r_{31} - g_{31} = 0 \tag{13}$$

in which we use the notations of Section 1, as well as $r_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\hat{\mathbf{c}}_j)$, that is given by (9). We also have

$$r_{31} = g_{31} + \sigma_{31}, \qquad \sigma_{31} = 2\,\frac{c_1 c_3 - \mathbf{c}^2 g_{13} - \tilde{c}^2}{1 + \mathbf{c}^2} \tag{14}$$

where $c_k = \hat{\mathbf{c}}_k(\mathbf{c})$ and $\tilde{c}^j = \omega c^j$. Note that unlike c^k, the components $\tilde{c}^k$ may be defined even if the $\hat{\mathbf{c}}_k$'s are coplanar and do not constitute a basis, as the vanishing

denominator in (1) is canceled.

Equation (13) has real solutions whenever its discriminant is non-negative

$$\omega^2 = |g| \geq r_{31}^2 - g_{31}^2 - 2g_{12}g_{23}(r_{31} - g_{31}) = \sigma_{31}^2 - 2\mathcal{G}^{31}\sigma_{31}$$

where $\mathcal{G}^{31} = g_{12}g_{23} - g_{31}$. Taking into account that g_{ik} is symmetric with diagonal elements equal to one, we may write the above inequality as

$$\Delta = \begin{vmatrix} 1 & g_{12} & r_{31} \\ g_{21} & 1 & g_{23} \\ r_{31} & g_{32} & 1 \end{vmatrix} = |g| - \sigma_{31}^2 + 2\mathcal{G}^{31}\sigma_{31} \geq 0 \tag{15}$$

which is guaranteed only for $g_{12} = g_{23} = 0$ (i.e., $\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_{1,3}$) and the solutions are

$$\tau_2^{\pm} = \frac{-\omega \pm \sqrt{\Delta}}{r_{31} + g_{31} - 2g_{12}g_{23}} = \frac{-\omega \pm \sqrt{\Delta}}{\sigma_{31} - 2\mathcal{G}^{31}}. \tag{16}$$

For the derivation of τ_1 and τ_3 we can use a linear algorithm - the trick here is to write (11) in three different ways

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_3, \mathbf{c} \rangle, \qquad \mathbf{c}_2 = \langle -\mathbf{c}_3, \mathbf{c}, -\mathbf{c}_1 \rangle, \qquad \mathbf{c}_3 = \langle \mathbf{c}, -\mathbf{c}_1, -\mathbf{c}_2 \rangle$$

and then multiply the kth equation on the left with $\hat{\mathbf{c}}_k^{\times}$, by which we derive three lower rank systems, each containing (in the regular case) a nontrivial relation derived by considering scalar product with $\mathbf{c}$. Thus we obtain

$$\tau_1^{\pm} = \frac{(\mathbf{c}^2 g_{23} - c_2 c_3 - \tilde{c}^1)\tau_2^{\pm}}{(c_1\tilde{c}^1 + c_2\tilde{c}^2 + g_{23}c_1 - g_{13}c_2)\tau_2^{\pm} + c_1 c_3 - \mathbf{c}^2 g_{13} - \tilde{c}^2}$$

$$\tau_3^{\pm} = \frac{(\mathbf{c}^2 g_{12} - c_1 c_2 - \tilde{c}^3)\tau_2^{\pm}}{(c_2\tilde{c}^2 + c_3\tilde{c}^3 + g_{12}c_3 - g_{13}c_2)\tau_2^{\pm} + c_1 c_3 - \mathbf{c}^2 g_{13} - \tilde{c}^2}. \tag{17}$$

With the aid of (9), the latter may be written in a more compact form as

$$\tau_1^{\pm} = \frac{(g_{32} - r_{32})\tau_2^{\pm}}{((g_{32} + r_{32})c_1 - (g_{31} + r_{31})c_2)\tau_2^{\pm} + r_{31} - g_{31}}$$

$$\tau_3^{\pm} = \frac{(g_{21} - r_{21})\tau_2^{\pm}}{((g_{21} + r_{21})c_3 - (g_{31} + r_{31})c_2)\tau_2^{\pm} + r_{31} - g_{31}} \tag{18}$$

which allows for rapid and accurate computations, but fails in the symmetric case $\mathcal{R}(\mathbf{n}, \pi) = 2\mathbf{n} \otimes \mathbf{n}^T - \mathcal{I}$, as we end up with an indeterminacy of the type $0 \times \infty$ in the first term of the denominator. Thus, for a *half-turn* $\varphi = \pi$ (i.e., when $\tau \to \infty$),

we apply directly *l'Hôpital*'s rule to obtain

$$\tau_1^\pm = \frac{(g_{23} - n_2 n_3)\,\tau_2^\pm}{(n_1 \tilde{n}^1 + n_2 \tilde{n}^2)\,\tau_2^\pm + n_1 n_3 - g_{13}}$$

$$\tau_2^\pm = \frac{-\omega \pm \sqrt{|g| + 4(g_{13} - n_1 n_3)(n_1 n_3 - g_{12} g_{23})}}{2(n_1 n_3 - g_{12} g_{23})} \tag{19}$$

$$\tau_3^\pm = \frac{(g_{12} - n_1 n_2)\,\tau_2^\pm}{(n_2 \tilde{n}^2 + n_3 \tilde{n}^3)\,\tau_2^\pm + n_1 n_3 - g_{13}}$$

where $\tilde{n}^j = \omega n^j$ (see the above definition of $\tilde{c}^j$).

Note that each rotation in the decomposition may happen to be a half-turn itself. Divergencies of the scalar parameters can easily be regularized by lifting back to the universal cover using (8), i.e., $\varphi = \pi \Rightarrow \zeta_0 = 0$, $\boldsymbol{\zeta} = \mathbf{n}$.

We may also consider decompositions with respect to only two axes

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1). \tag{20}$$

Following the same idea as before, we easily derive the equality

$$r_{21} = (\hat{\mathbf{c}}_2, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2, \hat{\mathbf{c}}_1) = g_{21} \tag{21}$$

which plays the role of necessary and sufficient condition for the existence of such decomposition. Next, we multiply (10) on the left by $\mathbf{c}^\times$ and consider dot products with $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ respectively to obtain (assuming $\tau_{1,2} \neq 0$)

$$\tau_1 = \frac{\tilde{c}^3}{g_{12} c_1 - c_2}, \qquad \tau_2 = \frac{\tilde{c}^3}{g_{12} c_2 - c_1}. \tag{22}$$

When the vectors $\hat{\mathbf{c}}_k$ constitute a basis, we can express directly the scalar parameters τ_k as functions of the contravariant components c^k of the compound vector parameter $\mathbf{c}$ - for that we first need to substitute

$$\sigma_{31} = 2\,\frac{g_{1i} g_{3k} c^i c^k - g_{ik} c^i c^k g_{13} - \omega c^2}{1 + g_{ik} c^i c^k} \tag{23}$$

in (15) and (16), while the other two scalar parameters are given by

$$\tau_1^\pm = \frac{(g_{ik} c^i c^k g_{23} - g_{2i} g_{3k} c^i c^k - \omega c^1)\tau_2^\pm}{(\omega(g_{1i} c^1 c^i + g_{2i} c^2 c^i) + g_{23} g_{1i} c^i - g_{13} g_{2i} c^i)\tau_2^\pm + g_{1i} g_{3k} c^i c^k - g_{ik} c^i c^k g_{13} - \omega c^2}$$

$$\tau_3^\pm = \frac{(g_{ik} c^i c^k g_{12} - g_{1i} g_{2k} c^i c^k - \omega c^3)\tau_2^\pm}{(\omega(g_{2i} c^2 c^i + g_{3i} c^3 c^i) + g_{12} g_{3i} c^i - g_{13} g_{2i} c^i)\tau_2^\pm + g_{1i} g_{3k} c^i c^k - g_{ik} c^i c^k g_{13} - \omega c^2}.$$

Similarly, we may express τ_k as explicit functions of the covariant components, i.e., the orthogonal projections c_k, which is straightforward.

4. Degenerate Solutions

There is a situation in which one cannot determine τ_1 and τ_3 independently from (17) due to lack of sufficient information. Topologically this phenomenon is explained by a singularity of the parametrization $\pi : \mathbb{RP}^2 \to \mathbb{T}^3$ resulting in the loss of a degree of freedom, while in the applications it is known as a *gimbal lock*. The condition for such degenerate type of solution is

$$\hat{\mathbf{c}}_3 = \pm \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1 \tag{24}$$

and in that case (11) takes the form

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\pm \tau_3 \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1)\mathcal{R}(\tau_2 \hat{\mathbf{c}}_2)\mathcal{R}(\tau_1 \hat{\mathbf{c}}_1)$$

which can also be written as

$$\mathcal{R}(\tau_2 \hat{\mathbf{c}}_2)\mathcal{R}(\tau_1 \hat{\mathbf{c}}_1) = \mathcal{R}(\mp \tau_3 \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1)\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c})\mathcal{R}(\mp \tau_3 \hat{\mathbf{c}}_1) \tag{25}$$

where for the last equality we use the well-known relation

$$\mathcal{R}(\mathbf{c})\,\mathcal{R}(\mathbf{a})\,\mathcal{R}(\mathbf{c})^{-1} = \mathcal{R}(\mathcal{R}(\mathbf{c})\,\mathbf{a}). \tag{26}$$

Multiplying both sides of (25) on the right with $\mathcal{R}(\pm \tau_3 \hat{\mathbf{c}}_1)$, we end up with

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\tau_2 \hat{\mathbf{c}}_2)\mathcal{R}(\langle \tau_1 \hat{\mathbf{c}}_1, \pm \tau_3 \hat{\mathbf{c}}_1 \rangle) = \mathcal{R}(\tau_2 \hat{\mathbf{c}}_2)\mathcal{R}(\tau_1' \hat{\mathbf{c}}_1) \tag{27}$$

where, if φ_k stands for the *Euler* angle about $\hat{\mathbf{c}}_k$, according to (10) we have

$$\tau_1' = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1 \tau_3} = \tan\left(\frac{\varphi_1 \pm \varphi_3}{2}\right). \tag{28}$$

We see that the problem is reduced to decomposition into a pair of rotations and since (27) guarantees that (21) is fulfilled[1], we may use (22) to obtain

$$\tau_2 = \frac{\tilde{c}^3}{g_{12}c_2 - c_1}, \qquad \tau_1' = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1 \tau_3} = \frac{\tilde{c}^3}{g_{12}c_1 - c_2}. \tag{29}$$

The first equation above determines τ_2 uniquely, but for τ_1 and τ_3 we end up with a one-parameter solution written in terms of the *Euler* angles φ_k as

$$\varphi_1 \pm \varphi_2 = 2 \arctan\left(\frac{\tilde{c}^3}{g_{12}c_1 - c_2}\right). \tag{30}$$

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[1]Otherwise the decomposition is impossible, as in this case we have $\Delta = -(r_{21} - g_{21})^2 \leq 0$.

5. The Hyperbolic Case

In this section we obtain the corresponding results in the hyperbolic case, i.e., for the three-dimensional restricted *Lorentz* group $\mathsf{SO}^+(2,1) \cong \mathsf{SU}(1,1)/\mathbb{Z}^2$. Choosing a basis of $\mathfrak{su}(1,1)$ in the form

$$\mathbf{e}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \qquad (31)$$

we easily identify the group $\mathsf{SU}(1,1)$ with the set of unit *split quaternions*

$$\zeta = \zeta_0 + \zeta_k \, \mathbf{e}^k, \qquad |\zeta| = \det \zeta = \zeta_0^2 - \zeta_1^2 - \zeta_2^2 + \zeta_3^2 = 1.$$

Expansion in the above basis allows for an explicit isometry $\mathbb{R}^{2,1} \to \mathfrak{su}(1,1)$

$$\mathbf{x} \to \mathrm{X} = \begin{pmatrix} \mathrm{i}x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & -\mathrm{i}x_3 \end{pmatrix}, \qquad \mathbf{x} \cdot \mathbf{x} = -\det \mathrm{X} = x_1^2 + x_2^2 - x_3^2.$$

The projection on $\mathsf{SO}^+(2,1)$ is given by the adjoint action $\mathrm{Ad}_\zeta : \ \mathrm{X} \to \zeta \, \mathrm{X} \, \zeta^{-1}$, which constitutes a norm-preserving automorphism. With the notation $\zeta = (\zeta_0, \boldsymbol{\zeta})$, $\boldsymbol{\zeta} \in \mathbb{R}^{2,1}$, the *Cartesian* coordinates of $\mathbf{x}$ are easily seen to be transformed by the pseudo-orthogonal matrix

$$\mathcal{R}_h(\zeta) = (\zeta_0^2 + \boldsymbol{\zeta}^2)\mathcal{I} - 2\,\boldsymbol{\zeta} \otimes \eta\,\boldsymbol{\zeta} + 2\zeta_0\,\boldsymbol{\zeta}^\curlywedge \qquad (32)$$

where $\eta = \mathrm{diag}(1,1,-1)$ is the flat metric in $\mathbb{R}^{2,1}$, $\mathcal{P}_\zeta = \boldsymbol{\zeta} \otimes \eta\,\boldsymbol{\zeta}$ is explicitly written as $\mathcal{P}\zeta_j^i = \eta_{jk}\zeta^i\zeta^k$ and $\boldsymbol{\zeta}^\curlywedge = \eta\,\boldsymbol{\zeta}^\times$, that will be denoted also as $\boldsymbol{\zeta} \curlywedge \boldsymbol{\xi} = \boldsymbol{\zeta}^\curlywedge \boldsymbol{\xi}$. A covariant definition of the tensor objects we work with allows for using $\times$ instead of $\curlywedge$ and $\boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^T$ for $\boldsymbol{\zeta} \otimes \eta\,\boldsymbol{\zeta}$ - switching from upper to lower indices and vice versa then naturally involves the metric η. However, we chose to make our notations as explicit as we can in order to avoid possible confusion.

Furthermore, we may introduce the *hyperbolic vector parameter* in the usual projective manner $\mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0}$ and write $\mathcal{R}_h$ with its help as

$$\mathcal{R}_h(\mathbf{c}) = \frac{(1 + \mathbf{c}^2)\mathcal{I} - 2\,\mathbf{c} \otimes \eta\,\mathbf{c} + 2\,\mathbf{c}^\curlywedge}{1 - \mathbf{c}^2}. \qquad (33)$$

The inverse transformation yields

$$\zeta_0^\pm = \pm(1 - \mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_0^\pm \mathbf{c} \qquad (34)$$

where the two signs correspond to different sheets of the cover.

From the multiplication rule for split quaternions we easily derive the composition law of hyperbolic vector parameters

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2 \cdot \mathbf{c}_1} \qquad (35)$$

and for a composition of three transformations $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$ we have

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + (\mathbf{c}_3 \cdot \mathbf{c}_2)\,\mathbf{c}_1 + \mathbf{c}_3 \curlywedge \mathbf{c}_2 + \mathbf{c}_3 \curlywedge \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1 + (\mathbf{c}_3 \curlywedge \mathbf{c}_2) \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_3 \cdot \mathbf{c}_2 + \mathbf{c}_3 \cdot \mathbf{c}_1 + \mathbf{c}_2 \cdot \mathbf{c}_1 + (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}.$$

It is easy to see that this construction constitutes a representation of $\mathsf{SO}^+(2,1)$.

One straightforward application is obtaining the *Euler decomposition*

$$\mathcal{R}_h(\mathbf{c}) = \mathcal{R}_h(\mathbf{c}_3)\mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1) \tag{36}$$

where $\mathbf{c} = \tau\,\mathbf{n}$ and $\mathbf{c}_k = \tau_k\,\hat{\mathbf{c}}_k$ are the vector parameters of the pseudo-rotations in the decomposition, $\mathbf{n}$ and $\hat{\mathbf{c}}_k$ - the corresponding *quasi-unit* vectors along the oriented axes and τ, τ_k - the *scalar parameters*. What we mean here by quasi-unit is the following: since the hyperbolic flat metric in *Minkowski* space allows non-zero vectors to have positive, negative and vanishing scalar square, it is not always possible to normalize with unit length. Instead, we have $\epsilon = \mathbf{n} \cdot \mathbf{n} = \pm 1$ in the *space-like*, respectively *time-like* case and $\epsilon = 0$ in the *isotropic* one. In order to normalize an isotropic vector, we resort to *Wick rotation*, taking *Euclidean* rather than hyperbolic scalar product, so if $\epsilon = 0$, we write $\mathbf{c} = \tau\mathbf{n}$, $(\mathbf{n}, \mathbf{n}) = 1$. Similar arguments hold for $\mathbf{c}_k$ and $\epsilon_k = \hat{\mathbf{c}}_k \cdot \hat{\mathbf{c}}_k$. Despite the evident analogy, a crucial difference from the *Euclidean* case is the explicit dependance on the geometric type of the invariant axis of the pseudo-rotation - that is why we have three versions of *Rodrigues'* formula

1. *Hyperbolic:* $\operatorname{Tr}\mathcal{R}_h(\zeta) > 3 \;\Leftrightarrow\; \zeta^2 > 0$ *(space-like)* $\;\Rightarrow\; \mathbf{c} = \tanh\dfrac{\varphi}{2}\,\mathbf{n}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \cosh\varphi\,\mathcal{I} + (1 - \cosh\varphi)\,\mathbf{n} \otimes \eta\,\mathbf{n} + \sinh\varphi\,\mathbf{n}^{\curlywedge}$$

2. *Elliptic:* $\operatorname{Tr}\mathcal{R}_h(\zeta) < 3 \;\Leftrightarrow\; \zeta^2 < 0$ *(time-like)* $\;\Rightarrow\; \mathbf{c} = \tan\dfrac{\varphi}{2}\,\mathbf{n}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \cos\varphi\,\mathcal{I} + (\cos\varphi - 1)\,\mathbf{n} \otimes \eta\,\mathbf{n} + \sin\varphi\,\mathbf{n}^{\curlywedge}$$

3. *Parabolic:* $\operatorname{Tr}\mathcal{R}_h(\zeta) = 3 \;\Leftrightarrow\; \zeta^2 = 0$ *(isotropic)* $\;\Rightarrow\; \mathbf{c} = \dfrac{\varphi}{2}\,\mathbf{n}$

$$\mathcal{R}_h(\mathbf{n}, \varphi) = \mathcal{I} + \varphi\,\mathbf{n}^{\curlywedge} - \frac{\varphi^2}{2}\,\mathbf{n} \otimes \eta\,\mathbf{n}, \qquad (\mathbf{n}, \mathbf{n}) = 1$$

which cover all transformations in the restricted group $\mathsf{SO}^+(2,1)$. One may, however, generalize to the whole *Lorentz* group $\mathsf{SO}(2,1)$, using a simple trick. The corresponding extension consists in allowing $\mathbf{c}^2 > 1$ for the hyperbolic case. Since this is not covered by the standard projection we use here, the above parametrization makes no sense, but (33) still does and we may denote

$$\zeta_0 = \mathrm{i}\sinh\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \mathrm{i}\cosh\frac{\varphi}{2}\,\mathbf{n} \qquad \Longrightarrow \qquad \mathbf{c} = \frac{\boldsymbol{\zeta}}{\zeta_0} = \coth\frac{\varphi}{2}\,\mathbf{n}$$

thus obtaining

$$\mathcal{R}_h(\mathbf{n}, \varphi) = -\cosh \varphi \, \mathcal{I} + (1 + \cosh \varphi) \, \mathbf{n} \otimes \eta \, \mathbf{n} - \sinh \varphi \, \mathbf{n}^\curlywedge \qquad (37)$$

that is valid for pseudo-rotations that do not preserve the orientation of time. Furthermore, we may introduce the notations

$$r_{ij} = \hat{\mathbf{c}}_i \cdot \mathcal{R}_h(\mathbf{c}) \, \hat{\mathbf{c}}_j, \qquad g_{ik} = \hat{\mathbf{c}}_i \cdot \hat{\mathbf{c}}_k, \qquad \omega = \hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2 \curlywedge \hat{\mathbf{c}}_3$$

and point that (36) leads to $r_{31} = \hat{\mathbf{c}}_3 \cdot \mathcal{R}_h(\tau_2 \hat{\mathbf{c}}_2) \, \hat{\mathbf{c}}_1$, from which we derive, according to (33), a quadratic equation for the middle scalar parameter τ_2

$$[\epsilon_2(r_{31} + g_{31}) - 2g_{12}g_{23}] \, \tau_2^2 - 2\omega\tau_2 + g_{31} - r_{31} = 0.$$

The latter has real roots whenever[2]

$$\Delta = \omega^2 + [\, \epsilon_2(r_{31} + g_{31}) - 2g_{12}g_{23}](r_{31} - g_{31}) = - \begin{vmatrix} \epsilon_1 & g_{12} & r_{31} \\ g_{21} & \epsilon_2 & g_{23} \\ r_{31} & g_{32} & \epsilon_3 \end{vmatrix} \geq 0 \qquad (38)$$

and they are explicitly given by

$$\tau_2^\pm = \frac{\omega \pm \sqrt{\Delta}}{\epsilon_2(r_{31} + g_{31}) - 2g_{12}g_{23}}. \qquad (39)$$

Now we use the familiar from the *Euclidean* case technique to obtain

$$\tau_1^\pm = \frac{(\mathbf{c}^2 g_{23} - c_2 c_3 + \tilde{c}^1)\tau_2^\pm}{(c_1 \tilde{c}^1 + c_2 \tilde{c}^2 + g_{23}c_1 - g_{13}c_2)\tau_2^\pm + c_1 c_3 - \mathbf{c}^2 g_{13} + \tilde{c}^2}$$

$$\tau_3^\pm = \frac{(\mathbf{c}^2 g_{12} - c_1 c_2 + \tilde{c}^3)\tau_2^\pm}{(c_2 \tilde{c}^2 + c_3 \tilde{c}^3 + g_{12}c_3 - g_{13}c_2)\tau_2^\pm + c_1 c_3 - \mathbf{c}^2 g_{13} + \tilde{c}^2} \qquad (40)$$

where $c_k = \hat{\mathbf{c}}_k(\mathbf{c})$ and $\tilde{c}^k = \omega c^k$ just as in Section 3.

Note that the above solutions may not be restricted to $\mathsf{SO}^+(2, 1)$, as can be seen in many examples. On the other hand, it is not possible to have a pure $\mathsf{SO}^+(2, 1)$ decomposition for a transformation of the type (37). As for the case of decompositions with respect to two axes, the condition $r_{21} = g_{21}$ is still relevant and the solutions are easily shown to be

$$\tau_1 = \frac{\tilde{c}^3}{\epsilon_1 c_2 - g_{12}c_1}, \qquad \tau_2 = \frac{\tilde{c}^3}{\epsilon_2 c_1 - g_{12}c_2}. \qquad (41)$$

We also have the gimbal lock condition $\hat{\mathbf{c}}_3 = \pm\mathcal{R}_h(\mathbf{c}) \, \hat{\mathbf{c}}_1$, which, if satisfied together with $r_{21} = g_{21}$, gives a one-parameter set of degenerate solutions

$$\tau_2 = \frac{\tilde{\rho}^3}{\epsilon_2 \rho_1 - \kappa_{12}\rho_2}, \qquad \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1 \tau_1 \tau_3} = \frac{\tilde{\rho}^3}{\epsilon_1 \rho_2 - \kappa_{12}\rho_1}. \qquad (42)$$

---

[2]Instead of the *Euclidean* $\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_{1,3}$, we found 21 configurations of axes that guarantee it.

Note that (40) is valid even if the vectors $\hat{\mathbf{c}}_k$ do not constitute a basis. If they do, however, it is possible to express τ_k explicitly from the components of $\mathbf{c}$, just as in Section 3, once we substitute in (38) and (39) the expressions

$$r_{31} + g_{31} = 2\,\frac{g_{13} - c_1 c_3 - \omega c^1}{1 - \mathbf{c}^2}, \qquad r_{31} - g_{31} = 2\,\frac{\mathbf{c}^2 g_{13} - c_1 c_3 - \omega c^1}{1 - \mathbf{c}^2}$$

and take into account that $c_i = g_{ik} c^k$, $c^i = g^{ik} c_k$, $\mathbf{c}^2 = c^i c_i = g_{ik} c^i c^k = g^{ik} c_i c_k$. On the other hand, one may easily write (40) in the more compact form

$$\tau_1^\pm = \frac{(r_{32} - g_{32})\,\tau_2^\pm}{((g_{32} + r_{32})c_1 - (g_{31} + r_{31})c_2)\,\tau_2^\pm + g_{31} - r_{31}}$$

$$\tau_3^\pm = \frac{(r_{21} - g_{21})\,\tau_2^\pm}{((g_{21} + r_{21})c_3 - (g_{31} + r_{31})c_2)\,\tau_2^\pm + g_{31} - r_{31}}. \tag{43}$$

These expressions fail in the limit $\tau \to \infty$, bit in this case we use *l'Hôpital*'s rule to obtain

$$\tau_1^\pm = \frac{(\epsilon g_{23} - n_2 n_3)\tau_2^\pm}{(n_1 \tilde{n}^1 + n_2 \tilde{n}^2)\tau_2^\pm + n_1 n_3 - \epsilon g_{13}}, \qquad \tau_3^\pm = \frac{(\epsilon g_{12} - n_1 n_2)\tau_2^\pm}{(n_2 \tilde{n}^2 + n_3 \tilde{n}^3)\tau_2^\pm + n_1 n_3 - \epsilon g_{13}}.$$

For details on the derivation of most of the results in this section, as well for the applications in special relativity and scattering theory, we refer to [2].

References

[1] Bauchau O., Trainelli L. and Bottaso C., *The Vectorial Parameterization of Rotation*, Nonlinear Dynamics **32** (2003) 71-92.

[2] Brezov D., Mladenova C. and Mladenov I., *Vector Parameters in Classical Hyperbolic Geometry*, J. Geom. Symmetry Phys. **30** (2013) 19-48.

[3] Brezov D., Mladenova C. and Mladenov I., *Vector Decompositions of Rotations*, J. Geom. Symmetry Phys. **28** (2012) 67-103.

[4] Diamond R., *On the Factorization of Rotations with Examples in Diffractometry*, Proc. R. Soc. Lond. A **428** (1990) 451-472.

[5] Fedorov F., *The Lorentz Group* (in Russian), Science, Moscow 1979.

[6] Kuvshinov V. and Tho N., *Local Vector Parameters of Groups, The Cartan Form and Applications to Gauge and Chiral Field Theory* (in Russian), Physics of Elementary Particles and the Nucleus **25** (1994) 603-648.

[7] Mladenova C., *Group Theory in the Problems of Modeling and Control of Multi-Body Systems*, J. Geom. Symmetry Phys. **8** (2006) 17-121.

[8] Mladenova C. and Mladenov I., *Vector Decomposition of Finite Rotations*, Rep. Math. Phys. **68** (2011) 107-117.

[9] Müller A., *Group Theoretical Approaches to Vector Parameterization of Rotations*, J. Geom. Symmetry Phys. **19** (2010) 43-72.

[10] Pina E., *Rotations with Rodrigues' Vector*, Eur. J. Phys. **32** (2011) 1171-1178.

A MODEL SUITABLE FOR NUMERICAL INVESTIGATION OF BEAM-SOLITON INTERACTION IN ELECTROSTATIC PLASMAS[*]

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Abstract. We derive a model suitable for computer simulations of a weak ion beam with solitons of the Korteweg-de Vries (KdV) equation. This type of interactions arise in experiments on soliton generation in double plasma devices and include soliton growth, damping, or decay. Our simplified model aims at capturing only the essential physics of these interactions. The model is formulated in the context of plasma physics in the electrostatic approximation. The bulk plasma is described by cold fluid ions and warm, massless electrons. The ion beam is included as a separate plasma species and is coupled to the bulk plasma through Poisson's equation. The derivation uses the Lagrangian of the system of plasma and beam and an expansion in small amplitude perturbations around an equilibrium. The Korteweg-de Vries equation arises from this expansion naturally. The model is thus applicable to general weakly non-linear ion-acoustic plasma waves, of which solitons are a particular case. A novel feature of our method is that it includes both the evolution of the wave and the perturbation while in previous analyses the perturbation is kept fixed. The computational advantages of such description other approaches, such as fluid description of both plasma and beam or all kinetic description, are that in the former case particle trapping cannot be fully simulated while in the latter case the computational time is much longer and the numerical noise is higher than in our hybrid approach.

1. Introduction

The Korteweg-de Vries (KdV) equation has been derived in multiple areas of physics [19], including plasma physics [4, 9, 23] (see also the review article [20], and references therein). Ion-acoustic solitons were first observed by Ikezi *et al.* [13] and subsequently confirmed by other experimental groups [2, 18]. In plasma physics experiments it is rare that when plasmas are excited only the phenomenon under investigation develops. As a rule, a variety of phenomena occur simultaneously and often their interaction affects, or even obscures, the targeted physics. For example, in double plasma experiment machines [2, 13, 18], the excitation of a soliton is accompanied by a burst of ions [1] and the two co-propagate and interact with each other [2, 18]. This interaction may lead to modifications of the soliton such as growth, damping, or decay. It is, therefore, of interest to investigate the more complex, simultaneous development of interacting waves (solitons) and beam particles.

The study of solitons and co-propagating (resonant) particles has been done theoretically [14, 16, 22] as well as numerically [2, 16, 17]. Theoretical studies are useful since explicit expressions for the damping rates may be obtained. However, a drawback is their restriction to either the linear stage or only the initial stages of the non-linear regime of interaction. Only exceptional cases of perturbative analysis of solitons yield themselves to non-linear treatment, one example being given by the case of soliton-soliton interactions [10, 11, 15]. A further drawback is that the time evolution of the perturbation itself is not taken into account. In contrast, numerical studies permit a more general investigation. One of the most prevalent and important non-linear kinetic effects is wave particle trapping, which necessarily includes the time evolution of the perturbation beam particles. While this effect is particularly difficult to tackle analytically, it is readily accessible from a numerical viewpoint. Yet, it is our opinion that this phenomenon has not been the subject of a comprehensive and systematical numerical study. It is the purpose of this paper to derive a model suitable for the numerical investigation of weakly non-linear ion-acoustic waves (e.g., solitons) and particle interaction in electrostatic plasmas. A novel and necessary feature in this model is the inclusion of the time evolution of both the wave and the perturbation (beam particles).

The two most common techniques of plasma simulations are fluid and kinetic [5, 12, 21]. Kinetic methods have the most general validity, i.e., comprehensive physics, but also the disadvantages of long computational times and high numerical noise; the latter may sometimes obscure importantly physics. In comparison, fluid simulations have fast computational times and very low numerical noise but have a limited validity, i.e., miss some relevant physics. For example, in the fluid picture the plasma is assumed to be in a local thermal equilibrium while this may be a good approximation for the bulk plasma, it is not necessarily true for the beam particles

of interest here. Therefore, the most advantageous approach to the simulation of a beam-soliton system is the hybrid fluid-kinetic, where the soliton is described by a fluid while the ion beam is described by a collection of particles. Such approach combines the advantages of fluid simulations–fast computational time and low noise–and those of kinetic models with the more comprehensive modeling of kinetic effects.

2. Background

Taniuti and Washimi [23] looked for weakly non-linear solutions of the following system of equations

$$\partial_t n + \partial_x(nu) = 0, \quad \partial_t u + u\partial_x u = E, \quad \partial_x n_e = -n_e E, \quad \partial_x E = n - n_e. \ (1)$$

In (1), n denotes the ion fluid density, u is the ion fluid velocity, n_e is the electron density, and E is the electric field. The system of equations (1) describes plasma in the electrostatic approximation. The first equation is the continuity equation for ions in the cold fluid approximation, the second is the momentum equation for the ions, the third equation is the momentum equation for the electron species of the plasma, assumed massless and in local thermal equilibrium (reduced to just the force balance condition), and the fourth equation in (1) is Poisson's equation. All variables are dimensionless, where the density is given in units of some characteristic density n_0, velocity is in units of $\sqrt{\kappa T_e/M}$ (ion-acoustic sound speed), with Boltzmann constant κ and constant electron temperature T_e, dimensions of length are in units of Debye length, $\sqrt{\kappa T_e/4\pi e^2 n_0}$, and electric potential is measured in units of $\kappa T_e/e$. Here M and e denote the ion mass and ion charge. The boundary condition is taken at $x \to \infty$

$$n = n_e = 1, \qquad u = 0. \tag{2}$$

The dispersion relation following from the linearized system (1) is $k^2 = \omega^2/(1 - \omega^2)$, where k is the wave number and ω is the wave frequency. This dispersion relation may be expanded for small ω as $k \approx \omega \left(1 + \frac{1}{2}\omega^2\right)$. The wave phase may be written as

$$kx - \omega t = (x - t)\omega + \frac{1}{2}\omega^3.$$

Defining μ as $\omega^2 = \epsilon\mu^2$, with a small parameter ϵ, we can write

$$kx - \omega t = \mu \left[\epsilon^{1/2}(x - t) + \frac{1}{2}\mu^2\epsilon^{3/2}x\right] \tag{3}$$

which suggests the coordinate transformation

$$\xi = \epsilon^{1/2}(x - t), \qquad \eta = \epsilon^{3/2}x. \tag{4}$$

After making this coordinate change in equations (1) and considering the first two orders in the expansion in ϵ, the KdV equation results [23].

3. Coupled System of Ion Beam and Plasma in the Weakly Non-linear Approximation

We now derive the main result of this paper, a model for a small ion beam interacting with weakly non-linear plasma waves. Our derivation proceeds from a variational principle. (A Hamiltonian perturbative derivation of KdV was given in [6]. We consider our method more general and more suitable as a computational starting point.) First, we note that the boundary condition (2) must be modified to include the ion beam; at $x \to \pm\infty$ we have

$$n = 1 \tag{5}$$
$$n_e = 1 + n_b \tag{6}$$
$$u = 0 \tag{7}$$

where n_b is a specified, constant equilibrium ion beam density. Next we note that in the electrostatic approximation the electric field E may be derived from an electric potential, $E = -\partial_x\phi$. Then the third equation in (1) may be solved and the boundary condition (6) used to yield the electron density

$$n_e(x) = (1 + n_b)\, e^{\phi(x)}. \tag{8}$$

Define a velocity potential Υ as

$$u = \partial_x \Upsilon. \tag{9}$$

We may write the following Lagrangian for the cold ion fluid, ion beam, and warm electrons

$$
\mathcal{L} = -\int \mathrm{d}x\; n(x) \left[\frac{1}{2}(\partial_x\Upsilon)^2 + \partial_t\Upsilon\right] + \frac{1}{2}\int \mathrm{d}x\, (\partial_x\phi)^2
$$
$$
- \int \mathrm{d}x\, n\phi + \int \mathrm{d}x\, (1 + n_b)e^{\phi(x)} + \sum_{j=1}^{N_p} w_j \frac{\dot{X}_j^2}{2} - \sum_{j=1}^{N_p} w_j \phi(X_j) \tag{10}
$$

with particle coordinates and velocities given by X_j and $\dot{X}_j$, respectively. The particle weight is defined as $w_j = n_b/N_p$. In fact, these computational particles may represent thousands of physical ions.

We write out the full set of equations following from the Lagrangian (10)

$$\frac{\delta \mathcal{L}}{\delta \Upsilon} \longrightarrow \partial_x (n \partial_x \Upsilon) + \partial_t n = 0 \quad \text{(continuity fluid equation)} \tag{11}$$

$$\frac{\delta \mathcal{L}}{\delta n} \longrightarrow -\frac{1}{2} (\partial_x \Upsilon)^2 - \dot{\Upsilon} - \phi(x) = 0 \quad \text{(momentum fluid equation)} \tag{12}$$

$$\frac{\delta \mathcal{L}}{\delta \phi} \longrightarrow -\partial_{xx}\phi - n(x) - \sum_{j=1}^{N_p} w_j \delta(x - X_j) + (1 + n_b)\mathrm{e}^{\phi} = 0$$

$$\text{(Poisson's equation)} \tag{13}$$

$$\frac{\delta \mathcal{L}}{\delta X_j} \longrightarrow -\ddot{X}_j - \partial_{X_j}\phi(X_j) = 0. \quad \text{(Newton's second law)}. \tag{14}$$

The system (11)–(14) is a coupled system of a fluid and kinetic plasma disturbances represented by the fluid quantities n and Υ, and N_p the number of particles at locations X_j. The fluid and particles interact with each other through electric field, which is found self-consistently (i.e., electric field time evolution is determined by all charges in the system, whose time evolution, in turn, is determined by the electric field) from equation (13). (Strictly speaking, the charge neutrality condition for the ion beam in equilibrium is not satisfied by a finite number of particles, N_p. This deficiency is removed when a computational grid is introduced; then, the computational particles are endowed with spatial extent and the charge neutrality condition may be satisfied exactly by a finite number of particles. For further details, please refer to [5, 8, 12] and the discussion in Section 4.) Such system is still very general in that the fluid and particle disturbances may be of very general form. We are interested only in small amplitude fluid perturbations and weak ion beams. Therefore, we will derive a system that captures these features explicitly. For this purpose, we first perform the change of variables (4) (which defines scaled time and space variables and a change to a moving reference frame, see equation (3)) and then expand the Lagrangian (10) around the equilibrium (uniform plasma density, zero fluid ion velocity, constant beam velocity, and zero electric potential) in the small parameter ϵ. Using the following relations

$$\frac{\partial}{\partial x} \longrightarrow \epsilon^{3/2} \frac{\partial}{\partial \eta} + \epsilon^{1/2} \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} \longrightarrow -\epsilon^{1/2} \frac{\partial}{\partial t} \tag{15}$$

$$\mathrm{d}x \longrightarrow \epsilon^{-3/2} \mathrm{d}\eta$$

the Lagrangian, after the change of variables and factoring out $\epsilon^{-3/2}$, takes the form

$$
\begin{aligned}
\mathcal{L} \;=\; & -\int \mathrm{d}\eta \; n \left(\frac{1}{2}\epsilon^2 \left(\frac{\partial \tilde{\Upsilon}}{\partial \eta}\right)^2 + \epsilon \frac{\partial \tilde{\Upsilon}}{\partial \eta}\frac{\partial \tilde{\Upsilon}}{\partial \xi} + \left(\frac{\partial \tilde{\Upsilon}}{\partial \xi}\right)^2 - \frac{\partial \tilde{\Upsilon}}{\partial \xi} \right) \\
& + \frac{1}{2}\int \mathrm{d}\eta \left(\epsilon^3 \left(\frac{\partial \phi}{\partial \eta}\right)^2 + \epsilon^2 \frac{\partial \phi}{\partial \eta}\frac{\partial \phi}{\partial \xi} + \epsilon \left(\frac{\partial \phi}{\partial \xi}\right)^2 \right) \\
& - \int \mathrm{d}\eta \; n\phi + \int \mathrm{d}\eta \; (1+\epsilon^2 \tilde{n}_b)e^{\phi} + \sum_{j=1}^{N_p} \tilde{w}_j \left(\frac{1}{2}\left(\frac{\mathrm{d}\Xi_j}{\mathrm{d}\xi}\right)^2 - \epsilon^2 \phi(\Xi_j) \right).
\end{aligned}
\tag{16}
$$

In deriving (16) we have assumed that the ion beam density is of order $n_b = \epsilon^2 \tilde{n}_b$ and we have used the scaled velocity potential $\tilde{\Upsilon} = \epsilon^{1/2}\Upsilon$. In addition, because of the stretching coordinate transformation, the particle weight scales as $\epsilon^{-3/2}$, which has been factored out from the full Lagrangian and the new particle weights become $\tilde{w}_j = \tilde{n}_b/N_p$.

We expand all variables as

$$
\begin{aligned}
n &\approx 1 + \epsilon\, n^{(1)} + \epsilon^2 n^{(2)} \tag{17} \\
\tilde{\Upsilon} &\approx \approx \epsilon \Upsilon^{(1)} + \epsilon^2 \Upsilon^{(2)} \tag{18} \\
\phi &\approx \epsilon\, \phi^{(1)} + \epsilon^2 \phi^{(2)} \tag{19} \\
\dot{\Xi}_j &\approx 1 + \epsilon^{3/2}\dot{\Xi}_j^{(1)}. \tag{20}
\end{aligned}
$$

The dot in equation (20) is a differentiation with respect to the scaled time ξ. Note that the ordering of the velocity in equation (20) matches the order of the scaled velocity potential, $\tilde{\Upsilon}$. After substitution of (17)–(20) into (16), we collect the terms of order ϵ^2 and ϵ^3. The result is

$$
\mathcal{L}^{(2)} = -\int \mathrm{d}\eta \left(\frac{1}{2}\left(\frac{\partial \Upsilon^{(1)}}{\partial \xi}\right)^2 - n^{(1)}\frac{\partial \Upsilon^{(1)}}{\partial \xi} \right) - \int \mathrm{d}\eta \; n^{(1)}\phi^{(1)} + \frac{1}{2}\int \mathrm{d}\eta \left(\phi^{(1)} \right)^2
\tag{21}
$$

and

$$
\mathcal{L}^{(3)} = -\int d\eta \left\{ \frac{\partial \Upsilon^{(1)}}{\partial \eta} \frac{\partial \Upsilon^{(1)}}{\partial \xi} + \frac{\partial \Upsilon^{(1)}}{\partial \xi} \frac{\partial \Upsilon^{(2)}}{\partial \xi} + n^{(1)} \left(\frac{1}{2} \left(\frac{\partial \Upsilon^{(1)}}{\partial \xi} \right)^2 - \frac{\partial \Upsilon^{(2)}}{\partial \xi} \right) \right.
$$
$$
\left. -n^{(2)} \frac{\partial \Upsilon^{(1)}}{\partial \xi} + \frac{1}{2} \left(\frac{\partial \phi^{(1)}}{\partial \xi} \right)^2 - \left(n^{(1)} \phi^{(2)} + n^{(2)} \phi^{(1)} \right) + \phi^{(1)} \phi^{(2)} \right. \tag{22}
$$
$$
\left. + \frac{1}{6} \left(\phi^{(1)} \right)^3 \right\} + \sum_{j=1}^{N_p} \tilde{w}_j \left(\frac{1}{2} \left(\dot{\Xi}_j^{(1)} \right)^2 - \int d\eta \ \phi^{(1)}(\eta) \delta(\eta - \Xi_j) \right).
$$

The beam of ions only contributes to $\mathcal{L}^{(3)}$, as desired by the assumed ordering. The last term in (22) was written explicitly identifying the particle as a delta function with support at Ξ_j and in Section 4 we indicate how to relax this assumption.

Next we derive the equations of motion. First, variation of $\mathcal{L}^{(2)}$ gives

$$
\frac{\delta \mathcal{L}^{(2)}}{\delta \Upsilon^{(1)}} \longrightarrow \frac{\partial^2 \Upsilon^{(1)}}{\partial \xi^2} - \frac{\partial n^{(1)}}{\partial \xi} = 0 \tag{23}
$$

$$
\frac{\delta \mathcal{L}^{(2)}}{\delta n^{(1)}} \longrightarrow \frac{\partial \Upsilon^{(1)}}{\partial \xi} - \phi^{(1)} = 0 \tag{24}
$$

$$
\frac{\delta \mathcal{L}^{(2)}}{\delta \phi^{(1)}} \longrightarrow -n^{(1)} + \phi^{(1)} = 0. \tag{25}
$$

From the three equations (23)–(25) we deduce

$$
n^{(1)} = \phi^{(1)}, \qquad \frac{\partial \Upsilon^{(1)}}{\partial \xi} = n^{(1)}. \tag{26}
$$

The second of the relations (26) was obtained by integrating (23) once in ξ and using the boundary conditions for the fluid velocity and density of the bulk plasma ions. Next, we vary the Lagrangian $\mathcal{L}^{(3)}$. Variation with respect to $\Upsilon^{(1)}$ gives

$$
\frac{\delta \mathcal{L}^{(3)}}{\delta \Upsilon^{(1)}} \longrightarrow 2 \frac{\partial^2 \Upsilon^{(1)}}{\partial \xi \partial \eta} + \frac{\partial^2 \Upsilon^{(2)}}{\partial \xi^2} + \frac{\partial}{\partial \xi} \left(n^{(1)} \frac{\partial \Upsilon^{(1)}}{\partial \xi} \right) - \frac{\partial n^{(2)}}{\partial \xi} = 0
$$

from which by an additional integration in ξ and using the boundary conditions we obtain

$$
2 \frac{\partial \Upsilon^{(1)}}{\partial \eta} + \frac{\partial \Upsilon^{(2)}}{\partial \xi} + n^{(1)} \frac{\partial \Upsilon^{(1)}}{\partial \xi} - n^{(2)} = 0. \tag{27}
$$

Variation with respect to $n^{(1)}$ gives

$$
\frac{\delta \mathcal{L}^{(3)}}{\delta n^{(1)}} \longrightarrow \frac{\partial \Upsilon^{(2)}}{\partial \xi} - \frac{1}{2} \left(\frac{\partial \Upsilon^{(1)}}{\partial \xi} \right)^2 - \phi^{(2)} = 0. \tag{28}
$$

Variation with respect to $\phi^{(1)}$ yields

$$\frac{\delta \mathcal{L}^{(3)}}{\delta \phi^{(1)}} \longrightarrow -\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - n^{(2)} + \phi^{(2)} + \frac{1}{2}\left(\phi^{(1)}\right)^2 - \sum_{j=1}^{N_p} \tilde{w}_j \, \delta(\eta - \Xi_j) = 0. \quad (29)$$

Finally, after a variation with respect to the particle coordinate Ξ_j, we have

$$\ddot{\Xi}_j + \frac{\partial \phi^{(1)}}{\partial \eta}(\Xi_j) = 0. \quad (30)$$

From equations (27)–(29) we can exclude all second order quantities. Adding equations (28) and (29) yields

$$\frac{\partial \Upsilon^{(2)}}{\partial \xi} - n^{(2)} - \frac{1}{2}\left(\frac{\partial \Upsilon^{(1)}}{\partial \xi}\right)^2 - \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{1}{2}\left(\phi^{(1)}\right)^2 - \sum_{j=1}^{N_p} \tilde{w}_j \, \delta(\eta - \Xi_j) = 0. \quad (31)$$

We see that from equations (27) and (31) we can solve for the second order quantity $\frac{\partial \Upsilon^{(2)}}{\partial \xi} - n^{(2)}$ while the two right-hand sides must be equal (compatibility condition)

$$\frac{1}{2}\left(\frac{\partial \Upsilon^{(1)}}{\partial \xi}\right)^2 + \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - \frac{1}{2}\left(\phi^{(1)}\right)^2 + \sum_{j=1}^{N_p} \tilde{w}_j \, \delta(\eta - \Xi_j) = -2\frac{\partial \Upsilon^{(1)}}{\partial \eta} - n^{(1)}\frac{\partial \Upsilon^{(1)}}{\partial \xi}. \quad (32)$$

Using relations (26), we can express all fluid quantities in equation (32) in terms of velocity potential

$$\frac{\partial \Upsilon^{(1)}}{\partial \eta} + \frac{1}{2}\left(\frac{\partial \Upsilon^{(1)}}{\partial \xi}\right)^2 + \frac{1}{2}\frac{\partial^3 \Upsilon^{(1)}}{\partial \xi^3} = -\sum_{j=1}^{N_p} \frac{\tilde{w}_j}{2}\delta(\eta - \Xi_j) \quad (33)$$

which is the KdV equation with a perturbation of a beam of ions (given by the right-hand side). The beam particles evolve according to

$$\ddot{\Xi}_j = -\frac{\partial^2 \Upsilon^{(1)}}{\partial \xi \partial \eta}(\Xi_j) \quad (34)$$

where relations (26) were used again. The coupled equations (33) and (34) represent the desired beam-plasma model. By using relations (26) we can rewrite the Lagrangian (22) to eliminate all but the velocity potential $\Upsilon^{(1)}$ and the particle coordinates Ξ_j

$$\mathcal{L}^{(3)} = -\int d\eta \left[\frac{\partial \Upsilon^{(1)}}{\partial \xi}\frac{\partial \Upsilon^{(1)}}{\partial \eta} + \frac{1}{3}\left(\frac{\partial \Upsilon^{(1)}}{\partial \xi}\right)^3 - \frac{1}{2}\left(\frac{\partial^2 \Upsilon^{(1)}}{\partial \xi^2}\right)^2\right]$$

$$+ \sum_{j=1}^{N_p} \tilde{w}_j \left[\frac{1}{2}\left(\dot{\Xi}_j^{(1)}\right)^2 - \int d\eta \, \frac{\partial \Upsilon^{(1)}}{\partial \xi}\delta(\eta - \Xi_j)\right]. \quad (35)$$

Variation of (35) yields KdV equation for the velocity [with $u^{(1)} = \partial_\xi \Upsilon^{(1)}$]

$$\frac{\partial u^{(1)}}{\partial \eta} + u^{(1)} \frac{\partial u^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 u^{(1)}}{\partial \xi^3} = -\sum_{j=1}^{N_p} \frac{\tilde{w}_j}{2} \frac{\partial \, \delta(\eta - \Xi_j)}{\partial \xi} \qquad (36)$$

which is just the ∂_ξ derivative of (33). The ξ dependence on the right-hand side of equation (36) is through Ξ_j. The other two fluid quantities, density and electric potential, satisfy the same equation (36).

We would like to stress the generality of the coupled equations (33) and (34). Previous authors have derived equations similar to (36) [16, 17] but their further analytical development required particular choices for the distribution function of the beam particles, which is a limitation of their approach. In addition, a fixed choice of the ion beam distribution function does not take into account its temporal modification due to the beam-wave interaction, which is a further limitation. In comparison, our model evolves the non-linear (soliton) as well as the beam particles. Since the ion beam distribution is simulated kinetically (with particles), it captures the correct evolution of the beam distribution function too.

4. Discussion and Conclusions

The model presented by Lagrangian (35) is the most convenient form for further reduction and computer simulations. We briefly outline the steps necessary to proceed, referencing the reader to the more detailed discussions in [5, 8, 12]. As it stands, the model is still an infinite degree-of-freedom (DOF) system and thus unsuitable yet for computer simulations. Two steps are necessary to reduce (35) to a finite DOF system: spatial discretization and time discretization. The two steps may either be performed simultaneously (however, the *action* then need be considered) or separately. The simpler approach is to consider the two steps separately. The variational principle allows for two general numerical approaches: use of finite differences to approximate the spatial derivatives of $\Upsilon^{(1)}$ in conjunction with an integration rule or the use of a truncated basis expansion. Either of these approaches reduces integrals of field quantities in (35) to finite sums. After the spatial discretization is performed, one varies the Lagrangian with respect to the field coefficients and particle positions to derive the equations of motion. Then time discretization must be done. The only restriction on a choice of a time integrating method is the possibility of encountering a numerical instability or high numerical dissipation. In our experience, second or fourth order accurate explicit Runge-Kutta methods perform satisfactory in wave-particle interaction simulations [7].

We note that as an alternative way to the spatial discretization of the Lagrangian described above, one could first obtain the continuous Euler-Lagrange equations

and then perform the spatial discretization. However, such procedure may introduce undesirable numerical errors (of the order of the accuracy of discretization); it is known that such errors may lead to loss of conservation properties of the original (continuous) system. In comparison, the approach described above allows to more fully utilize the relation between symmetries of the Lagrangian and conserved quantities in the discretized system. An illustration is provided by our continuous field Lagrangian (35): it does not contain explicit time dependence, hence, this system conserves total energy. Explicit time dependence is not introduced by the spatial discretization and therefore the resulting discrete system (time kept continuous) is guaranteed to also conserve energy.

A more subtle question we alluded to previously is our choice of a delta-function particle shape, which helped simplify the presentation. In kinetic simulations of plasmas via the so-called Particle-In-Cell (PIC) method [5, 12], particles are endowed with a spatial extent. Important advantages of using finite-size particles are the much lower numerical noise and the lack of certain numerical instabilities present in simulations with delta-function particles. In a recent publication [8], the author has revisited the concept of particle shapes. In particular, the connection of the particle shape and the numerical accuracy of the force on a particle is exhibited. The essence of the relevant results is the following. Take the case of a reduction of the continuous fields by a finite element basis. Finite elements [3] offer a consistent way of spatial discretization with increasing accuracy. Continuous quantities are solved for (and known) on a computational grid and a rule (e.g., polynomials) is given for finding the values of the solution between grid points. Therefore, if one represents the solution as sum over finite elements $\Psi_k(\eta)$

$$\Upsilon^{(1)}(\xi, \eta) = \sum_{k=1}^{N_g} U_k(\xi) \Psi_k(\eta) \tag{37}$$

the last term in equation (35) reduces to

$$\sum_{j=1}^{N_p} \sum_{k=1}^{N_g} \dot{U}_k \int \mathrm{d}\eta \ \Psi_k(\eta) S(\eta - \Xi_j) \tag{38}$$

where we have replaced the delta function with a more general (shape) function S. One can verify that for the simple choices of a top-hat function for S and linear finite elements, each particle contributes a certain amount of charge (in the PIC terminology, this is called the charge deposition rule) to not one but *three* grid points. Therefore, a particle is not a point in space but a "blob" of charge described by its centroid and velocity Ξ_j and $\dot{\Xi}_j$, respectively. By the use of such extended particles the quasi-neutrality condition of the plasma in equilibrium may be satisfied

exactly. We conclude this discussion by noting that Lagrangian finite elements offer a consistent way of interpolating the force (with a desired accuracy) from the grid to the particle location and for constructing charge deposition rules [8].

A natural further simplification of our model would be the special case of an ion beam interacting with a weakly coupled train of solitons, in a manner similar to that in [10, 11]. Of course, we no longer expect a nice reduction to another integrable system since the original system of beam and solitons is not integrable. However, such reduction should provide a much faster numerical solution since each soliton would now be described by only two parameters, a position and an amplitude. In addition, the numerical noise in such reduced system will be even lower than that of our hybrid model.

In conclusion, we have derived a model of a weak ion beam interacting with weakly non-linear solutions of the Korteweg-de Vries equation in the context of plasma physics in the electrostatic approximation. Particular advantages are the generality of our derivation from a Lagrangian starting point, the lower expected numerical noise in the hybrid fluid–particle approach, and the inclusion of kinetic effects of wave-particle interactions, including their non-linear stage. Further simplification of the model is suggested, which would apply to the interaction of a weakly coupled soliton train with a small ion beam. Interesting questions in these numerical investigations would be soliton growth and damping rates, soliton decay conditions, beam particles wave trapping, and the possibility of formation of new, quasi-stable structures of solitons and beam particles.

References

[1] Alexeff I., Jones W. and Lonngren K., *Excitation of Pseudowaves in a Plasma via a Grid*, Phys. Rev. Lett. **21** (1968) 878–881.

[2] Aziz F., Malik H., Enge S., Stollenwerk L. and Stroth U., *Experimental and Simulation Studies on Soliton Generation in a Double-Plasma Device: Role of Fast Ions*, Plasma Phys. Contr. F. **53** (2011) 065012.

[3] Becker E., Carey G. and Oden J., *Finite Elements: An Introduction* vol. 1, Prentice-Hall, Lebanon 1981.

[4] Berezin Y. and Karpman V., *Nonlinear Evolution of Disturbances in Plasma and Other Dispersive Media*, Soviet Physics JETP **24** (1967) 1049–1056.

[5] Birdsall C. and Langdon A., *Plasma Physics via Computer Simulations*. Plasma Physics Series, Institute of Physics Publishing, Bristol 1991.

[6] Broer L. and Sluijter F., *Stable Approximate Equations for Ion-Acoustic Waves*, Phys. Fluids **20** (1977) 1458–1460.

[7] Evstatiev E., Morrison P. and Horton W., *A Relativistic Beam-Plasma System with Electromagnetic Waves*, Phys. Plasmas **12** (2005) 072108.

[8] Evstatiev E. and Shadwick B., *Variational Formulation of Particle Algorithms for Kinetic Plasma Simulations*, arXiv:1210.3743 (2012).

[9] Gardner S. and Morikawa G., *Courant Institute of Mathematical Sciences Report # NYO-9082*, (unpublished) 1960.

[10] Gerdjikov V., Kaup D, Uzunov I. and Evstatiev E., *Asymptotic Behavior of N-Soliton Trains of the Nonlinear Schrödinger Equation*, Phys. Rev. Lett. **77** (1996) 3943–3946.

[11] Gerdjikov V., Uzunov I., Evstatiev E. and Diankov G., *Nonlinear Schrödinger Equation and N-Soliton Interactions: Generalized Karpman-Solov'ev Approach and the Complex Toda Chain*, Phys. Rev. E **55** (1997) 6039–6060.

[12] Hockney R. and Eastwood J., *Computer Simulation Using Particles*, Taylor & Francis, New York 1988.

[13] Ikezi H., Taylor R. and Baker D., *Formation and Interaction of Ion-Acoustic Solitions*, Phys. Rev. Lett. **25** (1970) 11–14.

[14] Karpman V., *The Effects of the Interaction Between Ion-Sound Solitons and Resonance Particles in a Plasma*, JETP **50** (1979) 695–701.

[15] Karpman V. and Solov'ev V., *A Perturbational Approach to the Two-Soliton Systems*, Physica D **3** (1981) 487–502.

[16] Karpman V., Lynov J., Michelsen P., Pècseli H., Rasmussen J. and Turikov V., *Modification of Plasma Solitons by Resonant Particles*, Phys. Rev. Lett. **43** (1979) 210–214.

[17] Karpman V., Lynov J., Michelsen P., Pècseli H., Rasmussen J. and Turikov V., *Modification of Plasma Solitons by Resonant Particles*, Phys. Fluids **23** (1980) 1782–1794.

[18] Lonngren K., Khaze M., Gabl E. and Bulson J., *On Grid Launched Linear and Nonlinear Ion-Acoustic Waves*, Plasma Physics **24** (1982) 1483, doi:10.1088/0032-1028/24/12/002.

[19] Miura R., *The Korteweg-de Vries Equation: A Survey of Results*, SIAM Review **18** (1976) 412–459.

[20] Pecseli H., *Solitons and Weakly Nonlinear Waves in Plasmas*, IEEE T. Plasma Sci. **13** (1985) 53–86.

[21] Tajima T., *Computational Plasma Physics*, Frontiers in Physics, Westview Press, Boulder 2004.

[22] Vandam J. and Taniuti T., *Nonlinear Ion Acoustic Waves with Landau Damping*, J. Phys. Soc. Jpn. **35** (1973) 897–906.

[23] Washimi H. and Taniuti T., *Propagation of Ion-Acoustic Solitary Waves of Small Amplitude*, Phys. Rev. Lett. **17** (1966) 996–998.

ON SOLITON INTERACTIONS FOR THE HIERARCHY OF A GENERALISED HEISENBERG FERROMAGNETIC MODEL ON $\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2))$ SYMMETRIC SPACE[*]

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Abstract. We consider an integrable hierarchy of nonlinear evolution equations (NLEE) related to linear bundle Lax operator L. The Lax representation is $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduced and can be naturally associated with the symmetric space $\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2))$. The simplest nontrivial equation in the hierarchy is a generalization of Heisenberg ferromagnetic model. We construct the N-soliton solutions for an arbitrary member of the hierarchy by using the Zakharov-Shabat dressing method with an appropriately chosen dressing factor. Two types of soliton solutions: quadruplet and doublet solitons are found. The one-soliton solutions of NLEEs with even and odd dispersion laws have different properties. In particular, the one-soliton solutions for NLEEs with even dispersion laws are *not* traveling waves while their velocities and amplitudes are time dependent. Calculating the asymptotics of the N-soliton solutions for $t \to \pm\infty$ we analyze the interactions of quadruplet solitons.

CONTENTS

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1. Introduction

The main object of present paper is the following coupled system of equations

$$\mathrm{i}u_t + u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0$$
$$\mathrm{i}v_t + v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0 \tag{1}$$

where the smooth functions $u : \mathbb{R}^2 \to \mathbb{C}$ and $v : \mathbb{R}^2 \to \mathbb{C}$ satisfy the algebraic constraint $|u|^2 + |v|^2 = 1$. The system (1) is a natural candidate to be a multi-component generalisation of the classical Heisenberg ferromagnetic equation. It is well known [32] that the Heisenberg ferromagnetic model is integrable in the sense of inverse scattering method (ISM). It has a Lax pair related to the algebra $\mathfrak{su}(2)$. Since the time the complete integrability of HF equations was discovered, many attempts for its generalization have been made [20–22]. A well known method [10, 12, 24, 26–31] to obtain new integrable nonlinear evolution equations (NLEE) is based on imposing certain algebraic reductions on generic Lax operators. Lax pairs associated to hermitian symmetric spaces represent a special interest in modern theory of integrable systems is study of NLEEs [1, 7, 8, 11] since the NLEEs they produce look relatively simple.

The system (1) is also integrable in the sense of ISM. Its Lax operators are associated with the symmetric space $SU(3)/S(U(1) \times U(2))$ with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction imposed on them [13, 15, 16].

The purpose of the present paper is to derive the soliton solutions for the integrable hierarchy of equations related to (1) and analyse the interactions between them. That is why this work is a natural continuation of our previous papers [13, 15, 16].

In Section 2 we start with some basic facts to be used further in the paper. Firstly we describe the hierarchy of nonlinear equations related to (1) in terms of recursion operators. Then we outline the spectral properties of the relevant Lax operator and formulate direct scattering problem. The spectrum of scattering operator L consists of a continuous and a discrete parts. As a result of the $\mathbb{Z}_2$ reductions L possesses

two configurations of discrete eigenvalues: generic ones, coming in quadruplets $\pm\lambda_k$, $\pm\lambda_k^*$ and purely imaginary ones coming as doublets $\pm i\kappa_j$.

In Section 3 we derive the one-soliton solutions for the NLEEs of the hierarchy. For this to be done we apply the Zakharov-Shabat dressing method [34–36, 38] with a rational dressing factor with two simple poles. Due to the action of reductions we have two types of one-soliton solutions: quadruplet solitons correspondinf to four eigenvalues and doublet ones corresponding to two eigenvalues respectively. We present explicit expressions for these two types of one-soliton solutions. In order to construct general multisoliton solutions we discuss two different purely algebraic constructions: by using a multiple pole dressing factor and by applying "one-soliton" dressing factors several times consecutively. It turns out that the properties of the one-soliton solutions to NLEEs with even and odd dispersion laws differ drastically. For example, the one-soliton solutions for NLEEs with even dispersion laws are *not* traveling waves. Even the doublet soliton of equation (1) exhibits two maxima (respectively minima) for $|u_1|$ (respectively for $|v_1|$) which first come closer to each other and then move away, one to ∞ and the other to $-\infty$ as time goes to $t \to \infty$. Their velocity, as well as their amplitudes are time dependent. These properties are similar to the ones of the boomerons and trappons discovered by Calogero and Degasperis [2–5]. At the same time the soliton solutions to the NLEEs with odd dispersion laws (e.g. the solutions of equation (19)) behave as standard solitons, i.e., they are traveling waves.

Section 4 is dedicated to interactions of quadruplet soliton solutions for the NLEE with odd dispersion laws. In order to do this we use the classical method of Zakharov and Shabat, see the monographs [32, 34] for a detailed exposition. Namely, we calculate the limits of the N-soliton solutions for $t \to \pm\infty$ assuming that all solitons move with different velocities. In this way we establish that the solitons preserve their velocities and amplitudes; the only effect of their interaction consists in shifts of the relative mass center and the phase of solitons. We provide explicit expressions for these shifts in terms of the poles μ_k of the dressing factors.

In Section 5 we briefly discuss the conservation laws of the NLEE and end up with some conclusions.

2. Preliminaries

In this section we shall expose in brief some basic facts on Lax operators and direct scattering problem for the integrable hierarchy of the equation (1). In doing this we shall use a gauge covariant formulation [14, 17–19].

2.1. Polynomial Lax Pair Related to $SU(3)/S(U(1) \times U(2))$

The NLEEs under consideration in this paper represent a zero curvature condition $[L, A] = 0$ for Lax operators L and A in the form

$$L(\lambda) = i\partial_x + \lambda L_1(x, t) \tag{2}$$

$$A(\lambda) = i\partial_t + \sum_{k=1}^{N} \lambda^k A_k(x, t) \tag{3}$$

where $\lambda \in \mathbb{C}$ is the so-called spectral parameter and the functions L_1 and A_k, $k = 1, \ldots, N$ take values in $\mathfrak{sl}(3, \mathbb{C})$. The Lax operators are subject to the following $\mathbb{Z}_2$ reductions

$$L^\dagger(\lambda^*) = -\breve{L}(\lambda), \qquad A^\dagger(\lambda^*) = -\breve{A}(\lambda) \tag{4}$$

$$\mathbf{C}L(-\lambda)\mathbf{C} = L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda) \tag{5}$$

where $\mathbf{C} = \text{diag}\,(1, -1, -1)$ and the operation $\breve{\ }$ is defined as given by the formula

$$\breve{L}(\lambda)\psi(x, t, \lambda) \equiv i\partial_x\psi(x, t, \lambda) - \lambda\psi(x, t, \lambda)L_1(x, t, \lambda).$$

Due to reduction (4) the matrix coefficients of the Lax pair are hermitian matrices. On the other hand the reduction (5) represents an action of Cartan's involutive automorphism which defines the symmetric space $SU(3)/S(U(1) \times U(2))$, see [23, 25]. It induces a $\mathbb{Z}_2$-grading in the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$

$$\mathfrak{sl}(3) = \mathfrak{sl}^0(3) \oplus \mathfrak{sl}^1(3), \qquad \mathfrak{sl}^\sigma(3) = \{X \in \mathfrak{sl}(3)\,;\, \mathbf{C}X\mathbf{C} = (-1)^\sigma X\}. \tag{6}$$

It is evident that $L_1, A_k \in \mathfrak{sl}^1(3)$ for k being an odd integer and $A_k \in \mathfrak{sl}^0(3)$ otherwise. This means that A_k for even k are block-diagonal matrices of the form

$$A_k = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

while L_1 and A_k for odd k have the complementary block structure. In particular, L_1 is written as

$$L_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}. \tag{7}$$

The potential L_1 is required to obey the following conditions

1. The eigenvalues of L_1 are $0, \pm 1$, i.e., the potential satisfies the characteristic equation $L_1^3 = L_1$.

2. The function $L_1(x, t) - L_\pm$ where

$$\lim_{x \to \pm\infty} L_1(x, t) = L_\pm = \begin{pmatrix} 0 & 0 & e^{i\phi_\pm} \\ 0 & 0 & 0 \\ e^{-i\phi_\pm} & 0 & 0 \end{pmatrix}, \qquad \phi_\pm \in \mathbb{R} \tag{8}$$

is a Schwartz type function, i.e., it is infinitely smooth and tends to 0 faster than any polynomial when $|x| \to \infty$.

The grading (6) means that any function X with values in $\mathfrak{sl}(3)$ can be split as follows

$$X = X^0 + X^1, \qquad X^{0,1} \in \mathfrak{sl}^{0,1}(3). \tag{9}$$

Let us define the Killing form for $\mathfrak{sl}(3)$ as follows

$$\langle X, Y \rangle = \operatorname{tr}(XY), \qquad X, Y \in \mathfrak{sl}(3).$$

Then each component $X^{0,1}$ splits into a term commuting with L_1 and its orthogonal complement with respect to the Killing form

$$X^0 = X^{0,\perp} + \kappa_0 L_2, \qquad L_2 = L_1^2 - \frac{2}{3}\mathbb{1}, \qquad \langle X^{0,\perp}, L_2 \rangle = 0 \tag{10}$$

$$X^1 = X^{1,\perp} + \kappa_1 L_1, \qquad \langle X^{1,\perp}, L_1 \rangle = 0. \tag{11}$$

As a simple consequence of condition 1 above L_1 and L_2 are normalized as follows

$$\langle L_1, L_1 \rangle = 2, \qquad \langle L_2, L_2 \rangle = \frac{2}{3}. \tag{12}$$

Therefore the coefficients κ_0 and κ_1 are given by the following equalities

$$\kappa_0 = \frac{3}{2}\langle X^0, L_2 \rangle, \qquad \kappa_1 = \frac{1}{2}\langle X^1, L_1 \rangle. \tag{13}$$

The zero curvature condition $[L, A] = 0$ for the pair (2), (3) leads to certain recurrence relations for the matrix coefficients of L and A, see [13]. Resolving them allows one to express A_k in terms of L_1 and its x-derivatives of order up to $N - k$. Since the maximal order term in the operator A must commute with L_1 there exists two options

$$\begin{aligned} &\text{a)} \quad A_N = c_{2p}L_2, &&\text{if} \quad N = 2p \\ &\text{b)} \quad A_N = c_{2p+1}L_1, &&\text{if} \quad N = 2p + 1 \end{aligned}$$

where c_{2p} and c_{2p+1} are constants. Then a more detailed analysis [13] shows that the NLEEs look as follows

$$\begin{aligned} &\text{a)} \quad \operatorname{iad}_{L_1}^{-1}L_{1,t} + \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^q L_2 + \sum_{q=0}^{p-1} c_{2q+1}(\Lambda_1\Lambda_2)^q \Lambda_1 L_1 = 0 \\ &\text{b)} \quad \operatorname{iad}_{L_1}^{-1}L_{1,t} + \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^q L_2 + \sum_{q=0}^{p} c_{2q+1}(\Lambda_1\Lambda_2)^q \Lambda_1 L_1 = 0. \end{aligned} \tag{14}$$

The integro-differential operators Λ_1 and Λ_2 appeared above are given by

$$\Lambda_1 = -\mathrm{i}\,\mathrm{ad}\,_{L_1}^{-1}\left(\pi\partial_x(\cdot) - \frac{1}{2}L_{1,x}\partial_x^{-1}\langle\partial_x(\cdot), L_1\rangle\right)$$

$$\Lambda_2 = -\mathrm{i}\,\mathrm{ad}\,_{L_1}^{-1}\left(\pi\partial_x(\cdot) - \frac{3}{2}L_{2,x}\partial_x^{-1}\langle\partial_x(\cdot), L_2\rangle\right)$$

(15)

where projection $\pi := \mathrm{ad}\,_{L_1}^{-1}\mathrm{ad}\,_{L_1}$ cuts all L_1-commuting parts off. The operator

$$\Lambda X := \begin{cases} \Lambda_1\Lambda_2 X, & X \in \mathfrak{sl}^0(3) \\ \Lambda_2\Lambda_1 X, & X \in \mathfrak{sl}^1(3) \end{cases}$$

is called a recursion operator. It can be viewed as an adjoint representation of the operator L. Its existence manifests the hierarchies associated with NLEE (non-linear equations, integrals of motion, simplectic forms etc) and thus plays a very important role in the theory of solitons.

Example 1. *Consider the simplest case when $N = 2$. Then the matrix coefficients of the second Lax operator A read*

$$A_2 = -\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & |u|^2 - 2/3 & u^*v \\ 0 & v^*u & |v|^2 - 2/3 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}$$

(16)

$$a = \mathrm{i}u_x + \mathrm{i}(uu_x^* + vv_x^*)u, \qquad b = \mathrm{i}v_x + \mathrm{i}(uu_x^* + vv_x^*)v.$$

(17)

This L-A pair produces the two-component system

$$\mathrm{i}u_t + u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0$$

$$\mathrm{i}v_t + v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0$$

(18)

we started our paper with (see (1)).

For completeness here we present another member of the hierarchy (14). It is the simplest NLEE corresponding to an odd dispersion law.

Example 2. *Consider the case when $\mathbf{f}(\lambda) = -8\lambda^3 J$, i.e., $c_3 = -8$, $c_2 = c_1 = 0$. Then the corresponding two-component system obtains the form*

$$u_t = 8u_{xxx} + 12(uu_x^* + vv_x^*)u_{xx} + r(u, v)u_x + s(u, v)u$$

$$v_t = 8v_{xxx} + 12(uu_x^* + vv_x^*)v_{xx} + r(u, v)v_x + s(u, v)v$$

(19)

where

$$r(u, v) = 3\left[4(|u_x|^2 + |v_x|^2) + 5(uu_x^* + vv_x^*)^2 + 6(uu_x^* + vv_x^*)_x\right]$$

$$s(u, v) = 3\left[2(uu_x^* + vv_x^*)_{xx} + 4(|u_x|^2 + |v_x|^2)_x + 5(uu_x^* + vv_x^*)_x^2\right].$$

Sometimes it is more convenient to deal with Lax operators written in canonical gauge. In this gauge the operator (2) looks as follows

$$\tilde{L}(\lambda) = g^{-1}Lg = \mathrm{i}\partial_x + U_0(x,t) + \lambda J, \qquad J = \mathrm{diag}\,(1,0,-1) \qquad (20)$$

where

$$g = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & -1 \\ u^* & \sqrt{2}v & u^* \\ v^* & -\sqrt{2}u & v^* \end{pmatrix}. \qquad (21)$$

The second Lax operator (3) is given by

$$\begin{aligned} \text{a)} \quad & \tilde{A}(\lambda) = \mathrm{i}\partial_t + \sum_{k=0}^{N-1}\lambda^k \tilde{A}_k(x,t) + c_N\lambda^N I, \qquad N = 2p \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (22) \\ \text{b)} \quad & \tilde{A}(\lambda) = \mathrm{i}\partial_t + \sum_{k=0}^{N-1}\lambda^k \tilde{A}_k(x,t) + c_N\lambda^N J, \qquad N = 2p+1 \end{aligned}$$

where $I = g^{-1}L_2\,g = \mathrm{diag}\,(1/3, -2/3, 1/3)$.

2.2. Direct Scattering Problem

In order to formulate a direct scattering problem for L, one needs to introduce the auxiliary spectral linear system

$$L(\lambda)\psi(x,t,\lambda) = \mathrm{i}\partial_x\psi(x,t,\lambda) + \lambda L_1(x,t)\psi(x,t,\lambda) = 0. \qquad (23)$$

Here ψ denotes a fundamental set of solutions or a fundamental solution for short. Since the operators (2) and (3) commute ψ also satisfies

$$A(\lambda)\psi(x,t,\lambda) = \left(\mathrm{i}\partial_t + \sum_{k=1}^{N}\lambda^k A_k(x,t)\right)\psi(x,t,\lambda) = \psi(x,t,\lambda)\mathbf{f}(\lambda) \qquad (24)$$

as well. The matrix-valued function

$$\mathbf{f}(\lambda) = \lim_{x\to\pm\infty} g_\pm^{-1}\sum_{k=1}^{N}\lambda^k A_k(x,t)g_\pm \qquad (25)$$

is called dispersion law of the nonlinear equation (14). The unitary matrix

$$g_\pm = \lim_{x\to\pm\infty} g(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2}\,\mathrm{e}^{\mathrm{i}\phi_\pm} & 0 \\ \mathrm{e}^{-\mathrm{i}\phi_\pm} & 0 & \mathrm{e}^{-\mathrm{i}\phi_\pm} \end{pmatrix}$$

diagonalizes the asymptotics $L_{1,\pm} = \lim_{x\to\pm\infty} L_1(x,t)$. It can be proven that the dispersion law of (14) reads

$$
\text{a)} \qquad \mathbf{f}(\lambda) = \sum_{q=0}^{p-1} c_{2q+1}\lambda^{2q+1} J + \sum_{q=1}^{p} c_{2q}\lambda^{2q} I
$$

$$
\text{b)} \qquad \mathbf{f}(\lambda) = \sum_{q=0}^{p} c_{2q+1}\lambda^{2q+1} J + \sum_{q=1}^{p} c_{2q}\lambda^{2q} I.
\tag{26}
$$

The dispersion law of the two-component system (18) is $-\lambda^2 I$ and that of (19) is $-8\lambda^3 J$. It is evident from (26) that $\mathbf{f}(\lambda)$ obeys the splitting

$$
\mathbf{f}(\lambda) = \mathbf{f}_0(\lambda)I + \mathbf{f}_1(\lambda)J
\tag{27}
$$

which is a result of the $\mathbb{Z}_2$ grading (6) of the Lie algebra $\mathfrak{sl}(3)$.

A special type of fundamental solutions are the so-called Jost solutions $\psi_\pm$ which are normalized as follows

$$
\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)\mathrm{e}^{-\mathrm{i}\lambda Jx}g_\pm^{-1} = \mathbb{1}.
\tag{28}
$$

Due to (25) one can show that the asymptotic behavior of $\psi_\pm$ do not depend on time and thus the definition is correct. The transition matrix

$$
T(t,\lambda) = [\psi_+(x,t,\lambda)]^{-1}\psi_-(x,t,\lambda)
\tag{29}
$$

is called scattering matrix. It can be easily deduced from relation (24) that the scattering matrix evolves with time according to the linear differential equation

$$
\mathrm{i}\partial_t T + [\mathbf{f}(\lambda), T] = 0
\tag{30}
$$

which is integrated straight away to give

$$
T(t,\lambda) = \mathrm{e}^{\mathrm{i}\mathbf{f}(\lambda)t}T(0,\lambda)\mathrm{e}^{-\mathrm{i}\mathbf{f}(\lambda)t}.
\tag{31}
$$

From now on the parameter t will be fixed and we shall omit it to simplify our notation. Due to reasons of simplicity we set $\phi_+ = \phi_- = 0$ as well.

The action of $\mathbb{Z}_2$-reductions (4), (5) imposes the following restrictions

$$
\begin{aligned}
\left[\psi_\pm^\dagger(x,\lambda^*)\right]^{-1} &= \psi_\pm(x,\lambda), & \left[T^\dagger(\lambda^*)\right]^{-1} &= T(\lambda) \\
\mathbf{C}\psi_\pm(x,-\lambda)\mathbf{C} &= \psi_\pm(x,\lambda), & \mathbf{C}T(-\lambda)\mathbf{C} &= T(\lambda)
\end{aligned}
\tag{32}
$$

on the Jost solutions and the scattering matrix.

The continuous spectrum of L fills up the real axis in the complex λ-plane. Thus the λ-plane is divided into two regions denoted by $\mathbb{C}_+$ (the upper half plane) and $\mathbb{C}_-$ (the lower half plane). These regions represent domains for fundamental solutions $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ to be analytic functions in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively

[16]. The fundamental analytic solutions (FAS) can be constructed by using Gauss factors in the decomposition of the scattering matrix

$$T(\lambda) = T^{\mp}(\lambda)D^{\pm}(\lambda)(S^{\pm}(\lambda))^{-1}. \tag{33}$$

The matrices S^{+} and T^{+} are upper triangular, S^{-} and T^{-} are lower triangular and $D^{\pm}$ are diagonal ones. Then χ^{+} and χ^{-} are expressed as follows

$$\chi^{\pm}(x,\lambda) = \psi_{-}(x,\lambda)S^{\pm}(\lambda) = \psi_{+}(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda). \tag{34}$$

Due to relations (34) the FAS can be interpreted as solutions to a local Riemann problem

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G(x,\lambda), \qquad G(\lambda) = (S^{-}(\lambda))^{-1}S^{+}(\lambda). \tag{35}$$

The established interrelation between the inverse scattering method and the Riemann problem plays an important role in constructing solutions to NLEEs through dressing method.

It can be shown that the reduction conditions (32) and equation (33) lead to the following demands on the Gauss factors

$$\left[S^{+}(\lambda^{*})\right]^{\dagger} = [S^{-}(\lambda)]^{-1}, \qquad \tilde{C}S^{\pm}(-\lambda)\tilde{C} = S^{\mp}(\lambda)$$

$$\left[T^{+}(\lambda^{*})\right]^{\dagger} = [T^{-}(\lambda)]^{-1}, \qquad \tilde{C}T^{\pm}(-\lambda)\tilde{C} = T^{\mp}(\lambda) \tag{36}$$

$$\left[D^{+}(\lambda^{*})\right]^{\dagger} = [D^{-}(\lambda)]^{-1}, \qquad \tilde{C}D^{\pm}(-\lambda)\tilde{C} = D^{\pm}(\lambda)$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Finally, combining all this information we see that the FAS obey the symmetry conditions

$$\left[\chi^{+}(x,\lambda^{*})\right] = [\chi^{-}(x,\lambda)]^{-1}, \qquad C\chi^{+}(x,-\lambda)C = \chi^{-}(x,\lambda). \tag{37}$$

3. Dressing Method and Soliton Solutions

As we mentioned in the previous section the inverse scattering method is tightly related to Riemann-Hilbert problem. The Riemann-Hilbert problem possesses two types of solutions: regular ones (without singularities) and singular ones. Singular solutions can be generated by dressing regular solutions with a factor which has prescribed singularities. The simplest types of singularities are first order poles and zeroes. It can be proven that they correspond to poles of the resolvent of L. Hence they are discrete eigenvalues of the Lax operator (2). The discrete eigenvalues of L form orbits of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$. There exist two types of orbits:

generic orbits containing quadruplets of eigenvalues $\{\pm\mu, \pm\mu^*\}$ and degenerate orbits consisting of two imaginary eigenvalues $\pm i\kappa$ (doublets).

There is a very deep connection between singular solutions to Riemann-Hilbert problem and soliton solutions to the corresponding nonlinear problem. In the present section we are going to analyze the soliton solutions to the system (14). For this to be done, we are going to apply the dressing method proposed in [38] and developed in [28, 29, 35, 36]. We demonstrate that the NLEE (14) has two types of one-soliton solutions: doublet soliton to be connected with two imaginary discrete eigenvalues of L and quadruplet soliton connected to four eigenvalues.

3.1. Rational Dressing

The dressing method is an indirect method for solving a NLEE possessing a Lax representation. This means that it allows one to generate a solution to the NLEE starting from a known one. Let us assume we know a solution

$$L_1^{(0)} = \begin{pmatrix} 0 & u_0 & v_0 \\ u_0^* & 0 & 0 \\ v_0^* & 0 & 0 \end{pmatrix}$$

of (14) and a fundamental solution $\psi_0(x, t, \lambda)$ of the auxiliary linear problems

$$L^{(0)}(\lambda)\psi_0 = i\partial_x\psi_0 + \lambda L_1^{(0)}\psi_0 = 0$$

$$A^{(0)}(\lambda)\psi_0 = i\partial_t\psi_0 + \sum_{k=1}^{N} \lambda^k A_k^{(0)}\psi_0 = 0. \tag{38}$$

Then one constructs another function $\psi_1(x, t, \lambda) = \Phi(x, t, \lambda)\psi_0(x, t, \lambda)$ to be a common solution to

$$L^{(1)}(\lambda)\psi_1 = i\partial_x\psi_1 + \lambda L_1^{(1)}\psi_1 = 0$$

$$A^{(1)}(\lambda)\psi_1 = i\partial_t\psi_1 + \sum_{k=1}^{N} \lambda^k A_k^{(1)}\psi_1 = 0 \tag{39}$$

where the potential

$$L_1^{(1)} = \begin{pmatrix} 0 & u_1 & v_1 \\ u_1^* & 0 & 0 \\ v_1^* & 0 & 0 \end{pmatrix}$$

is to be found. From (38) and (39) it follows that the dressing factor $\Phi(x,t,\lambda)$ satisfies the following equations

$$i\partial_x \Phi + \lambda L_1^{(1)} \Phi - \lambda \Phi L_1^{(0)} = 0 \tag{40}$$

$$i\partial_t \Phi + \sum_{k=1}^{N} \lambda^k A_k^{(1)} \Phi - \Phi \sum_{k=1}^{N} \lambda^k A_k^{(0)} = 0. \tag{41}$$

We also assume that the dressing factor is regular at $|\lambda| \to 0, \infty$. Then from (40) one can derive the following relation between $L_1^{(1)}$ and $L_1^{(0)}$

$$L_1^{(1)}(x,t) = \Phi(x,t,\infty) L_1^{(0)}(x,t) \Phi^\dagger(x,t,\infty). \tag{42}$$

This equation will play a central role in our further considerations since it allows one to generate a new solution to (14) from the given one $L_1^{(0)}$.

Due to the reduction conditions (4), (5) the dressing factor obeys the symmetries

$$\mathbf{C}\Phi(x,t,-\lambda)\mathbf{C} = \Phi(x,t,\lambda) \tag{43}$$

$$\Phi(x,t,\lambda)\Phi^\dagger(x,t,\lambda^*) = \mathbb{1}. \tag{44}$$

In order to obtain a nontrivial dressing we choose $\Phi(x,t,\lambda)$ as a rational function[1] of λ with a minimal number of simple poles. At first we shall consider the case when these poles are generic complex numbers. Hence the dressing factor looks as follows

$$\Phi(x,t,\lambda) = \mathbb{1} + \frac{\lambda M(x,t)}{\lambda - \mu} + \frac{\lambda \mathbf{C} M(x,t) \mathbf{C}}{\lambda + \mu} \tag{45}$$

where $\Re\mu \neq 0$, $\Im\mu \neq 0$. It is evident that the reduction condition (43) is fulfilled. On the other hand (44) leads to the conclusion that

$$\Phi^{-1}(x,t,\lambda) = \mathbb{1} + \frac{\lambda M^\dagger(x,t)}{\lambda - \mu^*} + \frac{\lambda \mathbf{C} M^\dagger(x,t) \mathbf{C}}{\lambda + \mu^*}. \tag{46}$$

The identity $\Phi(\lambda)\Phi^{-1}(\lambda) = \mathbb{1}$ must hold for any λ. Therefore after equating the residue at $\lambda = \mu^*$ to 0 one gets the equation

$$\left(\mathbb{1} + \frac{\mu^* M(x,t)}{\mu^* - \mu} + \frac{\mu^* \mathbf{C} M(x,t) \mathbf{C}}{\mu^* + \mu} \right) M^\dagger(x,t) = 0. \tag{47}$$

The rest of algebraic relations can be reduced to (47) due to the symmetry conditions (4), (5).

---

[1]If Φ is λ-independent then it does not depend on x and t either. Thus (42) produces simply a unitary transformation of $L_1^{(0)}$ which is not essential because of U(2) gauge symmetry of the model.

The residue M ought to be singular since otherwise it should be proportional to $\mathbb{1}$ and the dressing becomes trivial. It suffices to consider the case $\mathrm{rank}M = 1$. Then M can be decomposed in the following manner

$$M = |n\rangle\langle m|, \qquad |n\rangle = (n_1, n_2, n_3)^T, \qquad \langle m| = (m_1^*, m_2^*, m_3^*). \qquad (48)$$

After substituting this representation into (47) one derives a linear system for the three-vector $|n\rangle$

$$|m\rangle - \frac{\mu^*|n\rangle\langle m|m\rangle}{2\mathrm{i}\kappa} + \frac{\mu^*\mathbf{C}|n\rangle\langle m|\mathbf{C}|m\rangle}{2\omega} = 0 \qquad (49)$$

where we have used the notation $\omega = \Re\mu$, $\kappa = \Im\mu$. The solution of (49) reads

$$|n\rangle = \frac{1}{\mu^*}\left(\frac{\langle m|m\rangle}{2\mathrm{i}\kappa} - \frac{\langle m|\mathbf{C}|m\rangle}{2\omega}\mathbf{C}\right)^{-1}|m\rangle. \qquad (50)$$

The vector $|m\rangle$ is an element of the projective space $\mathbb{C}P^2$. Indeed, it is evident that a rescaling $|m\rangle \to h|m\rangle$ with any complex $h \neq 0$ does not change the matrix M. Taking into account the ansatz (45) one can rewrite (42) as

$$L_1^{(1)} = (\mathbb{1} + M + \mathbf{C}M\mathbf{C})L_1^{(0)}(\mathbb{1} + M + \mathbf{C}M\mathbf{C})^\dagger. \qquad (51)$$

Notice that the dressing procedure preserves the matrix structure of L since the factor $\mathbb{1} + M + \mathbf{C}M\mathbf{C}$ is a block-diagonal matrix.

We have expressed all quantities needed in terms of $|m\rangle$ and now it remains to find $|m\rangle$ itself. For that purpose we rewrite equations (40), (41) in the form

$$\Phi(x, t, \lambda)\left(\mathrm{i}\partial_x + \lambda L_1^{(0)}\right)\Phi^{-1}(x, t, \lambda) = \lambda L_1^{(1)}$$

$$\Phi(x, t, \lambda)\left(\mathrm{i}\partial_t + \sum_{k=1}^{N}\lambda^k A_k^{(0)}\right)\Phi^{-1}(x, t, \lambda) = \sum_{k=1}^{N}\lambda^k A_k^{(1)}. \qquad (52)$$

It is obviously satisfied at $\lambda = 0$. After equating the residues of (52) at $\lambda = \mu^*$ to 0 we obtain a set of the differential equations

$$\left(\mathbb{1} + \frac{\mu^* M}{\mu^* - \mu} + \frac{\mu^*\mathbf{C}M\mathbf{C}}{\mu^* + \mu}\right)\left(\mathrm{i}\partial_x + \mu^* L_1^{(0)}\right)|m\rangle = 0$$

$$\left(\mathbb{1} + \frac{\mu^* M}{\mu^* - \mu} + \frac{\mu^*\mathbf{C}M\mathbf{C}}{\mu^* + \mu}\right)\left(\mathrm{i}\partial_t + \sum_{k=1}^{N}(\mu^*)^k A_k^{(0)}\right)|m\rangle = 0. \qquad (53)$$

Taking into acount (47) the equations above can be reduced to

$$\left(\mathrm{i}\partial_x + \mu^* L_1^{(0)}(x, t)\right)|m(x, t)\rangle = h(x, t)|m(x, t)\rangle$$

$$\left(\mathrm{i}\partial_t + \sum_{k=1}^{N}(\mu^*)^k A_k^{(0)}(x, t)\right)|m(x, t)\rangle = h(x, t)|m(x, t)\rangle \qquad (54)$$

for some arbitrary function h. At this point we recall that the vectors in the decomposition (48) are not uniquely determined. Indeed, the operation $|n\rangle \rightarrow B^{-1}|n\rangle$ and $|m\rangle \rightarrow B^\dagger|m\rangle$ for any nondegenerate 3×3 matrix B produces another decomposition of M. It is not hard to see that it is always possible to choose B in such a way that $h \equiv 0$ is fulfilled. Thus from (54) it follows that $|m(x,t)\rangle$ is proportional to some fundamental solution $\psi_0(x, t, \lambda)$ of the bare linear problem, namely

$$|m(x,t)\rangle = \psi_0(x,t,\mu^*)|m_0\rangle \tag{55}$$

where $|m_0\rangle \in \mathbb{C}^3\backslash\{0\}$ is a constant vector of integration. The new solution $L_1^{(1)}$ of (14) and the solution $\psi_1(x, t, \lambda)$ of the corresponding linear system are parameterized by a complex number μ and a complex three-vector $|m_0\rangle$.

Thus we have proved the following

Proposition 3. *Let $L_1^{(0)}$ be a solution of (14) and $\psi_0(x, t, \lambda)$ be a common solution to (38). Let also μ be a complex number to fulfill $\Re \mu \neq 0$, $\Im \mu > 0$ and $|m_0\rangle \in \mathbb{C}^3\backslash\{0\}$. Then the matrix-valued function $L_1^{(1)}(x, t)$ defined by (51) where $M = |n\rangle\langle m|$ is determined by (50) and (55) is a solution to (14) as well. The corresponding fundamental solution $\psi_1(x, t, \lambda)$ of (39) is given by $\psi_1 = \Phi\psi_0$ where $\Phi(x, t, \lambda)$ is determined by (45), (48), (50) and (55).*

Let us now consider the case when the poles of the dressing factor are imaginary, i.e., we have

$$\Phi(x,t,\lambda) = \mathbb{1} + \lambda \left(\frac{M(x,t)}{\lambda - \mathrm{i}\kappa} + \frac{\mathbf{C}M(x,t)\mathbf{C}}{\lambda + \mathrm{i}\kappa} \right), \qquad \kappa \neq 0. \tag{56}$$

Then Φ^{-1} has the same poles as Φ and therefore the equality $\Phi\Phi^{-1} = \mathbb{1}$ already contains second order poles. In this case the natural requirement of vanishing of the matrix coefficients before $(\lambda - \mathrm{i}\kappa)^{-2}$ and $(\lambda - \mathrm{i}\kappa)^{-1}$ leads to the algebraic relations

$$M\mathbf{C}M^\dagger = 0 \tag{57}$$

$$\left(\mathbb{1} + M + \frac{\mathbf{C}M\mathbf{C}}{2} \right) \mathbf{C}M^\dagger\mathbf{C} + M \left(\mathbb{1} + \mathbf{C}M^\dagger\mathbf{C} + \frac{M^\dagger}{2} \right) = 0. \tag{58}$$

As before in order to obtain a nontrivial result M is required to be a degenerate matrix, i.e., the decomposition (48) holds true. Then relation (57) is rewritten as

$$\langle m|\mathbf{C}|m\rangle = 0. \tag{59}$$

The relation (58) in its turn can be easily reduced to the following linear system for three-vector $|n\rangle$

$$\left(\mathbb{1} + \frac{\mathbf{C}|n\rangle\langle m|\mathbf{C}}{2} \right) \mathbf{C}|m\rangle = \mathrm{i}\sigma|n\rangle \tag{60}$$

by introducing some auxiliary real function σ. That linear system allows one to express $|n\rangle$ through $\langle m|$ and σ, namely

$$|n\rangle = \left(\mathrm{i}\sigma - \frac{\langle m|m\rangle}{2}\mathbf{C}\right)^{-1}\mathbf{C}|m\rangle. \tag{61}$$

In order to find $|m\rangle$ and σ we turn back to the equations (52). Vanishing of the second order poles in (52) leads to the conclusion that

$$|m(x,t)\rangle = \psi_0(x,t,-\mathrm{i}\kappa)|m_0\rangle \tag{62}$$

where $|m_0\rangle$ is a constant nonzero three-vector. After substituting (62) into (59) and taking into account (4) one convinces himself that the components of the polarization vector $|m_0\rangle$ are no longer independent but satisfy the constraint

$$\langle m_0|\mathbf{C}|m_0\rangle = 0 \qquad \Leftrightarrow \qquad |m_{0,1}|^2 = |m_{0,2}|^2 + |m_{0,3}|^2. \tag{63}$$

The vanishing condition of the first order poles leads to some differential constraint on $\sigma(x,t)$ which is integrated to give

$$\sigma(x,t) = -\kappa\langle m_0|\psi^{-1}(x,t,\mathrm{i}\kappa)\dot{\psi}_0(x,t,\mathrm{i}\kappa)\mathbf{C}|m_0\rangle + \sigma_0 \tag{64}$$

where $\sigma_0 \in \mathbb{R}$ is a costant of integration.

Thus to calculate the soliton solution itself one just substitutes the result for $|n\rangle$ and $|m\rangle$ into M and uses formula (51). As it is seen the new solution is parametrized by the polarization vector $|m_0\rangle$, the real number σ_0 and the pole $\mathrm{i}\kappa$. All this can be formulated as the following

Proposition 4. *Let there be given a solution $L^{(0)}(x,t)$ to (14), a common solution $\psi_0(x,t,\lambda)$ to (38), real numbers $\kappa > 0$, σ_0 and a complex nonzero vector $|m_0\rangle$ satisfying (63). Then the function $L_1^{(1)}(x,t)$ determined by (51), (48), (61), (62) and (64) is a solution of the system (14) too. The solution $\psi_1(x,t,\lambda)$ of the dressed linear system (39) is given by $\psi_1 = \Phi\psi_0$ where Φ is defined by (56), (48), (61), (62) and (64).*

One can apply the dressing procedure repeatedly to build a sequence of exact solutions

$$L_1^{(0)} \xrightarrow{\Phi_1} L_1^{(1)} \xrightarrow{\Phi_2} \ldots \xrightarrow{\Phi_N} L_1^{(N)}. \tag{65}$$

More precisely this alternative procedure will be explained in Section 4.

3.2. Soliton Solutions

Let us apply the dressing procedure to the following seed solution

$$L_1^{(0)}(x,t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{66}$$

of equation (14). In this case a fundamental solution to (38) reads

$$\psi_0(x,t,\lambda) = \begin{pmatrix} \cos z(x,t) e^{\frac{i f_0(\lambda)t}{3}} & 0 & i \sin z(x,t) e^{\frac{i f_0(\lambda)t}{3}} \\ 0 & e^{\frac{-2 i f_0(\lambda)t}{3}} & 0 \\ i \sin z(x,t) e^{\frac{i f_0(\lambda)t}{3}} & 0 & \cos z(x,t) e^{\frac{i f_0(\lambda)t}{3}} \end{pmatrix} \qquad (67)$$

where $z(x,t) = \lambda x + \mathbf{f}_1(\lambda)t$. We recall that $\mathbf{f}_0(\lambda)$ and $\mathbf{f}_1(\lambda)$ are even and odd part of the dispersion law induced by the $\mathbb{Z}_2$ grading of $\mathfrak{sl}(3)$, see (27).

We are going to consider the generation of a quadruplet soliton first. In this case one uses factor (45). It is convenient to decompose the polarization vector $|m_0\rangle$ according to the eigensubspaces of the endomorphism ψ_0 (67)

$$|m_0\rangle = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (68)$$

where α, β, γ are arbitrary complex constants.

If the vector $|m_0\rangle$ is proportional to one of the eigenvectors of the endomorphism ψ_0, then the corresponding matrix M does not depend on the variables x and t (due to the projective nature of the vector $|m\rangle$) and the corresponding solution (51) is a simple unitary rotation of the constant solution $L_1^{(0)}$.

Thus elementary solitons correspond to vectors $|m_0\rangle$, belonging to essentially two-dimensional invariant subspaces of ψ_0, i.e., they correspond to polarization vectors with only one zero coefficient in the expansion (68). Let us consider each of these three cases in more detail.

Case i) $\alpha \neq 0, \beta \neq 0, \gamma = 0$

The one-soliton solution is given by

$$u_1(x,t) = 0$$

$$v_1(x,t) = \exp\left\{ 4i \arctan\left(\frac{\kappa \cos(2\omega x + 2\mathbf{f}_1^R(\mu)t + \phi_\alpha - \phi_\beta)}{\omega \cosh(2\kappa x + 2\mathbf{f}_1^I(\mu)t + \ln|\alpha/\beta|)} \right) \right\} \qquad (69)$$

where $\phi_\alpha = \arg \alpha$, $\phi_\beta = \arg \beta$. Here $\mathbf{f}_1^R(\lambda)$ and $\mathbf{f}_1^I(\lambda)$ are the real and the imaginary part of the polynomial $\mathbf{f}_1(\lambda)$ (respectively $\mathbf{f}_0^R(\lambda)$ and $\mathbf{f}_0^I(\lambda)$ stand for the real and imaginary part of $\mathbf{f}_0(\lambda)$ to be used later on). If the dispersion law of NLEE is an even polynomial ($\mathbf{f}_1(\lambda) \equiv 0$) then the function v_1 becomes stationary

$$v_1(x,t) = \exp\left\{ 4i \arctan\left(\frac{\kappa \cos(2\omega x + \phi_\alpha - \phi_\beta)}{\omega \cosh(2\kappa x + \ln|\alpha/\beta|)} \right) \right\}. \qquad (70)$$

A plot of that solution is presented in Fig. 1. It is easy to check that $u = 0$, $v = \exp(if(x))$ is an exact solution of (18) for any differentiable function $f(x)$
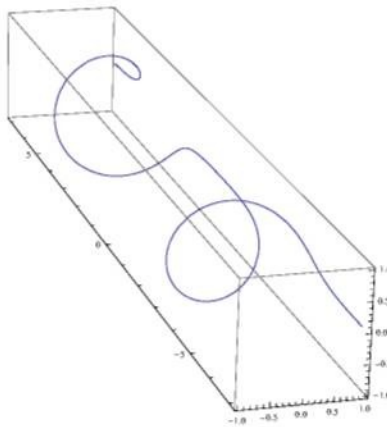
Figure 1. Real and imaginary parts of the component v_1 in 70 as a function of x. Here $\kappa = 1, \omega = 10^{-3}, \alpha = 1, \beta = 1 + i$.

tending to 0 when $x \rightarrow \pm\infty$. This resembles the case of the three-wave equation [33] where one wave of an arbitrary shape is an exact solution of the system and the two other waves are identically zero. The solution (70) has a simple spectral characterisation and an explicitly given analytic fundamental solution of the corresponding linear problem.

If the dispersion law contains odd powers of λ as well then the elementary soliton is no more stationary. For example in the case of equation (19) it reads

$$u_1(x, t) = 0, \qquad v_1(x, t) = \exp(4i \arctan \zeta_{\text{cub}}(x, t))$$

$$\zeta_{\text{cub}}(x, t) = \left[\frac{\kappa \cos 2\omega[x + 8(3\kappa^2 - \omega^2)t + (\phi_\alpha - \phi_\beta)/2\omega]}{\omega \cosh 2\kappa[x + 8(\kappa^2 - 3\omega^2)t + \ln|\alpha/\beta|/2\kappa]} \right]. \tag{71}$$

Case ii) $\alpha \neq 0, \beta = 0, \gamma \neq 0$

In this case the solution looks as follows

$$u_1(x, t) = \frac{4i\omega\kappa Q^*_{\text{gen}} \exp i\{\omega x + (\mathbf{f}_0^R(\mu) + \mathbf{f}_1^R(\mu))t + \phi_\alpha - \phi_\gamma\}}{(\omega - i\kappa)Q^2_{\text{gen}}}$$

$$v_1(x, t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q^2_{\text{gen}}} \tag{72}$$

where $\phi_\alpha = \arg \alpha$, $\phi_\gamma = \arg \gamma$ and

$$Q_{\text{gen}} = 2\omega e^{\kappa x + (\mathbf{f}_0^I(\mu) + \mathbf{f}_1^I(\mu))t + \ln|\alpha/\gamma|} + (\omega + i\kappa)e^{-\kappa x - (\mathbf{f}_0^I(\mu) + \mathbf{f}_1^I(\mu))t - \ln|\alpha/\gamma|}.$$
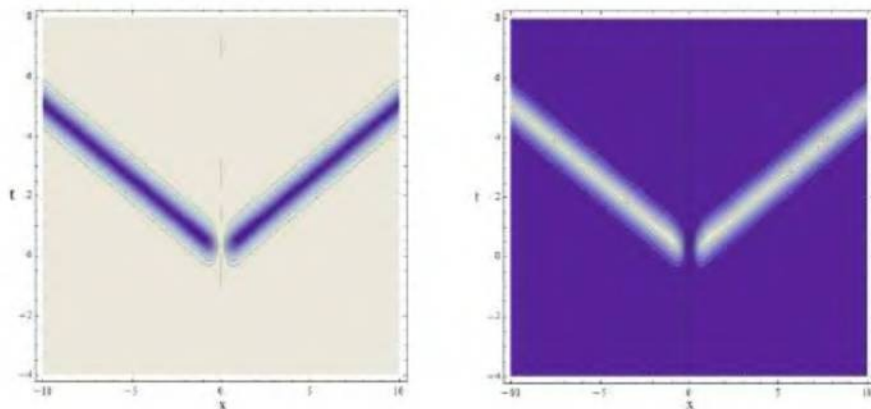
Figure 2. Contour plot of $|u_1|^2$ (left panel) and $|v_1|^2$ (right panel) for a generic soliton solution (73) as a function of x and t where $\alpha = \gamma = \kappa = \omega = 1$.

In particular, when $\mathbf{f}(\lambda) = -\lambda^2 I$, i.e., $\mathbf{f}_0(\lambda) = -\lambda^2$ and $\mathbf{f}_1(\lambda) = 0$ hold, we obtain a solution to (18)

$$u_1(x,t) = \frac{4i\omega\kappa Q^* \exp i\{\omega x + (\kappa^2 - \omega^2)t + \phi_\alpha - \phi_\gamma\}}{(\omega - i\kappa)Q^2}$$

$$v_1(x,t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q^2}$$

(73)

where

$$Q = 2\omega e^{\kappa(x-2\omega t)+\ln|\alpha/\gamma|} + (\omega + i\kappa)e^{-\kappa(x-2\omega t)-\ln|\alpha/\gamma|}.$$

Contour plots of $|u_1|^2$ and $|v_1|^2$ of the solutions (73) are shown in Fig. 2.

When the dispersion law is odd, say $\mathbf{f}_1(\lambda) = -8\lambda^3$, the quadruplet solution represents a travelling wave of the form

$$u_1(x,t) = \frac{4i\omega\kappa Q^* \exp i\omega[x + 8(3\kappa^2 - \omega^2)t + (\phi_\alpha - \phi_\gamma)/\omega]}{(\omega - i\kappa)Q^2}$$

$$v_1(x,t) = 1 - \frac{8\omega\kappa^2}{(\omega - i\kappa)Q^2}$$

(74)

where

$$Q = 2\omega e^{\kappa(x+8(\kappa^2-3\omega^2)t+\ln|\alpha/\gamma|/\kappa)} + (\omega + i\kappa)e^{-\kappa(x+8(\kappa^2-3\omega^2)t+\ln|\alpha/\gamma|/\kappa)}.$$

This is an elementary soliton for the cubic flow NLEE (19).

Case iii) $\alpha = 0, \beta \neq 0, \gamma \neq 0$

The solution now can be obtained from the solution in the case (ii), by changing $\alpha \to \beta$ and $x \to -x$.

In the cases ii) the solution (73) is a soliton of width $1/\kappa$ moving with velocity 2ω. The corresponding soliton in the case iii) moves with a velocity -2ω.

In the generic case, when all three constants are non-zero, the solution represents a nonlinear deformation of the above described solitons. For $\kappa > 0$ it may be viewed as a decay of unstable time independent soliton from the case i) into two solitons, corresponding to the cases ii) and iii) (see Fig. 2). For $\kappa < 0$, the solution is a fusion of two colliding solitons into a stationary one.

Let us now consider dressing by a factor with two imaginery poles (doublet case), i.e., $\mu = i\kappa$. There are two essentially different cases.

Case i) $\alpha \neq 0, \beta \neq 0, \gamma = 0$

From (63) it follows that $|m_{0,1}| = |m_{0,3}|$. It suffices to pick up $m_{0,1} = 1$ and the third component is $m_{0,3} = \exp(i\varphi)$, $\varphi \in \mathbb{R}$. The doublet solution reads

$$u_1(x,t) = 0, \qquad v_1(x,t) = \exp(4i \arctan \Xi_{\text{gen}}(x,t))$$

$$\Xi_{\text{gen}}(x,t) = \frac{\sigma_0 - 2\kappa(x + \dot{\mathbf{f}}_1(i\kappa)t)\sin\varphi}{\cosh 2(\kappa x + \mathbf{f}_1^I(i\kappa)t) + \sinh 2(\kappa x + \mathbf{f}_1^I(i\kappa)t)\cos\varphi}. \tag{75}$$

If the dispersion law of NLEE is even polynomial, i.e., $\mathbf{f}_1(\lambda) \equiv 0$, then v_1 becomes stationary

$$v_1(x,t) = \exp\left\{4i \arctan\left(\frac{\sigma_0 - 2\kappa x \sin\varphi}{\cosh 2\kappa x + \sinh 2\kappa x \cos\varphi}\right)\right\}. \tag{76}$$

Figure 3 presents the argument and the imaginary part of $v_1(x)$ in the stationary case as functions of x and the phase φ.

As in the quadruplet case if the dispersion law is an odd polynomial the doublet solution is time-depending. Let us consider the simplest example $\mathbf{f}_1(\lambda) = -8\lambda^3$ corresponding to equation (19). Now (75) obtains the form

$$u_1(x,t) = 0, \qquad v_1(x,t) = \exp(4i \arctan \Xi_{\text{cub}}(x,t))$$

$$\Xi_{\text{cub}}(x,t) = \frac{\sigma_0 - 2\kappa(x + 24\kappa^2 t)\sin\varphi}{\cosh 2\kappa(x + 8\kappa^2 t) + \sinh 2\kappa(x + 8\kappa^2 t)\cos\varphi}. \tag{77}$$

Case ii) Generic Doublet

Now let us assume $m_{0,2} \neq 0$. For simplicity we fix $m_{0,2} = 1$. Then the norms of $m_{0,1}$ and $m_{0,3}$ are interrelated through the equality

$$|m_{0,1}|^2 - |m_{0,3}|^2 = 1.$$
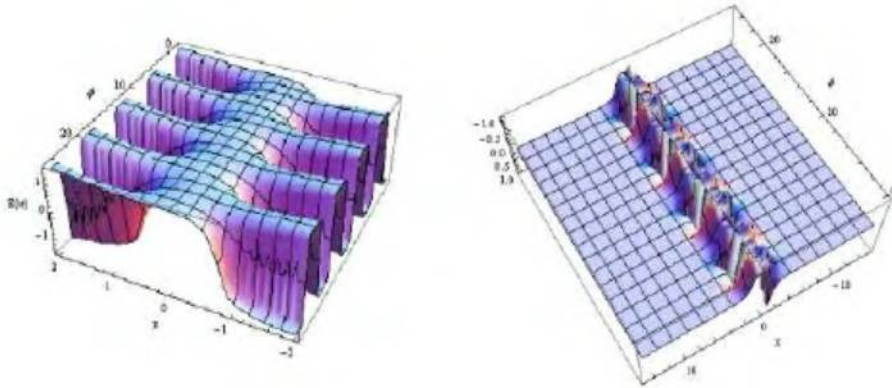
Figure 3. Plots of the argument (left panel) and $\operatorname{Im} v_1(x)$ (right panel) for the stationary solution (76) as a function of x and φ, $\kappa = \sigma = 1$.

This is why it proves to be convenient to parametrize them as follows

$$m_{0,1} = \cosh\theta_0 e^{i(\varphi_0 + \tilde\varphi)}, \qquad m_{0,3} = |\sinh\theta_0| e^{i(\varphi_0 - \tilde\varphi)} \tag{78}$$

where θ_0, φ_0 and $\tilde\varphi$ are arbitrary real numbers. Then the doublet soliton solution reads

$$u_1(x,t) = \frac{2\Delta^*}{\Delta^2} e^{i(\mathbf{f}_0(i\kappa)t + \varphi_0)} [\sinh\theta_+ \cos\tilde\varphi + i\sinh\theta_- \sin\tilde\varphi]$$

$$v_1(x,t) = 1 + \frac{2(2i\sigma - 1)}{\Delta} + \frac{4i\sigma(i\sigma - 1)}{\Delta^2} \tag{79}$$

where

$$\Delta(x,t) = \cosh^2\theta_+ \cos^2\tilde\varphi + \cosh^2\theta_- \sin^2\tilde\varphi - i\sigma$$

$$\sigma(x,t) = \sigma_0 + \kappa\dot{\mathbf{f}}_0^I(i\kappa)t + \kappa\left(x + \dot{\mathbf{f}}_1(i\kappa)t\right)\sinh 2\theta_0 \sin 2\tilde\varphi$$

$$\theta_\pm(x,t) = \kappa x + \mathbf{f}_1^I(i\kappa)t \pm \theta_0.$$

Let us consider the special case when the dispersion law is $-\lambda^2 I$. The solution (79) is significantly simplified if in addition one assumes that $m_{0,3}/m_{0,1} > 0$ ($\tilde\varphi = 0$). The result reads

$$u_1 = \frac{2\left(\cosh^2(\kappa x + \theta_0) + i(\sigma_0 - 2\kappa^2 t)\right)}{\left(\cosh^2(\kappa x + \theta_0) - i(\sigma_0 - 2\kappa^2 t)\right)^2} e^{i(\kappa^2 t + \varphi_0)}\sinh(\kappa x + \theta_0)$$

$$v_1 = \left(\frac{\cosh^2(\kappa x + \theta_0) + i(\sigma_0 - 2\kappa^2 t)}{\cosh^2(\kappa x + \theta_0) - i(\sigma_0 - 2\kappa^2 t)}\right)^2 \tag{80}$$

$$- \frac{2}{\left(\cosh^2(\kappa x + \theta_0) - i(\sigma_0 - 2\kappa^2 t)\right)^2}.$$
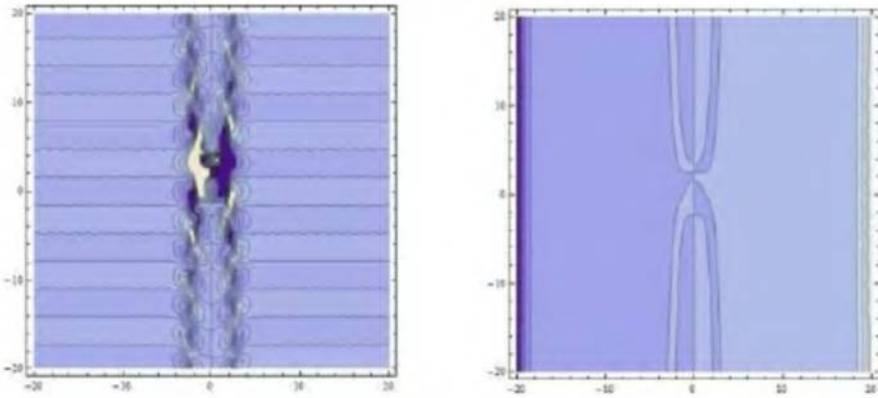
Figure 4. Contour plot of $\Re u_1(x,t)$ (left panel) and $\Re v_1(x,t)$ (right panel) for doublet soliton (81) as functions of x and t. Here $\kappa = 0$, $\sigma_0 = 5$ and $\theta_0 = 0$.

A plot of $\Re u_1(x,t)$ and $\Re v_1(x,t)$ is shown in Fig. 4.

It proves to be of some interest to consider the odd dispersion case as well. In the simplest nontrivial situation when $\mathbf{f}_1(\lambda) = -8\lambda^3$ (equation (19)) we have

$$
u_1 = \frac{2\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) + i\sigma_0\right)}{\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - i\sigma_0\right)^2}\, e^{i\varphi_0} \sinh(\kappa x + 8\kappa^3 t + \theta_0)
$$

$$
v_1 = \left(\frac{\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) + i\sigma_0}{\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - i\sigma_0}\right)^2 \tag{81}
$$

$$
- \frac{2}{\left(\cosh^2(\kappa x + 8\kappa^3 t + \theta_0) - i\sigma_0\right)^2}.
$$

We have assumed above that $\tilde{\varphi} = 0$.

Remark 5. *Let us make a few short remarks on the behaviour of doublet soliton (81). First of all it is evident that this **is not** a travelling wave solution. Moreover, as it is seen from* Fig. 5 *the component* $|u_1(x,t)|^2$ *has two symmetric maxima and one minimum at the origin (respectively* $|v_1(x,t)|^2$ *has two symmetric minima and one maximum at the origin). The value of the maximum of* $|u_1(x,t)|^2$ *(respectively the minimum of* $|v_1(x,t)|^2$*) first increases with time* $(\sigma(t) > 0)$ *and then decreases* $(\sigma(t) < 0)$*. The maxima positions of* u_1 *depend on* t *according to*

$$
\xi_0(t) = -\frac{\theta_0}{\kappa} + \frac{1}{\kappa}\ln\left(\sqrt{1 + \sqrt{1 + \sigma^2(t)}} + \sqrt[4]{1 + \sigma^2(t)}\right) \tag{82}
$$
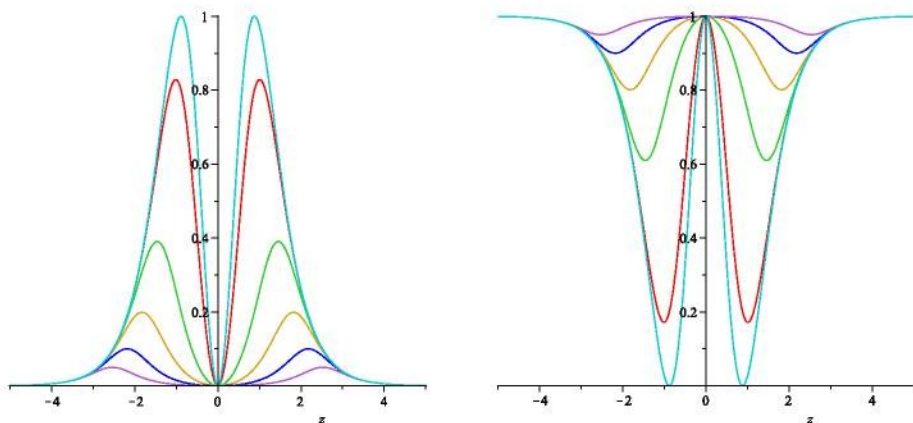
Figure 5. Contour plot of $|u_1(x,t)|^2$ (left panel) and $|v_1(x,t)|^2$ (right panel) for doublet soliton solution (81) as a function of x for several values of t: $t = 0, 1, 5, 10, 20, 40$.
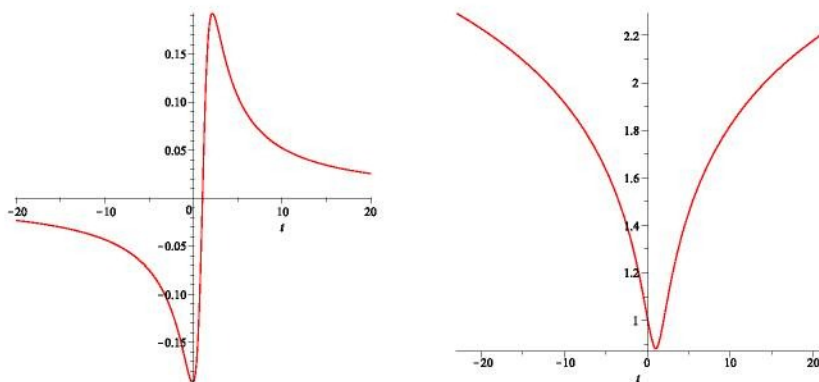
Figure 6. The soliton velocity $v(t)$ and position of the maxima $\xi_0(t)$ of solution (81) as a functions of t. Here $\kappa = \sigma_0 = 1$, $\theta_0 = 0$ and $\tilde{\varphi} = 0$.

where $\sigma(t) = \sigma_0 - 2\kappa^2 t$. *The soliton velocity* $v := \mathrm{d}\xi_0/\mathrm{d}t$ **is not constant** *but changes with* t *as given by*

$$v(t) = -\frac{2\kappa^2 t \sigma(t)}{1 + \sigma^2(t)} \frac{\sqrt[4]{1 + \sigma^2(t)}}{\sqrt{1 + \sqrt{1 + \sigma^2(t)}}}. \tag{83}$$

Such behavior resembles the boomerons and the trappons [4, 5]. *In* Fig. 6 *it is plotted the* t-*dependence of the soliton velocity.*

3.3. Multisoliton Solutions

As we have already mentioned the dressing procedure can be applied several times consequently. Thus after dressing the one-soliton solution one derives a two-soliton solution, after dressing the two-soliton solution one obtains a three-soliton solution and so on. Of course, in doing this one is allowed to apply either of dressing factors (45) and (56). Therefore the multisoliton obtained will be a certain combination of quadruplet and doublet solitons. Another way of derivation the multisoliton solution consists in using a dressing factor with a proper number of poles

$$
\Phi = \mathbb{1} + \lambda \sum_{k=1}^{N_1} \left(\frac{M_k}{\lambda - \mu_k} + \frac{\mathbf{C} M_k \mathbf{C}}{\lambda + \mu_k} \right) + \lambda \sum_{l=1}^{N_2} \left(\frac{P_l}{\lambda - \mathrm{i}\kappa_l} + \frac{\mathbf{C} P_l \mathbf{C}}{\lambda + \mathrm{i}\kappa_l} \right). \quad (84)
$$

As it follows from (84) the multisoliton solution obtained will be a mixture of N_1 quadruplet solitons and N_2 doublet ones. In order to determine the residues of Φ one follows basically the same steps as in the case of a two-poles dressing factor. Firstly, the identity $\Phi\Phi^{-1} = \mathbb{1}$ implies that the residues of Φ and Φ^{-1} fulfill some algebraic restrictions. For example we have

$$
\lim_{\lambda \to \mu_k} (\lambda - \mu_k)\Phi\Phi^{-1} = M_k \Phi_k^{-1} = 0, \qquad k = 1, \ldots, N_1 \quad (85)
$$

where

$$
\Phi_k^{-1} = \mathbb{1} + \mu_k \sum_{r=1}^{N_1} \left(\frac{M_r^\dagger}{\mu_k - \mu_r^*} + \frac{\mathbf{C} M_r^\dagger \mathbf{C}}{\mu_k + \mu_r^*} \right) + \mu_k \sum_{l=1}^{N_2} \left(\frac{P_l^\dagger}{\mu_k + \mathrm{i}\kappa_l} + \frac{\mathbf{C} P_l^\dagger \mathbf{C}}{\mu_k - \mathrm{i}\kappa_l} \right).
$$

Apart of this type of constraints we have another one originating from vanishing of the coefficients before the imaginery poles

$$
\lim_{\lambda \to \mathrm{i}\kappa_l} (\lambda - \mathrm{i}\kappa_l)^2 \Phi\Phi^{-1} = (\mathrm{i}\kappa_l)^2 P_l \mathbf{C} P_l^\dagger = 0, \qquad l = 1, \ldots, N_2 \quad (86)
$$

$$
\lim_{\lambda \to \mathrm{i}\kappa_l} \partial_\lambda [(\lambda - \mathrm{i}\kappa_l)^2 \Phi\Phi^{-1}] = \mathrm{i}\kappa_l \Theta_l \mathbf{C} P_l^\dagger \mathbf{C} + \mathrm{i}\kappa_l P_l \mathbf{C} \Theta_l^\dagger \mathbf{C} = 0 \quad (87)
$$

where

$$
\Theta_l = \mathbb{1} + \mathrm{i}\kappa_l \sum_{k=1}^{N_1} \left(\frac{M_k}{\mathrm{i}\kappa_l - \mu_k} + \frac{\mathbf{C} M_k \mathbf{C}}{\mathrm{i}\kappa_l + \mu_k} \right) + P_l + \frac{\mathbf{C} P_l \mathbf{C}}{2}
$$

$$
+ \mathrm{i}\kappa_l \sum_{s \neq l}^{N_2} \left(\frac{P_s}{\mathrm{i}(\kappa_l - \kappa_s)} + \frac{\mathbf{C} P_s \mathbf{C}}{\mathrm{i}(\kappa_l + \kappa_s)} \right).
$$

Vanishing of the rest of poles of $\Phi\Phi^{-1}$ leads to algebraic constraints which coincide with (85)–(87) due to the action of $\mathbb{Z}_2$ reductions.

Since $M_k(x,t)$ and $P_l(x,t)$ must be degenerate matrices one introduces their factorizations $M_k = |n_k\rangle\langle m_k|$ and $P_l = |q_l\rangle\langle p_l|$. Substituting it into (85)–(87) we reduce the first and the third constraint to linear systems for $|n_k\rangle$ and $|q_l\rangle$

$$|m_k\rangle = \sum_{r=1}^{N_1} \mathcal{B}_{rk}|n_r\rangle + \sum_{l=1}^{N_2} \mathcal{D}_{sk}|q_s\rangle$$

$$\mathbf{C}|p_l\rangle = \sum_{r=1}^{N_1} \mathcal{E}_{rl}|n_r\rangle + \sum_{s=1}^{N_2} \mathcal{F}_{sl}|q_s\rangle \tag{88}$$

where the matrix coefficients read

$$\mathcal{B}_{rk} := \mu_k^* \left(\frac{\langle m_r|m_k\rangle}{\mu_r - \mu_k^*} - \frac{\langle m_r|\mathbf{C}|m_k\rangle}{\mu_r + \mu_k^*}\mathbf{C} \right)$$

$$\mathcal{D}_{sk} := \mu_k^* \left(\frac{\langle p_s|m_k\rangle}{i\kappa_s - \mu_k^*} - \frac{\langle p_s|\mathbf{C}|m_k\rangle}{i\kappa_s + \mu_k^*}\mathbf{C} \right)$$

$$\mathcal{E}_{rl} := -i\kappa_l \left(\frac{\langle m_r|\mathbf{C}|p_l\rangle}{i\kappa_l - \mu_k} + \frac{\langle m_r|p_l\rangle}{i\kappa_l + \mu_k}\mathbf{C} \right), \qquad \mathcal{F}_{ss} := i\sigma_s - \frac{\langle p_s|p_s\rangle}{2}\mathbf{C}$$

$$\mathcal{F}_{sl} := \kappa_l \left(\frac{\langle p_s|\mathbf{C}|p_l\rangle}{\kappa_s - \kappa_l} - \frac{\langle p_s|p_l\rangle}{\kappa_s + \kappa_l}\mathbf{C} \right), \qquad s \neq l.$$

By inverting the linear system (88) we can express $|n_r\rangle$ and $|q_s\rangle$ through all $|m_k\rangle$, $|p_l\rangle$ and σ_l and that way determine the dressing factor in terms of the latter. The vectors $|m_k\rangle$ and $|p_l\rangle$ as well as the functions σ_l can be found from the natural requirement of vanishing of the poles in (52). The result reads

$$|m_k(x,t)\rangle = \psi_0(x,t,\mu_k^*)|m_{k,0}\rangle$$

$$|p_l(x,t)\rangle = \psi_0(x,t,-i\kappa_l)|p_{l,0}\rangle \tag{89}$$

$$\sigma_l(x,t) = -\kappa_l\langle p_{l,0}|\psi^{-1}(x,t,i\kappa_l)\dot{\psi}_0(x,t,i\kappa_l)\mathbf{C}|p_{l,0}\rangle + \sigma_{l,0}.$$

Analogously to the two-poles case the components of $|p_l\rangle$ are not independent. As a result of (86) that the following relations holds true

$$\langle p_l(x,t)|\mathbf{C}|p_l(x,t)\rangle = \langle p_{l,0}|\mathbf{C}|p_{l,0}\rangle = 0. \tag{90}$$

Thus we have proved that the dressing factor in the multiple poles case is determined if one knows the initial fundamental solution $\psi_0(x,t,\lambda)$. The multisoliton solution itself can be derived through the following formula

$$L_1^{(1)}(x,t) = \Phi(x,t,\infty)L_1^{(0)}(x,t)\Phi^\dagger(x,t,\infty) \tag{91}$$

where

$$\Phi(x,t,\infty) = \mathbb{1} + \sum_{k=1}^{N_1} (M_k + \mathbf{C}M_k\mathbf{C}) + \sum_{l=1}^{N_2} (P_l + \mathbf{C}P_l\mathbf{C}).$$

From all said above it follows that the algorithm for obtaining the multisoliton solution can be presented symbolically as follows

$$L_1^{(0)} \longrightarrow \{|m_k\rangle\}_{k=1}^{N_1}, \{|p_l\rangle\}_{l=1}^{N_2}, \{\sigma_l\}_{l=1}^{N_2}$$

$$\longrightarrow \{|n_k\rangle\}_{k=1}^{N_1}, \{|q_l\rangle\}_{l=1}^{N_2} \longrightarrow \{M_k\}_{k=1}^{N_1}, \{P_l\}_{l=1}^{N_2} \longrightarrow L_1^{(1)}.$$

4. Interactions of Quadruplet Solitons

In this section we aim to study the interactions of solitons we have derived. We shall restrict ourselves with quadruplet solitons for NLEEs with odd dispersion laws. This is the simplest case since the solitons are travelling wave-type solutions. The interactions of the other types of solitons require a special treatment and will be done elsewhere.

Our study will be based on the Zakharov-Shabat scheme [34] applied to the recursive procedure (65). Their approach consists in calculating the asymptotics of generic N-soliton solution for $t \to \pm\infty$ and establishing the pure elastic character of the interactions of generic soliton, i.e., solitons travelling at different velocities. The pure elastic character of the soliton interactions is demonstrated by the fact that for $t \to \pm\infty$ the N-soliton solution splits into a sum of N one soliton solutions preserving its amplitudes and velocities. The only effect of the interaction consists in shifting the center of mass and the initial phase of the solitons.

The one-soliton dressing factor corresponding to the quadruplet case with poles at $\pm\mu_k$ is given by

$$\Phi_k(x,t,\lambda,\mu_k) = \mathbb{1} + \frac{\lambda}{\lambda - \mu_k} M_k(x,t) + \frac{\lambda}{\lambda + \mu_k} \mathbf{C} M_k(x,t)\mathbf{C}. \qquad (92)$$

The residues $M_k(x,t) = |n_k\rangle\langle m_k|$ are determined by the following equalities

$$|n_k\rangle = \frac{1}{\mu_k^*}\left(\frac{\langle m_k|m_k\rangle}{2i\kappa_k} - \frac{\langle m_k|\mathbf{C}|m_k\rangle}{2\omega_k}\mathbf{C}\right)^{-1}|m_k\rangle$$

$$|m_k(x,t)\rangle = \psi_0(x,t,\mu_k)|m_{k0}\rangle, \qquad |m_{k0}\rangle = \begin{pmatrix} \alpha_k + \beta_k \\ \gamma_k \\ \alpha_k - \beta_k \end{pmatrix}. \qquad (93)$$

Let us now outline the alternative procedure for constructing the N-soliton solutions of the NLEE (14). The idea is to apply subsequently N times the the one-soliton dressing. For simplicity we assume that all N solitons are of quadruplet type. As a result the sequence of mappings (65) allows us to constructs a sequence

of Lax operators with potentials $L_1^{(k)}$, $k = 1, \ldots, N$ and eigenfunctions

$$\chi_{(k)}^{\pm}(x,t,\lambda) = \boldsymbol{\Phi}_k(x,t,\lambda,\mu_k)\boldsymbol{\Phi}_{k-1}(x,t,\lambda,\mu_{k-1})\ldots\boldsymbol{\Phi}_1(x,t,\lambda,\mu_1)$$

$$\times \psi_0(x,t,\lambda)\boldsymbol{\Phi}_{1,-}^{\dagger}(\lambda,\mu_1)\ldots\boldsymbol{\Phi}_{k-1,-}^{\dagger}(\lambda,\mu_{k-1})\boldsymbol{\Phi}_{k,-}^{\dagger}(\lambda,\mu_k) \tag{94}$$

where

$$\boldsymbol{\Phi}_{k,-}(\lambda,\mu_k) = \lim_{x \to -\infty} \boldsymbol{\Phi}_k(x,t,\lambda,\mu_k). \tag{95}$$

The dressing factors $\boldsymbol{\Phi}_k(x,t,\lambda,\mu_k)$ are constructed in analogy with (92) as follows

$$\boldsymbol{\Phi}_k(x,t,\lambda,\mu_k) = \mathbb{1} + \frac{\lambda}{\lambda-\mu_k}\mathbf{M}_k(x,t) + \frac{\lambda}{\lambda+\mu_k}\mathbf{C}\mathbf{M}_k(x,t)\mathbf{C}$$

$$\mathbf{M}_k(x,t) = \frac{1}{\mu_k^*}\left(\frac{\langle\mathbf{m}_k|\mathbf{m}_k\rangle}{2\mathrm{i}\kappa_k} - \frac{\langle\mathbf{m}_k|\mathbf{C}|\mathbf{m}_k\rangle}{2\omega_k}\mathbf{C}\right)^{-1}|\mathbf{m}_k\rangle\langle\mathbf{m}_k| \tag{96}$$

$$|\mathbf{m}_k\rangle = \Phi_{k-1}(x,t,\mu_k,\mu_{k=1})\ldots\Phi_1(x,t,\mu_k,\mu_1)|m_k\rangle.$$

Thus for the N-soliton potential we obtain

$$L_1^{(N)}(x,t) = \lim_{\lambda \to \infty} \chi_{(N)}^{\pm}(x,t,\lambda)L_1^{(0)}.\hat{\chi}_{(N)}^{\pm}(x,t,\lambda). \tag{97}$$

Next we recall that we are considering NLEE with odd dispersion laws (14b). Their one-soliton solutions are traveling waves and depend on $Z_k = x - V_k t$, where $V_k = 1/\kappa_k \operatorname{im} f_1(\mu_k)$. In particular, for the equation (19) $\mathbf{f}_1(\lambda) = -8\lambda^3$ and $V_k = 8(3\mu_k^2 - \kappa_k^2)$. Now let us to pick up the trajectory of the N-th soliton: $Z_N \equiv x - 2\omega_N t/3 = $ fixed and evaluate the asymptotics of $L_1^{(N)}(x,t)$ for $t \to \pm\infty$ for fixed Z_N. This will allow us to see what are the effects of the soliton interactions on the N-th soliton.

In what follows we will assume that all solitons move with different velocities, i.e., $V_j \neq V_k$ for $k \neq j$. It is natural to split the solitons in two groups

$$\mathcal{M}_+ \equiv \{V_k; V_k > V_N\}, \qquad \mathcal{M}_- \equiv \{V_k; V_j < \omega_N\} \tag{98}$$

i.e., the solitons belonging to $\mathcal{M}_+$ are moving faster than the N-th soliton, while the ones belonging to $\mathcal{M}_-$ are slower.

Now we are able to calculate the limits of $\boldsymbol{\Phi}_k(x,t,\lambda)$ for $t \to \pm\infty$ for fixed Z_N. To do this we firstly need to obtain the limits of the one-soliton dressing factor for

$x \to \pm\infty$. It can be verified that

$$
\Phi_{k,-}(\lambda, \mu_k) := \lim_{x \to -\infty} \Phi_k(x, t, \lambda) = \begin{pmatrix} c_k(\lambda) & 0 & -c_k'(\lambda), \\ 0 & 1 & 0 \\ -c_k'(\lambda) & 0 & c_k(\lambda) \end{pmatrix}
$$

$$(99)$$

$$
\Phi_{k,+}(\lambda, \mu_k) := \lim_{x \to \infty} \Phi_k(x, t, \lambda) = \begin{pmatrix} c_k(\lambda) & 0 & c_k'(\lambda) \\ 0 & 1 & 0 \\ c_k'(\lambda) & 0 & c_k(\lambda) \end{pmatrix}
$$

where

$$
c_k(\lambda) = \frac{\mu_k}{\mu_k^*} \frac{\lambda^2 - |\mu_k|^2}{\lambda^2 - \mu_k^2}, \qquad c_k'(\lambda) = -\frac{\mu_k}{\mu_k^*} \frac{\lambda(\mu_k - \mu_k^*)}{\lambda^2 - \mu_k^2}.
$$

Note that the asymptotics $\Phi_{k,\pm}(\lambda, \mu_k)$ do not depend upon the polarization vectors $|m_{k0}\rangle$ and that they commute for different values of λ. This allows us to describe explicitly the N-soliton interactions of quadruplet solitons.

The action of $\Phi_{k,\pm}(\lambda, \mu_k)$ on the polarization vectors produces the equalities

$$
\Phi_{k,\pm}(\lambda, \mu_k) \begin{pmatrix} \alpha_k + \beta_k \\ \gamma_k \\ \alpha_k - \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_k^{\pm} + \beta_k^{\pm} \\ \gamma_k \\ \alpha_k^{\pm} - \beta_k^{\pm} \end{pmatrix}
$$

$$(100)$$

$$
\frac{\alpha_k^{\pm}}{\alpha_k} = \frac{\mu_k}{\mu_k^*} \frac{\lambda \pm \mu_k^*}{\lambda \pm \mu_k}, \qquad \frac{\beta_k^{\pm}}{\beta_k} = \frac{\mu_k}{\mu_k^*} \frac{\lambda \mp \mu_k^*}{\lambda \mp \mu_k}.
$$

Next we have to evaluate the asymptotics of $|\mathbf{m}_k(x, t)\rangle$ when $t \to \pm\infty$ along the trajectory $Z_N(x, t) = \text{const.}$ This is done recursively using (99). Skipping all technical details here we get

$$
|\mathbf{m}_N(x, t)\rangle \underset{t \to \infty}{\sim} \prod_{j \in \mathcal{M}_+} \Phi_+(\mu_N, \mu_j) \prod_{j \in \mathcal{M}_-} \Phi_-(\mu_N, \mu_j) |m_N(x, t)\rangle
$$

$$(101)$$

$$
|\mathbf{m}_N(x, t)\rangle \underset{t \to -\infty}{\sim} \prod_{j \in \mathcal{M}_+} \Phi_-(\mu_N, \mu_j) \prod_{j \in \mathcal{M}_-} \Phi_+(\mu_N, \mu_j) |m_N(x, t)\rangle.
$$

Then from (100) and (101) one deduces that

$$
\frac{\alpha_N^+}{\alpha} = \prod_{k=1}^N \frac{\mu_k}{\mu_k^*} \prod_{j \in \mathcal{M}_+} A_{N,j} \prod_{j \in \mathcal{M}_-} B_{N,j}
$$

$$
\frac{\alpha_N^-}{\alpha} = \prod_{k=1}^N \frac{\mu_k}{\mu_k^*} \prod_{j \in \mathcal{M}_+} B_{N,j} \prod_{j \in \mathcal{M}_-} A_{N,j}
$$

$$
\frac{\beta_N^+}{\beta} = \prod_{k=1}^N \frac{\mu_k}{\mu_k^*} \prod_{j \in \mathcal{M}_+} B_{N,j} \prod_{j \in \mathcal{M}_-} A_{N,j} \tag{102}
$$

$$
\frac{\beta_N^-}{\beta} = \prod_{k=1}^N \frac{\mu_k}{\mu_k^*} \prod_{j \in \mathcal{M}_+} A_{N,j} \prod_{j \in \mathcal{M}_-} B_{N,j}
$$

$$
A_{N,j} = \frac{\mu_N + \mu_j^*}{\mu_N + \mu_j}, \qquad B_{N,j} = \frac{\mu_N - \mu_j^*}{\mu_N - \mu_j}.
$$

As a result we obtain that: i) the soliton interactions are purely elastic, and ii) their effect is shifts of the relative center of mass and the phase $\delta_N = \arg \alpha - \arg \beta$ of the solitons

$$
Z_N^\pm = Z_N \mp \sum_{j \in \mathcal{M}_+} z_{N,j} \pm \sum_{j \in \mathcal{M}_-} z_{N,j}
$$

$$
\delta_N^\pm = \delta_N \pm \sum_{j \in \mathcal{M}_+} \phi_{N,j} \mp \sum_{j \in \mathcal{M}_-} \phi_{N,j} \tag{103}
$$

$$
z_{N,j} = \frac{1}{2\kappa_N} (\ln |A_{N,j}| - \ln |B_{N,j}|), \quad \phi_{N,j} = \arg(A_{N,j}) - \arg(B_{N,j}).
$$

5. Integrals of Motion

Here we will sketch briefly the direct method for finding integrals of motion, introduced by Drinfel'd and Sokolov [6]. We will apply it to the system (18). In order to do that it proves to be technically more convenient to deal with the Lax pair (20), (22). We will use the transformation $\mathcal{P}(x, t, \lambda)$ that diagonalises simultaneously the Lax pair $\tilde{L}$ and $\tilde{A}$

$$
\mathcal{L} = \mathcal{P}^{-1} \tilde{L} \mathcal{P} = \mathrm{i}\partial_x + \lambda J + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \cdots
$$

$$
\mathcal{A} = \mathcal{P}^{-1} \tilde{A} \mathcal{P} = \mathrm{i}\partial_t + \lambda^2 I + \lambda \mathcal{A}_{-1} + \mathcal{A}_0 + \frac{\mathcal{A}_1}{\lambda} + \cdots . \tag{104}
$$

Here all matrix coefficients $\mathcal{L}_k$, $\mathcal{A}_{-1}$ and $\mathcal{A}_k$, $k = 0, 1, \ldots$ are diagonal. Using the asymptotic expansion for $\mathcal{P}(x, t, \lambda)$

$$\mathcal{P}(x, t, \lambda) = \mathbb{1} + \frac{p_1(x, t)}{\lambda} + \frac{p_2(x, t)}{\lambda^2} + \cdots \tag{105}$$

one can get a set of recurrence relations

$$U_0 + Jp_1 = \mathcal{L}_0 + p_1 J \tag{106}$$

$$\mathrm{i}p_{1,x} + U_0 p_1 + Jp_2 = \mathcal{L}_1 + p_1 \mathcal{L}_0 + p_2 J \tag{107}$$

$$\vdots$$

$$\mathrm{i}p_{k,x} + U_0 p_k + Jp_{k+1} = \mathcal{L}_k + p_{k+1} J + \sum_{m=0}^{k-1} p_{k-m} \mathcal{L}_m \tag{108}$$

$$\vdots$$

Here we assume that all coefficients p_l ($l = 1, 2, \ldots$) are off-diagonal matrices.

In order to solve the recursion relations above, we will split each relation into a diagonal and off-diagonal part. For example, treating this way the first relation above one gets

$$\mathcal{L}_0 = U_0^{\mathrm{d}}, \qquad U_0^{\mathrm{f}} = -[J, p_1] \tag{109}$$

where the superscripts d and f above denote projection onto diagonal and off-diagonal part of a matrix respectively. Taking into account the explicit form of U_0 for $\mathcal{L}_0$ we have

$$\mathcal{L}_0 = \frac{\mathrm{i}}{2}(uu_x^* + vv_x^*) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{110}$$

Thus as a density of our first integral we can choose: $\mathcal{I}_0 = u^* u_x + v^* v_x$. It represents momentum density of our system. For the stationary solutions (70) and (76) the momentum density is depicted on Fig. 7. It is evidential, that the integrals of motion are well localised function of x.

Similarly, for the second integral density one gets

$$\mathcal{I}_1 = |uu_x^* + vv_x^*|^2 + 4|uv_x - vu_x|^2.$$

In general, the k integral of motion can be calculated through the formula

$$\mathcal{L}_k = \left(U_0^{\mathrm{f}} p_k \right)^{\mathrm{d}}. \tag{111}$$

The matrix p_k in its turn is obtained from the following recursive formula

$$p_k = -\mathrm{ad}_J^{-1} \left(\mathrm{i}p_{k-1,x} + (U_0 p_{k-1})^{\mathrm{f}} - \sum_{m=0}^{k-1} p_{k-1-m} \mathcal{L}_m \right). \tag{112}$$
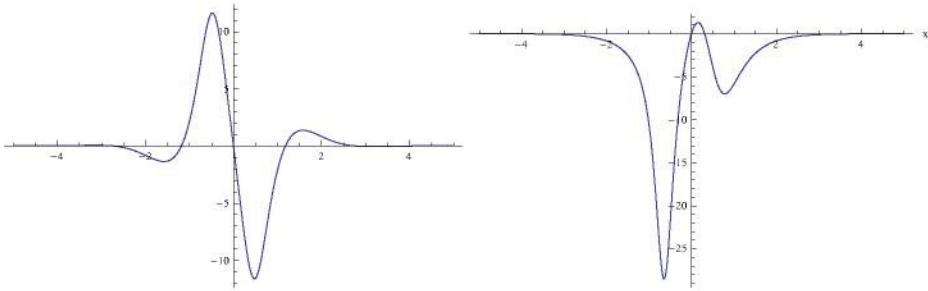
Figure 7. Plots of the density of the first integral of motion as a function of x evaluated on the stationary quadruplet soliton (70) for $\alpha = \beta = \gamma = \omega = 1$, $\delta = 0$ (left panel) and for the stationary doublet soliton (76) for $\kappa = \sigma_0 = 1$, $\phi = 0$ (right panel).

Note, that the zero curvature representation is gauge invariant, i.e., $[\mathcal{L}, \mathcal{A}] = 0$ is fulfilled. Since $[\mathcal{L}_k, \mathcal{A}_l] = 0$ the commutativity of $\mathcal{L}$ and $\mathcal{A}$ is equivalent to the following requirements

$$\partial_x \mathcal{A}_{-1} = 0, \qquad \partial_t \mathcal{L}_k - \partial_x \mathcal{A}_k = 0, \qquad k = 0, 1, \dots \qquad (113)$$

Hence $\mathcal{L}_k$ represent densities of the integrals of motion we are interested in.

6. Conclusions

The soliton solutions for a hirarchy of NLEEs related to the symmetric space $SU(3)/S(U(1) \times U(2))$ are constructed. In order to obtain the soliton solutions we have applied the dressing procedure with a two-poles dressing factor. It has been shown that there exist two types of one-soliton solutions: quadruplet solitons which are associated with four symmetrically located eigenvalues of L and doublet solitons which are associated with a pair of purely imaginary eigenvalues. This remarkable fact is a consequence of the simultaneous action of two $\mathbb{Z}_2$ reductions on the Lax pair. The properties of the elementary solitons depend crucially upon the symmetry properties the dispersion law. For example, if the dispersion law is an even polynomial then the elementary soliton of the first type will be stationary (see formula (70)) otherwise it is time-dependent (fomula (71)). In the case of the doublet type solitons the situation changes significantly – the components of the polarization vector $|m_0\rangle$ are no longer independent, see (63). This is why we have only two cases possible: generic case and a degenerate case. In the latter case the doublet soliton is stationary if $\mathbf{f}(\lambda)$ is an even polynomial, otherwise they are time-depending. In the generic case a new phenomenon arises. When the dispersion law is an even polynomial the soliton is not a travelling wave. Its behaviour resembles

that of trapons and boomerons – the soliton velocity is not fixed but varies with time.

We have described the quadruplet soliton interactions for NLEE with odd dispersion laws by calculating explicitly their asymptotics along the soliton trajectories in the generic case (different soliton velocities). The important result consisted in the following:

   i) the N-soliton interactions are purely elastic and always split into sequences of elementary two-soliton interactions

  ii) the effect of each two-soliton interaction consists in shifts of the relative center of mass and relative phases of each of the solitons

 iii) the corresponding shifts are different from the ones for the NLS and Heisenberg ferromagnetic equations.

Acknowledgements

References

[1] Athorne C. and Fordy A., *Generalised KdV and MKdV Equations Associated with Symmetric Spaces*, J. Phys. A **20** (1987) 1377-1386.
Athorne C. and Fordy A., *Integrable Equations in* $(2 + 1)$ *Dimensions Associated with Symmetric and Homogeneous Spaces*, J. Math. Phys. **28** (1987) 2018-2024.

[2] Calogero F. and Degasperis A., *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform I*, Nuovo Cimento B **32** (1976) 201-242.

[3] Calogero F. and Degasperis A., *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform II*, Nuovo Cimento B **39** (1976) 1-54.

[4] Calogero F. and Degasperis A., *Coupled Nonlinear Evolution Equations Solvable via the Inverse Spectral Transform and Solitons That Come Back*: *The Boomeron*, Lett. Nuovo Cimento **16** (1976) 425-433.

[5] Calogero F. and Degasperis A., *Novel Solution of the System Describing the Resonant Interaction of Three Waves*, Physica D **200** (2005) 242-256.

[6] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg-de Vries Type*, Sov. J. Math. **30** (1985) 1975-2036.

[7] Fordy A., *Derivative Nonlinear Schrödinger Equations and Hermitian Symmetric Spaces*, J. Phys. A **17** (1984) 1235-1245.

[8] Fordy A. and Kulish P., *Nonlinear Schrödinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427-443.

[9] Gerdjikov V., *Generalised Fourier Transforms for the Soliton Equations. Gauge Covariant Formulation*, Inv. Problems **2** (1986) 51-74.

[10] Gerdjikov V., *Algebraic and Analytic Aspects of N-wave Type Equations*, Contemp. Math. **301** (2002) 35-68, nlin.SI/0206014.

[11] Gerdjikov V., *On Spectral Theory of Lax Operators on Symmetric Spaces: Vanishing Versus Constant Boundary Conditions*, J. Geom. Symmetry Phys. **15** (2009) 1-41.

[12] Gerdjikov V., Grahovski G., Ivanov R. and Kostov N., *N-wave Interactions Related to Simple Lie Algebras. $\mathbb{Z}_2$-reductions and Soliton Solutions*, Inv. Problems **17** (2001) 999-1015.
    Gerdjikov V., Grahovski G. and Kostov N., *Reductions of N-wave Interactions Related to Simple Lie Algebras I: $\mathbb{Z}_2$-reductions*, J. Phys. A: Math. & Gen. **34** (2001) 9425-9461.

[13] Gerdjikov V., Grahovski G., Mikhailov A. and Valchev T., *Polynomial Bundles and Generalised Fourier Transforms for Integrable Equations on **A.III**-type Symmetric Spaces*, SIGMA **7** (2011) 096 (48 pp).

[14] Gerdjikov V. and Kulish P., *The Generating Operator for the $n \times n$ Linear System*, Physica D **3** (1981) 549-564.

[15] Gerdjikov V., Mikhailov A. and Valchev T., *Reductions of Integrable Equations on **A.III**-type Symmetric Spaces*, J. Phys. A: Math & Theor. **43** (2010) 434015 (13 pp).

[16] Gerdjikov V., Mikhailov A. and Valchev T., *Recursion Operators and Reductions of Integrable Equations on Symmetric Spaces*, J. Geom. Symmetry Phys. **20** (2010) 1-34.

[17] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics **748**, Springer, Berlin 2008.

[18] Gerdjikov V. and Yanovski A., *Gauge Covariant Theory of the Generating Operator I, The Zakharov-Shabat System*, Commun. Math. Phys. **103** (1986) 549-568.

[19] Gerdjikov V. and Yanovski A., *Gauge Covariant Formulation of the Generating Operator II, Systems on Homogeneous Spaces*, Phys. Lett. A **110** (1985) 53-58.

[20] Golubchik I. and Sokolov V., *Integrable Equations on $\mathbb{Z}$-graded Lie Algebras*, Theor. Math. Phys. **112** (1997) 1097-1103.

[21] Golubchik I. and Sokolov V., *Generalised Heisenberg Equations on $\mathbb{Z}$-graded Lie Algebras*, Theor. Math. Phys. **120** (1999) 1019-1025.

[22] Golubchik I. and Sokolov V., *Multicomponent Generalization of the Hierarchy of the Landau-Lifshitz Equation*, Theor. Math. Phys. **124** (2000) 909-917.

[23] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[24] Lombardo S. and Sanders J., *On the Classification of Automorphic Lie Algebras*, Commun. Math. Phys. **299** (2010) 793-824, arXiv:0912.1697v1 [nlin.SI].

[25] Loos O., *Symmetric Spaces,* vol. I: *General Theory,* vol. II: *Compact Spaces and Classification*, Benjamin, New York 1969.

[26] Mikhailov A., *On the Integrability of Two-dimensional Generalization of the Toda Lattice*, Lett. J. Exper. Theor. Phys. **30** (1979) 443-448.

[27] Mikhailov A., *Reductions in Integrable Systems. The Reduction Groups*, Lett. J. Exper. Theor. Phys. **32** (1980) 187-192.

[28] Mikhailov A., *The Reduction Problem and the Inverse Scattering Method*, Physica D **3** (1981) 73-117.

[29] Mikhailov A., *The Landau-Lifschitz Equation and the Riemann Boundary Problem on a Torus*, Phys. Lett. A **92** (1982) 51-55.

[30] Mikhailov A. and Lombardo S., *Reductions of Integrable Equations: Dihedral Group*, J. Phys. A **37** (2004) 7727-7742.

[31] Mikhailov A. and Lombardo S., *Reduction Groups and Automorphic Lie Algebras*, Commun. Math. Phys. **258** (2005) 179-202.

[32] Takhtadjan L. and Faddeev L., *The Hamiltonian Approach to Soliton Theory*, Springer, Berlin 1987.

[33] Zakharov V. and Manakov S., *Exact Theory of Resonant Interaction of Wave Packets in Nonlinear Media* (in Russian), INF preprint **74-41** Novosibirsk, 1974, pp 1-52.
Zakharov V. and Manakov S., *On the Theory of Resonant Interaction of Wave Packets in Nonlinear Media* (in Russian), Zh. Exp. Teor. Fiz. **69** (1975) 1654-1673.
Zakharov V. and Manakov S., *Asymptotic Behavior of Nonlinear Wave Systems Integrable by the Inverse Scattering Method* (in Russian), Zh. Exp. Teor. Fiz. **71** (1976) 203-215.

[34] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons*: *the Inverse Scattering Method*, Plenum, New York 1984.

[35] Zakharov V. and Mikhailov A., *Relativistically Invariant 2-dimensional Field Theory Models Integrable by Means of the Inverse Scattering Method* (in Russian), Zh. Exp. Teor. Fiz. **74** (1978) 1953-1973.

[36] Zakharov V. and Mikhailov A., *On the Integrability of Classical Spinor Models in Two-dimensional Space-time*, Commun. Math. Phys. **74** (1980) 21-40.

[37] Zakharov V. and Shabat A., *Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media*, Soviet Physics-JETP **34** (1972) 62-69.

[38] Zakharov V. and Shabat A., *A Scheme for Integrating Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform I* (in Russian), Funct. Anal. Appl. **8** (1974) 43-53.
Zakharov V. and Shabat A., *Integration of Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform II* (in Russian), Funct. Anal. Appl. **13** (1979) 13-22.

ON SOLITON EQUATIONS WITH $\mathbb{Z}_h$ AND $\mathbb{D}_h$ REDUCTIONS: CONSERVATION LAWS AND GENERATING OPERATORS[*]

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Abstract. The Lax representations for the soliton equations with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions are analyzed. Their recursion operators are shown to possess factorization properties due to the grading in the relevant Lie algebra. We show that with each simple Lie algebra one can relate r fundamental recursion operators Λ_{m_k} and a master recursion operator Λ generating NLEEs of MKdV type and their Hamiltonian hierarchies. The Wronskian relations are formulated and shown to provide the tools to understand the inverse scattering method as a generalized Fourier transform. They are also used to analyze the conservation laws of the above mentioned soliton equations.

CONTENTS

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1. Introduction

This paper is a natural continuation of our articles [21], where a gauge covariant approach to the generating operator was proposed for the case of the Zakharov-Shabat system and [22], where the results obtained in [21] have been generalized to the cases of the so-called Caudrey-Beals-Coifman system. Here we consider the nonlinear evolution equations (NLEEs), solvable through the inverse scattering method (ISM) for the linear system

$$
L\psi(x,t,\lambda) \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + U(x,t,\lambda) \right) \psi(x,t,\lambda) = 0
$$
$$
U(x,t,\lambda) = U_0(x,t) + \lambda U_1, \qquad U_0(x,t) = [U_1, Q(x,t)].
$$

(1)

If we choose U_1 to be constant diagonal matrix with real eigenvalues the system (1) is the generalized Zakharov-Shabat system [42, 45]. It allows one to solve the class of N-wave equations which have important physical applications.

As has been shown by Shabat [37] the difficulties of solving the inverse scattering problem for (1) could be overcome if we are able to reduce the solution of the relevant nonlinear evolution equation (NLEE) to a Riemann-Hilbert problem (RHP)

$$
\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda)G(\lambda,t), \qquad \lambda \in \mathbb{R}.
$$

(2)

In the above $\chi^\pm(x,t,\lambda)$ are the fundamental analytic solutions (FAS), that is, fundamental solutions of (1) that allow analytic extension in λ for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$ – the upper and lower half-plane.

In particular, using this fact, Zakharov and Shabat developed the well known dressing method, which became the most efficient method for constructing the soliton solutions of the NLEEs we speak about.

Later Caudrey, Beals and Coifman [3, 4, 6] studied the inverse scattering problem for a more general system which has also the form of (1) but now the eigenvalues of U_1 were assumed to be complex. They succeeded to construct the FAS which now have analyticity properties only inside certain sectors in the complex λ-plane and the RHP is formulated on the rays separating these sectors.

The results of Caudrey-Beals-Coifman were extended from $\mathfrak{sl}(n)$ to any semisimple Lie algebra in any faithful representation [22]. In [22] we have assumed that

$Q(x,t)$ is an arbitrary element of a semisimple complex Lie algebra $\mathfrak{g}$ of rank r and U_1 is a regular element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Applying the inverse scattering method (ISM) to the CBC system one can integrate generic NLEEs. The simplest of them are analogous to the N-wave equations but with complex-valued 'group velocities'. Until now we know nothing about physical applications of such NLEEs, though they are completely integrable Hamiltonian systems whose action-angle variables have been constructed in [5].

The situation changes if we apply the reduction group technique [32] and request that the Lax operator possesses $\mathbb{Z}_h$ or $\mathbb{D}_h$ symmetry. The importance of such Lax operators became clear due to the pioneer paper of Mikhailov [32] where he discovered the integrability of the two-dimensional Toda field theories (2DTFT). Mikhailov was the first to show that the ISM for $\mathbb{Z}_h$ and $\mathbb{D}_h$-reduced Lax operators can be reduced to a RHP on a set of $2h$ rays starting from the origin and closing angles π/h. Next, using the Zakharov-Shabat dressing method [45], he constructed the soliton solutions of the 2DTFT and proved that there are gaps in the sequences of integrals of motion.

Soon the results of [32] were generalized to any simple Lie algebra and draw a large attention, see [13,33–35] and the references therein. Of course, each 2DTFT is just one representative from a hierarchy of NLEEs that are related to the corresponding Lax operator.

One of the aims of the present paper is to provide a tool to study the other members of these hierarchies. To this end we have to reformulate most of the results we have for the general system to the special but important case when the Lax operator is subject to additional group of symmetries, among which are $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries proposed by Mikhailov [32].

Our second aim is to generalize the AKNS method [1, 26] and to construct the recursion operators for the NLEE with $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries. The recursion operators (see the monographs [7,20,27] and [1,8,12,14–16,19,21,22,24,26]) are an effective tool to generate both the NLEEs and their Hamiltonian hierarchies [15, 16, 19, 28, 29]. We confirm and generalize previous results on recursion operators of reduced systems [11, 13, 18, 23, 38, 39], especially their factorization properties.

In Section 2 we give some Lie algebraic preliminaries and introduce the notion of Mikhailov reduction group [32]. In Section 3 we outline the spectral theory for the Lax operators, introduce their FAS and the scattering data. Section 4 demonstrates that the FAS satisfy a local RHP on a set of $2h$ rays passing through the origin and closing angles π/h [32]. Section 5 is dedicated to the calculation of the recursion operators introduced by [1]. However, the AKNS method needs generalization, because we are dealing with a $\mathbb{Z}_h$-reduced Lax pair that take values in the graded algebra. The recursion operators that are obtained have substantially new structure as compared to the AKNS ones. The recursion operators now factorizes into a product of h more elementary operators. For the first time such factorization has been observed studying a particular case in [11]. We also show that to each simple Lie algebra of rank r one can relate r fundamental recursion operators $\mathbf{\Lambda}_{m_k}$ and

a master recursion operators Λ generating both NLEEs of MKdV type and their Hamiltonian hierarchies. In Section 6 we derive three types of Wronskian relations for the Lax operator $\tilde{L}$. The first type allows one to interpret the mapping $\mathcal{F}$: $\mathcal{M} \rightarrow \mathcal{T}$ from the manifold of allowed potentials $\mathcal{M}$ to the set of scattering data $\mathcal{T}$ as a generalized Fourier transform (GFT). It allows also to introduce the so-called 'squared solutions' of $\tilde{L}$, called also generalized exponentials or adjoint solutions. The second type of Wronskian relations permits to establish that the same GFT allows one to analyze the mapping $\delta\mathcal{F}$: $\delta\mathcal{M} \rightarrow \delta\mathcal{T}$ between the variations of the potentials and the relevant variations of scattering data. The third type of Wronskian relations is useful for the analysis of the conservation laws. Section 7 briefly comments on the locality of the integrals of motion and on the Hamiltonian structures of the NLEEs.

2. Preliminaries

We assume that the reader is familiar with the basic facts from the theory of simple Lie algebras [25]. In what follows by $\mathfrak{h}$ is denoted a fixed Cartan subalgebra of $\mathfrak{g}$, Δ is the root system of $\mathfrak{g}$, $\alpha \in \Delta$ are the roots, $A \equiv \{\alpha_1, \ldots, \alpha_r\}$ is the set of the simple roots – naturally, the rank of $\mathfrak{g}$ is r. Suppose that E_α, $\alpha \in \Delta$ and H_k, $k = 1, \ldots, r$ denotes the standard Cartan-Weyl basis in $\mathfrak{g}$, see e.g. [25]. Then as well known the commutation relations of the Cartan-Weyl basis have the form

$$[H, E_\alpha] = \alpha(H)E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}, \qquad N_{\alpha,\beta} = \begin{cases} \neq 0 & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{if } \alpha + \beta \notin \Delta. \end{cases} \qquad (3)$$

Let us outline some important facts about the graded Lie algebras and more specifically – how one can introduce bases in $\mathfrak{g}^{(k)}$, see below.

In this article we shall consider grading defined by the Coxeter automorphism, that is $C \in \text{Aut}\,\mathfrak{g}$, $C^h = \mathbb{1}$ and h is the Coxeter number. Obviously the eigenvalues of C are ω^k, $k = 0, 1, \ldots, h - 1$, where $\omega = \exp(2\pi i/h)$. To each eigenvalue there corresponds a linear subspace $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ determined by

$$\mathfrak{g}^{(k)} \equiv \left\{ X \, ; \, X \in \mathfrak{g}, \quad C(X) = \omega^k X \right\}.$$

We than have $\mathfrak{g} = \overset{h-1}{\underset{k=0}{\oplus}} \mathfrak{g}^{(k)}$ and $\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)}\right] \subset \mathfrak{g}^{(k+n)}$, where $k + n$ is taken modulo h. This of course turns $\mathfrak{g}$ into a graded algebra.

Remark 1. *In fact, as we shall see below, one can view the potentials of the Lax operators as elements of Kac-Moody type $\widehat{\mathfrak{g}}_C$ whose elements are*

$$X(\lambda) = \sum_k X_k \lambda^k, \qquad X_k \in \mathfrak{g}^{(k)}$$

where the sum runs contains only a finite number of terms. For reasons that will become clear later we shall assume that the element U_1 from (1) *belongs to $\mathfrak{g}^{(1)}$.*

Remark 2. *Note also that the Killing form between any two elements $X_{(k)} \in \tilde{\mathfrak{g}}^{(k)}$ and $Y_{(k)} \in \tilde{\mathfrak{g}}^{(m)}$ may be non-vanishing only if $k + m = 0 \mod (h)$.*

2.1. The Coxeter Automorphism as Cartan Subgroup Element

We shall consider two realizations of the Coxeter automorphism and start with a realization of as element from the Cartan subgroup

$$C = \exp\left(\frac{2\pi i}{h} H_{\hat{\rho}}\right), \qquad (\hat{\rho}, \alpha_j) = 1 \tag{4}$$

where

$$\hat{\rho} = \sum_{j=1}^{r} \frac{(\alpha_j, \alpha_j)}{2} \omega_j \tag{5}$$

and ω_j are the fundamental weights of $\mathfrak{g}$. For the classical Lie algebras the vectors $\hat{\rho}$ take the form

$$A_r: \quad \hat{\rho} = \sum_{s=0}^{r} \left(\left[\frac{r}{2}\right] - s\right) e_{s+1}, \qquad B_r: \quad \hat{\rho} = \sum_{s=0}^{r-1} (r - s) e_{s+1}$$

$$C_r: \quad \hat{\rho} = \sum_{s=0}^{r-1} \left(r - s - \frac{1}{2}\right) e_{s+1}, \qquad D_r: \quad \hat{\rho} = \sum_{s=0}^{r-1} (r - s - 1) e_{s+1} \tag{6}$$

where e_k are as usual orthonormal vectors. Obviously $(\hat{\rho}, \alpha_j) = 1$ for $j = 1, \ldots, r$. With the above choices we have

$$\mathfrak{g}^{(0)} \equiv \mathfrak{h}, \qquad \mathfrak{g}^{(k)} \equiv \operatorname{span}\{E_\alpha \,;\, \operatorname{ht}\alpha = k \mod (h)\} \tag{7}$$

where $\operatorname{ht}\alpha$ is the height of the root α. In other words, if $\alpha = \sum_{k=1}^{r} m_k \alpha_k$, then $\operatorname{ht}\alpha = \sum_{k=1}^{r} m_k$.

In this case the matrix U_1 that appears in the Lax operator will be denoted by J and is given by

$$J = \sum_{s=0}^{r} E_{\alpha_s} \tag{8}$$

where α_k, $k = 1, \ldots, r$ are the simple roots of $\mathfrak{g}$ and α_0 is the minimal root. We will use it applying additional normalization

$$\langle J, J^T \rangle = 1, \qquad J^T = \sum_{s=0}^{r} E_{-\alpha_s} \tag{9}$$

where $\langle X, Y \rangle$ denotes the Killing form applied to X and Y. The characteristic equations for J are as follows

$$
\begin{array}{llll}
A_r & z^h - 1 = 0, & B_r & (z^h - 1)z = 0 \\
C_r & z^h - 1 = 0, & D_r & (z^h - 1)z^2 = 0
\end{array}
\tag{10}
$$

where h is the Coxeter number of the algebra, see Table 1. In what follows we will need not only the basis in each of $\mathfrak{g}^{(k)}$, but also the orthogonal splitting

$$
X_k \in \mathfrak{g}^{(k)}, \qquad X_k = X_k^{\perp} + X_k^{\parallel}
\tag{11}
$$

where

$$
[X_k^{\parallel}, J] = 0, \qquad \langle X_k^{\perp}, X_k^{\parallel} \rangle = 0.
\tag{12}
$$

Table 1. The Coxeter numbers, the exponents and the minimal roots of the classical series of Lie algebras, see [25].

Algebra	Coxeter number	exponents	minimal root
$A_r \simeq \mathfrak{sl}(r+1)$	$r+1$	$1, 2, 3, \ldots, r$	$e_{r+1} - e_1$
$B_r \simeq \mathfrak{so}(2r+1)$	$2r$	$1, 3, 5, \ldots, 2r-1$	$-e_1 - e_2$
$C_r \simeq \mathfrak{sp}(2r)$	$2r$	$1, 3, 5, \ldots, 2r-1$	$-2e_1$
$D_r \simeq \mathfrak{so}(2r)$	$2r-2$	$1, 3, 5, \ldots, 2r-3, r-1$	$-e_1 - e_2$
G_2	6	$1, 5$	
F_4	12	$1, 5, 7, 11$	

We will also need ad_J^{-1}. Since J is not diagonal this will present technical difficulty which can be overcome by using the characteristic equation for ad_J, see [21]. As for the splitting (11), for the classical series A_r, B_r and C_r it can be found using the fact that any matrix from the algebra commuting with J (9) is a polynomial of J of maximal order h. In particular, the powers of J commute with J. For A_r it is enough to use J^k, $k = 1, \ldots, r$. Then for $\mathfrak{g} \simeq \mathfrak{sl}(r+1)$ we have

$$
X_k = X_k^{\perp} + X_k^{\parallel}, \qquad X_k^{\parallel} = c_k^{-1} J^k \langle X, J^{h-k} \rangle, \qquad h = r + 1
\tag{13}
$$

where $c_k = \langle J^k, J^{h-k} \rangle$. For the classical series B_r and C_r we have

$$
\begin{array}{ll}
X_k^{\parallel} = 0, & \text{if } k = 2s \text{ is not an exponent} \\
X_k^{\parallel} = c_k^{-1} J^k \langle X_k, J^{h-k} \rangle, & \text{if } k = 2s - 1 \text{ is an exponent.}
\end{array}
\tag{14}
$$

The case of the series D_r will be treated separately elsewhere, because for it there are some specifics due to the fact that $r - 1$ is an exponent, see Table 1. For the algebra D_r, in case $r = 2p + 1$ the even number $2p$ is as exponent while in the case $r = 2p$ the odd number $2p - 1$ is a double valued exponent.

2.2. The Coxeter Automorphism as Weyl Group Element

Now we shall consider another realization of the Coxter automorphism. Technically it is easier to handle the case $\mathfrak{g} \simeq \mathfrak{sl}(r + 1)$, so we shall start with it. The Coxeter automorphism $\tilde{C}$ can be realized as

$$\tilde{C}(X) = KXK^{-1}, \qquad K = \sum_{s=1}^{h} E_{s+1,s}, \qquad (E_{k,m})_{jp} = \delta_{kj}\delta_{mp}. \qquad (15)$$

Here and below all indexes are understood modulo h, so that the last term with $s = h$ in (15) equals $E_{h,1}$. The calculations are much simpler if we introduce a convenient basis in $\mathfrak{g}^{(k)}$, namely

$$J_s^{(k)} = \sum_{j=1}^{h} \omega^{kj} E_{j,j+s}, \qquad K J_s^{(k)} K^{-1} = \omega^k J_s^{(k)}. \qquad (16)$$

Obviously, $J_s^{(k)}$ satisfies the commutation relations

$$\left[J_s^{(k)}, J_l^{(m)} \right] = \left(\omega^{ms} - \omega^{kl} \right) J_{s+l}^{(k+m)}. \qquad (17)$$

Another option is to use the dihedral realization of the Coxeter automorphism $\tilde{C}$

$$\tilde{C} = C_1 C_2 \qquad (18)$$

where

$$C_1 = \prod_{\alpha \in A_1} S_\alpha, \qquad C_2 = \prod_{\beta \in A_2} S_\beta. \qquad (19)$$

In the above A_0 and A_1 are subsets of the set of simple roots $A = \{\alpha_1, \ldots \alpha_r\}$ of $\mathfrak{g}$ such that

$$A_1 \cup A_2 = A, \qquad (\alpha_j, \alpha_k) = 0, \qquad (\beta_j, \beta_k) = 0, \quad \text{for } j \neq k \qquad (20)$$

for all $\alpha_j, \alpha_k \in A_0$ and $\beta_j, \beta_k \in A_1$. Also, by S_α is denoted the Weyl reflection related to the root α, i.e., $S_\alpha \vec{x} = \vec{x} - \dfrac{2(\vec{x}, \alpha)}{(\alpha, \alpha)}\alpha$, where $(\,,\,)$ is the canonical inner product in the Euclidean space $\mathbb{E}^r$. For the classical series A_r, B_r and C_r of Lie algebras

$$\begin{aligned} A_1 &= \{\alpha_2, \alpha_4, \ldots, \alpha_{2p}\}, & A_2 &= \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}, & \text{if } r = 2p \\ A_1 &= \{\alpha_2, \alpha_4, \ldots, \alpha_{2p}\}, & A_2 &= \{\alpha_1, \alpha_3, \ldots, \alpha_{2p+1}\}, & \text{if } r = 2p+1 \end{aligned} \qquad (21)$$

and for the D_r series

$$\begin{aligned} A_1 &= \{\alpha_2, \alpha_4, \ldots, \alpha_{2p-2}\}, & A_2 &= \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}, \alpha_{2p},\}, & \text{if } r = 2p \\ A_1 &= \{\alpha_2, \alpha_4, \ldots, \alpha_{2p}, \alpha_{2p+1}\}, & A_2 &= \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}, & \text{if } r = 2p+1. \end{aligned} \qquad (22)$$

In the above we have used the standard notations for the sets of simple roots $\{\alpha_1, \ldots, \alpha_r\}$, see [25].

It is natural to call the set of roots $\mathcal{O}_\alpha \equiv \{\alpha, \tilde{C}\alpha, \ldots, \tilde{C}^{h-1}\alpha\}$ the orbit of $\tilde{C}$ passing through the root α. The special choice for $\tilde{C}$ (18) has the advantage to split the root system Δ into r orbits as follows

$$\Delta = \bigcup_{\alpha \in A_1} \mathcal{O}_\alpha \cup \bigcup_{\beta \in A_2} \mathcal{O}_{-\beta} = \bigcup_{\alpha \in A_1} \mathcal{O}_{-\alpha} \cup \bigcup_{\beta \in A_2} \mathcal{O}_\beta. \tag{23}$$

This allows us to introduce two bases in $\mathfrak{g}$ compatible with the grading

$$\tilde{\mathfrak{g}}^{(k)} \equiv \operatorname{span}\{\mathcal{E}_\alpha^{(k)}, \mathcal{E}_{-\beta}^{(k)}, \mathcal{H}_j^{(k)}; \ \alpha \in A_0, \ \beta \in A_1\}$$

$$\tilde{\mathfrak{g}}^{(k)} \equiv \operatorname{span}\{\mathcal{E}_{-\alpha}^{(k)}, \mathcal{E}_\beta^{(k)}, \mathcal{H}_j^{(k)}; \ \alpha \in A_0, \ \beta \in A_1\} \tag{24}$$

$$\mathcal{E}_\alpha^{(k)} = \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sk} \tilde{C}^s(E_\alpha), \qquad \mathcal{H}_j^{(k)} = \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sk} \tilde{C}^s(H_j).$$

Remark 3. *Note that $\mathcal{H}_j^{(k)}$ is non-vanishing only if k is an exponent of $\mathfrak{g}$. That also means that each $\tilde{\mathfrak{g}}^{(k)}$ has at most one-dimensional intersection with the Cartan subalgebra and the only exceptions take place for the algebras of the series D_{2r} which have $2r - 1$ as double valued exponent. These cases will be considered elsewhere.*

We can pick U_1 to be equal to $\mathcal{J}$ where

$$\mathcal{J} = H_{e_1} + \sum_{s=1}^{p-1} \left(\omega^s H_{e_{2s}} + \omega^{-s} H_{e_{2s+1}} \right), \qquad \text{for } A_{2p}, B_{2p+1}, C_{2p+1}, D_{2p}$$

$$\tag{25}$$

$$\mathcal{J} = H_{e_1} + \sum_{s=1}^{p-1} \left(\omega^s H_{e_{2s}} + \omega^{-s} H_{e_{2s+1}} \right) + \omega^p H_{e_{2p}}, \quad \text{for } A_{2p-1}, B_{2p}, C_{2p}, D_{2p+1}.$$

Choosing $\mathcal{J}$ as in (25) we have

$$\tilde{C}(\mathcal{J}) = \omega \mathcal{J}, \qquad C_1(\mathcal{J}) = \mathcal{J}^*, \qquad C_2(\mathcal{J}) = \omega \mathcal{J}^* \tag{26}$$

where $\omega = \exp(2\pi \mathrm{i}/h)$. In addition

$$C_1(\mathcal{E}_\alpha^{(0)}) = \mathcal{E}_{-\alpha}^{(0)}, \qquad C_2(\mathcal{E}_\beta^{(0)}) = \mathcal{E}_{-\beta}^{(0)}$$

$$C_2(\mathcal{E}_\alpha^{(0)}) = C_1(\mathcal{E}_\alpha^{(0)}), \qquad C_1(\mathcal{E}_{-\beta}^{(0)}) = C_2(\mathcal{E}_{-\beta}^{(0)}) \tag{27}$$

$$(\mathcal{E}_\alpha^{(s)})^\dagger = \mathcal{E}_{-\alpha}^{(s)}, \qquad C_i(\mathcal{E}_\alpha^{(s)}) = \mathcal{E}_{w_i(\alpha)}^{(h-s)}, \qquad i = 1, 2.$$

Below we will need the commutation relations between $\mathcal{J}$ and the $\mathcal{E}_\beta^{(s)}$. They can be derived as follows

$$
\begin{aligned}
[\mathcal{J}, \mathcal{E}_\alpha^{(p)}] &= \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sp} [\mathcal{J}, \tilde{C}^s(E_\alpha)] = \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sp} \tilde{C}^s \left([\tilde{C}^{-s}(\mathcal{J}), E_\alpha] \right) \\
&= \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sp-s} \tilde{C}^s \left([\mathcal{J}, E_\alpha] \right) = \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sp-s} \alpha(\mathcal{J}) \tilde{C}^s (E_\alpha) \\
&= \alpha(\mathcal{J}) \mathcal{E}_\beta^{(p+1)}
\end{aligned}
\tag{28}
$$

and similarly

$$
[\mathcal{J}, \mathcal{E}_{-\beta}^{(p)}] = -\beta(\mathcal{J}) \mathcal{E}_\beta^{(p+1)}.
\tag{29}
$$

The rest of the commutation relations between the basic elements in $\tilde{\mathfrak{g}}^{(k)}$ may be a bit complicated to derive though of course one can do it using the equations (3) and (24) and take into account the fundamental property of the coefficients $N_{\alpha,\beta}$, namely $N_{\alpha,\beta} = N_{\tilde{C}(\alpha),\tilde{C}(\beta)}$. However, one can approach the problem differently if we notice that $\tilde{C}$ has the same set of eigenvalues as C, which as we mentioned are equal to the powers of $\omega = \exp(2\pi \mathrm{i}/h)$. Hence $\tilde{C}$ and C are related by a similarity transformation

$$
\tilde{C} = u_0^{-1} C u_0
\tag{30}
$$

where u_0 is some constant element of the corresponding Lie group. Therefore the basis in $\tilde{\mathfrak{g}}^{(k)}$ (24) can be obtained from the basis in $\mathfrak{g}^{(k)}$ (7) via this transformation.

2.3. Mikhailov's Reduction Group

Mikhailov's reduction group is a finite group, which must have two realizations: i) as a subgroup of the group of automorphisms Aut $_\mathfrak{g}$ of the algebra $\mathfrak{g}$ and ii) as a subgroup of the conformal transformations Conf of the complex λ-plane. In what follows we shall use the Coxeter automorphism $C^h = \mathbb{1}$ and the involutions $C_1^2 = \mathbb{1}$, or $C_2^2 = \mathbb{1}$ for realizations of the groups $\mathbb{Z}_2$, $\mathbb{Z}_h$, $\mathbb{D}_h$ acting on $\mathfrak{g}$. Note that for a given realization in $\mathfrak{g}$ one may have inequivalent realizations in Conf, that is why we use the indexes $1, 2$ to distinguish different cases. More specifically, the automorphisms C, C_1 and C_2 listed below lead to the following reductions for the matrix-valued functions

$$
\begin{array}{llll}
1) & C(U(\lambda)) = U(\omega\lambda), & C(V(\lambda)) = V(\omega\lambda) & \\
2) & C_1(\tilde{U}^\dagger(\lambda^*)) = \tilde{U}(\lambda), & C_1(\tilde{V}^\dagger(\lambda^*)) = \tilde{V}(\lambda) & (31) \\
3) & C_2(\tilde{U}^\dagger((\lambda\omega)^*)) = \tilde{U}(\lambda). & C_2(\tilde{V}^\dagger((\lambda\omega)^*)) = \tilde{V}(\lambda). &
\end{array}
$$

The above restrictions naturally extend to restrictions on the FAS, the scattering matrix $T(\lambda, t)$, the spectral data of the Lax operator etc., see below.

3. Lax Pairs and NLEEs

Now we are in position to outline the Lax pairs with $\mathbb{Z}_h$- and $\mathbb{D}_h$- reductions and the relevant NLEEs that result in. First we shall use the realization of the Coxeter automorphism as an element of the Cartan subgroup. We have

$$L\chi \equiv i\frac{\partial \chi}{\partial x} + (q(x,t) - \lambda J)\chi(x,t,\lambda) = 0$$

$$q(x) = \sum_{j=1}^{r} q_j(x,t)H_j \in \mathfrak{h}, \qquad J = \sum_{\text{ht }\alpha=1} E_\alpha. \tag{32}$$

Note that the root height should be understood modulo h, so in J along with the generators corresponding to the simple roots we have to add also the generator corresponding to the minimal root whose height is $-h+1 = 1 \mod (h)$.

The best known NLEE's of the above type are the well known two-dimensional Toda field theories, discovered by Mikhailov [32]. They attracted a lot of attention, see [9, 30, 32, 34, 35] and the numerous references therein. Their Lax representation $[L, M_{\text{Tft}}] = 0$ involves an L-operator as in equation (32) with $q(x,t) = 2i\phi_x(x,t) \in \mathfrak{h}$ and M-operator of the form

$$M_{\text{Tft}}\chi \equiv i\frac{\partial \chi}{\partial x} + \frac{i}{\lambda}V_{-1}(x,t)\chi(x,t,\lambda) = 0$$

$$V_{-1}(x,t) = \sum_{k=0}^{r} e^{2(\vec{\phi},\alpha_k)}E_{-\alpha_k} \in \mathfrak{g}^{(h-1)} \tag{33}$$

where $\vec{\phi}(x,t)$ is the vector in the Euclidean space, that corresponds to the Cartan element $\phi(x,t)$ in $\mathfrak{h}$. The corresponding equations take the form

$$\frac{\partial^2 \vec{\phi}}{\partial x \partial t} = \sum_{k=0}^{r} \alpha_k e^{2(\vec{\phi},\alpha_k)}. \tag{34}$$

These equations have been studied in detail, so we will turn our attention to the other members in their hierarchies. For that reason below we will consider M-operators that are polynomial in λ of order N. The compatibility of the Lax pair requires in particular $[\mathcal{J}, \mathcal{K}_N] = 0$, which is possible only if $N = m_k + ph$ where m_k is an exponent of the algebra $\mathfrak{g}$, see Table 1, and p is any integer. Therefore

$$M_{(k)}\chi \equiv i\frac{\partial \chi}{\partial t} + \left(\sum_{s=1}^{N} \lambda^{N-s}V_s(x,t) - \lambda^N K_N\right)\chi(x,t,\lambda) = 0$$

$$V_s(x) = \sum_{\text{ht }\beta=N-s} v_{s;\beta}(x,t)E_\beta \in \mathfrak{g}^{(N-s)}, \qquad s > 1, \qquad K_N \in \mathfrak{g}^{(N)}. \tag{35}$$

Remark 4. *As one can see, even for algebras of low rank and for small values of p the order N of the NLEE grows rather quickly, see* Table 2.

Table 2. The orders of the NLEE's with $N = ph + m_k$ for the classical Lie algebras of rank up to 4 and $p = 1$ and $p = 2$.

N	A_2	B_2, C_2	G_2	
$h + m_k$	$4, 5$	$5, 7$	$7, 11$	
$2h + m_k$	$7, 8$	$9, 11$	$13, 17$	
	A_3	B_3, C_3		
$h + m_k$	$5, 6, 7$	$7, 9, 11$		
$2h + m_k$	$9, 10, 11$	$13, 15, 17$		
	A_4	B_4, C_4	D_4	F_4
$h + m_k$	$6, 7, 8, 9$	$9, 11, 13, 15$	$7, 9, 9, 11$	$13, 17, 19, 23$
$2h + m_k$	$11, 12, 13, 14$	$17, 19, 21, 23$	$13, 15, 15, 17$	$25, 29, 31, 35$

Now let us consider the Lax operators

$$\tilde{L}\tilde{\chi} \equiv i\frac{\partial \tilde{\chi}}{\partial x} + \tilde{U}(x, t, \lambda)\tilde{\chi}(x, t, \lambda) = 0$$

$$\tilde{M}\tilde{\chi} \equiv i\frac{\partial \tilde{\chi}}{\partial t} + \tilde{V}(x, t, \lambda)\tilde{\chi}(x, t, \lambda) = 0 \tag{36}$$

where

$$\tilde{U}(x, t, \lambda) = Q(x, t) - \lambda \mathcal{J}$$

$$Q(x) = \sum_{j=1}^{r} q_j(x, t) u_0^{-1} H_j\, u_0 = \sum_{\alpha \in A_1} q_\alpha \mathcal{E}_\alpha^{(0)} + \sum_{\beta \in A_2} q_\beta \mathcal{E}_{-\beta}^{(0)} \tag{37}$$

and $C_1(Q^\dagger(x, t)) = Q(x, t)$, i.e.,

$$q_\alpha^*(x, t) = q_\alpha(x, t), \qquad q_\beta^*(x, t) = q_\beta(x, t). \tag{38}$$

The potential of the operator $\tilde{M}$ takes the form

$$\tilde{V}(x, t, \lambda) = \sum_{s=1}^{N} \lambda^{N-s} \tilde{V}_s(x, t, \lambda) - \lambda^N \mathcal{K}_N$$

$$\tilde{V}_s(x, t) = \sum_{\alpha \in A_1} v_{s,\alpha}(x, t)\mathcal{E}_\alpha^{(N-s)} + \sum_{\beta \in A_2} v_{s,\beta}(x, t)\mathcal{E}_{-\beta}^{(N-s)} + \tilde{V}_s^{\parallel}(x, t)$$

$$V_N(x) = \sum_{j=1}^{r} v_{N,j}(x, t) H_j \in \mathfrak{h} \tag{39}$$

$$\mathcal{J} = u_0^{-1} J u_0 \in \mathfrak{h}, \quad \mathcal{K}_N = u_0^{-1} K_N u_0 \in \mathfrak{h}, \quad s = 1, \ldots, N - 1.$$

The above form of the $\tilde{L}$-$\tilde{M}$ pair is obtained from L (32) and M (35) by taking a similarity transformation with u_0, see equation (30), that diagonalizes simultaneously J and K_N. In equation (39) $\tilde{V}_s(x,t)^{\|} \in \mathfrak{h}$ and is non-vanishing only if s is an exponent (of course mod (h)) of the corresponding algebra.

Now we can formulate a $\mathbb{D}_h$-reduction group. It is generated by the the involutions:

$$
\begin{aligned}
C_1(\tilde{U}^\dagger(x,t,\lambda^*)) &= \tilde{U}(x,t,\lambda), & C_2(\tilde{U}^\dagger(x,t,(\lambda\omega)^*)) &= \tilde{U}(x,t,\lambda) \\
C_1(\tilde{V}^\dagger(x,t,\lambda^*)) &= \tilde{V}(x,t,\lambda), & C_2(\tilde{V}^\dagger(x,t,(\lambda\omega)^*)) &= \tilde{V}(x,t,\lambda).
\end{aligned}
\tag{40}
$$

Imposing both reductions (40) we obtain that $\tilde{L}$ is a subject also of the $\mathbb{Z}_h$-reduction

$$
\tilde{C}(\tilde{U}(x,t,\lambda)) = \tilde{U}(x,t,\omega\lambda), \qquad \tilde{C}(\tilde{V}(x,t,\lambda)) = \tilde{V}(x,t,\omega\lambda).
\tag{41}
$$

Thus this realization of the reductions effectively gives rise to the additional conditions on q_α and q_β (38) so our reduction group is $\mathbb{D}_h$. In order to check that V_s also satisfy the reduction conditions one needs to use equation (38) above.

3.1. The Spectral Properties of the Lax Operator

In our paper [22] we extended the construction of Caudrey-Beals-Coifman to any simple Lie algebra and constructed the FAS of generalized Zakharov-Shabat systems whose U_1 has complex eigenvalues. Imposing the $\mathbb{Z}_h$-reduction we obtain the Lax operator (36). Particular cases of such operators have been considered in [23, 38].

The Jost solutions $\tilde{\psi}(x,t,\lambda)$ and $\tilde{\phi}(x,t,\lambda)$ and the scattering matrix $T(\lambda,t)$ of $\tilde{L}$ (36) exist for large class of potentials, in particular for potentials on compact support and are determined uniquely by the following conditions

$$
\lim_{x\to\infty} \tilde{\psi}(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \tilde{\phi}(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} = \mathbb{1}
$$
$$
T(\lambda,t) = \tilde{\psi}^{-1}(x,t,\lambda)\tilde{\phi}(x,t,\lambda).
\tag{42}
$$

When the potential is on compact support both Jost solutions, as well as the scattering matrix are rational functions of λ.

It can be shown that the continuous spectrum of $\tilde{L}$ lies on those lines in $\mathbb{C}$ on which one or more of the entries of $\mathrm{e}^{-i\lambda\mathcal{J}x}$ oscillate. In other words the continuous spectrum of $\tilde{L}$ is determined by the condition λ belongs to the continuous spectrum $\mathcal{S}_{\mathrm{cont}}$ if and only if $\mathrm{Im}\,\lambda\alpha(\mathcal{J}) = 0$ for some root α of the algebra $\mathfrak{g}$. Using the explicit form of $\mathcal{J}$ from equation (25) one derives that the continuous spectrum of L fills up the set of rays

$$
\mathcal{S}_{\mathrm{cont}} \equiv \bigcup_{\nu=1}^{2h} l_\nu, \qquad l_\nu \equiv \{\lambda;\ \arg\lambda = -\frac{\pi}{2} + (\nu-1)\frac{\pi}{h}, \quad \nu = 1,\ldots,2h\}.
\tag{43}
$$

These rays split $\mathbb{C}$ into $2h$ sectors Ω_ν situated between the rays l_ν and $l_{\nu+1}$. In addition, for each fixed value of ν one finds that the set of roots

$$\Delta_\nu \equiv \{\alpha \,;\, \operatorname{Im} \lambda\alpha(\mathcal{J}) = 0 \text{ for } \lambda \in l_\nu\} \tag{44}$$

forms a root system for a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$.

With the above choice of $\mathcal{J}$ one can check that the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ associated with the rays l_1 and l_2 respectively, have as root systems Δ_1 and Δ_2 where

$$\Delta_1 \equiv \{\pm\alpha \,;\, \alpha \in A_1\}, \quad \Delta_2 \equiv \{\pm\beta \,;\, \beta \in A_2\}. \tag{45}$$

Since any pair of roots $\alpha_i, \alpha_j \in A_0$ and $\beta_i, \beta_j \in A_1$ are orthogonal, then each of the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are direct sums of $\mathfrak{sl}(2)$ subalgebras. The $\mathbb{Z}_h$ reduction condition allows one to check that the subalgebras related to the other rays are obtained from $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by acting with the Coxeter automorphism

$$\Delta_{2\nu+1} \equiv \{\pm\tilde{C}^{\nu-1}\alpha \,;\, \alpha \in A_1\}, \qquad \Delta_{2\nu} \equiv \{\pm\tilde{C}^{\nu-1}\beta \,;\, \beta \in A_2\}. \tag{46}$$

Remark 5. *The scattering matrix $T(\lambda, t)$ of the scattering problem L for $\lambda \in l_\nu$ takes values in the subgroup $\mathcal{G}_\nu$ whose Lie algebra is the subalgebra $\mathfrak{g}_\nu$.*

The next step is to construct FAS $\chi_\nu(x, t, \lambda)$ which retain their analyticity properties inside the sector Ω_ν, closed between the rays l_ν and $l_{\nu+1}$, in the case when the potential is not on compact support. Skipping the details (see [22]) we note that these FAS are related to the Jost solutions by

$$\begin{aligned}
\tilde{\chi}_\nu(x, t, \lambda) &= \tilde{\phi}(x, t, \lambda)S_\nu^+(\lambda, t) \\
&= \tilde{\psi}(x, t, \lambda)T_\nu^-(\lambda, t)D_\nu^+(\lambda), \qquad \lambda \in l_\nu e^{+i0} \\
\tilde{\chi}_\nu(x, t, \lambda) &= \tilde{\phi}(x, t, \lambda)S_{\nu+1}^-(\lambda, t) \\
&= \tilde{\psi}(x, t, \lambda)T_{\nu+1}^+(\lambda, t)D_{\nu+1}^-(\lambda), \quad \lambda \in l_{\nu+1}e^{-i0}
\end{aligned} \tag{47}$$

where the factors $S_\nu^\pm(\lambda, t)$, $T_\nu^\pm(\lambda, t)$ and $D_\nu^\pm(\lambda)$ are related to $T(\lambda, t)$ by its Gauss decomposition

$$T(\lambda, t) = T_\nu^+(\lambda, t)D_\nu^+(\lambda)\hat{S}_\nu^+(\lambda, t) = T_\nu^+(\lambda, t)D_\nu^-(\lambda)\hat{S}_\nu^-(\lambda, t), \qquad \lambda \in l_\nu. \tag{48}$$

More specifically, using for each of the $\mathfrak{sl}(2)$-subalgebras the Cartan-Weyl basis $E_\alpha, E_{-\alpha}, H_\alpha$ or $E_\beta, E_{-\beta}, H_\beta$ respectively we have

$$S_\nu^\pm(\lambda, t) = \exp \boldsymbol{s}_\nu^\pm(\lambda, t), \quad T_\nu^\pm(\lambda, t) = \exp \boldsymbol{\tau}_\nu^\pm(\lambda, t), \quad D_\nu^\pm(\lambda) = \exp \boldsymbol{d}_\nu^\pm(\lambda)$$

where

$$s_{2\nu-1}^\pm(\lambda, t) = \sum_{\alpha \in \Delta_{2\nu-1}^+} \sigma_{2\nu-1;\alpha}^\pm(\lambda, t) E_{\pm\alpha}, \quad s_{2\nu}^\pm(\lambda, t) = \sum_{\beta \in \Delta_{2\nu}^+} \sigma_{2\nu;\beta}^\pm(\lambda, t) E_{\pm\beta}$$

$$\tau_{2\nu-1}^\pm(\lambda, t) = \sum_{\alpha \in \Delta_{2\nu-1}^+} \tau_{2\nu-1;\alpha}^\pm(\lambda, t) E_{\pm\alpha}, \quad \tau_{2\nu}^\pm(\lambda, t) = \sum_{\beta \in \Delta_{2\nu}^+} \tau_{2\nu;\beta}^\pm(\lambda, t) E_{\pm\beta} \quad (49)$$

$$d_{2\nu-1}^+(\lambda) = \sum_{\alpha \in \Delta_{2\nu-1}^+} d_{2\nu-1;\alpha}^\pm H_\alpha, \quad\quad d_{2\nu}^+(\lambda) = \sum_{\beta \in \Delta_{2\nu}^+} d_{2\nu;\beta}^\pm H_\beta.$$

While the triangular Gauss factors $S_\nu^\pm(\lambda, t)$ and $T_\nu^\pm(\lambda, t)$ exist only for $\lambda \in l_\nu$, the diagonal Gauss factors $D_\nu^+(\lambda)$ and $D_{\nu+1}^-(\lambda)$ allow analytic extension inside the whole sector Ω_ν.

As mentioned above, the reduction conditions (31) on the Lax pair impose constraints on the scattering data as follows

i) the $\mathbb{Z}_h$ reduction

$$S_{2\nu+1}^\pm(\lambda, t) = C^{\nu-1} S_1^\pm(\omega^{\nu-1}\lambda, t), \quad\quad T_{2\nu+1}^\pm(\lambda, t) = C^{\nu-1} T_1^\pm(\omega^{\nu-1}\lambda, t)$$
$$S_{2\nu}^\pm(\lambda, t) = C^{\nu-1} S_2^\pm(\omega^{\nu-1}\lambda, t), \quad\quad T_{2\nu}^\pm(\lambda, t) = C^{\nu-1} T_2^\pm(\omega^{\nu-1}\lambda, t) \quad (50)$$
$$D_{2\nu+1}^\pm(\lambda) = C^{\nu-1} D_1^\pm(\omega^{\nu-1}\lambda), \quad\quad D_{2\nu}^\pm(\lambda) = C^{\nu-1} D_2^\pm(\omega^{\nu-1}\lambda).$$

ii) the first $\mathbb{Z}_2$-reduction acts on the complex λ-plane by $\lambda \to \lambda^*$. This means that it acts on the sectors as $\Omega_\nu \to \Omega_{h-\nu+1}$ and on the rays as $l_\nu \to l_{h+2-\nu}$. On the Gauss factors of $T(\lambda, t)$ it acts in the following way

$$C_1(S_\nu^{+,\dagger}(\lambda, t)) = \hat{S}_{h-\nu+2}^-(\lambda^*, t), \quad\quad C_1(D_\nu^{+,\dagger}(\lambda)) = \hat{D}_{h-\nu+2}^-(\lambda^*)$$
$$C_1(T_\nu^{-,\dagger}(\lambda, t)) = \hat{T}_{h-\nu+2}^+(\lambda^*, t). \quad\quad (51)$$

Consequently the coefficients $\tau_{2\nu-1,\alpha}^\pm(\lambda, t), \tau_{2\nu,\beta}^\pm(\lambda, t), \sigma_{2\nu-1,\alpha}^\pm(\lambda, t)$ and $\sigma_{2\nu,\beta}^\pm(\lambda, t)$ are related as follows

$$\sigma_{2\nu-1,\alpha}^-(\lambda, t) = -\sigma_{h-2\nu+1,C_1(\alpha)}^{+,*}(\lambda^*, t), \quad\quad \lambda \in l_{2\nu-1}, \quad \alpha \in \Delta_{2\nu-1}^+$$
$$\tau_{2\nu,\beta}^+(\lambda, t) = -\tau_{h-2\nu+2,C_1(\beta)}^{-,*}(\lambda^*, t), \quad\quad \lambda \in l_{2\nu}, \quad\quad \beta \in \Delta_{2\nu}^+. \quad (52)$$

iii) the second $\mathbb{Z}_2$-reduction acts on λ as $\lambda \to \lambda^*\omega^{-1}$. This means that it acts on the sectors $\Omega_\nu \to \Omega_{h-\nu-1}$ and on the rays as $l_\nu \to l_{h-\nu}$. The action on the Gauss factors is then given by

$$C_2(S_\nu^{+,\dagger}(\lambda, t)) = \hat{S}_{h-\nu}^-(\omega^{-1}\lambda^*, t), \quad C_2(D_\nu^{+,\dagger}(\lambda)) = \hat{D}_{h-\nu}^-(\omega^{-1}\lambda^*)$$
$$C_2(T_\nu^{-,\dagger}(\lambda, t)) = \hat{T}_{h-\nu}^+(\omega^{-1}\lambda^*, t) \quad\quad (53)$$

and the coefficients $\tau^{\pm}_{2\nu-1,\alpha}(\lambda, t)$, $\tau^{\pm}_{2\nu,\beta}(\lambda, t)$, $\sigma^{\pm}_{2\nu-1,\alpha}(\lambda, t)$ and $\sigma^{\pm}_{2\nu,\beta}(\lambda, t)$ are related by

$$
\begin{aligned}
\sigma^{-}_{2\nu-1,\alpha}(\lambda, t) &= -\sigma^{+,*}_{h-2\nu-3,C_2(\alpha)}(\omega^{-1}\lambda^*, t), \qquad \lambda \in l_{2\nu-1}, \quad \alpha \in \Delta^{+}_{2\nu-1} \\
\tau^{+}_{2\nu,\beta}(\lambda, t) &= -\tau^{-,*}_{h-2\nu-1,C_2(\beta)}(\omega^{-1}\lambda^*, t), \qquad \lambda \in l_{2\nu}, \qquad \beta \in \Delta^{+}_{2\nu}.
\end{aligned}
\tag{54}
$$

3.2. The Time Evolution of the Scattering Data

The Lax representation with L and M as in (32), (35) allows one to solve a system of NLEEs for $q_j(x, t)$. We will give examples of such systems in the next sections. Here we just note that the Lax representation determines the t-dependence of the scattering matrix (and its Gauss factors) as follows

$$
\begin{aligned}
\mathrm{i}\frac{\partial T_\nu}{\partial t} - \lambda^N [\mathcal{K}_N, T_\nu(\lambda, t)] &= 0, \qquad \mathrm{i}\frac{\partial S^{\pm}_\nu}{\partial t} - \lambda^N [\mathcal{K}_N, S^{\pm}_\nu(\lambda, t)] = 0 \\
\mathrm{i}\frac{\partial T^{\pm}_\nu}{\partial t} - \lambda^N [\mathcal{K}_N, T^{\pm}_\nu(\lambda, t)] &= 0, \qquad\qquad\qquad\quad \mathrm{i}\frac{\partial D^{\pm}_\nu}{\partial t} = 0.
\end{aligned}
\tag{55}
$$

Since these equations can be immediately solved, thus one finds the evolution in time. In particular, the last equations in (55) show that the functions $D^{\pm}_\nu(\lambda)$ are time-independent, i.e., they can be viewed as generating functionals of the integrals of motion of the corresponding NLEEs.

4. The Inverse Scattering Problem and the Riemann-Hilbert Problem

Of course, finding the time evolution of the Gauss factors is only a step towards finding the solutions for the corresponding NLEE. One should be able to construct from the Gauss factors the solutions, a process called Inverse Scattering Transform (IST). We shall outline here how one can do it reducing the IST for the GZS system to a local Riemann-Hilbert problem (RHP). Indeed, on the ray l_ν we have

$$
\xi_\nu(x, t, \lambda) = \xi_{\nu-1}(x, t, \lambda) G_\nu(x, t, \lambda), \qquad \lambda \in l_\nu
$$

$$
G_\nu(x, t, \lambda) = \mathrm{e}^{-\mathrm{i}(\lambda \mathcal{J} x + \lambda^N \mathcal{K}_N t)} G_{0,\nu}(\lambda) \mathrm{e}^{\mathrm{i}(\lambda \mathcal{J} x + \lambda^N \mathcal{K}_N t)}
\tag{56}
$$

where $G_{0,\nu}(\lambda) = \hat{S}^{-}_\nu S^{+}_\nu(\lambda, t)\big|_{t=0}$ and $\xi_\nu, \xi_{\nu-1}$ are functions analytic in the sectors $\Omega_\nu, \Omega_{\nu-1}$. The collection of all relations (56) for $\nu = 1, 2, \ldots, 2N$ together with the condition

$$
\lim_{\lambda \to \infty} \xi_\nu(x, t, \lambda) = \mathbb{1}
\tag{57}
$$

can be viewed as a local RHP with canonical normalization posed on the collection of rays $\Sigma \equiv \{l_\nu\}^{2N}_{\nu=1}$. The canonical normalization implies that each solution of the RHP possesses asymptotic expansion of the form

$$
\xi_\nu(x, t, \lambda) = \exp\left(\sum_{s=1}^{\infty} \lambda^{-s} Q_s(x, t) \right)
\tag{58}
$$

where $Q_s(x,t)$ are elements of the Lie algebra $\mathfrak{g}$. The $\mathbb{Z}_h$-reduction means that $Q_s(x,t) \in \mathfrak{g}^{(k)}$. Quite straightforwardly one can prove that if $\xi_\nu(x,\lambda)$ is a solution of the RHP (56), (57) then $\chi_\nu(x,\lambda) = \xi_\nu(x,\lambda)e^{-i\lambda\mathcal{J}x}$ is a FAS of $\tilde{L}$ with potential

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda\left(\mathcal{J} - \xi_\nu(x,t,\lambda)\mathcal{J}\hat{\xi}_\nu(x,t,\lambda)\right) = [\mathcal{J}, Q_1(x,t)]. \qquad (59)$$

In what follows we will consider two classes of solutions to the RHP: i) the class of regular solutions which have no singularities in their sectors of analyticity, ii) the class of singular solutions, which allow both poles and zeros[1] in their regions of analyticity.

Each regular solution of the RHP (56), (57) is determined uniquely by the sewing functions $G_\nu(x,t,\lambda)$, which due to (50) also satisfy the $\mathbb{Z}_h$ reduction condition. Therefore it is enough to know the sewing functions on the rays l_1 and l_2 in order to calculate the whole set of sewing functions $G_\nu(x,t,\lambda)$. Thus the minimal set of scattering data that determines the regular solution $\xi^\nu(x,t,\lambda)$ are given by

$$\mathcal{T} \equiv \{\sigma_{1;\alpha}^\pm(\lambda,t);\ \alpha \in A_1,\ \lambda \in l_1\} \cup \{\sigma_{2;\beta}^\pm(\lambda,t);\ \beta \in A_2,\ \lambda \in l_2\}. \qquad (60)$$

In other words the set $\mathcal{T}$ contains r functions of λ and t, each defined on a certain ray and from $\mathcal{T}$ one must recover r scalar functions defined on the real axis – the potential $Q(x,t)$, see equation (38). The singular solutions to the RHP and the related soliton solutions of the corresponding NLEE can be derived using the the dressing Zakharov-Shabat method [32,45]. Starting from the trivial solution of the RHP we obtain explicit rational solutions of the RHP.

5. The Recursion Operators and the NLEEs

One of the important steps in the theory is the explicit derivation of the relevant NLEEs. Below we assume that $N \mod (h)$ is an exponent of $\mathfrak{g}$. We will follow [1] and construct recurrent relations for calculating the coefficients $\tilde{V}_s(x,t)$ in the $\tilde{M}$-operator (35) in terms of $Q(x,t)$ and its derivatives. Doing this it is natural to use the condition that $\tilde{L}$ and $\tilde{M}$ commute *identically* with respect to λ. Equating to zero all the coefficients of the positive powers in λ in $[\tilde{L}, \tilde{M}]$ we obtain a set of recurrent relations for the functions $\tilde{V}_s(x,t)$. Note, that due to the $\mathbb{Z}_h$ reduction, the potentials of both $\tilde{L}$ and $\tilde{M}$ take values in the graded algebra, which requires generalization of the AKNS method and a substantially new structure of the recursion

---

[1] by a zero of $\xi^\nu(x,t,\lambda)$ at $\lambda_{\nu,k} \in \Omega_\nu$ here we mean that $\det \xi^\nu(x,t,\lambda_{\nu,k}) = 0$.

operators. Thus we have

$$
\begin{aligned}
\lambda^{N+1} : && [\mathcal{J}, \mathcal{K}_N] &= 0 \\
\lambda^{N} : && [\mathcal{J}, \tilde{V}_1(x,t)] + [Q(x,t), \mathcal{K}_N] &= 0 \\
\lambda^{N-s} : && \mathrm{i}\frac{\partial \tilde{V}_s}{\partial x} + [Q(x,t), \tilde{V}_s(x,t)] - [\mathcal{J}, \tilde{V}_{s+1}(x,t)] &= 0 \\
\lambda^{0} : && -\mathrm{i}\frac{\partial Q}{\partial t} + \mathrm{i}\frac{\partial \tilde{V}_N}{\partial x} + [Q(x,t), \tilde{V}_N(x,t)] &= 0.
\end{aligned}
\tag{61}
$$

The first of the above equations is satisfied identically. The second can be resolved as

$$
\tilde{V}_1(x,t) = \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)
\tag{62}
$$

where $\mathrm{ad}_{\mathcal{J}} \tilde{X} \equiv [\mathcal{J}, X]$. The operator $\mathrm{ad}_{\mathcal{J}}$ obviously has a kernel, and its inverse $\mathrm{ad}_{\mathcal{J}}^{-1}$ is defined only on its image. Thus there naturally appear the necessity to split each of the coefficients $\tilde{V}_s(x,t)$ into 'orthogonal' and 'parallel' parts

$$
\tilde{V}_s(x,t) = \tilde{V}_s^{\perp}(x,t) + \tilde{V}_s^{\parallel}(x,t), \quad [\tilde{V}_s^{\parallel}(x,t), \mathcal{J}] = 0, \quad \langle \tilde{V}_s^{\perp}(x,t), \mathcal{H} \rangle = 0.
\tag{63}
$$

where $\mathcal{H}$ is any element of the Cartan subalgebra. Since $\tilde{V}_s(x,t) \in \mathfrak{g}^{(N-s)}$ in fact we need to split each of the subspaces $\mathfrak{g}^{(N-s)}$ into

$$
\mathfrak{g}^{(N-s)} = \mathfrak{g}^{(N-s)\perp} \oplus \mathfrak{g}^{(N-s)\parallel}
$$

$$
V_s = V_s^{\perp} + V_s^{\parallel}, \quad V_s^{\parallel} = \begin{cases} 0 & \text{if } s \text{ is not an exponent} \\ c_s^{-1}\mathcal{J}^s\langle V_s, \mathcal{J}^{h-N+s}\rangle & \text{if } s \text{ is an exponent} \end{cases}
\tag{64}
$$

where $c_s = \langle \mathcal{J}^{h-N+s}s, \mathcal{J}^{N-s} \rangle$.

The above formulae hold true for the classical series of algebras A_r, B_r and C_r while the D_r series requires more care and will be discussed elsewhere.

Note that we can always fix up the gauge so that $Q(x,t) \equiv Q^{\perp}(x,t)$. Then we have

$$
\tilde{V}_1(x,t) = \sum_{\alpha \in A_1} \frac{\alpha(\mathcal{K}_N)}{\alpha(\mathcal{J})} q_\alpha(x,t)\mathcal{E}_\alpha^{(N-1)} + \sum_{\beta \in A_2} \frac{\alpha(\mathcal{K}_N)}{\beta(\mathcal{J})} q_\beta(x,t)\mathcal{E}_{-\beta}^{(N-1)}.
\tag{65}
$$

In doing this we used the commutation relations (28).

Equation (65) provides the initial condition for the recurrent relations. They are determined from the third line of (61) where we must insert the splitting of $V_s(x,t)$ and $V_{s+1}(x,t)$ according to (64).

5.1. The Case of A_r

Let us consider first the A_r series, for which all numbers $1, 2, \ldots, r$ are exponents. Then we obtain the following two equations

$$\tilde{V}^{\perp}_{s+1}(x,t) = \operatorname{ad}\,{}^{-1}_{\mathcal{J}} \left(\mathrm{i}\frac{\partial \tilde{V}^{\perp}_s}{\partial x} + [Q(x,t), \tilde{V}^{\perp}_s] + [Q(x,t), \tilde{V}^{\parallel}_s] \right)$$

$$\mathrm{i}\left\langle \frac{\partial \tilde{V}^{\parallel}_s}{\partial x}, \mathcal{J}^s \right\rangle = \langle [Q(x,t), \tilde{V}^{\perp}_s]\mathcal{J}^s \rangle. \tag{66}$$

Integrating formally the second one we obtain

$$\tilde{V}^{\parallel}_s(x,t) = \mathrm{i}(\partial_x)^{-1}_{\pm} \left([Q(x,t), \tilde{V}^{\perp}_s] \right)^{\parallel}$$
$$= \mathrm{i}c_s^{-1}\mathcal{J}^{N-s}\partial_x^{-1}\langle [Q(x,t), \tilde{V}^{\perp}_s], \mathcal{J}^{h-N+s} \rangle + \tilde{v}_{s0} \tag{67}$$

where $(\partial_x)^{-1}_{\pm}\cdot = \displaystyle\int^x_{\pm\infty} \mathrm{d}y \cdot$ and $\tilde{v}_{s0} = \mathrm{const}$. Then we can write down the formal solution of the recurrent relations in the form

$$\tilde{V}^{\perp}_{s+1}(x,t) = \Lambda^{\pm}_{N-s}\tilde{V}^{\perp}_s(x,t) + \tilde{v}_{s0}[Q(x,t), \mathcal{J}^{N-s}]$$

$$\Lambda^{\pm}_{N-s}X^{\perp}_s = \operatorname{ad}\,{}^{-1}_{\mathcal{J}} \left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}_s}{\partial x} + [Q(x,t), \tilde{X}^{\perp}_s] \right.$$

$$\left. + \mathrm{i}c_s^{-1}[Q(x,t), \mathcal{J}^{N-s}](\partial_x)^{-1}_{\pm}\langle [Q(y,t), X^{\perp}_s(y)]\mathcal{J}^{h-N+s} \rangle \right). \tag{68}$$

Further, taking for simplicity $\tilde{v}_{s0} = 0$ the solution to these recursion relation is

$$\tilde{V}_s(x,t) = \Lambda^{\pm}_{N-s+1}\Lambda^{\pm}_{N-s+2}\cdots\Lambda^{\pm}_{N-1}\operatorname{ad}\,{}^{-1}_{\mathcal{J}}[\mathcal{K}_N, Q(x,t)] \tag{69}$$

for $s = 2, \ldots, N$. The corresponding NLEEs can be written in compact form as

$$\mathrm{i}\operatorname{ad}\,{}^{-1}_{\mathcal{J}}\frac{\partial Q}{\partial t} - f_N\Lambda_0\Lambda^{\pm}_1\Lambda^{\pm}_2\cdots\Lambda^{\pm}_{N-1}\operatorname{ad}\,{}^{-1}_{\mathcal{J}}[\mathcal{K}_N, Q(x,t)] = 0 \tag{70}$$

where

$$\Lambda_0\tilde{X}^{\perp}_{2p+1} = \operatorname{ad}\,{}^{-1}_{\mathcal{J}} \left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}_{2p+1}}{\partial x} + [Q(x,t), \tilde{X}^{\perp}_{2p+1}] \right). \tag{71}$$

The dispersion law of the NLEE (70) is $f_N\lambda^N$. In the simplest cases $N = 2$ and $N = 3$ we get

$$\tilde{V}_N(x,t) = \Lambda^{\pm}_1\operatorname{ad}\,{}^{-1}_{\mathcal{J}}[\mathcal{K}_2, Q(x,t)], \quad \tilde{V}_N(x,t) = \Lambda^{\pm}_1\Lambda^{\pm}_2\operatorname{ad}\,{}^{-1}_{\mathcal{J}}[\mathcal{K}_3, Q(x,t)]$$

$$\mathcal{K}_2 = f_2\mathcal{J}^2, \qquad\qquad\qquad \mathcal{K}_3 = f_3\mathcal{J}^3. \tag{72}$$

The recursion operators $\Lambda_{N-s}^{\pm}$ can be obtained as the restriction of the operators $\Lambda_{\pm}$ which are the recursion operators for potential without any restrictions

$$
\begin{aligned}
\Lambda^{\pm}\tilde{X}^{\perp} = \mathrm{ad}\,_{\mathcal{J}}^{-1} & \left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}}{\partial x} + [Q(x,t),\tilde{X}_s^{\perp}] \right. \\
& \left. + \mathrm{i}\sum_{s=1}^{r} c_{N-s}^{-1}[Q(x,t),\mathcal{J}^s](\partial_x)_{\pm}^{-1}\langle[Q(y,t),\tilde{X}_s^{\perp}(y)],\mathcal{J}^{h-s}\rangle \right)
\end{aligned}
\tag{73}
$$

by restricting them onto the subspaces $\mathfrak{g}^{(N-s)}$. Indeed, as readily seen, $\Lambda_{\pm}$ maps $\mathfrak{g}^{(s)}$ into $\mathfrak{g}^{(s-1)}$. However, practically it is much better to have expressions where the grading can be observed explicitly.

We end this subsection by giving an example of integrable NLEE known as the $\mathbb{Z}_h$-reduced derivative NLS equation. The Lax operator $\tilde{L}$ is parametrized by

$$
Q(x,t) = \sum_{j=1}^{N-1}\psi_j(x,t)J_j^{(0)}, \qquad \mathcal{J} = -a\omega^{-1/2}J_0^{(1)}.
\tag{74}
$$

The $\tilde{M}$-operator is quadratic in λ with

$$
\begin{aligned}
V_1(x,t) &= \sum_{k=1}^{N}v_{1,k}(x,t)J_j^{(1)}, \qquad v_{1,p} = -\frac{b}{a}\omega^{(p+1)/2}\cos\left(\frac{p\pi}{N}\right)\psi_p(x,t) \\
V_0(x,t) &= \sum_{k=1}^{N-1}v_{0,k}(x,t)J_j^{(0)}, \qquad V_2 = -bJ_0^{(2)}
\end{aligned}
\tag{75}
$$

where

$$
v_{0,p} = \gamma\left(\mathrm{i}\cotan\frac{p\pi}{N}\,\psi_{p,x} - \sum_{k+s=p}^{N}\psi_k\psi_s(x,t) \right), \qquad \gamma = \frac{b\omega}{a^2}.
\tag{76}
$$

The λ-independent term in the Lax representation vanishes whenever the functions ψ_k satisfy the $\mathbb{Z}_h$-reduced derivative NLS equation [13, 17]

$$
\frac{\partial q_k}{\partial t} + \gamma\cotan\left(\frac{\pi k}{N}\right)\frac{\partial^2 q_k}{\partial x^2} - \gamma\sum_{p=1}^{N-1}\frac{\partial}{\partial x}(q_p q_{k-p}) = 0, \qquad k = 1, 2, \ldots, r.
\tag{77}
$$

Its dispersion law is $\gamma\lambda^2$.

5.2. The Case of B_r and C_r

Consider now the series B_r and C_r. For them all the odd numbers $1, 3, \ldots, 2r-1$ are exponents. Note also that now N must be odd: $N = 2k + 1$. Again we obtain

two equations for each s

$$\tilde{V}^{\perp}_{s+1}(x,t) = \mathrm{ad}\,_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{V}^{\perp}_s}{\partial x} + [Q(x,t),\tilde{V}^{\perp}_s] + [Q(x,t),\tilde{V}^{\|}_s]\right)$$

$$\mathrm{i}\left\langle\frac{\partial \tilde{V}^{\|}_s}{\partial x}, \mathcal{J}^s\right\rangle = \langle[Q(x,t),\tilde{V}^{\perp}_s]\mathcal{J}^s\rangle \tag{78}$$

but this time they have to be considered separately for even $s = 2p$ and odd $s = 2p + 1$ values of s. Indeed, $\tilde{V}_{2p} \in \mathfrak{g}^{(2k-2p+1)}$ and therefore $\tilde{V}^{\|}_{2p}$ are nontrivial, while $\tilde{V}_{2p+1} \in \mathfrak{g}^{(2k-2p)}$ and therefore $\tilde{V}^{\|}_{2p+1} = 0$. Applying similar technique as above we obtain

$$\tilde{V}^{\perp}_{2p+2}(x,t) = \Lambda_0\tilde{V}^{\perp}_{2p+1}(x,t)$$

$$\tilde{V}^{\perp}_{2p+1}(x,t) = \Lambda^{\pm}_{N-2p}\tilde{V}^{\perp}_{2p}(x,t) + \tilde{v}_{2p,0}[Q(x,t),\tilde{\mathcal{J}}^{N-2p}] \tag{79}$$

where

$$\Lambda_0\tilde{X}^{\perp}_{2p+1} = \mathrm{ad}\,_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}_{2p+1}}{\partial x} + [Q(x,t),\tilde{X}^{\perp}_{2p+1}]\right) \tag{80}$$

$$\Lambda^{\pm}_{N-2p}\tilde{X}^{\perp}_{2p} = \mathrm{ad}\,_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}_{2p}}{\partial x} + [Q(x,t),\tilde{X}^{\perp}_{2p}]\right.$$

$$\left.+\frac{1\mathrm{i}}{c_{2s}}[Q(x,t),\mathcal{J}^{N-2p}](\partial_x)^{-1}_{\pm}\langle[Q(y,t),\tilde{X}^{\perp}_{2p}(y)],\mathcal{J}^{h-N+2p}\rangle\right). \tag{81}$$

The formal solutions for $\tilde{V}_s(x,t)$ in terms of the recursion operators, using again for simplicity $\tilde{v}_{s0} = 0$ is

$$\tilde{V}_{2p}(x,t) = \Lambda_0\Lambda^{\pm}_{N-2p+2}\Lambda_0\Lambda^{\pm}_{N-2p+4}\cdots\Lambda_0\Lambda^{\pm}_{N-2}\Lambda_0\mathrm{ad}\,_{\mathcal{J}}^{-1}[\mathcal{K}_N,Q(x,t)]$$

$$\tilde{V}_{2p+1}(x,t) = \Lambda^{\pm}_{N-2p}\Lambda_0\Lambda^{\pm}_{N-2p+2}\Lambda_0\cdots\Lambda^{\pm}_{N-2}\Lambda_0\mathrm{ad}\,_{\mathcal{J}}^{-1}[\mathcal{K}_N,Q(x,t)] \tag{82}$$

for $p = 2,\ldots,k$, $N = 2k+1$. The corresponding NLEEs can be written implicitly as

$$\mathrm{i}\mathrm{ad}\,_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_N\Lambda_0\tilde{V}_N(x,t) = 0 \tag{83}$$

or more explicitly

$$\mathrm{i}\mathrm{ad}\,_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_N\Lambda_0\Lambda^{\pm}_1\Lambda_0\Lambda^{\pm}_3\Lambda_0\cdots\Lambda^{\pm}_{N-2}\Lambda_0\mathrm{ad}\,_{\mathcal{J}}^{-1}[\mathcal{K}_N,Q(x,t)] = 0. \tag{84}$$

In the simplest cases $N = 3$ and $N = 5$ we get

$$\tilde{V}_3(x,t) = \Lambda_1^{\pm}\Lambda_0\text{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_3, Q(x,t)], \qquad \mathcal{K}_3 = f_3\mathcal{J}^3$$

$$\tilde{V}_5(x,t) = \Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0\text{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_5, Q(x,t)], \qquad \mathcal{K}_5 = f_5\mathcal{J}^5. \tag{85}$$

Then the corresponding NLEEs

$$\text{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_3\Lambda_0\Lambda_1^{\pm}\Lambda_0\text{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^3, Q(x,t)] = 0$$

$$\text{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_5\Lambda_0\Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0\text{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^5, Q(x,t)] = 0 \tag{86}$$

will be systems of differential equations of order 3 and 5 respectively for the r independent functions q_α and q_β.

We will call Λ_0 and Λ_{2k-1} elementary recursion operators. Along with them we will introduce r fundamental recursion operators

$$\boldsymbol{\Lambda}_{(1)} = \Lambda_0, \qquad\qquad \boldsymbol{\Lambda}_{(3)} = \Lambda_0\Lambda_1\Lambda_0$$

$$\boldsymbol{\Lambda}_{(m_k)} = \Lambda_0\Lambda_1\Lambda_0\cdots\Lambda_{m_k-2}\Lambda_0, \qquad k = 1,\ldots r. \tag{87}$$

Each of these recursion operators generates an MKdV-type of NLEE

$$\text{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_{m_k}\boldsymbol{\Lambda}_{m_k}\text{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^{m_k}, Q(x,t)] = 0 \tag{88}$$

where $m_k = 2k - 1$ is an exponent of $\mathfrak{g}$. The equation (88) is a system of r equations whose highest order derivative with respect to x equals m_k. Each of them is a simplest member of a hierarchy of NLEE generated by the master recursion operator

$$\boldsymbol{\Lambda}_{\pm} = \Lambda_0\Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0\cdots\Lambda_{h-1}^{\pm}\Lambda_0 \tag{89}$$

namely

$$\text{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_{m_k+hp}\boldsymbol{\Lambda}^p\boldsymbol{\Lambda}_{m_k}\text{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^{m_k}, Q(x,t)] = 0, \qquad p = 1, 2,\ldots. \tag{90}$$

The corresponding M-operators are polynomials in λ of degree $m_k + ph$.

Remark 6. *The equation (88) with $m_k = 1$ is in fact linear evolution equation. However its hierarchy (90) starting with $p = 1$ is nontrivial.*

Remark 7. *The algebra $\mathfrak{so}(3)$ is of rank 1 and its Coxeter number is $h = 2$. It has only one exponent equal to 1. Thus it has only one hierarchy. The MKdV equation is of the form (90) with $p = 1$.*

Again the recursion operators Λ_0 and $\Lambda^{\pm}_{N-2p}$ can be obtained from the operators $\Lambda_{\pm}$ for the general case

$$
\begin{aligned}
\Lambda^{\pm}\tilde{X}^{\perp} = \operatorname{ad}^{-1}_{\mathcal{J}} &\left(\mathrm{i}\frac{\partial \tilde{X}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}^{\perp}_s] \right. \\
&\left. + \mathrm{i}\sum_{s=1}^{r} c^{-1}_{N-s}[Q(x,t), \mathcal{J}^s](\partial_x)^{-1}_{\pm}\langle[Q(y,t), \tilde{X}^{\perp}_s(y)], \mathcal{J}^{h-s}\rangle \right)
\end{aligned}
\tag{91}
$$

by restricting them onto the subspace $\mathfrak{g}^{(N-2p)}$. Note that $\Lambda_{\pm}$ maps $\mathfrak{g}^{(s)}$ into $\mathfrak{g}^{(s-1)}$. The simplest NLEE we obtain in the above way would be of MKdV-type, i.e., this would be systems of r equations which contains third derivative with respect to x. We end this Section recalling briefly the equivalence of the inverse scattering problem for $\tilde{L}$ to the RHP (56), (57). As we mentioned above the solution of the RHP allows the asymptotic expansion (58) which can be used to prove the relation

$$
\begin{aligned}
W_{\nu,N} = \xi_{\nu}\mathcal{J}^N\hat{\xi}_{\nu}(x,t,\lambda) = \mathcal{J}^N + \sum_{s=1}^{\infty}\frac{1}{s!}\operatorname{ad}^s_{\mathcal{Q}}\mathcal{J} \\
= \mathcal{J}^N + \lambda^{-1}\operatorname{ad}_{Q_1}\mathcal{J}^N + \lambda^{-2}\left(\operatorname{ad}_{Q_2}\mathcal{J}^N + \operatorname{ad}^2_{Q_1}\mathcal{J}^N\right) + \cdots
\end{aligned}
\tag{92}
$$

where $\mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty}\lambda^{-s}Q_s(x,t)$. It is easy to check that i) for $N = m_k + s_0 h$, where m_k is an exponent of $\mathfrak{g}$, we have that $W_{\nu,N}(x,t,\lambda) \in \mathfrak{g}^{(m_k)}$ is analytic function of λ in the sector Ω_{ν}; ii) the right hand sides of the expansions (92) in fact do not depend on ν so we will skip the index ν in what follows; iii) we can split $Y_N = \lambda^N W_N(x,t,\lambda)$ into

$$
\begin{aligned}
\lambda^N W_N(x,t,\lambda) &= (Y_N(x,t,\lambda))_+ + (Y_N(x,t,\lambda))_- \\
(Y_N(x,t,\lambda))_+ &= \lambda^N\mathcal{J} - \sum_{s=1}^{N}\lambda^{N-s}\tilde{V}_s(x,t) \\
(Y_N(x,t,\lambda))_- &= -\sum_{s=N+1}^{\infty}\lambda^{N-s}\tilde{V}_s(x,t).
\end{aligned}
\tag{93}
$$

The importance of the splitting (93) is demonstrated by the next Lemma.

Lemma 8. *The generating functional of the M-operators of the $\mathbb{Z}_h$ and $\mathbb{D}_h$-reduced NLEEs are provided by $Y_N(x,t,\lambda)$ where N can take the values $N = m_k + s_0 h$ and m_k is an exponent of the algebra $\mathfrak{g}$. The corresponding M-operator takes the form*

$$
M_N \equiv \mathrm{i}\frac{\partial}{\partial t} - (Y_N(x,t,\lambda))_+
\tag{94}
$$

and the corresponding NLEEs *can be written in the form*

$$
\mathrm{i}\frac{\partial Q}{\partial t} + [\mathcal{J}, V_{N+1}] = 0.
\tag{95}
$$

Proof: It is easy to check, using the relation between ξ_ν and χ_ν

$$\xi_\nu(x,t,\lambda) = \chi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} \tag{96}$$

that $Y_N(x,t,\lambda)$ is a solution to the equation

$$\mathrm{i}\frac{\partial Y_N}{\partial x} + [Q - \lambda\mathcal{J}, Y_N(x,t,\lambda)] = 0. \tag{97}$$

Therefore the compatibility condition $[\tilde{L}, \tilde{M}] = 0$ with $\tilde{M}$ chosen as in (94) gives

$$\begin{aligned}
\mathrm{i}\frac{\partial(Y_N)_+}{\partial t} &- \mathrm{i}\frac{\partial Q}{\partial t} + [Q - \lambda\mathcal{J}, (Y_N)_+] \\
&= -\mathrm{i}\frac{\partial(Y_N)_-}{\partial t} - \mathrm{i}\frac{\partial Q}{\partial t} - [Q - \lambda\mathcal{J}, (Y_N)_+] \\
&= -\mathrm{i}\frac{\partial Q}{\partial t} + [\mathcal{J}, \tilde{V}_N] + \mathcal{O}(\lambda^{-1}).
\end{aligned} \tag{98}$$

But by definition L and M are polynomial in λ. The lemma is proved. ∎

6. The Wronskian Relations and the Effects of Reduction

6.1. The Mapping $\mathcal{F}$

We start with the Wronskian relations

$$\left(\hat{\tilde{\chi}}_\nu \mathcal{K} \tilde{\chi}_\nu(x,t,\lambda) - \mathcal{K}\right)\Big|_{x=-\infty}^{\infty} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}y \, \left(\hat{\tilde{\chi}}_\nu [\mathcal{K}, Q(y,t)] \tilde{\chi}_\nu(y,t,\lambda)\right). \tag{99}$$

Let us outline the technicalities in deriving the basic relations describing the mapping $\mathcal{F}$. In doing this it will be enough to consider these relations for the two rays l_1 and l_2. The left hand sides of (99) take the form

$$\begin{aligned}
\mathrm{i}(\hat{D}_1^+ \hat{T}_1^- \mathcal{K}(T_1^- D_1^+) - \hat{S}_1^+ \mathcal{K} S_1^+), &\qquad \text{for } \lambda \in l_1 \\
\mathrm{i}(\hat{D}_2^- \hat{T}_2^+ \mathcal{K}(T_2^+ D_2^-) - \hat{S}_2^- \mathcal{K} S_2^-), &\qquad \text{for } \lambda \in l_2.
\end{aligned} \tag{100}$$

Next we will multiply both sides of (99) by E_γ and apply the Killing form. After this operation the right hand side of (99) acquires the form

$$\int_{-\infty}^{\infty} \mathrm{d}y \, \left\langle \hat{\tilde{\chi}}_\nu [\mathcal{K}, Q(y,t)] \tilde{\chi}_\nu(x,t,\lambda), E_\gamma \right\rangle$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y \, \langle [\mathcal{K}, Q(y,t)], e_{\nu,\gamma}(x,t,\lambda) \rangle = -[[[\mathcal{K}, \mathrm{ad}\,_{\mathcal{J}} Q(y,t)], e_{\nu,\gamma}(y,t,\lambda)]] \tag{101}$$

where we have introduced the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ and the skew-scalar product $[\![X,Y]\!]$ as follows

$$e_{\nu,\gamma}(x,t,\lambda) = (\tilde{\chi}_\nu(x,t,\lambda)E_\gamma\hat{\tilde{\chi}}_\nu)^\perp$$

$$[\![X,Y]\!] = \int_{-\infty}^{\infty} dx \ \langle X(x), [\mathcal{J}, Y(y)] \rangle .$$

(102)

Then using equations (49) after some calculations we obtain the following relations

$$\sigma_{1,\alpha}^+(\lambda,t) = \frac{i}{\alpha(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,-\alpha}(x,t,\lambda)]\!], \quad \lambda \in l_1$$

$$\sigma_{2,\beta}^-(\lambda,t) = \frac{1}{i\beta(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,\beta}(x,t,\lambda)]\!], \quad \lambda \in l_2$$

$$\tilde{\tau}_{1,\alpha}^-(\lambda,t) = \frac{i}{\alpha(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,\alpha}(x,t,\lambda)]\!], \quad \lambda \in l_1$$

$$\tilde{\tau}_{2,\beta}^+(\lambda,t) = \frac{1}{i\beta(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,-\beta}(x,t,\lambda)]\!], \quad \lambda \in l_2$$

(103)

where $\alpha \in A_1$, $\beta \in A_2$ and

$$\tilde{\tau}_{1,\alpha}^-(\lambda,t) = \tau_{1,\alpha}^-(\lambda,t)e^{(\alpha,\alpha)d_{1,\alpha}^+}, \qquad \tilde{\tau}_{2,\beta}^+(\lambda,t) = \tau_{2,\beta}^+(\lambda,t)e^{-(\beta,\beta)d_{2,\beta}^-}.$$

These relations are fundamental for the analysis of the mapping $\mathcal{F}$ between the space of allowed potentials $Q(x,t)$ and the minimal set of scattering data. The main conclusion from them is that $\mathcal{F}$ has the meaning of generalized Fourier transform in which the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ play the role of generalized exponents. Of course one must prove that $e_{\nu,\gamma}(x,t,\lambda)$ form complete set of functions in the space of allowed potentials. This can be done applying the contour integration method to a certain Green functions. This will be done elsewhere. Here we remark that the 'squared solutions' are eigenfunctions of the recursion operators. To state this more precisely we write down each 'squared solution' as sum of its projections according to the grading of $\mathfrak{g}$

$$e_{\nu,\gamma}(x,t,\lambda) = \sum_{s=0}^{h-1} e_{\nu,\gamma}^{(s)}(x,t,\lambda), \qquad e_{\nu,\gamma}^{(s)}(x,t,\lambda) \in \mathfrak{g}^{(s)}$$

(104)

and each of this projections should be split into orthogonal and parallel part as in equation (63)

$$e_{\nu,\gamma}^{(s)}(x,t,\lambda) = e_{\nu,\gamma}^{(s),\perp}(x,t,\lambda) + e_{\nu,\gamma}^{(s),\|}(x,t,\lambda), \qquad e_{\nu,\gamma}^{(2s),\|}(x,t,\lambda) = 0$$

$$e_{\nu,\gamma}^{(2s+1),\|}(x,t,\lambda) = \frac{1}{c_s} \mathcal{J}^{2s+1} \left\langle \mathcal{J}^{h-2s-1}, e_{\nu,\gamma}^{(2s+1)}(x,t,\lambda) \right\rangle .$$

(105)

It is also easy to check that the squared solutions satisfy the equation

$$i \frac{\partial e_{\nu,\gamma}}{\partial x} + [Q(x), e_{\nu,\gamma}(x,t,\lambda)] - \lambda[\mathcal{J}, e_{\nu,\gamma}(x,t,\lambda)] = 0.$$

(106)

Inserting the splitting (104) into (106) we get

$$\mathrm{i}\,\frac{\partial e^{(s)}_{\nu,\gamma}}{\partial x} + [Q(x), e^{(s)}_{\nu,\gamma})(x,t,\lambda)] - \lambda[\mathcal{J}, e^{(s-1)}_{\nu,\gamma})(x,t,\lambda)] = 0 \qquad (107)$$

$s = 0, 1, \ldots, h - 1$. Next we insert the splitting (105) and express the parallel parts of the squared solutions through the orthogonal ones. The calculations are similar to the ones in Subsection 5.1. Skipping the details we obtain

$$\Lambda_0 e^{(2k),\perp}_{\nu,\gamma}(x,t,\lambda) = \lambda e^{(2k-1),\perp}_{\nu,\gamma}(x,t,\lambda)$$

$$\Lambda^{\pm}_{2k-1} e^{(2k-1),\perp}_{\nu,\gamma}(x,t,\lambda) = \lambda e^{(2k),\perp}_{\nu,\gamma}(x,t,\lambda) - \frac{a^{\pm}_k}{c_k}\mathrm{ad}\,_{\mathcal{J}}^{-1}[Q(x), \mathcal{J}^{2k-1}] \qquad (108)$$

where

$$a^{\pm}_k = \lim_{x\to\pm\infty}\left\langle \mathcal{J}^{h-2k+1}, e^{(2k-1)}_{\nu,\gamma}(x,t,\lambda)\right\rangle. \qquad (109)$$

If we choose ν and γ in such a way, that the constants $a^{\pm}_k = 0$ we find that $e^{(h-1),\perp}_{\nu,\gamma}(x,t,\lambda)$ are eigenfunctions of the master recursion operator (89)

$$\Lambda_{\pm} e^{(h-1),\perp}_{\nu,\gamma}(x,t,\lambda) = \lambda^h e^{(h-1)}_{\nu,\gamma}(x,t,\lambda). \qquad (110)$$

6.2. The Mapping $\delta\mathcal{F}$

The mapping between the variations of the potential and the variation of the scattering data is based on the following Wronskian relation

$$\left(\mathrm{i}\hat{\tilde{\chi}}_{\nu}\delta\tilde{\chi}_{\nu}(x,t,\lambda)\right)\Big|^{\infty}_{x=-\infty} = -\int_{-\infty}^{\infty} \mathrm{d}y\,\hat{\tilde{\chi}}_{\nu}\delta Q(y,t)\tilde{\chi}_{\nu}(y,t,\lambda). \qquad (111)$$

Its left hand side on the rays l_1 and l_2 is given by

$$\begin{aligned}\mathrm{i}(\hat{D}^+_1\hat{T}^-_1\delta(T^-_1 D^+_1) - \hat{S}^+_1\delta S^+_1), &\qquad \text{for } \lambda \in l_1 \\ \mathrm{i}(\hat{D}^-_2\hat{T}^+_2\delta(T^+_2 D^-_2) - \hat{S}^-_2\delta S^-_2), &\qquad \text{for } \lambda \in l_2.\end{aligned} \qquad (112)$$

Next we multiply both sides of (111) by E_γ and apply the Killing form. After this operation the right hand side of (111) acquires the form

$$\int_{-\infty}^{\infty} \mathrm{d}y\left\langle \hat{\tilde{\chi}}_{\nu}\delta Q(y,t)\tilde{\chi}_{\nu}(y,t,\lambda), E_\gamma\right\rangle$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y\,\langle\delta Q(y,t), e_{\nu,\gamma}(y,t,\lambda)\rangle = -[[\mathrm{ad}\,_{\mathcal{J}}^{-1}Q(y,t), e_{\nu,\gamma}(y,t,\lambda)]] \qquad (113)$$

where we have used the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ and the skew-scalar product $[\![X,Y]\!]$ introduced above. As a result we get

$$
\begin{aligned}
\delta\sigma_{1,\alpha}^{+}(\lambda,t) &= i[\![\operatorname{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,-\alpha}(x,t,\lambda)]\!], & \lambda\in l_1,\ \alpha\in A_1 \\
\delta\sigma_{2,\beta}^{-}(\lambda,t) &= i[\![\operatorname{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,\beta}(x,t,\lambda)]\!], & \lambda\in l_2,\ \beta\in A_2 \\
\delta'\tau_{1,\alpha}^{-}(\lambda,t) &= -i[\![\operatorname{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,\alpha}(x,t,\lambda)]\!], & \lambda\in l_1,\ \alpha\in A_1 \\
\delta\tau_{2,\beta}^{+}(\lambda,t) &= -i[\![\operatorname{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,-\beta}(x,t,\lambda)]\!], & \lambda\in l_2,\ \beta\in A_2.
\end{aligned}
\tag{114}
$$

where

$$
\delta'\tau_{1,\alpha}^{-}(\lambda,t) = \delta\tau_{1,\alpha}^{-}(\lambda,t)e^{(\alpha,\alpha)d_{1,\alpha}^{+}}, \qquad \delta'\tau_{2,\beta}^{+}(\lambda,t) = \delta\tau_{2,\beta}^{+}(\lambda,t)e^{-(\beta,\beta)d_{2,\beta}^{-}}.
$$

Thus we conclude that the mapping $\delta\mathcal{F}$ also has the meaning of a generalized Fourier transform based on the same 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ as generalized exponents. This mapping and the formulae (114) are very important for analyzing the Hamiltonian properties of the relevant NLEEs.

We can derive useful relations also by multiplying both sides of (111) by $H_\alpha^\vee$ and applying the Killing form. The result is

$$
\langle\hat{D}_{2\nu-1}^{+}\delta D_{2\nu-1}^{+}, H_\alpha^\vee\rangle = i\int_{-\infty}^{\infty} dy\,\langle\delta Q(y,t)h_{2\nu-1,\alpha}^\vee(y,t,\lambda)\rangle, \quad \lambda\in l_{2\nu-1}
$$

$$
\langle\hat{D}_{2\nu}^{-}\delta D_{2\nu}^{-}, H_\beta^\vee\rangle = i\int_{-\infty}^{\infty} dy\,\langle\delta Q(y,t)h_{2\nu-1,\beta}^\vee(y,t,\lambda)\rangle, \quad \lambda\in l_{2\nu}
\tag{115}
$$

where $h_{\nu,\alpha}^\vee(x,t,\lambda) = \tilde{\chi}_\nu H_\alpha^\vee \hat{\tilde{\chi}}_\nu(x,t,\lambda)$ and $\alpha\in C^{\nu-1}A_1$ and $\beta\in C^{\nu-1}A_2$. Putting $\nu=1$ we have

$$
\begin{aligned}
\delta d_{1,\alpha}^{+}(\lambda) &= -i[\![\operatorname{ad}_{\mathcal{J}}\delta Q(x), h_{1,\alpha}^\vee(x,t,\lambda)]\!], & \lambda\in l_1,\ \ \alpha\in A_1 \\
\delta d_{2,\beta}^{-}(\lambda) &= -i[\![\operatorname{ad}_{\mathcal{J}}\delta Q(x), h_{1,\beta}^\vee(x,t,\lambda)]\!], & \lambda\in l_2,\ \ \beta\in A_2.
\end{aligned}
\tag{116}
$$

7. The Conservation Laws and Hamiltonian Structures

In order to treat the question of the conservation laws we need to introduce yet another type of Wronskian relations. They have the form

$$
\left(i\hat{\tilde{\chi}}_\nu\dot{\tilde{\chi}}_\nu(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} = \int_{x=-\infty}^{\infty} dx\,\left(\hat{\tilde{\chi}}_\nu\mathcal{J}\chi_\nu(x,t,\lambda) - \mathcal{J}\right) \tag{117}
$$

where by 'dot' we denote the derivative with respect to λ.

The reason for considering these Wronskian relations is that they are related with the factors $D_\nu^{\pm}$ and from here to the conservation laws. Indeed, the left hand side

of equation (117) is expressed through the scattering data of L as follows

$$
\begin{aligned}
\left(\mathrm{i}\hat{\tilde{\chi}}_1 \dot{\tilde{\chi}}_1(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} &= \hat{D}_1^{\pm}\dot{D}_1^{\pm}, \qquad \lambda \in l_1 \mathrm{e}^{\pm \mathrm{i}0} \\
\left(\mathrm{i}\hat{\tilde{\chi}}_2 \dot{\tilde{\chi}}_2(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} &= \hat{D}_2^{\pm}\dot{D}_2^{\pm}, \qquad \lambda \in l_2 \mathrm{e}^{\mp \mathrm{i}0}.
\end{aligned}
\tag{118}
$$

In order to evaluate the functions $d_{1,\alpha}^{\pm}(\lambda)$ and $d_{2,\beta}^{\pm}(\lambda)$ we can use the Killing form

$$
\begin{aligned}
\dot{d}_{1,\alpha_j}^{\pm}(\lambda) &= \langle \hat{D}_1^{\pm}\dot{D}_1^{\pm}, H_{\alpha_j}^{\vee}\rangle, \qquad \lambda \in l_1 \mathrm{e}^{\pm \mathrm{i}0}, \qquad \alpha_j \in A_1 \\
\dot{d}_{2,\beta_j}^{\pm}(\lambda) &= \langle \hat{D}_2^{\pm}\dot{D}_2^{\pm}, H_{\beta_j}^{\vee}\rangle, \qquad \lambda \in l_2 \mathrm{e}^{\pm \mathrm{i}0}, \qquad \alpha_j \in A_2
\end{aligned}
\tag{119}
$$

where $H_{\alpha_j}^{\vee}$ and $H_{\beta_j}^{\vee}$ are dual to H_{α_j} and H_{β_j}

$$
\langle H_{\alpha_j}^{\vee}, H_{\alpha_k}\rangle = \delta_{jk}, \qquad \langle H_{\beta_j}^{\vee}, H_{\beta_k}\rangle = \delta_{jk}.
\tag{120}
$$

Thus we obtain

$$
\begin{aligned}
\dot{d}_{1,\alpha_j}^{\pm}(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}x \left(\langle \hat{\chi}_1 \mathcal{J} \chi_1(x,t,\lambda), H_{\alpha_j}^{\vee}\rangle - \langle \mathcal{J}, H_{\alpha_j}^{\vee}\rangle \right) \\
\dot{d}_{2,\beta_j}^{\pm}(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}x \left(\langle \hat{\chi}_2 \mathcal{J} \chi_2(x,t,\lambda), H_{\beta_j}^{\vee}\rangle - \langle \mathcal{J}, H_{\beta_j}^{\vee}\rangle \right).
\end{aligned}
\tag{121}
$$

The analyticity properties of $D_k^{\pm}(\lambda)$ allow one to reconstruct them from the sewing function $G(\lambda)$ (56) and from the locations of their simple zeros and poles but we are not going to treat these questions here.

It is well known, see for example [9], that the evolution equations related to the L operators we consider, (32), (36), possess $r = \mathrm{rank}\,\mathfrak{g}$ series of conservation laws. We will present below the formulae for the conservation laws obtained through the theory of the recursion operators. Their advantage, comparing with the formulae obtained via another approaches, is that they are compact in and give us the possibility to understand which of the conservation laws trivialize if we have reductions. We are speaking below about the linear problem (36). So the constant element in it is $\mathcal{J}$ in (36)). The Cartan subalgebra that is relevant to the corresponding L will be denoted by $\mathfrak{h}$, it equals $\ker(\mathrm{ad}_{\mathcal{J}})$. Its orthogonal complement $((\ker(\mathrm{ad}_{\mathcal{J}}))^{\perp}$ will be denoted by $\bar{\mathfrak{g}}$ and the orthogonal projection on it by π_0. The potential function in (36) Q, it takes values in $\bar{\mathfrak{g}}$. The conservation laws are closely related to the adjoint solutions $h_{\nu,\alpha}^{\vee}(x,t,\lambda) = \tilde{\chi}_\nu H_\alpha^{\vee}\hat{\tilde{\chi}}_\nu(x,t,\lambda)$, which are defined above after the formula (115), or more generally, to the solutions $h_{\nu,H}(x,t,\lambda) = \tilde{\chi}_\nu H \hat{\tilde{\chi}}_\nu(x,\lambda)$, $H \in \mathfrak{h}$, $x,t \in \mathbb{R}$, $\lambda \in \Omega_\nu$. Even more precisely, relevant to us are the projections of these functions, namely $h_{\nu,H}^a(x,\lambda) = \pi_0 h_{\nu,H}(x,\lambda)$. Here of course $\tilde{\chi}_\nu(x,\lambda)$ is a FAS to the CBC system analytic in the sector Ω_ν. We have obvious analogs of the functions entering (121) which are defined for arbitrary $H \in \mathfrak{h}$, let us denote them by $d_{\nu,H}^{\pm}(\lambda)$. They are analytic in the sectors Ω_ν and for their λ-derivatives

we have analogs of the relations (121)

$$\dot{d}^{\pm}_{\nu,H}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \; \left(\langle \hat{\tilde{\chi}}_\nu \mathcal{J} \tilde{\chi}_\nu(x,t,\lambda), H \rangle - \langle \mathcal{J}, H \rangle \right). \tag{122}$$

In the sectors Ω_ν the functions $d^{\pm}_{\nu,H}(\lambda)$ have the following asymptotic behavior

$$d^{\pm}_{\nu,H}(\lambda) = \sum_{k=1}^{\infty} d^{\pm}_{H,k} \lambda^{-k}, \qquad |\lambda| \gg 1. \tag{123}$$

One can prove that actually the coefficients in the asymptotic expansion do not depend on the sector so we denote them by $d_{H,s}$.

The analytic and asymptotic properties of the functions $h^a_{\nu,H}(x,\lambda)$ for large $|\lambda|$ are crucial for the derivation of the conservation laws and they can be found in analogy with the case when the constant element in the operator L is real. However, though the final formulae we obtain are the same as in the real case and the main steps in the calculations are the same, there are some difficulties to overcome. We cannot go into more detais so we shall just sketch the main steps in this calculation and present the final results.

The first way to obtain the coefficients $d_{H,s}$ is to use the Wronskian-type relations (117) and the analytic properties of the functions $h^a_{\nu,H} = \pi_0 \tilde{\chi}_\nu H \hat{\tilde{\chi}}_\nu$, $H \in \mathfrak{h}$. In this way one is able to link the expansions of q over the adjoint solutions (obtained using the map $\mathcal{F}$, see the Wronskian relations (99)) with the functions $h^a_{\nu,H}$. Next one needs to use asymptotic formulae for $h^a_{\nu,H}$ and from there to calculate the quantities $d_{H,s}$. Thus one can obtain the formula

$$d_{H,s} = \frac{1}{s} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], \Lambda^s_{\pm} \mathrm{ad}^{-1}_{\mathcal{J}}[H, Q] \rangle \mathrm{d}y, \qquad s = 1, 2, \ldots \tag{124}$$

where $\Lambda_{\pm}$ are the recursion operators for the case when on q are not imposed any conditions. One can prove that these conservation laws have local densities and are in involution with respect to a hierarchy of symplectic forms, see below.

The second way of obtaining the conservation laws is to use the Wronskian type relations involving the variation of the potential δQ, see (111). In this way we get the formula

$$\delta d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \langle \delta Q, \Lambda^{s-1}_{\pm} \mathrm{ad}^{-1}_{\mathcal{J}}[H, Q] \rangle \mathrm{d}x, \qquad s = 1, 2, \ldots \tag{125}$$

which is more popular in another form. In order to obtain it, let us identify the space $\mathcal{M}_{\mathcal{J}}$ consisting of Schwartz-type functions on the line with values in $\mathfrak{h}^{\perp} = \bar{\mathfrak{g}}$ and

its dual $\mathcal{M}_{\mathcal{J}}^*$ through the bilinear form

$$\langle\langle X, Y\rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle X(x), Y(x)\rangle \mathrm{d}x.$$

In other words, we shall consider the elements from $\mathcal{M}_{\mathcal{J}}^*$ as generalized functions (distributions) and a generalized function, say ξ, will be written as

$$\langle\langle \xi, Y\rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle \xi(x), Y(x)\rangle \mathrm{d}x, \qquad Y(x) \in \mathcal{M}_{\mathcal{J}}.$$

As a matter of fact the generalized functions we have are regular, that is represented by locally Lebesgue integrable functions over $\mathbb{R}$, and even most of them belong to $\mathcal{M}_{\mathcal{J}}$. Taking into account the identification for the differentials of the conservation laws we get

$$\mathrm{d}d_{H,s} = -\mathrm{i}\Lambda_{\pm}^{s-1}\mathrm{ad}_{\mathcal{J}}^{-1}[H, Q] \tag{126}$$

and hence

$$\mathrm{d}d_{H,s} = \Lambda_{\pm}\mathrm{d}d_{H,s-1}, \qquad s = 2, 3, \ldots. \tag{127}$$

The above relations in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ are called *Lenart relations*, see [2], so we shall call them Lenart-type relations or Lenart chains.

In fact one can prove that with the above identification $\mathrm{d}d_{\nu,H} = \mathrm{i}h_{\nu,H}^a$ which explains why the functions $h_{\nu,H}^a$ are so important in the study of the conservation laws. Using (127) and the Poincaré lemma for closed one-forms one gets another formula

$$d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{0}^{1} \langle Q, \Lambda_{\pm}^{s-1}|_{(\zeta Q)} \, \mathrm{ad}_{\mathcal{J}}^{-1}[H, \zeta Q]\rangle \mathrm{d}\zeta \tag{128}$$

where by $\Lambda_{\pm}|_{(\zeta Q)}$ is denoted the recursion operator in which Q is substituted by $Q' = \zeta Q$. This formula can be obtained also directly. Indeed, let us consider instead of the potential q the potential $Q' = \zeta Q$ where $0 \le \zeta \le 1$ is a real parameter. Let us consider variation of the potential Q' of the form $\delta Q' = Q\delta\zeta$. Then (125) implies

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \langle Q, \Lambda_{\pm}^{s-1}|_{Q \to Q'}\mathrm{ad}_{\mathcal{J}}^{-1}[H, Q']\rangle \mathrm{d}x, \quad s = 1, 2, \ldots. \tag{129}$$

Integrating over ζ between 0 and 1 and taking into account that for $\zeta = 0$ we have $Q' = 0$, $d_{H,s} = 0$ and for $\zeta = 1$ we have $Q' = Q$ we obtain the formula (128).

When one calculates the hierarchy of conservation laws the last form of the conservation laws can be a real advantage as the expressions become more and more

complicated when s increases and in it $\Lambda_\pm$ enters with power $s - 1$ while in (124) $\Lambda_\pm$ enters with power s.

When one has $\mathbb{Z}_h$ reductions the above formulae for the conservation laws remain true and naturally the conservation laws continue to have local densities. However, one can observe that some of them trivialize, that is they become identically zero. Indeed, since ad $_\mathcal{J}^{-1}$ takes $\overline{\mathfrak{g}}^{(k)}$ into $\overline{\mathfrak{g}}^{(k-1)}$, $\Lambda_\pm^m$ takes functions with values in $\overline{\mathfrak{g}}^{(k)}$ to functions with values in $\overline{\mathfrak{g}}^{(k-m)}$ so if $H \in \mathfrak{h}^{(k)} = \mathfrak{g}^{(k)} \cap \mathfrak{h}$ (k must be exponent of course) then unless $k - s = 0 \,(\mathrm{mod}(\mathrm{h}))$ the expression (124) is identically zero. Of course, the same conclusion is obtained if one uses the expression (128). Then assumming that $0 \le k \le h - 1$ for example in the hierarchies (124) 'survive' only the following integrals of motion

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], (\Lambda_\pm^p)^n \Lambda_\pm^{h-k} \mathrm{ad}\,_\mathcal{J}^{-1}[H, Q] \rangle \mathrm{d}y, \quad n = 1, 2, \ldots \quad (130)$$

or, if one prefers the notation through $\Lambda_\pm$, as for example in (110), from the hierarchy of integrals of motion remain only

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], \Lambda_\pm^n (Q[H, k]) \rangle \mathrm{d}y, \quad n = 1, 2, \ldots$$

$$\tag{131}$$

$$Q[H, k] = \Lambda_\pm^{h-k} \mathrm{ad}\,_\mathcal{J}^{-1}[H, Q].$$

One sees that $Q[H, k]$ takes values in $\mathfrak{g}^{(h-1)}$ and the role of the recursion operator is played now by $\Lambda_\pm$. The observation that in case of reductions $\mathbb{Z}_h$ reductions have gaps in the conservation laws sequencie has been made in [32].

Maybe we must say here that we went a bit too quickly into the discussion of the conservation laws. In fact we ought to discuss first equation (126) and the Lenart chains. One can see that when we have reductions (126) should be modified because when one identifies a linear functional over $\mathfrak{g}^{(0)}$ with an element from $\mathfrak{g}$ this element should belong to $\mathfrak{g}^{(0)}$, because $\mathfrak{g}^{(0)}$ is orthogonal to $\mathfrak{g}^{(k)}$ for $k \neq 0$. So in case we have reductions (126) should be replaced by

$$\mathrm{d}d_{H,s} = -\mathrm{i}p_0 \Lambda_\pm^{s-1} \mathrm{ad}\,_\mathcal{J}^{-1}[H, Q] \tag{132}$$

where p_0 is the orthogonal projection on $\mathfrak{g}^{(0)}$. Thus one sees again that if $H \in \mathfrak{h}^{(k)}$ then unless $k - s = 0 \,(\mathrm{mod}(\mathrm{h}))$ the above expression is identically zero. It is also readily seen that the Lenart chains generate from a given nontrivial conservation law a nontrivial one if one acts not with $\Lambda_\pm$ but with $\Lambda_\pm^h$, or if one prefers by $\Lambda_\pm$ as for example in (110). As an illustration we give the two first integrals of motion of the $\mathbb{Z}_h$-DNLS equation

$$I_{1,1}^{(1)} = \frac{1}{2\omega} \int_{-\infty}^{\infty} \mathrm{d}x \sum_{p=1}^{n} \psi_p \psi_{n-p}(x, t) \tag{133}$$

$$I_{1,1}^{(2)} = \frac{1}{2\omega^2} \int_{-\infty}^{\infty} dx \left\{ \sum_{p=1}^{n} i \cotan\left(\frac{\pi p}{n}\right) \left(\frac{d\psi_p}{dx}\psi_{n-p} - \psi_p \frac{d\psi_{n-p}}{dx}\right) \right.$$

$$\left. - \frac{2}{3} \sum_{p+k+l=n} \psi_p \psi_k \psi_l(x,t) \right\}. \quad (134)$$

Let us mention very briefly the hierarchies of the Hamiltonian structures for the soliton equations we study. Two approaches are possible here. One is based on the hierarchy of Poisson structures, the other on the hierarchy of symplectic structures. The Poisson structures are easier to construct in the general case of non-restricted systems, see for example [29]. From the other side, if one has reductions then one must calculate the corresponding Dirac brackets. The symplectic structures seem more complicated to construct but offer the advantage that the restrictions are immediate, provided they do not degenerate. It is well-known that the equations without any reductions possess a hierarchy of Hamiltonian structures with symplectic forms Ω_m that can be written as follows

$$\Omega_m(X,Y) = \int_{-\infty}^{+\infty} \langle X, \Lambda_{\pm}^m \mathrm{ad}\,_{\mathcal{J}}^{-1}(Y)\rangle dx \quad (135)$$

where $X(x), Y(x)$ are smooth functions with values in $\overline{\mathfrak{g}}$ – the orthogonal complement of $\mathfrak{h}$, see [20] and the numerous citations therein. However, due to the fact that q takes values in $\overline{\mathfrak{g}}^{(0)}$ some of these structures degenerate. Indeed, if X, Y take values in $\overline{\mathfrak{g}}^{(0)}$ then $\mathrm{ad}\,_{\mathcal{J}}^{-1} Y$ takes values in $\overline{\mathfrak{g}}^{(-1)}$, $\Lambda_{\pm}^m \mathrm{ad}\,_{\mathcal{J}}^{-1} Y$ takes values in $\overline{\mathfrak{g}}^{(-m-1)}$ and unless $m + 1 = 0 \pmod{h}$ the form is identically zero. Let $m = kh - 1$. Then

$$\Omega_m(X,Y) = \int_{-\infty}^{+\infty} \langle X, (\Lambda_{\pm}^h)^k \Lambda_{\pm}^{-1} \mathrm{ad}\,_{\mathcal{J}}^{-1}(Y)\rangle dx = \Omega_{-1}(X, (\Lambda_{\pm}^h)^k(Y)) \quad (136)$$

again demonstrating that the operator that generates the symplectic structures is now $\mathbf{\Lambda}_{\pm} = \Lambda_{\pm}^h$. This fact has been also proved with geometric methods in [40] using the theory of the so-called Poisson-Nijenhuis manifolds.

8. Conclusions

As it is well-known, the generic integrable NLEEs related with the generalized Zakharov-Shabat system and its generalization – the CBC system defined on a semisimple Lie algebra of rank r possesses a number of interesting properties. Among them are: they possess a hierarchy of Hamiltonian structures, r series of conservation laws etc. Both the Hamiltonian hierarchy and the series of conservation laws are generated by a certain recursion operator (called also Λ-operator)

[1, 11, 14, 15, 19, 20]. Its spectral properties and the expansions over the eigenvectors of Λ have deep applications to the theory of the corresponding NLEEs, see e.g. [20]. The case when we have reductions is of big importance and in this case we have some specifics, see [11], which must be taken care of when studying the properties of the corresponding Λ-operator. In this article we analysed $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions. In case we have these type of reductions and we have NLEE represented in a Lax form $[L, M] = 0$ with $L\psi = 0$ being a CBC type system with reductions, the coefficient in front of the leading power of λ in the operator M see (35) should be an exponent of the corresponding algebra $\mathfrak{g}$. Since 2 is an exponent only for the series A_r, there would be no new examples of $\mathbb{Z}_h$-reduced Nonlinear Schrödinger type equations. However, 3 is an exponent for all the algebras from the classical series. Therefore choosing M in the Lax representation to be cubic in λ one gets series of KdV type equations and the simplest of them are already known, see [8, 9, 38]. The theory we develop applies to these type of equations and of course to those corresponding to exponents larger than 2 and 3.

Our results show the specific way the generating operator factorizes in case of reductions. Along with the elementary recursion operators – the restriction of the standard Λ on $\mathfrak{g}^{(k)}$, we have introduced fundamental recursion operators $\Lambda_{(m_k)}$ and the master recursion operator Λ. The operators $\Lambda_{(m_k)}$ generate the MKdV-type NLEEs, and combined with Λ – the hierarchy of their Hamiltonian structures. Thus we have outlined the effects of the reduction group on the recursion operators. Important questions, that will be answered in next publications concern the spectral theory of the recursion operators, see [14–16, 19, 22] and their geometrical properties, see [31, 40, 41, 46] and the monograph [20].

Acknowledgements

References

[1] Ablowitz A., Kaup D., A., Newell A. and Segur H., *The Inverse Scattering Transform – Fourier Analysis for Nonlinear Problems*, Studies in Appl. Math. **53** (1974) 249–315.

[2] Adler M., *On a Trace Functional for Formal Pseudo-Differential Operators and the Symplectic Structure of the Korteweg-de Vries Equations*, Inv. Math. **50** (1979) 219–248.

[3] Beals R. and Coifman R., *Scattering and Inverse Scattering for First Order Systems*, Commun. Pure & Appl. Math. **37** (1984) 39–90.

[4] Beals R. and Coifman R., *Inverse Scattering and Evolution Equations*. Commun. Pure & Appl. Math. **38** (1985) 29–42.

[5] Beals R. and Sattinger D, *On the Complete Integrability of Completely Integrable Systems*, Comm. Math. Phys. **138** (1991) 409–436.

[6] Caudrey P., *The Inverse Problem for the Third Order Equation* $u_{xxx} + q(x)u_x + r(x)u = -i\zeta^3 u.$, Phys. Lett. A **79A** (1980) 264–268;
The Inverse Problem for a General $n \times n$ *Spectral Equation*, Physica D **6** (1982) 51–66.

[7] Calogero F. and Degasperis A., *Spectral Transform and Solitons*, North Holland, Amsterdam 1982.

[8] Chvartatskyi O. and Sydorenko Yu., *Matrix Generalizations of Integrable Systems with Lax Integro-Differential Representations*, nlin.SI ArXiv1212.3444.

[9] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg-de Vries Type*, Sov. J. Math. **30** (1985) 1975–2036.

[10] Faddeev L. and Takhtadjan A., *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin 1987.

[11] Fordy A. and Gibbons J., *Factorization of Operators II*, J. Math. Phys. **22** (1981) 1170–1175.

[12] Fordy A. and Kulish P., *Nonlinear Schrödinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[13] Gerdjikov V., Z_N*–Reductions and New Integrable Versions of Derivative Nonlinear Schrödinger Equations*, In: Nonlinear Evolution Equations: Integrability and Spectral Methods, A. Fordy, A. Degasperis and M. Lakshmanan (Eds), Manchester University Press, Manchester 1981, pp. 367–372.

[14] Gerdjikov V., *On the Spectral Theory of the Integro-differential Operator* Λ*, Generating Nonlinear Evolution Equations*, Lett. Math. Phys. **6** (1982) 315–324.

[15] Gerdjikov V., *Generalised Fourier Transforms for the Soliton Equations. Gauge Covariant Formulation*, Inverse Problems **2** (1986) 51–74.

[16] Gerdjikov V., *Algebraic and Analytic Aspects of* N*-wave Type Equations*. Contemporary Mathematics **301** (2002) 35–68, nlin.SI/0206014.

[17] Gerdjikov V., *Derivative Nonlinear Schrödinger Equations with* $\mathbb{Z}_N$ *and* $\mathbb{D}_N$*– Reductions*, Romanian Journal of Physics **58** (2013) (in press).

[18] Gerdjikov V., Grahovski G., Ivanov R. and Kostov N., N*-Wave Interactions Related to Simple Lie Algebras–* $\mathbb{Z}_2$*-Reductions and Soliton Solutions*, Inverse Problems **17** (2001) 999–1015.

[19] Gerdjikov V. and, Kulish P., *The Generating Operator for the* $n \times n$ *Linear System*, Physica D **3** (1981) 549–564.

[20] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics **748**, Springer, Berlin 2008.

[21] Gerdjikov V. and Yanovski A., *Gauge Covariant Formulation of the Generating Operator. I. The Zakharov–Shabat System*, Phys. Lett. A **103** (1984) 232–236.

[22] Gerdjikov V. and Yanovski A., *Completeness of the Eigenfunctions for the Caudrey-Beals-Coifman System*, J. Math. Phys. **35** (1994) 3687–3725.

[23] Grahovski G., *On The Reductions and Scattering Data for the CBC System*, Geometry, Integrability and Quantization **3** (2002) 262–277.

[24] Gürses M., Karasu A. and Sokolov V. *On Construction of Recursion Operators From Lax Representation*, J. Math. Phys. **40** (1999) 6473, doi:10.1063/1.533102 (18 pages)

[25] Helgasson S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic, New York 1978.

[26] Kaup D. and Newell A., *Soliton Equations, Singular Dispersion Relations and Moving Eigenvalues*, Adv. Math. **31** (1979) 67–100.

[27] Konopelchenko B., *Nonlinear Integrable Equations: Recursion Operators, Group Theoretical and Hamiltonian Structures of Soliton Equations*, Lecture Notes in Physics **270**, Springer, Berlin 1987.

[28] Kulish P. and Reyman A., *The Hierarchy of Symplectic Forms for the Schrödinger Equation and for the Dirac Equation* (in Russian), Sci. Notes of LOMI seminars **77** (1978) 134–147 .

[29] Kulish P. and Reiman A., *Hamiltonian Structure of Polynomial Bundles* (in Russian), Sci. Notes. LOMI Seminars **123** (1983) 67–76; Translated in J. Sov. Math. **28** (1985) 505–513.

[30] Kuperschmidt B. and Wilson G., *Conservation Laws and Symmetries of Generalized Sine-Gordon Equations*, Commun. Math. Phys. **81** (1981) 189–202.

[31] Ludu A., *Differential Geometry of Moving Surfaces and its Relation to Solitons*, J. Geom. Symmetry Phys. **21** (2011) 1–28, doi:10.7546/jgsp-21-2011-1-28

[32] Mikhailov A., *The Reduction Problem and the Inverse Scattering Problem*, Physica D **3** (1981) 73–117.

[33] Mikhailov, A., Olshanetzky, M. and Perelomov, A., *Two Dimensional Generalized Toda Lattice*, Commun. Math. Phys. **79** (1981) 473–490.

[34] Olive D. and Turok N., *The Toda Lattice Field Theory Hierarchies and Zero-curvature Conditions in Kac-Moody Algebras*, Nucl. Phys. B **265** (1986) 469–484.

[35] Olshanetzky M. and Perelomov A., *Classical Integrable Systems Related to Lie Algebras*, Phys. Repts. **71** (1983) 313–404.

[36] Schmid R., *Infinite Dimensional Lie Groups with Applications to Mathematical Physics*, J. Geom. Symmetry Phys. **1** (2004) 54-120, doi:10.7546/jgsp-1-2004-54-120

[37] Shabat A., *The Inverse Scattering Problem for a System of Differential Equations* (in Russian), Functional Annal. & Appl. **9** (1975) 75-78;
Shabat A., *The Inverse Scattering Problem* (in Russian), Diff. Equations **15** (1979) 1824–1834.

[38] Valchev T., *On the Kaup-Kupershmidt Equation. Completeness Relations for the Squared Solutions*, In: Proc. 9-th International Conference on Geometry, Integrability and Quantization, I. Mladenov and M. de Leon (Eds), Softex, Sofia 2008, pp. 308–319.

[39] Valchev T., *On Generalized Fourier Transform for Kaup-Kupershmidt Type Equations*, J. Geom. Symmetry Phys. **19** (2010) 73–86, doi:10.7546/jgsp-19-2010-73-86

[40] Yanovski A., *Geometry of the Recursion Operators for Caudrey-Beals-Coifman System in the Presence of Mikhailov $\mathbb{Z}_p$ Reductions*, J. Geom. Symmetry Phys. **25** (2012) 77–97.

[41] Yanovski A., *Geometric Interpretation of the Recursion Operators for the Generalized Zakharov-Shabat System in Pole Gauge on the Lie Algebra A_2*, J. Geom. Symmetry Phys. **23** (2011) 97–111. doi:10.7546/jgsp-23-2011-97-111

[42] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Consultants Bureau, New York 1984.

[43] Zakharov V. and Manakov S., *The Theory of Resonant Interaction of Wave Packets in Nonlinear Media* (in Russian), Sov. Phys. JETP **69** (1975) 1654–1673.

[44] Zakharov V. and Mikhailov A., *Relativistically Invariant Two-dimensional Models of Field Theory Which are Integrable by Means of the Inverse Scattering Problem Method*, Zh. Eksp. Teor. Fiz. **74** (1978) 1953–1973.

[45] Zakharov V. and Shabat A., *A Scheme for Integrating Nonlinear Evolution Equations of Mathematical Physics by the Inverse Scattering Method I*, Funkts. Anal. Prilozhen. **8** (1974) 43–53.

[46] Yanovski A. and Vilasi G., *Geometric Theory of the Recursion Operators for the Generalized Zakharov–Shabat System in Pole Gauge on the Algebra $\mathfrak{sl}(n, \mathbb{C})$ with and without Reductions*, SIGMA **8** (2012) 087, 23 pp.

ON THE PERSISTENCE PROPERTIES OF THE CROSS-COUPLED CAMASSA-HOLM SYSTEM[*]

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Abstract. In this paper we examine the evolution of solutions, of a recently-derived system of cross-coupled Camassa-Holm equations, that initially have compact support. The analytical methods which we employ provide a full picture for the persistence of compact support for the momenta. For the solutions of the system itself, the answer is more convoluted, and we determine when the compactness of the support is lost, replaced instead by an exponential decay rate.

1. Introduction

This paper is concerned with the persistence of compact support in solutions to a recently derived cross-coupled Camassa-Holm (CCCH) equation [7], which is given by

$$m_t + 2v_x m + v m_x = 0, \qquad n_t + 2u_x n + u n_x = 0 \qquad (1)$$

where $m = u - u_{xx}$ and $n = v - v_{xx}$. This system generalises the celebrated Camassa-Holm (CH) equation [1], since for $u = v$ the system (1) reduces to two copies of the CH equation

$$m_t + 2u_x m + u m_x = 0.$$

The CH equation models a variety of phenomena, including the propagation of unidirectional shallow water waves over a flat bed [1, 8, 12, 16, 17]. The CH equation

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possesses a very rich structure, being an integrable infinite-dimensional Hamiltonian system with a bi-Hamiltonian structure and an infinitely many conservation laws [1, 4, 15]. It also has a geometric interpretation as a re-expression of the geodesic flow on the diffeomorphism group of the circle [14]. One of the most interesting features of the CH equation, perhaps, is the rich variety of solutions it admits. Some solutions exist globally, whereas others exist only for a finite length of time, modelling wave breaking [3, 6].

The CCCH equation can be derived from a variational principle as an Euler-Lagrange system of equations for the Lagrangian

$$l(u, v) = \int_{\mathbb{R}} (uv + u_x v_x) \, \mathrm{d}x.$$

Alternatively it can be formulated as a two-component system of Euler-Poincaré (EP) equations in one dimension on $\mathbb{R}$ as follows

$$\partial_t m = -\mathrm{ad}^*_{\delta h/\delta m} m = -(vm)_x - mv_x \qquad \text{with} \qquad v := \frac{\delta h}{\delta m} = K * n$$

$$\partial_t n = -\mathrm{ad}^*_{\delta h/\delta n} n = -(un)_x - nu_x \qquad \text{with} \qquad u := \frac{\delta h}{\delta n} = K * m$$

with $K(x, y) = \frac{1}{2}\mathrm{e}^{-|x-y|}$ being the Green function of the Helmholtz operator, and h being the Hamiltonian defined via the convolution in the spatial variable

$$h(n, m) = \int_{\mathbb{R}} n \, K * m \mathrm{d}x = \int_{\mathbb{R}} m \, K * n \, \mathrm{d}x.$$

This Hamiltonian system has two-component singular momentum map [13]

$$m(x, t) = \sum_{a=1}^{M} m_a(t) \, \delta(x - q_a(t)), \qquad n(x, t) = \sum_{b=1}^{N} n_b(t) \, \delta(x - r_b(t)).$$

The $M = N = 1$ case is very simple for analysis [7]. If the initial conditions are $m_1(0) > 0$ and $n_1(0) > 0$ then one observes the so-called *waltzing* motion. It could be noted that for half of the waltzing period (half cycle) the two types of peakons exchange momentum amplitudes - see Fig. 1. The explicit solutions as well as other examples with waltzing peakons and compactons are given in [7].

The aim of this study is to analyse the persistence of compact support for solutions of the system (1). In particular, we will examine whether the solution m, n, and in turn u, v, of (1), which initially have compact support, will continue to have that property as they evolve. Solutions of the system which have compact support can be viewed as localized disturbances, and whether a "disturbance" which is initially localized propagates with a finite, or infinite speed, is a matter of great interest. We will see that some solutions will remain compactly supported at all future times of their existence, while others solution display an infinite speed of propagation and
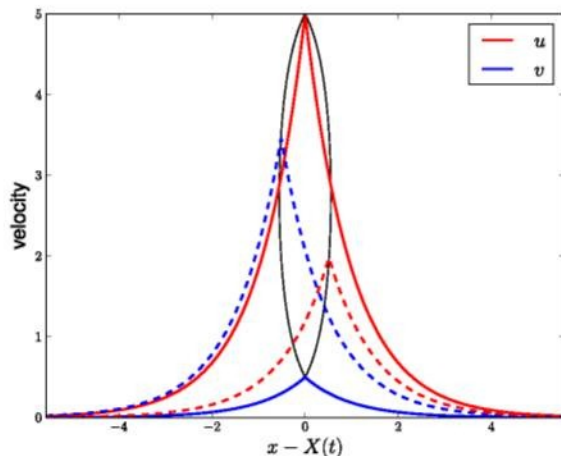
Figure 1. Plot showing velocity fields of a peakon-peakon pair with $m_1(0) = 10$, $n_1(0) = 1$ (solid lines). The dotted path indicates the subsequent path of the two peaks in the frame travelling at the particles mean velocity. For these initial conditions the total period for one orbit of the cycle is $T = 3.6$. Also shown is the form of the two peakons at subsequent times $t = 0.45 + 1.8n$, $n \in \mathbb{Z}$.

instantly lose their compact support. These results have analogues in the case of CH equation [2, 9, 11].

2. Preliminaries

We may express equation (1) in terms of u and v as follows

$$
\begin{aligned}
u_t - u_{xxt} + 2v_x u - 2v_x u_{xx} + vu_x - vu_{xxx} = 0 \\
v_t - v_{xxt} + 2u_x v - 2u_x v_{xx} + uv_x - uv_{xxx} = 0.
\end{aligned}
\tag{2}
$$

From this form of the equations one observes that there are no terms with self-interaction (e.g. uu_x, $u_x u_{xx}$, uu_{xxx} etc.) which justifies the name 'cross-coupled'. If $p(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$ and so $p * m = u$, $p * n = v$. Indeed,

$$
u(x) = \frac{1}{2}e^{-x} \int_{-\infty}^{x} e^{y} m(y) \, dy + \frac{1}{2}e^{x} \int_{x}^{\infty} e^{-y} m(y) \, dy
\tag{3}
$$

$$
u_x(x) = -\frac{1}{2}e^{-x} \int_{-\infty}^{x} e^{y} m(y) \, dy + \frac{1}{2}e^{x} \int_{x}^{\infty} e^{-y} m(y) \, dy.
\tag{4}
$$

In other words, if we denote by $I_1(x)$ and $I_2(x)$ the integrals appearing in the first and the second term of (3), we have

$$u = I_1 + I_2, \qquad u_x = -I_1 + I_2. \tag{5}$$

Applying the convolution operator to equation (1) we can cast it in the form of a conservation law

$$(u + v)_t + \partial_x (uv + p * (2uv + u_x v_x)) = 0, \qquad x \in \mathbb{R},\, t \geq 0. \tag{6}$$

Thus $L = u + v$ is a density of the conserved momentum $\int (m + n)\mathrm{d}x$. The representation (6) agrees with the CH reduction when $u = v$, cf. [9].

The Hamiltonian

$$H = \int (uv + u_x v_x)\mathrm{d}x$$

(in terms of u and v) is of course another conserved quantity, the 'energy' of the system, see more details in [7].

One can directly observe that (1) can be complexified in a natural way if the variables u, v are assumed complex, while the independent variables x, t are still real. Such a complexified system is remarkable with the fact that it admits the obvious reduction $u = \bar{v}$ which leads to a single scalar complex equation

$$u_t - u_{xxt} + 2\bar{u}_x u - 2\bar{u}_x u_{xx} + \bar{u}u_x - \bar{u}u_{xxx} = 0. \tag{7}$$

This is a geodesic equation for a complex H^1 metric, given by the Hamiltonian $H = \frac{1}{2} \int (|u|^2 + |u_x|^2)\mathrm{d}x$.

Of course, if one reverts to real dependent variables by putting $u = r + \mathrm{i}s$ then (7) leads to the coupled system

$$\begin{aligned}
r_t - r_{xxt} + 2(rr_x + ss_x) - 2(r_x r_{xx} + s_x s_{xx}) - (rr_{xxx} + ss_{xxx}) = 0 \\
s_t - s_{xxt} + r_x s - rs_x - 2(r_x s_{xx} - s_x r_{xx}) - (rs_{xxx} - sr_{xxx}) = 0.
\end{aligned} \tag{8}$$

Unless it is explicitly specified that the variables (u, v) are complex, we assume that they are real.

3. Results

In the following we let $T = T(u_0, v_0) > 0$ to denote the maximal existence time of the solutions $u(x, t), v(x, t)$ to the system (1) with the given initial data $u_0(x)$ and $v_0(x)$.

3.1. Persistence of Compact Support for the Momenta

For the following, the flow prescribed by the system (1) is given by the two families of diffeomorphisms $\{\varphi(\cdot,t)\}_{t\in[0,T)}$, $\{\xi(\cdot,t)\}_{t\in[0,T)}$ as follows

$$\begin{aligned}
\varphi_t(x,t) &= v(\varphi(x,t),t), & \varphi(x,0) &= x \\
\xi_t(x,t) &= u(\xi(x,t),t), & \xi(x,0) &= x.
\end{aligned} \tag{9}$$

Solving (9), we get

$$\varphi_x(x,t) = e^{\int_0^t v_x(\varphi(x,s),s)\mathrm{d}s} \quad \text{and} \quad \xi_x(x,t) = e^{\int_0^t u_x(\xi(x,s),s)\mathrm{d}s} > 0 \tag{10}$$

hence $\varphi(\cdot,t)$ and $\xi(\cdot,t)$ are increasing functions.

Lemma 1. *Assume that u_0 and v_0 are such that $m_0 = u_0 - u_{0,xx}$ and $n_0 = v_0 - v_{0,xx}$ are nonnegative (nonpositive) for $x \in \mathbb{R}$. Then $m(x,t)$ and $n(x,t)$ remain nonnegative (nonpositive) for all $t \in [0,T)$.*

Proof: It follows from (1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}m(\varphi(x,t),t)\varphi_x^2(x,t) = m_t\varphi_x^2 + m_x\varphi_t\varphi_x^2 + 2m\varphi_x\varphi_{xt}$$

$$= (m_t + 2v_x m + v m_x)\varphi_x^2 = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}n(\xi(x,t),t)\xi_x^2(x,t) = n_t\xi_x^2 + n_x\xi_t\xi_x^2 + 2m\xi_x\xi_{xt}$$

$$= (n_t + 2u_x n + u n_x)\xi_x^2 = 0.$$

Therefore

$$m(\varphi(x,t),t)\varphi_x^2(x,t) = m_0(x), \qquad n(\xi(x,t),t)\xi_x^2(x,t) = n_0(x). \tag{11}$$

Now, since $m_0(x), n_0(x)$ are nonnegative (nonpositive) then $m(x,t)$ and $n(x,t)$ remain nonnegative (nonpositive) for all $t \in [0,T)$. ∎

Lemma 2. *Assume that u_0 is such that $m_0 = u_0 - u_{0,xx}$ has compact support, say contained in the interval $[\alpha_{m_0}, \beta_{m_0}]$, then for any $t \in [0,T)$, the function $x \mapsto m(x,t)$ has compact support contained in the interval $[\varphi(\alpha_{m_0},t), \varphi(\beta_{m_0},t)]$ for all $t \in [0,T)$. Similarly, if $n_0 = v_0 - v_{0,xx}$ has compact support, then the function $x \mapsto n(x,t)$ is compactly supported for all $t \in [0,T)$.*

Proof: From (11) and from the assumption that $m_0(x)$ is supported in the compact interval $[\alpha_{m_0}, \beta_{m_0}]$, it follows directly that $m(\cdot,t)$ are compactly supported, with support contained in the interval $[\varphi(\alpha_{m_0},t), \varphi(\beta_{m_0},t)]$, for all $t \in [0,T)$. Similar reasoning applies to n_0. ∎

Relation (11) represents the conservation of momentum in the physical variables cf. discussion in [7].

3.2. On the Evolution of (u, v)

In this subsection we are going to examine the general behaviour of the solution (u, v) of (1) which is initially compactly supported. The following theorem provides us with some information about the asymptotic behavior of the solution as it evolves over time – in general, the solution has an exponential decay as $|x| \to \infty$ for all future times $t \in [0, T)$.

Theorem 3. *Let (u, v) be a nontrivial solution of* (1)*, with maximal time of existence $T > 0$, and which is initially compactly supported on an interval $\mathcal{I}_0 = [\alpha_{u_0}, \beta_{u_0}] \times [\alpha_{v_0}, \beta_{v_0}]$. Then we have*

$$u(x, t) = \begin{cases} \frac{1}{2} E_+^u(t) e^{-x} & \text{for } x > \xi(\beta_{u_0}, t) \\ \frac{1}{2} E_-^u(t) e^{x} & \text{for } x < \xi(\alpha_{u_0}, t) \end{cases} \tag{12}$$

$$v(x, t) = \begin{cases} \frac{1}{2} E_+^v(t) e^{-x} & \text{for } x > \varphi(\beta_{v_0}, t) \\ \frac{1}{2} E_-^v(t) e^{x} & \text{for } x < \varphi(\alpha v_0, t) \end{cases} \tag{13}$$

where α, β are defined in (14) *below, and $E_-^u, E_+^u, E_-^v, E_+^v$ are continuous functions, with $E_+^u(0) = E_+^v(0) = E_-^u(0) = E_-^v(0) = 0$.*

Proof: Firstly, if (u_0, v_0) is initially supported on the compact interval $\mathcal{I}_0 = [\alpha_{u_0}, \beta_{u_0}] \times [\alpha_{v_0}, \beta_{v_0}]$ then so is m_0 too, and from the proof Lemma 2 it follows that $(m(\cdot, t), n(\cdot, t))$ is compactly supported, with its support contained in the interval $\mathcal{I}_t = [\xi(\alpha, t), \xi(\beta, t)] \times [\varphi(\alpha, t), \varphi(\beta, t)]$ for fixed $t \in [0, T)$. Here

$$\alpha = \min\{\alpha_{u_0}, \alpha_{v_0}\}, \qquad \beta = \max\{\beta_{u_0}, \beta_{v_0}\}. \tag{14}$$

We use the relation $u = p * m$ to write

$$u(x) = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{y} m(y) \, \mathrm{d}y + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-y} m(y) \, \mathrm{d}y$$

and then we define

$$E_+^u(t) = \int_{\xi(\alpha, t)}^{\xi(\beta, t)} e^{y} m(y, t) \, \mathrm{d}y \quad \text{and} \quad E_-^u(t) = \int_{\xi(\alpha, t)}^{\xi(\beta, t)} e^{-y} m(y, t) \mathrm{d}y. \tag{15}$$

We have

$$u(x, t) = \frac{1}{2} e^{-x} E_+^u(t), \qquad x > \xi(\beta, t)$$

$$u(x, t) = \frac{1}{2} e^{x} E_-^u(t), \qquad x < \xi(\alpha, t) \tag{16}$$

and therefore from differentiating (16) we get directly

$$\frac{1}{2}\mathrm{e}^{-x}E_+^u(t) = u(x,t) = -u_x(x,t) = u_{xx}(x,t), \qquad x > \xi(\beta,t)$$

$$\frac{1}{2}\mathrm{e}^{x}E_-^u(t) = u(x,t) = u_x(x,t) = u_{xx}(x,t), \qquad x < \xi(\alpha,t).$$

Since $u(\cdot,0)$ is supported in the interval $[\alpha,\beta]$, we have $E_+^u(0) = E_-^u(0) = 0$, as we can see by taking integration by parts and taking into account that the boundary terms vanish. ∎

Corollary 4. *If in addition $m_0(x)$ and $n_0(x)$ are everywhere nonnegative (non-positive), then the solution (u,v) (if nontrivial) loses its compactness immediately.*

Proof: Indeed, in order for a nontrivial solution to remain with compact support one needs that $E_\pm^u(t) \equiv 0$, $E_\pm^v(t) \equiv 0$ for all $t \in [0,T]$. However from Lemma 1 it follows that $m(x,t)$ and $n(x,t)$ remain everywhere nonnegative (nonpositive) and thus the quantities $E_\pm^u(t)$, $E_\pm^v(t)$ defined e.g. in (15) are positive (negative) for all $t \in (0,T]$ in the case we have nontrivial solution. ∎

From (6) we know that $L = u + v$ is a density of a conserved quantity and as such it deserves a special attention. From Theorem 3 one can find the asymptotics of L as $x \to \pm\infty$ as

$$L \to \frac{1}{2}E_\pm(t)\mathrm{e}^{-|x|}$$

where $E_\pm \equiv E_\pm^u + E_\pm^v$. Since the nature of the solution that we expect is several coupled 'waltzing' waves, i.e., the maximum elevations of $u(x,t)$ and $v(x,t)$ increase and decrease with time in the waltzing process. In other words the functions $E_\pm^u(t)$ and $E_\pm^v(t)$ are in general non-monotonic functions of t. However in some cases a monotonic property holds for the conserved density L.

Theorem 5. *If (u,v) is an initially compactly supported solution and in addition $m_0(x)$ and $n_0(x)$ are everywhere nonnegative (nonpositive), then the quantity $E_+(t)$ is a monotonically increasing function and $E_-(t)$ is a monotonically decreasing function.*

Proof: Indeed, from Lemma 1 it follows that the functions $m(x,t)$ and $n(x,t)$ remain everywhere nonnegative (nonpositive) and from the explicit form of the inverse Helmholtz operator $u(x,t)$ and $v(x,t)$ remain everywhere nonnegative (nonpositive). Since $m(\cdot,t)$ is supported in the interval $[\xi(\alpha,t),\xi(\beta,t)]$, for each fixed t, the derivative is given by

$$\frac{\mathrm{d}E_+^u(t)}{\mathrm{d}t} = \int_{\xi(\alpha,t)}^{\xi(\beta,t)} \mathrm{e}^y m_t(y,t)\mathrm{d}y = \int_{-\infty}^{\infty} \mathrm{e}^y m_t(y,t)\mathrm{d}y.$$

Similarly, if we define

$$E_+^v(t) = \int_{\varphi(\alpha,t)}^{\varphi(\beta,t)} e^y m(y,t)\, \mathrm{d}y \quad \text{and} \quad E_-^v(t) = \int_{\varphi(\alpha,t)}^{\varphi(\beta,t)} e^{-y} m(y,t)\mathrm{d}y$$

then $E_+^v(0) = E_-^v(0) = 0$ and

$$\frac{\mathrm{d}E_+^v(t)}{\mathrm{d}t} = \int_{-\infty}^{\infty} e^y n_t(y,t)\mathrm{d}y.$$

From (2) and integration by parts we have

$$\frac{\mathrm{d}E_+(t)}{\mathrm{d}t} = \int_{-\infty}^{\infty} e^y (m_t(y,t) + n_t(y,t))\, \mathrm{d}x = -\int_{\mathbb{R}} e^x \left[2v_x(u - u_{xx}) \right.$$

$$+ v(u - u_{xx})_x + 2u_x(v - v_{xx}) + u(v - v_{xx})_x \big]\, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} e^y (2uv + u_x v_x)\, \mathrm{d}y, \quad t \in [0,T)$$

where all boundary terms after integration by parts vanish, since the functions $m(\cdot,t)$, $n(\cdot,t)$ have compact support and $u(\cdot,t)$, $v(\cdot,t)$ decay exponentially at $\pm\infty$, for all $t \in [0,T)$. Using (5) for $u = I_1^u + I_2^u$, $u_x = -I_1^u + I_2^u$, $v = I_1^v + I_2^v$, $v_x = -I_1^v + I_2^v$, and noticing that all integrals $I_{1,2}^{u,v}$ are all nonnegative (nonpositive), we have that

$$2uv + u_x v_x = 3I_1^u I_1^v + I_2^u I_1^v + I_1^u I_2^v + 3I_2^u I_2^v$$

and thus

$$\frac{\mathrm{d}E_+(t)}{\mathrm{d}t} > 0. \tag{17}$$

Similarly, we have

$$\frac{\mathrm{d}E_-(t)}{\mathrm{d}t} = \int_{-\infty}^{\infty} e^{-y} (m_t(y,t) + n_t(y,t))\, \mathrm{d}x$$

$$= -\int_{-\infty}^{\infty} e^{-y} (2uv + u_x v_x)\, \mathrm{d}y < 0, \quad t \in [0,T) \tag{18}$$

for analogous reasons as before. ∎

3.3. Evolution in the Case $u = \bar{v}$ when Initially Functions are Compactly Supported

Some analytical results can be established in the case $u = \bar{v}$, for example one can prove immediately the analogue of Theorem 5.

Theorem 6. *If $u = \bar{v}$ is initially compactly supported, then $E_- = (E_-^u + E_-^v)(t)$ is a decreasing function, with $E_-(0) = 0$, and $E_+(t)$ is increasing, with $E_+(0) = 0$.*

Proof: Follows the lines of the proof of Theorem 5. In our case $2uv + u_x v_x = 2|u|^2 + |u_x|^2 \geq 0$ and for nontrivial solutions this expresion is positive at some point. ∎

The following Lemma is proved by making extensive use of relation (3).

Lemma 7 ([9]).. *Let (u, v) be a solution of system (1), and suppose u is such that $m = u - u_{xx}$ has compact support. Then, for each fixed time $0 < t < T$, u has compact support if and only if*

$$\int_{\mathbb{R}} e^x m(x) \, \mathrm{d}x = \int_{\mathbb{R}} e^{-x} m(x) \, \mathrm{d}x = 0. \tag{19}$$

The equivalent relation holds for the functions v and n.

We now establish a relation which is satisfied by solutions of (1) whose support remains compact throughout their evolution. This relation will have profound implications for solutions (u, v) of (1) which have a direct relation to each other, as we shall see in Corollary (9).

Theorem 8. *Let us assume that the functions u_0, v_0 have compact support, and let $T > 0$ be the maximal existence time of the solutions $u(x, t), v(x, t)$ which are generated by this initial data. If, for every $t \in [0, T)$, the function $x \mapsto (u(x, t), v(x, t))$ has compact support, then*

$$\int_{\mathbb{R}} e^x \left(2uv + u_x v_x\right) \, \mathrm{d}x = \int_{\mathbb{R}} e^{-x} \left(2uv + u_x v_x\right) \, \mathrm{d}x = 0 \ \text{ for } \ t \in [0, T). \tag{20}$$

Proof: By the assumptions of this theorem, Lemma 7 applies. Using (1) and differentiating the left hand side of (19) with respect to t we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} e^x \left(m + n\right) \, \mathrm{d}x = - \int_{\mathbb{R}} e^x \left(2v_x m + v m_x + 2u_x n + u n_x\right) \, \mathrm{d}x$$

$$= \int_{\mathbb{R}} e^x \left(2uv + u_x v_x\right) \, \mathrm{d}x = 0$$

similarly to the proof of Theorem 5. The final equality follows from the fact that identity (19) holds for all $t \in [0, T)$, according to Lemma 7.
Similarly, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} e^{-x} \left(m + n\right) \, \mathrm{d}x = - \int_{\mathbb{R}} e^{-x} \left(2uv + u_x v_x\right) \, \mathrm{d}x = 0. \tag{21}$$

Therefore,

$$\int_{\mathbb{R}} e^x \left(2uv + u_x v_x\right) \, \mathrm{d}x = \int_{\mathbb{R}} e^{-x} \left(2uv + u_x v_x\right) \, \mathrm{d}x = 0, \qquad t \in [0, T). \tag{22}$$

The expression under the integral on the right hand side of this relation must be identically zero by (19). This completes the proof. ∎

Corollary 9. *Let us suppose that $u(x,t) = \bar{v}(x,t)$. Then the only solution (u,v) of* (1), *i.e.,* (7) *is compactly supported over a positive time interval is the trivial solution $u \equiv v \equiv 0$. That is to say, any non-trivial solution (u,v) of* (7) *which is initially compactly supported instantaneously loses this property, and so has an infinite propagation speed.*

Proof: The statement follows directly from the relations in (22).   ■

3.4. Global Solutions for Nonnegative m_0, n_0

From (3) and (4) it follows that

$$u(x,t) + u_x(x,t) = \mathrm{e}^x \int_x^\infty \mathrm{e}^{-y} m(y,t) \, \mathrm{d}y. \tag{23}$$

Thus the nonnegativity of $m(x,t)$, $n(x,t)$ are ensures $u_x(x,t) \geq -u(x,t)$ and similarly $v_x(x,t) \geq -v(x,t)$, preventing blowup in finite time, because the solution (u,v) is uniformly bounded as long as it exists.

Blowup however might be possible if $m(x,0)$, $n(x,0)$ take both positive and negative values.

4. Conclusions

In the presented study we analysed the behavior of the solutions of the CCCH system when m, n are initially compactly supported and (i) initially u, v everywhere nonpositive/nonnegative (ii) $u = \bar{v}$. In both cases the result is that the compactness property is lost immediately, i.e., for any time $t > 0$. Asymptotically the solutions decay exponentially to zero, such that $u + v$ decays to zero monotonically. The exponential decay is already observed in the case of the peakon solutions, where m, n are supported only at finite number of points.

5. Acknowledgments

References

[1] Camassa R. and Holm D., *An Integrable Shallow Water Equation with Peaked Solitons*, Phys. Rev. Lett. **71** (2008) 1661–1664, ArXiv: patt-sol/9305002.
Camassa R., Holm D. and Hyman J., *A New Integrable Shallow Water Equation*, Adv. Appl. Mech. **31** (1994) 1–33.

[2] Constantin A., *Finite Propagation Speed for the Camassa-Holm Equation*, J. Math. Phys. **46** (2005) 023506 (4 pp).

[3] Constantin A. and Escher J., *Wave Breaking for Nonlinear Nonlocal Shallow Water Equations*, Acta Mathematica **181** (1998) 229–243.

[4] Constantin A., Gerdjikov V. and Ivanov R., *Inverse Scattering Transform for the Camassa-Holm Equation*, Inv. Problems **22** (2006) 2197–2207, arXiv:nlin/0603019v2 [nlin.SI].

[5] Constantin A. and Ivanov R., *On an Integrable Two-Component Camassa-Holm Shallow Water System*, Phys. Lett. A **372** (2008) 7129–7132.

[6] Constantin A. and McKean H., *A Shallow Water Equation on the Circle*, Commun. Pure Appl. Math. **52** (1999) 949–982.

[7] Cotter C., Holm D., Ivanov R. and Percival J., *Waltzing Peakons and Compacton Pairs in a Cross-Coupled Camassa-Holm Equation*, J. Phys. A **44** (2011) 265205 (28pp).

[8] Dullin H., Gottwald G. and Holm D., *An Integrable Shallow Water Equation with Linear and Nonlinear Dispersion*, Phys. Rev. Lett. **87** (2001) 194501 (4pp), arXiv: nlin.CD/0104004.
Dullin H., Gottwald G. and Holm D., *Camassa-Holm, Korteweg-de Vries-5 and Other Asymptotically Equivalent Eequations for Shallow Water Waves*, Fluid Dyn. Res. **33** (2003) 73–95.
Dullin H., Gottwald G. and Holm D., *On Asymptotically Equivalent Shallow Water Wave Equations*, Physica D **190** (2004) 1–14.

[9] Henry D., *Compactly Supported Solutions of the Camassa-Holm Equation*, J. Nonlin. Math. Phys. **12** (2005) 342–347.

[10] Henry D., *Persistence Properties for a Family of Nonlinear Partial Differential Equations*, Nonlinear Anal. **70** (2009) 1565–1573.

[11] Henry D., *Compactly Supported Solutions of a Family of Nonlinear Partial Differential Equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A **15** (2008) 145–150.

[12] Holm D. and Ivanov R., *Smooth and Peaked Solitons of the Camassa-Holm Equation and Applications*, J. Geom. Symmetry Phys. **22** (2011) 13-49.

[13] Holm D. and Marsden J., *Momentum Maps and Measure-Valued Solutions (Peakons, Filaments and Sheets) for the EPDiff Equation*, In: The Breadth of Symplectic and Poisson Geometry, Progr. Math. **232**, J. Marsden and T. Ratiu (Eds), Birkhäuser, Boston 2004, pp 203–235.

[14] Holm D., Marsden J. and Ratiu T., *The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories*, Adv. Math. **137** (1998) 1–81, arXiv: chao-dyn/9801015.
Holm D., Marsden J. and Ratiu T., *Euler-Poincaré Models of Ideal Fluids with Nonlinear Dispersion*, Phys. Rev. Lett. **349** (1998) 4173–4176.

[15] Ivanov R., *Extended Camassa-Holm Hierarchy and Conserved Quantities*, Z. Naturforsch. A **61** (2006) 133–138.

[16] Ivanov R., *Water Waves and Integrability*, Philos. Trans. R. Soc. Lond. Ser. A **365** (2007) 2267–2280.

[17] Johnson R., *Camassa-Holm, Korteweg-de Vries and Related Models for Water Waves*, J. Fluid Mech. **455** (2002) 63–82.

FOUR POINTS LINEARIZABLE LATTICE SCHEMES[*]

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Abstract. We provide conditions for a lattice scheme defined on a four points lattice to be linearizable by a point transformation. We apply the obtained conditions to a symmetry preserving difference scheme for the Burgers potential introduced by Dorodnitsyn and show that it is not linearizable.

1. Introduction

In a recent article [4] we extended to lattice equations the theorems introduced by Bluman and Kumei [2] for proving the linearizability of nonlinear Partial Differential Equations (PDEs) (for a recent extended review see [1]) based on the analysis of the symmetry properties of linear PDEs.

Here we extend the results of [4] to the case of a lattice scheme, i.e., when the lattice is not a priory given but it is defined by an equation so as to be able to perform a symmetry preserving discretization of a PDE.

In Section 2 we prove a theorem characterizing the symmetries of linear difference schemes on four lattice points and in Section 3 we apply it to find conditions under which a nonlinear difference scheme is linearizable. These conditions are then applied to the symmetry preserving discretization of the Burgers potential.

2. Symmetries of Linear Schemes

In this Section we define a difference scheme and provide the symmetry conditions under which such a scheme is linearizable. To do so in a definite way we limit ourselves to the case when the equation and the lattice are defined on four points

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in the plane, i.e., we consider one scalar equation for a continuous function of two (continuous) variables: $u_{m,n} = u(x_{m,n}, t_{m,n})$ defined on four lattice points.
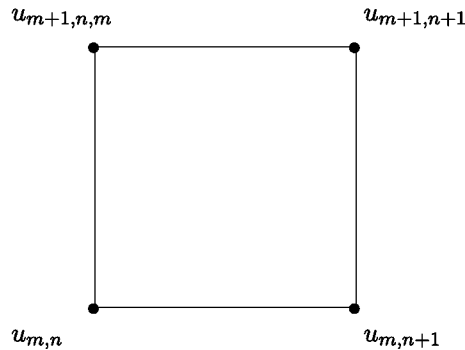
Figure 1. The $\mathbb{Z}^2$ square-lattice where the equation is defined.

2.1. The Difference Scheme

As we consider one scalar equation for a continuous function of two (continuous) variables, a lattice will be a set of points P_i, lying in the plane $\mathbb{R}^2$ and stretching in all directions with, a priori, no boundaries. The points P_i in $\mathbb{R}^2$ will be labeled by two discrete labels $P_{m,n}$. The Cartesian coordinates of the point $P_{m,n}$ will be $(x_{m,n}, t_{m,n})$ with $-\infty < m < \infty$, $-\infty < n < \infty$. The value of the dependent variable in the point $P_{m,n}$ will be denoted $u_{m,n} \equiv u(x_{m,n}, t_{m,n})$.

A difference scheme will be a set of b equations relating the values of $\{x, t, u\}$ in a finite number of points. We start with one 'reference point' $P_{m,n}$ and define a finite number of points $P_{m+i,n+j}$ in the neighborhood of $P_{m,n}$. They must lie on two different sets of curves, two of which will be intersecting in $P_{m,n}$. Thus, the difference scheme will have the form

$$E_a\bigg(\{x_{m+i,n+j}, t_{m+i,n+j}, u_{m+i,n+j}\} \bigg) = 0, \qquad 1 \le a \le b$$

$$-i_1 \le i \le i_2, \qquad -j_1 \le j \le j_2, \qquad i_1, i_2, j_1, j_2 \in \mathbb{Z}^{\ge 0}. \tag{1}$$

The situation is illustrated on Fig. 2 in the case of a 7 points lattice. Our convention is that x increases as m grows, t increases as n grows (i.e., $x_{m+1,n} - x_{m,n} \equiv h_1 > 0$, $t_{m,n+1} - t_{m,n} \equiv h_2 > 0$). The scheme on Fig. 2 could be used e.g. to approximate a differential equation of third order in x, second in t.

The value of b, the maximum number of different equations we consider, depends on the kind of problems we are considering. Starting from the reference point $P_{m,n}$ and a given number of neighboring points, it must be possible to calculate the values of $\{x, t, u\}$ in all points in a unique way. This requires a minimum of
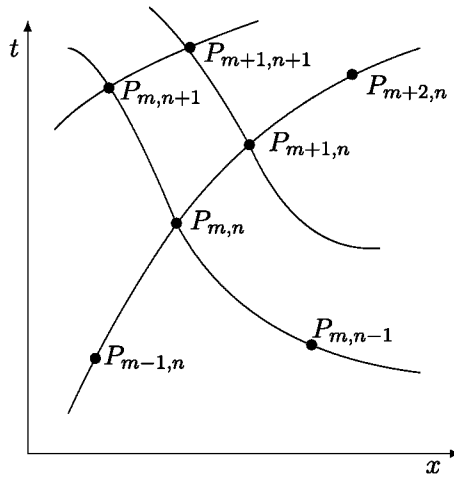
Figure 2. Points on a two dimensional lattice.

three equations to calculate the independent variables (x, t) in two directions and the dependent variable u in all points. With one dependent variable in $\mathbb{R}^2$, at most we can set $b = 5$. Of the five equations in (1), four determine completely the lattice, one the difference equation. If we choose $b = 3$ than two define the lattice and one the difference equation and we are solving an initial value problem when both the equation and the lattice are defined from given initial conditions. If a continuous limit exists, (1) represent a PDE in two variables. The equations determining the lattice will reduce to identities (like $0 = 0$).

As an example of difference scheme, let us consider the simplest and most standard lattice, namely a uniformly spaced orthogonal lattice and a difference equation approximating the linear heat equation on this lattice. The five equations (1) in this case are

$$x_{m+1,n} - x_{m,n} = h_1, \qquad t_{m+1,n} - t_{m,n} = 0 \tag{2}$$

$$x_{m,n+1} - x_{m,n} = 0, \qquad t_{m,n+1} - t_{m,n} = h_2 \tag{3}$$

$$\frac{u_{m,n+1} - u_{m,n}}{h_2} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(h_1)^2} \tag{4}$$

where h_1 and h_2 are two constants.

This example is simple as the lattice equations can be solved explicitly to give

$$x_{m,n} = h_1 m + x_0, \qquad t_{m,n} = h_2 n + t_0. \tag{5}$$

The usual choice is $x_0 = t_0 = 0$, $h_1 = h_2 = 1$ and then x is simply identified with m, t with n. The above example however suffices to bring out several points
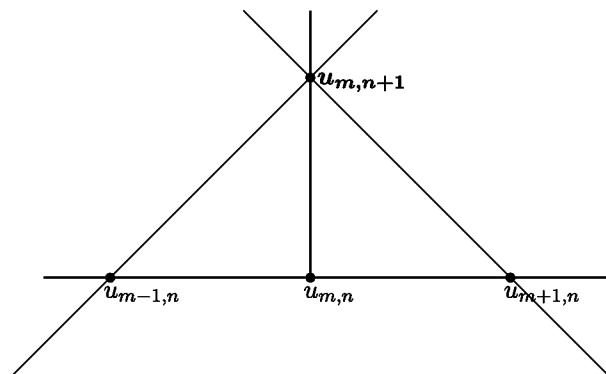
Figure 3. Four points in the case of the heat equation.

1. Four equations are needed to describe completely the lattice but in this case there is a compatibility condition. In the whole generality two equations are sufficient and provide the lattice starting from some initial conditions.
2. Four points are needed for equations of second order in x, first in t. Only three figure in the lattice equation, namely $P_{m+1,n}, P_{m,n}$ and $P_{m,n+1}$. To get the fourth point, $P_{m-1,n}$, we shift m down by one unit the equations (2-4).
3. An independence condition is needed to be able to solve for $x_{m+1,n}, t_{m+1,n}$, $x_{m,n+1}, t_{m,n+1}$ and $u_{m,n+1}$.

We need the more complicated two index notation to describe arbitrary lattices and to formulate the symmetry algorithm.

2.2. Symmetries of the Difference Scheme

We are interested in point transformations of the type

$$\tilde{x} = F_\lambda(x, t, u), \qquad \tilde{t} = G_\lambda(x, t, u), \qquad \tilde{u} = H_\lambda(x, t, u) \tag{6}$$

where λ is a group parameter, such that when (x, t, u) satisfy the system (1) then $(\tilde{x}, \tilde{t}, \tilde{u})$ satisfy the same system. The transformation acts on the entire space (x, t, u), at least locally, i.e., in some neighborhood of the reference point $P_{m,n}$, including all points $P_{m+i,n+j}$ figuring in equation (1). That means that the same functions F, G and H determine the transformation of all points. The transformations (6) are generated by the vector field

$$\hat{X} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u. \tag{7}$$

The symmetry algebra of the system (1) is the Lie algebra of the local symmetry group of local point transformations. An infinitesimal symmetry (7) is a symmetry

of (1) if (1) is invariant under a transformation (6). To check it we must prolong the action of the vector field $\hat{X}$ from the reference point $(x_{m,n}, t_{m,n}, u_{m,n})$ to all points figuring in the system (1). Since the transformations are given by the same functions F, G and H at all points, the prolongation of the vector field (7) is obtained simply by evaluating the functions ξ, τ and ϕ at the corresponding points. In other words, we have

$$
\mathrm{pr}\,\hat{X} = \sum_{m,n} \Big[\xi(x_{m,n}, t_{m,n}, u_{m,n}) \partial_{x_{m,n}}
$$
$$
+ \tau(x_{m,n}, t_{m,n}, u_{m,n}) \partial_{t_{m,n}} + \phi(x_{m,n}, t_{m,n}, u_{m,n}) \partial_{u_{m,n}} \Big] \quad (8)
$$

where the summation is over all points figuring in the system (1). The invariance requirement is formulated in terms of the prolonged vector field as

$$
\mathrm{pr}\hat{X}\, E_a \,|_{E_c=0} = 0, \qquad 1 \le a, c \le b. \tag{9}
$$

Just as in the case of PDE's [6], we can turn equation (9) into an algorithm for determining the symmetries, i.e., finding the coefficients in vector field (7) [5].

2.3. Symmetries of a Linear Partial Difference Scheme

To be able to linearize a difference scheme using the knowledge of its symmetries we must be able to characterize the symmetries of a linear scheme. To do so in this subsection we prove a theorem on the structure of the symmetries of a linear partial difference scheme.

Theorem 1. *Necessary and sufficient conditions for the three difference equations* $\mathfrak{E}_{m,n} = 0$, $\mathfrak{F}_{m,n} = 0$ *and* $\mathfrak{G}_{m,n} = 0$ *defined on four points* $\{(m,n), (m+1,n), (m,n+1), (m+1,n+1)\}$ *for a scalar function* $u_{m,n}(x_{m,n}, t_{m,n})$ *and the lattice variables* $x_{m,n}$ *and* $t_{m,n}$ *to be linear is that they are invariant with respect to the following infinitesimal generator*

$$
\hat{X}_{m,n} = v_{m,n}\partial_{u_{m,n}} + \chi_{m,n}\partial_{x_{m,n}} + \eta_{m,n}\partial_{t_{m,n}} \tag{10}
$$

where the discrete functions $v_{m,n}$, $\chi_{m,n}$, $\eta_{m,n}$ *satisfy three linear equations, i.e.,* $v_{m+1,n+1} = \mathfrak{e}_{m,n}$, $\chi_{m+1,n+1} = \mathfrak{f}_{m,n}$ *and* $\tau_{m+1,n+1} = \mathfrak{g}_{m,n}$. *The functions* $\mathfrak{e}, \mathfrak{f}$ *and* $\mathfrak{g}$ *depend just on the functions* $(v_{m,n}, \chi_{m,n}, \eta_{m,n})$ *in the points* $(m,n), (m+1,n)$

and $(m, n+1)$ *and are given by*

$$\mathfrak{e}_{0,0} = a_1 v_{0,0} + a_2 v_{0,1} + a_3 v_{1,0} + a_4 \chi_{0,0} + a_5 \chi_{0,1} + a_6 \chi_{1,0} + a_7 \eta_{0,0}$$
$$+ a_8 \eta_{0,1} + a_9 \eta_{1,0}$$

$$\mathfrak{f}_{0,0} = b_1 v_{0,0} + b_2 v_{0,1} + b_3 v_{1,0} + b_4 \chi_{0,0} + b_5 \chi_{0,1} + b_6 \chi_{1,0} + b_7 \eta_{0,0}$$
$$+ b_8 \eta_{0,1} + b_9 \eta_{1,0} \tag{11}$$

$$\mathfrak{g}_{0,0} = c_1 v_{0,0} + c_2 v_{0,1} + c_3 v_{1,0} + c_4 \chi_{0,0} + c_5 \chi_{0,1} + c_6 \chi_{1,0} + c_7 \eta_{0,0}$$
$$+ c_8 \eta_{0,1} + c_9 \eta_{1,0}$$

where $a_1, \ldots, c_9$ *depend only on the lattice indices and where, here and in the following, for the sake of simplicity we set in any discrete variable on the square* $z_{m+i,n+j} = z_{i,j}$. *The linear equations* $\mathfrak{E}_{m,n} = 0$, $\mathfrak{F}_{m,n} = 0$ *and* $\mathfrak{G}_{m,n} = 0$ *have the form*

$$u_{1,1} = a_1 u_{0,0} + a_2 u_{0,1} + a_3 u_{1,0} + a_4 x_{0,0} + a_5 x_{0,1} + a_6 x_{1,0} + a_7 t_{0,0}$$
$$+ a_8 t_{0,1} + a_9 t_{1,0}$$

$$x_{1,1} = b_1 u_{0,0} + b_2 u_{0,1} + b_3 u_{1,0} + b_4 x_{0,0} + b_5 x_{0,1} + b_6 x_{1,0} + b_7 t_{0,0}$$
$$+ b_8 t_{0,1} + b_9 t_{1,0} \tag{12}$$

$$t_{1,1} = c_1 u_{0,0} + c_2 u_{0,1} + c_3 u_{1,0} + c_4 x_{0,0} + c_5 x_{0,1} + c_6 x_{1,0} + c_7 t_{0,0}$$
$$+ c_8 t_{0,1} + c_9 t_{1,0}.$$

Proof: To prove this Theorem we require that a generic PΔE $F_{m,n} = 0$, depending on a scalar function $u_{m,n}(x_{m,n}, t_{m,n})$ and the lattice variables $x_{m,n}$ and $t_{m,n}$ in the four points $\{(m, n), (m+1, n), (m, n+1), (m+1, n+1)\}$, i.e., 12 variables, be invariant under the prolongation of (10), as given by (8). The invariance condition (9), when $\xi_{m,n}(x, t, u) = \chi_{m,n}$, $\tau_{m,n}(x, t, u) = \eta_{m,n}$ and $\phi_{m,n}(x, t, u) = v_{m,n}$ implies that $F_{m,n}$ should depend on a set of 11 independent invariants of $v_{m,n}$, $\chi_{m,n}$ and $\eta_{m,n}$

$$\begin{aligned}
L_1 &= v_{0,0} u_{0,1} - v_{0,1} u_{0,0}, & L_2 &= v_{0,0} u_{1,0} - v_{1,0} u_{0,0} \\
L_3 &= v_{0,0} u_{1,1} - \mathfrak{e}_{0,0} u_{0,0}, & L_4 &= v_{0,0} x_{0,1} - \chi_{0,1} u_{0,0} \\
L_5 &= v_{0,0} x_{1,0} - \chi_{1,0} u_{0,0}, & L_6 &= v_{0,0} x_{1,1} - \mathfrak{f}_{0,0} u_{0,0} \\
L_7 &= v_{0,0} t_{0,1} - \eta_{0,1} u_{0,0}, & L_8 &= v_{0,0} t_{1,0} - \eta_{1,0} u_{0,0} \\
L_9 &= v_{0,0} t_{1,1} - \mathfrak{g}_{0,0} u_{0,0}, & L_{10} &= v_{0,0} x_{0,0} - \chi_{0,0} u_{0,0} \\
L_{11} &= v_{0,0} t_{0,0} - \eta_{0,0} u_{0,0}. &&
\end{aligned} \tag{13}$$

As $F_{m,n}$ should not depend on the functions $(v_{m,n}, \chi_{m,n}, \eta_{m,n})$ in the points (m, n), $(m+1, n)$ and $(m, n+1)$ we have nine constraints given by the equations $\partial F_{m,n}/\partial v_{m+i,n+j} = 0$, $\partial F_{m,n}/\partial \chi_{m+i,n+j} = 0$ and $\partial F_{m,n}/\partial tau_{m+i,n+j} = 0$ with $(i, j) = (0, 0), (0, 1), (1, 0)$. These are first order partial differential equations for the function $F_{m,n}$ with respect to the 11 invariants which we can solve on

the characteristics to define three invariants

$$
\begin{aligned}
K_1 = {}& v_{0,0}\{u_{1,1} - [\mathfrak{e}_{0,0,v_{0,1}}u_{0,1} + \mathfrak{e}_{0,0,v_{1,0}}u_{1,0} + \mathfrak{e}_{0,0,\chi_{0,0}}x_{0,0} + \mathfrak{e}_{0,0,\chi_{0,1}}x_{0,1} \\
& + \mathfrak{e}_{0,0,\chi_{1,0}}x_{1,0} + \mathfrak{e}_{0,0,\eta_{0,0}}t_{0,0} + \mathfrak{e}_{0,0,\eta_{0,1}}t_{0,1} + \mathfrak{e}_{0,0,\eta_{1,0}}t_{1,0}]\} - u_{0,0}\{\mathfrak{e}_{0,0} \\
& - [\mathfrak{e}_{0,0,v_{0,1}}v_{0,1} + \mathfrak{e}_{0,0,v_{1,0}}v_{1,0} + \mathfrak{e}_{0,0,\chi_{0,0}}\chi_{0,0} + \mathfrak{e}_{0,0,\chi_{0,1}}\chi_{0,1} + \mathfrak{e}_{0,0,\chi_{1,0}}\chi_{1,0} \\
& + \mathfrak{e}_{0,0,\eta_{0,0}}\eta_{0,0} + \mathfrak{e}_{0,0,\eta_{0,1}}\eta_{0,1} + \mathfrak{e}_{0,0,\eta_{1,0}}\eta_{1,0}]\} \\
K_2 = {}& v_{0,0}\{u_{1,1} - [\mathfrak{f}_{0,0,v_{0,1}}u_{0,1} + \mathfrak{f}_{0,0,v_{1,0}}u_{1,0} + \mathfrak{f}_{0,0,\chi_{0,0}}x_{0,0} \\
& + \mathfrak{f}_{0,0,\chi_{1,0}}x_{1,0} + \mathfrak{f}_{0,0,\eta_{0,0}}t_{0,0} + \mathfrak{f}_{0,0,\eta_{0,1}}t_{0,1} + \mathfrak{f}_{0,0,\eta_{1,0}}t_{1,0}]\} - u_{0,0}\{\mathfrak{f}_{0,0} \\
& - [\mathfrak{f}_{0,0,v_{0,1}}v_{0,1} + \mathfrak{f}_{0,0,v_{1,0}}v_{1,0} + \mathfrak{f}_{0,0,\chi_{0,0}}\chi_{0,0} + \mathfrak{f}_{0,0,\chi_{0,1}}\chi_{0,1} \\
& + \mathfrak{f}_{0,0,\chi_{1,0}}\chi_{1,0} + \mathfrak{f}_{0,0,\eta_{0,0}}\eta_{0,0} + \mathfrak{f}_{0,0,\eta_{0,1}}\eta_{0,1} + \mathfrak{f}_{0,0,\eta_{1,0}}\eta_{1,0}]\} \\
K_3 = {}& v_{0,0}\{u_{1,1} - [\mathfrak{g}_{0,0,v_{0,1}}u_{0,1} + \mathfrak{g}_{0,0,v_{1,0}}u_{1,0} + \mathfrak{g}_{0,0,\chi_{0,0}}x_{0,0} + \mathfrak{g}_{0,0,\chi_{0,1}}x_{0,1} \\
& + \mathfrak{g}_{0,0,\chi_{1,0}}x_{1,0} + \mathfrak{g}_{0,0,\eta_{0,0}}t_{0,0} + \mathfrak{g}_{0,0,\eta_{0,1}}t_{0,1} + \mathfrak{g}_{0,0,\eta_{1,0}}t_{1,0}]\} - u_{0,0}\{\mathfrak{g}_{0,0} \\
& - [\mathfrak{g}_{0,0,v_{0,1}}v_{0,1} + \mathfrak{g}_{0,0,v_{1,0}}v_{1,0} + \mathfrak{g}_{0,0,\chi_{0,0}}\chi_{0,0} + \mathfrak{g}_{0,0,\chi_{0,1}}\chi_{0,1} \\
& + \mathfrak{g}_{0,0,\chi_{1,0}}\chi_{1,0} + \mathfrak{g}_{0,0,\eta_{0,0}}\eta_{0,0} + \mathfrak{g}_{0,0,\eta_{0,1}}\eta_{0,1} + \mathfrak{g}_{0,0,\eta_{1,0}}\eta_{1,0}]\}.
\end{aligned}
\tag{14}
$$

By construction the three invariants K_i, $i = 1, 2, 3$ are independent and the three equations $\mathfrak{E}_{m,n} = 0$, $\mathfrak{F}_{m,n} = 0$ and $\mathfrak{G}_{m,n} = 0$ must be defined in terms of them. The three invariants K_3, L_3 and M_3 still depend on the functions $(v_{m,n}, \chi_{m,n}, \eta_{m,n})$ in the points (m, n), $(m + 1, n)$ and $(m, n + 1)$ while they should depend just on the variables $(u_{m,n}, x_{m,n}, t_{m,n})$ in the points (m, n), $(m+1, n)$, $(m, n+1)$ and $(m + 1, n + 1)$. The derivatives F_{m,n,K_i}, $i = 1, 2, 3$ will satisfy a set of nine linear equations whose coefficients will form a matrix $\mathfrak{A}$ 9x3. The matrix $\mathfrak{A}$ can have rank 3, 2 or 1. In the case of rank 3 we have $F_{m,n,K_i} = 0$, $i = 1, 2, 3$, i.e., the function $F_{m,n}$ does not depend on the 3 invariants. If the rank of $\mathfrak{A}$ is 2 or 1 we can have at most two independent invariants. If we want to have three invariants we need to require that the coefficients of the matrix $\mathfrak{A}$ be zero, i.e., defining $\alpha_1 = v_{0,0}$, $\alpha_2 = v_{0,1}$, $\alpha_3 = v_{1,0}$, ..., $\alpha_9 = \eta_{1,0}$ we have $\frac{\partial K_p}{\partial \alpha_q} = 0$ $p = 1, 2, 3$, $q = 1, \ldots, 9$. The equations $\frac{\partial K_p}{\partial \alpha_q} = 0$ are linear homogeneous expressions in the veriables $u_{i,j}$, $x_{i,j}$ and $t_{i,j}$ with coefficients depending on $v_{i,j}$, $\chi_{i,j}$ and $\eta_{i,j}$, for appropriate values of i and j. Consequently (12). Than $\frac{\partial K_p}{\partial \alpha_q} = 0$ turn out to be a set of 159 overdetermined partial differential equations for the functions $\mathfrak{e}_{m,n}$, $\mathfrak{f}_{m,n}$ and $\mathfrak{g}_{m,n}$ whose solution (11) is obtained using Maple. It depends on 27 integration constants which must be set equal zero if (12) does not depend on $v_{i,j}$, $\chi_{i,j}$ and $\eta_{i,j}$. ∎

A few remarks can be derived from Theorem 1 and must be stressed.

Remark 1. *The equation for $u_{m,n}$ and those for the lattice variables $x_{m,n}$ and $t_{m,n}$ are independent, however the functions appearing in the symmetry* (10) *do not satisfy equations independent from those satisfied by the lattice scheme. In fact these symmetries correspond to independent superposition laws for the equation and the lattice.*

Remark 2. *If the linear equation for $u_{m,n}$ is autonomous than the coefficients $\{a_4, \ldots, a_9\}$ are zero. The variable $v_{m,n}$ will satisfy a similar equation but the lattice equations can depend linearly on $u_{m,n}$.*

Remark 3. *The proof of* Theorem 1 *does not depends on the position of the four lattice points considered, i.e.,* $\{(m,n), (m+1,n), (m,n+1), (m+1,n+1)\}$. *The same result is also valid if the four points are put on the triangle shown in* Fig. 3, *i.e.,* $\{(m,n), (m+1,n), (m-1,n), (m,n+1)\}$.

3. Linearizable Nonlinear Schemes

In this article each equation of a difference scheme is an equation for the continuous variable $u_{m,n}$, $x_{m,n}$ and $t_{m,n}$. If the equations for the lattice variables, $x_{m,n}$ and $t_{m,n}$, are solvable we get

$$x_{m,n} = \mathcal{X}(m, n, c_0, c_1, \ldots), \qquad t_{m,n} = \mathcal{T}(m, n, d_0, d_1, \ldots) \qquad (15)$$

and then the remaining equation for the variable $u_{m,n}$ depends explicitly on n and m, on the integration constants contained in (15) and turns out to be an algebraic, maybe transcendental, equation of $u_{m,n}$ in the various lattice points involved in the equation. So the difference scheme reduce to a non autonomous equation on a fixed lattice and for its linearization we can apply the results of [4].

If the equations for the lattice are not solvable the difference scheme can be thought as a system of coupled equations for the variables $u_{m,n}$, $x_{m,n}$ and $t_{m,n}$ on a fixed lattice. In this way we can apply to the equations of the scheme the results of [4] and, taking into account the results of the previous section, we can propose the following linearizability theorem

Theorem 2. *A nonlinear difference scheme* (1) *involving $i_1 + i_2$ different points in the m index and $j_1 + j_2$ in the n index for a scalar function $u_{m,n}$ of a 2–dimensional space of coordinates $x_{m,n}$ and $t_{m,n}$ will be linearizable by a point transformation*

$$w_{m,n}(y_{m,n}, z_{m,n}) = f(x_{m,n}, t_{m,n}, u_{m,n})$$
$$y_{m,n} = g(x_{m,n}, t_{m,n}, u_{m,n}), \qquad z_{m,n} = k(x_{m,n}, t_{m,n}, u_{m,n}) \qquad (16)$$

to a linear difference scheme of the kind of (12) *for* $w_{m,n}$, $y_{m,n}$ *and* $z_{m,n}$ *if it possesses a symmetry generator*

$$\hat{X} = \xi(x,t,u)\partial_x + \phi(x,t,u)\partial_t + \psi(x,t,u)\partial_u$$

$$\xi(x,u) = \alpha(x,t,u)y, \quad \phi(x,t,u) = \beta(x,t,u)z, \quad \psi(x,t,u) = \gamma(x,t,u)w \tag{17}$$

with α, β *and* γ *given functions of their arguments and* y, z *and* w *an arbitrary solution of* (11).

In the following we will consider the application of this theorem to a difference scheme which one would hope that it is linearizable as is a symmetry preserving discretization of a linearizable PDE, the Burgers potential equation [6].

3.1. Application

We consider here the discretization of the Burgers potential presented by Dorodnitsyn *et al* [3] and show that, even if it is reducible by a point transformation to the discrete scheme of the heat equation, it is not linearizable by a point transformation. As a consequence we have also that the symmetry preserving discretization of the heat equation presented by Dorodnitsyn *et al* is not a linear difference scheme. The symmetry preserving discretization of the Burgers potential is given by the following scheme

$$\frac{\Delta x}{\tau} = \frac{1}{h^+ + h^-}\left[\frac{h^-}{h^+}(w_+ - w) + \frac{h^+}{h^-}(w - w_-)\right] \tag{18}$$

$$e^{\hat{w} - w - \frac{\Delta^2 x}{2\tau}} = 1 + \frac{2\tau}{(h^+)^2}\left[\frac{w_+ - w}{h^+} - \frac{w - w_-}{h^-}\right] \tag{19}$$

$$\tau = t_{m,n+1} - t_{m,n}, \qquad t_{m+1,n} = t_{m-1,n} = t_{m,n} = t \tag{20}$$

where τ is a constant and

$$w = w_{m,n}(x_{m,n}, t_{m,n}), \quad \hat{w} = w_{m,n+1}, \quad w_- = w_{m-1,n}, \quad w_+ = w_{m+1,n}$$

$$\Delta x = x_{m,n+1} - x_{m,n}, \quad h^+ = x_{m+1,n} - x_{m,n}, \quad h^- = x_{m,n} - x_{m-1,n}.$$

Equations (18), (19) are written in terms of the discrete invariants $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ on the stencil defined in terms of $(\tau, x, \Delta x, h^+, h^-, w, \hat{w}, w_+, w_-)$ of the finite point symmetries of continuous Burgers potential equation

$$w_t = w_{xx} - \frac{1}{2}w_x^2 \tag{21}$$

$$\hat{X}_1 = \partial_t, \qquad \hat{X}_2 = \partial_x, \quad \hat{X}_3 = t\partial_x + x\partial_t, \quad \hat{X}_4 = 2t\partial_t + x\partial_x$$

$$\hat{X}_5 = \partial_w, \qquad \hat{X}_6 = t^2\partial_t + tx\partial_x + \left(\frac{1}{2}x^2 + t\right)\partial_w \tag{22}$$

$$\mathcal{I}_1 = \frac{H^+}{h^-}, \qquad \mathcal{I}_2 = \frac{\tau^{1/2}}{h^+}\mathrm{e}^{\frac{1}{2}(w-\hat{w})+\frac{\Delta^2 x}{4\tau}}$$

$$\mathcal{I}_3 = \frac{1}{4}\frac{h^{+2}}{\tau} + \frac{h^{+2}}{h^+ + h^-}\left[\frac{w_+ - w}{h^+} - \frac{w - w_-}{h^-}\right] \tag{23}$$

$$\mathcal{I}_4 = \Delta x\frac{h^+}{\tau} - \frac{2h^+}{h^+ + h^-}\left[\frac{h^-}{h^+}(w_+ - w) + \frac{h^-}{h^+}(w - w_-)\right]$$

and goes into it in the continuous limit.

Equations (18, 19) are related to a symmetry preserving discretization of the heat equation for $u_{m,n}$ by the point transformation $w_{m,n} = -2\log(u_{m,n})$. However it is not completely obvious if (18, 19) are reducible to a linear discrete equation, i.e., if it possess, as its continuous counterpart (21), an infinite dimensional symmetry $\hat{X} = u(x,t)e^{-w}\partial_w$ with $u(x,t)$ solution of the linear heat equation $u_t = u_{xx}$.

We can apply on the lattice scheme (18, 19, 20) the symmetry generator

$$\hat{X} = \psi(x,t,w)u\partial_w + \phi(x,t,w)s\partial_t + \xi(x,t,w)y\partial_x \tag{24}$$

with (x,t,w) satisfying (18, 19, 20) while (y,s,u) are solutions of the linear scheme prescribed by Theorem 1

$$u_{m,n+1} = a_1 u_{m,n} + a_2 u_{m-1,n} + a_3 u_{m+1,n} + a_4 y_{m,n} + a_5 y_{m-1,n} + a_6 y_{m+1,n}$$
$$+ a_7 s_{m,n} + a_8 s_{m-1,n} + a_9 s_{m+1,n}$$
$$y_{m,n+1} = c_1 u_{m,n} + c_2 u_{m-1,n} + c_3 u_{m+1,n} + c_4 y_{m,n} + c_5 y_{m-1,n} + c_6 y_{m+1,n}$$
$$+ c_7 s_{m,n} + c_8 s_{m-1,n} + c_9 s_{m+1,n} \tag{25}$$
$$s_{m,n+1} = b_1 u_{m,n} + b_2 u_{m-1,n} + b_3 u_{m+1,n} + b_4 y_{m,n} + b_5 y_{m-1,n} + b_6 y_{m+1,n}$$
$$+ b_7 s_{m,n} + b_8 s_{m-1,n} + b_9 s_{m+1,n}$$

where $(a_j, b_j, c_j, j = 1,\ldots,9)$ are parameters at most depending on n and m. By a long and tedious calculation carried out using a symbolic calculation program we get that

$$\psi(x,t,w) = \psi_0(t) + \psi_1(t)x + \psi_2(t)x^2$$
$$\phi(x,t,w) = \phi_0(t) + \phi_1(t)x + \phi_2(t)x^2 \tag{26}$$
$$\xi(x,t,w) = \xi_0(t) + \xi_1(t)x.$$

Introducing (26) into the determining equations for the symmetries of the discrete Burgers potential scheme (18, 19, 20) we get 1672 equations for the functions $(\psi_j(t), \phi_j(t), \xi_j(t), j = 0,1,2)$ depending on the coefficients $(a_j, b_j, c_j, j = 1,\ldots,9)$. 168 of those equations do not depend on the coefficients $(a_j, b_j, c_j, j = 1,\ldots,9)$ and on $(\psi_j(t+\tau), \phi_j(t+\tau), \xi_j(t+\tau), j = 0,1,2)$ and can be solved imposing that $\tau \neq 0$ we get $\psi_j(t) = 0$ for $j = 0,1,2$, $\phi_k = 0$ for $k = 1,2$

and $\xi_k = 0$ for $k = 0, 1$. Introducing this result in the remaining 1508 equations, we get the following 9 equations

$$b_1\phi_0(t + \tau) = b_2\phi_0(t + \tau) = b_3\phi_0(t + \tau) = b_4\phi_0(t + \tau) = b_5\phi_0(t + \tau)$$
$$= b_6\phi_0(t + \tau) = \phi_0(t) - b_7\phi_0(t + \tau) = b_8\phi_0(t + \tau) = b_9\phi_0(t + \tau) = 0.$$

If we require $\phi_0(t)$ be not identically null, the coefficients b_j, $j = 1, \ldots 6, 8, 9$ must be all zero and $b_7 \neq 0$. As a consequence $\phi_0(t) = b_7^{-n}\bar{\phi}$, with $\bar{\phi}$ an arbitrary constant. In this case we have a symmetry generator $\hat{X} = b_7^{-m}s\partial_t$ which is a consequence of the linearity of (20). So we can conclude that the Burgers potential scheme (18, 19) is not linearizable and that the corresponding discretization of the heat equation [3] is not given by a linear scheme. The linearity of the lattice equation for $t_{m,n}$ (20) is confirmed by the presence of the symmetry $\hat{X} = b_7^{-n}s\partial_t$.

Acknowledgements

References

[1] Bluman G., Cheviakov A. and Anco S., *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York 2010.

[2] Bluman G. and Kumei S., *Symmetry-Based Algorithms to Relate Partial Differential Equations I. Local Symmetries*, Eur. J. Appl. Math. **1** (1990) 189–216.
Bluman G. and Kumei S., *Symmetry-Based Algorithms to Relate Partial Differential Equations II. Linearization by Nonlocal Symmetries*, Eur. J. Appl. Math. **1** (1990) 217–223.

[3] Dorodnitsyn V., *Applications of Lie Groups to Difference Equations*, CRC Press, Boca Raton 2011.

[4] Levi D. and Scimiterna C., *Linearization Through Symmetries for Discrete Equations*, J. Phys. A: Math. & Theor. (in press).

[5] Levi D. and Winternitz P., *Continuous Symmetries of Difference Equations*, J. Phys. A: Math. & Gen. **39** (2006) R1–R63.

[6] Olver P., *Applications of Lie Groups to Differential Equations*, 2nd Edn, Springer, New York 1993.

CLASSICAL AND QUANTUM SYMMETRIES REDUCTION AND INTEGRABILITY[*]

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Abstract. Completely integrable systems always admit more alternative Hamiltonian descriptions. The geometrical formulation of quantum systems shows that similar conclusions hold true also for quantum systems. In addition, the description of quantum systems on Hilbert manifolds, e.g., the complex projective space, shows that not only quantum systems admit alternative Hamiltonian descriptions, they also admit alternative linear descriptions.

1. Introduction

In his *Lectures on Dynamics* [7], Jacobi starts with the problem of integrating the differential equations of motion. He explicitly says: In *Mécanique Analytique one finds everything related to the problem of setting up and transforming the differential equations, but very little on their integration.*

He goes on to elaborate what we nowadays call the Hamilton-Jacobi theory and elaborates on constants of the motion and symmetries.

The aim of our paper is to present a more general point of view in which the Hamilton-Jacobi theory is only an instance of the general procedure of integrating a system by reducing it to a *normal form*. In this respect we follow the view point of Birkhoff, all dynamical systems in the same orbit of the diffeomorphism

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group enjoy the same properties, therefore to study the integration problem one may select a particular representative of the equivalence class and consider it as a "normal form".

In this picture, the Hamilton-Jacobi procedure becomes a way to reduce a given Hamiltonian system to a normal form by replacing the full diffeomorphism group with the subgroup of canonical transformation necessary to achieve the transition to the normal form, the particular trnsformation is found by means of a generating function, solution of the Hamilton-Jacobi equation associated with the starting Hamiltonian functions, the comparison Hamiltonian and the one we want to tranform.

By using the well known fact that Quantum Mechanics can be treated as a Hamiltonian system (on an infinite or finite dimensional manifold as the case may be), we maintain that most of our arguments apply both to Classical and Quantum Mechanics.

For Quantum Mechanics we should bear in mind that the carrier space (space of pure states) is the complex projective space $\mathbb{PH}$ associated to the Hilbert space $\mathbb{H}$. In this framework, the geometric structures pertinent to the standard treatment are a Riemannian structure, a Poisson structure and a complex structure. A compatibility condition among them qualifies the carrier space as a Kähler manifold. In this respect, i.e., from the point of view of geometric structures, quantum mechanics requires additional structures with respect to the symplectic structure of Hamiltonian classical dynamics. Of course the most difficult aspects of quantum mechanics have to do with the infinite dimensionality of the carrier space and the fact that infinite dimensional differential geometry is much less advanced than the finite dimensional one. The most serious problem in the generic infinite dimensional situation is that unbounded operators are not continuous, therefore all our assumptions about differentiability cannot be applied without further qualifications which should be made case-by-case.

Nevertheless, the structural aspects, what we may call synthetic as opposed to analytics will be essentially the same.

To avoid technicalities we shall mainly restrict to finite dimensional carrier spaces.

To exhibit the variety of aspects emerging from the view point of Birkhoff we primarily deal with linear systems, the evolution associated with a differential equation will be our main concern, additional geometric structures on the carrier space will not be postulated at a "kinematical level" but derived by solving equations defined by the dynamics we start with.

By using symmetries and constants of the motion it is possible to derive nonlinear systems as reduction of linear ones. Of course, most of the properties valid for the linear situation will be inherited by the nonlinear one when they are compatible with the reduction procedure.

Many aspects will be illustrated by means of examples instead of formulating and proving general theorems. A more extensive treatment of these issues can be found in [4] and references therein.

Organization of the paper is as follows:

- The Geometrical Formulation of Quantum Mechanics
- Linear Systems on Vector Spaces (invariant structures: Poisson, symplectic, Lagrangian)
- Bihamiltonian Systems
- From linear to nonlinear (what happens of the superposition rule, Riccati equation on $\mathbb{PH}$).

2. Geometrical Formulation of Quantum Mechanics

The observation that quantum theory could be described in the language of symplectic mechanics may be traced back in the work of Segal and Mackey [8, 11]. More recently, several authors have been investigating the further developments of geometrical quantum mechanics, in doing so this methodology has unveiled new aspects and insights into the workings of the quantum world the way we understand it now, see [4] and references therein.

One particular aspect which may be connected with the existence of nonlinear transformations connecting linear descriptions seem to us worth of notice.

Let us start by reviewing very briefly how quantum mechanics is usually formulated.

With any physical system we associate a complex Hilbert space $\mathbb{H}$, the choice of a linear carrier space and a linear equation of motion is usually motivated [3] by the need to incorporate interference. The probabilistic interpretation of quantum mechanics requires the restriction to norm-one-vectors; to preserve the probability one usually restricts the evolution to be unitary, the differential equation of motion will then be associated with a skew-Hermitian "infinitesimal generator", written as iH with H Hermitian operator. Observables are identified with Hermitian operators which are usually thought of as the real elements of the C^*-algebra of bounded operators acting on $\mathbb{H}$.

If we use Dirac's bra and ket notation, we find that pure physical states are associated with rank-one projectors, this association depends on the specific Hermitian product we are using

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

From the Schrödinger differential equation

$$\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = \frac{H}{\mathrm{i}\hbar}|\psi\rangle$$

one derives the equation of motion for rank-one projectors

$$\frac{\mathrm{d}\rho_\psi}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[H, \rho_\psi\right].$$

Using the linearity of the equation, it is possible to extend this equation to convex linear combinations

$$\rho = \sum_j p_j \rho_{\psi_j}$$

with $p_j \geq 0$, $\sum_j p_j = 1$, to get the equation of motion on density states

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[H, \rho\right].$$

If we use also complex linear combinations we arrive at equations of motion for generic operators

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[H, A\right]$$

which represent equations of motion in the Heisenberg picture.

It is now clear that the dynamics on the complex projective space ceases to be linear, the sum of two rank-one projectors is no more a rank-one projector.

To remedy this situation, after all interference phenomena should be treated also on this nonlinear space, we have to write down a composition law for pure states which is inner, out of two pure states gives another pure state.

Such a composition law may be given the following form [4, 9], out of ϱ_1 and ρ_2 we get

$$\rho = |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + c_1 c_2^* \frac{\rho_1 \rho_o \rho_1}{\sqrt{\mathrm{Tr}\ (\rho_1 \rho_o \rho_2 \rho_0)}} + h.c.$$

with the understanding that $\left|c_1^2\right|$ and $\left|c_2^2\right|$ are required to satisfy $\rho^2 = \rho$, $\mathrm{Tr}\ \rho = 1$. The reason we have inserted this composition law by means of fiducial state ρ_0 is due to the circumstance that, when written in homogeneous coordinates for the complex projective space, Schrödinger equation becomes a Riccati-type equation and the given composition is exactly the one we would get by composing solutions by means of the "harmonic ratio" [2]. To illustrate how Riccati equation emerges from Schrï£¡dinger equation on the space of state vectors, we consider the two-dimensional case, $\mathbb{H} = \mathbb{C}^2$. Let us introduce an orthonormal basis $|e_1\rangle, |e_2\rangle$,

$\langle e_j | e_k \rangle = \delta_{jk}$, and define associated coordinates, by setting $z_1(\psi) = \langle e_1 | \psi \rangle$, $z_2(\psi) = \langle e_2 | \psi \rangle$. In this basis we write Schrödinger equation as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} = \frac{H}{\mathrm{i}\hbar} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}$$

whith H being the Hermitian matrix

$$\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}.$$

In this way we find

$$\dot{z}_1 = \frac{1}{\mathrm{i}\hbar}(h_{11}z_1 + h_{12}z_2), \qquad \dot{z}_2 = \frac{1}{\mathrm{i}\hbar}(h_{21}z_1 + h_{22}z_2)$$

along the corresponding complex conjugate equations for $\bar{z}_1, \bar{z}_2$.

We introduce homogeneous coordinates to implement the probabilistic interpretation, according to which the physical state is associated with the ray defined by $\lambda \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}$ with λ any complex number different from zero, we have $\xi = z_1/z_2$ and obtain

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{h_{21}}{\mathrm{i}\hbar} + \frac{h_{22} - h_{11}}{\mathrm{i}\hbar}\xi - \frac{h_{12}}{\mathrm{i}\hbar}\xi^2.$$

This Riccati equation has composition law

$$\frac{\xi(t) - \xi_1(t)}{\xi(t) - \xi_2(t)} = k\frac{\xi_1(t) - \xi_3(t)}{\xi_2(t) - \xi_3(t)}$$

in terms of solutions $\xi_1(t), \xi_2(t), \xi_3(t)$ and initial conditions determining k. This composition law says we can still describe interference phenomena even though the equation of motion is not linear. Perhaps, we should also remark that the lack of completeness of the vector field representing the evolution, equation of motion, is an artifact of the coordinates we have introduced to describe the space of pure states S^2. The actual dynamics is associated with a one-parameter group of transformations preserving the Kähler structure on the complex projective space $\mathbb{CP}^1 \equiv S^2$. Schrödinger equation on $\mathbb{H} = \mathbb{C}^n$ may be written in a similar form

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{H}{\mathrm{i}\hbar}\psi = \begin{vmatrix} H_1 & V \\ V^\dagger & H_2 \end{vmatrix}\psi.$$

Separating the upper $n-1$ components of ψ denoted by ξ from the n−th one η i.e., $\psi = \begin{vmatrix} \xi \\ \eta \end{vmatrix}$, Schrödinger equation may be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = H_1\xi + V\eta$$
$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = V^\dagger\xi + H_2\eta.$$

If we now set $z = \xi/\eta$, in analogy with the two-dimensional case, we get

$$\dot{z} = V + H_1 z - z H_2 - z V^\dagger z.$$

This is again a Riccati-type equation, but now z ia an $(n-1)$ −component vector, as is V, while H_2 is a single real variable.

As the carrier space is no more linear, we cannot consider operators anymore, thus we shall replace them with expectation value functions

$$e_A\left([\psi]\right) = \frac{\langle \psi \,|\, A\psi \rangle}{\langle \psi \,|\, \psi \rangle} = \operatorname{Tr} \rho_\psi A.$$

When the operator A is hermitian these expectation value functions are called Kälherian functions. They are completely characterized by the property that the Hamiltonian vector fields associated with them are also Killing vector fields. Here the Poisson structure and the Riemannian tensors are those arising from projecting the imaginary and the real part of the Hermitian tensor from the complex Hilbert space to the space of rays, or complex projective space, when expressed in contravariant form.

In summary, the standard formulation of quantum mechanics, when written in terms of pure states gives rise to a geometrical formulation on the manifold of rays, the complex projective space. Equations of motion are described by a Hamiltonian vector field which is also a Killing vector field preserving the complex structure.

In conclusion the equations of motion of a quantum system are represented by a vector field Γ on a Hilbert manifold M, for physical motivations connected with the probabilistic interpretation, the vector field is required to be Hamiltonian and Killing

3. Classical Dynamics

In principle it would be possible to formulate also classical dynamics on the Hilbert space of square integrable functions on phase space with the Liouville measure associated with the symplectic volume ω^n. Now, infinitesimal generators of the evolution may be arbitrary differential operators of arbitrary order (indeed, they could even be pseudo-differential operators). The non local character of quantum mechanics is encoded in the non local product of expectation-value functions and the corresponding derivations associated with higher order differential operators.

In classical mechanics, described on phase space, we may write the equations of motion in the form

$$i\frac{\partial f\left(q, p, t\right)}{\partial t} = \hat{H} f\left(q, p, t\right)$$

with $\hat{H}$ the Liouville operator

$$\hat{H} = -\mathrm{i}\frac{\partial H}{\partial p}\frac{\partial}{\partial q} + \mathrm{i}\frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

and H is the Hamiltonian function. In general, to restrict the equation to states, we replace $f(p, q)$ with $\rho(p, q)$ which is a probability density on phase space. It is possible to introduce the Hilbert space of square integrable functions on phase space with respect to the Liouville measure ω^n, ω being the symplectic structure. the Liouville probability density may now be written as $\rho(p, q) = \psi^*(p, q)\,\psi(p, q)$. It should be stressed that in this picture the operators $\hat{q}$ and $\hat{p}$ commute differently than in quantum mechanics. all operators are first order operators while are derivations for the standard point-wise product and therefore their exponentiations will be one-parameter groups of automorphisms of the local point-wise product. The motivation for introducing this Hilbert space point of view for classical mechanics was to study the ergodicity issues and the understanding of Lyapunov exponents. This was attempted by Koopman [6].

In this picture, the one-parameter group of transformations is an automorphism group of the point wise product on functions, i.e., of the local product.

In the quantum case, the one-parameter group of "unitary transformations" on the complex projective space does not generate automorphisms of the point wise product of Kälherian functions but of the non local product corresponding to the operator product, namely

$$e_A \star e_B([\psi]) = \mathrm{Tr}\,\rho_\psi A \cdot B = e_{AB}([\psi]).$$

It should be remarked, however, that the requirement of irreducibility of the representation for the canonical commutation relations requires that quantum mechanics be formulated on the space of square integrable functions on a Lagrangian submanifold of the phase space of the classical system.

In any case, our considerations allow us to say that a classical dynamical system is represented by a vector field Γ on a carrier space M endowed with a symplectic structure ω such that $i_\Gamma\omega = -\mathrm{d}H$. Thus, unlike for quantum case, the vector field describing the dynamics is Hamiltonian but is not required to be a Killing vector field.

Now, we can consider the problem of integrating a dynamical vector field Γ on a manifold M.

As Birkhoff [1] pointed out, all dynamical systems in the orbit of $\mathrm{Diff}(M)$ passing through Γ will share the same properties of Γ, therefore we might study a particularly relevant representative of the equivalence class to unveil the properties of each one of them in the orbit, what would be called a "normal form".

To be concrete, we shall carry on this analysis in the simple case of linear systems.

4. Linear Structures and Linear Vector Fields

Our starting point is a pair (Γ, M), a vector field Γ representing the dynamics and a carrier manifold M, thought of as a space of states.

A linear structure on M is characterized by a vector field Δ having the following properties:

1. There exists one and only one point, a critical point for Δ, such that $\Delta(m_0){=}0$
2. The eigenvalue problem

$$L_\Delta h = 0.h$$

   has only trivial solutions on each connected component of M
3. The eigenvalue problem

$$L_\Delta f = f$$

   has as many functionally independent solutions $(f_1, f_2, ..., f_n)$ as the dimension of the carrier space M
4. Δ is a complete vector field on M.

Such a vector field Δ is also called a dilation vector field.

Proposition 1. *A connected manifold M, possessing a dilation vector field Δ, may be endowed with a vector space structure.*

The statement follows by selecting a family of independent solutions for the eigenvalue equation

$$L_\Delta f = f$$

say $(f_1, f_2, ..., f_n)$ with $f_i \epsilon \mathfrak{F}(\mathrm{M})$. We may define a composition law on points of M by setting

$$(\lambda_1 m_1 + \lambda_2 m_2)(f) = \lambda_1 f(m_1) + \lambda_2 f(m_2), \qquad m_1, m_2 \in M$$

where f belongs to the linear span of $(f_1, f_2, ..., f_n)$. Notice that $(\lambda \cdot m)(f) = \lambda \cdot f(m)$ gives again a point in M because of the completeness assumption on Δ. By using $(f_1, f_2, ..., f_n)$ as a coordinate system for the whole of M, we have that $\mathrm{d}f_j(\Delta) = f_j$ implies $f_j(m_0) = 0$ and $\Delta = f_j \partial/\partial f_j$.

Thus the point m_0 is the null vector of the vector space structure on M. It is now clear that any other vector field in the orbit $\mathrm{Diff}(M)$ passing through Δ will give rise to another linear structure.

The subgroup of diffeomorphisms preserving Δ will be the group of linear transformations, $\mathrm{GL}(M, \Delta)$, which is isomorphic to $\mathrm{GL}(n, \mathbb{R})$ therefore alternate linear structures are parametrized by $\mathrm{Diff}(M)/\mathrm{GL}(M, \Delta)$.

Having defined a linear structure, it is possible to define linear vector fields

Definition 2. *A vector field Γ is linear with respect to Δ if $[\Gamma, \Delta] = 0$.*

Proposition 3. *Any linear vector field* Γ *is represented by a class of matrices defining an orbit of* $\mathrm{GL}\,(n,\mathbb{R})$.

Proof: Consider a particular independent set of solutions $L_\Delta f_j = f_j, j = 1, 2, ..., n$. We have

$$L_\Gamma L_\Delta f_j = L_\Gamma f_j = \gamma_j^k f_k, \qquad \gamma_j^k \in \mathbb{R}.$$

If we choose a different basis of solutions, we associate a different matrix to Γ, all of them will be related by the matrix $\left\|T_k^i\right\|$ taking from one basis to another, therefore they are elements in the orbit $T\gamma T^{-1}$ with $T \in \mathrm{GL}\,(n,\mathbb{R})$. The vector field Γ may be represented as a linear homogeneous differential operator

$$\Gamma = f_k \gamma_j^k \frac{\partial}{\partial f_j}.$$

The equation of motion associated with Γ are written, in the selected basis,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = f_k \gamma_j^k \frac{\partial h}{\partial f_j}.$$

In finite dimensions, a solution is provided by the exponentiation of the representative matrix γ, *i.e.,*

$$\vec{x}\,(t) = \mathrm{e}^{t\gamma}\vec{x}\,(0).$$

$\blacksquare$

Even though for each initial condition we have found the solution, it is clear that many questions like conservation, stability, periodic orbits and many others cannot be easily answered from the given form.

As matter of fact more can be said if we reduce γ to some normal form. For instance, we may use Schur decomposition to write $\gamma = N + S$, with N a nilpotent matrix and S semisimple, $[N, S] = 0$.

The commutativity property gives rise to the "composition of independent motions"

$$\mathrm{e}^{tN} \cdot \mathrm{e}^{tS} = \mathrm{e}^{t(N+S)}.$$

As N is nilpotent, e^{tN} reduces to a polynomial in t, of degree equal one-less nilpotency index.

We may derive in general that the commutant of γ, say $[\gamma]'$, the algebra of matrices commuting with γ contains all powers of γ and is Abelian if all eigenvalues are not degenerate. If γ admits degenerate eigenvalues, the algebra of symmetries will not be Abelian.

An interesting question from our point of view is the existence of constants of motion.

From our experience coming from Hamiltonian formalism, where there is a connection between symmetries and constants of the motion, we could investigate the

possibility that our dynamical vector field preserves quadratic forms or bilinear maps.

In geometrical languages solutions of the equation $L_\Gamma \mathcal{T} = 0$ would give all invariant tensor field $\mathcal{T}$.

Of particular relevance are tensor fields of rank two, perhaps decomposed into symmetric and skewsymmetric ones.

For instance $\Gamma = \Lambda (\mathrm{d} H)$, i.e., the existence of an Hamiltonian description in terms of Poisson tensor Λ and the quadratic hamiltonian H, when written in terms of matrices, would give $\gamma = \Lambda H$, where Λ and H are the representative matrices for the Poisson tensor and the Hamiltonian function respectively. We would have

$$\Lambda = \Lambda_{jk} \frac{\partial}{\partial f_j} \wedge \frac{\partial}{\partial f_k}, \qquad H = \frac{1}{2} H^{jk} f_j f_k$$

so that at matrix level $\gamma_j^k = \Lambda_{jm} H^{mk}$.

Thus, the existence of Hamiltonian descriptions for the linearizable vector field Γ amounts to the decomposition of the representative matrix as the product of a skew-symmetric times a symmetric matrix.

For the generic case we can immediately give a necessary and sufficient condition for such a decomposition.

Proposition 4. *If γ has minimal degeneracy, a necessary and sufficient condition for the decomposition $\gamma = \Lambda \cdot H$, is that all odd powers of γ are traceless.*

Proof: In one direction the choice is obvious because $\mathrm{Tr}\, \gamma = \mathrm{Tr}\, \gamma^\dagger$ and $(\Lambda \cdot H)^\dagger = -H \cdot \Lambda$. The general proof may be found in [5]. ∎

What is relevant for our considerations is the part that all matrices in the orbit

$$T\gamma T^{-1} = \gamma \left(T\Lambda T^\dagger \right) \left(T^\dagger \right)^{-1} HT^{-1} = \left(T\Lambda T^\dagger \right) \left(T^{-1} \right)^\dagger HT^{-1}$$

have also the same property, i.e., they are Hamiltonian, factorizable with respect to different Poisson structures and different Hamiltonians.

Of course, of particular significance is the family of transformations for which $T\gamma T^{-1} = \gamma$, $\left(T\Lambda T^\dagger \right) \neq \Lambda$, i.e., symmetries for γ which are not canonical.

These particular symmetries will carry one Hamiltonian description into an alternative one. We find that Hamiltonian systems admitting symmetries which are not canonical will always be bi-Hamiltonian.

From our considerations it should be clear that there are linear systems with a large group of symmetries which may admit no constants of the motion.

For instance this is the case for the dilation vector field Δ itself. The equation $L_\Delta h = 0$ admits only trivial solutions while the symmetry group is the full $\mathrm{GL}\,(n, \mathbb{R})$.

A generic linear Hamiltonian system will have all powers γ^k as symmetries but only the odd powers γ^{2k+1} will generate canonical symmetries. Thus, any linear Hamiltonian system always admits alternative Hamiltonian descriptions.

However, this way of generating alternative Hamiltonian descriptions will not exhaust the family of alternative ones. For instance the two-dimensional isotropic Harmonic Oscillator has different decompositions with H positive definite (the standard one) or with signature $(+ + --)$ which arises from

$$\Lambda = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} - \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2}, \qquad H = \frac{1}{2} \left(p_1^2 + q_1^2 - p_2^2 - q_2^2 \right).$$

Obviously this factorization cannot be related to the standard one by similarity transformation.

Remark 5. *When the Poisson tensor is not degenerate, we can invert it and define a symplectic structure. When this symplectic structure is exact, we can consider a symplectic potential and use it to define a cotangent bundle structure on M. From the Hamiltonian, if transversal to fibers, with a nondegenerate Hessian, we can go to the Lagrangian description. It is therefore clear that the chain of steps Poisson-Symplectic-Lagrangian puts more and more restrictions on the dynamical vector field we are considering, and correspondingly on the transformation group we may use to find "normal forms".*

A final comment is in order when the orbits of our dynamical evolution are all bounded. Going back to the decomposition of $\gamma = N + S$, it is clear that for bounded orbits N must be zero and S should admit only purely imaginary eigenvalues if they are not vanishing.

Thus, not vanishing purely imaginary eigenvalues require M to have even dimension. As a matter of fact, in this situation, M maybe endowed with a complex structure so that it becomes isomorphic with $\mathbb{C}^n$, Λ will represent the imaginary part of an Hermitian structure and in all we can prove the following proposition.

Proposition 6. *A complex linear vector field Γ generates a flow $\phi_t : \mathbb{C}^n \to \mathbb{C}^n$ preserving same Hermitian scalar product h, i.e., $\phi_t^\star h = h$, iff any one of the following equivalent conditions is satisfied.*

1. *$H = H^\dagger$, where the adjoint is taken with respect to the scalar product defined by h, i.e., $L_\Gamma h = 0$*

2. *H is diagonalizable and has a real spectrum*

3. *all orbits $\mathrm{e}^{-iHt}\psi$ are bounded sets, for any initial condition ψ.*

When moving to infinite dimensions, one may try to use a similar procedure, however the corresponding separation of H holds true only for a special class of operators. Therefore one has to use a different approach. Further details can be found in [10].

Acknowledgements

References

[1] Birkhoff G., *Dynamical Systems*, Am. Math. Soc., Providence 1927.

[2] Carinena J., Gracia X., Marmo G., Martinez E., Munoz-Lecanda M. and Roman-Roy N., *Geometric Hamilton-Jacobi Theory*, Int. J. Geom. Meth. Mod. Phys. **3** (2006) 1417-1458.

[3] Dirac P., *The Principles of Quantum Mecahanics*, Clarendon Press, Oxford 1958.

[4] Ercolessi E., Marmo G. and Morandi G., *From the Equations of Motion to the Canonical Commutation Relations*, La Rivista del Nuovo Cimento, Serie V **33** (2010) 401-590.

[5] Giordano M., Marmo G. and Rubano C., *The Inverse Problem in Hamiltonian Formalism: Integrability of Linear Hamiltonian Fields*, Inverse Problems **9** (1993) 443-467.

[6] Koopman B., *Hamiltonian Systems and Transformations in Hilbert Space*, Proc. Natl. Acad. Sci. **17** (1931) 315-318.

[7] Jacobi C., *Jacobi's Lectures on Dynamics* (delivered at the University of Konigsberg in the Winter Semester 1842-1843 and According to the Notes Prepared by C. Brockardt), Hindustan Book Agency, Gurgaon 2009.

[8] Mackey G., *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York 1963.

[9] Man'ko V., Marmo G., Sudarshan E. and Zaccaria F., *Inner Composition Law of Pure States as a Purification of Impure States*, Phys. Lett. A **273** (2000) 31-36.

[10] Marmo G., Simoni A. and Ventriglia F., *Quantum Systems and Alternative Unitary Descriptions*, Int. J. Mod. Phys. A **19** (2004) 2561-2578.

[11] Segal I., *Mathematical Problems of Relativistic Physics*, J. Math. Phys. **1** (1960) 468-488.

PARAMETRIC REPRESENTATIONS OF SO(n) ORTHOGONAL MATRICES FOR THE PURPOSES OF QUADRATIC STABILITY ANALYSIS

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Abstract. The group SO(n) is of a great interest in physics and mechanics because of its numerous applications to problems of monitoring of unknown nonlinear systems. The present paper treats the basic theory of this group for the purposes of quadratic stability analysis of cognitive control systems. It is shown that any transformation of the group SO(n) may be presented as a product of plane transformations in clear analytical forms, appropriate for practical applications. The approach presented here is inspired by the close analogy of plane rotations with the vector-parameterization of the SO(3) group.

1. Introduction

The group SO(n) is a generalization of the SO(3) rotation group acting in $\mathbb{R}^n$. Since $\mathbb{R}^3$ space is the real space where one lives, and where all laws of classical mechanics are valid, the experience with the investigations of the motions in $\mathbb{R}^3$ is helpful to study the motions in higher dimensions. Here is the place to stress the special attention on the group SO(3) since it is a very important in modelling and control of a mechanical system in $\mathbb{R}^3$ and its kinematical description [2]. It is well known that the rigid-body motion in $\mathbb{R}^3$ is described by the Euclidean group E(3), and that the SO(3) group cannot be avoided in the representation of orientations. The appropriate parameterization of SO(3) is one of the most important practical problem in mechanics because it has a great influence over the overall efficiency of all methods [3], [17]. Angular velocity or momentum information is required

by the most control strategies. It could be obtained using the derivatives of various orientation parameters like Euler and Bryant angles, Euler or Cayley-Klein parameters, quaternions [2], [5], etc., or the so called vector-parameter (called also Rodrigues or Gibbs vector [14]), which as an element of a Lie group, has a very nice and clear properties and simplifies the treatment of many problems [11], [13]. It is worth to be mentioned that there is an analogy between the rigid body description through vector-parameter and this one realized on the base of screw operators. The intrinsic mathematical formalism in physical rigid body motions description is presented with the use of affine geometry together with Lie group theory and it is used for description of the kinematic pairs.

After introducing the vector-parameter in connection with representation theory of the SO(4) Lorentz group in the special relativity theory [4], different group parameterizations of the rotational motion for higher dimensions and their after–effects are investigated. There is also a great interest in using SO(n) for $n \geq 4$ in the theory of elementary particles. In the general case (when n is big) the expressions for the orthogonal matrices are very complicated, but as it is shown latter in the paper, in every group SO(n) may be found a subgroup of transformations which may be parameterized in a simple and universal way which does not depend on the dimension n of the vector space.

Plane rotations appear in many classical and quantum mechanical analysis which lead to considerations of the spectrum and eigenvectors of either 3×3 or 4×4 real symmetric matrices. The objective is: given a symmetric 3×3 matrix A, construct a *diagonalizing rotation matrix* O such that

$$O^T A O = \Lambda = \text{Diag}\left[\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}\right] \tag{1}$$

where T denotes the matrix transposition, $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of the matrix A and σ is an element of the group Σ_3 of permutations of the three element set $1, 2, 3$.

The story starts with the Jacobi's method for solving the eigenvalue problem for a concrete 8×8 symmetric matrix that arises in his studies on dynamics. Jacobi diagonalizes the above matrix by performing a sequence of orthogonal similarity transformations and his method is relevant and effective in all dimensions (see [9] for numerical counterpart). Each transformation is a plane rotation, chosen so that the induced similarity diagonalizes some 2×2 principal submatrix, moving the weight of the annihilated elements onto the diagonal. Performing the same procedure in lower dimensions has a lot of specificity. E.g., using the isomorphism between 4×4 orthogonal matrices and algebra of quaternions [1] present a construction of an orthogonal similarity that acts directly on 2×2 blocks and diagonalizes a 4×4 symmetric matrix. This problem appears in many concrete

situations in physics, mechanics, crystallography, elasticity, hydromechanics, ro-
botics, etc., where one has to deal with various symmetric matrices.

Having in mind the abundance of contexts in physics, mechanics, crystallography,
elasticity, hydromechanics, robotics, etc., where symmetric matrices appear, this
problem is useful in any concrete situation, as it happens in *quadratic stabilization
of cognitive systems*.

Cognitive capability into automatic control system designed for stabilization prob-
lem is a quite interesting subject of research at recent time. Different interpretations
of cognition exist. The point of view in cognitive science is that cognition can be
treated as a computational process operating on representational structures. A cog-
nition oriented method in accordance with the characteristics of cognitive control
systems is quite appropriate for investigation of unknown nonlinear discrete-time
systems. Assuming that the system states are fully measurable and measurements
are free of noise, the proposed method can realize quadratic stabilization. The
suggested stabilization method requires neither the information about the systems
dynamical structure nor the knowledge about system physical behaviors. All the
information necessary for stabilizing the unknown system is gained during the in-
teraction of the controller with the unknown system to be controlled. The core of
the problem is the data-driven quadratic stability criterion, which is taken as the
expert knowledge in the proposed control method. More details concerning this
criterion may be found in the paper [20] as well as in the references in it. The
criterion is based on the geometrical interpretation of Quadratic Lyapunov Func-
tions (QLF) [7] and transforms the quadratic stability criterion into the problem
of judging emptiness of a polyhedrial cone, which is identical to solving a max-
min optimization problem. The focus is establishing a data-driven online stability
monitoring method suitable for unknown discrete-time nonlinear systems [8]. The
proposed stability condition shows that the existence of a QLF can only be guaran-
teed if the observed system trajectory can be mapped with one certain orthogonal
matrix at every time instant into a negative halfspace, which is equivalent to the fact
that the corresponding polar cone of the mapped data has a non-empty intersection
with the positive real space.

The paper is organized as follows: An expert knowledge in cognition-oriented con-
trol approach is proposed after some introduction words clarifying the aim of the
paper. The problem of data-driven stability analysis is involved, the diagonal qua-
dratic Lyapunov function (DQLF) and it is presented the relations between DQLF
and QLF. This is our motivation to investigate further in the paper how to obtain
the $SO(n)$ matrices which guarantee cognition-oriented quadratic stabilization of
unknown nonlinear systems. In the beginning, an useful algorithm for numerical
parametrical presentation of $SO(n)$ group is presented. Further, the general theory

of SO(n) group is given. In particular, any element of the group SO(n) is presented as a product of not more than $[n/2]$ plane transformations and it is noted that the theory of plane rotations, reviewed in the text, is analogous to the vector-parameterization of the rotation group SO(3), that is treated in a separate section. The analytical form of $n \times n$ orthogonal matrices is presented in the last section.

2. Data-Driven Stability Criterion as Expert Knowledge in Cognition - Oriented Stabilization

A data-driven stability criterion is used as the expert knowledge in the cognition-oriented control approach. The term *data-driven* is used to characterize the class of methods that appear in recent years in the field of system analysis and control. The data-driven methods use only measured data of the target system to solve system analysis and control problems, thereby possessing the advantages when a sufficiently precise model is hard to be built, unlike the widely used model-based methods which rely to a large extent on a precise mathematical model.

This feature makes the data-driven stability judgment method more suitable to serve as expert knowledge in cognition-oriented stabilization. In cognition-oriented stabilization, a precise representation of the dynamics of an unknown system cannot be obtained before the control but refined from the interaction between the controller and the plant. This fact means the representation of the plant dynamics is dynamically changing and should, if necessary, be partitioned into several local models. On the other hand, in model-based stability analysis, the stability can be judged by finding a common Lyapunov function of these different local models, which is usually difficult to be solved algorithmically especially when the local model is a nonlinear one.

In contrast, the system dynamics which is contained in the system trajectory, because the data-driven method concentrates on the system trajectory, is taken as a whole without being partitioned into local models. In this way the problem of finding a common Lyapunov function is avoided. From this perspective, it can be seen that a data-driven stability criterion is inherently suitable for stability judgment in cognition-based stabilization, i.e. it is suitable for quadratic stability assessment of an arbitrary unknown nonlinear system. In this aspect a necessary and sufficient condition for determining the existence of a Quadratic Lyapunov Function (QLF) for currently measured system trajectories is proposed. This problem is handled by using of the geometrical links of QLFs with convex cones [20]

2.1. Problem Definition of Data-Driven Stability Analysis

The discrete-time nonlinear system concerned in this thesis has the form of

$$x\left(k+1\right) = F(x(k)) \tag{2}$$

with $f(\cdot) : \Omega \rightarrow \mathbb{R}^n$ a mapping from a compact set $\Omega \subset \mathbb{R}^n$ into the n - dimensional real vector space $\mathbb{R}^n$, and with the system state vector x belonging to the region Ω. The definition for the quadratic stability for such systems can be stated as follows

Quadratic Stability: The discrete-time nonlinear system (2) is defined to be quadratic stable if there exists a positive definite Hermitian matrix P such that the first-order difference of the function $V(x(k)) = x(k)^T P x(k)$ along the solution of system (2) satisfies

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = V(f(x(k))) - V(x(k)) \leq 0.$$

Correspondingly, the function $V(x(k)) = x(k)^T P x(k)$ is named as the *Quadratic Lyapunov Function* (QLF). If in addition P is diagonal, $V(x(k))$ is named as *Diagonal Quadratic Lyapunov Function* (DQLF) and the related system (3) is defined to be diagonally quadratic stable.

In the data-driven context, the existence of a QLF cannot be determined by using the analytical form of $f(x)$ because it is unknown. Suppose that the system (2) be fully observable and the system states be measured without noise. At the time instant $t = r$, the data set containing r consecutive measurements of system states can be denoted as

$$\mathcal{X}_r = \{x(1), \ldots, x(r)\} . \tag{3}$$

The task of online stability judgment in this contribution is defined as to determine the existence of a QLF directly from the data set (3) instead of a mathematical description of $f(x)$ at every time instant. The system is judged as quadratic stable if and only if a QLF can be found based on the measured data.

2.2. Relations Between DQLF and QLF

Searching a QLF $V(x) = x(k)^T P x(k)$ is equal to searching a suitable positive definite matrix P. The complete set of the matrix P in QLF can be mapped to surjectively from the special orthogonal group $\mathrm{SO}(n)$ (defined in the next section) and the conventional topology of n - dimensional positive real vector space $\mathbb{R}^n_+$. This mapping can be defined as

$$(O, d) \mapsto P : P = O^T \mathrm{Diag}\,[d]\, O \tag{4}$$

where $O \in \mathrm{SO}(n)$ and d is a real vector in $\mathbb{R}^n_+$. It can be concluded that no QLF exists if no element over the complete set $\mathrm{SO}(n) \times \mathbb{R}^n_+$ because the mapping (4) is surjective, can be found and to construct a QLF, and vice versa. Therefore, the existence of a QLF can be determined by searching through the special orthogonal group $\mathrm{SO}(n)$ and the conventional topology of $\mathbb{R}^n_+$.

A left-multiplication with the orthogonal matrix O given in (4) of the both sides of

the concerned discrete-time system (2), $x\left(k+1\right) = f\left(x\left(k\right)\right)$ leads to the transformed system

$$z\left(k+1\right) = g(z\left(k\right)), \qquad z\left(k\right) = O\,x\left(k\right), \qquad g(z\left(k\right)) = O\,f(x\left(k\right)). \quad (5)$$

If the discrete-time system (2) has a QLF $V(x) = x\left(k\right)^T P\,x\left(k\right)$, according to the definition of QLF it can be obtained that $\Delta V(x) = x\left(k+1\right)^T P\,x\left(k+1\right) - x\left(k\right)^T P\,x\left(k\right) < 0$. Correspondingly, taking the function $V_z(z) = z(k)^T D\,z(k)$ as the Lyapunov function candidate for the transformed system, with $D = \text{Diag}[d]$, given in the mapping (4), we have

$$\begin{aligned}
\Delta V_z(z) &= z\left(k+1\right)^T D\,z\left(k+1\right) - z\left(k\right)^T d\,z\left(k\right) \\
&= z(k+1)^T O^T D\,O\,z\left(k+1\right) - z\left(k\right)^T O^T d\,O\,z\left(k\right) \qquad (6) \\
&= x\left(k+1\right)^T P\,x\left(k+1\right) - x\left(k\right)^T P\,x\left(k\right) < 0
\end{aligned}$$

which shows that the function $V_z(z)$ is a DQLF for the transformed system (5). On the other hand, if the transformed system (5) had a DQLF, it can be proven similarly to the above discussion that the system (2) has a QLF, which is composed by the orthogonal matrix in the system transformation and the diagonal matrix in the DQLF of system (5). From this discussion it can be seen that a QLF of the system (2) is equivalent to the DQLF of its corresponding transformed system (5), which is stated as the following

Lemma 1. *If the discrete-time system (2) has a QLF $V\left(x\left(k\right)\right) = x\left(k\right)^T P\,x\left(k\right)$, then there exists an orthogonal matrix O such that the transformed system (5) possesses a DQLF as $V_d(z\left(k\right)) = z\left(k\right)^T D\,z\left(k\right)$, where $D = O\,P\,O^T$ and $z\left(k\right) = O\,x\left(k\right)$, and vice versa.*

The orthogonal matrix O used in the above system transformation is exactly the same as the orthogonal matrix used in the mapping (4). Since the searching a QLF is equivalent to searching all through the combinations in the special orthogonal group SO(n) and the conventional topology of $\mathbb{R}^n_+$, it can be concluded that to search a QLF within SO(n) $\times \mathbb{R}^n_+$ for the system (2) is equivalent to searching a DQLF within $\mathbb{R}^n_+$ for the system (5) transformed with every element in the special orthogonal group SO(n). This fact not only shows that the above mentioned lemma gives the both necessary and sufficient condition, but also provides us an idea to determine the existence of a QLF for a discrete-time nonlinear system: if no DQLF exists for every possible orthogonal transformation of the concerned system, the concerned system has no QLF and is correspondingly not quadratic stable.

2.3. Numerical Algorithm for Parametric Presentation of SO(n) Matrices

In this section an useful algorithm [18] for numerical parametrical presentation of SO(n) matrices is presented for the purposes of quadratic stability analysis of cognitive - oriented control [20].

The algorithm is based on generating of random variables and the method of Gram - Schmidt orthogonalization. It consists of the following main moments

An orthogonal matrix is defined as

$$\Phi = [\Phi_1 \Phi_2 \ldots \Phi_n] \tag{7}$$

where

$$
\begin{aligned}
\Phi_1 &= v_1(\theta_1) \\
\Phi_i &= (Q_1 Q_2 \ldots Q_i) v_i(\theta_i), \qquad i = 2, \ldots, n-2 \\
\Phi_n &= Q_1 Q_2 \ldots Q_{n-1}.
\end{aligned} \tag{8}
$$

The definition of the parameters in (8) can be stated as follows

- θ_i - $(n-1) \times 1$ dimensional vector containing $n-1$ parameters

$$
\theta_i = \begin{bmatrix} \theta_{i1} \\ \theta_{i2} \\ \ldots \\ \theta_{i(n-1)} \end{bmatrix}_{(n-1)\times 1} \tag{9}
$$

- $v_i(\theta_i) - (n-i+1) \times 1$ dimensional vector

$$
v_i(\theta_i) = \begin{bmatrix} \sin \theta_{i1} \\ \cos \theta_{i1} \sin \theta_{i2} \\ \ldots \\ (\prod_{j=1}^{n-i-1} \cos \theta_{ij}) \sin \theta_{i(n-1)} \\ \prod_{j=1}^{n-i} \cos \theta_{ij} \end{bmatrix}_{(n-i+1)\times 1} \tag{10}
$$

- Q_i is a $(n-i+1) \times (n-i)$ matrix, constructed by the following procedure:
  1. After $v_i(\theta_i)$ being obtained, construct *randomly* $n-i+1$ vectors b_j, $j = 2, \ldots, n-i+1$, in such a way that $v_i(\theta_i)$ and these constructed vectors form a base in $\mathbb{R}^{n-i+1}$, denoted as

$$
B_i = [v_i(\theta_i), b_2, \ldots, b_{n-i+1}]_{(n-i+1)\times(n-i+1)}. \tag{11}
$$

  2. Execute Gram-Schmidt procedure to the base B_i to obtain the orthonormal base in $\mathbb{R}^{n-i}$. Introducing the notation $g_i = v_i(\theta_i)$, do the following calculation

$$
g_{j+1} = b_{j+1} = \sum_{j=1}^{j} \frac{<b_{j+1}, g_j>}{<g_j, g_j>}, \quad q_j = \frac{g_{j+1}}{\| g_{j+1} \|}, \quad j = 1, 2, \ldots, n-i. \tag{12}
$$

Then the orthonormal base is obtained as $[v_i(\theta_i), q_1, \ldots, q_{n-1}]$.
3. The matrix Q_i can be obtained as

$$Q_i = [q_1, \ldots, q_n - i]. \tag{13}$$

Since this algorithm contains many numerical operations and requires a lot of computational time, further in the paper, we present an analytical form for presenting SO(n) matrices.

3. General Remarks on SO(n) Group

3.1. The Lie Group SO(n)

A *Lie group* is a set G such that: 1) G is a group; 2) G is a smooth manifold; 3) the group operations of composition and inversion are smooth maps of G into itself relative to the manifold structure defined in 2).

SO(n) is a Lie group. The matrices in SO(n) present the rotational motions and as a set are defined as follows

$$\mathsf{SO}(n) = \{O \in \mathrm{Mat}(n, \mathbb{R}) \; ; \; \det O = 1, \; OO^T = I\} \tag{14}$$

where $\mathrm{Mat}(n, \mathbb{R})$ is the group of $n \times n$ matrices with elements in $\mathbb{R}$ together with its Lie algebra (i.e., its infinitesimal generators) consisting of the skew-symmetric $n \times n$ matrices. If A belongs to the Lie algebra of SO(n), the matrix $I - A$ is invertible (see [12], [13]). The Hamilton - Cayley formula provides in general the connection between the Lie algebra and the group, and therefore every orthogonal $n \times n$ matrix $O \in$ SO(n) (real or complex) can be written in the form [5], [19], [16]

$$O = O(A) = (I + A)(I - A)^{-1}$$
$$= (2I - (I - A))(I - A)^{-1} = 2(I - A)^{-1} - I. \tag{15}$$

It follows that this map can be easily inverted and in this way one obtains the matrix $A = (O - I)(O + I)^{-1}, \quad A^T = (O^T - I)(O^T + I)^{-1}, \;$ or

$$A = \frac{1}{2}(A - A^T) = (O - O^T)(2I + O + O^T)^{-1}. \tag{16}$$

The last is fulfilled provided that $\det(I + O) \neq 0$, i.e., $|I + O| \neq 0$, which is satisfied since one has $O + I = 2(I - A)^{-1}$ and $|O + I| = 2^n |I - A|^{-1}$. Hence, $|O + I| = 0$ only when the elements of the matrix A are very large. For the matrix $(I - A)^{-1}$ we may write the Neumann series, namely: $(I - A)^{-1} = I + A + A^2 + A^3 + \ldots.$ According to the theorem of Hamilton - Cayley every matrix is a root of its characteristic polynomial, which degree is equal to the order

n of the matrix. Hence, the n-th and higher degrees of the matrix O are expressed through the lower ones

$$O = a_1 A^{n-1} + a_2 A^{n-2} + \ldots a_{n-1} A + a_n. \tag{17}$$

The number of the independent parameters of the matrix $A = A_{ij}$ is $n(n-1)/2$ and they are parameters which define the special orthogonal transformation SO(n). Generally, the matrix O may be presented as $O = f(A)/f(-A)$, where $f(A)$ is any bounded function for which $f(A) \neq f(-A)$ and $|f(A)| \neq 0$. For example, such function is $f(A) = \exp A/2$ and therefore $O = \exp A$ (for more details see e.g. [15]).

From the general theory we have $A\psi = \lambda\psi$, where ψ and λ are respectively the eigenvector and the eigenvalue. Then we may present the determinant of the matrix $\lambda I - A$ in the well-known polynomial form

$$|\lambda - A| = \lambda^n - a_1 \lambda^{n-1} + a_2 \lambda^{n-2} - \ldots + (-1)^n a_n = 0 \tag{18}$$

where the following equations are valid

$$a_1 = \lambda_1 + \lambda_2 + \ldots + \lambda_n = A_{11} + A_{22} + \ldots + A_{nn} = A_t \tag{19}$$

$$a_n = \lambda_1 \lambda_2 \ldots \lambda_n \tag{20}$$

$$a_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \ldots + \lambda_{n-1} \lambda_n = \frac{1}{2}[(\lambda_1 + \lambda_2 + \ldots + \lambda_n)^2$$
$$-(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2)] \tag{21}$$

$$A_t^k = (A^k)_t = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k \tag{22}$$

$$a_2 = \frac{1}{2}[(A_t)^2 - (A^2)_t], \qquad A_t \text{ means the trace of the matrix } A. \tag{23}$$

For a matrix A of second order one has: $A^2 + A_t A + |A| = 0$. Analogically, identities for higher order matrices may be obtained, and having in mind the Hamilton - Cayley equation for the skew-symmetric matrices, they can be simplified significantly, and in the special cases of $n = 2, 3, 4, 5, 6$ look like

$$A^2 + |A| = 0 \tag{24}$$

$$A^3 - \frac{1}{2}(A^2)_t A = 0 \tag{25}$$

$$A^4 - \frac{1}{2}(A^2)_t A^2 + |A| = 0 \tag{26}$$

$$A^5 - \frac{1}{2}(A^2)_t A^3 + \frac{1}{2}[((A^2)_t)^2 - (A^4)_t]A = 0 \tag{27}$$

$$A^6 - \frac{1}{2}(A^2)_t A^4 + \frac{1}{2}[((A^2)_t)^2 - (A^4)_t]A^2 - |A| = 0. \tag{28}$$

3.2. Plane Orthogonal Transformations

The matrix $O = O(A)$ in (15) may be presented in the following way

$$O = O(A) = 2 \sum_{r=0}^{k-1} b_{2r}(I - A)A^{2(k-1-r)} - I, \qquad n = 2k$$

$$O = O(A) = I + 2 \sum_{r=0}^{k-1} b_{2r}(I - A)A^{2(k-r)+1}, \qquad n = 2k + 1 \tag{29}$$

where

$$b_{2r} = (\sum_{m=0}^{r} a_{2m})/(\sum_{m=0}^{k} a_{2m}). \tag{30}$$

Here

$$a_{2m} = (-1)^m S^m = (-1)^m \sum_{i_r=1}^{k} \lambda_{i_1}^2 \lambda_{i_2}^2 \ldots \lambda_{i_r}^2 \ldots \lambda_{i_m}^2$$

$$= -[a_{2(m-1)}s_1 + a_{2(m-2)}s_2 + \ldots + a_2 s_{m-1} + s_m] \tag{31}$$

$$i_1 \neq i_2 \neq \ldots \neq i_r \ldots \neq i_m$$

are the coefficients of the characteristic (minimal) equation of a skew-symmetric $n \times n$ matrix of a general type, expressed by elementary symmetrical polynomials s_m via the squares of the eigenvalues $\lambda_{2i} = -\lambda_{2i+1}, \ i = 1, 2, \ldots k$

$$s_m = \sum_{i=1}^{k} \lambda_i^{2m} = \frac{1}{2}(A^{2m})_t, \qquad m = 0, 1, \ldots, k. \tag{32}$$

It is easy to be seen that the equations (29)-(32) are alternative general form of those given in the previous section.

The skew-symmetric matrix A may be written in the form of linear combinations

$$A = \sum_{i=1}^{m} A_{0i} = \sum_{i=1}^{m} \alpha_i(z'^i.z^i - z^i.z'^i), \qquad m \leq N = \frac{n(n-1)}{2} \tag{33}$$

of the plane matrices

$$A_0 = \alpha(z'.z - z.z'), \qquad A_{0[kl]} = -A_{0[lk]} = \alpha(z'_k.z_l - z_k.z'_l) \tag{34}$$

where $z.y = (z_k y_l), k, l = 1, 2, \ldots, n$ means diadic matrix composed of n-dimensional (real or complex when SO(n, $\mathbb{C}$) is considered) vectors z and y - the so called divisors of the matrix A_0, and α is a normalizing coefficient.

Using some tensor algebra we may introduce the matrix of rang $n-2$, the so called dual skew-symmetric matrix $A^\times$ of A as follows

$$A^\times_{i_1 i_2 \dots i_{n-2}} = \epsilon_{i_1 i_2 \dots i_{n-2} i_{n-1} i_n} A_{i_{n-1} i_n}, \qquad i_k = 1, 2, \dots n \tag{35}$$

where $\epsilon_{i_1 i_2 \dots i_{n-2} i_{n-1} i_n}$ are the Levi - Civita symbols in n - dimensional space by which one defines also the product $AB^\times$

$$(AB^\times)_{i_1 i_2 \dots i_{n-3} j} = \epsilon_{i_1 \dots i_n} A_{i_{n-2} i_{n-1}} B_{i_n j}. \tag{36}$$

Then the necessary and sufficient condition the matrix A to be a simple one, i.e., $A = -A^T$ is equivalent the matrix products given below to be equal to the zero matrix, namely

$$A \, A^\times = A^\times \, A = 0. \tag{37}$$

In a similar way, one can define the product $A^\times B^\times$ and to obtain the relations

$$AB + B^\times A^\times = \frac{1}{2}(AB)_t, \qquad A^2 + (A^\times)^2 = \frac{1}{2}(A^2)_t. \tag{38}$$

From the second equation of (38) follows the minimal equation

$$A_0^3 = \frac{1}{2}(A_0^2)_t \, A_0, \qquad \frac{1}{2}(A_0^2)_t = s_1 = \lambda^2. \tag{39}$$

The inverse statement is also valid: every $n \times n$ matrix $A = -A^T$, which obeys to (39) and for which $s_1 \neq 0$ is simple. In this manner, we may conclude that the conditions (39) and $s_1 \neq 0$ are necessary and sufficient for the matrix A to be a simple one.

The orthogonal transformation $O_0 = O(A_0)$ (see the equations 29), which is defined through the simple matrix A_0, which fulfills (39) may be written in the general universal form

$$O(A_0) = O_0 = I + \frac{2}{1 - s_1}(A_0 + A_0^2), \quad 1 - s_1 \neq 0, \quad s_1 = \frac{1}{2}(A_0^2)_t \tag{40}$$

which coincides with the form of any transformation of the group SO(3). For the matrix $O_0 = O(A_0)$, from (40) is valid the minimal equation

$$O_0^3 = (\gamma - 1)(O_0^2 - O_0) + I, \qquad \gamma = 4 - n + (O_0)_t \tag{41}$$

and conversely, the matrix $A = -A^T$, defining the transformation $O = O(A)$ which obeys to (39) is simple. In this case the general relation (16) becomes

$$A = A_0 = (O_0 - O_0^T)(2I + O_0 + O_0^T)^{-1} = (O_0 - O_0^T)/(4 - n + (O_0)_t). \tag{42}$$

Definition 2. *The orthogonal transformation $O = O(A_0)$ defined in (40) via the skew-symmetric matrix - parameters A_0 is a plane rotation.*

This name is natural because the vectors z, z' through which the matrix A_0 is expressed according to (34), define some plane in n - dimensional space. The product of two plane orthogonal transformations is also a plane one

$$O'_0 O_0 = O(A'_0)O(A_0) = O(A''_0) = O''_0. \tag{43}$$

In this case the matrices A_0, A'_0 have to satisfy the condition

$$A_0 A'^{\times}_0 + A'_0 A^{\times}_0 = 0 \tag{44}$$

and since

$$(AB^{\times} + BA^{\times})_{i_1 i_2 ... i_{n-3} j} = \frac{1}{2}\delta_{i_1 j}(AB^{\times})_{m i_2 ... i_{n-3} m} \tag{45}$$

where $\delta_{i_1 j}$–Cristoffel symbol, we may write

$$(A_0 A'^{\times}_0)_{j i_2 ... i_{n-3} j} = (A'^{\times}_0 A_0)_{j i_2 ... i_{n-3} j} = 0. \tag{46}$$

Having in mind the relation (42), for the resultant matrix - parameter $A'' = A''_0$ the following expression is obtained

$$A''_0 = < A'_0, A_0 > = \frac{A'_0 + A_0 + [A'_0, A_0]}{1 + \frac{1}{2}(A'_0 A_0)_t} \tag{47}$$

which is analogical to the composition law of the vector - parameters of the SO(3) group.

If we set for α in (34) to be equal to $\alpha = (z^2 + zz')^{-1}$, at the conditions $z^2 = z'^2 \neq 0$, $1 - s_1 \neq 0$, we get

$$A_0 = \frac{z'.z - z.z'}{z^2 + zz'}, \qquad s_1 = \frac{zz' - z^2}{zz' + z^2}. \tag{48}$$

If we substitute (48) in (40), we obtain the matrix of so called plane orthogonal transformation

$$O_0 = O(A_0) = I - 2e.e + 2e'_0.e'_0 = O[z', z] \tag{49}$$

which realizes the transition

$$O[z', z]\, z = z'. \tag{50}$$

Here

$$e_0 = \frac{z}{\sqrt{z^2}}, \quad e'_0 = \frac{z'}{\sqrt{z'^2}}, \quad e = \frac{z + z'}{\sqrt{(z + z')^2}}, \quad e_0^2 = e_0'^2 = e^2 = 1. \tag{51}$$

The matrix $O[z', z]$ from (49) as well as every plane transformation is a product of two transformations of symmetry

$$O_0 = (I - 2e_2.e_2)(I - 2e_1.e_1), \qquad A_0 = (e_2.e_1 - e_1.e_2)/e_1 e_2 \tag{52}$$

and it corresponds to those unit vectors e_1 and e_2

$$e_1 = e_0, \qquad e_2 = e \qquad \text{or} \qquad e_1 = e, \qquad e_2 = e_0 \tag{53}$$

for which the equation (50) is fulfilled. The transformation of symmetry $I - 2e.e$ defines reflection according to a hyperplane orthogonal to the unit vector e and $(I - 2e.e)^2 = I$, $|I - 2e.e|^2 = 1$ are valid.

Matrix $O[z', z]$ in (49) may be found as one of the solutions of (50). The product of two plane transformations $O'' = O(A'') = O(A_0')O(A_0) = O[z'', \tilde{z}]O[z', z]$ has the structure of (52) when $\tilde{z} = z'$. Obviously, the case $\tilde{z} = z'$ is one of the most simple realization of the condition (44), which in the general case leads to the statement that between the four vectors $z, z', \tilde{z}, z''$, defining the simple matrices A_0 and A_0' in (48), not more than three are independent. That is why, if we consider the set of all simple matrices, every two of which satisfy (44), then the corresponding family of the plane rotations will be closed with respect to the operation (47) and defined in some subspace of three independent vectors of the n-dimensional space. This is just the situation in $\mathbb{R}^3$, where every transformation of the group of rotations SO(3) is plane. For example, all 3×3 rotation matrices in Euler angles are plane.

It is obvious, that the family of the plane transformations, defined through the set of all commutative simple matrices is given in the subspace of two independent vectors - in some plane. According to a theorem of Cartan [6], every transformation of the group SO(n) in n - dimensional vector space may be presented as a product of even numbers $\neq n$ transformations of symmetry. As far as the matrix $O[z', z]$ is a product of two transformations of symmetry, than any transformation of the group SO(n) may be presented as a product not greater than $[n/2]$ plane transformations. Naturally, this procedure has non-unique character. It has to be also noted that the theory of plane rotations given here relies heavily on the analogy with vector-parameterization of the rotation group SO(3), which we present on purpose in the section that follows.

4. Vector Representation of Rotation Motions

Let us consider the special orthogonal group SO(3) presenting the rotation motions

$$\text{SO}(3) = \{O \in \text{Mat}(3, \mathbb{R}) \; ; \; \det O = 1, \quad OO^T = I\} \tag{54}$$

where $\text{Mat}(3, \mathbb{R})$ is the group of 3×3 real matrices together with its Lie algebra (i.e., its infinitesimal generators) consisting of the real skew-symmetrical 3×3 matrices. Again here is valid that if A belongs to the Lie algebra of SO(3), the matrix $I - A$ is invertible, and the Hamilton - Cayley transformation given in (15) is used. As an exception in the three-dimensional space, there exists a map (actually isomorphism) between vectors and skew-symmetric matrices, i.e., if $c \in \mathbb{R}^3$, we have $c \to c^\times$, where $c^\times$ is the corresponding skew–symmetric matrix.

Then we may write the SO(3) matrix in the form [11]

$$O = O(c) = (I + c^\times)(I - c^\times)^{-1} = \frac{(1 - c^2)I + 2\,c \otimes c + 2\,c^\times}{1 + c^2} \qquad (55)$$

and consider it as a mapping from $\mathbb{R}^3$ to SO(3) for which the smooth inverse is

$$c^\times = \frac{O - O^T}{1 + \mathrm{tr}(O)}. \qquad (56)$$

Here I is the 3×3 identity matrix, $c \otimes c$ means diadic, $\mathrm{tr}(O)$ is the trace of the matrix O and "T" is the symbol for transposition of a matrix. The formula above provides us with an explicit parameterization of SO(3). The vector c is called *vector-parameter*. It is parallel to the axis of rotation and its module $\|c\|$ is equal to $\tan(\alpha/2)$, where α is the angle of rotation. The so defined vector-parameters form a Lie group with the following composition law

$$c' = \langle c_1, c_2 \rangle = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1.c_2}. \qquad (57)$$

The symbol "$\times$" means cross product of vectors. Every component of c can take all values from $-\infty$ to $+\infty$ without any restrictions, which is a great advantage compared with the obvious asymmetry in the Eulerian parameterization. The vector $c \equiv 0$ corresponds to the identity matrix $O(0) \equiv I$ and $-c$ produces the inverse rotation $O(-c) \equiv O^{-1}(c)$. Conjugating with elements from the SO(3) group leads to linear transformations in the vector-parameter space

$$O(c)\,O(c')\,O^{-1}(c) = O(c'')$$

where $c'' = O(c)\,c' = O_c\,c'$. Such a parameterization in the Lie group theory is called natural. It is worth mentioning also that no other parameterization possesses neither this property nor a manageable superposition law. This parameterization of SO(3) is known also as Gibbs' vector or Rodrigues' vector [14]. Some authors call it vector of finite rotations. Vector representation of rotations in three-dimensional space $\mathbb{R}^3$ is a subject of considerations of many authors, but we are the first in the literature using this parameterization as a Lie group with its nice group properties [11]. As for considering rotation problems of a rigid body and spacecrafts, it is used later also in modeling and control of open-loop mechanical systems like manipulators, vehicle devices, biomechanical systems [12]. Important properties of

the composition law of the vector-parameter group are

$$O(c')\,O(c) \;=\; O(c''), \qquad c'' = \langle c',\,c\rangle = \frac{c' + c + c' \times c}{1 - c'.\,c}$$

$$\langle c,\,0\rangle \;=\; \langle 0,\,c\rangle = c, \qquad \langle c,\,-c\rangle = 0, \qquad \langle c',\,c\rangle \neq \langle c,\,c'\rangle$$

$$\langle\langle a,\,b\rangle,\,c\rangle \;=\; \langle a,\,\langle b,\,c\rangle\rangle = \langle a,\,b,\,c\rangle, \qquad (O_a\,O_b)\,O_c = O_a(O_b\,O_c) \quad (58)$$

$$-\langle a,\,b\rangle \;=\; \langle -b,\,-a\rangle, \qquad -\langle a,\,b,\,c\rangle = \langle c,\,-b,\,-a\rangle$$

$$O^{-1}(\langle a, b\rangle) \;=\; (O(a)\,O(b))^{-1}$$

$$\;=\; O^{-1}(b)\,O^{-1}(a) = O(-b)\,O(-a) = O(\langle -b,\,-a\rangle).$$

From the alternative expression of tne Cayley formula

$$O = (I - A)(I + A)^{-1} \tag{59}$$

we obtain the composition law in the case when c_1 is the first rotation

$$c' = \langle c_2, c_1\rangle = \frac{c_1 + c_2 - c_1 \times c_2}{1 - c_1.c_2}. \tag{60}$$

Generally said, we have

$$c'_+ = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1.c_2}, \qquad c'_- = \frac{c_1 + c_2 - c_1 \times c_2}{1 - c_1.c_2}.$$

The both vector-parameters c_+ and c_- correspond to the composition of two vectors in inverse order, that are symmetrically situated according to the plane defined by c_1 and c_2. This part is a proof for the strong analogy of plane orthogonal transformations in n - dimensional space with the vector-parameterization of the group $SO(3)$.

5. Analytical Form of $n \times n$ Rotation Matrix

A block diagonal Givens matrix $R_i(\phi) \in \mathbb{R}^{n \times n}$ has the form [10]

$$R_i(\phi) = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & (\cos(\phi))_{i,i} & (-\sin(\phi))_{i,i+1} & 0 \\ 0 & (\sin(\phi))_{i+1,i} & (\cos(\phi))_{i+1,i+1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}, \quad 0 \leq \phi \leq 2\pi \tag{61}$$

for $i \in 1, 2,, \ldots, n-1$. As can be seen, the Givens matrix $R_i(\phi)$ involves only two coordinates that are affected by the rotation angle ϕ whereas the other directions, which correspond to eigenvalues 1, are unaffected by the rotation matrix. In dimension n there are $n-1$ Givens rotation matrices of the type (61). Composed they can generate a $n \times n$ matrix $R(\phi)$ according to

$$R(\phi) = R_1(\phi)R_2(\phi)\ldots R_{n-1}(\phi). \tag{62}$$

It is clear that the choice of the matrix $R(\phi)$ is a special one, when the angles in matrices $R_i(\phi)$ are chosen to be equal.

An explicit presentation of $R(\phi)$ is of the form

$$
\begin{bmatrix}
\cos(\phi) & -\cos(\phi)\sin(\phi) & \cdots & \cdots & (-1)^n\cos(\phi)\sin^{n-2}(\phi) & (-1)^{n+1}\sin^{n-1}(\phi) \\
\sin\phi & \cos^2(\phi) & -\cos^2(\phi)\sin(\phi) & \cdots & (-1)^{n-1}\cos^2(\phi)\sin^{n-3}(\phi) & (-1)^n\cos(\phi)\sin^{n-2}(\phi) \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & -\cos^2(\phi)\sin(\phi) & \vdots \\
\vdots & & \ddots & \ddots & \cos^2(\phi) & -\cos(\phi)\sin(\phi) \\
0 & \cdots & \cdots & 0 & \sin(\phi) & \cos(\phi)
\end{bmatrix}
$$

The matrix (62) is almost an upper triangular matrix but with $\sin(\phi)$ on the first subdiagonal and the $(n-2) \times (n-2)$ submatrix starting on position $(2,2)$ is a Toeplitz upper triangular matrix. (A Toeplitz matrix T or diagonal - constant matrix, is a matrix in which each descending diagonal from left to right is constant, i.e., $T_{i,j} = T_{i+1,j+1}$).

Composed the Givens rotations can transform the basis of the space to any other frame in the space. The matrix $R(\phi)$ fulfills the properties $\det R(\phi) = 1$ and $R(\phi)R^T(\phi) = I_n$, and the property $R(\phi = 0) = I_n$ holds. When n is odd, the matrix $R(\phi)$ will have an eigenvalue 1 and the remaining eigenvalues are pairs of complex conjugates, whose product is 1. The last is valid also when n is even. Consequently, the matrix $R(\phi)$ is a rotation matrix and obviously orthogonal. We may conclude that every rotation matrix when expressed in a suitable coordinate systems, partitions into independent rotations of two-dimensional subspace like in (61).

6. Conclusion

This study is provoked from the fact that the group parameterizations of the rotational motions in higher dimensions and their after–effects are of a great interest nowadays because of many applications in different scientific areas. The motivation of the present investigation is how to obtain the SO(n) matrices in analytical form which guarantee cognition-oriented quadratic stabilization of unknown nonlinear systems. The paper is an interplay of the theory of SO(n) Lie group, the plane representations of any SO(n) and the analogy with the vector-parameterization of the rotation group in the three-dimensional space SO(3).

7. Acknowledgement

References

[1] Altmann S., *Rotations, Quaternions, and Double Groups*, Clarendon, Oxford 1972.

[2] Angeles J., *Rational Kinematics*, Springer, New York 1988.

[3] Borri M., Trainelli L. and Bottaso C., *On Representations and Parameterizations of Motion*, Multibody System Dynamics **4** (2000) 129-193.

[4] Bogush A. and Fedorov F., *Vector-Parameterization of the Complex Lorentz Group* (in Russian), Theor. & Math. Physics **13** (1972) 67-74.

[5] Bottema O. and Roth B., *Theoretical Kinematics*, North-Holland, Amsterdam 1979.

[6] Cartan E., *The Theory of Spinors*, Dover, New York 1966.

[7] Cheng D., Guo L. and Huang J., *On Quadratic Lyapunov Functions*, IEEE Trans. Automatic Control **48** (2003) 885-890.

[8] Cheng D., Hu X. and Shen T., *Analysis and Design of Nonlinear Control Systems*, Springer, Berlin 2010.

[9] Golub G. and van der Vorst H., *Numerical Progress in Eigenvalue Computation in the 20–th Century*, See also the web page: http://citeseer.nj.nec.com/golub99numerical.html, 1999.

[10] Klein A. and Spreij P., *Transformed Statistical Distance Measures and the Fisher Infromation Matrix*, Linear Algebra Appl. (2012), http://dx.doi.org/10.1016/j.Iaa.2012.03.002 (in press).

[11] Mladenova C., *An Approach to Description of a Rigid Body Motion*, Comp. Rend. Bulg. Acad. Sci. **12** (1985) 1657-1660.

[12] Mladenova C., *Group Theory in the Problems of Modeling and Control of Multi-Body Systems*, J. Geom. Symmetry Phys. **8** (2006) 17-121.

[13] Mladenova C., *The Rotation Group and its Mechanical Applications. Multibody Systems*, VDM Verlag, Saarbrucken 2011.

[14] Piña E., *Rotations with Rodrigues' Vector*, European J. Phys. **32** (2011) 1171-1178.

[15] Rohan R-A. and Andrica D., *Some Remarks on the Exponential Map on the Group SO(n) and SE(n)*, Geometry, Integrability & Quantization **14** (2013) 160-175.

[16] Schaub H., Tsiotras P. and Junkins J., *Principal Rotation Representations of Proper $N \times N$ Orthogonal Matrices*, Int. J. Eng. Sci. **33** (1995) 2277-2295.

[17] Schuster M., *A Survey of Attitude Representations*, J. Astron. Sci. **41** (1993) 439-517.

[18] Tang Y. and Li J. P., *A New Algorithm of ICA:Using the Parametrized Orthogonal Matrices of Any Dimensions*, Acta Automatica Sinica **1** (2008) 31-39.

[19] Tsiotras P., Junkins J. and Schaub H., *Higher Order Cayley Transforms with Applications to Attitude Representations*, In: Proc. Aerodynamics Specialists Conference San Diego, CA 1996, pp. 29–31.

[20] Zhang Y. and Söffker D., *A Data-Driven On-line Stability Monitoring Method for Unknown Discrete-Time Nonlinear Systems*, In: Proc. 50th Conference on Decision and Control and European Control Conference (CDC - ECC), Orlando, Florida 2011, pp. 7356-7361.

ON VENTCEL'S TYPE BOUNDARY CONDITION FOR LAPLACE OPERATOR IN A SECTOR[*]

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Abstract. This paper deals with classical solutions of the Dirichlet-Ventcel boundary value problem (BVP) for the Laplace operator in bounded sector in the plane having opening of the corresponding angle $\varphi_0 > 0$. Ventcel BVP is given by second order differential operator on the boundary satisfying Lopatinksii condition there. As the boundary is non smooth, two different cases appear: $\frac{\pi}{\varphi_0}$ is irrational and $\frac{\pi}{\varphi_0}$ is an integer. At first we prove uniqueness result via the maximum principle and then existence of the classical solution. To do this we apply two different approaches: the machinery of the small denominators and the concept of Green function.

1. Introduction

This paper deals with existence and uniqueness of the classical solution for the Laplace operator equipped with Ventcel's type boundary condition in a bounded sector in the plane. Ventcel boundary conditions are second order differential conditions appearing in asymptotic models proposed by Feller and Ventcel [4], [7, 8] (interpreted as a surface diffusion). The opening of the angle $\varphi_0 > 0$ with vertex at the origin is such that $\pi/\varphi_0 \notin \mathbb{Q}$ or $\frac{\pi}{\varphi_0} \in \mathbb{N}$, $\mathbb{Q}$ being the set of rational numbers and $\mathbb{N}$ standing for the set of positive integers. At first we state the problem and prove uniqueness result (comparison principle) via the maximum principle for elliptic equations. Our second step is to prove existence result for classical solutions. To do this we apply the machinery of small denominators. Another approach is from the theory of ordinary differential equations (see Section 3 of the paper).

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Under different conditions imposed on φ_0 we prove existence of a C^2 solution in the bounded domain as well regularity results, including C^∞ solutions. The solution is found in the form of convergent series in $r^m \sin(\frac{n\pi\varphi}{\varphi_0})$, $m, n \in \mathbb{N}$. As Laplace operator is C^∞ and even analytic hypoelliptic, the main difficulties are in proving regularity up to the boundary. Laplace-Dirichlet-Ventcel problem in a disc, in a ring and in a bounded smooth domain were studied via pseudo-differential operators approach in [2].

To be more precise, we shall mention that in 1951 M. Vishik studied in a bounded smooth domain Ω the following BVP

$$\Delta u = F \quad \text{in} \quad \Omega, \qquad -\Delta' u + \frac{\partial u}{\partial n} = f \quad \text{on} \quad \partial\Omega \tag{1}$$

where Δ' is the Laplace-Beltrami operator on $\partial\Omega$ and n is the unit outer normal to $\partial\Omega$. He proved that (1) is a Fredholm BVP of index 0, i.e., it possesses finite-dimensional kernel and co-kernel of the same dimension. Difficulties appear when $\partial\Omega$ has singular points (corner ponts in the plane, dihedral angles, conical points in the multidimensional case). There are a lot of investigations on the subject by V. Kondratiev, P. Grisvard, B.-W. Schulze and his collaborators and many others. We concentrate in our paper to the Dirichlet-Ventcel problem in $\mathbb{R}^2$ in a sector (corner domain).

2. Formulation and Proof of the Main Results

2.1. Comparison Principle

In this Subsection we shall formulate Ventcel's BVP and we shall prove a Comparison principle which guarantees the uniqueness of the solution.

The Ventcel's boundary value problem in the sector

$$S_R = \{(r, \varphi) \,;\, 0 < r < R, \quad 0 < \varphi < \varphi_0\} \in \mathbb{R}^2$$

for Laplace operator is given by

$$\Delta u = f \in C^0(\bar{S}_R), \quad \left(-\frac{\partial^2 u}{\partial \varphi^2} + \alpha \frac{\partial u}{\partial n} + \beta u\right)\Bigg|_{r=R} = 0, \ u|_{\varphi=0} = u|_{\varphi=\varphi_0} = 0. \tag{2}$$

In polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

where the solution $u \in C^2(\bar{S}_R)$, the constants $\alpha > 0$, $\beta > 0$ and $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ is the unit outer normal to the arc $\varphi \in (0, \varphi_0)$, $r = R$ of the boundary of S_R.

We propose below several useful results to be used further on.

1. Let $\alpha \geq 0$, $\beta \neq 0$ and $u = r^{\alpha} \sin(\beta \varphi)$. Then $\Delta u = (\alpha^2 - \beta^2)r^{\alpha-2}\sin(\beta \varphi)$. Therefore, $\alpha = \beta \Rightarrow \Delta u = 0$, $\alpha = 1 \Rightarrow \Delta u = (1 - \beta^2)\sin(\beta\varphi)\, r^{-1}$.

2. Let $\alpha \geq 2$. Then $\Delta(r^{\alpha}\log r \sin(\alpha\varphi)) = 2\alpha r^{\alpha-2}\sin(\alpha\varphi)$.

3. $\sin(\beta\varphi) = 0 \iff \beta = \frac{k\pi}{\varphi_0}$, $k = \pm 1, \pm 2, \ldots$.

To simplify the things we assume $\alpha = \beta = 1$.

Proposition 1 (Comparison principle). *Suppose that $\Delta u \geq 0$ in $\bar{S}_R$, $u \in C^2(\bar{S}_R)$, $u|_{\varphi=0} \leq 0$, $u|_{\varphi=\varphi_0} \leq 0$ and $-\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial u}{\partial r} + u \leq 0$ on $r = R$. Then u does not attain positive maximum in $\bar{S}_R$.*

Proof: Put $\max_{\bar{S}_R} u = u(P_0) = M$. If $P_0 \in S_R$ then $u \equiv$ const and therefore $M \leq 0$. Assume now $P_0 \in \partial S_R$. If $P_0 \in \{\varphi = 0\} \cup \{\varphi = \varphi_0\} \Rightarrow u(P_0) = M \leq 0$, while $P_0 \in \{r = R, 0 < \varphi < \varphi_0\}$ implies $-\frac{\partial^2 u}{\partial \varphi^2}(P_0) \geq 0$ (maximum attained in an interior point of the arc, $\frac{\partial}{\partial \varphi}$ being tangential to the arc) and $\frac{\partial \varphi}{\partial n}(P_0) > 0$ (Hopf maximum principle [5]). Thus

$$-\frac{\partial^2 u}{\partial \varphi^2}(P_0) + \frac{\partial \varphi}{\partial n}(P_0) + u(P_0) > 0$$

which leads to contradiction. So $u(P) \leq 0$ in $\bar{S}_R$. ∎

Corollary 2. *The solution of (2) is uniquely determined in $C^2(\bar{S}_R)$.*

In fact, $-u$ verifies too the conditions of Proposition 1.

Proposition 3. *Consider Ventcel's boundary value problem (2) with*

$$f = r^{k+1} A(r, \varphi) \qquad A \in C^2(\bar{S}_R) \qquad A(r, 0) = A(r, \varphi_0) = 0, \qquad k \in \mathbb{N}$$

and assume that the solution u is such that

$$u = r^2 D(r, \varphi), \qquad D \in C^2(\bar{S}_R), \qquad D(r, 0) = D(r, \varphi_0) = 0$$

$$\frac{\pi}{\varphi_0} = k + 2 + \lambda, \qquad 0 < \lambda < 1.$$

Then there exists a constant $A > 0$ such that for each $\varepsilon > 0$

$$|u| \leq \frac{A}{\varepsilon} r^{\frac{\pi}{\varphi_0} - \varepsilon} \sin(\frac{\pi\varphi}{\varphi_0}), \qquad 0 \leq r \leq R, \qquad 0 \leq \varphi \leq \varphi_0.$$

Certainly, we can write $C(\varepsilon) = A/\varepsilon$.

Proof: Having in mind that $\sin(\frac{\pi\varphi}{\varphi_0}) > 0$ for $\varphi \in (0, \varphi_0)$, $A(r, 0) = A(r, \varphi_0) = 0$ and l'Hospital rule we can write

$$f = r^{k+1} B(r, \varphi) \sin(\frac{\pi\varphi}{\varphi_0}), \qquad B(r, \varphi) \in C^0(\bar{S}_R)$$

respectively $u = r^2 E(r, \varphi) \sin\left(\frac{\pi\varphi}{\varphi_0}\right)$, $E(r, \varphi) \in C^0(\bar{S}_R)$. Define now the auxiliary function

$$u_1(r, \varphi) = C(\varepsilon) r^{\frac{\pi}{\varphi_0} - \varepsilon} \sin(\frac{\pi\varphi}{\varphi_0}) = C(\varepsilon) r^{k+2+\lambda-\varepsilon} \sin(\frac{\pi\varphi}{\varphi_0}), \quad 0 < \varepsilon < \lambda < 1$$

where the constant $C(\varepsilon) > 0$ will be found further on. Then

$$\Delta u_1 = -\varepsilon C \left(\frac{2\pi}{\varphi_0} - \varepsilon\right) r^{\frac{\pi}{\varphi_0} - \varepsilon - 2} \sin(\frac{\pi\varphi}{\varphi_0})$$

and

$$\Delta(u - u_1) = \left[\varepsilon C \left(\frac{2\pi}{\varphi_0} - \varepsilon\right) r^{k+\lambda-\varepsilon} + r^{k+1} B(r, \varphi)\right] \sin(\frac{\pi\varphi}{\varphi_0})$$

$$= \sin(\frac{\pi\varphi}{\varphi_0}) r^{k+\lambda-\varepsilon} \left[\varepsilon C(\frac{2\pi}{\varphi_0} - \varepsilon) + r^{1+\varepsilon-\lambda} B(r, \varphi)\right].$$

Having in mind that

$$r^{1+\varepsilon-\lambda} |B(r, \varphi)| \leq R^{1+\varepsilon-\lambda} C_1, \qquad C_1 = \max_{\bar{S}_R} |u|$$

for $0 \leq r \leq R, 0 \leq \varphi \leq \varphi_0, 0 < 1 + \varepsilon - \lambda < 1$ we get that $\Delta(u - u_1) \geq 0$ if

$$C(\varepsilon) \geq \frac{C_1 R^{1+\varepsilon-\lambda}}{\varepsilon \left(\frac{2\pi}{\varphi_0} - \varepsilon\right)}, \qquad C_1 = \text{const} > 0. \tag{3}$$

On the other hand, $u_1|_{\varphi=0, \varphi=\varphi_0} = u|_{\varphi=0, \varphi=\varphi_0} = 0$ and on $\{r = R\}$

$$u - u_1 \leq R^2 \left(C_2 - C(\varepsilon) R^{k+\lambda-\varepsilon}\right) \sin(\frac{\pi\varphi}{\varphi_0}), \qquad C_2 = \max_{\bar{S}_R} |E| > 0.$$

Therefore,

$$u - u_1 \leq 0 \qquad \text{on} \qquad \partial S_R \qquad \text{if} \qquad C(\varepsilon) \geq \frac{C_2}{R^{k+\lambda-\varepsilon}}. \tag{4}$$

Combining (3), (4) and the comparison principle to the Dirichlet problem for Δ operator in S_R we get $u \leq u_1$ in $\bar{S}_R$. Similar considerations for $-u$ leads to $-u \leq u_1 \Rightarrow |u| \leq u_1$ in $\bar{S}_R$. In other words, if f vanishes of order $k + 1$ with respect to r and u vanishes of order 2 with respect to r, then u vanishes of order $k + 2$ in r. Moreover, $\partial_r^{k+2} u \in C^{\lambda-\varepsilon}(\bar{S}_R)$, $C^{\lambda-\varepsilon}$, $\varepsilon > 0$ being the corresponding Hölder class (see also [6]). ∎

2.2. Existence of Solution via Small Denominators

This Subsection deals with the existence of a classical solution of the BVP (2) via the method of the small denominators. The angle φ_0 is such that $\frac{\pi}{\varphi_0} > 2$ is an irrational number.

We assume that $0 < \varphi_0 < \frac{\pi}{2}$ as then $r^{\frac{\pi}{\varphi_0}} \in C^2[0, R]$, while $\varphi_0 > \frac{\pi}{2} \Rightarrow r^{\frac{\pi}{\varphi_0}} \notin C^2[0, R]$. We remind that each real $x = [x] + \{x\}$, $[x]$ being the integer part of x, $0 \leq \{x\} < 1$. We shall find a solution of (2) for right-hand side

$$f(r, \varphi) = \sum_{m \geq 2} \sum_{n=1}^{\infty} B_{mn} r^{m-2} \sin(\frac{n\pi\varphi}{\varphi_0}). \tag{5}$$

To do this we suppose that $B_{mn} = \mathcal{O}(\frac{1}{(m^2+n^2)^s})$ and apply Cauchy integral test to the double series (5) with $R \leq 1$. Then (5) is absolutely and uniformly convergent if

$$\int\int_{x^2+y^2 \geq 1/4} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^s} < \infty \iff s > 1.$$

Thus $f \in C^0(\bar{S}_R)$. Moreover, $f \in C^1(\bar{S}_R)$ for $s > \frac{3}{2}$ etc.

We split the proof of the solvability of (2) into two parts: $u = u_1 + u_2$, where

$$\Delta u_2 = f \quad \text{in} \quad S_R, \qquad u_2|_{\varphi=0} = u_2|_{\varphi=\varphi_0} = 0 \tag{6}$$

$$\Delta u_1 = 0 \quad \text{in} \quad S_R, \qquad u_1|_{\varphi=0} = u_1|_{\varphi=\varphi_0} = 0. \tag{7}$$

Evidently $Lu_1 = -Lu_2$ for $r = R$, where

$$L = -\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial r} + 1$$

is the boundary operator in (2).

We look for the non unique solution of (6) in the form

$$u_2(r, \varphi) = \sum_{m \geq 2} \sum_{n=1}^{\infty} A_{mn} r^m \sin(\frac{n\pi\varphi}{\varphi_0}) \tag{8}$$

i.e., if

$$\frac{\pi}{\varphi_0} \notin \mathbb{Q}, \qquad \frac{\pi}{\varphi_0} > 2 \tag{9}$$

then $m \neq n\frac{\pi}{\varphi_0} \notin \mathbb{Q}$, for all $m, n \in \mathbb{N}$ and therefore

$$\Delta u_2 = \sum_{m \geq 2} \sum_{n=1}^{\infty} A_{mn} \left(m^2 - \frac{n^2\pi^2}{\varphi_0^2} \right) r^{m-2} \sin(\frac{n\pi\varphi}{\varphi_0}) \equiv f.$$

Consequently, $A_{mn}(m^2 - \frac{n^2\pi^2}{\varphi_0^2}) = B_{mn}$, i.e.,

$$u_2(r, \varphi) = \sum_{m \geq 2} \sum_{n \geq 1} \frac{B_{mn}}{m^2 - \frac{n^2\pi^2}{\varphi_0^2}} r^m \sin(\frac{n\pi\varphi}{\varphi_0}). \tag{10}$$

As what concerns (7) we take

$$u_1(r, \varphi) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\varphi_0}} \sin(\frac{n\pi\varphi}{\varphi_0}) \tag{11}$$

beacause $\Delta u_1 = 0$. Here A_n are unknown coefficients and one can easily see that for $r = R$

$$Lu_1 = \sum_{n=1}^{\infty} A_n \left(\frac{n^2\pi^2}{\varphi_0^2} + \frac{n\pi}{\varphi_0} R^{-1} + 1 \right) R^{\frac{n\pi}{\varphi_0}} \sin(\frac{n\pi\varphi}{\varphi_0})$$

$$= -\sum_{m \geq 2} \sum_{n \geq 1} \frac{B_{mn}}{m^2 - \frac{n^2\pi^2}{\varphi_0^2}} \left(\frac{n^2\pi^2}{\varphi_0^2} + mR^{-1} + 1 \right) R^m \sin(\frac{n\pi\varphi}{\varphi_0})$$

i.e.,

$$A_n = -\sum_{m=2}^{\infty} \frac{B_{mn}}{m^2 - \frac{n^2\pi^2}{\varphi_0^2}} \left(\frac{n^2\pi^2}{\varphi_0^2} + mR^{-1} + 1 \right) \frac{R^{m-\frac{n\pi}{\varphi_0}}}{\frac{n^2\pi^2}{\varphi_0^2} + \frac{n\pi}{\varphi_0} R^{-1} + 1}. \tag{12}$$

To simplify the things let $R \leq 1$. Then with $C = \text{const} > 0$

$$|A_n| \leq C \sum_{m=2}^{\infty} \frac{1}{(m^2 + n^2)^{s-1/2}|(m - \frac{n\pi}{\varphi_0})|n^2 R^{\frac{n\pi}{\varphi_0}}}$$

and

$$|u_1(r, \varphi)| \leq C \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} \frac{(\frac{r}{R})^{\frac{n\pi}{\varphi_0}}}{n^2(m^2 + n^2)^{s-1/2}|(m - \frac{n\pi}{\varphi_0})|}, \quad \frac{r}{R} \leq 1, n \geq 1. \tag{13}$$

Now we shall use the approach of the small denominators having many applications in the celestial mechanics (see for example [1]). There it is proved the following result.

Lemma 4 ([1], Chapter 3, §12). *Let $\sigma > 0$. Then for almost each real μ there exists a constant $K(\sigma, \mu) > 0$ such that*

$$\left| \mu - \frac{p}{q} \right| \geq \frac{K}{|q|^{2+\sigma}}, \qquad \text{for all} \quad p, q \neq 0, \quad p, q \in \mathbb{Z}. \tag{14}$$

Evidently, $\mu \notin \mathbb{Q}$.

A result similar to (14) was proved by Liouville in 1844 for each algebraic number (non-rational). Actually A. Thue improved the theorem of Liouville in 1908, while in 1921 C. Siegel and in 1955 K. Roth obtained the optimal in some sense results on the subject. Moreover, Liouville found effective examples of transcendental numbers, known as Liouville transcendental numbers [3]. Unfortunately, algebraic numbers form a countable set, i.e., they have zero Lebesgue measure. The real numbers μ verifying (14) have a full Lebesgue measure and almost each transcendental number is a solution of (14). Liouville numbers do not satisfy (14). We shall suppose that $\frac{\pi}{\varphi_0}$ is a solution of (14), i.e.,

$$\left| \frac{m}{n} - \frac{\pi}{\varphi_0} \right| \geq \frac{K}{n^{2+\sigma}}, \quad \text{for all} \quad m, n \in \mathbb{N}. \iff \left| m - \frac{n\pi}{\varphi_0} \right| \geq \frac{K}{n^{1+\sigma}}. \quad (15)$$

According to (13), (15)

$$|u_1(r,\varphi)| \leq \frac{C}{K} \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} \frac{n^{1+\sigma}}{(m^2+n^2)^{s-1/2}}, \quad n^{1+\sigma} \leq C_0(n^2+m^2)^{\frac{1+\sigma}{2}}. \quad (16)$$

The double series in the right hand side of (15) is convergent if $s > 2 + \frac{\sigma}{2}$, $\sigma > 0$ and $u_1 \in C^0(\bar{S}_R)$. The differentiability of u_1, u_2 is shown similarly.

This way we come to

Theorem 5. *Consider the Ventcel boundary value problem* (2) *with right-hand side* f *satisfying* (5), $\frac{\pi}{\varphi_0}$ *verifying* (9), (15). *Then for each sufficiently large* $s \gg \frac{\sigma}{2}$ *there exists a unique* $C^2(\bar{S}_R)$ *solution of* (2) *that can be written in the form* $u = u_1 + u_2$ *and* u_1, u_2 *satisfies* (6), *respectively* (7).

Remark 6. *The solution* u *is given by*

$$u(r,\varphi) = \sum_{m \geq 2} \sum_{n=1}^{\infty} \frac{B_{mn}}{m^2 - n^2 \frac{\pi^2}{\varphi_0^2}} r^m \sin(\frac{n\pi\varphi}{\varphi_0}) + \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\varphi_0}} \sin(\frac{n\pi\varphi}{\varphi_0}).$$

Evidently, $u_1 \in C^\infty(\bar{S}_R) \iff A_n = 0$, $n \in \mathbb{N}$. *The condition* (9) *implies that if* $A_1 = \ldots = A_{k-1} = 0$, *then* $u_1 \in C^{[\frac{k\pi}{\varphi_0}]}$, $\frac{k\pi}{\varphi_0} > 2k$. *As* Δ *is hypoelliptic,* $f \in C^\infty(S_R) \Rightarrow u \in C^\infty(S_R)$. *In general,* $f \in C^\infty(\bar{S}_R) \not\Rightarrow u \in C^\infty(\bar{S}_R)$.

2.3. Existence of Solution for $\varphi_0 = \pi$

We shall discuss here the existence of a classical solution of (2) in the case $\frac{\pi}{\varphi_0} \in \mathbb{N}$ and more specifically, $\pi = \varphi_0$.

As we mentioned before, another case to be investigated is $\frac{\pi}{\varphi_0} \in \mathbb{N}$. Let $\frac{\pi}{\varphi_0} = 1$, i.e., $\varphi_0 = \pi$. Again we look for $u = \tilde{u}_1 + \tilde{u}_2$

$$\tilde{u}_1 = \sum_{n=1}^{\infty} A_n r^n \sin(n\varphi).$$

Evidently, $\Delta \tilde{u}_1 = 0$. For appropriate A_n the harmonic function $\tilde{u}_1 \in C^{\infty}(\bar{S}_R)$. We know from the beginning of the paper that

$$\tilde{u}_2 = \sum_{n \neq m \geq 2} \sum_{n=1}^{\infty} A_{mn} r^m \sin(n\varphi) + \sum_{n=m=2}^{\infty} A_{nn} r^n \log\ r \sin(n\varphi) \equiv u_3 + u_4$$

implying

$$\Delta \tilde{u}_2 = \sum_{n \neq m \geq 2} \sum_{n=1}^{\infty} A_{mn}(m^2 - n^2) r^{m-2} \sin(n\varphi) + \sum_{n=m=2}^{\infty} A_{nn} 2n.r^{n-2} \sin(n\varphi)$$

$$= f \equiv \sum_{m \geq 2} \sum_{n=1}^{\infty} B_{mn} r^{m-2} \sin(n\varphi)$$

i.e.,

$$A_{mn} = \frac{B_{mn}}{m^2 - n^2} \quad \text{for} \quad m \neq n, \qquad A_{nn} = \frac{B_{nn}}{2n}.$$

Consequently

$$|A_{mn}| \leq \frac{\text{const}}{(m^2 + n^2)^{s+1/2}}, \quad \text{for} \quad m \neq n, \qquad |A_{nn}| \leq \frac{\text{const}}{n^{2s+1}}.$$

If A_{mn} are rapidly decreasing faster than any polynomial of $(m^2 + n^2)^s$, $s \in \mathbb{N}$, s-arbitrary, we have that $u_3 \in C^{\infty}(\bar{S}_R)$. The function $r^n \log r \notin C^{\infty}([0, R])$ and $u_4 \in C^{\infty}(\bar{S}_R)$ iff $A_{nn} = 0$ for each $n \geq 2$, i.e., if $B_{nn} = 0$ for $n \geq 2$. We conclude that $C^{\infty}(\bar{S}_R)$ solution u of (2) eventually exists in the case $\pi = \varphi_0$ if the right-hand side $f \in C^{\infty}(\bar{S}_R)$ satisfies infinitely many compatibility conditions $B_{nn} = 0$, $n \geq 2$. Due to the small denominators we have the effect of loss of regularity of the corresponding solution u of (2).

3. Some Generalizations of the Previous Results

Consider in $T_{R_1 R_2} = S_{R_2} \setminus \bar{S}_{R_1}$, $R_2 > R_1 > 0$ the following BVP

$$\Delta u = f, \qquad -\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial u}{\partial n} + u|_{S_{R_1} \cup S_{R_2}} = 0$$

$$u|_{\varphi=0} = u|_{\varphi=\varphi_0} = 0, \qquad \frac{\pi}{\varphi_0} \notin \mathbb{Q}.$$

(17)

Then

$$\frac{\partial}{\partial n}\bigg|_{S_{R_2}} = \frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial n}\bigg|_{S_{R_1}} = -\frac{\partial}{\partial r}.$$

Repeating the proof of Proposition 1 we conclude that (17) with $f = 0$ possesses the unique solution $u \equiv 0$ in $C^2(\bar{T}_{R_1 R_2})$. As above, we are looking for f, u in the form (22). Thus

$$u_n'' + \frac{1}{r}u_n' - \frac{n^2\pi^2}{\varphi_0^2 r^2}u_n = f_n(r), \qquad R_1 < r < R_2$$

$$M_1(u)|_{r=R_1} = -u_n'(R_1) + \left(1 + \frac{n^2\pi^2}{\varphi_0^2}\right)u_n(R_1) = 0 \qquad (18)$$

$$M_2(u)|_{r=R_2} = u_n'(R_2) + \left(1 + \frac{n^2\pi^2}{\varphi_0^2}\right)u_n(R_2) = 0.$$

The boundary value problem (18) is simpler to deal with as

$$r^{\frac{\pi}{\varphi_0}} \in C^{[\frac{\pi}{\varphi_0}]}([0,R]) \setminus C^{[\frac{\pi}{\varphi_0}]+1}([0,R])$$

$r^{-\frac{\pi}{\varphi_0}}$ is unbounded near 0, while $r^{\pm\frac{\pi}{\varphi_0}} \in C^\infty([R_1, R_2])$. Certainly, the general solution of (18) is given by (we drop the subindex n and put $\omega = \frac{n\pi}{\varphi_0}$)

$$u = C_1 r^\omega + C_2 r^{-\omega} + \bar{u} \qquad (19)$$

C_1, C_2 being arbitrary constants and $\bar{u}$ is some solution of the nonhomogeneous equation (18). Therefore

$$\begin{aligned}
C_1 M_1(r^\omega) + C_2 M_1(r^{-\omega})|_{r=R_1} &= -M_1(\bar{u})|_{r=R_1} \\
C_2 M_2(r^\omega) + C_2 M_2(r^{-\omega})|_{r=R_2} &= -M_2(\bar{u})|_{r=R_2}.
\end{aligned} \qquad (20)$$

The determinant δ of the linear with respect to C_1, C_2 system is

$$\delta = (R_1 R_2)^{-1}\left(\frac{R_2}{R_1}\right)^\omega \left[\left(\frac{R_1}{R_2}\right)^{2\omega} \frac{(1+\omega^2)R_2 - \omega}{(1+\omega^2)R_2 + \omega} \cdot \frac{(1+\omega^2)R_1 - \omega}{(1+\omega^2)R_1 + \omega} - 1\right]$$

$$\times ((1+\omega^2)R_2 + \omega)((1+\omega^2)R_1 + \omega) \neq 0 \quad (21)$$

for $\omega > 0, 0 < \frac{R_1}{R_2} < 1$.

If we write again the subindex n we easily see that

$$\delta_n \sim -\omega_n^4 \left(\frac{R_2}{R_1}\right)^{\omega_n} \qquad \text{for} \qquad \omega_n = \frac{n\pi}{\varphi_0} \to \infty.$$

Again we omit the details.

4. Concluding Remarks

We can generalize the existence result to the boundary value problem (2) looking for f, respectively u in the following form

$$f(r, \varphi) = \sum_{n=1}^{\infty} f_n(r) \sin \frac{n\pi}{\varphi_0} \varphi \qquad u(r, \varphi) = \sum_{n=1}^{\infty} u_n(r) \sin(\frac{n\pi\varphi}{\varphi_0}) \qquad (22)$$

and supposing that f, u belong to some Hölder classes. Having in mind that

$$f_n(r) = \frac{2}{\varphi_0} \int_0^{\varphi_0} f(r, \Theta) \sin(\frac{n\pi\Theta}{\varphi_0}) \mathrm{d}\Theta$$

we see that if $f \in C^{k,\alpha}(\bar{S}_R)$ then $f_n \in C^{k,\alpha}([0, R])$, $0 < \alpha < 1$. Putting (22) in (2) we get that $u_n(r)$ should satisfy

$$u_n'' + \frac{1}{r} u_n' - \frac{n^2\pi^2}{r^2\varphi_0^2} u_n = f_n \quad \text{in } S_R$$

$$M u_n \equiv u_n'(R) + \left(1 + \frac{n^2\pi^2}{\varphi_0^2}\right) u_n(R) = 0. \qquad (23)$$

For the sake of simplicity denote $\omega = \frac{n\pi}{\varphi_0} > 0$ and drop the indexes n in (23). Let $0 < R \le 1$. The standard Euler substitution $r = \mathrm{e}^t \iff -\infty < t \le t_0 = \ln R \le 0$ transforms the equation (23) into

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \omega^2 u = \mathrm{e}^{2t} f_1(t), \qquad f_1(t) \equiv f(\mathrm{e}^t), \qquad -\infty < t \le t_0 \qquad (24)$$

having the bounded solution for $t \to -\infty$

$$u = C_1 \mathrm{e}^{\omega t} + \bar{u}(t), \qquad C_1 = \text{const} \qquad (25)$$

and $\bar{u}$ being some bounded solution of (24). Thus, $u = C_1 r^\omega + \bar{u}(\ln r)$.
The change $t - t_0 = z \le 0 \Rightarrow z = \ln \frac{r}{R}$ transforms (24) into

$$\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}z^2} - \omega^2 \bar{u} = \mathrm{e}^{2t_0} \mathrm{e}^{2z} f_1(t_0 + z) \equiv f_2(z), \qquad z \ge 0. \qquad (26)$$

The function $U(z) = \sinh(\omega z)/\omega$ satisfies the Cauchy problem $U'' - \omega^2 U = 0$, in which $z \le 0$, $U(0) = 0$, $U'(0) = 1$. Consequently

$$\bar{u}(z) = \int_0^z \frac{\sinh(\omega(z - \xi))}{\omega} f_2(\xi) \mathrm{d}\xi.$$

The change $\xi = \ln \frac{\lambda}{R}$, $z = \ln \frac{r}{R}$ in the previous integral leads to

$$\bar{u}\left(\ln \frac{r}{R}\right) = \frac{1}{\omega} \int_R^r \sinh\left(\omega \ln \frac{r}{\lambda}\right) f(\lambda) \lambda \mathrm{d}\lambda$$
$$= \frac{1}{2\omega}\left[r^\omega \int_R^r \lambda^{-\omega+1} f(\lambda)\mathrm{d}\lambda - r^{-\omega} \int_R^r \lambda^{\omega+1} f(\lambda)\mathrm{d}\lambda\right]. \tag{27}$$

The kernel of (23) contains $r^{-\omega}\int_R^0 \lambda^{\omega+1} f(\lambda)\mathrm{d}\lambda$ and we conclude that we can take

$$\bar{u} = \frac{1}{2\omega}\left[r^\omega \int_R^r \lambda^{-\omega+1} f(\lambda)\mathrm{d}\lambda - r^{-\omega} \int_0^r \lambda^{\omega+1} f(\lambda)\mathrm{d}\lambda\right]. \tag{28}$$

Evidently

$$r^{-\omega}\left|\int_0^r \lambda^{\omega+1} f(\lambda)\mathrm{d}\lambda\right| \leq \text{const} \times r^2$$

while there are the possibilities $0 < \omega < 2$, $\omega \geq 2$, for the first integral in the right-hand side of (27), guaranteeing its convergence, respectively divergence for $r \to 0$. Supposing $\omega \geq 2$ (divergence to ∞) we can apply l'Hospital rule to obtain

$$\lim_{r \to 0} r^\omega \int_R^r \lambda^{-\omega+1} f(\lambda)\mathrm{d}\lambda = 0.$$

More precise results concerning the behavior of $\bar{u}$ for $r \to 0$ that take into account $f \in C^{k,\alpha}([0, R])$ and eventual vanishing of f at 0 can be obtained by using Taylor formula in Hölder classes

$$f(r) = f(0) + \frac{r}{1!}f'(0) + \ldots + \frac{r^k}{k!}f^{(k)}(0) + \mathcal{O}(r^{k+\alpha}), \qquad r \to 0. \tag{29}$$

Going back to (23) we have that

$$C_1 M(r^\omega)|_{r=R} = -M(\bar{u})|_{r=R}$$

i.e., the constant C_1 from (25) is uniquely determined by the equality

$$C_1 = \frac{-M(\bar{u})|_{r=R}}{R^\omega(1 + \omega^2 + \omega R^{-1})}. \tag{30}$$

We do not use in this approach series, small denominators etc., but we do not discuss the problem of the convergence of the series (22). The restriction in working in Hölder classes in r are weaker than the restrictions imposed on the power series in r. We do not enter into technical details here.

References

[1] Arnold V., *Additional Chapters of the Theory of Ordinary Differential Operators* (in Russian), Nauka, Moscow 1978.

[2] Bonnaillie-Noël, Dambrine M., Hérau F. and Vial G., *On Generalized Ventcel's Type Boundary Conditions for Laplace Operator in a Bounded Domain*, SIAM J. Math. Anal. **42** (2010) 931-945.

[3] Bouchschtab A., *Number Theory* (in Russian), Moscow 1960.

[4] Feller W., *Generalized Second Order Differential Operators and Their Lateral Conditions*, Illinois J. Math. **1** (1957) 459-504.

[5] Gilbarg D. and Trudinger N., *Elliptic Partial Differential Operators of Second Order*, Grundlehren der Math. Wissenshaften **224**, Springer, Berlin 1983.

[6] Kondratiev V. and Landis E., *Qualitative Theory of Second Order Linear Partial Differential Operators* (in Russian), Itogi Nauki i Techniki **32**, Moscow 1988.

[7] Ventcel A., *Semigroups of Operators That Correspond to a Generalized Differential Operator of Second Order*, Dokl. Akad. Nauk SSSR **111** (1956) 269-272.

[8] Ventcel A., *On Boundary Conditions for Multidimensional Diffusion Processes*, Theor. Probab. Appl. **4** (1959) 164-177.

DELAUNAY SURFACES IN TERMS OF WEIERSTRASSIAN FUNCTIONS

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Abstract. Strangely enough (in view of the long time since their original discovery) the description of the Delaunay surfaces via the Weierstrassian functions is absent in the literature. Here we have filled this gap by providing this missing explicit parameterization along with some comments about the alternative parameterization in terms of elliptic integrals.

1. Delaunay Surfaces

Almost two centuries ago the French mathematician Delaunay [3] has classified all surfaces of revolution in $\mathbb{R}^3$ with a constant mean curvature. The respective (and exhaustive) list includes planes, cylinders, spheres, catenoids, unduloids and nodoids. In an Appendix to that paper Sturm characterized these surfaces variation-
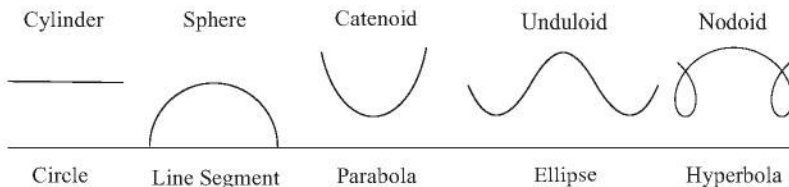

Figure 1: The profile curves of the Delaunay's surfaces obtained by rolling the conics listed below them.

ally and their profile curves (meridians) as the traces of the foci of non-degenerated conics rolling along a line in the plane (cf Fig.1).

We refer to the book by D'Arcy Thompson [10, Chapter 3] for a nice essay on the appearance of these surfaces in nature and to Eells [4] for a clear exposition and pointing out some deep connections with problems in geometry and mechanics.

For the sake of completeness we present here some well-known results about regular surfaces of revolution. First of all, we fix an orthonormal basis in $\mathbb{R}^3$ and assume that the x-axis coincides with the axis of revolution and that the profile curve $z = z(x)$ specifying the meridional section of the Delaunay surface lies in the XOZ plane. In these settings any point on our surface of revolution is given by the vector-valued function

$$\mathbf{x} = (x, z(x)\cos v, z(x)\sin v), \qquad x \in \mathbb{R}, \quad v \in [0, 2\pi).$$

By making use of the above parameterization and the machinery of the classical differential geometry (cf [9]), one can easily calculate the first and the second fundamental forms and after that the two principle curvatures. Namely, one obtains the so called meridional

$$\kappa_\mu = -\frac{z''(x)}{(1 + z'(x)^2)^{3/2}}$$

and parallel

$$\kappa_\pi = \frac{1}{z(x)\sqrt{1 + z'(x)^2}}$$

curvatures of the surface, where $z'(x) \equiv \mathrm{d}z/\mathrm{d}x$. By these, the mean (meaning average) curvature H of the surfaces is found to be

$$H := \frac{1}{2}(\kappa_\mu + \kappa_\pi) = \frac{-z(x)z''(x) + z'(x)^2 + 1}{2z(x)\left(1 + z'(x)^2\right)^{3/2}}. \tag{1}$$

The above expression is the starting point for all considerations to follow. Before that we will mention that a few alternative parameterizations of the Delaunay surfaces can be found in [5,7,8] and [2].

By the very definition it is clear that the axially-symmetric surfaces could have a constant mean curvature which is either negative, zero or positive. Let us consider first the case of the surfaces with a zero mean curvature. Imposing this condition on the expression for H in (1) leads to the equation

$$z(x)z''(x) - z'(x)^2 - 1 = 0. \tag{2}$$

In order to solve it, let us notice that the independent variable is not present in equation (2) and therefore it can be reduced to the first order equation. Assuming

that the smooth inverse function $x = x(z)$ exists, this can be achieved by the sub-stitution $z'(x(z)) = \phi(z)$ which implies also the identity $z'' = \phi\phi'$ and therefore we end up with the equation

$$z\phi(z)\phi'(z) - \phi^2(z) - 1 = 0.$$

Being an equation with separable variables it is not a problem to solve it and its solution can be written down as

$$\sqrt{1 + \phi^2(z)} = \lambda z, \qquad \lambda = \text{const}.$$

Going back to the original variables one easily gets the sought solution of the initial equation (2) in the form

$$z(x) = \lambda \cosh(\frac{x}{\lambda}).$$

The so obtained curve can be immediately recognized as a catenary while the gen-erated axially-symmetric surface as a catenoid (cf [9]).

The remaining two cases of surfaces with positive, respectively negative mean cur-vatures will be considered in parallel and following the tradition (cf [3]) we will assume that $H = \pm\frac{1}{2a}$ and $a > 0$. Elaborating a little bit further the expression for H obtained in (1) one can see that it could be rewritten in the form

$$\frac{2az(x)}{\sqrt{1 + z'(x)^2}} \pm z(x)^2 = \text{const}. \tag{3}$$

Let us consider first the case when the constant on the right-hand side of the above equation is zero. Then, it is not difficult to see that the solution of (3) is either a line that is parallel to the symmetry axis or the circle

$$x^2 + z^2 = 4a^2.$$

When rotated they generate respectively a cylinder and a sphere, which are other representatives of the class of the surfaces under consideration.

So, from now on we will assume that the constant on the right-hand side of (3) is different from zero and for definiteness we will denote it by $\pm b^2$ in order to distinguish the strictly positive and negative cases. It is not hard to see also that in these cases the differential equation (3) can be rewritten in the form

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\mp z^2 - b^2}{\sqrt{4a^2z^2 - (z^2 \pm b^2)^2}}. \tag{4}$$

As was shown by Sturm in the Appendix to Delaunay's paper [3] (see also [9]) this equation describes the roulette of an ellipse, the undulary, and the roulette of a hyperbola, the nodary, for the upper, respectively the lower sign.

2. Undulary, Nodary and Weierstrassian Functions

In order to parameterize the undulary and the nodary, the roulettes introdused in the preceding section, we recast the equation (4) in a system of two equations

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \mp z^2 - b^2$$

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \sqrt{-z^4 + 2(2a^2 \mp b^2)z^2 - b^4}$$

$$(5)$$

where $u \in \mathbb{R}$ is a new variable, a parameter of the roulette. The second one of these two equations is directly solvable in elliptic functions by inverting the integral

$$u + C_1^{\pm} = \int\limits_{c}^{z} \frac{\mathrm{d}\tau}{\sqrt{-\tau^4 + 2(2a^2 \mp b^2)\tau^2 - b^4}} \qquad (6)$$

where c is an arbitrary root of the polynomial

$$f(\tau) = -\tau^4 + 2(2a^2 \mp b^2)\tau^2 - b^4 \qquad (7)$$

and $C_1^{\pm} = \mathrm{const}$. Here and henceforth the upper indexes plus and minus refer to the undulary and the nodary, respectively.

As it is easily seen, for $a > |b| > 0$, all roots of the polynomial $f(\tau)$ are real numbers and can be written explicitly, namely

$$\tau_1 = \tau_1^{\pm} = -a - \sqrt{a^2 \mp b^2}, \qquad \tau_2 = \tau_2^{\pm} = -a + \sqrt{a^2 \mp b^2}$$
$$\tau_3 = \tau_3^{\pm} = a - \sqrt{a^2 \mp b^2}, \qquad \tau_4 = \tau_4^{\pm} = a + \sqrt{a^2 \mp b^2}.$$

As lower limits c in the integrals (6) we choose the respective greatest roots of the polynomial (7), i.e.,

$$c = \tau_4 = \tau_4^{\pm} = a + \sqrt{a^2 \mp b^2}.$$

By making two successive substitutions we remove the quartic and the quadratic terms in the polynomial [11]. At first we write

$$\tau = \frac{1 + ct}{t}, \qquad z = \frac{1 + c\xi}{\xi}$$

to obtain

$$u + C_1^{\pm} = \int\limits_{\xi}^{\infty} \frac{\mathrm{d}t}{\sqrt{(8a^2c \mp 4b^2c - 4c^3)t^3 + (4a^2 \mp 2b^2 - 6c^2)t^2 - 4ct - 1}}.$$

For the second substitution we take

$$t = \frac{6s - 2a^2 \pm b^2 + 3c^2}{12a^2c \mp 6b^2c - 6c^3}, \qquad \xi = \frac{6\eta - 2a^2 \pm b^2 + 3c^2}{12a^2c \mp 6b^2c - 6c^3}$$

so that we arrive at the quadrature in the Weierstrassian form

$$u + C_1^{\pm} = \int\limits_{\eta}^{\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}}$$

where

$$g_2 = g_2^{\pm} = \frac{4}{3}(a^4 \mp a^2 b^2 + b^4), \qquad g_3 = g_3^{\pm} = -\frac{4}{27}(2a^6 \mp 3a^4 b^2 - 3a^2 b^4 \pm 2b^6)$$

are the so called invariants of $f(\tau)$. Hence, we have the Weierstrassian elliptic $\wp$-function [11]

$$\eta = \wp(u + C_1^{\pm}) \equiv \wp(u + C_1^{\pm}; g_2, g_3)$$

and returning back to the original variables we can write finally the solutions

$$z^{\pm}(u) = \frac{c\left(6\wp(u + C_1^{\pm}) + 10a^2 \mp 5b^2 - 3c^2\right)}{6\wp(u + C_1^{\pm}) - 2a^2 \pm b^2 + 3c^2}. \tag{8}$$

Now, we can solve by direct integration the first equation in (5) which produces

$$\begin{aligned} x^{\pm}(u) \ = \ & 2c^2(\mp 2a^2 + b^2 \pm c^2)J_1(u + C_1^{\pm}) \\ & \mp c^2(\mp 2a^2 + b^2 \pm c^2)^2 J_2(u + C_1^{\pm}) - (b^2 \pm c^2)u + C_2^{\pm} \end{aligned} \tag{9}$$

where we have made use of the integrals (cf [1, 6])

$$J_k(u) = \int \frac{\mathrm{d}u}{\left[\wp(u) - \wp(\mathring{u})\right]^k}, \qquad k = 1, 2$$

that can be expressed in terms of the complete set of Weierstrassian functions $\wp(u)$, $\sigma(u)$ and $\zeta(u)$

$$J_1(u) \ = \ \frac{1}{\wp'(\mathring{u})}\left(2\zeta(\mathring{u})u + \ln\frac{\sigma(u - \mathring{u})}{\sigma(u + \mathring{u})}\right)$$

$$J_2(u) \ = \ -\frac{1}{\wp'^2(\mathring{u})}\left(\wp''(\mathring{u})J_1(u) + 2\wp(\mathring{u})u + \zeta(u - \mathring{u}) + \zeta(u + \mathring{u})\right).$$

Here $\mathring{u}$ denotes the argument of $\wp(\cdot)$ which produces the value $\frac{1}{6}(2a^2 \mp b^2 - 3c^2)$, $C_2^{\pm} = \text{const}$, and $\wp'(\mathring{u}) \equiv \mathrm{d}\wp(u)/\mathrm{d}u_{|u=\mathring{u}}$, etc. Thus, we obtain the profile curves of the unduloids and the nodoids, i.e., the undulary and the nodary, parameterized as specified by the equations (8) and (9) in terms of the Weierstrassian functions (see Fig. 2 and Fig. 3).

Figure 2: An open part of the unduloid (right) and its profile curve (left) generated by formulas (8) and (9) and parameters $r = 6$, $R = 25$.

3. Alternative Parameterization and `Mathematica`®

Another parameterization of the profile curves of the considered two types of Delaunay surfaces, the undulary and the nodary, is ensured by the ordinary circular sine-function and the elliptic integrals of the first $F(\cdot, k)$ and second kind $E(\cdot, k)$ (more details for the unduloids can be found in [5])

$$
\begin{aligned}
x^{\pm}(u) &= \pm rF\left(\frac{\mu u}{2} - \frac{\pi}{4}, k\right) + RE\left(\frac{\mu u}{2} - \frac{\pi}{4}, k\right) \qquad (10) \\
z^{\pm}(u) &= z(u) = \sqrt{m \sin \mu u + n}
\end{aligned}
$$

where

$$
\mu = \frac{2}{R + r}, \qquad k^2 = \frac{R^2 - r^2}{R^2}, \qquad m = \frac{R^2 - r^2}{2}, \qquad n = \frac{R^2 + r^2}{2}.
$$
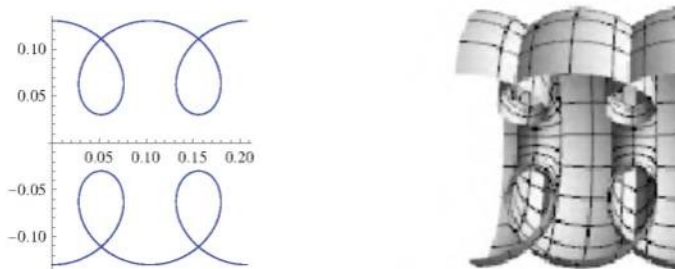
Figure 3: An open part of the nodoid (right) and its profile curve (left) generated by formulas (8) and (9) and parameters $r = 0.03$, $R = 0.13$.

The two parameters r and R which appear in the above formulas are positive, i.e., $r > 0$, $R > 0$, and it is assumed that $R > r$. A closer inspection of the formulas presented in (10) shows that these parameters have a simple geometrical meaning – they are equal to the minimum and the maximum of $z(x)$, i.e., $z_{\min} = r$, $z_{\max} = R$. From (4) we observe also that each one of the four roots of the polynomial coincide

with either $\pm z_{\min}$ or $\pm z_{\max}$. As a result we can deduce

$$a = \frac{R \pm r}{2}, \qquad b = \pm\sqrt{Rr}.$$

Using the computer program `Mathematica`®, we have compared the graphs of undularies and nodaries, obtained by the above two parameterizations (8)-(9) and (10) for different values of $r = z_{\min}$ and $R = z_{\max}$. In order to be ensured identical initial conditions, $x^{\pm}(0) = 0$, $z^{\pm}(0) = z_{\max}$, the following choices for the integration constants were made: for the undulary

$$C_1^+ = -4\omega, \qquad C_2^+ = 4(b^2 + c^2)\omega$$

and for the nodary, $C_1^- = C_2^- = 0$, where ω is the real-valued half-period of the $\wp$-function. The result of the full coincidence of the graphs will be discussed elsewhere.

References

[1] Abramowitz M. and Stegun I. (Eds), *Handbook of Mathematical Functions*, Dover, New York 1972.

[2] Bendito E., Bowick M. and Medina A., *A Natural Parameterization of the Roulettes of the Conics Generating the Delaunay Surfaces*, J. Geom. Symmetry Phys. **33** (2014) 27-45.

[3] Delaunay C., *Sur la surface de revolution dont la courbure moyenne est constante*, J. Math. Pures et Appliquées **6** (1841) 309 – 320.

[4] Eells J., *The Surfaces of Delaunay*, Math. Intelligencer **9** (1987) 53–57.

[5] Hadzhilazova M., Mladenov I. and Oprea J., *Unduloids and Their Geometry*, Arch. Mathematicum **43** (2007) 417 – 429.

[6] Jahnke E., Emde F. and Lösch F., *Tafeln Höherer Funktionen*, Teubner, Stuttgart 1960.

[7] Mladenov I., *New Solutions of the Shape Equation*, Eur. Phys. J. B **29** (2002) 327-330.

[8] Mladenov I., *Delaunay Surfaces Revisited*, C. R. Bulg. Acad. Sci. **55** (2002) 19-24.

[9] Oprea J., *Differential Geometry and its Applications*, Mathematical Association of America, Washington D.C. 2007.

[10] Thompson D., *On Growth and Form*, Cambridge University Press, Cambridge 1961.

[11] Whittaker E. and Watson G., *A Course of Modern Analysis*, Cambridge University Press, Cambridge 1927.

MAPPING BETWEEN NONLINEAR SCHRÖDINGER EQUATIONS WITH REAL AND COMPLEX POTENTIALS[*]

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Abstract. A mapping between the stationary solutions of nonlinear Schrödinger equations with real and complex potentials is constructed and a set of exact solutions with real energies are obtained for a large class of complex potentials. As specific examples we consider the case of dissipative periodic soliton solutions of the nonlinear Schrödinger equation with complex potential.

1. Introduction

Nonlinear wave phenomena with time evolutions governed by non hermitian Hamiltonians are presently attracting a great interest both from the theoretical and the applicative point of view. The non hermiticity is in general due to the presence of a complex potential in the Hamiltonian accounting for typical dissipative and amplification effects met in classical and quantum contexts [5, 12]. In particular, dissipative solitons [4] of the nonlinear Schrödinger (NLS) equation with periodic complex potentials have been extensively investigated during the past years in connections with the propagation of light in nonlinear optical fibers with periodic modulations of the complex refractive index [13, 18]. Recently similar studies were done for matter wave solitons of Bose-Einstein condensates (BEC) trapped in absorbing optical lattices [1, 7] and in the presence of three body interatomic interactions [3]. In the linear context, the recent discovery [6] that the Schrödinger eigenvalue problem with complex potentials that are invariant under the combined parity and time reversal symmetry (so called $\mathcal{PT}$-potentials), may have fully real

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spectrum, has raised interest also in view of possible connection with the theory of quantum dissipative systems [10]. Complex potentials with $\mathcal{PT}$-symmetry are presently investigated in nonlinear optics [11] where it has been demonstrated that nonlinear media with *linear* damping and amplifications that are $\mathcal{PT}$-symmetric can support stable stationary localized and periodic states [14]. Also, quite recently, physical systems with $\mathcal{PT}$-symmetry have been successfully implemented in real experiments [9, 15, 17]. Solutions of the NLS equation with a complex potential which belong to the real part of the spectrum (real energies or real chemical potentials) can exist, however, for generic complex potentials and it is therefore of interest to characterize them in general, independently from the $\mathcal{PT}$-symmetry.

The aim of the present paper is to show how one can systematically construct stationary solutions of the complex nonlinear Schrödinger equation via a mapping between real and complex NLS equations. The problem is formulated in terms of a nonlocal eigenvalue problem which involves only real potentials, whose eigenfunctions and eigenvalues fix amplitudes and energies of the stationary solutions of the complex NLS equation, respectively. The complex potentials and the phases of the solutions are also determined self-consistently through the mapping. To illustrate our approach we discuss the case of the NLS equation with different complex potentials for which we construct periodic dissipative solitons in the form of elliptic functions.

The paper is organized as follows. In Section 2 we introduce model equations and illustrate the mapping used to determine the solutions. In Section 3 we show how to construct exact solutions of the NLS with periodic complex potentials while in the last section the main results of the paper will be briefly summarized.

2. Model Equations and Mapping

The model equation we consider is the NLS equation with real and complex potentials both of linear and nonlinear types, e.g.

$$\mathrm{i}\psi_t = -\frac{1}{2}\psi_{xx} + (V_l(x) + \mathrm{i}W_l(x))\psi + (\sigma + V_{nl}(x) + \mathrm{i}W_{nl}(x))|\psi|^2\psi. \quad (1)$$

The case of the linear Schrodinger equation (e.g. $\sigma = V_{nl} = W_{nl} = 0$) can be used as an example of quantum dissipative system. In the nonlinear case the above equation can appear in connection with several interesting phenomena including light propagation in photonic crystals and Bose-Einstein condensates. Due to the possibility of different physical applications we shall keep equation (1) in normalized form, looking for stationary solutions of the type

$$\psi(x,t) = A(x)\mathrm{e}^{\mathrm{i}\theta(x)}\mathrm{e}^{-\mathrm{i}\omega t} \quad (2)$$

with the amplitude $A(x)$ and phase $\theta(x)$ as real functions. Substituting this expression into equation(1), we obtain the system of equations

$$\omega A + \frac{1}{2}A_{xx} - \sigma A^3 - \frac{A}{2}(\theta_x)^2 - V_l A - V_{nl}A^3 = 0 \qquad (3)$$

$$\frac{1}{2}A\theta_{xx} + A_x\theta_x - W_l A - W_{nl}A^3 = 0. \qquad (4)$$

These equations can be easily separated. In this respect notice that by multiplying equation (4) by A and integrating it twice one obtains

$$\frac{1}{2}\theta(x) = B_2 + \int_{-\infty}^{x} \frac{B_1 + F(y)}{A^2(y)}\mathrm{d}y \qquad (5)$$

with

$$F(y) = \int_{-\infty}^{y} \left[W_l(z) + W_{nl}(z)A^2(z)\right] A^2(z)\mathrm{d}z \qquad (6)$$

and B_1, B_2 integration constants. By substituting equation (5) into equation (3) we obtain the following nonlinear eigenvalue problem for the real amplitude A

$$\left\{-\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + 2\left(\frac{F(x)}{A^2}\right)^2\right\} A = \omega A \qquad (7)$$

where the integration constants B_1, B_2, have been fixed to zero for simplicity. Note that for stationary solutions equation (7) is completely equivalent to equation (1) in the sense that any solution of (7) gives a stationary solution of (1) with the phase fixed by (5). Also note that the dependence on the complex potentials in the eigenvalue problems comes through the function F and for an arbitrary $F(x)$ (e.g. arbitrary complex potentials) the problem can become singular. It is possible, however, to construct potentials W_l and W_{nl} (e.g. functions F) so that the solutions of (7) are regular. This establishes a mapping between stationary solutions of the NLS equation with real potentials and stationary solution of equation (1) with the phase given by (5). In this respect, one can take F in general to be an analytical function of A^2 and derivatives e.g. $F(x) \equiv F(A^2, (A^2)_x, ...)$. In the simplest case F can be taken of the form

$$F(x) = \frac{1}{2}C_n A^{n+2}, \qquad n = 0, 1, 2... \qquad (8)$$

with C_n constants to be determined. Equation (7) then reduces to the following NLS real eigenvalue problem

$$\left\{-\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + \frac{C_n^2}{2}A^{2n}\right\} A = \omega A \qquad (9)$$

which can be solved analytically for particular forms of the potentials V_l, V_{nl}, or numerically with high accuracy (using for example the self-consistent method discussed in [16]) for generic real potentials. In the following we therefore assume

that the real amplitudes A and frequencies ω for given V_l and V_{nl} are exactly obtained from (9), either analytically or numerically.

On the other hand from equation (8) one can characterize the complex potentials which support such solutions. Using equation (6) we have indeed that equation (8) is satisfied if the amplitude A is related to W_l and W_{nl} by the relation

$$W_l + W_{nl}A^2 = C_n \left(\frac{n+2}{2n}\right) \frac{\mathrm{d}A^n}{\mathrm{d}x} \tag{10}$$

and from equations (5), (8), one gets that the phase is given by

$$\theta(x) = C_n \int_{-\infty}^{x} A^n \mathrm{d}y. \tag{11}$$

Note that in this case equation (10) allows to relate the constant C_n to the amplitude of the solution, A_0, and the amplitudes W_{0l}, W_{0nl}, of the linear and nonlinear complex potentials, respectively. In particular, for the case $W_{nl} = 0$ we have that

$$C_n = \frac{2}{n+2} \frac{W_{0l}}{A_0^n}, \qquad W_{0nl} = 0 \tag{12}$$

while for $W_l = 0$ one obtains

$$C_n = \frac{2}{n+2} \frac{W_{0nl}}{A_0^{n-2}}, \qquad W_{0l} = 0. \tag{13}$$

It is worth to note that while the case $n = 1$ leads to a pure cubic NLS eigenvalue problem, the case $n > 1$ introduces higher order nonlinearities in equation (9) which can however be eliminated by redefying the linear real potential as

$$V_l = \tilde{V}_l - \frac{C_n^2}{2} A^{2n} \tag{14}$$

or the nonlinear real potential as

$$V_{nl} = \tilde{V}_{nl} - \frac{C_n^2}{2} A^{2n-2} \tag{15}$$

(or a combination of both). Also notice that equations (2), (10) - (13) allow to map solutions of the real eigenvalue problem (9) into solutions of the NLS equation (1) with the corresponding complex potentials determined as in (10). It is clear that this approach can be extended to functions of the type

$$F(x) = \frac{1}{2} \sum_{n=0}^{k} C_n A^{n+2}, \qquad k = 0, 1, 2\ldots \tag{16}$$

In this case coefficient C_n are self-consistently determined from the real eigenvalue problem

$$\left\{ -\frac{1}{2}\frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl})A^2 + \frac{1}{2}\left(\sum_n C_n A^n\right)^2 \right\} A = \omega A \qquad (17)$$

and complex potentials and phase are given by

$$W_l + W_{nl}A^2 = \frac{1}{A^2}\frac{\mathrm{d}F}{\mathrm{d}x} \qquad (18)$$

$$\theta(x) = \sum_n C_n \int_{-\infty}^x A^n \mathrm{d}y. \qquad (19)$$

Note that the sum in equation (16) can include infinite terms and to have a map between real and complex NLS equations it is necessary to subtract higher order nonlinearities from the real linear and nonlinear potentials as done in equations (14)-(15). Finally we remark that if the functions A_x/A, A_{xx}/A, ... are bounded, the expression (16) can be further generalized as

$$F(x) = \sum_{n,m}^k C_{n,m}\frac{\mathrm{d}^m A^{n+2}}{\mathrm{d}x^m} \qquad (20)$$

with $C_{n,m}$ suitable constants and with the complex potentials determined as (18). In all these cases a map between solutions of the real eigenvalue problem (17) and solutions of the NLS equation (1) is constructed.

The mapping guarantees that the constructed solutions always have real energies and may be therefore of physical interest. We finally remark that a similar approach based on a priori fixing of the solution and a posteriori determination of the complex potential, has been considered also in [2, 8], although not in terms of a mapping between stationary solutions of NLS equations. In the following we illustrate how the mapping works on some specific example.

3. Nonlinear Schrödinger Equation with Complex Potentials

3.1. Case $n = 1$

Let us consider first the simplest ansatz (8) with $n = 1$. We fix the nonlinearity to be attractive ($\sigma < 0$) and restrict to linear complex potentials (i.e., $W_{nl} = V_{nl} = 0$) and with linear potential of the form $V_l = V_{0l}\,\mathrm{cn}^2(x,k)$. In this case the real eigenvalue problem (9)

$$\left[-\frac{1}{2}\frac{\partial}{\partial x^2} + V_{0l}\,\mathrm{cn}(x,k)^2 + \left(\sigma + \frac{C_1^2}{2}\right)A^2 \right] A = \omega A \qquad (21)$$

admits the following exact solutions in terms of elliptic functions

a) $A(x) = A_0 \operatorname{cn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \qquad \omega = \frac{1 - 2k^2}{2} \tag{22}$$

b) $A(x) = A_0 \operatorname{sn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{C_1^2 - 2|\sigma|}}, \qquad \omega = \frac{1 + k^2}{2} + V_{0l} \tag{23}$$

c) $A(x) = A_0 \operatorname{dn}(x, k)$

$$A_0 = \pm \frac{1}{k} \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \qquad \omega = \frac{k^2}{2} - 1 + V_{0l}\left(1 - \frac{1}{k^2}\right). \tag{24}$$

Similar solutions can be constructed for the case of a repulsive nonlinearity $\sigma > 0$ with linear potentials of the form $V_l = V_{0l} \operatorname{sn}^2(x, k)$. In this case we have

d) $A(x) = A_0 \operatorname{cn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(V_{0l} - k^2)}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{1 - 2k^2}{2} + V_{0l} \tag{25}$$

e) $A(x) = A_0 \operatorname{sn}(x, k)$

$$A_0 = \pm \sqrt{\frac{2(k^2 - V_{0l})}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{1 + k^2}{2} \tag{26}$$

f) $A(x) = A_0 \operatorname{dn}(x, k)$

$$A_0 = \pm \frac{1}{k} \sqrt{\frac{V_{0l} - k^2}{C_1^2 + 2\sigma}}, \qquad \omega = \frac{k^2}{2} - 1 + \frac{V_{0l}}{k^2}. \tag{27}$$

Using the above mapping we can readily construct the stationary solutions of the corresponding complex NLS with linear complex potentials given by

$$W_l = \frac{3}{2} C_1 A_x, \qquad C_1 = \frac{2}{3} \frac{W_{0l}}{A_0} \tag{28}$$

and with the phase given by $\theta(x) = C_1 \int_{-\infty}^{x} A(y)\mathrm{d}y$. Thus, for example, from the solution a) we get

$$A = A_0 \mathrm{cn}(x,k), \qquad\qquad V_l(x) = V_{0l}\,\mathrm{cn}^2(x,k)$$

$$A_0 = \frac{\sqrt{9\left(k^2+V_{0l}\right)+2W_{0l}^2}}{3\sqrt{|\sigma|}}, \qquad \omega = \frac{1-2k^2}{2} \qquad\qquad (29)$$

$$W_l = -W_{0l}\,\mathrm{sn}(x)\mathrm{dn}(x), \qquad\qquad \theta(x) = \frac{2W_{0l}}{3k}\arccos(\mathrm{dn}(x)).$$

In similar manner one proceeds with the other solutions above. It is also clear that exact solutions of this type can be constructed also for other types of linear elliptic potentials (we omit them for brevity).

3.2. Case $n = 2$

As a further application of the ansatz (8) we consider the case $n = 2$ for which the mappings involves higher order nonlinearities. We assume as before that $V_{nl} = W_{nl} = 0$. In order to balance the quintic nonlinearity in equation (9), the potential V_l must be taken as in equation (14). We take $\tilde{V}_l = V_{0l}\,\mathrm{cn}^2(x,k)$ and consider a solution of the form $A(x) = A_0\,\mathrm{cn}(x,k)$. One can then check that this is a solution of (9) with

$$V_l(x) = V_{0l}\mathrm{cn}^2(x,k) - \frac{C_2^2}{2}A_0^4\,\mathrm{cn}^4(x,k) \qquad\qquad (30)$$

if $A_0^2 = V_{0l} + k^2$ and $\omega = \frac{1-2k^2}{2}$. From the mapping we then have that

$$C_2 = \frac{W_{0l}}{2A_0^2} = \frac{W_{0l}}{2(V_0+k^2)}$$

$$W_l(x) = 2C_2 A A_x = -W_{0l}\,\mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}(x) \qquad\qquad (31)$$

and the phase is

$$\theta(x) = x - \frac{x}{k^2} + \frac{1-k^2+k^2\mathrm{cn}^2(x,k)}{k^2\mathrm{dn}^2(x,k)}E(\mathrm{am}(x,k),k). \qquad\qquad (32)$$

As a further example of $n = 2$ we consider the case of pure nonlinear optical lattices, i.e., $V_l = W_l = 0$. Fixing $\tilde{V}_{nl} = 0$ and looking for solutions of the type $A(x) = A_0\mathrm{cn}(x,k)$, we have from equation (15) that that

$$V_{nl}(x) = -\frac{C_2^2}{2}A_0^2\,\mathrm{cn}^2(x,k). \qquad\qquad (33)$$

with C_2 fixed according to equation (13) as $C_2 = W_{0nl}/2$. One can easily check that this is indeed a solution of the eigenvalue problem (9) if

$$\omega = \frac{1}{2} - k^2, \qquad A_0 = \frac{k}{\sqrt{|\sigma|}} \qquad (34)$$

(we consider $\sigma < 0$). From the mapping we have that this is also a solution of the NLS with the complex part of the nonlinear potential fixed according to equation (10) as

$$W_{nl}(x) = 2C_2\frac{A_x}{A} = -W_{0nl}\frac{\mathrm{sn}(x,k)\mathrm{dn}(x,k)}{\mathrm{cn}(x,k)}. \qquad (35)$$

For the cases $n > 2$ one can proceed in similar manner.

3.3. General Case

Let us now consider an example with the more general ansatz (16). To this regard we take $F(x) = \frac{1}{2}(C_0 + C_2 A^2)A^2$ and look for solutions of the form $A(x) = A_0\mathrm{dn}(x,k)$. Let us fix the linear potentials as $V_l = W_l = 0$ and the real nonlinear potential as $V_{nl} = V_{0nl} - \alpha_2^2 A^2$ with V_{0nl} a constant and with $\alpha_n = \frac{C_n}{\sqrt{2}}, n = 0, 2$ (notice that we fixed all coefficients for $n \neq 0, 2$, equal to zero). By substituting these expressions into the real eigenvalue problem we find that $A(x)$ is indeed a solution if

$$\omega = \alpha_0^2 + \frac{k^2}{2} - 1, \qquad \alpha_0 = -\frac{1 + A_0^2(\sigma + V_{0nl})}{2A_0^2\alpha_2}. \qquad (36)$$

Thus, for example, if we fix $V_{0nl} = 2/k^2$, $\alpha_2 = -1/k$ and consider $\sigma = 1$ (repulsive interactions), we have

$$V_{nl} = \frac{2 - \mathrm{dn}^2(x,k)}{k^2} = \frac{1}{k^2} + \mathrm{sn}^2(x,k)). \qquad (37)$$

From equation (36) we have

$$\omega = \sigma - 1 + \frac{1}{A_0^2} + \frac{k^2\left[(1 + 2\sigma A_0^2) + (\sigma^2 + 2)A_0^4\right]}{4A_0^4} + \frac{1}{k^2}$$

$$\alpha_0 = \frac{1}{k} + \frac{k(1 + \sigma A_0^2)}{2A_0^2} \qquad (38)$$

and from (16) we get the function F as

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_2 A^2)A^2 \\ &= \frac{A_0^2\mathrm{dn}^2(x,k)}{2\sqrt{2}k}\left[2 + \frac{k^2}{A_0^2}(1 + \sigma A_0^2) - 2A_0^2\mathrm{dn}^2(x,k)\right]. \end{aligned}$$

Substituting into equations (18) we finally get the complex potential as

$$W_{nl} = \sqrt{2}k\frac{\mathrm{sn}(x,k)\mathrm{cn}(x,k)}{\mathrm{dn}^3(x,k)} \times \left[2 - \frac{1}{A_0^2} - \frac{k^2}{2A_0^4}(1 + \sigma A_0^2) - 2k^2\mathrm{sn}^2(x,k)\right]. \quad (39)$$

The phase of the solution can be readily obtained from equation (19). Notice that in the case $\sigma = 1, A_0 = 1$, this solution coincides with the one derived in [2] with a slightly different approach. We remark that the above solutions of the complex NLS equations not only have real energies but are also stable (not shown for brevity) under time evolution.

4. Conclusions

In conclusion we have demonstrated the possibility to construct stationary solutions of the linear and nonlinear Schrodinger equation with complex potentials via a mapping with stationary solutions of the NLS equation with suitable real potentials. In particular we showed that by means of this mapping it is possible to construct sets of exact solutions with real energies for different types of complex potentials. The presented approach can be applied to other types equations, including the linear Schrödinger equation describing quantum dissipative oscillators, and the NLS equation with arbitrary higher order nonlinearities, as it will be discussed elsewhere.

Acknowledgements

References

[1] Abdullaev F., Gammal A., da Luz H. and Tomio L., *Dissipative Dynamics of Matter-Wave Solitons in a Nonlinear Optical Lattice,* Phys. Rev. A **76** (2007) 043611-1-9.

[2] Abdullaev F., Konotop V., Salerno M. and Yulin A., *Dissipative Periodic Waves, Solitons, and Breathers of the Nonlinear Schrödinger Equation with Complex Potentials,* Phys. Rev. E **82** (2010) 056606-1-6.

[3] Abdullaev F. and Salerno M., *Gap-Townes Solitons and Localized Excitations in Low-Dimensional Bose-Einstein Condensates in Optical Lattices,* Phys. Rev. A **72** (2005) 033617-1-12.

[4] Akhmediev N. and Ankiewicz A. (Eds), *Dissipative Solitons,* Lecture Notes in Physics **661**, Springer, Berlin 2005.

[5] Aranson I. and Kramer L., *The World of the Complex Ginzburg-Landau Equation,* Rev. Mod. Phys. **74** (2002) 99-143.

[6] Bender C., Boettcher S., *Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry,* Phys. Rev. Lett. **80** (1998) 5243-5246.

[7] Bludov Yu. and Konotop V., *Nonlinear Patterns in Bose-Einstein Condensates in Dissipative Optical Lattices,* Phys. Rev. A **81** (2010) 013625-1-8.

[8] Brazhnyi V. and Konotop V., *Theory of Nonlinear Matter Waves in Optical Lattices,* Mod. Phys. Lett. B **18** (2004) 627-651.

[9] Guo A., Salamo G., Duchesne D., Morandotti R., Volatier-Ravat M., Aimez V., Siviloglou G., and Christodoulides D., *Observation of PT-Symmetry Breaking in Complex Optical Potentials,* Phys. Rev. Lett. **103** (2009) 093902-1-4.

[10] See e.g. the special issue # 32: The Physics of Non-Hermitian Operators, J. Phys. A **39** (2006).

[11] Longhi S., *Bloch Oscillations in Complex Crystals with PT Symmetry,* Phys. Rev. Lett. **103** (2009) 123601-1-4.

[12] Muga J., Palao J., Navarro B. and Egusquiza I., *Complex Absorbing Potentials,* Phys. Rep. **395** (2004) 357-426.

[13] Musslimani Z., Markis K., El-Ganainy R. and Christodoulides D., *Optical Solitons in PT Periodic Potentials,* Phys. Rev. Lett. **100** (2008) 030402-1-4.

[14] Musslimani Z., Markis K., El-Ganainy R. and Christodoulides D., *Analytical Solutions to a Class of Nonlinear Schrödinger Equations with PT-Like Potentials,* J. Phys. A **41** (2008) 244019 .

[15] Reuter C., Makris K., El-Ganainy R., Christodoulides D., Segev M. and Kip D., *Observation of Parity-Time Symmetry in Optics,* Nature Phys. **6** (2010) 192-195.

[16] Salerno M., Laser Physics, *Macroscopic Bound States and the Josephson Effect in Bose-Einstein Condensates in Optical Lattices* **14** (2005) 620-625.

[17] Schindler J., Li A., Zheng M., Ellis F. and Kottos T., *Experimental Study of Active LRC Circuits with PT Symmetries,* Phys. Rev. A **84** (2011) 040101-1-5.

[18] Staliunas K., Herrero R. and Vilaseca R., *Subdiffraction and Spatial Filtering due to Periodic Spatial Modulation of the Gain-Loss Profile,* Phys. Rev. A **80** (2009) 013821-1-6.

REPULSIVE GRAVITY OF LIGHT BEAMS[*]

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Abstract. The analysis of some exact solutions of Einstein equations, describing gravitational waves produced by light, shows that there can be repulsion between light beams. This is due only to the spin-1 character of the solutions and not to the introduction of other inputs external to General Relativity. Cosmological relativistic jets give an example of a such phenomenon.

1. Introduction

In this paper we study the possibility of a repulsive behavior in gravitational interaction, in particular between two light beams, due to the spin properties of some exact solutions of Einstein equations. More precisely we observe that there are exact solutions with spin-1 which describe gravitational waves produced by light sources and give rise to a repulsive behavior. The arguments which actually attribute spin-2 to gravitational waves, relies on the assumption that the solutions be Fourier expandable and, as a consequence have no spin-1 components. However there exist [7, 23, 31, 32] (see Sections 2 and 3) physically meaningful solutions of Einstein equations which although not Fourier expandable are nonetheless finite energy solutions. An application of this result (see Section 4) is concerned with the numerical data related to cosmological relativistic jets.

Section 2 describes a family of exact solutions of Einstein equations, representing gravitational waves generated by a light beam or, more generally, by massless particles, and their physical properties.

Section 3 contains some historical remarks and with the Tolman, Ehrenfest, Podolsky [33] and Wheeler [36] results on gravitational repulsive behavior.

Section 4 deals with cosmological relativistic jets.

2. The Gravitational Interaction of Light

2.1. Geometric Properties

In previous papers [7–10, 27–29, 34, 35] a family of exact solutions g of Einstein field equations, representing the gravitational wave generated by a beam of light, has been explicitly written

$$g = 2f(\mathrm{d}x^2 + \mathrm{d}y^2) + \mu \left[(w(x,y) - 2q)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}q \right] \tag{1}$$

where $\mu(x,y) = A\Phi(x,y) + B$ (with $\Phi(x,y)$ a harmonic function and A, B numerical constants), $f(x,y) = (\nabla\Phi)^2 \sqrt{|\mu|}/\mu$, and $w(x,y)$ is a solution of the *Euler-Darboux-Poisson equation*

$$\Delta w + (\partial_x \ln |\mu|) \partial_x w + (\partial_y \ln |\mu|) \partial_y w = \rho$$

where $T_{\mu\nu} = \rho\delta_{\mu 3}\delta_{\nu 3}$ is representing the energy-momentum tensor and Δ is the Laplace operator in the $(x,y)-$plane.

Previous metrics are invariant for the non Abelian Lie agebra $\mathcal{G}_2$ of Killing fields

$$X = \frac{\partial}{\partial p}, \qquad Y = \mathrm{e}^p \frac{\partial}{\partial q}$$

with

$$[X,Y] = Y, \qquad g(Y,Y) = 0$$

generating a two-dimensional distribution $\mathcal{D}$ whose orthogonal distribution $\mathcal{D}^\perp$ is integrable.

In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above family is locally diffeomorphic to a subclass of Peres solutions and, by using the transformation

$$p = \ln |u|, \qquad q = uv$$

can be written in the form

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + \frac{w}{u^2}\mathrm{d}u^2 \tag{2}$$

with $\Delta w(x,y) = \rho$, and has the Lorentz invariant *Kerr-Schild* form

$$g_{\mu\nu} = \eta_{\mu\nu} + V k_\mu k_\nu, \qquad k_\mu k^\mu = 0.$$

2.2. Physical Properties

2.2.1. Wave Character

The wave character and the polarization of these gravitational fields has been analyzed in many ways. For example, the Zel'manov criterion [37] was used to show that these are gravitational waves and the propagation direction was determined by using the Landau-Lifshitz pseudo-tensor. However, the algebraic Pirani criterion is easier to handle since it determines both the wave character of the solutions and the propagation direction at once. Moreover, it has been shown that, in the vacuum case, the two methods agree. To use this criterion, the Weyl scalars must be evaluated according to Petrov classification [24].

In the Newmann-Penrose formulation [22] of Petrov classification, we need a *tetrad* basis with two real null vector fields and two real spacelike (or two complex null) vector fields. Then, if the metric belongs to type $\mathbb{N}$ of the Petrov classification, it is a gravitational wave propagating along one of the two real null vector fields (Pirani criterion). Let us observe that ∂_x and ∂_y are spacelike real vector fields and ∂_v is a null real vector but ∂_u is not. With the transformation

$$x \mapsto x, \qquad y \mapsto y, \qquad u \mapsto u, \qquad v \mapsto v + \frac{w(x,y)}{2u}$$

whose Jacobian is equal to one, the metric (2) becomes

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}w(x,y)\mathrm{d}\ln|u|. \tag{3}$$

Since ∂_x and ∂_y are spacelike real vector fields and ∂_u and ∂_v are null real vector fields, the above set of coordinates is the right one to apply for the Pirani's criterion. Since the only nonvanishing components of the Riemann tensor, corresponding to the metric (3), are

$$R_{iuju} = \frac{2}{u^3}\partial_{ij}^2 w(x,y), \qquad i,j = x,y$$

these gravitational fields belong to Petrov type $\mathbb{N}$ [37]. Then, according to the Pirani's criterion, previous metric does indeed represent a gravitational wave propagating along the null vector field ∂_u.

It is well known that linearized gravitational waves can be characterized entirely in terms of the linearized and gauge invariant Weyl scalars. The non vanishing Weyl scalar of a typical spin-2 gravitational wave is Ψ_4. Metrics (3) also have as non vanishing Weyl scalar Ψ_4.

2.2.2. Spin

Besides being an exact solution of the Einstein equations, the metric (3) is (for $w/u^2 \ll 1$), also a solution of linearized Einstein equations, thus representing a perturbation of Minkowski metric $\eta = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2$

(where $u = (z - t)/\sqrt{2}, \quad v = (z + t)/\sqrt{2}$) with the perturbation, generated by a light beam or by a photon wave packet moving along the z-axis, given by

$$h = \mathrm{d}w(x, y)\mathrm{d}\ln|z - t|$$

whose non vanishing components are

$$h_{0,1} = -h_{13} = -\frac{w_x}{(z - t)}, \qquad h_{0,2} = -h_{23} = -\frac{w_y}{(z - t)}.$$

2.2.3. The Energy-Momentum Tensor

A transparent method to determine the spin of a gravitational wave is to look at its physical degrees of freedom, i.e., the components which contribute to the energy [11]. One should use the Landau-Lifshitz (pseudo)-tensor t^μ_ν which, in the asymptotically flat case, agrees with the Bondi flux at infinity [9]. It is worth to remark that the canonical and the Landau-Lifchitz energy-momentum pseudo-tensors are true tensors for Lorentz transformations. Thus, any Lorentz transformation will preserve the form of these tensors and this allows to perform the analysis according to the Dirac procedure. A globally square integrable solution $h_{\mu\nu}$ of the wave equation is a function of $r = k_\mu x^\mu$ with $k_\mu k^\mu = 0$.

With the choice $k_\mu = (1, 0, 0, -1)$, we get for the energy density t^0_0 and the energy momentum t^3_0 the following result

$$16\pi t^0_0 = \frac{1}{4}(u_{11} - u_{22})^2 + u^2_{12}, \qquad t^0_0 = t^3_0$$

where $u_{\mu\nu} \equiv \mathrm{d}h_{\mu\nu}/\mathrm{d}r$. Thus, the physical components which contribute to the energy density are $h_{11} - h_{22}$ and h_{12}. Following the analysis of Dirac, we see that they are eigenvectors of the infinitesimal rotation generator $\mathcal{R}$, in the plane $x - y$, belonging to the eigenvalues $\pm 2i$. The components of $h_{\mu\nu}$ which contribute to the energy thus correspond to spin-2.

In the case of the prototype of spin-1 gravitational waves (3), both Landau-Lifchitz energy-momentum pseudo-tensor and Bel-Robinson energy-momentum tensor [3, 4, 26] single out the same wave components and we have

$$\tau^0_0 \sim c_1(h_{0x,x})^2 + c_2(h_{0y,x})^2, \qquad t^0_0 = t^3_0$$

where c_1 and c_2 are constants, so that the physical components of the metric are h_{0x} and h_{0y}. Following the previous analysis one can see that these two components are eigenvectors of $i\mathcal{R}$ belonging to the eigenvalues ± 1. In other words, metrics (3), which are not pure gauge since the Riemann tensor is not vanishing, represent spin-1 gravitational waves propagating along the $z-$axis at light velocity.

2.2.4. Summarizing

Globally square integrable spin-1 *gravitational waves propagating on a flat background are always pure gauge* and spin–1 *gravitational waves which are not globally square integrable are not pure gauge.*

It is always possible to write metric (3) in an apparently transverse gauge [31] but since these coordinates are no more harmonic this transformation is not compatible with the linearization procedure.

What truly distinguishes spin-1 from spin-2 gravitational waves is the fact that in the spin-1 case the Weyl scalar has a non trivial dependence on the transverse coordinates (x, y) due to the presence of the harmonic function. This could led to observable effects on length scales larger than the *characteristic length scale* where the harmonic function changes significantly.

Indeed, the Weyl scalar enters in the geodesic deviation equation implying a non standard deformation of a ring of test particles breaking the invariance under of π rotation around the propagation direction. Eventually, one can say that there should be distinguishable effects of spin-1 waves at suitably large length scales.

It is also worth to stress that the results of Aichelburg and Sexl, Felber and van Holten [1, 13, 17] suggest that the sources of asymptotically flat $pp-$waves (which have been interpreted as spin-1 gravitational waves [7, 9]) repel each other. Thus, in a field theoretical perspective, "$pp-$ gravitons" must have spin-1.

2.3. Gravitoelectrodynamics

Hereafter the spatial part of four-vectors will be denoted in bold and the standard symbols of three-dimensional vector calculus will be adopted.

Metric (3) can be written in the gravitoelectromagnetic form

$$g = (2\Phi^{(g)} - 1)\mathrm{d}t^2 - 4(\mathbf{A}^{(g)}.\mathrm{d}\mathbf{r})\mathrm{d}t + (2\Phi^{(g)} + 1)\mathrm{d}\mathbf{r}.\mathrm{d}\mathbf{r} \tag{4}$$

where

$$\mathbf{r} = (x, y, z), \qquad 2\Phi^{(g)} = h_{00}, \qquad 2A_i^{(g)} = -h_{0i}.$$

2.3.1. Gravito-Lorentz Gauge

In terms of $\Phi^{(g)}$ and $\mathbf{A}^{(g)}$ the harmonic gauge condition reads

$$\frac{\partial \Phi^{(g)}}{\partial t} + \frac{1}{2}\nabla \cdot \mathbf{A}^{(g)} = 0 \tag{5}$$

and, once the gravitoelectric and gravitomagnetic fields are defined in terms of g-potentials, as

$$\mathbf{E}^{(\mathbf{g})} = -\nabla\Phi^{(g)} - \frac{1}{2}\frac{\partial\mathbf{A}^{(g)}}{\partial t}, \qquad \mathbf{B}^{(\mathbf{g})} = \nabla \wedge \mathbf{A}^{(g)}$$

one finds that the linearized Einstein equations resemble Maxwell equations. Consequently, being the dynamics fully encoded in Maxwell-like equations, this formalism describes the physical effects of the vector part of the gravitational field.

2.3.2. Gravito-Faraday Tensor Field

Gravitational waves can be also described in analogy with electromagnetic waves, the gravitoelectric and the gravitomagnetic components of the metric being

$$E_\mu^{(g)} = F_{\mu 0}^{(g)}, \qquad B^{(g)\mu} = -\varepsilon^{\mu 0 \alpha \beta} F_{\alpha \beta}^{(g)}/2$$

where

$$F_{\mu\nu}^{(g)} = \partial_\mu A_\nu^{(g)} - \partial_\nu A_\mu^{(g)}, \qquad A_\mu^{(g)} = -h_{0\mu}/2 = (-\Phi^{(g)}, \mathbf{A}^{(g)}).$$

3. Tolman-Erhenfest-Podolsky Problem

3.1. History

The interest in repulsive gravity, or *antigravity* as it was usually called, goes back to the fifty's [18, 19, 21].

The general point of view was that since gravitational interaction is mediated by a spin-2 particle, it can only be attractive and thus, to obtain a repulsive behavior, some other ingredient is required. The idea was then to explore the possibility of repulsive matter-antimatter gravity, but within the old quantum field theories there was no room for such a possibility.

The main arguments, reviewed in [21], were of various kinds including violation of energy conservation and disagreement with experiments of the Eötvös type due to the effects of antigravity on the vacuum polarization diagrams of atoms.

More recently however, within the context of modern quantum field theories, it was proven that those arguments were no longer sufficient to exclude repulsive effects and the interest in antigravity increased again. For example, in [16] it was shown that in supergravity and string theory, due to dimensional reduction, the effective four-dimensional theory of gravity may show repulsive aspects because of the appearance of spin-1 graviphotons.

3.2. Photon-Photon Scattering

Photon-photon scattering can occur through the creation and annihilation of virtual electron-positron pairs and may even lead to collective photon phenomena. Photons also interact gravitationally but the gravitational scattering of light by light has been much less studied.

Purely general relativistic treatments of electromagnetic wave interactions have been made resulting in exact solutions [14, 15], but these calculations are different

from pure scattering processes and do not address the interaction at single photon level.

It is not clear to what extent, calculations of the gravitational cross-section using QFT methods are consistent with classical GR. First studies go back to Tolman, Ehrenfest and Podolsky 1931 and, later, to Wheeler 1955 [33, 36] who analysed the gravitational field of light beams and the corresponding geodesics in the linear approximation of Einstein equations. They discovered that null rays behave differently according to whether they propagate parallel or antiparallel to a steady, long, straight beam of light, but they did not provide a physical explanation of this fact.

Later, Barker, Bathia and Gupta [2], following a previous analysis of Barker, Gupta and Haracz [6], analyzed in QED the photon-photon interaction through the creation and annihilation of a virtual graviton in the center-mass system and they found that the interaction is eight times the "Newtonian" value plus a polarization dependent repulsive contact interaction and also obtained the gravitational cross sections for various photon polarization states.

Results of Tolman, Ehrenfest, Podolsky, Wheeler were clarified in part in [12], in the setting of classical pure General Relativity, using an approach based on a generalization to null rays of the gravitoelectromagnetic Lorentz force of linearized gravity.

They also extended the analysis to the realm of exact pp-wave solutions of the Einstein equations. Later, photon-photon scattering due to self-induced gravitational perturbations on a Minkowski background has been also analyzed by Brodin, Eriksson and Marklund [5] solving the Einstein-Maxwell system perturbatively to third order in the field amplitudes and confirming the dependence of differential gravitational cross section on the photon polarizations.

3.3. Geodesic Motion

The geodesic motion of a *massive particle* moving with four-velocity $v^\mu = (1, \underline{v})$, $|\mathbf{v}| \ll 1$, in a light beam gravitational field characterized by gravitoelectric $\mathbf{E}^{(g)}$ and gravitomagnetic $\mathbf{B}^{(g)}$ fields, is determined (at first order in $|\mathbf{v}|$) by the *acceleration*

$$\mathbf{a}^{(g)} = -\mathbf{E}^{(g)} - 2\mathbf{v} \wedge \mathbf{B}^{(g)}.$$

The geodesic motion of a *massless particle* moving with velocity $v^\mu = (1, \mathbf{v})$, $|\mathbf{v}| = 1$, in the light beam gravitational field, parallel(anti) to z-axis ($v_j = \pm\delta_{j3}$) is slightly different

$$\mathbf{a}^{(g)} = -2\left(\mathbf{E}^{(g)} + \mathbf{v} \wedge \mathbf{B}^{(g)}\right).$$

There are two contributions, one by the light beam, which is the source of gravity, and the other by the test photon.

Since the gravitoelectric and gravitomagnetic fields corresponding to our metric are given by

$$\mathbf{E}^{(g)} = (w_x, w_y, 0)/4u^2, \qquad \mathbf{B}^{(g)} = (w_y, -w_x, 0)/4u^2$$

the 'gravitational acceleration' of a massless particle will be

$$\mathbf{a}^{(g)} = -[w_x(1 - v_z)\mathbf{i} + w_y(1 - v_z)\mathbf{j} + (w_x v_x + w_y v_y)\mathbf{k}]/2u^2. \qquad (6)$$

The velocity $\mathbf{v}$ of a photon is determined by the null geodesics equations

$$(h - 1) - 2hv_z + (h + 1)v_z^2 = 0$$

which has two solutions

$$v_z = 1, \qquad v_z = \frac{h - 1}{h + 1} = \frac{w - u^2}{w + u^2}.$$

If the photon propagates parallel to the light beam, $v = (0, 0, 1)$, then

$$\mathbf{a}^{(g)} = 0$$

and there is not attraction or repulsion (see also [38]).

If the photon does not propagate parallel to the light beam, the velocity will be $v = (h - 1)/(h + 1)$, then

$$\mathbf{a}^{(g)} = -\nabla w/2\left(w + u^2\right)$$

and the force turns out to be attractive.

Thus, the lack of attraction found by Tolman, Ehrenfest, Podolsky comes out also from the analysis of the geodesical motion of a massless spin-1 test particle in the strong gravitational field of the light, neglecting however the gravitational field generated by that particle. An exhaustive answer could derive only determining the gravitational field generated by two photons, each one generating spin-1 gravitational waves. However, since helicity seems to play for photons the same role that charge plays for charged particles, two photons with the same helicity should repel one another. This repulsion turns out to be very weak and cannot be certainly observed in laboratory but it could play a relevant role at cosmic scale and could give not trivial contributions to the dark energy. Thus, together with gravitons (spin-2), one may postulate the existence of graviphotons (spin-1) and graviscalar (spin-0).

4. Relativistic Jets

Relativistic jets are extremely powerful jets of plasma emerging from presumed massive objects at the centers of some active radio galaxies and quasars. Their lengths can reach several thousand or even hundreds of thousands of light years. Among the different types of astrophysical jets, the most energetic ones are potential candidates to give rise to emission of gravitational waves. For example,

highly relativistic jets should be associated with some sources of gamma ray bursts (GRBs) [25]. The impact of an ultra relativistic jet over the space-time metric can be studied starting from the extreme situation where the velocity of the particles in the beam is assumed to be equal to the velocity of light. The jet is then represented by a beam of null particles. For a flow of radiation of a null electromagnetic (*em*) field along the z-axis, the (*em*) energy-momentum tensor macroscopic components $T_{\mu\nu} = F_{\mu\alpha}F_\nu^\alpha + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ reduce to

$$T_{00} = \frac{\rho}{z - ct}, \qquad T_{03} = T_{30} = -\frac{\rho}{z - ct}, \qquad T_{33} = \frac{\rho}{z - ct}$$

where $\rho = \left(E^2 + B^2\right)/2$ represents the amplitude of the field, i.e., the density of radiant energy at point of interest. They are just the components in the coordinates (t, x, y, z) of the energy-momentum tensor $T = \rho \mathrm{d}u^2$ of Section 2.

We assume then that the energy density is a constant ρ_0 within a certain radius $0 \le r = \sqrt{x^2 + y^2} \le r_0$ and vanishes outside. Thus, the source represents a cylindrical beam with width r_0 and constitutes a simple generalization of a single null particle.

Introducing back the standard coupling constant of Einstein tensor with matter energy-momentum tensor, we have

$$\Delta w(x, y) = \frac{8\pi G}{c^4}\rho. \tag{7}$$

The cylindrical symmetry implies that $w(x, y)$ will depend only on the distance r from the beam. A solution $w(r)$ of Poisson equation (7) satisfying the continuity condition at $r = r_0$ can be easily written as

$$w(r) = \frac{4\pi G}{c^4}\rho_0 r^2, \qquad r \le r_0 \tag{8}$$

$$w(r) = \frac{8\pi G}{c^4}\rho_0 r_0^2 \left[\ln\left(\frac{r}{r_0}\right) + \frac{1}{2}\right], \qquad r > r_0 \tag{9}$$

or also

$$w(r) = \frac{4\pi G\rho_0}{c^4}r_0^2 W(r) \tag{10}$$

with $W(r) = r^2/r_0^2$ or $W(r) = 1 + \ln\left(\frac{r}{r_0}\right)^2$ depending on whether $r < r_o$ or $r > r_0$.

Thus, a photon moving antiparallel and external to the beam will experience at the space-time point (t, x, y, z) a transversal gravitational attraction expressed by

$$\mathbf{a}^{(g)}(t, x, y, z) = -\frac{16\pi G}{c^4}\rho_0 r_0^2 \frac{\mathbf{r}}{r^2(z - ct)^2} \tag{11}$$

where the speed of light c has been reintroduced and the retardation is automatically accounted for. As a consequence of spin-1 of our wave and of QFT a photon

moving parallel and external to the beam will experience at the space-time point (t, x, y, z) a transversal gravitational repulsion given by

$$\mathbf{a}^{(g)}(t, x, y, z) = \frac{16\pi G}{c^4} \rho_0 r_0^2 \frac{\mathbf{r}}{r^2 (z - ct)^2}. \tag{12}$$

For jets which start with a small opening angle $\theta_0 \leq 10^{-3} - 10^{-4}$ [25], it can be assumed that the width of the beam remains constant during the first stage of the jet expansion [20] and, for a beam-length $L = c\tau \approx 10^6 - 10^7$ Km (a typical jet lasts $\tau \approx 10 - 100$ s), will be of the order of $r_0 = L\theta_0 \approx 10^2 - 10^3$ km. The energy is of the order of $E = \approx 10^{44} - 10^{45} J$, so that $\rho_0 = E/L \approx 10^{37} - 10^{39} J/$km.

Replacing these values in equation (12) and taking $G/c^4 \approx 10^{-44} N^{-1}$, we obtain for the transversal acceleration per unit length

$$a^{(g)}(t, x, y, z) = \frac{10^{-5}}{r^2 (z - ct)^2} \text{ cm}^{-1}$$

where $r = \sqrt{x^2 + y^2}$ and z are the distances, expressed in cm, between the source and the point of interest and t the observation time.

Conclusions

Repeating the above calculations for a laser beam in an interferometer of LIGO or VIRGO type, in the formula above we would get a factor of 10^{-50} instead of 10^{-5}. Then, the repulsion (as well as the attraction) turns out to be very weak. However it could play a relevant role at cosmic scale and could give not trivial contributions to the dark energy.

At this point, together with gravitons (spin-2), one could postulate the existence of graviphotons (spin-1) and of graviscalar (spin-0) too. Through coupling to fermions, they might give forces depending on the barion number. These fields might give [30] two (or more) Yukawa type terms of different signs, corresponding to repulsive graviphoton exchange and attractive graviscalar exchange (range $\gg 200m$). However, much more work must be done for a better understand of the role played by the gravitational field of the electromagnetic radiation and/or of null particles beams in the evolution of the universe.

Acknowledgements

References

[1] Aichelburg A. and Sexl R., *On the Gravitational Field of a Massless Particle,* Gen. Rel. Grav. **2** (1971)303-312.

[2] Barker B., Bhatia M. and Gupta S., *Gravitational Scattering of Light by Light,* Phys. Rev. **158** (1967)1498-1499. Erratum: Barker B., Manjit S., Bhatia M., and Gupta S., *Gravitational Scattering of Light by Light,* Phys. Rev. **162** (1967) 1750-1750.

[3] Bel L., *Sur la radiation gravitationnelle,* C.R. Acad. Sci. (Paris) **247** (1958) 1094-1096.

[4] Bel L., *Introduction d'un tenseur du quatrieme order,* C.R. Acad. Sci. (Paris) **248** (1959) 1297-1300.

[5] Brodin G., Eriksson D. and Maklund M., *Graviton Mediated Photon-Photon Scattering in General Relativity,* Phys. Rev. D **74** (2006) 124028-124032.

[6] Barker B., Gupta S. and Haracz R., *One-Graviton Exchange Interaction of Elementary Particles,* Phys. Rev. **149** (1966) 1027-1032.

[7] Canfora F., Vilasi G. and Vitale P., *Nonlinear Gravitational Waves and Their Polarization,* Phys. Lett. **545** (2002) 373-378.

[8] Canfora F. and Vilasi G., *Spin-1 Gravitational Waves and Their Natural Sources,* Phys. Lett. B **585** (2004) 193-199.

[9] Canfora F., Vilasi G. and Vitale P., *Spin-1 Gravitational Waves,* Int. J. Mod. Phys. B **18** (2004) 527-540.

[10] Canfora F., Parisi L. and Vilasi G., *Nonlinear Gravitational Waves, Their Polarization and Realistic Sources,* Theor. Math. Phys. **152** (2007) 1069-1080.

[11] Dirac PAM, *General Theory of Relativity,* Wiley, New York 1975.

[12] Faraoni V. and Dumse R., *The Gravitational Interaction of Light: From Weak to Strong Fields,* Gen. Rel. Grav. **31** (1999) 91-105.

[13] Felber F., *Exact Antigravity-Field Solutions of Einstein's Equation,* arXiv0803.2864; Felber F., *Dipole Gravity Waves From Unbound Quadrupoles,* arXiv1002.0351.

[14] Ferrari V. and Ibanez J., *Gravitational Interaction of Massless Particles,* Phys. Lett. A **141** (1989) 233-237.

[15] Ferrari V., Pendenza P. and Veneziano G., *Beamlike Gravitational Waves and Their Geodesics,* Gen. Rel. Grav. **20** (1988) 1185-1191.

[16] Fabbrichesi M. and Roland K., *Strong Anti-Gravity,* Nucl. Phys. B **388** (1992) 539-569.

[17] van Holten J., *The Gravitational Field of a Light Wave,* arXiv0808.0997.

[18] Morrison P. and Gold T., In: *Essays on Gravity, Nine Winning Essays of the Annual Award* (1949 − 1958) *of the Gravity Research Foundation,* Gravity Research Foundation, New Boston 1958, pp 45-50.

[19] Morrison P., *Approximate Nature of Physical Symmetries,* Am. J. Phys. **26** (1958) 358-368.

[20] de Rey Neto E. and de Araujo J., Aguiar O, *A Gravitational Shock Wave Generated by a Beam of Null Matter in Quadratic Gravity*, Class. Quant. Grav. **21** (2003) 1479-1488.

[21] Nieto M. and Goldman T., *The Arguments Against 'Antigravity' and the Gravitational Acceleration of Antimatter*, Phys. Rep. **205** (1991) 221-281 (Note: Errata issued in 1992 in volume **216**).

[22] Penrose R., *A Spinor Approach to General Relativity*, Ann. Phys. **10** (1960) 171-201.

[23] Peres A., *Some Gravitational Waves*, Phys. Rev. Lett. **3** (1959) 571-572.

[24] Petrov A., *Einstein Spaces*, Pergamon Press, New York 1969.

[25] Piran T., *The Physics of Gamma-Ray Bursts*, Rev. Mod. Phys. **76** (2004) 1143-2010; Sari R., Piran T. and Halpern J., *The Physics of Gamma-Ray Bursts*, Ap. J. **519** (1999) L17-L20; Piran T., *The Physics of Gamma-Ray Bursts*, Phys. Rep. **333** (2000) 529-553; Mészáros P., *The Physics of Gamma-Ray Bursts*, Progress of Theoretical Physics Supplement, **136** (1999) 300-320.

[26] Robinson I., *On the Bel-Robinson Tensor*, Class. Quantum Grav. **14** (1997) A331-A333.

[27] Sparano G., Vilasi G. and Vinogradov A., *Gravitational Fields with a non Abelian Bidimensional Lie Algebra of Symmetries*, Phys. Lett. B **513** (2001) 142-146.

[28] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves I. Local Aspects*, Diff. Geom. Appl. **16** (2002) 95-120.

[29] Sparano G., Vilasi G. and Vinogradov A., *Vacuum Einstein Metrics with Bidimensional Killing Leaves II. Global Aspects*, Diff. Geom. Appl. **17** (2002) 15-35.

[30] Stacey F., Tuck G., and Moore G., *Quantum Gravity: Observational Constraints on a Pair of Yukawa Terms*, Phys. Rev. D **36** (1987) 2374-2380.

[31] Stephani H., *General Relativity: An Introduction to the Theory of the Gravitational Field*, Cambridge University Press, Cambridge 1996.

[32] Stephani H., Kramer D., MacCallum M., Honselaers C. and Herlt E., *Exact Solutions of Einstein Field Equations*, Cambridge University Press, Cambridge 2003.

[33] Tolman R., Ehrenfest P. and Podolsky B., *On Gravitational Field Produced by Light*, Phys. Rev. **37** (1931) 602-615.

[34] Vilasi G., *Gravitational Waves as Exact Solutions of Einstein Field Equations*, J. Phys. Conference Series **87** (2007) 012017-012029.

[35] Vilasi S, Sparano G. and Vilasi G., *On the Gravity of Light*, Class. Quantum Grav. **28** (2011) 195014-195025.

[36] Wheeler J., *Geons*, Phys. Rev. **97** (1955) 511-536.

[37] Zakharov V., *Gravitational Waves in Einstein's Theory*, Halsted Press, New York 1973.

[38] Zee A., *Quantum Field Theory in a Nutshell* (2nd Edn), Princeton University Press, Princeton 2010.

ROLE OF THE NONLINEAR COUPLING IN THE COLLISION DYNAMICS OF QUASI-PARTICLES GOVERNED BY VECTOR NLSE

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Abstract. We present a conservative fully implicit scheme using complex arithmetic for the Coupled Nonlinear Schrödinger Equations (CNLSE) which allows us to reduce the computational time fourfold. In this work we investigate collisions of solitons with no time frequency of the carrier wave in the initial configuration. We obtain various results numerically and investigate the role of nonlinear coupling on the quasi-particle dynamics. For nontrivial but moderate nonlinear coupling parameter, we find that the polarization of the system changes, but no other effects are present. For moderate and large values of the nonlinear coupling parameter, additional solitons are created during the collision of the initial ones. These seem to be new effects, not reported in the literature.

CONTENTS

1. Introduction

The investigation of soliton supporting systems is of great importance both for the applications and for the fundamental understanding of the phenomena associated with propagation of solitons. Recently, elaborate models such as Coupled Nonlinear Schrödinger Equations (CNLSE) appeared in the literature (see, e.g. [6, 8]). They involve more parameters and have richer phenomenology but, as a rule, are not fully integrable and require numerical approaches. The non-fully-integrable models possess as a rule three conservation laws: for (wave) "mass", (wave) momentum, and energy and these have to be faithfully represented by the numerical scheme.

An implicit scheme of Crank-Nicolson type was first proposed for the single NLS in the extensive numerical treatise [11]. The concept of the internal iterations was first applied to CNLSE in [4] and extended in [9] in order to ensure the implementation of the conservation laws on difference level within the round-off error of the calculations. The CNLSE is investigated numerically also in [5]. Here, we follow generally the works [4, 9] but focus on a new complex-variable implementation of the conservative scheme. This allows us to invert five-diagonal matrices (albeit complex-valued) while the real-valued scheme requires the inversion of nine-diagonal matrices [4,9]. To this end, we generalize the computer code for Gaussian elimination with developed earlier pivoting for real-valued algebraic systems in [3]. This gives a significant advantage in the efficiency of the algorithm. The numerical validation of the new code includes comparisons with [4,9] which show that the complex-numbers implementation of the scheme gives identical results with the real-numbers codes but is approximately four times as efficient. Several featuring examples of interacting solitons in CNLSE are elaborated.

2. Coupled Nonlinear Schrödinger Equations

In optics, the most popular model is the cubic Schrödinger equation which describes the single-mode wave propagation in a fiber [1,2]. It has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{1}$$

where $i = \sqrt{-1}$ and $\psi(x,t)$ is a complex-valued wave function. Depending on the sign of coefficient α, the localized solutions of equation (1) are either the hyperbolic secants (bright solitons) or hyperbolic tangents (dark solitons). Since the fibers also allow propagation of multiple "orthogonal" modes, a multi-component version of equation (1) has been actively investigated during the last decade.

A general form of the Coupled Nonlinear Schrodinger Equations (CNLSE) reads

$$i\psi_t = \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi + \gamma\psi + \Gamma\phi = 0$$
$$i\,\phi_t = \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi + \gamma\phi + \Gamma\psi = 0$$

$$(2)$$

where β is the dispersion coefficient and α_1 describes the self-focusing of a signal for pulses in birefringent media. Complex-valued coefficients γ and Γ are responsible for the linear coupling between the two equations. Respectively α_2 governs the nonlinear coupling between the equations. It is interesting to note that when $\alpha_2 = 0$, the nonlinear coupling is not present despite the fact that "cross-terms" proportional to α_1 appear in the equations. In fact, when $\gamma = \Gamma = \alpha_2 = 0$, the solution of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE, equation (1) with nonlinearity coefficient $\alpha = 2\alpha_1$. The coefficient α_2 is called sometimes "cross-phase modulation" and its value (when $\alpha_2 \neq 0$) plays role in defining the elliptic, circular and linear polarizations. In this case, integrability is lost, and numerical methods are to be used to study the evolution of the system.

Here is to be mentioned that two main versions of equation (1) appear in the literature. In the first one the sign of the time derivative is positive (as in equation (1), which is the most popular version in nonlinear optics), and in the other – the sign is negative. Unlike the parabolic equations, changing the sign does not make the equation incorrect in the sense of Hadamard. Hence it does not really make difference which version will be used.

Functions ψ and ϕ have various interpretations in the context of optic pulses including the amplitudes of x and y polarizations in a birefringent nonlinear planar waveguide, pulsed wave amplitudes of left and right circular polarizations, etc. The quantity γ is called normalized birefringence, and Γ is the relative propagation constant. The presence of the two new parameters, γ and Γ, in equations (2) makes the phenomenology of the system (2) much richer. In particular, they allow to study the phenomena such as "self-dispersion", "cross-dispersion", and dissipation, etc. (see [9] and the literature cited therein).

For $\Gamma = \gamma = 0$, equation (2) is alternatively called the Gross-Pitaevskii equation or an equation of Manakov-Type. It was solved analytically for the case $\alpha_2 = 0, \beta = \frac{1}{2}$ by Manakov [7] via inverse scattering transform who generalized an earlier result by Zakharov and Shabat [13,14] for the scalar cubic NLSE (i.e., equation (2)$_\psi$ with $\phi(x, t) = 0$).

Let us define mass, M, (pseudo)momentum, P, and energy, E, of the wave system as follows

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta} \int_{-\infty}^{\infty} \left(|\psi|^2 + |\phi|^2 \right) \mathrm{d}x \tag{3}$$

$$P \stackrel{\text{def}}{=} - \int_{-\infty}^{\infty} \Im(\psi \bar{\psi}_x + \phi \bar{\phi}_x) \mathrm{d}x \tag{4}$$

$$E \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathcal{H} \mathrm{d}x$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta \left(|\psi_x|^2 + |\phi_x|^2 \right) - \frac{\alpha_1}{2} (|\psi|^4 + |\phi|^4) \tag{5}$$

$$- (\alpha_1 + 2\alpha_2) \left(|\phi|^2 |\psi|^2 \right) - \gamma \left(|\psi|^2 + |\phi|^2 \right) - 2\Gamma \Re(\bar{\phi}\psi)$$

where $\mathcal{H}$ is the Hamiltonian density of the system. Note that the factor $\frac{1}{2\beta}$ is a matter of definition and is added for the sake of further convenience. In the same vein the signs in the expression of the energy are up to a definition and the choice in the present paper is based on considerations of further convenience when the quasi-particles are considered. It is readily proved that these quantities are either conserved on the solutions of equation (2), or a balance law holds, namely

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = \mathcal{H}\big|_{x=L_2} - \mathcal{H}\big|_{x=-L_1}, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0 \tag{6}$$

where $-L_1$ and L_2 are the left end and the right end of the interval under consideration. For asymptotic boundary conditions the requirement $\psi, \phi = 0$ at infinity entails the requirement that the spatial derivatives also vanish. As a result, the Hamiltonian density vanishes at infinities and the balance law for the pseudomomentum becomes a conservation law.

We assume that for each of the functions ϕ, ψ the initial condition is of the form of a single propagating soliton, namely

$$\psi(x,t), \phi(x,t) = A \operatorname{sech}\left[b(x - X - ct)\right] \exp\left\{ i \left[\frac{c}{2\beta}(x - X) - nt \right] \right\}$$

$$b^2 = \frac{1}{\beta}\left(n + \gamma + \frac{c^2}{4\beta} \right), \qquad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \qquad u_c = \frac{2n\beta}{c} \tag{7}$$

where X is the spatial position (center of soliton) where the modulus soliton has maximum, c is the phase speed, and n is the carrier frequency. Respectively b^{-1} is a measure of the support of the localized wave.

In this paper we investigate the evolution of systems of waves which in the initial moment of time are superpositions of solitons of type of equation (7) and which evolve according to system equations (2).

To solve this problem numerically, we use the conservative scheme in complex arithmetic described in Appendix A. If one is to construct a numerical algorithm, the above conservation laws have to be embodied in the scheme in order to faithfully represent the physics of the problem. We use different number of points in spatial direction, typically of order of 8000-20000 points.

The parametric space of the problem is multidimensional, and it is impossible to exhaust the different ranges in a single paper. We focus our attention here on the effect of the nonlinear coupling and set $\Gamma = \gamma = 0$. We also fix $\beta = 1$, because, in fact, the independent variable x can be scaled by β and the latter is not an independent parameter. For the predominant set of numerical experiments, we choose initial solitons which are moving envelopes over standing wave, i.e., $n = 0$. Similarly to the dispersion parameter β, the nonlinearity parameter α can be absorbed in the amplitude of the solitons and can be held fixed. Thus the parameter to be varied is α_2, and more specifically, its ratio to α_1. For definiteness, we fix $\alpha_1 = 1$.

The aim of our work is to understand better the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

3. Weak Interaction

Before proceeding to investigating the role of $\alpha_2 \neq 0$, we computed the solution for $\alpha_2 = 0$. As expected no interaction between the two components of the vector soliton was observed, which confirms that only α_2 governs the nonlinear effects, not the full coefficient $(\alpha_1 + 2\alpha_2)$. As should have been expected, our computations showed that for $\alpha_2 = 0$ there was no interaction between the two orthogonal modes ψ and ϕ, despite of the fact that $\alpha_1 \neq 0$ means that terms proportional to $|\psi|^2$ are present in the equation for ϕ, and *vice versa*.

We begin our study with the case of relatively small α_2. We chose for the phase speeds of solitons $c_l = 1$ and $c_r = -0.5$ which does not restrict us very much because in absence of linear coupling, $\gamma = \Gamma = 0$, one can change the phase speed, but still obtains the same results provided that α_1 is also changed. The selected values for the phase speeds give for the amplitudes of the initial solitons the following

$$A_l \equiv A_\psi = \frac{\sqrt{2}}{2} \approx 0.7075, \qquad A_r \equiv A_\phi = \frac{\sqrt{2}}{4} \approx 0.3537. \qquad (8)$$

According to the analytical expression from Appendix B, the masses of the two quasi-particles are $M_l \equiv M_\psi = 1$ and $M_r \equiv M_\phi = 0.5$. Respectively the total pseudomomentum is $c_l M_l - c_r M_r = 0.75$. Since in the initial moment of time the two QPs are strictly $90°$ polarized we have only one of the amplitudes A_ψ, A_ϕ not equal to zero. Then equation (18) can be applied to the left and right solitons separately to get that

$$E_l^k = \frac{1}{2}c_l^2 M_l = 0.5, \quad E_l^p = \frac{2}{3}\left(A_l^2 b_l - \frac{A_l^4}{b_l}\right) = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{2}\right) = -\frac{1}{6} \approx -0.1667$$

$$E_r^k = \frac{1}{2}c_r^2 M_r = \frac{1}{16} = 0.0625$$

$$E_r^p = \frac{2}{3}\left(A_r^2 b_r - \frac{A_r^4}{b_r}\right) = \frac{2}{3}\left(\frac{1}{32} - \frac{1}{16}\right) = -\frac{1}{48} \approx -0.0208$$

$$E_l = E_l^k + E_l^p = \frac{1}{3} \approx 0.3333, \qquad E_r = E_r^k + E_r^p = \frac{1}{24} \approx 0.0408$$

$$E = E_l + E_r = \frac{3}{8} = 0.375$$

where the superscripts stand for "kinetic" and "potential" energies.

Note that the actual values obtained from the initial condition after being discretized on the chosen grid, are

$$M_l = 1.0000000, \; M_r = 0.50000000, \; P = 0.74921909, \; E = 0.37462784.$$

The small deviations for P and E of order of 0.1% are the effect of the truncation error. Since the scheme is conservative, the above values are the one which are kept constant during the time stepping.

We found that the interaction between the two components, ψ, ϕ, is insignificant for $\alpha_2 < 1$. Fig. 1, shows the case $\alpha_2 = 2$ when for the first time an appreciable cross signal is excited as a result of the interaction of the main solitons.

As it is the case with various other soliton problems (e.g., KdV, Boussinesq, Sine-Gordon), the quasi-particle of lesser energy suffers more from the interaction. In order to elucidate the process of interaction, we present in Fig. 2 the actual wave profiles for both real and imaginary parts of the ψ and ϕ solutions for several time stages during the interaction. It is clearly seen that there is intricate (but smooth!) interaction between the real and imaginary parts. If one monitors just the absolute values, then the profiles do not appear smooth during the interaction.

Now we concentrate on the trajectories of the quasi-particles after the collision. Fig. 3 shows the result of the numerical calculations. The "center" of a QP is defined as the point of maximum of either ψ or ϕ depending on which one was
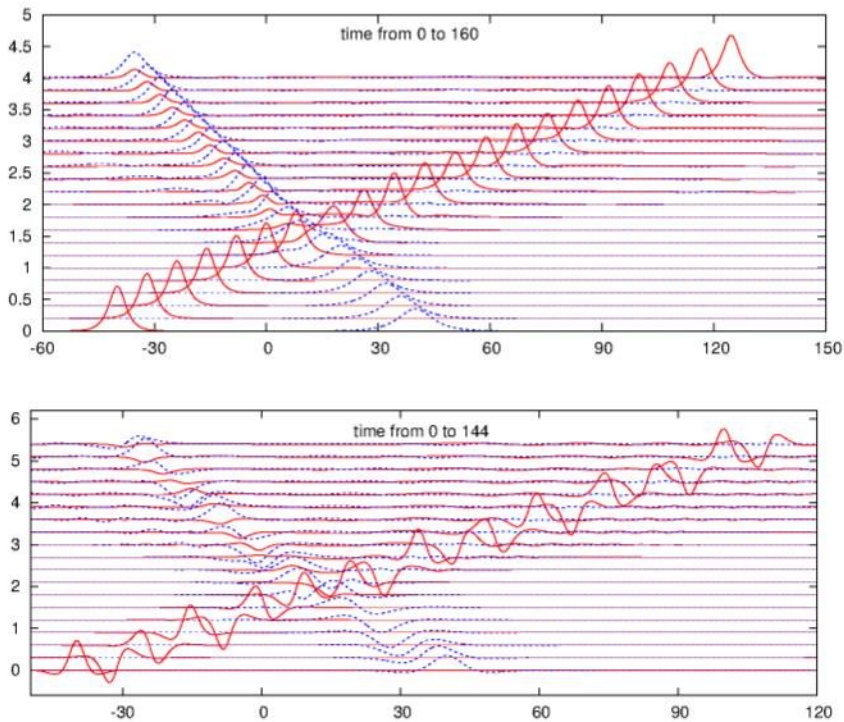
Figure 1. Head-on collision for $c_{\text{left}} = 1$, $c_{\text{right}} = -0.5$; $\alpha_2 = 2$.
Up: $|\psi|$, $|\phi|$. Down: $\Re\psi$, $\Re\phi$.

present in the initial condition for this particular QP. We chose to track the composite quasi-particle via the center for the shape of the function that was predominant in the initial moment. In doing so we were guided by the natural assumption that during the interaction additional orthogonal components will be excited, but the deformation of the main components will not be so drastic, so as to make them disappear completely.

The most important observation for CNLSE is that the speeds of QPs change after collision. Note that in the case of KdV, sG, and NLSE, the interactions result solely in phase shift, while the phase speeds are strictly recovered after the interaction. As shown in Fig. 3, the larger quasi-particle experiences lesser impact which is the case with all soliton model equations reported in the literature. The excited orthogonal signal is very small (as testified by Fig. 1), and the speed changes very little: increasing from 1 to 1.0239. At the same time, the smaller quasi-particle *decreases* its speed to 0.4218. It is a distinctly new feature of CNLSE not observed in the other models. It is connected to the fact that the cross-modulation enhances the excitability of the system. Under these circumstances, the phase shift have
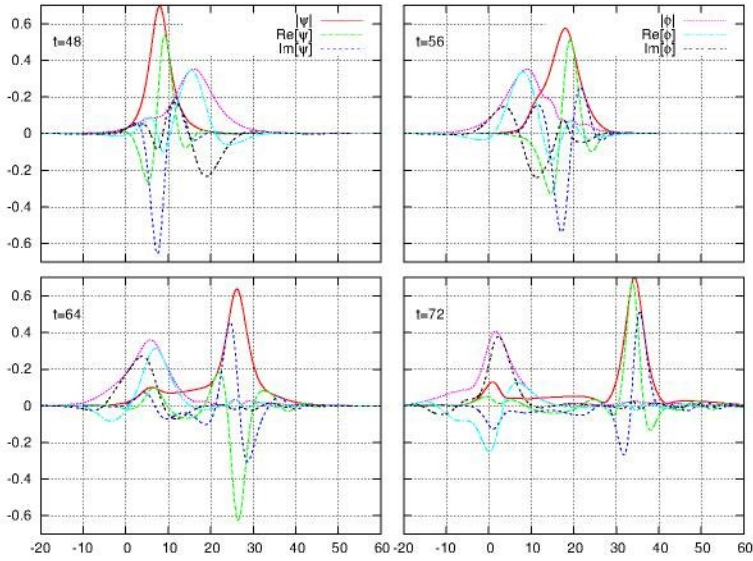
Figure 2. Wave profiles in the cross-section of interaction.

no analytical analog and we will define it as the difference between the actual starting point of a quasi-particle and the starting point of a QP that coincides with the morphed QP after the interaction. In this sense, the phase shift of the smaller soliton is $40-32 = 8$ (because the terminal trajectory is given by $x = 32-0.4218t$. Respectively, the phase shift of the larger QP is 0.8.

As already mentioned when discussing Fig. 2, the modulus does not show the intricate mechanisms of the interaction. Although, the moduli of the two orthogonal modes show the individuality of each QP, the fact that each mode has real and imaginary part is of utter importance during the interaction itself. A rather unexpected feature of the interaction appears to be the change of the carrier frequency. The reader should be reminded here that we chose the initial value of $n = 0$ which allows interpretation for a QP as a moving envelop over a standing wave. If we "ride" the QP which means to introduce a moving coordinate system $x = ct$ we would observe a temporal oscillation with frequency $\omega = c^2/2$ in the moving frame. Yet the modulus will have almost constant amplitude equal to the current maximum. Note that the real and imaginary parts of the solution will oscillate in the moving frame. If, after the interaction, the carrier frequency changes to some $n \neq 0$, the temporal oscillation in the moving frame will have frequency different from the apparent frequency of a single QP when considered in its moving frame.

In Fig. 4 a) we show the temporal frequency in the moving frame of the larger QP (initial phase speed $c_l = 1$). In the time interval before the collision, the frequency
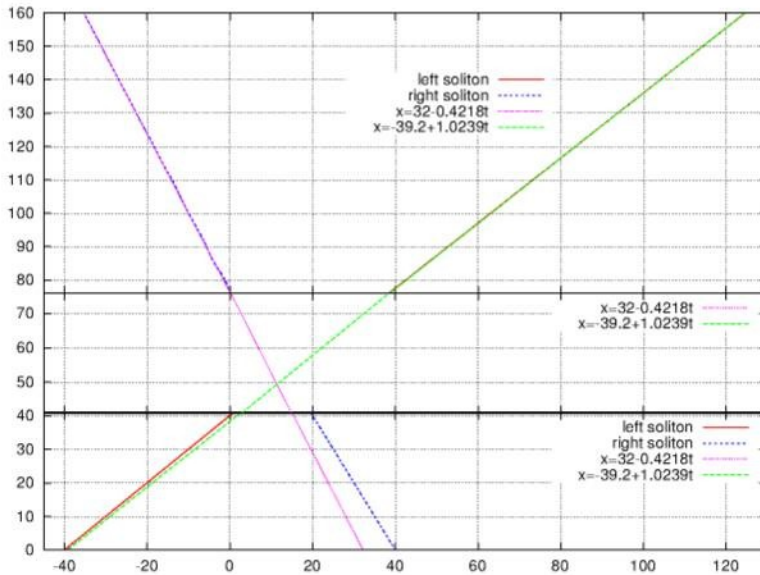
Figure 3. Trajectories of the quasi-particles form Fig. 1.

is equal to $0.5 = 1^2/2$ within the truncation error. During the cross-section of the interaction, there is brief attempt at creation of an orthogonal signal adjacent to the lager QP, but in the long term, the former disappears, and the QP continues as a single soliton with given phase speed and carrier frequency almost equal to zero. After the solitons recover from the collision, the left soliton is already in the right half of the spatial domain and is moving with phase speed 1.0239 (see Fig. 3). At the same, time the direct inspection of the frequency data gives us a frequency in the moving frame 0.482. Since the solitons preserve the general structure of being envelops over some carrier frequency, then for the right-going soliton, the frequency in the absolute coordinate system is related to the frequency in the moving frame as $n = 1.0239^2/2 - 0.482 = 0.0422$ (see Table 1 a)). This means that the right-going soliton acquired some positive carrier frequency as a result of the interaction. This value is not very large, which is another confirmation of the fact that the larger soliton "suffers" insignificantly during the interaction for moderate values of the cross-modulation parameter α_2. The amplitude also does not change very much: it goes down from $\sqrt{2}/2 \approx 0.707$ to 0.680. Because the excited orthogonal soliton has very small amplitude, the polarization after the interaction is virtually equal to the initial polarization of $90°$. In the same fashion we find that the small ϕ-soliton excited with the right-going ψ-soliton has frequency in the moving frame $n_m = 1.83$ which gives for the carrier frequency in the absolute coordinate
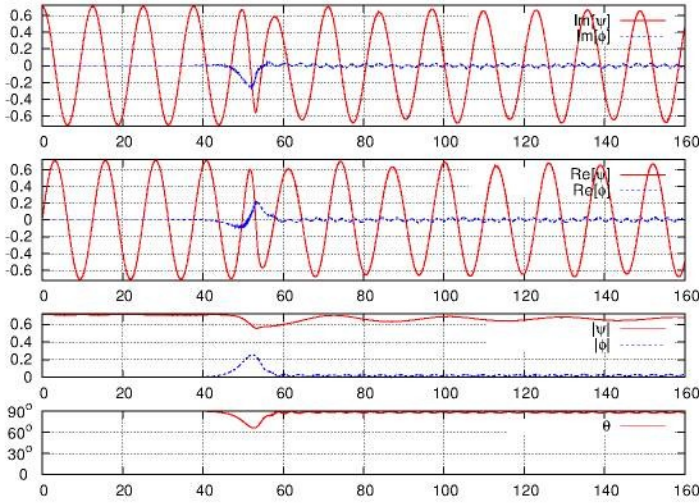
a) soliton moving to the right.
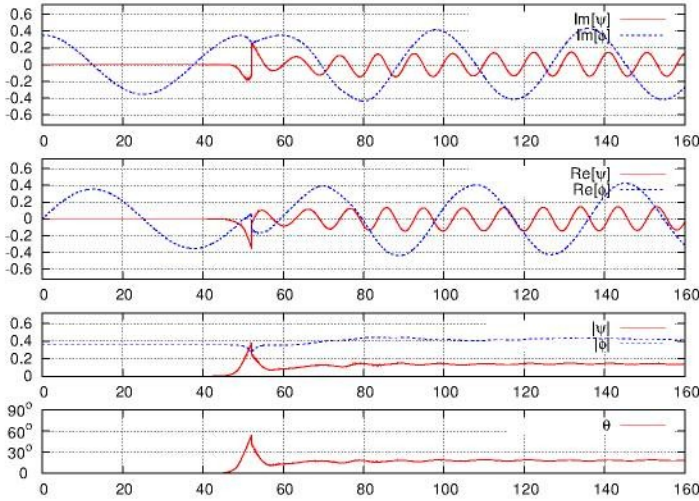
b) soliton moving to the left.

Figure 4. Temporal behavior of solitons in their moving frames.

system $n = 1.024^2/2 - 1.83 = -1.306$, i.e., the satellite ϕ-soliton has negative carrier frequency.

The situation with the smaller QP is quite different. As shown in Fig. 4 b), the smaller QP suffers more from the interaction in the sense that the orthogonal signal that is excited is of much larger amplitude. Respectively, the initial polarization of
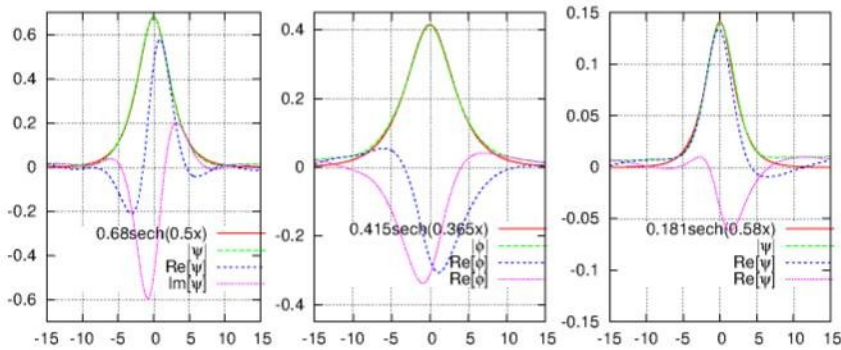
Figure 5. The localized shapes in their moving frames for time $t = 160$. Left: the ψ-mode of the right-going QP. Middle: the main ϕ-mode of the left-going QP. Right: the excited ψ mode for the left going QP.

$0°$ changes to $26°$ after the interaction. The excited orthogonal soliton has not only appreciable amplitude but also much smaller frequency in comparison with the excited soliton for the larger QP. Thus the smaller QP after the interaction becomes quite a different creature: it has what is called elliptic polarization (see [10]). The elliptically polarized soliton has different carrier frequencies and supports for the two components.

In the end, we try to identify the shapes of the QPs after the interaction. Fig. 5 shows the results for the shapes of the QPs in the moving frames for the last moment of time $t = 160$.

The left panel of Fig. 5 presents the shape attained by the larger QP. A best fit with a *sech* function allows us to judge whether it has a shape similar to the analytical solution. The larger QP has $b \approx 0.5$ which is similar to the original support parameter, but the amplitude of the main soliton is actually smaller than the original amplitude, at the time when the phase speed is larger. Clearly, the shape to which the larger QP morphed after the interaction does not comply with the formulas equation (7). The issues connected with polarization require a special investigation and will be pursued elsewhere. The most important conclusion here is that the larger QP preserves its identity undergoing only a slight deformation.

The middle panel of Fig. 5 shows the ϕ-component of the QP which moves to the left and did actually have a nontrivial ϕ-amplitude in the initial moment of time. Finally, the third panel of Fig. 5 shows the excited by the interaction ψ-soliton that travels to the left with the ϕ soliton. The situation with the smaller QP is also not described by equation (7), because the supports of the *sech*-shapes for the ψ- and ϕ-solitons are different. To the limitation of our knowledge, an analytical

solution with different supports for the two orthogonal solitons is not available in the literature. In [10], the shapes of the two components of the elliptically polarized soliton are found numerically and they agree with the shapes found in the present work.

Table 1. Quasi-Particles (QPs) for $c_l = 1, c_r = -0.5$ when $\alpha_2 = 2$.

a) Best-fit Parameters.

soliton	collision	component	n	c	n_m	A	b
right going	before	ψ (given)	0.0	1.	0.5	0.707	0.5
	after	ψ	0.0422	1.0239	0.482	0.680	0.5
		ϕ	-1.306		1.83	0.031	N/A
left going	before	ϕ (given)	0.0	0.5	0.125	0.354	0.25
	after	ϕ	-0.079	0.4218	0.168	0.415	0.365
		ψ	-0.554		0.643	0.181	0.580

b) Properties of QPs before and after the collision.

soliton	M	c	E_k	E_p	E	P
Before collision						
left (right going)	1.0	1.0	0.5000	-0.1667	0.3333	1.00
right (left going)	0.5	-0.5	0.0625	-0.0208	0.0417	-0.25
total	1.5	0.5[†]	0.5625	-0.1875	0.3750	0.75
After collision						
right (right going)	0.9258	1.0239	0.4854	-0.1343	0.3511	0.9480
left (left going)	0.5283	-0.4218	0.0470	-0.0804	-0.0334	-0.2230
total	1.4541	0.4986[†]	0.5324	-0.2147	0.3177	0.7250[†]

[†] the speed of the center of mass $c = P_{\text{total}}/M_{\text{total}}$.

We organize the above results in the Table 1 a) where the main parameters of QPs before and after the interaction are presented.

On the basis of the best-fit parameters, as identified in Table 1 a), the mechanical characteristics of the QP after the interaction can be computed. In Table 1 b) we present these properties before and after the collision. The latter are computed using the formulas form the Appendix B in which the best-fit parameters from Table 1 a) are introduced. It is seen that the QPs emerge with different phase speeds, masses, energies, and pseudomomenta. They are surrounded by a pool of small oscillations that also have mass, energy and pseudomomentum. Since, the total energy of the system of quasi-particles and oscillations is strictly conserved by our scheme, the mismatch between the values of the main characteristics before and after the collision can be attributed to the effect of the radiation. It is interesting to note, that the total mass of the quasi-particles is slightly decreased from 1.5 to 1.454, which means that the mass lost in the oscillations is 0.056. The total energy is decreased down to 0.3177 from the initial value of 0.375. This means that the oscillations carried away not just part of the mass, but also part of the energy. For

the kinetic energy we find (see Table 1 b)) that it is reduced after the collision, as it should have been expected. The interesting observation is that the potential energy of the QPs becomes more negative after the collision which results in even larger decrease of the total energy. We can identify the potential energy as the "internal" energy of the particles.

Finally, we mention that the pseudomomentum of the system of quasi-particles is reduced from 0.75 to 0.725 as a result of the collision. This means that the radiation also evacuates some 3% of the wave momentum.

4. Moderate Interaction

For consistency we keep the grid parameters the same and the values of the initial phase speeds are once again $c_{\text{left}} = 1$, $c_{\text{right}} = -0.5$ and increase the cross-modulation parameter $\alpha_2 = 6$.
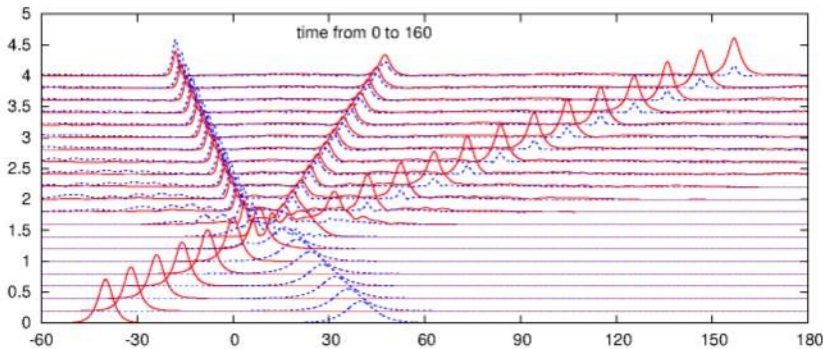
Figure 6. Head-on collision for $c_{\text{left}} = 1$, $c_{\text{right}} = -0.5$; $\alpha_2 = 6$. The two type of lines present $|\psi|$ and $|\phi|$, respectively.

Increasing the latter to 4 does not change qualitatively the dynamics of QP. The difference from the case $\alpha_2 = 2$ is mostly in the increased amplitude of the excited orthogonal solitons, especially the ψ-soliton accompanying the smaller left-going ϕ soliton. In addition the support of the left-going soliton becomes shorter, and the soliton further slows down. The qualitative change of the dynamics is observed for $\alpha_2 = 6$. As shown in Fig. 6, apart from the increased amplitudes of the excited accompanying components, a third QP appears as a result of the interaction.

The general tendency that the larger QP acquires larger velocity, and the smaller QP acquires smaller phase speed and becomes much narrower is preserved, but now a third QP appears after the collision. There was some small hint at this effect even for the weak interaction, but now it is more pronounced. It is interesting that
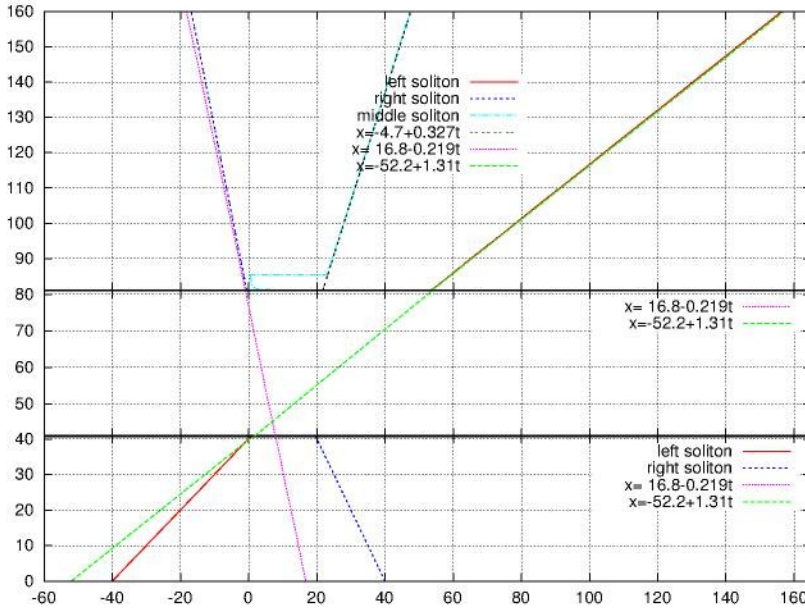
Figure 7. Trajectories of the quasi-particles form Fig. 6.

the third QP is moving to the right and its appearance did not seem to slow the right going QP. In fact, the latter acquires even faster phase speed (see Fig. 7).

The kinetic energy of the larger QP increases after the interaction while the internal energy of the smaller particle decreases so much that it becomes negative (to a smaller extent this effect is observed also for the weak interaction, $\alpha_2 = 2$). In a sense, the internal energy of the smaller QP is converted to kinetic energy of the larger particle and is also used to create a new QP between the two main QPs. We can call this "recoil effect".

Now, we investigate the motion of the solitons in their moving frames. The results are similar to the case $\alpha_2 = 2$, but the effect of the increase of the frequency in the moving frame is much more pronounced (see Table 2). The most interesting observation is that for the newly-born QP (the third soliton) the amplitudes of the ψ-component is slightly larger than the ϕ-component, but the carrier frequency of the latter is much higher. This situation is depicted in Fig. 8. The third QP has somewhat larger support than the left-going one. Similarly to the latter, the former exhibits almost equal amplitudes for ψ and ϕ solitons and almost equal supports. Thus, the third QP looks closer to the analytical solution of type of equation (7).

Following the procedures described in the previous section for finding the best-fit parameters we compiled a similar information for the QPs and is presented in Table 2.
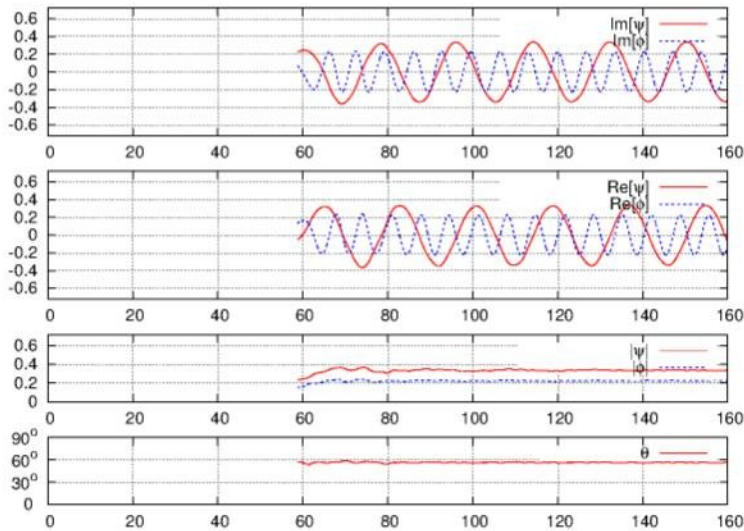
Figure 8. Temporal behavior in its moving frame of the newly-created QP after it appears for $t \approx 60$ for the case from Fig. 6.

On can see from Table 2 that the energy of the left-going QP is negative, $E = -0.4020$ as computed on the basis of the best-fit parameters. This raises the question of the reliability of the result. In order to verify the latter, we clipped the region around the left-going QP and computed numerically the energy and the other characteristics directly from the available profile. We have found that the directly evaluated energy is -0.3878 which confirms the validity of the best-fit formulas. Respectively, for the directly computed mass we got $M = 0.413$ which is in very good agreement with the best-fir result of 0.4231 (see the respective entry of Table 2.

A natural question arises here about the symmetry of the interaction, namely what will happen if the initial configuration of the solitons is perfectly symmetric. The expectation is that the third QP will stay in the origin of the coordinate system, i.e., a standing soliton should be born. Indeed, the computations confirmed this supposition (see Fig. 9).

The fact that the third QP is a standing soliton is confirmed by Fig. 10 where one sees that the real and the imaginary parts of the solution are strictly synchronized. Table 3 presents the details about the properties of the three QPs for this case.

The interesting observation here is that the polarization of the standing soliton is linear with $\theta = 45°$ which means that the amplitudes of the ψ- and ϕ-components are equal. At the same time the polarization of the moving solitons is elliptic.

Table 2. Quasi-Particles (QPs) for $c_l = 1, c_r = -0.5$ when $\alpha_2 = 6$.

a) Best-fit Parameters.

soliton	collision	component	n	c	n_m	A	b
right going	before	ψ (given)	0.0	1.	0.5	0.707	0.5
	after	ψ	0.4028	1.31	0.4553	0.62	0.58
		ϕ	-1.8738		2.7318	0.158	1.231
left going	before	ϕ (given)	0.0	-0.5	0.125	0.354	0.25
	after	ϕ	-0.9894	-0.219	1.0134	0.59	1.1
		ψ	-2.6497		2.6737	0.4	1.5
newly born (middle)	before	none	0.0	0.0	0.0	0.0	0.0
	after	ϕ	-0.8315	0.327	0.885	0.23	0.86
		ψ	-0.2935		0.347	0.34	0.63

b) Properties of QPs before and after the collision.

soliton	M	c	E_k	E_p	E	P
Before collision						
left (right going)	1.0	1.0	0.5000	-0.1667	0.3333	1.00
right (left going)	0.5	-0.5	0.0625	-0.0208	0.0417	-0.25
total	1.5	$0.5^{\dagger}$	0.5625	-0.1875	0.3750	0.75
After collision						
right (right going)	0.6830	1.31	0.5861	-0.1848	0.4013	0.8948
left (left going)	0.4231	-0.219	0.0102	-0.4122	-0.4020	-0.0927
middle (newly born)	0.2450	0.327	0.0131	-0.0797	-0.0666	0.0801
total	1.3512	$0.6529^{\dagger}$	0.6093	-0.6767	-0.0673	0.8822

† the speed of the center of mass $c = P_{\text{total}}/M_{\text{total}}$.
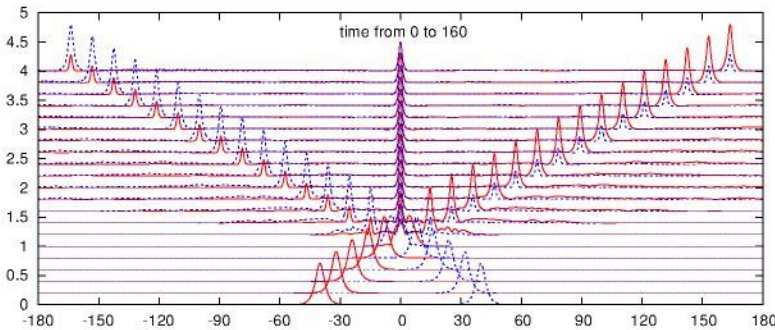
Figure 9. $\alpha_2 = 6, c_l = -c_r = 1$.

5. Strong Interaction

Clearly, the increase of the interaction parameter α_2 makes the phenomenology of the interaction much richer. We went even further and investigated also the
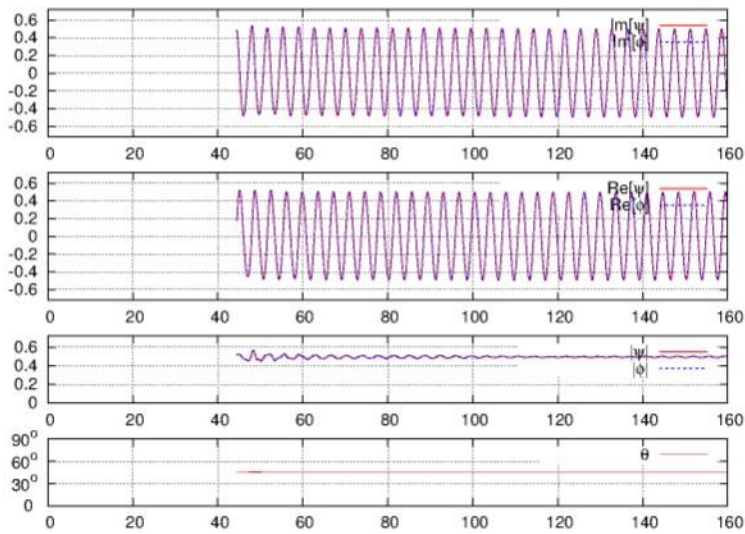
Figure 10. Evolution of the standing soliton in the origin of the coordinate system for the case from Fig. 9.

case $\alpha_2 = 10$ with the same grid parameters and the same phase speeds. Fig. 11 presents the interaction for the moduli of the solutions. The qualitative difference with the case of moderate interaction is that now two new solitons are born after the collision. The shapes of the new solitons are not very intuitive. The one that goes to the right is much more prolate (with longer support), while the one going to the left is steeper.
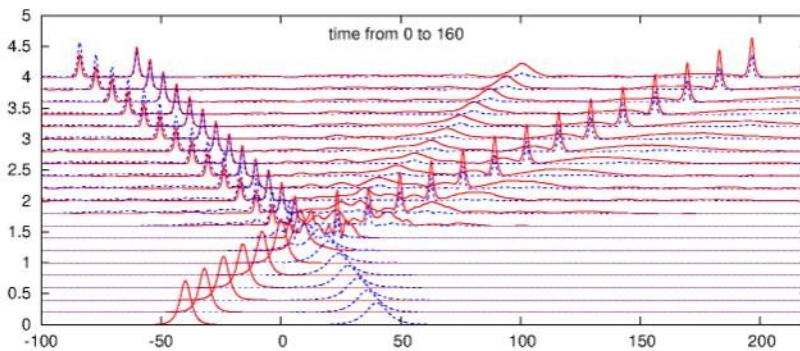
Figure 11. $\alpha_2 = 10$, $c_l = 1$, $c_r = -0.5$.

Table 3. Quasi-Particles (QPs) for $c_l = 1, c_r = -1$ when $\alpha_2 = 6$.

a) Best-fit Parameters.

soliton	collision	component	n	c	n_m	A	b
right going	before	ψ (given)	0.0	1	0.5	0.707	0.5
	after	ψ	-0.1981	1.325	1.0759	0.7971	0.87
		ϕ	-5.4054		6.2832	0.2766	1.71
left going	before	ϕ (given)	0.0	-1	0.5	0.7071	0.5
	after	ϕ	-0.1981	-1.325	1.0759	0.7971	0.87
		ψ	-5.4054		6.2872	0.2766	1.71
standing	before	none	0.0	0.0	0.0	0.0	0.0
	after	ϕ	-1.6755	0	1.6755	0.4982	1.31
		ψ	-1.6755		1.6755	0.4982	1.31

b) Properties of QPs before and after the collision.

soliton	M	c	E_k	E_p	E	P
Before collision						
left (right going)	1.0	1.0	0.5000	-0.1667	0.3333	1.00
right (left going)	1.0	-1.0	0.5	-0.1667	0.3333	-1.00
total	2.0	$0^\dagger$	1.0	-0.3333	0.6667	0.0
After collision						
right (right going)	0.775	1.325	0.6804	-0.5091	0.1713	1.027
left (left going)	0.775	-1.325	0.6804	-0.5091	0.1713	-1.027
standing	0.3729	0.0	0.0	-0.4443	-0.4443	0.0
total	1.9229	$0.0^\dagger$	1.3608	-1.4624	-0.1017	0.0

† the speed of the center of mass $c = P_{\text{total}}/M_{\text{total}}$.

The details of the interaction are presented in Table 5. To follow more than one newly born QP is algorithmically complicated and we left out the issue of the carrier frequency of the fourth QP (the right newly born one).

The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP. As already mentioned, the left going secondary soliton is much steeper and this results into negative potential energy commensurate with the potential energies of the main left and right solitons. Yet, the right-going secondary soliton is a rather different creature. Its support is much larger and, as a result, the stored elastic strain is smaller, manifesting itself into less negative potential energy. As a result the total energy of this QP is positive, albeit small while the left-going ones have negative total energies. When α_2 is large, the system is much more excitable than for moderate and small values of α_2 and the considerable deformations of the profile during a collision can become the seeds of shapes that can evolve into quite different QPs during the cross-section of the collision. As above mentioned, the total energy of the QPs is radically different from the total energy of the initial wave profile. The

Table 4. Integral Characteristics of Excitations.

	Left forerunner	Between QP-2 and QP-3	Between QP-3 and QP-4	Right forerunner
M	0.0431	0.0269	0.05908	0.1969
P	0.0082	0.0083	0.07669	0.4087
E	0.2215	0.0044	0.0045	0.7225

differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.

In order to confirm this conjecture we consider the final time stage from Fig. 11, clip away the main QPs and calculate the energy of the wave profile that is left in the reduced region. As one can see, the amplitude of the radiation that dwells in the mentioned region is rather small. Yet the energy of the oscillations is very large, particularly, the kinetic energy. For the case depicted in Fig. 11 we have computed the different integral characteristics (mass, pseudomomentum, and energy) of the excitations that appear between the main QPs and to the left and to the right of them (the forerunners). We enumerate the different QPs from left to right, namely the leftmost soliton is QP-1, and the rightmost is QP-4. The results are organized in Table 4 for the different regions, save the small interval between QP-1 and QP-2 whose characteristics are negligible.

The masses and the pseudomomenta of the excitations are commensurate with their relative importance for the amplitudes of the total wave profile. The important result in Table 4 is that the predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region. Strictly speaking the forerunners are not QPs because the shapes of their envelops evolve in time. Yet they are localized "creatures" and have similar properties to the QPs as far as energy, mass, and pseudomomenta are concerned.

Finally, we add together the energies of all four QPs and the excitations alike and get results which are within 20% of the total energy as conserved of the scheme. This means that the large negative energies of the QPs are the result of the evacuation of energy by the forerunners and is not a numerical effect. This kind of transformation is a physical effect that is connected with the excitability of the system and is not present in the case of a single Schrödinger equation. Energy transformation is a specific trait of the coupled system considered here.

6. Conclusion

In this paper we develop a complex-arithmetic implementation of a conservative difference scheme for Coupled Nonlinear Schrödinger Equations (CNLSE). To this end a special solver for Gaussian elimination with pivoting is developed for inverting five-diagonal complex-valued matrices, which is a generalization of the solver created earlier by one of the authors. The new solver allows us to use grids of considerable sizes (up to 20000 grid points) and small time increments and to obtain thus a reliable approximation.

The algorithm is validated by the mandatory numerical tests involving doubling the mesh size and halving the time increment, as well as by the direct comparison in couple of cases with the real-arithmetic schemes [4,9]. The advantages of the new scheme are that the band of the matrix is twice smaller and that the overall number of unknowns is also twice smaller. Our numerical tests show that, as expected, it performs four times faster than the scheme from [4,9].

The new tool developed here allowed us to investigate physically important sets of parameters of the CNLSE and to investigate the role of nonlinear coupling in the quasi-particle dynamics. The nonlinear coupling results in changing the original polarization of the two signals from vertical/horizontal to a generally slanted one. This means that although the initial conditions are nontrivial for only one of the functions in each of the initial locations, after the interaction both functions acquire nontrivial amplitudes in both locations.

We consider as an initial profile the superposition of solitons with linear polarizations, one of them having only ψ-component, and the other - only ϕ-component. Then this initial profile is allowed to evolve according to the system of Coupled Schrödinger equations (CNLSE) and the results of the collision depend mostly on nonlinear coupling parameter α_2 (cross-modulation parameter) on the dynamics of quasi-particles (QP). We have found that for $\alpha_2/\alpha_1 < 4$ the collision of the two initial solitons of linear polarization produces only two solitons that are of different polarization. The smaller soliton suffers more from the interaction and its polarization becomes elliptic. In addition the new QPs that are born after the collisions have slightly different phase speeds.

For moderate value of the cross-modulation parameter, $\alpha_2/\alpha_1 = 6$, we have found that an additional QP is born which propagates in the direction of the faster QP, while the initially smaller QP considerably reduces its phase speed. The initially faster QP becomes even faster. The effect of the nonlinearity is so profoundly felt that even the faster QP becomes elliptically polarized, although to a smaller extent than the slower QP and the new-born QP. If the initial QPs have the same phase speeds, the new-born QP is a standing soliton with linear polarization of 45%. In the moderate case, we observed that the energies of the initially slower QP and the

Table 5. Quasi-Particles (QPs) for $c_l = 1, c_r = -0.5$ when $\alpha_2 = 10$.

a) Best-fit Parameters.

soliton	collision	component	n	c	n_m	A	b
right going	before	ψ (given)	0.0	1.	0.5	0.707	0.5
	after	ψ	-0.5221	1.6625	1.904	0.6509	1.3
		ϕ	-4.9012		6.2832	0.3718	2.0
left going	before	ϕ (given)	0.0	-0.5	0.125	0.354	0.25
	after	ϕ	-1.0351	-0.85	1.3963	0.5709	1.25
		ψ	-3.8276		4.1888	0.3628	1.758
middle left going	before	none	0.0	0.0	0.0	0.0	0.0
	after	ϕ	-2.7726	-0.725	3.0354	0.428	1.6
		ψ	-1.6412		1.904	0.4992	1.4
middle right going	before	none	0.0	0.0	0.0	0.0	0.0
	after	ϕ	N/A	0.7875	N/A	0.092	0.65
		ψ	N/A		N/A	0.2363	0.345

b) Properties of QPs before and after the collision.

soliton	M	c	E_k	E_p	E	P
Before collision						
left (right going)	1.0	1.0	0.5000	-0.1667	0.3333	1.00
right (left going)	0.5	-0.5	0.0625	-0.0208	0.0417	-0.25
total	1.5	0.5[†]	0.5625	-0.1875	0.3750	0.75
After collision						
right (right going)	0.3950	1.6625	0.5459	-0.5408	0.0051	0.6567
left (left going)	0.3356	-0.85	0.1212	-0.4360	-0.3148	-0.2853
middle (right going)	0.1749	0.7875	0.0542	-0.0162	0.038	0.1377
middle (left going)	0.2925	-0.725	0.0769	-0.4677	-0.3908	-0.2121
total	1.198	0.2479[†]	0.7982	-1.4607	-0.6625	0.297

[†] the speed of the center of mass $c = P_{\text{total}}/M_{\text{total}}$.

new-born QP are negative. This is due to the fact that they are steeper (with smaller support) and the potential energy (which is the stored elastic energy) becomes very negative and dominates the kinetic energy. This means that part of the energy is taken wavy by the after the interaction by the excited radiation.

Finally, we consider the case of large value of cross-modulation parameter, say $\alpha_2/\alpha_1 = 10$ and find that the dynamics changes even more radically. Now two new QPs are born, accompanying the two initial QPs which are also radically transformed in the sense of polarization and energy. Now all four QPs have elliptic polarizations and negative total energies. In additions fast forerunners with large positive (mostly kinetic) anergy are born which preserves the total energy of

the system. In other words the radiation in form of forerunners evacuates positive energy leaving the system of QPs with negative total energy.

These results are illustrated graphically and the data about properties of the QPs are organized in tables.

7. Acknowledgements

Appendix A. Conservative Difference Scheme in Complex Arithmetic

Creating and validating a numerical scheme based on complex-number arithmetic is important for the future application of the conservative-scheme approach in two spatial dimensions. The system for the real and imaginary parts of the wave function has an intricate form that precludes using operator splitting in the real-valued version of the algorithm. Without splitting the needed computational resources in 2D are enormous which makes the approach rather unpractical. In this instance, the present approach can be a good basis development of a 2D numerical scheme based on operator splitting.

For the sake of selfcontainedness of the paper we present here the scheme developed in [12]. Consider a uniform mesh in the interval $[-L_1, L_2]$

$$x_i = (i-1)h, \qquad h = (L_1 + L_2)/(N-1) \qquad \text{and} \quad t^n = n\tau$$

where N is the total number of grid points in the interval and τ is the time increment. Respectively, ψ_i^n and ϕ_i^n denote the value of the ψ and ϕ at the ith spatial point and time stage t^n. Clearly, $n = 0$ refers to the initial conditions.

Our purpose is to create a difference scheme that is not only convergent (consistent and stable), but also reflects the conservation laws equation (6). A scheme that

satisfies them reads

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1} - \psi_i^n}{\tau} =& \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\right) \\
&+ \frac{\psi_i^{n+1} + \psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right) + (\alpha_1 + 2\alpha_2)\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right)\right] \\
&\hspace{2cm} + \frac{\gamma}{2}(\psi_i^n + \psi_i^{n+1}) + \frac{\Gamma}{2}(\phi_i^n + \phi_i^{n+1})
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\mathrm{i}\frac{\phi_i^{n+1} - \phi_i^n}{\tau} =& \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\right) \\
&+ \frac{\phi_i^{n+1} + \phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right) + (\alpha_1 + 2\alpha_2)\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right)\right] \\
&\hspace{2cm} + \frac{\gamma}{2}(\phi_i^n + \phi_i^{n+1}) + \frac{\Gamma}{2}(\psi_i^n + \psi_i^{n+1}).
\end{aligned} \tag{10}
$$

We have proved that the scheme equations (9),(10) conserves the discrete analogs of mass and energy, from (6). Namely, for all $n \geq 0$, we have

$$
M^n = \sum_{i=2}^{N-1}\left(|\psi_i^n|^2 + |\phi_i^n|^2\right) = \text{const}
$$

$$
\begin{aligned}
E^n =& \sum_{i=2}^{N-1}\frac{\beta}{2h^2}\left(|\psi_{i+1}^n - \psi_i^n|^2 + |\phi_{i+1}^n - \phi_i^n|^2\right) - \frac{\alpha_1}{4}\left(|\psi_i^n|^4 + |\phi_i^n|^4\right) \\
&- \frac{\alpha_1 + 2\alpha_2}{2}\left(|\psi_i^n|^2|\phi_i^n|^2\right) - \frac{\gamma_r^\phi}{2}|\phi_i^n|^2 - \frac{\gamma_r^\psi}{2}|\psi_i^n|^2 - \Gamma\Re[\overline{\phi}_i^n \psi_i^n] = \text{const}.
\end{aligned}
$$

The scheme equations (9), (10) cannot be implemented directly, because it is non-linear with respect to the variables ψ_i^{n+1} and ϕ_i^{n+1}. We follow the idea of [4] and introduce internal iterations, namely

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1} - \psi_i^n}{\tau} =& \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1,k+1} - 2\psi_i^{n+1,k+1} + \psi_{i+1}^{n+1,k+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\right) \\
&+ \frac{\psi_i^{n+1,k} + \psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2\right)\right. \\
&\left. + (\alpha_1 + 2\alpha_2)\left(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2\right)\right] \\
&+ \frac{\gamma}{2}(\psi_i^n + \psi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\phi_i^n + \phi_i^{n+1,k+1})
\end{aligned} \tag{11}
$$

$$i\frac{\phi_i^{n+1,k+1} - \phi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1,k+1} - 2\phi_i^{n+1,k+1} + \phi_{i+1}^{n+1,k+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\right)$$
$$+\frac{\phi_i^{n+1,k} + \phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2\right)\right.$$
$$+(\alpha_1 + 2\alpha_2)\left(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2\right)\right]$$
$$+\frac{\gamma}{2}(\phi_i^n + \phi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\psi_i^n + \psi_i^{n+1,k+1}).$$

(12)

Now for the current iteration of the unknown functions (superscript $n + 1, k + 1$) we have an implicitly coupled system of two tridiagonal systems with complex coefficients. The coupling here is essential. Without it, it is not possible to ensure the absolute stability of the scheme.

We conduct the internal iterations (repeating the calculations for the same time step $(n+1)$ with increasing value of the superscript k until convergence, i.e., when both the following criteria are satisfied

$$\max_i |\psi_i^{n+1,k+1} - \psi_i^{n+1,k}| \leq 10^{-12} \max_i |\psi_i^{n+1,k+1}|$$
$$\max_i |\phi_i^{n+1,k+1} - \phi_i^{n+1,k}| \leq 10^{-12} \max_i |\phi_i^{n+1,k+1}|.$$

(13)

After the internal iterations converge, one gets $\psi^{n+1} \equiv \psi^{n+1,k+1}$ and $\phi^{n+1} \equiv \phi^{n+1,k+1}$ which are the solutions of the nonlinear implicte scheme, equations (9), (10) . We mention here that for physically reasonable time increments τ the number of internal iterations needed for convergence is four to six, which is a small price to pay to have fully implicit, nonlinear and conservative scheme.

As mentioned above, the linearized scheme equations (11), (12) is inextricably coupled. In order to be solved, the respective two tridiagonal linear algebraic systems are to be recast as a single five-diagonal system (see for details [12]. For the inversion of the five-diagonal $2N \times 2N$ matrix we created an algorithm based on Gaussian elimination with pivoting which is a generalization of a similar algorithm for real system developed in [3]. The details of the new algorithm and the FORTRAN code will be published elsewhere. As it should have been expected, the computational time needed for the scheme with complex arithmetic to complete the calculations for a given set of parameters is four times shorter than the scheme with real arithmetic.

The scheme was thoroughly validated through the standard numerical tests involving halving the spacing and time increment. The results confirm the second order of accuracy in space and time. Another crucial validation is possible, through direct comparison with the results [4, 9]. We did repeat a couple of the more involved cases using both the scheme with real arithmetic and the presented here scheme

with complex arithmetic. The results with the same scheme parameters turn out to indistinguishable within the order of round-off error.

Appendix B. Analytic Expressions for Conserved Properties

Let us assume here that the two components of the vector soliton are moving together without changing their relative position. Then for both ψ and ϕ components we have the same phase speed, c, but the amplitudes, A, size of support, b, and frequencies, n can be different. A propagating soliton of this type is described by the following formulas

$$\psi = A_\psi \text{sech}[b_\psi(x - ct)] \exp\left[i\left(\frac{c}{2\beta}x - n_\psi t\right)\right]$$

$$\phi = A_\phi \text{sech}[b_\phi(x - ct)] \exp\left[i\left(\frac{c}{2\beta}x - n_\phi t\right)\right]. \tag{14}$$

When the compound solution, the vector (ψ, ϕ), propagates steadily, the above parameters are related as in equation (7)

For solution of type equation (14) the mass is given by the following formula

$$M = \frac{1}{2\beta}\int_{-\infty}^{\infty}(|\psi|^2 + |\phi|^2)\mathrm{d}x = \frac{1}{2\beta}A_\psi^2\int_{-\infty}^{\infty}\text{sech}^2[b_\psi(x - ct)]\mathrm{d}x$$

$$+\frac{1}{2\beta}A_\phi^2\int_{-\infty}^{\infty}\text{sech}^2[b_\phi(x - ct)]\mathrm{d}x = \frac{1}{2\beta}\frac{A_\psi^2}{b_\psi}\tanh[b_\psi(x - ct)]\Big|_{-\infty}^{\infty} \tag{15}$$

$$+\frac{1}{2\beta}\frac{A_\phi^2}{b_\phi}\tanh[b_\phi(x - ct)]\Big|_{-\infty}^{\infty} = \frac{1}{\beta}\left(\frac{A_\psi^2}{b_\psi} + \frac{A_\phi^2}{b_\phi}\right).$$

For the wave momentum (pseudomomentum) we get in a similar fashion

$$P = \int_{-\infty}^{\infty}\Im[\psi\bar{\psi}_x + \phi\bar{\phi}_x]\mathrm{d}x$$

$$= \int_{-\infty}^{\infty}\left(\frac{A_\psi^2 c}{2\beta}\text{sech}^2[b_\psi(x - ct)] + \frac{A_\phi^2 c}{2\beta}\text{sech}^2[b_\phi(x - ct)]\right)\mathrm{d}x$$

$$= \frac{1}{\beta}\left(\frac{A_\psi^2}{b_\psi} + \frac{A_\phi^2}{b_\phi}\right)c \equiv Mc. \tag{16}$$

The momentum of the quasi-particle is exactly the product of the mass and the phase speed.

Now, for the energy we get

$$
\begin{aligned}
E = \int\limits_{-\infty}^{\infty} & \left[\beta(|\psi_x|^2 + |\phi_x|^2) - \frac{1}{2}\alpha_1 \left(|\psi|^4 + |\phi|^4 \right) \right. \\
& \left. - (\alpha_1 + 2\alpha_2)(|\phi|^2|\psi|^2) - \gamma(|\psi|^2 + |\phi|^2) - 2\Gamma\mathcal{R}(\bar{\psi}\psi) \right] \mathrm{d}x \\
= \frac{2}{3}&\beta(A_\psi^2 b_\psi + A_\phi^2 b_\phi) + \frac{c^2}{2\beta}\left(\frac{A_\psi^2}{b_\psi} + \frac{A_\phi^2}{b_\phi} \right) - \frac{2}{3}\alpha_1 \left(\frac{A_\psi^4}{b_\psi} + \frac{A_\phi^4}{b_\phi} \right) \\
& - (\alpha_1 + 2\alpha_2)A_\psi^2 A_\phi^2 \int\limits_{-\infty}^{\infty} \mathrm{sech}^2[b_\psi(x - ct)]\mathrm{sech}^2[b_\phi(x - ct)]\mathrm{d}x \\
& - 2\left[\gamma\left(\frac{A_\psi^2}{b_\psi} + \frac{A_\phi^2}{b_\phi} \right) + 2\Gamma\frac{A_\psi^2}{b_\psi} \right].
\end{aligned}
$$

In the above formula one of the integrals cannot be found analytically directly unless $b_\psi = b_\phi$. Yet, we can have a reasonable approximation after we note that

$$
\mathrm{sech}^2[b_\psi(x - ct)]\mathrm{sech}^2[b_\phi(x - ct)]\mathrm{d}x
$$

$$
= \frac{4}{\{\cosh[(b_\psi + b_\phi)(x - ct)] + \cosh[(b_\psi - b_\phi)(x - ct)]\}^2}
$$

and that for the cases treated in this work $b_\psi \approx b_\phi$, i.e., $|b_\phi - b_\psi| \ll |b_\phi + b_\psi|$. In such a case, in the region where $\cosh[(b_\psi - b_\phi)(x - ct)]$ changes rapidly, one can assume that $\cosh[(b_\psi - b_\phi)(x - ct)] \sim 1$. Then

$$
\begin{aligned}
\int\limits_{-\infty}^{\infty} & \frac{4\mathrm{d}x}{\{\cosh[(b_\psi + b_\phi)(x - ct)] + \cosh[(b_\psi - b_\phi)(x - ct)]\}^2} \\
& \approx \int\limits_{-\infty}^{\infty} \frac{4\mathrm{d}x}{[1 + \cosh(b_\psi + b_\phi)(x - ct)]^2} \tag{17} \\
& = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^4\left[\frac{1}{2}(b_\psi + b_\phi)(x - ct) \right]} = \frac{8}{3}\frac{1}{b_\psi + b_\phi}.
\end{aligned}
$$

The accuracy of this formula can easily be verified for a couple of specific values of b_ψ and b_ϕ for which the original integral can also be found analytically. For specific ratios b_ϕ/b_ψ, both the approximate and exact integrals can be represented as κ/b_ψ, where κ is a different coefficient for the different cases. The results for

several different ratios b_ϕ/b_ψ are presented in Table 6. Clearly, up to $b_\phi = 2b_\psi$ the approximation is very reasonable and can be used to get an approximate analytical expression for the integral and compare the energy to the numerically obtained value.

Table 6. Comparison of the values for κ as obtained for the approximate and exact integrals for several ratios b_ϕ/b_ψ, for which the the exact integral can be solved analytically.

b_ϕ/b_ψ	1	1.25	1.3333	1.5	1.6666	2
approx.	1.3333	1.1852	1.1428	1.06667	1	0.8889
exact	1.3333	1.1805	1.1355	1.05333	0.9805	0.8584
error [%]	0	0.40	0.64	1.27	1.99	3.55

Finally, under the above assumption that the scales of the supports for the two components are not very different, we have the following analytical approximation of the energy

$$
E \approx \frac{c^2}{2}\left[\frac{1}{\beta}\left(\frac{A_\psi^2}{b_\psi}+\frac{A_\phi^2}{b_\phi}\right)\right] + \frac{2}{3}\beta(A_\psi^2 b_\psi + A_\phi^2 b_\phi) - \frac{2}{3}\alpha_1\left(\frac{A_\psi^4}{b_\psi}+\frac{A_\phi^4}{b_\phi}\right)
$$
$$
- \frac{8}{3}(\alpha_1+2\alpha_2)\frac{A_\psi^2 A_\phi^2}{b_\psi+b_\phi} - 2\left[\gamma\left(\frac{A_\psi^2}{b_\psi}+\frac{A_\phi^2}{b_\phi}\right)+2\Gamma\frac{A_\psi^2}{b_\psi}\right]. \quad (18)
$$

The term $\frac{c^2}{2}\left[\frac{1}{\beta}\left(\frac{A_\psi^2}{b_\psi}+\frac{A_\phi^2}{b_\phi}\right)\right] \equiv \frac{Mc^2}{2}$ can be called the "kinetic energy" of the quasi-particle, while the rest of the terms can be called "internal energy" of the quasi-particle.

References

[1] Agrawal G., *Nonlinear Fiber Optics*, 2nd Edn, Academic Press, San Diego 1989.

[2] Agrawal, G., *Fiber-Optic Communication Systems*, 2nd ed., Wiley, New York 1997.

[3] Christov C., *Gaussian Elimination with Pivoting for Multidiagonal Systems*, Internal Report **4**, University of Reading, UK 1994.

[4] Christov C., Dost S., and Maugin G., *Inelasticity of Soliton Collisions in System of Coupled NLS Equations*, Physica Scripta **50** (1994) 449–454.

[5] Ismail M. and Taha T., *Numerical Simulation of Coupled Nonlinear Schroödinger Equation*, Math. Comp. Simulat. **56** (2001) 547–562.

[6] Marcuse D., Manyuk C. and Wai P., *Application of the Manakov-PMD Equation to Studies of Signal Propagation in Optical Fibers with Randomly Varying Birefringence*, J. Lightw. Technol. **15** (1997) 1735–1746.

[7] Manakov S., *On The Theory of Two-Dimensional Stationary Self-Focusing of Electromagnetic Waves*, Soviet Physics JETP **38** (1974) 248–253.

[8] Menyuk C. and Marks B., *Interaction of Polarization Mode Dispersion and Nonlinearity in Optical Fiber Transmission Systems*, J. Lightw. Technol. **24** (2006) 2806–2826.

[9] Sonnier W. and Christov C., *Strong Coupling of Schrödinger Equations: Conservative Scheme Approach*, Math. Computers Simulat. **69** (2005) 514–525.

[10] Sophocleous C. and Parker D., *Pulse Colliasions and Polarization Conversion for Optical Fibers*, Opt. Commun. **112** (1994) 214–224.

[11] Taha T. and Ablovitz M., *Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. II. Numerical, Schroödinger Equation*, J. Comp. Phys. **55** (1984) 203–230.

[12] Todorov M. and Christov C., *Conservative Scheme in Complex Arithmetic for Coupled Nonlinear Schrödinger Equations*, Discrete and Continuous Synamical Systems, Supl. 2007, pp 982–992.

[13] Zakharov V. and Shabat A., *Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-modulation of Waves in Nonlinear Media*, Soviet Physics JETP **34** (1972) 62–69.

[14] Zakharov V. and Shabat A., *Interaction Between Solitons in a Stable Medium*, Soviet Physics JETP **37** (1973) 823–828.

ON THE QUADRATIC BUNDLES RELATED TO HERMITIAN SYMMETRIC SPACES[*]

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Abstract. Here we develop the direct scattering problem for quadratic bundles associated to Hermitian symmetric spaces. We adapt the dressing method for quadratic bundles which allows us to find special solutions to multicomponent derivative Schrödinger equation for instance. The latter is an infinite dimensional Hamiltonian system possessing infinite number of integrals of motion. We demonstrate how one can derive them by block diagonalization of the corresponding Lax pair.

CONTENTS

1. Introduction

The modern period in the history of integrable systems started with the discovery of the inverse scattering transform (IST) by Gardner, Greene, Kruskal and Miura [7] who solved the Cauchy problem for the Korteweg-de Vries equation. Ever since

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that time the applications of IST increased tremendously – from purely discrete equations to multidimensional partial differential equations [1,22].

Historically the first nonlinear evolution equations (NEEs) solved by means of IST were associated with the scattering operator

$$L(\lambda) = i\partial_x + Q(x, t) - \lambda\sigma_3 \tag{1}$$

where $\lambda \in \mathbb{C}$ is an external parameter called *spectral parameter* and

$$Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ \pm q^*(x, t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The family of operators of this type operator is known as a linear bundle due to its dependence on λ. Since that time the scheme of IST has been extended to matrix Lax operators with a polynomial [6, 21] and even rational λ-dependence [28, 40]. The first step in this direction was done by Mikhailov and Kuznetsov [23, 26] who proved the integrability of the two-dimensional Thirring model. The problem of integrability can be reduced to the study of the following quadratic bundle Lax operator

$$L(\lambda) = i\partial_x - \frac{1}{2}|q|^2\sigma_3 + \lambda Q(x, t) - \lambda^2\sigma_3 \tag{3}$$

where

$$Q = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}. \tag{4}$$

Another equation with a physical application [29, 30] was considered by Kaup and Newell [20] who introduced the Lax operator

$$L(\lambda) = i\partial_x + \lambda Q(x, t) - \lambda^2\sigma_3 \tag{5}$$

where $Q(x, t)$ is again in the form (4). This allowed them to solve the derivative nonlinear Schrödinger equation (DNSE)

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0 \tag{6}$$

and to find integrals of motion for it. The study of DNSE was continued by Gerdjikov, Kulish and Ivanov [14] who developed the generalised Fourier interpretation for DNSE in terms of generating operators, squared solutions etc., have found the action-angle variables for it and thus proved its complete integrability. Later Gerdjikov and Ivanov [10, 11] carried out an exhaustive study of the generic quadratic bundle

$$L(\lambda) = i\partial_x + U_0(x, t) + \lambda U_1(x, t) - \lambda^2\sigma_3 \tag{7}$$

where $U_0(x, t)$ is an arbitrary 2×2 matrix while $U_1(x, t)$ has zero diagonal elements. In the latter papers the existence of Riemann-Hilbert problem with canonical normalization was exploited and its importance was clarified.

Another fruitful idea in the soliton theory is to search for multicomponent equations integrable by means of IST. This trend was pioneered by Zakharov and Manakov [25, 38] who derived a three-wave system and a two-component counterpart of the nonlinear Schrödinger equation. For that purpose they used a 3×3-matrix analogue of the Lax operator (2). Soon it became clear that Lax pairs can be related to homogeneous and symmetric spaces in a very natural way [2, 4, 5]. In [4] Fordy derived multicomponent versions of DNSE related to different Hermitian symmetric spaces amongst which is the following one

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \frac{2\mathrm{i}}{m+1} \left(\left(\mathbf{q}^T \mathbf{q}^* \right) \mathbf{q} \right)_x = 0 \tag{8}$$

where $\mathbf{q} : \mathbb{R}^2 \to \mathbb{C}^m$, $m \geq 2$ is a smooth function. The multicomponent NEEs related to symmetric spaces attracted attention again [12, 13] as a result of recent studies on Bose-Einstein condensates [18, 24, 33].

The aim of the current paper is to build the foundations of the theory of quadratic bundles associated with Hermitian symmetric spaces. In order to do this we are going to use a gauge covariant approach [15]. This will allow us to treat in a uniform manner any quadratic bundle regardless of the structure of the underlying symmetric space.

The paper is organised as follows. In Section 2 we give some basic preliminary facts on quadratic bundles associated with Hermitian symmetric spaces. After introducing the main object of study we proceed with developing the direct scattering problem and discuss the spectral properties of the respective scattering operator. In Section 3 we adapt Zakharov-Shabat dressing method for the case quadratic bundles of the mentioned type. This method allow us to derive particular solutions of multicomponent DNSE. The form of dressing factor depends crucially on the structure of symmetric space. Section 4 is dedicated to Hamiltonian interpretation of DNSE. We prove that there exist infinite number of integrals of motion and present a general recursion formula. In doing this we make use of the method of (block) diagonalization of Lax pair proposed in [3]. Section 5 contains a summary of our results and some additional remarks.

2. Quadratic Bundles Related to Hermitian Symmetric Spaces

The current section is preliminary in nature. Its purpose is to provide an introduction to the direct scattering theory of quadratic bundles related to Hermitian symmetric spaces. In doing this we shall follow some well-known ideas from soliton theory [15, 39].

Firstly we are going to shed light on the relation that exists between Hermitian symmetric spaces and quadratic bundles. Let G/H be a Hermitian symmetric space,

i.e., G is assumed to be a connected simple Lie group[1] and H $\subset$ G is a stabilizer of a typical point $p \in$ G, see [16] for more detailed explanations. The Lie algebra $\mathfrak{g}$ corresponding to the Lie group G obeys the splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{9}$$

where $\mathfrak{h}$ is the subalgebra corresponding to the Lie subgroup H and the subspace $\mathfrak{m}$ represents its complement in $\mathfrak{g}$. Since the homogeneous space G/H is symmetric the following relations

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{10}$$

hold true as well. In other words $\mathfrak{g}$ is $\mathbb{Z}_2$-graded and the subspaces $\mathfrak{h}$ and $\mathfrak{m}$ are eigensubspaces

$$\mathfrak{h} = \{X \in \mathfrak{g} \; ; \; \mathbf{C}X\mathbf{C} = X\}, \qquad \mathfrak{m} = \{X \in \mathfrak{g} \; ; \; \mathbf{C}X\mathbf{C} = -X\}$$

of the adjoint action of an involutive automorphism $X \mapsto \mathbf{C}X\mathbf{C}$ (Cartan's involutive automorphism).

Let us now consider the Lax pair

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J \tag{11}$$

$$A(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{2N} \lambda^k A_k(x,t) \tag{12}$$

where $\lambda \in \mathbb{C}$ is spectral parameter while $Q(x,t)$, J and A_k belong to the Lie algebra $\mathfrak{g}$. Let L and A be subjects to the $\mathbb{Z}_2$ reduction conditions [27, 28]

$$L^\dagger(\lambda^*) = \tilde{L}(\lambda), \qquad A^\dagger(\lambda^*) = \tilde{A}(\lambda) \tag{13}$$

$$\mathbf{C}L(-\lambda)\mathbf{C} = L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda) \tag{14}$$

where tilde operation is defined as follows

$$\tilde{L}(\lambda)\psi = -\mathrm{i}\partial_x\psi + \lambda\psi(Q - \lambda J)$$

for $\psi : \mathbb{R}^2 \to \mathbb{C}^n$ being a smooth function and * stands for complex conjugation. As a result of (13) all coefficients above become Hermitian matrices while the latter reduction implies that $J, A_{2k}(x,t) \in \mathfrak{h}$ while $Q(x,t), A_{2k-1}(x,t) \in \mathfrak{m}$. This way L and A become compatible with $\mathbb{Z}_2$-grading of $\mathfrak{g}$ and thus following [4] we say that the Lax operators are related to the symmetric space G/H.

Remark 1. *It is always possible to pick up J in such a way that $\mathfrak{h}$ coincide with the centralizer $C_J = \{X \in \mathfrak{g} \; ; \; [X, J] = 0\}$ of J. We shall assume that this is done since this will simplify significantly some of our further considerations.*

---

[1]We shall deal with matrix Lie groups and Lie algebras only.

Example 2. *Let us consider as a simple illustration a quadratic bundle related to the symmetric space* $SU(m + 1)/S(U(1) \times U(m))$, $m \geq 2$. *In this case* $\mathbf{C} = \text{diag}(1, -1 \ldots, -1)$ *and the subspace* $\mathfrak{h}$ *consists of all* $(m + 1) \times (m + 1)$ *block diagonal anti-Hermitian traceless matrices of the form*

$$
\begin{pmatrix}
* & 0 & \ldots & 0 \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & *
\end{pmatrix}
$$

while $\mathfrak{m}$ *consists of all anti-Hermitian matrices with complementary block structure, namely*

$$
\begin{pmatrix}
0 & * & \ldots & * \\
* & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & 0 & \ldots & 0
\end{pmatrix} .
$$

In particular, the potential Q *is given by*

$$
Q(x, t) = \begin{pmatrix}
0 & q_1(x, t) & \ldots & q_m(x, t) \\
q_1^*(x, t) & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
q_m^*(x, t) & 0 & \ldots & 0
\end{pmatrix} .
\tag{15}
$$

The subalgebra $\mathfrak{h}$ *coincides with* C_J *if* $J = \text{diag}(m, -1, \ldots, -1)$. *The compatibility condition of the operators* (11) *and* (12) *for* $N = 2$ *(i.e., the quadratic flow) produces exactly the multicomponent* DNSE *we mentioned in the previous section (see formula* (8)).

Example 3. *Another example worthy to mention here is given by a quadratic bundle related to the symmetric space* $SO(2r + 1)/SO(2) \times SO(2r - 1)$, $r \geq 2$. *Now Cartan's involution is given by* $\mathbf{C} = \text{diag}(-1, 1 \ldots, 1, -1)$. *The subalgebra* $\mathfrak{h}$ *therefore consists of all anti-Hermitian matrices of the form*

$$
\begin{pmatrix}
* & 0 & \ldots & 0 & 0 \\
0 & * & \ldots & * & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & * & \ldots & * & 0 \\
0 & 0 & \ldots & 0 & *
\end{pmatrix}
$$

while $\mathfrak{m}$ *contains all block off-diagonal matrices*

$$\begin{pmatrix} 0 & * & \ldots & * & 0 \\ * & 0 & \ldots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \ldots & 0 & * \\ 0 & * & \ldots & * & 0 \end{pmatrix}.$$

The constant element $J \in \mathfrak{h}$ *now is chosen as follows* $J = \mathrm{diag}(1, 0, \ldots, 0, -1)$ *while the potential* Q *is given by*

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^T(x,t) & 0 \\ \mathbf{q}^*(x,t) & 0 & s_0\mathbf{q}(x,t) \\ 0 & \mathbf{q}^\dagger(x,t)s_0 & 0 \end{pmatrix} \tag{16}$$

for some smooth function $\mathbf{q} : \mathbb{R}^2 \to \mathbb{C}^{2r-1}$. *The presence of the* $2r - 1 \times 2r - 1$ *matrix* $(s_0)_{ij} = (-1)^{i-1}\delta_{i\,2r-j}$ *takes into account that* Q *is an element of the orthogonal algebra* $\mathfrak{so}(2r + 1)$.

The compatibility condition of the Lax pair (11), (12) *in the quadratic flow case* $(N = 2)$ *is equivalent to the multicomponent DNSE*

$$i\mathbf{q}_t + \mathbf{q}_{xx} + i[2(\mathbf{q}^T\mathbf{q}^*)\mathbf{q} - (\mathbf{q}^T s_0\mathbf{q}^*)s_0\mathbf{q}^*]_x = 0. \tag{17}$$

In order to make the spectral problem well-defined we must impose certain boundary conditions on the potential Q. We shall restrict ourselves to the simplest case of zero boundary conditions

$$\lim_{x \to \pm\infty} Q(x,t) = 0. \tag{18}$$

To be more specific we require that each matrix element of Q is a function of the Schwartz type for $x \in \mathbb{R}$. Moreover, we assume that Q is such that the corresponding Lax operator has a *finite number* of discrete eigenvalues.

The spectrum of the scattering operator L is determined by its resolvent $R(\lambda)$ defined by the equality

$$L(\lambda) \circ R(\lambda) = \mathbb{1} \tag{19}$$

where $\circ$ stands for operator composition. It follows from (19) that the resolvent is an integral operator of the form

$$(R(\lambda) F)(x,t) = \int_{-\infty}^{\infty} \mathcal{R}(x,y,t,\lambda)F(y)\mathrm{d}y \tag{20}$$

for $F : \mathbb{R} \to \mathbb{C}^n$ being any continuous function. The kernel $\mathcal{R}(x,y,t,\lambda)$ is assumed to be continuous with respect to variables x and y and its domain in the spectral λ-plane is complementary to the spectrum of L. More specifically the pole singularities of $\mathcal{R}$, if exist, correspond to discrete eigenvalues of L while the

locus of points in the λ-plane for which the boundary $\lim_{x, y \to \pm \infty} \mathcal{R}(x, y, t, \lambda)$ does not exist determines the continuous part of the spectrum [8, 15]. We shall convince ourselves that the latter requirement is reduced to the following one by the condition

$$\operatorname{Im} \lambda^2 J = 0.$$

In all examples we shall encounter later on in the text J is a real matrix hence the continuous spectrum is simply the real and imaginary axis of the Cartesian frame in $\mathbb{C}$. On the other hand due to reductions (13) and (14) the discrete eigenvalues of L are sorted into certain discrete orbits of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [15,27,28]. Indeed, the resolvent obeys the symmetries

$$R^\dagger(\lambda^*) = \tilde{R}(\lambda) \qquad \Longrightarrow \qquad \mathcal{R}^\dagger(x, y, t, \lambda^*) = \mathcal{R}(y, x, t, \lambda) \quad (21)$$

$$\mathbf{C}R(-\lambda)\mathbf{C} = R(\lambda) \qquad \Longrightarrow \qquad \mathbf{C}\mathcal{R}(x, y, t, -\lambda)\mathbf{C} = \mathcal{R}(x, y, t, \lambda) \quad (22)$$

where

$$(\tilde{R}(\lambda)F)(x, t) = \int_{-\infty}^{\infty} F(y)\mathcal{R}(y, x, t, \lambda)\mathrm{d}y.$$

It is immediately seen from the above relations that if μ is a pole of $R(.)$ then $-\mu$ and $\pm\mu^*$ are poles as well. Therefore the eigenvalues of L go into quadruples (each quadrant in $\mathbb{C}$ contains the same number of eigenvalues).

Let us now consider the auxiliary linear problem

$$\mathrm{i}\partial_x \psi(x, t, \lambda) + \lambda(Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0. \tag{23}$$

The function ψ is fundamental solution to (23) and takes values in the Lie group G. Since L and A commute any fundamental solution ψ satisfies

$$A(\lambda)\psi(x, t, \lambda) = \psi(x, t, \lambda)f(\lambda) \tag{24}$$

as well. The quantity

$$f(\lambda) = \lim_{x \to \pm\infty} \sum_{k=1}^{N} \lambda^k A_k(x, t) \tag{25}$$

is called dispersion law. It labels the specific NEE amongst the integrable hierarchy, i.e., all equations to share the same Lax operator L. It is therefore an essential feature of the integrable $1 + 1$ dimensional equations.

Next we introduce a special type of fundamental solutions, namely Jost solutions $\psi_\pm$. They are defined through the following equality

$$\lim_{x \to \pm\infty} \psi_\pm(x, t, \lambda)\mathrm{e}^{\mathrm{i}\lambda^2 Jx} = \mathbb{1}$$

as any two fundamental solutions are related with Jost solutions. The transition matrix

$$T(t, \lambda) = (\psi_+(x, t, \lambda))^{-1}\psi_-(x, t, \lambda) \tag{26}$$

is called scattering matrix. It is not hard to see that the time evolution of T is driven by the dispersion law through the linear equation

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \tag{27}$$

which is easily integrated to give

$$T(t, \lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t}T(0, \lambda)\mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

Equation (27) is a linearized version of the corresponding NEE. This important fact underlies the method of IST for integration of NEEs [1, 15, 39]. From now on shall skip the time dependance in order to simplify our notation.

Like in the case of a quadratic bundle related to $\mathfrak{sl}(2)$ [11, 20] the Jost solutions are defined on the continuous spectrum of L only. To see this one introduces the auxiliary functions $\xi_\pm = \psi_\pm \mathrm{e}^{\mathrm{i}\lambda^2 Jx}$ which satisfy the linear equation

$$\mathrm{i}\partial_x \xi_\pm + \lambda Q\xi_\pm - \lambda^2[J, \xi_\pm] = 0$$

with boundary condition

$$\lim_{x \to \pm\infty} \xi_\pm(x, \lambda) = \mathbb{1}.$$

Equivalently $\xi_\pm$ are solutions to the following Voltera type integral equations

$$\xi_\pm(x, \lambda) = \mathbb{1} + \mathrm{i}\lambda \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}\lambda^2 J(x-y)}Q(x)\xi_\pm(y, \lambda)\mathrm{e}^{\mathrm{i}\lambda^2 J(x-y)}\mathrm{d}y. \tag{28}$$

Outside of the continuous spectrum of L (i.e., $\lambda^2 \notin \mathbb{R}$) there always exist at least one increasing exponential factor to make the integral divergent. This is why $\xi_\pm$ as well as $\psi_\pm$ *can not* be analytically extended outside of the continuous spectrum. A more detailed analysis however shows that there exists a solution χ^+ to be analytic in the first and third quadrant in $\mathbb{C}$ and another denoted by χ^- analytic in the second and forth quadrant. The fundamental analytic solutions are related to the Jost solutions through

$$\chi^\pm(x, \lambda) = \psi_-(x, \lambda)S^\pm(\lambda) = \psi_+(x, \lambda)T^\mp(\lambda)D^\pm(\lambda) \tag{29}$$

where all matrix factors introduced above appear in the LDU decomposition

$$T(\lambda) = T^\mp(\lambda)D^\pm(\lambda)(S^\pm(\lambda))^{-1}$$

of the scattering matrix. In fact this is a generalization of the usual LDU decomposition since all matrices involved here have a block structure compatible with the splitting (9) of the Lie algebra $\mathfrak{g}$. For example when dealing with symmetric

spaces of the type $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ we have the following

$$S^+(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{s}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{t}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix}$$

$$S^-(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{s}_-(\lambda) & \mathbb{1}_n \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{t}_-(\lambda) & \mathbb{1}_n \end{pmatrix}$$

$$D^+(\lambda) = \begin{pmatrix} d_m^+(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^+(\lambda) \end{pmatrix}, \qquad D^-(\lambda) = \begin{pmatrix} d_m^-(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^-(\lambda) \end{pmatrix}$$

where $\mathbf{s}_\pm(\lambda)$ and $\mathbf{t}_\pm(\lambda)$ are $n \times m$ complex matrices and $d_m^\pm(\lambda)$ are $m \times m$ and $d_n^\pm(\lambda)$ are $n \times n$ complex matrices respectively. All these quantities can be expressed in terms of matrix elements of T, see [9] for instance.

It is clear from (29) that the fundamental analytic solutions are interrelated through

$$\chi^+(x, \lambda) = \chi^-(x, \lambda) G(\lambda), \qquad \lambda \in \mathbb{R} \cup i\mathbb{R}$$

for $G(\lambda) = (S^-(\lambda))^{-1} S^+(\lambda)$. Thus $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ can be viewed as solutions[2] to a local Riemann-Hilbert factorization problem in λ-plane for continuous spectrum of L being the boundary contour. As we shall see in next section this fact is important for elaboration of dressing method.

The reductions (13) and (14) impose certain symmetry conditions on the Jost solutions, the scattering matrix and fundamental analytic solutions. Here is a list of these

$$\left[\psi_\pm^\dagger(x, \lambda^*)\right]^{-1} = \psi_\pm(x, \lambda), \qquad \left[T^\dagger(\lambda^*)\right]^{-1} = T(\lambda)$$

$$\mathbf{C}\psi_\pm(x, -\lambda)\mathbf{C} = \psi_\pm(x, \lambda), \qquad \mathbf{C}T(-\lambda)\mathbf{C} = T(\lambda) \tag{30}$$

$$\left[\chi^+(x, \lambda^*)\right]^\dagger = [\chi^-(x, \lambda)]^{-1}, \qquad \mathbf{C}\chi^\pm(x, -\lambda)\mathbf{C} = \chi^\pm(x, \lambda).$$

One important application of the fundamental analytic solutions is in the spectral theory of the scattering operator. To see this let us define the function

$$\mathcal{R}(x, y, \lambda) = \begin{cases} \mathcal{R}^+(x, y, \lambda), & \operatorname{Im} \lambda^2 > 0 \\ \mathcal{R}^-(x, y, \lambda), & \operatorname{Im} \lambda^2 < 0 \end{cases} \tag{31}$$

where

$$\mathcal{R}^\pm(x, y, \lambda) = \pm i\chi^\pm(x, \lambda)\Theta^\pm(x - y)\left(\chi^\pm(y, \lambda)\right)^{-1}. \tag{32}$$

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[2]Strictly speaking solutions to a Riemann-Hilbert problem are the functions $\chi^\pm e^{i\lambda^2 Jx}$ since they have the proper normalization as $|\lambda| \to \infty$.

In the above equation $\Theta^{\pm}$ is a matrix-valued function expressed in terms of Heaviside's step function and certain constant projectors. For instance, for the symmetric space $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(m))$ we have

$$\Theta^{\pm}(x-y) = \theta(\pm(y-x))P - \theta(\pm(x-y))(\mathbb{1} - P) \tag{33}$$

for P being a constant projector of the form $P = \mathrm{diag}(1, 0, \ldots, 0)$.

It directly follows from (29) that the asymptotic behavior of the fundamental analytic solutions for $\lambda^2 \in \mathbb{R}$ are given by

$$
\begin{aligned}
\chi^{\pm}(x, \lambda) &\xrightarrow[x \to \infty]{} \mathrm{e}^{-\mathrm{i}\lambda^2 Jx} S^{\pm}(\lambda) \\
\chi^{\pm}(x, \lambda) &\xrightarrow[x \to -\infty]{} \mathrm{e}^{-\mathrm{i}\lambda^2 Jx} T^{\mp}(\lambda) D^{\pm}(\lambda).
\end{aligned}
\tag{34}
$$

Taking into account (34) we see that for $\lambda^2 \in \mathbb{R}$ the operator $R(\lambda)$ becomes asymptotically unbounded. Hence the integral (20) does not converge as $R(\lambda)$ is not defined. As we have mentioned this condition determines the continuous part of the spectrum of the scattering operator L. On the other hand when $\lambda^2 \notin \mathbb{R}$ it is the role of the projector P to ensure that $\mathcal{R}^{\pm}$ decreases exponentially as $x, y \to \pm\infty$. P is therefore implicitly related to the structure of the underlying symmetric space, i.e., the $\mathbb{Z}_2$-grading of the corresponding Lie algebra.

We shall state without proof the following important theorem

Theorem 4. *The function $\mathcal{R}$ defined through* (31) *and* (32) *is an integral kernel of the resolvent of L, i.e., the equality*

$$L(\lambda)\mathcal{R}(x, y, \lambda) = \delta(x - y) \tag{35}$$

holds true. The kernel $\mathcal{R}$ is a mesomorphic function in $\mathbb{C}$ with a finite number of poles $\{\pm\mu_k, \pm\mu_k^*\}_{k=1}^l$ to form the discrete spectrum of the scattering operator L. The continuous part of the spectrum coincides with the real and the imaginary axis in the spectral λ-plane.

3. Special Solutions to DNSE

In this section we are going to integrate multicomponent DNSE related to symmetric spaces, i.e., find their particular solutions. There are different approaches for integration of such nonlinear evolution equations, see [15, 17, 31]. It is our belief that the dressing technique proposed by Zakharov-Shabat [41] and developed further in [19, 40] provides a very convenient and powerful tool to solve multicomponent evolution equations associated with homogeneous or symmetric spaces. Being purely algebraic in nature the dressing method takes into account the algebraic structures (if present) underlying the Lax pairs and in that way offers a uniform approach to a variety of integrable nonlinear problems. This is why our main purpose here is to

adapt the dressing method in case of quadratic bundles. This will allow us to find easily the soliton solutions to DNSE.

3.1. Dressing Method

As we saw in the previous section the inverse scattering method can be reduced to a matrix Riemann problem on the λ-plane. This remarkable fact underlies one of the formulations of the dressing method [15,39] in terms of one-parameter families of gauge transforms of the Lax pair. The dressing method allows one to integrate a given NEE indirectly, i.e., starting from a known solution one obtains another one. Let $Q_0(x, t)$ be a known solution to a DNSE related to some Hermitian symmetric space. It plays the role of a potential for the linear problem

$$L_0\psi_0 = i\partial_x\psi_0 + \lambda(Q_0 - \lambda J)\psi_0 = 0. \tag{36}$$

Let us apply the gauge transform

$$g : \psi_0(x, t, \lambda) \to \psi_1(x, t, \lambda) = g(x, t, \lambda)\psi_0(x, t, \lambda) \tag{37}$$

to the fundamental solution ψ_0. Under the assumption of g-covariance of the linear problem, i.e., the dressed function ψ_1 is a fundamental solution to the linear problem

$$L_1\psi_1 = i\partial_x\psi_1 + \lambda(Q_1 - \lambda J)\psi_1 = 0 \tag{38}$$

where $Q_1(x, t)$ is some other potential to be found, we deduce that the dressing factor g satisfies

$$i\partial_x g + \lambda Q_1\, g - \lambda g Q_0 - \lambda^2[J, g] = 0. \tag{39}$$

Similarly, by comparing the two linear problems

$$A_0(\lambda)\psi_0 = i\partial_t\psi_0 + \sum_{k=1}^{2N} \lambda^k A_k^{(0)}\psi_0 = \psi_0 f(\lambda)$$

$$A_1(\lambda)\psi_1 = i\partial_t\psi_1 + \sum_{k=1}^{2N} \lambda^k A_k^{(1)}\psi_1 = \psi_1 f(\lambda)$$

we obtain another differential equation for g, namely

$$i\partial_t g + \sum_{k=1}^{2N} \lambda^k A_k^{(1)} g - g\sum_{k=1}^{2N} \lambda^k A_k^{(0)} = 0. \tag{40}$$

The gauge transform (37) acts on any fundamental solution including the Jost solutions. To ensure that the dressing procedure leads to Jost solutions to (38) one has to modify (37) into

$$\psi_{0,\pm} \to \psi_{1,\pm} = g\psi_{0,\pm}\, g_\pm^{-1}, \qquad g_\pm = \lim_{x \to \pm\infty} g. \tag{41}$$

This results in the following transformation law for the scattering matrix

$$T_0 \to T_1 = g_+ \, T_0 \, g_-^{-1}. \tag{42}$$

The fundamental analytic solutions in their turn are dressed through the formula

$$\chi_1^{\pm} = g \chi_0^{\pm} g_-^{-1}. \tag{43}$$

Using (43) it is seen that the resolvent kernel $\mathcal{R}_0$ for the bare operator L_0 is transformed into

$$\mathcal{R}_1(x, y, t, \lambda) = g(x, t, \lambda) \mathcal{R}_0(x, y, t, \lambda) [g(y, t, \lambda)]^{-1}. \tag{44}$$

Formula (44) shows that even if the bare kernel $\mathcal{R}_0$ might have not any singular points while the dressed one could – these are singularities introduced by the dressing factor and/or its inverse. The new singular points contribute to the discrete spectrum of the dressed operator L_1. As we discussed in the previous section these points could not be arbitrary. Another way to see this is to write down the symmetry conditions fulfilled by the dressing factor. Indeed, due to the $\mathbb{Z}_2$ reductions (30) we have

$$\left[g^{\dagger}(x, t, \lambda^*) \right]^{-1} = g(x, t, \lambda) \tag{45}$$

$$\mathbf{C} g(x, t, -\lambda) \mathbf{C}^{-1} = g(x, t, \lambda). \tag{46}$$

Relation (46) implies that if μ is a singularity of g so is $-\mu$ while from (45) we deduce that $\pm \mu^*$ are singularities for g^{-1}. This proves that the singularities of the resolvent go in quadruples which resonates to our statement from the previous section.

In order to proceed further we need to make some additional assumptions for the structure of the dressing factor. It is evident from (39) and (40) that if g does not depend on λ then it is simply a constant. On the other hand the connection between the inverse scattering method and Riemann-Hilbert problem implies that the dressing factor has to be divergent as $|\lambda| \to \infty$ to ensure that dressed solutions $\chi_1^{\pm}$ have the proper λ-asymptotics. So to obtain nontrivial results we should pick up a dressing factor possessing certain number of singularities. For the sake of simplicity we shall restrict ourselves with dressing factors having simple poles only. Such a factor can be presented as follows

$$g(x, t, \lambda) = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k} \left(\frac{B_k(x, t)}{\lambda - \mu_k} + \frac{\mathbf{C} B_k(x, t) \mathbf{C}}{\lambda + \mu_k} \right), \qquad \mu_k^2 \notin \mathbb{R}. \tag{47}$$

According to (46) its inverse looks as follows

$$g^{-1} = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k^*} \left(\frac{B_k^{\dagger}}{\lambda - \mu_k^*} + \frac{\mathbf{C} B_k^{\dagger} \mathbf{C}}{\lambda + \mu_k^*} \right). \tag{48}$$

After multiplying (39) by g^{-1}/λ, and then taking the limit as $|\lambda| \to \infty$ we get the following interrelation between the seed solution Q_0 and the dressed one

$$Q_1 = AQ_0A^\dagger + \sum_{k=1}^{l} [J, B_k - \mathbf{C}B_k\mathbf{C}]A^\dagger \tag{49}$$

where

$$A = \mathbb{1} + \sum_{k=1}^{l} \frac{1}{\mu_k}(B_k + \mathbf{C}B_k\mathbf{C}). \tag{50}$$

So Q_1 is completely determined if we know the residues B_k. The power of the dressing method consists in the fact that B_k can be expressed in terms of fundamental solutions to (36) (and its λ - derivatives) only. To see this we shall analyse the identity

$$gg^{-1} = \mathbb{1}. \tag{51}$$

Since (51) holds identically with respect to λ it gives rise to certain algebraic relations for the residues of g. The form of these relations depends crucially on whether a part of the poles of g and its inverse coincide or not. This is why we shall consider two examples which are more or less representative ones.

Example 5. *Let us consider the case of a quadratic bundle associated with symmetric space* $\mathrm{SU}(m + 1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(m))$. *Then the simplest choice for* g *is*

$$g(x, t, \lambda) = \mathbb{1} + \frac{\lambda B(x,t)}{\mu(\lambda - \mu)} + \frac{\lambda \mathbf{C}B(x,t)\mathbf{C}}{\mu(\lambda + \mu)}, \qquad \mu^2 \notin \mathbb{R} \tag{52}$$

and formula (49) *simplifies into*

$$Q_1 = AQ_0A^\dagger + [J, B - \mathbf{C}B\mathbf{C}]A^\dagger \tag{53}$$

where

$$A = \mathbb{1} + \frac{1}{\mu}(B + \mathbf{C}B\mathbf{C}).$$

After calculating the residue at $\lambda = \mu$ *in* (51) *we obtain the algebraic relation*

$$B \left(\mathbb{1} + \frac{\mu B^\dagger}{\mu^*(\mu - \mu^*)} + \frac{\mu \mathbf{C}B^\dagger\mathbf{C}}{\mu^*(\mu + \mu^*)} \right) = 0. \tag{54}$$

If B *is invertible then* (54) *implies that it is proportional to* $\mathbb{1}$. *In order to obtain nontrivial dressing we assume* B *is degenerate. Hence there exist two rectangular* $(m + 1) \times k$ *matrices* $X(x, t)$ *and* $F(x, t)$ *such that* $B = XF^T$. *Then* (54) *is reduced to an algebraic equation for* X *whose solution reads*

$$X = \frac{\mu}{\mu^*} \left(\frac{F^T F^*}{\mu - \mu^*} - \frac{F^T \mathbf{C}F^*}{\mu + \mu^*}\mathbf{C} \right)^{-1} F^*. \tag{55}$$

The factor F can be found from differential equation (39). Evaluating the residue at $\lambda = \mu$ and taking into account (54) leads to the differential equation

$$\mathrm{i}\partial_x F^T - F^T(\mu Q_0 - \mu^2 J) = 0. \tag{56}$$

Therefore we have

$$F^T(x) = F_0^T[\psi_0(x,\mu)]^{-1} \tag{57}$$

where ψ_0 is any fundamental solution to (36) defined in a vicinity of μ and F_0 is a constant matrix. What remains is to recover the time evolution. For this to be done we analyse equation (40) in the same way we did with (39). The residue of (40) at the point μ gives rise to a differential equation for F in the form

$$\mathrm{i}\partial_t F^T - F^T \sum_{k=1}^{2N} \lambda^k A_k^{(0)} = 0. \tag{58}$$

After taking into account (57) and (40) we deduce that the matrix F_0 evolves with time according to equation

$$\mathrm{i}\partial_t F_0^T - F_0^T f(\mu) = 0 \tag{59}$$

where $f(\lambda)$ is the dispersion law of the nonlinear equation. Thus in order to derive the time dependence for the dressed potential one does the following substitution

$$F_0^T \to F_0^T \mathrm{e}^{-\mathrm{i}f(\mu)t}. \tag{60}$$

In the previous example the poles of the dressing factor and its inverse were distinct. As we shall see in next example this is not always possible to achieve. This results in a more complicated procedure to find the residues of g.

Example 6. *Let us consider now quadratic bundles related to* **BD.I** *Hermitian symmetric spaces. Then apart of (45) and (46) the dressing factor must obey the orthogonality condition*

$$g^T S g = S \tag{61}$$

where S is the matrix involved in the definition of the orthogonal group. To meet the requirements of all reductions we pick up g in the form

$$g = \mathbb{1} + \frac{\lambda B}{\mu(\lambda - \mu)} + \frac{\lambda \mathbf{C}B\mathbf{C}}{\mu(\lambda + \mu)} + \frac{\lambda S B^* S}{\mu^*(\lambda - \mu^*)} + \frac{\lambda \mathbf{C}S B^* S\mathbf{C}}{\mu^*(\lambda + \mu^*)} \tag{62}$$

while its inverse looks as follows

$$g^{-1} = \mathbb{1} + \frac{\lambda B}{\mu(\lambda - \mu)} + \frac{\lambda \mathbf{C}B\mathbf{C}}{\mu(\lambda + \mu)} + \frac{\lambda S B^* S}{\mu^*(\lambda - \mu^*)} + \frac{\lambda \mathbf{C}S B S\mathbf{C}}{\mu^*(\lambda + \mu^*)}. \tag{63}$$

Relation (49) *now looks as follows*

$$Q_1 = AQ_0A^\dagger + [J, B + SB^*S - \mathbf{C}B\mathbf{C} - S\mathbf{C}B^*\mathbf{C}S]A^\dagger \qquad (64)$$

for A in the form

$$A = \mathbb{1} + \frac{1}{\mu}(B + \mathbf{C}B\mathbf{C}) + \frac{1}{\mu^*}S(B^* + \mathbf{C}B^*\mathbf{C})S \cdot \qquad (65)$$

The identity (51) *now leads to a couple of algebraic conditions for B, namely*

$$BSB^T = 0 \qquad (66)$$
$$BS\Omega^T S + \Omega SB^T S = 0 \qquad (67)$$

where

$$\Omega = \mathbb{1} + \frac{\mathbf{C}B\mathbf{C}}{2\mu} + \frac{\mu SB^*S}{\mu^*(\mu - \mu^*)} + \frac{\mu\mathbf{C}SB^*S\mathbf{C}}{\mu^*(\mu + \mu^*)} \cdot$$

Equations (66) *and* (67) *imply that B is a degenerate, i.e., it is decomposable into* $B = XF^T$ *for* $X(x,t)$ *and* $F(x,t)$ *being* $m \times k$ *rectangular matrices. Relations* (66) *and* (67) *can be rewritten in terms of F and X to give*

$$F^T SF = 0, \qquad \Omega SF = X\alpha \qquad (68)$$

for $\alpha(x,t)$ *being some appropriately chosen* $k \times k$ *skew-symmetric matrix. In the simplest case* $k = 1$ *it simply vanishes and* (68) *obtains the form*

$$SF = a\mathbf{C}X + bSX^* + c\,\mathbf{C}SX^* \qquad (69)$$

where we have introduced

$$a = -\frac{F^T\mathbf{C}SF}{2\mu}, \qquad b = -\frac{\mu F^\dagger F}{\mu^*(\mu - \mu^*)}, \qquad c = -\frac{\mu F^\dagger \mathbf{C}F}{\mu^*(\mu + \mu^*)} \cdot$$

Due to the $\mathbb{Z}_2$ *symmetries the algebraic relations derived at the other three poles read*

$$\mathbf{C}SF = aX + b\mathbf{C}SX^* + cSX^* \qquad (70)$$
$$F^* = a^*\mathbf{C}SX^* + b^*X + c^*\mathbf{C}X \qquad (71)$$
$$\mathbf{C}F^* = a^*SX^* + b^*\mathbf{C}X + c^*X. \qquad (72)$$

The system of equations (69)–(72) *is regarded as a linear system for the factor* X *(as well as for* SX^*, $\mathbf{C}X$ *and* $\mathbf{C}SX^*$*). After performing elementary manipulations we get*

$$X = \frac{1}{\Delta}\left(\Delta_d SF + \Delta_a \mathbf{C}SF + \Delta_b F^* + \Delta_c \mathbf{C}F^*\right) \qquad (73)$$

where

$$\Delta_d = a^*(bc^* + cb^*), \qquad\qquad \Delta_a = a^*(|a|^2 - |b|^2 - |c|^2)$$

$$\Delta_b = b|b|^2 - b|a|^2 - b^*c^2, \qquad\qquad \Delta_c = c|c|^2 - c|a|^2 - c^*b^2$$

$$\Delta = |a|^4 - 2|ac|^2 - 2|ab|^2 + |b|^4 - (b^*)^2c^2 - b^2(c^*)^2 + |c|^4.$$

Thus we have expressed X through F. It is not hard to be verified that the formula (57) holds in this case too.

In order to recover the time evolution one follows the same steps as in the previous example. By doing this one can convince himself that the rule (60) is still valid.

3.2. Soliton Solutions

Here apply the general results from the previous one to evaluate the simplest class of solutions – the one-soliton solutions. We shall focus our attention to the vector DNSE related to $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(m))$, see (8).

To derive the one-soliton solution we set $Q_0 = 0$. As a fundamental solution to (36) we can pick up the plane wave $\exp(-i\lambda^2 Jx)$. Then in the case when $\mathrm{rank}\,B = 1$ F becomes a column vector of the form

$$F(x,t) = \begin{pmatrix} e^{mi\mu^2 x} F_{0,1} \\ e^{-i\mu^2 x} F_{0,2} \\ \vdots \\ e^{-i\mu^2 x} F_{0,m+1} \end{pmatrix}. \tag{74}$$

After substituting (74) into (55) and then into (53) we get the reflectionless potential to be

$$q_1^{j-1}(x) = (Q_1)_{1j}(x) = 2i(m+1) \sum_{l=2}^{m+1} \frac{\rho \sin(2\varphi)e^{-i\sigma_l(x)}e^{\theta_l(x)}}{e^{-2i\varphi} + \sum_{p=2}^{m+1} e^{2\theta_p(x)}}$$

$$\times \left(\delta_{jl} - 2i\sin(2\varphi) \frac{e^{\theta_j(x)+\theta_l(x)}e^{i(\delta_j - \delta_l - 2\varphi)}}{e^{-2i\varphi} + \sum_{p=2}^{m+1} e^{2\theta_p(x)}} \right). \tag{75}$$

We have used above the notation

$$\theta_p(x) = (m+1)\rho^2 \sin(2\varphi)x - \xi_{0,p}, \qquad\qquad p = 2, \ldots, m+1$$

$$\sigma_p(x) = (m+1)\rho^2 \cos(2\varphi)x + \delta_1 - \delta_p - \varphi, \qquad \mu = \rho \exp(i\varphi)$$

$$\xi_{0,p} = \ln|F_{0,1}/F_{0,p}|, \qquad \delta_1 = \arg F_{0,1}, \qquad \delta_p = \arg F_{0,p}.$$

In order to obtain the one-soliton solution from (76) one needs to recover the time dependence. Taking into account that for (8) $f(\lambda) = -(m+1)\lambda^4 J$ formula (60) leads to the following correspondence

$$\xi_{0,p} \to \xi_{0,p} - 2(m+1)\rho^4 \sin(4\varphi)t$$

$$\delta_1 \to \delta_1 + 2m\rho^4 \cos(4\varphi)t, \qquad \delta_p \to \delta_p - 2\rho^4 \cos(4\varphi)t. \tag{76}$$

Remark 7. *Let us consider the simplest possible case which occurs when* $m = 1$. *Then the dressing factor* (52) *obtains the form*

$$g = \mathbb{1} + \frac{\lambda B}{\mu(\lambda - \mu)} + \frac{\lambda \sigma_3 B \sigma_3}{\mu(\lambda + \mu)}. \tag{77}$$

According to (53) *the reflectionless potential can be written as follows*

$$q_1(x) = \frac{4\mathrm{i}\rho \sin(2\varphi)\mathrm{e}^{-\mathrm{i}\sigma(x)}\mathrm{e}^{\theta(x)}\left[\mathrm{e}^{2\mathrm{i}\varphi} + \mathrm{e}^{2\theta(x)}\right]}{\left[\mathrm{e}^{-2\mathrm{i}\varphi} + \mathrm{e}^{2\theta(x)}\right]^2} \tag{78}$$

where

$$\theta(x) = 2\rho^2 \sin(2\varphi)x - \xi_0, \qquad \xi_0 = \ln|F_{0,1}/F_{0,2}|$$

$$\sigma(x) = 2\rho^2 \cos(2\varphi)x - \delta_0, \qquad \delta_0 = \delta_2 - \delta_1 - 3\varphi.$$

To obtain the one-soliton solution for DNSE (6) *we should recover the time dependence in* (78) *by using the rule*

$$\xi_0 \to \xi_0 - 4\rho^4 \sin(4\varphi)t, \qquad \delta_0 \to \delta_0 - 2\rho^4 \cos(4\varphi)t.$$

This way we have just reproduced the Kaup-Newell soliton obtained in [20].

It is clear that by dressing (76) once again one is able to construct a two-soliton solution and so on. Proceeding this way one can generate step by step the multi-soliton solutions

$$Q_0 \to Q_1 \to \ldots \to Q_l \to \ldots$$

Another way to do this is by using a dressing factor with an appropriate number of simple poles, namely

$$g = \mathbb{1} + \sum_{k=1}^{l} \frac{\lambda}{\mu_k}\left(\frac{B_k}{\lambda - \mu_k} + \frac{\mathbf{C}B_k\mathbf{C}}{\lambda + \mu_k}\right), \qquad \mu_k^2 \notin \mathbb{R}. \tag{79}$$

Then the multisoliton solution can be derived from the formula (49) by setting $Q_0 = 0$. As before the residues of g can be presented as a product of two rectangular matrices X_k and F_k. A detailed analysis, quite similar to what we did before, shows that the factor F_k are expressed through the a fundamental solution to the bare linear problem as follows

$$F_k^T(x) = F_{0,k}^T[\psi_0(x, \mu_k)]^{-1}. \tag{80}$$

On the other hand the factors X_k are solutions to the linear system

$$F_k^* = \sum_{j=1}^{l} \frac{\mu_k^*}{\mu_j} \left(X_j \frac{F_j^T F_k^*}{\mu_j - \mu_k^*} - \mathbf{C} X_j \frac{F_j^T \mathbf{C} F_k^*}{\mu_j + \mu_k^*} \right). \tag{81}$$

By solving it one is able to find the residues B_k and then derive the reflectionless potential. To recover the time dependence one should apply the same considerations as in the Example 5. The result is given by the rule

$$F_{k,0}^T \rightarrow F_{k,0}^T \mathrm{e}^{-\mathrm{i} f(\mu_k) t}$$

which is a natural generalization of correspondence (60).

4. Integrals of Motion

As it was shown in [4] the multicomponent DNSEs related to Hermitian symmetric spaces can be viewed as infinite dimensional Hamiltonian systems whose Hamiltonian is connected to the curvature tensor of the corresponding symmetric space. In this section we aim to describe analytically the conserved densities of integrals of motion for multicomponent DNSEs. For this to be done we are going to use the method of diagonalization of Lax pair proposed by Drinfel'd and Sokolov [3]. This will allow us to derive a general formula generating the conserved quantities in a recursive manner.

We shall start with some general remarks on quadratic bundles related to arbitrary symmetric spaces. Then in order to obtain more concrete results we shall consider two examples referring to symmetric spaces of the type **A.III** and **BD.I**, see [16]. Let us consider the quadratic bundle Lax pair

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J \tag{82}$$

$$A(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{2N} A_k(x,t)\lambda^k \tag{83}$$

which is related to a Hermitian symmetric space G/H. This means that the potential Q as well as A_{2j-1}, $j = 1, \dots, N$ take values in $\mathfrak{m} \in \mathfrak{g}$ while J and A_{2j} take values in the subalgebra $\mathfrak{h}$ (see the beginning of Section 2 for detailed explanations). In accordance with the discussion in Section 2 we pick up J in such a way that its centralizer coincides with $\mathfrak{h}$.

Let

$$\mathcal{P}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{\infty} p_k(x,t)\lambda^{-k} \tag{84}$$

be a one-parameter family of gauge transformations[3] acting on the fundamental solutions to the linear problem (23) as follows

$$\psi(x, t, \lambda) \to \tilde{\psi}(x, t, \lambda) = (\mathcal{P}(x, t, \lambda))^{-1} \psi(x, t, \lambda).$$

The Lax pair (82) and (83) is transformed into

$$\tilde{L} = \mathcal{P}^{-1} L \mathcal{P} = \mathrm{i}\partial_x - \lambda^2 J + \lambda \mathcal{L}_{-1} + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \frac{\mathcal{L}_2}{\lambda^2} + \cdots \qquad (85)$$

$$\tilde{A} = \mathcal{P}^{-1} A \mathcal{P} = \mathrm{i}\partial_t + \sum_{k=1}^{2N} \lambda^k \mathcal{A}_{-k} + \mathcal{A}_0 + \frac{\mathcal{A}_1}{\lambda} + \frac{\mathcal{A}_2}{\lambda^2} + \cdots \qquad (86)$$

Let us now assume that $\mathcal{L}_k, \mathcal{A}_k \in \mathfrak{h}$, i.e., they are block diagonal matrices. As we shall see in next examples for certain diagonal matrix elements (or traces of diagonal blocks) of $\mathcal{L}_k$ and $\mathcal{A}_k$ the commutator in the zero curvature representation

$$\mathrm{i}\partial_t \mathcal{L}_k - \mathrm{i}\partial_x \mathcal{A}_k + \sum_j [\mathcal{A}_j, \mathcal{L}_{k-j}] = 0, \qquad k = -1, 0, \ldots \qquad (87)$$

vanishes. Thus (87) reduces to continuity equation, i.e., the corresponding elements (or traces of blocks) of $\mathcal{L}_k$ are local conserved densities. Apart of local conserved densities there exist nonlocal ones related to matrix elements for which the commutator does not vanish.

To find the conserved densities we simply substitute (82) and (84) into (85) and then compare coefficients before the same powers of λ. In result we get the following system of recurrence relations

$$\mathcal{L}_{-1} = Q - [J, p_1] \qquad (88)$$

$$\mathcal{L}_0 + p_1 \mathcal{L}_{-1} = Q p_1 - [J, p_2] \qquad (89)$$

$$\mathcal{L}_1 + p_1 \mathcal{L}_0 + p_2 \mathcal{L}_{-1} = \mathrm{i} p_{1,x} + Q p_2 - [J, p_3] \qquad (90)$$

$$\cdots$$

$$\mathcal{L}_k + \sum_{j=1}^{k+1} p_j \mathcal{L}_{k-j} = \mathrm{i} p_{k,x} + Q p_{k+1} - [J, p_{k+2}] \qquad (91)$$

$$\cdots$$

---

[3]Strictly speaking the gauge transformation $\mathcal{P}$ takes values in G and one should use an expansion of the form

$$\mathcal{P}(x, t, \lambda) = \exp\left(\sum_{k=1}^{\infty} \mathcal{P}_k(x, t) \lambda^{-k} \right), \qquad \mathcal{P}_k(x, t) \in \mathfrak{g}$$

instead of (84). But since we deal with matrix Lie groups and Lie algebras the expansion (84) is correctly defined. Of course, one should keep in mind that $p_k(x, t)$ are neither group nor algebra elements – they are arbitrary matrices. This choice of expansion parameters although not quite aesthetic from theoretical point of view is very useful from purely practical one, since it will significantly simplify our further calculations.

In order to resolve it we need to introduce the following projector

$$\Pi_J = \mathrm{ad}\,_J^{-1}\mathrm{ad}\,_J, \qquad (\mathrm{ad}\,_J^{-1}X)_{rs} = \frac{X_{rs}}{J_r - J_s}, \qquad r \neq s$$

which cuts off the corresponding block-diagonal parts of matrices. Thus extracting the block diagonal part from the first recurrence relation we see that $\mathcal{L}_{-1}$ does not contribute to the integrals of motion while the off-block diagonal part reads

$$Q = [J, p_1]. \tag{92}$$

To fix the existing ambiguity we assume that the matrices p_j, $j = 1, 2, \dots$ do not have block diagonal parts. Then (92) allows one to write

$$p_1 = \mathrm{ad}\,_J^{-1}Q. \tag{93}$$

To obtain a nonzero conserved density one considers relation (89) which splits into

$$\mathcal{L}_0 = (\mathbb{1} - \Pi_J)\,Qp_1 = (\mathbb{1} - \Pi_J)\left(Q\,\mathrm{ad}\,_J^{-1}Q\right) \tag{94}$$

$$p_2 = \mathrm{ad}\,_J^{-1}\Pi_J(Qp_1) = \mathrm{ad}\,_J^{-1}\Pi_J\left(Q\,\mathrm{ad}\,_J^{-1}Q\right). \tag{95}$$

Proceeding in the same way with the general recursion relation (91) we get the following result

$$\mathcal{L}_k = (\mathbb{1} - \Pi_J)\left(Qp_{k+1} - \sum_{j=1}^{k+1}p_j\mathcal{L}_{k-j}\right), \qquad k = 1, 2, \dots \tag{96}$$

$$p_{k+2} = \mathrm{i}\mathrm{ad}\,_J^{-1}p_{k,x} + \mathrm{ad}\,_J^{-1}\Pi_J\left(Qp_{k+1} - \sum_{j=1}^{k+1}p_j\mathcal{L}_{k-j}\right). \tag{97}$$

Formula (96) allows us to find the conserved density contained in $\mathcal{L}_k$ algorithmically.

In order to interpret DNSE as a Hamiltonian equation one needs to introduce a Poisson structure. Let

$$F([Q(x,t)]) = \int_{-\infty}^{\infty}\mathcal{F}([Q(x,t)])\mathrm{d}x$$

be a functional of the potential Q and its x-derivatives. The variational derivative $\delta F/\delta Q$ is a matrix whose matrix elements are defined by the equality

$$\left(\frac{\delta F}{\delta Q}\right)_{rs} = \frac{\delta F}{\delta Q_{rs}}.$$

For any two functionals F and G the simplest Poisson bracket[4] for DNSE reads

$$\{F, G\} = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{tr} \left(\frac{\delta F}{\delta Q} \frac{\partial}{\partial x} \frac{\delta G}{\delta Q^T} \right).$$ (98)

In order to be more specific let us illustrate our results with two examples.

Example 8. *Consider the symmetric space* $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(m))$. *Then taking into account formula* (15) *for* p_1 *we get*

$$p_1(x, t) = \frac{1}{m+1} \begin{pmatrix} 0 & \mathbf{q}^T(x, t) \\ -\mathbf{q}^*(x, t) & 0 \end{pmatrix}$$ (99)

where $\mathbf{q}(x, t)$ *is a complex* m-*vector. According to* (94) *and* (95) *the coefficient* $\mathcal{L}_0$ *is given by*

$$\mathcal{L}_0 = \frac{1}{m+1} \begin{pmatrix} -\mathbf{q}^T \mathbf{q}^* & 0 \\ 0 & \mathbf{q}^* \mathbf{q}^T \end{pmatrix}$$ (100)

while p_2 *vanishes. Thus as an integral density one can choose* $\mathfrak{I}_1 = \mathbf{q}^\dagger \mathbf{q}$. *The general recursion formula* (96) *in its turn simplifies into*

$$\mathcal{L}_k = Q p_{k+1}$$ (101)

where p_k *can be found from the equality*

$$p_k = \mathrm{ad}_J^{-1} \left(\mathrm{i} p_{k-2, x} - \sum_{j=1}^{k-2} p_j \mathcal{L}_{k-2-j} \right).$$ (102)

Taking into account (101) *and* (102) *it is evident that* $\mathcal{L}_1 = 0$. *Thus next nonzero integral density* $\mathfrak{I}$ *is connected to the matrix* $\mathcal{L}_2(x, t)$. *The result reads*

$$\mathfrak{I}_2 = \mathrm{i} \mathbf{q}^\dagger \mathbf{q}_x - \frac{1}{m+1} (\mathbf{q}^\dagger \mathbf{q})^2.$$ (103)

It is not hard to be checked that it represents the Hamiltonian density $\mathcal{H}$ *for the multicomponent* DNSE (8) *provided the Poisson bracket is defined as in* (98). *The* DNSE *can be written down in a Hamiltonian form as follows*

$$q_{k, t} = \partial_x \frac{\partial \mathcal{H}}{\partial q_k^*}, \qquad k = 1, \ldots, m.$$ (104)

The results we have just obtained can be summarized in the following theorem:

Theorem 9. *All matrices* $\mathcal{L}_k$ *corresponding to odd indices vanish while the rest are generated by formulae* (101) *and* (102).

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[4]In fact, there is a whole infinite hierarchy of Poisson brackets introduced by appropriate recursion operator.

Proof: We already saw that $\mathcal{L}_{-1} = \mathcal{L}_1 = 0$. So the statement of the theorem follows immediately from (101) and (102) after performing elementary induction. ∎

Example 10. *Let us now examine the case when the Hermitian symmetric space is of the type* $\mathrm{SO}(2r+1)/\mathrm{SO}(2) \times \mathrm{SO}(2r-1)$. *The potential in this case is given by* (16) *and the coefficient* p_1 *reads*

$$p_1 = \begin{pmatrix} 0 & \mathbf{q}^T & 0 \\ -\mathbf{q}^* & 0 & s_0\mathbf{q} \\ 0 & -\mathbf{q}^\dagger s_0 & 0 \end{pmatrix}. \tag{105}$$

According to formulae (94) *and* (95) *we have*

$$\mathcal{L}_0 = \begin{pmatrix} -\mathbf{q}^T\mathbf{q}^* & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbf{q}^*\mathbf{q}^T - s_0\mathbf{q}\mathbf{q}^\dagger s_0 & \mathbf{0} \\ 0 & \mathbf{0}^T & \mathbf{q}^\dagger\mathbf{q} \end{pmatrix} \tag{106}$$

$$p_2 = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{0}^T & \mathbf{q}^T s_0\mathbf{q} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{q}^\dagger s_0\mathbf{q} & \mathbf{0}^T & 0 \end{pmatrix}. \tag{107}$$

Hence the first conserved density $\mathfrak{I}_1 = \mathbf{q}^\dagger\mathbf{q}$ *formally coincides with that in the previous case. It is not hard to be verified that* $\mathcal{L}_1 = 0$ *so next conserved density is obtained from* $\mathcal{L}_2$. *Substituting all quantities needed in* (96) *leads to the following result*

$$\mathfrak{I}_2 = i\mathbf{q}^\dagger\mathbf{q}_x - \left(\mathbf{q}^\dagger\mathbf{q}\right)^2 + \frac{1}{2}\left|\mathbf{q}^T s_0\mathbf{q}\right|^2. \tag{108}$$

This is the Hamiltonian density of DNSE (17) *provided the Poisson structure is picked up as in* (98).

5. Conclusions

In the present paper we have studied some general properties of quadratic bundles related to arbitrary Hermitian symmetric spaces. In particular, we have introduced all basic notions like Jost solutions, scattering matrix, fundamental analytic solutions etc., required to formulate direct scattering problem. Using the fundamental analytic solutions we have constructed the resolvent of the scattering operator and discussed its properties which determine the spectrum of the scattering operator L.

We have adapted the Zakharov-Shabat dressing technique to quadratic bundles of the afore-mentioned type. Though the method itself is not sensitive to the symmetric space type (more precisely to its structure), the form of the dressing factor may vary from one symmetric space to another. For example in the case of **A.III** symmetric spaces one can use the two-pole dressing factor (52) while for **BD.I** this is not possible any more – one needs to use a four poles factor, see formula (62).

By applying the dressing method we have derived the one-soliton solution to the multicomponent DNSE related to **A**.**III** and discussed how one can construct multisoliton solutions. These results generalize the classical ones by Kaup and Newell [20] for the scalar DNSE – the latter can be obtained by using a dressing factor of the form (77). Similarly, one can derive the soliton solutions for DNSE related to other symmetric spaces, say symmetric spaces of the series **BD**.**I**. However, this requires much more technical efforts due to the complicated form of the dressing factor (62) .

Since multicomponent DNSE are infinite dimensional Hamiltonian systems there exist at least one integral of motion for them – the Hamiltonian itself. We have proved in the previous section that in fact there are infinite number of conserved quantities associated with multicomponent DNSE and we have derived a general recursion formula which allows one to generate them. For that purpose we have applied the method of block-diagonalization of Lax pair. As a simple illustration we have evaluated the first two integrals of motion in the case of the symmetric spaces $\mathrm{SU}(m + 1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(m))$ and $\mathrm{SO}(2r + 1)/\mathrm{SO}(2) \times \mathrm{SO}(2r - 1))$. The second integrals of motion represent the Hamiltonian of the multicomponent DNSE (8) and (17) respectively provided the Poisson bracket is defined as in (98). All this underlies the proof of the complete integrability of the multicomponent DNSE in the sense of Liouville-Arnol'd, i.e., the construction of symplectic basis and action-angle variables. To do this one needs to develop the generalized Fourier transform interpretation of IST by introducing squared solutions (adjoint solutions) and recursion operator [15, 34–36]. All this is a matter of a future study.

The results presented in the paper could be extended in several ways. Firstly, one can study complete quadratic bundles

$$L(\lambda) = \mathrm{i}\partial_x + U_0(x, t) + \lambda U_1(x, t) - \lambda^2 J \qquad (109)$$

where $U_0(x, t)$ splits into a diagonal and off-diagonal part, $U_1(x, t)$ is strictly off-diagonal and J is a diagonal matrix. In general the bundle (109) can not be associated with symmetric spaces unless U_0 contains block diagonal part only and U_1 has a block structure complementary to U_0 (otherwise symmetry conditions (14) will be violated). As it is expected the theory of such bundles becomes more complicated than that of bundles related to symmetric spaces.

We have been dealing in this paper with solutions satisfying zero boundary conditions (the so-called trivial background solutions). These represent the simplest class of solutions to the NEE. On the other hand finding nontrivial background solutions is of current interest even for classical integrable equations like the scalar nonlinear Schrödinger equation [32, 37]. Hence extending the results presented here for potentials satisfying more complicated boundary conditions is another meaningful direction of further developments.

Acknowledgements

References

[1] Ablowitz M. and Clarkson P., *Soliton, Nonlinear Equations and Inverse Scattering*, London Math. Society Lecture Notes Series **149**, Cambridge Univ. Press, Cambridge 1991.

[2] Athorne C. and Fordy A., *Generalised KdV and MKdV Equations Associated with Symmetric Spaces*, J. Phys. A **20** (1987) 1377–1386.

[3] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg-de Vries Type*, Sov. J. Math. **30** (1985) 1975–2036.

[4] Fordy A., *Derivative Nonlinear Schrödinger Equations and Hermitian Symmetric Spaces*, J. Phys. A: Math. & Gen. **17** (1984) 1235–1245.

[5] Fordy A. and Kulish P., *Nonlinear Schrödinger Equations and Simple Lie Algebras*, Commun. Math. Phys. **89** (1983) 427–443.

[6] Gadzhiev I., Gerdzhikov V. and Ivanov M., *Hamiltonian Structures of Nonlinear Evolution Equations Associated with a Polynomial Bundle*, J. Sov. Math. **37** (1987) 1186–1194.

[7] Gardner C., Greene J., Kruskal M. and Miura R., *Method for solving the Korteweg-de Vries Equation*, Phys. Rev. Lett. **19** (1967) 1095–1097.

[8] Gerdjikov V., *On Spectral Theory of Lax Operators on Symmetric Spaces: Vanishing Versus Constant Boundary Conditions*, J. Geom. Symmetry Phys. **15** (2009) 1–41.

[9] Gerdjikov V., *Algebraic and Analytic Aspects of N-wave Type Equations*, Contemp. Math. **301** (2002) 35–68, nlin.SI/0206014.

[10] Gerdjikov V. and Ivanov M., *The Quadratic Bundle of General Form and the Nonlinear Evolution Equations I. Expansions over the "Squared" Solutions – Generalized Fourier Transform*, Bulg. J. Phys. **10** (1983) 13–26.

[11] Gerdjikov V. and Ivanov M., *The Quadratic Bundle of General Form and the Nonlinear Evolution Equations. II. Hierarchies of Hamiltonian Structures*, Bulg. J. Physics **10** (1983) 130–143.

[12] Gerdjikov V., Grahovski G., Mikhailov A. and Valchev T., *Polynomial Bundles and Generalised Fourier Transforms for Integrable Equations on **A.III**-type Symmetric Spaces*, SIGMA **7** (2011) 096 (48 pp).

[13] Gerdjikov V., Kostov N. and Valchev T., *Solutions of Multi-component NLS Models and Spinor Bose-Einstein Condensates*, Physica D **238** (2009) 1306–1310.

[14] Gerdjikov V., Kulish P. and Ivanov M., *Quadratic Bundle and Nonlinear Equations*, TMF **44** (1980) 342–357.

[15] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics **748**, Springer, Berlin 2008.

[16] Helgasson S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[17] Hirota R., *The Direct Method in Soliton Theory*, Cambridge Univ. Press, New York 2004.

[18] Ieda J., Miyakawa T. and Wadati M., *Exact Analysis of Soliton Dynamics in Spinor Bose-Einstein Condensates*, Phys. Rev Lett. **93** (2004) 194102.

[19] Ivanov R., *On the Dressing Method for the Generalised Zakharov-Shabat System*, Nuc. Phys. B **694** (2004) 509–524.

[20] Kaup D. and Newell A., *An Exact Solution for a Derivative Nonlinear Schrödinger Equation*, J. Math. Phys. **19** (1978) 798–801.

[21] Konopelchenko B., *The Polynomial Spectral Problem of Arbitrary Order: A General Form of the Integrable Equations and Bäcklund Transformations*, J. Phys. A: Math. & Gen. **14** (1981) 3125–3141.

[22] Konopelchenko B., *Introduction to Multidimensional Integrable Equations: The Inverse Spectral Transform in $2+1$ Dimensions*, Series: Plenum Monographs in Nonlinear Physics, Spinger, Berlin 1993.

[23] Kuznetsov E. and Mikhailov A., *On the Complete Integrability of the Two-dimensional Classical Thirring Model*, TMF **30** (1977) 303–314.

[24] Li L., Li Z., Malomed B., Mihalache D. and Liu W., *Exact Soliton Solutions and Nonlinear Modulation Instability in Spinor Bose-Einstein Condensates*, Phys. Rev. A **72** (2005) 033611.

[25] Manakov S., *Contribution to the theory of Two-dimensional Stationary Self-focusing of Electromagnetic Waves*, JETP **56** (1973) 505–516.

[26] Mikhailov A. *Integrability of the Two-dimensional Thirring Model*, JETP Lett. **23** (1976) 356–358.

[27] Mikhailov A., *Reductions in Integrable Systems. The Reduction Groups*, JETF Lett. **32** (1980) 187–192.

[28] Mikhailov A., *The Reduction Problem and the Inverse Scattering Method*, Physica D **3** (1981) 73–117.

[29] Mio K., Ogino T., Minami K. and Takeda S., *Modified Nonlinear Schrödinger Equation for Alfvén Waves Propagating along the Magnetic Field in Cold Plasmas*, J. Phys. Soc. Japan **41** (1976) 265–271.

[30] Mjølhus E., *On the Modulational Instability of Hydromagnetic Waves Parallel to the Magnetic Field*, J. Plasma Physics **16** (1976) 321–334.

[31] Polyanin A., Zaitsev V. and Zhurov A., *Method for Solving Nonlinear Equations of Mathematical Physics and Mechanics* (In Russian), FizMatLit, Moscow 2005.

[32] Tajiri M. and Watanabe Y., *Breather Solutions to the Focusing Nonlinear Schrödinger Equation*, Phys. Rev. E **57** (1998) 3510–3519.

[33] Uchiyama M., Ieda J. and Wadati M., *Multicomponent Bright Solitons in $F = 2$ Spinor Bose-Einstein Condensates*, J. Phys. Soc. Japan **76** (2007) 74005–74010.

[34] Yanovski A., *Recursion Operators and Bi-Hamiltonian Formulations of the Landau-Lifshitz Equation Hierarchies*, J. Phys. A: Math. & Gen. **39** (2006) 2409-2433.

[35] Yanovski A., *On the Recursion Operators for the Gerdjikov, Mikhailov and Valchev System*, JMP **52** (2011) 082703(14pp).

[36] Yanovski A., *Geometry of the Recursion Operators for Caudrey-Beals- Coifman System in the Presence of Mikhailov Type Z_p Reductions*, J. Geom. Symmetry Phys. **25** (2012) 77–97.

[37] Zakharov V. and Gelash A., *Soliton on Unstable Condensate*, (2011) 4 pages nlin.SI:1109.0620.

[38] Zakharov V. and Manakov S. *Resonant Interaction of Wave Packets in Nonlinear Media*, JETP Lett. **18** (1973) 413–417.

[39] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Plenum, New York 1984.

[40] Zakharov V. and Mikhailov A., *On the Integrability of Classical Spinor Models in Two-dimensional Space-time*, Commun. Math. Phys. **74** (1980) 21–40.

[41] Zakharov V. and Shabat A., *Integration of Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform II* (in Russian), Funct. Anal. Appl. **13** (1979) 13–22.

ZAKHAROV-SHABAT SYSTEM WITH CONSTANT BOUNDARY CONDITIONS. REFLECTIONLESS POTENTIALS AND END POINT SINGULARITIES

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Abstract. We consider scalar defocusing nonlinear Schrödinger equation with constant boundary conditions. We aim here to provide a self contained pedagogical exposition of the most important facts regarding integrability of that classical evolution equation. It comprises the following topics: direct and inverse scattering problem and the dressing method.

CONTENTS

1. Introduction

The integrability of the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0 \tag{1}$$

where $q : \mathbb{R}^2 \to \mathbb{C}$ is an infinitely smooth function and the subscripts mean partial differentiations, was discovered in the pioneer papers by Zakharov and Shabat [35, 36]. Historically, it was the second nonlinear evolution equations (NLEE) solved by means of the inverse scattering method after Gardner, Greene, Kruskal and Muira solved the Korteweg-de Vries equation [10, 11, 25] and proved its complete integrability. NLS has numerous applications in physics and mathematics. In nonlinear optics it models quasi-monochromatic wave packets propagating in nonlinear media [1] while in plasma physics it describes Langmuir waves in plasma [9, 19]. Another application of NLS is in fluid mechanics [31] where it appears in the context of deep water gravity waves. NLS also occurs in classical differential geometry of curves moving in three dimensional Euclidean space [26].

Despite the fact that NLS and its multicomponent counterparts have been thoroughly investigated [1, 7, 8, 14, 17, 22, 29, 34] they are still an attractive subject to study [6, 15, 18, 28, 32] due to recently established applications of the multicomponent NLS in Bose-Einstein condensation, see for example [16, 20, 30] and references therein. There is also increasing interest in derivation and study of nontrivial background solutions to NLS [2, 4, 5, 18]. The latter are claimed to model extreme physical phenomena like rogue (freak) waves [3, 27]. All this motivated the authors of the present paper to summarize the most important results about scalar NLS with constant boundary conditions.

In the following we shall deal with the repulsive (or defocusing) NLS equation [36]

$$\mathrm{i}q_t + q_{xx} - 2\left(|q|^2 - \rho^2\right)q = 0, \qquad \rho \in \mathbb{R}_+ \tag{2}$$

where the extra linear term introduced above ensures that the x-asymptotics of q

$$\lim_{x \to \pm\infty} q(x, t) = \rho \mathrm{e}^{\mathrm{i}\theta_\pm}, \qquad \theta_\pm \in [0, 2\pi) \tag{3}$$

are time independent. The purpose of the present paper is to give an accessible summary of results on inverse scattering method applied to the NLS equation (2) with boundary conditions (3). In doing this the we shall try, following [13, 14, 29], to provide a self-contained pedagogical exposition.

The paper is organized as follows. In Section 2 we introduce basic notions and facts needed for our further considerations. We outline the construction of the Jost solutions and of the fundamental analytic solutions (FAS) of L. In Section 3 we derive the singularities of the Jost solutions and the FAS in the vicinity of the points of the continuous spectrum of L. Sections 4 and Section 5 are dedicated to

two methods for integration of a given integrable NLEE. These are the Gelfand-Levitan-Marchenko integral equations and the Zakharov-Shabat's dressing technique. Taking into account the form of Lax pair and the boundary conditions for $q(x,t)$ specified in (3) one can fit both of these methods to effectively generate special solutions, dark solitons in particular. In Section 6 we introduce the kernel of the resolvent of L expressed in terms of the FAS and derive its singularities at the end points of the spectrum.

2. Preliminaries. Zakharov-Shabat Spectral Problem with Constant Boundary Conditions

In this section we are going to discuss some basic properties of the Zakharov-Shabat (AKNS) auxiliary linear system

$$i\partial_x \Psi(x,t,\lambda) + (Q(x,t) - \lambda\sigma_3)\Psi(x,t,\lambda) = 0 \tag{4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter and $\sigma_3 = \text{diag}(1,-1)$ is one of Pauli matrices. The potential

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -q^*(x,t) & 0 \end{pmatrix} \tag{5}$$

where $*$ stands for complex conjugation, is assumed to obey constant boundary conditions

$$\lim_{x\to\pm\infty} Q(x,t) = Q_\pm = \begin{pmatrix} 0 & q_\pm \\ -q_\pm^* & 0 \end{pmatrix}, \qquad q_\pm = \rho e^{i\theta_\pm}, \qquad \theta_\pm \in [0,2\pi). \tag{6}$$

Matrix-valued function Ψ is viewed as an arbitrary fundamental solution to (4), i.e., its columns are given by two linearly independent solutions to the Zakharov-Shabat linear system. It is our aim here to present the inverse scattering method (ISM) for (4). In doing this we are going to formulate direct scattering problem for Zakharov-Shabat systems with constant boundary conditions and analyze spectral properties of scattering operator

$$L(\lambda) = i\partial_x + Q(x,t) - \lambda\sigma_3 \tag{7}$$

introduced in (4). Our considerations shall mostly follow the ideas and the notation used in [12, 13, 17].

The linear system (4) with boundary conditions (6) underlies the inverse scattering method application to the defocusing NLS equation (2). It is equivalent to zero curvature condition $[L(\lambda), A(\lambda)] = 0$ for $L(\lambda)$ given by (7) and the second operator being in the form

$$A(\lambda)\Psi(x,t,\lambda) = i\partial_t \Psi + (V_0 + 2\lambda Q - 2\lambda^2\sigma_3)\Psi(x,t,\lambda) = \Psi(x,t,\lambda)f(\lambda)$$

$$V_0 = \left(\rho^2 - |q|^2\right)\sigma_3 + \frac{i}{2}[\sigma_3, Q_x]. \tag{8}$$

The function $f(\lambda)$ will be calculated below; it determines the dispersion law of the NLS.

Due to intrinsic $U(1)$ symmetry of equation (2) we can set one of the phases in the asymptotic values of Q to be zero. Thus from now on we shall fix $\theta_- = 0$ and will denote the other phase simply by θ in order to simplify our notation.

The space of allowed potentials $\mathcal{M}_{\rho,\theta}$, i.e., the set of all smooth matrix-valued functions of the form (5) satisfying boundary condition (6) for ρ and θ being fixed, **is not** a linear space. However, the difference of any two allowed potentials $Q_1(x,t)$ and $Q_2(x,t) \in \mathcal{M}_{\rho,\theta}$ is such that $Q_1(x,t) - Q_2(x,t)$ vanishes as $x \to \pm\infty$. Thus we can obtain the whole space $\mathcal{M}_{\rho,\theta}$ by adding up to a fixed allowed potential any potential satisfying vanishing boundary conditions.

The form of the potential implies that it obeys the following symmetry condition

$$\sigma_3 Q^\dagger(x,t)\sigma_3 = Q(x,t) \tag{9}$$

where $\dagger$ stands for Hermitian conjugation. Following the concepts by Mikhailov [23, 24] one can interpret (9) in terms of a finite reduction group acting on the set of solutions $\{\Psi(x,t,\lambda)\}$ to the Zakharov-Shabat system. In our case the reduction group is $\mathbb{Z}_2$ acting on the fundamental solutions in the following way

$$\sigma_3 \hat{\Psi}^\dagger(x,t,\lambda^*)\sigma_3 = \Psi(x,t,\lambda) \tag{10}$$

where $\Psi(x,t,\lambda)$ is any of the Jost solutions of (4) and the 'hat' means the matrix inverse

$$\hat{\Psi}(x,t,\lambda) = [\Psi(x,t,\lambda)]^{-1}.$$

It proves to be convenient to introduce a gauge transform denoted by $\varphi_\pm(\lambda)$ to diagonalize the asymptotic values of $Q(x,t) - \lambda\sigma_3$ when $x \to \pm\infty$, i.e., we have

$$\varphi_\pm(\lambda)(Q_\pm - \lambda\sigma_3)\hat{\varphi}_\pm(\lambda) = -j(\lambda)\sigma_3, \qquad j(\lambda) = \sqrt{\lambda^2 - \rho^2} \tag{11}$$

where $\varphi_\pm$ are given by

$$\varphi_\pm(\lambda) = \frac{1}{\sqrt{2j(\lambda)(\lambda + j(\lambda))}} \begin{pmatrix} \lambda + j(\lambda) & -q_\pm \\ -q_\pm^* & \lambda + j(\lambda) \end{pmatrix}. \tag{12}$$

Thus the spectral parameter λ lives in a two-sheet Riemann surface $\mathcal{S} \equiv \mathcal{S}_+ \cup \mathcal{S}_-$ associated with $j(\lambda)$. To construct $\mathcal{S}$ one cuts the complex plane from $-\infty$ to $-\rho$ and from ρ to ∞ along real axis. The sheets $\mathcal{S}_+$ and $\mathcal{S}_-$ are determined by

$$\mathcal{S}_+ : \operatorname{Im} j(\lambda) > 0, \qquad \mathcal{S}_- : \operatorname{Im} j(\lambda) < 0. \tag{13}$$

The transform $\varphi_\pm(\lambda)$ allows one to define Jost solutions through the equality below

$$\lim_{x\to\infty} \psi(x,t,\lambda)\hat{E}_+(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,t,\lambda)\hat{E}_-(x,\lambda) = \mathbb{1} \tag{14}$$

where

$$E_\pm(x,\lambda) = \hat{\varphi}_\pm(\lambda)\mathrm{e}^{-\mathrm{i}j(\lambda)\sigma_3 x} \tag{15}$$

are solutions to the equation

$$i\partial_x E_\pm + (Q_\pm - \lambda\sigma_3)E_\pm = 0.$$

It straightforwardly follows from (15) that $\psi(x,t,\lambda)$ and $\phi(x,t,\lambda)$ are unimodular matrices. The transition matrix

$$T(t,\lambda) = \hat{\psi}(x,t,\lambda)\phi(x,t,\lambda) \tag{16}$$

between the Jost solutions is called scattering matrix. As a result of the reduction (10) we deduce that the scattering matrix obeys the symmetry

$$\sigma_3\hat{T}^\dagger(t,\lambda^*)\sigma_3 = T(t,\lambda). \tag{17}$$

From the compatibility of the linear problems (4) and (8) we find that the time evolution of T is determined by

$$T(t,\lambda) = e^{if(\lambda)t}T(0,\lambda)e^{-if(\lambda)t} \tag{18}$$

where the dispersion law $f(\lambda)$ for the NLS is given by

$$f(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) = -2\lambda j(\lambda)\sigma_3. \tag{19}$$

Further on in text the variable t will not be essential so we shall omit it.

Let us now discuss the spectral properties of the scattering operator. Generally speaking the spectrum of $L(\lambda)$ consists of a continuous and a discrete part. The operator (7) is equivalent to a self-adjoint eigenvalue problem

$$\mathcal{L}\Psi \equiv i\sigma_3\frac{\partial\Psi}{\partial x} + \sigma_3 Q(x)\Psi(x,\lambda) = \lambda\Psi(x,\lambda) \tag{20}$$

and therefore its spectrum must be on the real axis. The continuous part of its spectrum is determined by equality Im $j(\lambda) = 0$, i.e., it coincides with set $\mathbb{R}_\rho = (-\infty,-\rho) \cup (\rho,\infty)$. The discrete eigenvalues of $L(\lambda)$ are simple and must belong to $(-\rho,\rho)$. The Jost solutions and the scattering matrix are defined for $\lambda \in \mathbb{R}_\rho$ only. To see this one needs to introduce the auxiliary functions

$$\begin{aligned}
\xi_+(x,\lambda) &= \varphi_+(\lambda)\psi(x,\lambda)\hat{E}_+(x,\lambda)\hat{\varphi}_+(\lambda) \\
\xi_-(x,\lambda) &= \varphi_-(\lambda)\phi(x,\lambda)\hat{E}_-(x,\lambda)\hat{\varphi}_-(\lambda)
\end{aligned} \tag{21}$$

which satisfy the differential equation

$$i\partial_x\xi_\pm + \tilde{Q}_+(x,\lambda)\xi_\pm - j(\lambda)[\sigma_3,\xi_\pm] = 0 \tag{22}$$

where

$$\tilde{Q}_\pm(x,\lambda) = \varphi_\pm(\lambda)(Q(x) - Q_\pm)\hat{\varphi}_\pm(\lambda).$$

Equivalently, $\xi_\pm$ can be viewed as solutions to the Volterra type integral equation

$$\xi_\pm(x,\lambda) = \mathbb{1} + i\int_{\pm\infty}^x e^{ij(\lambda)\sigma_3(y-x)}\tilde{Q}_\pm(y,\lambda)\xi_\pm(y,\lambda)e^{-ij(\lambda)\sigma_3(y-x)}dy. \tag{23}$$

Let us consider now the function ξ_+. It is easily seen from the integral equation that the second column of ξ_+ is analytic on $\mathcal{S}_+$ hence the second column of $\psi(x, \lambda)$ is analytic on $\mathcal{S}_+$. Similarly, the first column of $\phi(x, \lambda)$ are analytic on $\mathcal{S}_+$ while the second one as well as the first column of $\psi(x, \lambda)$ are analytic on $\mathcal{S}_-$. That is why we will denote the Jost solutions by

$$\psi(x, \lambda) = ||\psi^-(x, \lambda), \psi^+(x, \lambda)||, \qquad \phi(x, \lambda) = ||\phi^+(x, \lambda), \phi^-(x, \lambda)|| \quad (24)$$

where the superscript $+$ (respectively $-$) refers to the analyticity properties of the corresponding column on the sheet $\mathcal{S}_+$ (respectively on $\mathcal{S}_-$).

As a result of the above considerations one can construct another pair of solutions χ^+ and χ^- which are analytic in $\mathcal{S}_+$ and $\mathcal{S}_-$ respectively. This is done using the formulae below

$$\chi^+(x, \lambda) \equiv ||\phi^+(x, \lambda), \psi^+(x, \lambda)|| = \phi(x, \lambda)S^+(\lambda) = \psi(x, \lambda)T^-(\lambda)$$
$$\chi^-(x, \lambda) \equiv ||\psi^-(x, \lambda), \phi^-(x, \lambda)|| = \phi(x, \lambda)S^-(\lambda) = \psi(x, \lambda)T^+(\lambda) \quad (25)$$

where

$$S^+(\lambda) = \begin{pmatrix} 1 & -b^*(\lambda) \\ 0 & a(\lambda) \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ b(\lambda) & 1 \end{pmatrix} \quad (26)$$

$$S^-(\lambda) = \begin{pmatrix} a^*(\lambda) & 0 \\ -b(\lambda) & 1 \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} 1 & b^*(\lambda) \\ 0 & a^*(\lambda) \end{pmatrix}. \quad (27)$$

The triangular matrices $S^\pm(\lambda)$ and $T^\pm(\lambda)$ are LU-decomposition of the scattering matrix $T(\lambda)$

$$T(\lambda) = \begin{pmatrix} a(\lambda) & b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix} = T^-(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)\hat{S}^-(\lambda). \quad (28)$$

Due to reduction (10) it is seen that χ^+ and χ^- satisfy relation

$$\sigma_3 \left[\hat{\chi}^+(x, \lambda^*) \right]^\dagger \sigma_3 = \chi^-(x, \lambda). \quad (29)$$

The fundamental analytic solutions satisfy the following interrelation

$$\chi^-(x, \lambda) = \chi^+(x, \lambda)G(\lambda), \qquad \lambda \in \mathbb{R}_\rho. \quad (30)$$

This is a manifestation of the fact that the fundamental analytic solutions satisfy local Riemann-Hilbert problem [17, 29, 34].

It also follows from (25)–(27) that

$$\det \chi^+(x, \lambda) = a(\lambda). \quad (31)$$

Therefore $a(\lambda)$ is an analytic function on the whole sheet $\mathcal{S}^+$. In what follows we will need to know the structure of χ^+ and χ^- and their inverse in the vicinity

of discrete eigenvalues $\{\lambda_j\}_{j=1}^{N}$ of the operator $L(\lambda)$. The discrete eigenvalues are simple zeroes of $a(\lambda)$, i.e., in vicinity of λ_j we have the following Taylor expansion

$$a(\lambda) = (\lambda - \lambda_j)\left(\dot{a}_j + \frac{1}{2}\ddot{a}_j(\lambda - \lambda_j) + \cdots\right) \qquad (32)$$

where dot stands for the differentiation with respect to λ. Due to (31) at any point λ_j the columns of $\chi^+(x, \lambda_j)$ are proportional to each other, i.e., there exists $b_j \in \mathbb{C}$ such that

$$\phi_j^+(x) = b_j \psi_j^+(x), \qquad \phi_j^-(x) = \frac{1}{b_j^*}\psi_j^-(x) \qquad (33)$$

where $\phi_j^{\pm}(x) = \phi^{\pm}(x, \lambda_j$ and $\psi_j^{\pm}(x) = \psi^{\pm}(x, \lambda_j$. Thus we find

$$\begin{aligned}
\chi_j^+(x, \lambda_j) &= \psi_j^+(x)(b_j, 1) = \phi_j^+(x)(1, 1/b_j) \\
\chi_j^-(x, \lambda_j) &= \psi_j^-(x)(1, -b_j^*) = \phi_j^-(x)(-1/b_j^*, 1).
\end{aligned} \qquad (34)$$

In what follows we will need similar formulae also for the inverse of $\chi^{\pm}(x, \lambda)$ for $\lambda \simeq \lambda_j$. It is easy to see that

$$\hat{\chi}^+(x, \lambda) = \frac{1}{a(\lambda)}\begin{pmatrix}\tilde{\psi}^+ \\ -\tilde{\phi}^+\end{pmatrix}, \qquad \hat{\chi}^-(x, \lambda) = \frac{1}{a^*(\lambda)}\begin{pmatrix}\tilde{\phi}^- \\ -\tilde{\psi}^-\end{pmatrix} \qquad (35)$$

where the operation 'tilde' applied to the vector $\begin{pmatrix}y_1 \\ y_2\end{pmatrix}$ maps it onto the row $(y_2, -y_1)$. Thus we obtain

$$\begin{aligned}
\hat{\chi}^+(x, \lambda) &\underset{\lambda \to \lambda_j}{\simeq} \begin{pmatrix}1 \\ -b_j\end{pmatrix}\frac{\tilde{\psi}_j^+}{(\lambda - \lambda_j)\dot{a}_j} \simeq \begin{pmatrix}-1/b_j \\ 1\end{pmatrix}\frac{\tilde{\phi}_j^+}{(\lambda - \lambda_j)\dot{a}_j} \\
\hat{\chi}^-(x, \lambda) &\underset{\lambda \to \lambda_j}{\simeq} \begin{pmatrix}1 \\ -1/b_j^*\end{pmatrix}\frac{\tilde{\psi}_j^-}{(\lambda - \lambda_j)\dot{a}_j^*} \simeq -\begin{pmatrix}b_j^* \\ 1\end{pmatrix}\frac{\tilde{\phi}_j^-}{(\lambda - \lambda_j)\dot{a}_j^*}.
\end{aligned} \qquad (36)$$

Given the potential $Q(x)$ one can obtain the Jost solutions uniquely by solving the integral equations (23). The Jost solutions in turn determine uniquely the scattering matrix $T(\lambda)$. $Q(x)$ contains one independent complex-valued function $q(x)$ of x. Thus it is natural to expect that only one of the coefficients of $T(\lambda)$ for $\lambda \in \mathbb{R}_\rho$, will be independent.

At the same time the matrix elements of $T(\lambda)$ (28) are determined by the complex-valued functions $a(\lambda)$ and $b(\lambda)$ and their complex conjugate, satisfying the condition $|a|^2 - |b|^2 = 1$. It is important that $a(\lambda)$ (respectively $a^*(\lambda)$) are analytic functions of λ for $\lambda \in \mathcal{S}_+$ (respectively $\lambda \in \mathcal{S}_-$). This fact allows one to determine $a(\lambda)$ using its values on the cuts $\mathbb{R}_\rho$ and the set of its zeroes λ_j. We assume that $a(\lambda)$ has a finite number of zeroes λ_j; it is well known that $a(\lambda)$ may have

only simple zeroes [13, 14, 29]. Skipping the details we introduce two equivalent minimal sets of scattering data

$$\mathcal{T}_1 \equiv \mathcal{T}_{1,c} \cup \mathcal{T}_{1,d}, \qquad \mathcal{T}_2 \equiv \mathcal{T}_{2,c} \cup \mathcal{T}_{2,d} \tag{37}$$

where

$$\mathcal{T}_{1,c} \equiv \{\rho(\lambda), \quad \lambda \in \mathbb{R}_\rho\}, \qquad \mathcal{T}_{1,d} \equiv \{C_j, \lambda_j\}_{j=1}^N$$
$$\mathcal{T}_{2,c} \equiv \{\tau(\lambda), \quad \lambda \in \mathbb{R}_\rho\}, \qquad \mathcal{T}_{2,d} \equiv \{M_j, \lambda_j\}_{j=1}^N. \tag{38}$$

The reflection coefficients $\rho(\lambda)$ and $\tau(\lambda)$ and the coefficients C_j and M_j are given by

$$\rho(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \qquad \tau(\lambda) = -\frac{b^*(\lambda)}{a(\lambda)}, \qquad C_j = \frac{b_j}{\dot{a}_j}, \qquad M_j = \frac{1}{b_j \dot{a}_j}. \tag{39}$$

Now we need to compute the asymptotics of the FAS solutions for large values of $|\lambda|$. The results are summarized in the tables below

	$\mathrm{Im}\,\lambda > 0, \ \lambda \in \mathcal{S}_+$	$\mathrm{Im}\,\lambda < 0, \ \lambda \in \mathcal{S}_+$	
$\chi^+(x,\lambda)\mathrm{e}^{\mathrm{i}j(\lambda)\sigma_3 x}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$-\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta_+} \\ \mathrm{e}^{-\mathrm{i}\theta_-} & 0 \end{pmatrix}$	(40)
$a^+(\lambda)$	1	$\mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}$	

	$\mathrm{Im}\,\lambda > 0, \ \lambda \in \mathcal{S}_-$	$\mathrm{Im}\,\lambda < 0, \ \lambda \in \mathcal{S}_-$	
$\chi^-(x,\lambda)\mathrm{e}^{\mathrm{i}j(\lambda)\sigma_3 x}$	$-\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta_-} \\ \mathrm{e}^{-\mathrm{i}\theta_+} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(41)
$a^-(\lambda)$	$\mathrm{e}^{-\mathrm{i}(\theta_+ - \theta_-)}$	$1.$	

The functions $b^\pm(\lambda)$ in general do not admit analytic continuations, however if one considers special functional classes one can argue that [17, 29]

$$\lim_{|\lambda| \to \infty} b(\lambda) = 0.$$

3. The End Points of the Spectrum

The Lax operator L with constant boundary condition is one of the basic examples of ordinary differential operator whose continuous spectrum multiplicity varies. Obviously the continuous spectrum on the rays $\mathbb{R}_\rho \equiv (-\infty, -\rho) \cup (\rho, \infty)$ has multiplicity 2, while on the lacuna $(-\rho, \rho)$ it has vanishing multiplicity.

In the limit $\lambda \to \lambda\varepsilon\rho$ the matrices $\varphi_\pm$ become singular. So in order to introduce the Jost solutions we will need regularized definitions, see below. Besides, we have to

take into account that the eigenfunctions of $L_{\varepsilon\rho,\mathrm{as}}$ are given by

$$
L_{\pm\mathrm{as}}|_{\lambda=\varepsilon\rho}E_{\varepsilon\rho} = 0, \qquad E_{\varepsilon\rho} = \begin{pmatrix} 1 & \mathrm{i}\rho x \mathrm{e}^{\mathrm{i}\theta_\pm} \\ \varepsilon\rho\mathrm{e}^{-\mathrm{i}\theta_\pm} & \mathrm{i}\rho\varepsilon x + 1 \end{pmatrix} \tag{42}
$$

where $\varepsilon = \pm 1$. The inverse of $E_{\varepsilon\rho}$ is

$$
\hat{E}_{\varepsilon\rho} = \begin{pmatrix} \mathrm{i}\rho\varepsilon x + 1 & -\rho x \mathrm{e}^{\mathrm{i}\theta_\pm} \\ -\varepsilon\rho\mathrm{e}^{\mathrm{i}\theta_\pm} & 1 \end{pmatrix}. \tag{43}
$$

Thus the Jost solutions for $\lambda = \varepsilon\rho$ may be defined by

$$
\lim_{\lambda\to\varepsilon\rho}\lim_{x\to\infty}\hat{E}_{\varepsilon\rho}\psi(x,\varepsilon\rho) = \mathbb{1}, \qquad \lim_{\lambda\to\varepsilon\rho}\lim_{x\to-\infty}\hat{E}_{\varepsilon\rho}\phi(x,\varepsilon\rho) = \mathbb{1}. \tag{44}
$$

This definition shows that the Jost solutions and the FAS may develop singularities at the end points of the spectrum. This definition is valid for the generic case, because the Lax operator L will have two linearly independent eigenfunctions also at the end points $\lambda = \varepsilon\rho$. Along with the generic case we will consider also the possibility of virtual eigenvalues at the end points. In general both possibilities have been analyzed in [29]. We will present slightly different approach which is gauge covariant.

Skipping the details we collect the formulae, giving the asymptotics of the Jost solutions, the FAS and the scattering matrix for $\lambda \to \varepsilon\rho$.

3.1. Generic Case – Asymptotics

The Jost solutions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ develop singularities at the end points of the spectrum, which are consequence of the singularity of $E_\pm(x,\lambda)$. Indeed, for $\lambda \to \varepsilon\rho$ we have

$$
E_\pm(x,\lambda) = \sqrt{\frac{\rho}{j(\lambda)}}\left(E_{\varepsilon,0} + \frac{j(\lambda)}{2\rho}E_{\varepsilon,1} + \mathcal{O}(j^2)\right)\mathrm{e}^{-\mathrm{i}j(\lambda)x\sigma_3}
$$

$$
E_{\varepsilon,0} = \begin{pmatrix} 1 & -\mathrm{e}^{\mathrm{i}\theta_\pm} \\ -\mathrm{e}^{-\mathrm{i}\theta_\pm} & 1 \end{pmatrix}, \qquad E_{\varepsilon,1} = \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\theta_\pm} \\ \mathrm{e}^{-\mathrm{i}\theta_\pm} & 1 \end{pmatrix}. \tag{45}
$$

In the vicinity of end points $\lambda \simeq \varepsilon\rho$ the Jost solutions become

$$
\psi^\pm(x,\lambda) = \sqrt{\frac{\rho}{2j(\lambda)}}\left(\psi^\pm_{\varepsilon\rho,0}(x) + j(\lambda)\psi^\pm_{\varepsilon\rho,1}(x) + \mathcal{O}(j^2)\right)
$$

$$
\phi^\pm(x,\lambda) = \sqrt{\frac{\rho}{2j(\lambda)}}\left(\phi^\pm_{\varepsilon\rho,0}(x) + j(\lambda)\phi^\pm_{\varepsilon\rho,1}(x) + \mathcal{O}(j^2)\right) \tag{46}
$$

$$a_{\varepsilon\rho} = \frac{a_{\varepsilon\rho,0}}{j(\lambda)} + a_{\varepsilon\rho,1} + \mathcal{O}(j), \qquad b_{\varepsilon\rho} = -\varepsilon\frac{\mathrm{i}b_{\varepsilon\rho,0}}{j(\lambda)} + b_{\varepsilon\rho,1} + \mathcal{O}(j) \tag{47}$$

$$\psi_{\varepsilon\rho}^-(x) = \varepsilon\mathrm{i}\psi_{\varepsilon\rho}^+(x), \qquad \phi_{\varepsilon\rho}^+(x) = \varepsilon\mathrm{i}\phi_{\varepsilon\rho}^-(x)$$

i.e.,

$$\lim_{\lambda\to\pm\rho} \frac{a_{\varepsilon\rho}}{b_{\varepsilon\rho}} = \varepsilon\mathrm{i}.$$

Similar relations can be derived also for the asymptotic values of the FAS, namely

$$\chi_{\varepsilon\rho}^+(x,\lambda) \equiv ||\phi_{\varepsilon\rho}^+(x), \psi_{\varepsilon\rho}^+(x)||$$

$$= \sqrt{\frac{\rho}{2j(\lambda)}} \left(\chi_{\varepsilon\rho,0}^+(x) + j(\lambda)\chi_{\varepsilon\rho,1}^+(x) + \mathcal{O}(j^2) \right)$$

$$\chi_{\varepsilon\rho}^-(x,\lambda) \equiv ||\psi_{\varepsilon\rho}^-(x), \phi_{\varepsilon\rho}^-(x)|| \tag{48}$$

$$= \sqrt{\frac{\rho}{2j(\lambda)}} \left(\chi_{\varepsilon\rho,0}^-(x) + j(\lambda)\chi_{\varepsilon\rho,1}^-(x) + \mathcal{O}(j^2) \right)$$

as well as for their inverse

$$\hat{\chi}_{\varepsilon\rho}^+(x,\lambda) = \sqrt{\frac{2j(\lambda)}{\rho}} \left(\hat{\chi}_{\varepsilon\rho,0}^+(x) - j(\lambda)\hat{\chi}_{\varepsilon\rho,0}^+(x)\chi_{\varepsilon\rho,1}^+(x)\hat{\chi}_{\varepsilon\rho,0}^+(x) + \mathcal{O}(j^2) \right)$$

$$\hat{\chi}_{\varepsilon\rho}^-(x,\lambda) = \sqrt{\frac{2j(\lambda)}{\rho}} \left(\hat{\chi}_{\varepsilon\rho,0}^-(x) - j(\lambda)\hat{\chi}_{\varepsilon\rho,0}^-(x)\chi_{\varepsilon\rho,1}^-(x)\hat{\chi}_{\varepsilon\rho,0}^-(x) + \mathcal{O}(j^2) \right) \tag{49}$$

where $\hat{\chi}_{\varepsilon\rho,0}^{\pm}(x) = (\chi_{\varepsilon\rho,0}^-(x))^{-1}$.

3.2. Virtual Eigenvalue – Asymptotics

In this case the Jost solutions become degenerate for $\lambda = \varepsilon\rho$, i.e., the matrices $\psi_{\varepsilon\rho,0}(x,\lambda)$ are $\phi_{\varepsilon\rho,0}(x,\lambda)$ become degenerate

$$\psi_{\varepsilon\rho,0}(x) = \sqrt{\frac{\rho}{2j(\lambda)}}\psi_{\varepsilon\rho,0}^-(x)(1,\varepsilon\mathrm{i}), \quad \phi_{\varepsilon\rho,0}(x) = \sqrt{\frac{\rho}{2j(\lambda)}}\phi_{\varepsilon\rho,0}^+(x,\lambda)(1,\varepsilon\mathrm{i}) \tag{50}$$

Then both $a(\lambda)$ and $b(\lambda)$ are regular for $\lambda \to \pm\rho$

$$\lim_{\lambda\to\varepsilon\rho} a_{\varepsilon\rho} = a_{\varepsilon\rho,1}, \qquad \lim_{\lambda\to\varepsilon\rho} b_{\varepsilon\rho} = b_{\varepsilon\rho,1}. \tag{51}$$

Similarly we can analyze the behavior of the FAS in the vicinity of the end points of the spectrum. The results are

$$
\chi_{\varepsilon\rho}^+(x) = \sqrt{\frac{\rho}{2j(\lambda)}}\,\psi_{\varepsilon\rho}^+(x)(b_{\varepsilon\rho},1) = \sqrt{\frac{\rho}{2j(\lambda)}}\,\phi_{\varepsilon\rho}^+(x)(1,1/b_{\varepsilon\rho})
$$

$$
\chi_{\varepsilon\rho}^-(x,\lambda_j) = \sqrt{\frac{\rho}{2j(\lambda)}}\,\psi_{\varepsilon\rho}^-(x)(1,-b_{\varepsilon\rho}^*) = \sqrt{\frac{\rho}{2j(\lambda)}}\,\phi_{\varepsilon\rho}^-(x)(-1/b_{\varepsilon\rho}^*,1).
$$

(52)

and

$$
\hat{\chi}^+(x,\lambda) \underset{\lambda\to\varepsilon\rho}{\simeq} \sqrt{\frac{\rho}{2j(\lambda)}}\begin{pmatrix}1\\-b_{\varepsilon\rho}\end{pmatrix}\tilde{\psi}_{\varepsilon\rho}^+ \simeq \sqrt{\frac{\rho}{2j(\lambda)}}\begin{pmatrix}-1/b_{\varepsilon\rho}\\1\end{pmatrix}\tilde{\phi}_{\varepsilon\rho}^+
$$

$$
\hat{\chi}^-(x,\lambda) \underset{\lambda\to\varepsilon\rho}{\simeq} -\sqrt{\frac{\rho}{2j(\lambda)}}\begin{pmatrix}1/b_{\varepsilon\rho}^*\\1\end{pmatrix}\tilde{\psi}_{\varepsilon\rho}^- \simeq \sqrt{\frac{\rho}{2j(\lambda)}}\begin{pmatrix}1\\b_{\varepsilon\rho}\end{pmatrix}\tilde{\phi}_{\varepsilon\rho}^-.
$$

(53)

We will use this in Section 6 below for analyzing the singularities of the resolvent at the end points of the spectrum.

4. Gelfand-Levitan-Marchenko Equations

An effective method to derive soliton solutions and the corresponding eigenfunctions is based upon the Gelfand-Levitan-Marchenko integral equations (GLM). Here we just briefly outline the basic facts of derivation of GLM [13, 17, 29].

Let us consider the linear problems

$$
L_0\Psi_0 \equiv i\partial_x\Psi_0 + (Q_0 - \lambda\sigma_3)\Psi_0 = 0 \tag{54}
$$

$$
L\Psi \equiv i\partial_x\Psi + (Q - \lambda\sigma_3)\Psi = 0 \tag{55}
$$

where both potentials are of the form (5) and have equal asymptotic values as $x \to \infty$, i.e.,

$$
\lim_{x\to\infty} Q_0(x) = \lim_{x\to\infty} Q(x) = Q_+. \tag{56}
$$

The Jost solutions of (54) and (55) by definition obey the equalities

$$
\lim_{x\to\infty} \psi_0(x,\lambda)\hat{E}_+(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,\lambda)\hat{E}_+(x,\lambda) = \mathbb{1}. \tag{57}
$$

We assume now that Q_0 is known and we shall refer to it as bare (or seed) potential, while the other one, to be found, will be called dressed potential

Remark 1. *It is not possible for that the bare potential and the dressed one share the same asymptotic values both at $x \to -\infty$ and $x \to \infty$. For the derivation of* GLM *it suffices to have consistency of just one of the asymptotics, say as $x \to \infty$.*

The dressed Jost solution can be expressed from the bare one through the integral transformation

$$\psi(x, \lambda) = \psi_0(x, \lambda) + \int_\infty^x dy\, \Gamma_+(x, y)\psi_0(y, \lambda)$$

$$\phi(x, \lambda) = \phi_0(x, \lambda) + \int_{-\infty}^x dy\, \Gamma_-(x, y)\phi_0(y, \lambda)$$

(58)

where the integral kernels $\Gamma_\pm$ satisfy

$$\lim_{y \to \infty} \Gamma_+(x, y) = 0, \qquad \lim_{y \to -\infty} \Gamma_-(x, y) = 0.$$

(59)

In order for transformation (58) to be consistent the kernel must satisfy certain differential constraints. Indeed, after substituting (58) into (55) and taking into account that the bare Jost solution fulfils (54) one obtains the following relations

$$i\frac{\partial \Gamma_\pm(x, y)}{\partial x} + i\sigma_3 \frac{\partial \Gamma_\pm(x, y)}{\partial y}\sigma_3 + Q(x)\Gamma_\pm(x, y) - \sigma_3\Gamma_\pm(x, y)\sigma_3 Q_0(y) = 0$$

$$Q(x) - Q_0(x) + i(\Gamma_\pm(x, x) - \sigma_3\Gamma_\pm(x, x)\sigma_3) = 0.$$

(60)

Obviously the solutions of (60) will be parametrized by the scattering data of both operators L_0 and L.

Let us develop this idea first considering the Jost solution $\psi(x, \lambda)$ and the transformation operator with kernel $\Gamma_+(x, y)$. This transformation operator maps the Jost solution $\psi_0(x, \lambda)$ of the 'naked' operator L_0 into the Jost solution $\psi(x, \lambda)$ of the 'dressed' operator L. Each of these operators has its own scattering data (38): reflection coefficients $\rho_0(\lambda)$ and $\rho(\lambda)$ and sets of discrete eigenvalues

$$\mathcal{D}_0 \equiv \{\lambda_{0j} : a_0(\lambda_{0j}) = 0\}_{j=1}^{N_0}, \qquad \mathcal{D} \equiv \{\lambda_j : a(\lambda_j) = 0\}_{j=1}^N.$$

Note that some of the eigenvalues of L_0 may coincide with the eigenvalues of L. The generic solution of (60) can be presented in the form

$$\Gamma_+(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}_\rho} d\lambda \left(c_+(\lambda)\psi^+(x, \lambda)\hat{\psi}_0^+(y, \lambda) - c_+^*(\lambda)\psi^-(x, \lambda)\hat{\psi}_0^-(y, \lambda) \right) \sigma_3$$

$$- \sum_{\lambda_j \in \mathcal{D}} c_j \psi_j^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j \in \mathcal{D}_0} c_{0j} \psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3$$

(61)

where $\psi_{0j}^+(x) = \psi_0^+(x, \lambda_j)$ and $\psi_j^+(x) = \psi^+(x, \lambda_j)$. The coefficients c_j, c_{0j} and the function $c(\lambda)$ can be expressed in terms of the scattering data, e.g. $c(\lambda) = \rho(\lambda) - \rho_0(\lambda)$.

By $\psi_0^\pm(x, \lambda)$ and $\psi^\pm(x, \lambda)$ we mean the corresponding columns of Jost solutions (see (24)) to (54) and (55) respectively, while $\tilde{\psi}_j(x)$ and $\tilde{\psi}_{0,j}(x)$ are row vectors

of inverse Jost solutions (see (35)). Obviously they satisfy

$$i\partial_x \psi_0^{\pm}(x) + (Q_0(x) - \lambda\sigma_3)\,\psi_0^{\pm}(x, \lambda) = 0$$
$$i\partial_x \psi^{\pm}(x) + (Q(x) - \lambda\sigma_3)\,\psi^{\pm}(x, \lambda) = 0$$

(62)

and

$$i\partial_x \tilde{\psi}_0^{\pm}(x) - \tilde{\psi}_0^{\pm}(x, \lambda)\,(Q_0(x) - \lambda\sigma_3) = 0$$
$$i\partial_x \tilde{\psi}^{\pm}(x) - \tilde{\psi}^{\pm}(x, \lambda)\,(Q(x) - \lambda\sigma_3) = 0$$

(63)

respectively. Therefore $\psi_{0j}^+(x)$, $\psi_j^+(x)$ and $\tilde{\psi}_{0j}^+(x)$, $\tilde{\psi}_j^+(x)$ will be solutions to (62) and (63) with $\lambda = \lambda_{0j}$ and $\lambda = \lambda_j$ respectively.

Taking into account that from equation (58) there follows

$$\psi^{\pm}(x, \lambda) = \psi_0^{\pm}(x, \lambda) + \int_{\infty}^{x} \mathrm{d}y\, \Gamma_+(x, y)\psi_0^{\pm}(y, \lambda)$$
$$\psi_j^{\pm}(x) = \psi_{0j}^{\pm}(x) + \int_{\infty}^{x} \mathrm{d}y\, \Gamma_+(x, y)\psi_{0j}^{\pm}(y)$$

(64)

we can easily rewrite equation (61) as the well GLM equation

$$\Gamma_+(x, y) + F_+(x, y) + \int_{\infty}^{x} \mathrm{d}z\, \Gamma_+(x, z)F_+(z, y) = 0$$

(65)

where

$$F_+(x, y) = \frac{-1}{2\pi} \int_{\mathbb{R}_\rho} \mathrm{d}\lambda\, \left(c(\lambda)\psi_0^+(x, \lambda)\hat{\psi}_0^+(y, \lambda) - c^*(\lambda)\psi_0^-(x, \lambda)\hat{\psi}_0^-(y, \lambda) \right)\sigma_3$$
$$+ \sum_{\lambda_j \in \mathcal{D}} c_j \psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3 - \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}\psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3.$$

(66)

This is the most general form of the GLM equation which relates two Lax operators L and L_0 with generic choice for their spectral data. If one knows the Jost solutions of the 'naked' operator L_0 then solving the GLM equation one can construct the Jost solutions of the 'dressed' operator L. However, the Jost solutions of L_0 for generic spectral data (i.e., non-vanishing $\rho_0(\lambda)$) can not be evaluated explicitly. There is however, a special class of potential – the so-called reflectionless potentials, that can be constructed explicitly by solving the GLM equation

Indeed, let us now assume that the $Q_0(x, t) = Q_+$. Then $\psi_0(x, \lambda) = E_+(x, \lambda)$ and the kernel F is degenerate, i.e., we have $c(\lambda) = c_0(\lambda) = 0$, $N_0 = 0$ and $N = 1$. In this special case GLM equations reduces to a set of linear algebraic equations and one can construct the solution explicitly. For $N = 1$ equations (61) and (64) lead to the following result for the dressed solutions

$$\psi_1^+(x) = \left(1 - c_1 \int_{\infty}^{x} \left(\tilde{\psi}_{0,1}^+(y)|\psi_{0,1}^+(y) \right) \right)^{-1} \psi_{0,1}^+(x).$$

(67)

Taking into account the explicit form of $E_+(x, \lambda)$ (see equalities (12) and (15)) one can perform the integration above to obtain

$$\psi_1^+(x) = \frac{1}{1 + V_0(x)} \psi_{0,1}^+(x), \qquad V_0(x) = \frac{c_1 \rho e^{-2\sqrt{\rho^2 - \lambda_1^2} x} e^{i\theta}}{2i(\rho^2 - \lambda_1^2)}. \qquad (68)$$

After substituting (68) into (61) for the transformation operator kernel we get

$$\Gamma_+(x, y) = \frac{\rho \sin \theta_1 V_0(x) e^{\sqrt{\rho^2 - \lambda_1^2}(x-y)}}{1 + V_0(x)} \begin{pmatrix} 1 & e^{i(\theta - \theta_1)} \\ e^{-i(\theta - \theta_1)} & 1 \end{pmatrix}. \qquad (69)$$

Thus we have all information needed to find the dressed potential. Making use of relation (60) one derives the following result

$$\begin{aligned} Q(x) &= Q_+ + ic_1\sigma_3 \left[\psi_1^+(x)\tilde{\psi}_{0,1}^+(x), \sigma_3 \right] \\ &= \frac{\rho}{1 + V_0(x)} \begin{pmatrix} 0 & e^{i\theta}\left(1 + e^{-2i\theta_1}V_0(x)\right) \\ -e^{-i\theta}\left(1 + e^{2i\theta_1}V_0(x)\right) & 0 \end{pmatrix}. \end{aligned} \qquad (70)$$

We remind that $\lambda_1 + i\sqrt{\rho^2 - \lambda_1^2} = \rho e^{i\theta_1}$ and one can pick up the discrete eigenvalue λ_1 in such a way that $\theta = 2\theta_1$ is fulfilled.

Now we shall construct the one-soliton eigenfunctions. In order to do so we substitute (69) into (58) and perform the integration required. The result reads

$$\psi(x, \lambda) = \begin{pmatrix} A - f_1 \frac{V_0}{1 + V_0} & e^{i\theta}\left(B + f_2 \frac{V_0}{1 + V_0} e^{-i\theta_1}\right) \\ e^{-i\theta}\left(B - f_1 \frac{V_0}{1 + V_0} e^{i\theta_1}\right) & A + f_2 \frac{V_0}{1 + V_0} \end{pmatrix} e^{-ij(\lambda)\sigma_3 x} \qquad (71)$$

where

$$\begin{aligned} f_1 &= \frac{\sqrt{\rho^2 - \lambda_1^2}(A + Be^{-i\theta_1})}{ij(\lambda) + \sqrt{\rho^2 - \lambda_1^2}}, & A(\lambda) &= \sqrt{\frac{\lambda + j(\lambda)}{2j(\lambda)}} \\ f_2 &= \frac{\sqrt{\rho^2 - \lambda_1^2}(A + Be^{i\theta_1})}{ij(\lambda) - \sqrt{\rho^2 - \lambda_1^2}}, & B(\lambda) &= \sqrt{\frac{\lambda - j(\lambda)}{2j(\lambda)}}. \end{aligned} \qquad (72)$$

After an elementary transformation of expressions above the dressed Jost solution are rewritten as follows

$$\psi(x, \lambda) = \left\{ \mathbb{1} + \frac{i\sqrt{\rho^2 - \lambda_1^2}V_0(x)}{(\lambda - \lambda_1)(1 + V_0(x))} \begin{pmatrix} 1 & -e^{i\theta/2} \\ e^{-i\theta/2} & -1 \end{pmatrix} \right\} E_+(x, \lambda). \qquad (73)$$

Similarly, one can use the second transformation operator $\Gamma_-(x, y)$ in equation (58) connecting the other pair of Jost solutions $\phi(x, \lambda)$ and $\phi_0(x, \lambda)$, and apply the same considerations as before. The dual' GLM equation is

$$\Gamma_-(x, y) + F_-(x, y) + \int_{-\infty}^{x} dz\, \Gamma_-(x, z)F_-(z, y) = 0. \qquad (74)$$

If we choose a generic kernel

$$F_-(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}_\rho} d\lambda \, \left(c_-(\lambda)\phi_0^+(x,\lambda)\hat{\phi}_0^+(y,\lambda) - c_-^*(\lambda)\phi_0^-(x,\lambda)\hat{\phi}_0^-(y,\lambda) \right) \sigma_3$$
$$- \sum_{\lambda_j \in \mathcal{D}} c_j^- \phi_{0j}^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}^- \phi_{0j}^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 \tag{75}$$

then its generic solution of (60) can be presented in the form

$$\Gamma_-(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}_\rho} d\lambda \, \left(c_-(\lambda)\phi^+(x,\lambda)\hat{\phi}_0^+(y,\lambda) - c_-^*(\lambda)\phi^-(x,\lambda)\hat{\phi}_0^-(y,\lambda) \right) \sigma_3$$
$$- \sum_{\lambda_j \in \mathcal{D}} c_j^- \phi_j^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}^- \phi_j^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 \tag{76}$$

where $\phi_{0j}^+(x) = \phi_0^+(x,\lambda_j)$ and $\phi_j^+(x) = \phi^+(x,\lambda_j)$. The coefficients c_j^-, c_{0j}^- and the function $c_-(\lambda)$ again can be expressed in terms of the scattering data, e.g. $c_-(\lambda) = \tau(\lambda) - \tau_0(\lambda)$.

If we take the simplest nontrivial kernel with $N = 1$ and $Q_0 = Q_-$ then the final result for the dressed potential reads

$$Q(x) = Q_+ + ic_1\sigma_3 \left[\phi_1^+(x)\tilde{\phi}_{0,1}^+(x), \sigma_3 \right]$$
$$= \frac{\rho}{1+\tilde{V}_0(x)} \begin{pmatrix} 0 & 1 + e^{i\theta}\tilde{V}_0(x) \\ -\left(1 + e^{-i\theta}\tilde{V}_0(x)\right) & 0 \end{pmatrix} \tag{77}$$

where

$$\tilde{V}_0(x) = \frac{c_1\rho e^{2\sqrt{\rho^2 - \lambda_1^2}x}}{2i(\rho^2 - \lambda_1^2)}. \tag{78}$$

The corresponding Jost solution $\phi(x,\lambda)$ is given by

$$\phi(x,\lambda) = \left\{ \mathbb{1} - \frac{i\sqrt{\rho^2 - \lambda_1^2}\tilde{V}_0(x)}{(\lambda - \lambda_1)(1+\tilde{V}_0(x))} \begin{pmatrix} 1 & -e^{i\theta/2} \\ e^{-i\theta/2} & -1 \end{pmatrix} \right\} E_-(x,\lambda). \tag{79}$$

Let us now calculate the dressed scattering matrix

$$T(\lambda) = \lim_{x\to-\infty} \psi(x,\lambda)\phi(x,\lambda) = \text{diag}\left(a(\lambda), 1/a(\lambda)\right) \tag{80}$$

where

$$a(\lambda) = \frac{A - Be^{i\theta_1}}{A - Be^{-i\theta_1}} = \frac{\lambda + j - \lambda_1 - ij_1}{\lambda + j - \lambda_1 + ij_1}. \tag{81}$$

The dressed fundamental analytic solutions are constructed through equalities

$$\chi^+(x,\lambda)e^{ij\sigma_3 x} = u(x,\lambda) \begin{pmatrix} A & Be^{i\theta} \\ B & A \end{pmatrix} \tag{82}$$

where the dressing factor reads

$$u(x,\lambda) = \left\{ \mathbb{1} + \frac{f_2(A - Be^{i\theta_1})}{(1 + V_0(x))} \begin{pmatrix} 1 \\ e^{-i\theta_1} \end{pmatrix} \left(1, \ V_0 e^{i(\theta - \theta_1)} \right) \right\} e^{-ij\sigma_3 x}. \tag{83}$$

5. Dressing Method

The dressing method is another indirect method to solve a nonlinear evolution equation, i.e., it allows one to find a particular solution from a known one. For this to be done one uses substantially the existence of Lax representation and the connection between ISM and Riemann-Hilbert problem [17, 21, 34, 37].

Let us consider once again the auxiliary linear problems (54) and (55). We shall denote by g the dressing transform $\Psi_0 \to \Psi$. By comparing (54) and (55) we see that g satisfies

$$i\partial_x g + Qg - gQ_0 - \lambda[\sigma_3, g] = 0. \tag{84}$$

We are going to use a dressing factor in the form

$$g(x,\lambda) = \mathbb{1} + \frac{A(x)}{\lambda - \lambda_1}, \qquad \lambda_1 \in (-\rho, \rho). \tag{85}$$

Due to reduction (10) the dressing factor obeys the following symmetry condition

$$\sigma_3 \hat{g}^\dagger(x, \lambda^*)\sigma_3 = g(x, \lambda). \tag{86}$$

Hence the inverse factor $\hat{g}$ reads

$$\hat{g}(x,\lambda) = \mathbb{1} + \frac{\sigma_3 A^\dagger(x)\sigma_3}{\lambda - \lambda_1}. \tag{87}$$

Let us evaluate the limit $|\lambda| \to \infty$ in equation (84). As a result we obtain a relation between bare potential Q_0 and dressed one Q, namely

$$Q = Q_0 + [\sigma_3, A]. \tag{88}$$

Thus we can find Q if we only know the residue A. As it turns out the latter can be expressed in terms of a fundamental solution of the bare linear problem (54). In order to find A we consider the identity $g\hat{g} = \mathbb{1}$ which gives rise to the following algebraic relations

$$A\sigma_3 A^\dagger = 0 \tag{89}$$

$$A + \sigma_3 A^\dagger \sigma_3 = 0. \tag{90}$$

It follows from (89) that $A(x)$ is a degenerate matrix so there exist two vectors $X(x)$ and $F(x)$ such that $A = XF^T$. Substituting this decomposition into (89) we obtain

$$F^T \sigma_3 F^* = 0. \tag{91}$$

From (89), (90) and (91) it follows that

$$X = -\mathrm{i}\frac{\sigma_3 F^*}{\alpha} \tag{92}$$

for some real function α to be determined further on. F and α can be found if one considers equation (84). We shall skip all technical details here and give the final result. Both F and α are expressed through a solution Ψ_0 to the bare linear problem in the following way

$$F^T(x) = F_0^T \hat{\Psi}_0(x, \lambda_1) \tag{93}$$

$$\alpha(x) = \mathrm{i}F_0^T \hat{\Psi}_0(x, \lambda_1)\partial_\lambda|_{\lambda=\lambda_1}\Psi_0(x, \lambda)C_0(\lambda_1)\sigma_3 F_0^* + \alpha_0 \tag{94}$$

where the two-vector[1] F_0 as well as $\alpha_0 \in \mathbb{R}$ are constants of integration. The constant matrix $C_0(\lambda_1)$ appears in the reduction condition (10), namely

$$C_0(\lambda) = \hat{\Psi}_0(x, \lambda)\sigma_3\hat{\Psi}_0^\dagger(x, \lambda^*)\sigma_3. \tag{95}$$

The final step in our considerations consists in recovering the time evolution in all quantities. To achieve this one can use the following rule

$$F_0^T \rightarrow F_0^T \mathrm{e}^{-\mathrm{i}f(\lambda_1)t}$$
$$\alpha_0 \rightarrow \alpha_0 - F_0^T\frac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda_1)C_0(\lambda_1)\sigma_3 F_0^* t. \tag{96}$$

Let us recall that $f(\lambda)$ is the dispersion law of the NLEE under consideration. Formulae (96) are derived by analyzing the equation

$$\mathrm{i}g_t + Vg - gV^{(0)} = 0 \tag{97}$$

where $V^{(0)}$ and V are involved in the Lax operators

$$A_0(\lambda) = \mathrm{i}\partial_t + V^{(0)}(x, t, \lambda)$$
$$A(\lambda) = \mathrm{i}\partial_t + V(x, t, \lambda)$$

respectively. Let us illustrate this general scheme in the following example.

Example 2. *Let us consider the case when the bare solution is constant, i.e., $q_0 = \rho$. Then the corresponding fundamental solution is given by*

$$\Psi_0(x, \lambda) = \hat{\varphi}_0(\lambda)\mathrm{e}^{-\mathrm{i}j(\lambda)\sigma_3 x} \tag{98}$$

where $\varphi_0(\lambda)$ is given by equation (12). For our purposes it suffices to pick up the polarization vector F_0 as follows $F_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the two-vector F and the

---

[1]The vector F_0 is usually called polarization vector.

scalar function α acquire the form

$$F(x) = \frac{e^{-ij(\lambda_1)x}}{\sqrt{2j(\lambda_1)(\lambda_1 + j(\lambda_1))}} \begin{pmatrix} -\rho \\ \lambda_1 + j(\lambda_1)) \end{pmatrix}$$

$$\alpha(x) = -\frac{\rho e^{-2ij(\lambda_1)x}}{2j^2(\lambda_1)} + \alpha_0. \tag{99}$$

Finally substitution of all these results into (88) leads to

$$q(x) = \rho \frac{1 + e^{2i\theta_1} e^{2\sqrt{\rho^2 - \lambda_1^2}(x - x_0)}}{1 + e^{2\sqrt{\rho^2 - \lambda_1^2}(x - x_0)}}$$

$$\tan\theta_1 = \frac{\sqrt{\rho^2 - \lambda_1^2}}{\lambda_1}, \qquad x_0 = \frac{1}{2\sqrt{\rho^2 - \lambda_1^2}} \ln \frac{2(\rho^2 - \lambda_1^2)\alpha_0}{\rho} \tag{100}$$

for the reflectionless potential. It coincides with the result obtained in the previous section, see (70). In order to recover the t-dependence in (100) we have to replace x with $x + 2\lambda_1 t$. Thus we have

$$q(x,t) = \rho \frac{1 + e^{2i\theta_1} e^{2\sqrt{\rho^2 - \lambda_1^2}(x + 2\lambda_1 t - x_0)}}{1 + e^{2\sqrt{\rho^2 - \lambda_1^2}(x + 2\lambda_1 t - x_0)}}. \tag{101}$$

*This is the well-known dark soliton for the defocusing NLS [13, 29]. It is immediately seen that asymptotic value of the dark soliton as $x \to -\infty$ coincide with that of vacuum solution while the other asymptotic differs, i.e., Q and Q_0 belong to different phase spaces, $\mathcal{M}_{2\theta_1}$ and $\mathcal{M}_0$ respectively. This means that in general the dressing procedure **does not** respect the constant boundary conditions. This motivates one to introduce a more general space*

$$\mathcal{M}_\rho = \bigcup_\theta \mathcal{M}_{\rho,\theta}$$

which is dressing invariant. The the dressing procedure describe above will map $\mathcal{M}_{\rho,\theta}$ into $\mathcal{M}_{\rho,\theta+2\theta_1}$. $\square$

In order to find more complicated solutions one can pursue either of the following two ways: apply the discussed procedure to the solution already dressed and thus generate a sequence of solutions; or use a multiple poles dressing factor in the form

$$g(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k(x,t)}{\lambda - \lambda_k}, \qquad \lambda_k \in \mathbb{R}. \tag{102}$$

In the latter case the dressed solution can be obtained through the following relation

$$Q = Q_0 + \sum_{k=1}^{N} [\sigma_3, A_k]. \tag{103}$$

As before in order to find residues of the dressing factor we consider the algebraic relations:

$$A_k \sigma_3 A_k^\dagger = 0 \tag{104}$$

$$A_k \sigma_3 \Omega_k^\dagger \sigma_3 + \sigma_3 A_k^\dagger \sigma_3 \Omega_k = 0 \tag{105}$$

where

$$\Omega_k(x,t) = \mathbb{1} + \sum_{j \neq k} \frac{A_j(x,t)}{\lambda_k - \lambda_j}.$$

Relation (104) means that each residue $A_k(x,t)$ is a degenerate matrix hence there exists couples of vectors X_k and F_k, $k = 1, \ldots, N$ such that $A_k = X_k F_k^T$. Due to (104) the components of F_k are not independent but satisfy the relations

$$F_k^T \sigma_3 F_k^* = 0. \tag{106}$$

Relation (105) can be reduced to

$$\Omega_k \sigma_3 F_k^* = \mathrm{i}\alpha_k X_k \tag{107}$$

for some matrices $\alpha_k(x)$, yet to be determined. The system (107) can be viewed as a linear system for the vectors X_k

$$\sigma_3 F_k^* = \sum_{j=1}^{N} \mathcal{B}_{kj} X_j \tag{108}$$

where

$$\mathcal{B}_{kk} = \mathrm{i}\alpha_k, \qquad \mathcal{B}_{kj} = \frac{F_j^T \sigma_3 F_k^*}{\lambda_j - \lambda_k}, \quad k \neq j.$$

This allows us to express X_k in terms of F_k and α_k Similarly to the one-pole case the F_k and α_k are expressed through a seed solution as follows

$$F_k^T(x) = F_{k,0}^T \hat{\Psi}_0(x, \lambda_k) \tag{109}$$

$$\alpha_k(x) = \mathrm{i} F_k^T(x) \partial_\lambda \Psi_0(x, \lambda_k) C_0(\lambda_k) \sigma_3 F_{k,0}^* + \alpha_{k,0} \tag{110}$$

where $F_{k,0}$ and $\alpha_{k,0}$ are integration constants. Finally in order to recover the time dependence one uses the following formulae

$$F_{k,0}^T \rightarrow F_{k,0}^T \mathrm{e}^{-\mathrm{i} f(\lambda_k) t} \tag{111}$$

$$\alpha_{k,0} \rightarrow \alpha_{k,0} - F_{k,0}^T \frac{\mathrm{d} f}{\mathrm{d}\lambda}(\lambda_k) C_0(\lambda_k) \sigma_3 F_{k,0}^* t. \tag{112}$$

6. FAS and the Resolvent of L

The FAS play important role in analyzing the spectral properties of the Lax operator. Here we will demonstrate that the FAS can be used to construct the kernel of the resolvent of L.

Let us now show how the resolvent $R(\lambda)$ can be expressed through the FAS of $L(\lambda)$. Indeed, let us write down $R(\lambda)$ in the form

$$R(\lambda)f(x) = \int_{-\infty}^{\infty} R(x, y, \lambda)f(y) \tag{113}$$

where the kernel $R(x, y, \lambda)$ of the resolvent is given by

$$R^{\pm}(x, y, \lambda) = \frac{1}{i}\chi^{\pm}(x, \lambda)\Theta^{\pm}(x - y)\hat{\chi}^{\pm}(y, \lambda). \tag{114}$$

Here

$$\begin{aligned}
\Theta^+(x - y) &= \operatorname{diag}\left(-\theta(y - x), \theta(x - y)\right) \\
\Theta^-(x - y) &= \operatorname{diag}\left(\theta(x - y), -\theta(y - x)\right).
\end{aligned} \tag{115}$$

Theorem 3. *Let $Q(x) \in \mathcal{M}_\theta$ and let $-\rho < \lambda_j < \rho$ be the simple zeroes of the $a(\lambda)$. Then*

1. *$R^{\pm}(x, y, \lambda)$ is an analytic function of λ for $\lambda \in \mathcal{S}_{\pm}$ having pole singularities at λ_j*

2. *$R^{\pm}(x, y, \lambda)$ is a kernel of a bounded integral operator for $\lambda \in \mathcal{S}_{\pm}$*

3. *$R(x, y, \lambda)$ is uniformly bounded function for $\lambda \in \mathbb{R}_\rho$ and provides a kernel of an unbounded integral operator*

4. *$R^{\pm}(x, y, \lambda)$ satisfy the equation*
$$L(\lambda)R^{\pm}(x, y, \lambda) = \mathbb{1}\delta(x - y) \tag{116}$$

5. *If $Q(x)$ is such that for $\lambda \to \varepsilon\rho$ the FAS have generic behavior then the kernel of resolvent is regular for $\lambda \to \varepsilon\rho$*

6. *If $Q(x)$ is such that for $\lambda \to \varepsilon\rho$ $a(\lambda)$ and $b(\lambda)$ remain finite then $R^{\pm}(x, y, \lambda)$ behaves like $1/j(\lambda)$ for $\lambda \to \varepsilon\rho$.*

Proof: 1. is obvious from the fact that $\chi^{\pm}(x, \lambda)$ are the FAS of $L(\lambda)$

2. Assume that $\lambda \in \mathcal{S}_+$ and consider the asymptotic behavior of $R^+(x, y, \lambda)$ for $x, y \to \infty$. Equation (115) can be rewritten as

$$R^+(x, y, \lambda) = \frac{1}{i}X^+(x, \lambda)e^{-ij(\lambda)\sigma_3 x}\Theta^+(x - y)e^{ij(\lambda)\sigma_3 y}\hat{X}^+(y, \lambda) \quad (117)$$

where $X^+(x, \lambda) = \chi^+(x, \lambda)e^{ij(\lambda)\sigma_3 x}$. Note that due to equations (25)–(27) the functions $X^+(x, \lambda)$ are bounded for $\lambda \in \mathcal{S}_+$ where $\operatorname{Im} j(\lambda) > 0$. But for $\operatorname{Im} j(\lambda) > 0$ both exponential factors in (117) fall off exponentially for $x, y \to \infty$. All other possibilities are treated analogously.

3. For $\lambda \in \mathbb{R}_\rho$ the arguments of 2) can not be applied because the exponents in the right hand side of (117) Im $j(\lambda) = 0$ only oscillate. Thus we conclude that $R^\pm(x, y, \lambda)$ for $\lambda \in \mathbb{R}_\rho$ is only a bounded function and thus the corresponding operator $R(\lambda)$ is an unbounded integral operator.

4. The proof of equation (116) follows from the fact that $L(\lambda)\chi^+(x, \lambda) = 0$ and

$$\partial_x \Theta^\pm(x - y) = \mp \mathbb{1}\delta(x - y). \tag{118}$$

5. From equations (35), (47) and (48) there follows that

$$\lim_{\lambda \to \varepsilon\rho} R^+(x, y, \lambda) = \frac{1}{2\mathrm{i}a_{\varepsilon\rho,0}} \chi^+_{\varepsilon\rho,0}(x)\Theta^+(x - y)\tilde{\chi}^+_{\varepsilon\rho,0}(y) \tag{119}$$

where $\tilde{\chi}^+_{\varepsilon\rho,0}(y)$ is a non-degenerate constant matrix. The respective limit $\lim_{\lambda \to \varepsilon\rho} R^-(x, y, \lambda)$ is treated analogously.

6. In the case of virtual eigenvalue we use equations (35), (50) and (51). Now the result is

$$
\begin{aligned}
\lim_{\lambda \to \varepsilon\rho} R^+(x, y, \lambda) &\simeq \frac{1}{2\mathrm{i}j(\lambda)} \psi^+_{\varepsilon\rho,0}(x)\tilde{\phi}^+_{\varepsilon\rho,0}(y) + \mathcal{O}(1) \\
\lim_{\lambda \to \varepsilon\rho} R^-(x, y, \lambda) &\simeq \frac{1}{2\mathrm{i}j(\lambda)} \phi^+_{\varepsilon\rho,0}(x)\tilde{\psi}^+_{\varepsilon\rho,0}(y) + \mathcal{O}(1).
\end{aligned}
\tag{120}
$$

∎

It is well known that applying the contour integration method on the kernel of the resolvent one can prove the completeness relation for the Jost solutions [13, 22]. From the above theorem it is obvious that, if the potential $Q(x)$ satisfies the virtual eigenvalue condition, then there will be additional terms in this relations.

7. Conclusion

Here we have outlined the construction of the Jost solutions and the FAS for the Zakharov-Shabat system L with constant boundary conditions. We also calculated their singularities at the end points $\pm\rho$ of the continuous spectrum of L. We also demonstrated the derivation of the reflectionless potentials of L and the dark soliton solutions for the relevant NLS equation (2) using first the GLM approach and second – the dressing Zakharov-Shabat method. Finally we constructed the kernel of the resolvent of L and proved that in the regular case $R^\pm(x, y, \lambda)$ are regular for $\lambda \to \varepsilon\rho$, while in the virtual soliton case $R^\pm(x, y, \lambda)$ develop pole singularities for $\lambda \to \varepsilon\rho$. The explicit form of the resolvent can be used to derive the completeness relation for the Jost solutions. Our result shows that in the virtual soliton case this relation will contain additional term corresponding to a discrete eigenvalue at $\varepsilon\rho$.

The results can be used also to derive the generalized Fourier transform, i.e., the expansions over the 'squared solutions' of L. This will be published elsewhere.

Acknowledgements

References

[1] Ablowitz M and Segur H., *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia 1981.

[2] Akhmediev N., Ankiewicz A. and Soto-Crespo J., *Rogue Waves and Rational Solutions of the Nonlinear Schrödinger Equation*, Phys. Rev. E **80** (2009) 026601.

[3] Akhmediev N., Ankiewicz A. and Taki M., *Waves that Appear from Nowhere and Disappear without a Trace*, Phys. Letts A **373** (2009) 675–678.

[4] Aktosun T., Demontis F. and van der Mee C., *Exact Solutions to the Focusing Nonlinear Schrödinger Equation*, Inv. Problems **23** (2007) 2171–2195.

[5] Ankiewicz A., Clarkson P. and Akhmediev N., *Rogue Waves, Rational Solutions, the Patterns of their Zeros and Integral Relations*, J. Phys. A: Math. & Theor. **43** (2010) 122002 (9pp).

[6] Atanasov V. and Gerdjikov V., *On the Multi-Component Nonlinear Schrödinger Equation with Constant Boundary Conditions*, In: Gravity, Astrophysics and Strings'05, P. Fiziev and M. Todorov (Eds), "St. Kliment Ohridsky" University Press, Sofia (2006), pp 22–36 arXiv:0803.3400.

[7] Bullough R. and Caudrey P., *Solitons*, Springer, London 1980.

[8] Dodd R., Eilbeck J., Gibbon J. and Morris H., *Solitons and Nonlinear Wave Equations*, Academic Press, London 1982.

[9] Fried B. and Ichikawa Y., *On the Nonlinear Schrödinger Equation for Langmuir Waves*, J. Phys. Soc. Japan **33** (1972) 789–792.

[10] Gardner C., *Korteweg-de Vries Equation and Generalizations. IV. The Korteweg-de Vries Equation as a Hamiltonian System*, J. Math. Phys. **12** (1971) 1548–1551.

[11] Gardner C., Greene J., Kruskal M. and Miura R., *Method for Solving the Korteweg-de Vries Equation*, Phys. Rev. Lett. **19** (1967) 1095–1097.

[12] Gerdjikov V., *On Spectral Theory of Lax Operators on Symmetric Spaces: Vanishing Versus Constant Boundary Conditions*, J. Geom. Symmetry Phys. **15** (2009) 1–41.

[13] Gerdjikov V. and Kulish P., *Completely Integrable Hamiltonian Systems Connected with the Nonself-adjoint Dirac Operator*(In Russian), Bulg. J. Phys. **5** 337–349, (1978); English version: JINR preprint E2-11394, Dubna (1978).

[14] Gerdjikov V. and Kulish P., *On the Multicomponent Nonlinear Schrödinger Equation in the Case of Non-vanishing Boundary Conditions*, Sci. Notes of LOMI Seminars **131** (1983) 34–46.

[15] Gerdjikov V. and Kostov N., *Multi-Component Nonlinear Schrödinger Equation on Symmetric Spaces with Constant Boundary Conditions. I*, J. Geom. Symmetry Phys. **19** (2010) 1–28.

[16] Gerdjikov V., Kostov N. and Valchev T., *Solutions of Multi-component NLS Models and Spinor Bose-Einstein Condensates*, Physica D **238** (2009) 1306–1310.

[17] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics **748**, Springer, Berlin 2008.

[18] Hone A., *Crum Transformation and Rational Solutions of the Non-Focusing Nonlinear Schrödinger Equation*, J. Phys. A Math. & Gen. **30** (1997) 7473–7483.

[19] Ichikawa Y., Imamura T. and Tanuiti T., *Nonlinear Wave Modulation in Collisionless Plasma*, J. Phys. Soc. Japan **33** (1972) 189–197 .

[20] Ieda J., Miyakawa T. and Wadati M., *Exact Analysis of Soliton Dynamics in Spinor Bose-Einstein Condensates*, Phys. Rev Lett. **93** (2004) 194102.

[21] Ivanov R., *On the Dressing Method for the Generalised Zakharov-Shabat System*, Nuc. Phys. B **694** (2004) 509–524.

[22] Konotop V. and Vekslerchik V. *Direct Perturbation Method for Dark Solitons*, Phys. Rev. E **49** 2397–2407 (1994).

[23] Mikhailov A., *Reductions in Integrable Systems. The Reduction Groups*, Lett. in Jour. of Exper. and Theor. Phys. **32** (1980) 187–192.

[24] Mikhailov A., *The Reduction Problem and the Inverse Scattering Method*, Physica D **3** (1981) 73–117.

[25] Miura R., Gardner C. and Kruskal M., *Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion*, J. Math. Phys. **9** (1968) 1204–1209.

[26] Rogers C. and Schief W., *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, Cambridge Univ. Press, Cambridge 2002.

[27] Shrira V. and Geogjaev V., *What Makes the Peregrine Soliton so Special as a Prototype of Freak Waves?*, J. Eng. Math. **67** (2010) 11–22.

[28] Tajiri M. and Watanabe Y., *Breather Solutions to the Focusing Nonlinear Schrödinger equation*, Phys. Rev. E **57** (1998) 3510–3519.

[29] Takhtadjan L. and Faddeev L., *The Hamiltonian Approach to Soliton Theory*, Springer, Berlin 1987.

[30] Uchiyama M., Ieda J. and Wadati M., *Multicomponent Bright Solitons in $F = 2$ Spinor Bose-Einstein Condensates*, J. Phys. Soc. Japan **76** (7) (2007) 74005.

[31] Zakharov V., *Stability of Periodic Waves of Finite Amplitude on the Surface of a Deep Fluid*, Appl. Mech. Tech. Phys. **9** (1968) 86–94.

[32] Zakharov V. and Gelash A., *Soliton on Unstable Condensate*, nlin.SI:1109.0620.

[33] Zakharov V. and Manakov S., *On the Complete Integrability of the Nonlinear Schrödinger Equation* (In Russian), TMP **19** (1974) 332–343.

[34] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Plenum, New York 1984.

[35] Zakharov V. and Shabat A., *Exact Theory of Two-dimensional Self-focusing and One-dimentional Modulation of Waves in Nonlinear Media* (In Russian), Sov. Phys JETF **34** (1972) 62–69.

[36] Zakharov V. and Shabat A., *On the Interaction of Solitons in Stable Medium* (In Russian), Sov. Phys. JETF **37** (1973) 823–828.

[37] Zakharov V. and Shabat A., *Integration of Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Transform II* (in Russian), Funct. Anal. Appl. **13** (1979) 13–22 .

ANALYTIC REPRESENTATION OF A CLASS OF AXIALLY SYMMETRIC WILLMORE SURFACES

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Abstract. The surfaces providing local extrema to the so-called Willmore functional, which assigns to each surface its total squared mean curvature, are frequently referred to as the Willmore surfaces. The corresponding Euler-Lagrange equation is usually called Willmore equation. The present work is concerned with a special class of axially symmetric solutions to the Willmore equation, which are solutions of a simpler ordinary differential equation. An analytic representation of the corresponding Willmore surfaces is given in terms of Jacobi elliptic functions and elliptic integrals.

1. Introduction

In 1965, T. Willmore proposed (see references [8, 9]) to study the surfaces (widely known nowadays as the Willmore surfaces) that provide extremum to the functional

$$\mathcal{W} = \int_{\mathcal{S}} H^2 \mathrm{d}A \tag{1}$$

(often called the Willmore functional), which assigns to each surface $\mathcal{S}$ its total squared mean curvature H. Here $\mathrm{d}A$ denotes the area element of the surface. The corresponding Euler-Lagrange equation (frequently referred to as the Willmore equation in the current literature) read

$$\Delta H + 2(H^2 - K)H = 0 \tag{2}$$

where Δ is the Laplace-Beltrami operator on the surface $\mathcal{S}$ and K is its Gaussian curvature.

Actually, according to Nitsche [3, 4] the history of this variational problem can be traced about two centuries back to the memoir by Siméon Denis Poisson published in 1812 and that by Marie-Sophie Germain appeared in 1821 where the functional (1) was proposed as the bending energy of elastic shells. Approximately a century later, in 1923, an equivalent variational problem was studied by Thomsen [5] in the context of conformal geometry. Finally, let us remark that the Willmore functional (1) and the Willmore equation (2) are invariant under the conformal transformations of the three-dimensional Euclidean space, see references [2, 6].

2. Rotationally–Invariant Solutions of the Willmore Equation

Let a surface $\mathcal{S}$ immersed in the three-dimensional Euclidean space $\mathbb{R}^3$ be given as the graph of a function $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, i.e.,

$$\mathcal{S} = \operatorname{graph}(w) = \{(x^1, x^2, w(x^1, x^2)) \subset \mathbb{R}^3; (x^1, x^2) \in \Omega \subset \mathbb{R}^2\}$$

which is supposed to be continuous and to possess as many derivatives as may be required on the domain Ω. Let us take x^1, x^2 to serve as Gaussian coordinates on the surface $\mathcal{S}$. Then, relative to this coordinate system, the components of the first fundamental tensor $g_{\alpha\beta}$, second fundamental tensor $b_{\alpha\beta}$, and alternating tensor $\varepsilon^{\alpha\beta}$ of $\mathcal{S}$ are given by the expressions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + w_\alpha w_\beta, \qquad b_{\alpha\beta} = g^{-1/2} w_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = g^{-1/2} e^{\alpha\beta} \qquad (3)$$

where

$$g = \det(g_{\alpha\beta}) = 1 + (w_1)^2 + (w_2)^2 \qquad (4)$$

$\delta_{\alpha\beta}$ is the Kronecker delta symbol ($\delta_{11} = \delta_{22} = 1$, $\delta_{12} = \delta_{21} = 0$) and $e^{\alpha\beta}$ is the alternating symbol ($e^{11} = e^{22} = 0$, $e^{12} = -e^{21} = 1$). Moreover, the contravariant components $g^{\alpha\beta}$ of the first fundamental tensor read

$$g^{\alpha\beta} = g^{-1} \delta^{\alpha\beta} + \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} w_\mu w_\nu = g^{-1} \left(\delta^{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_\mu w_\nu \right). \qquad (5)$$

Here and in what follows: Greek indices have the range $1, 2$, and the usual summation convention over a repeated index is employed, $w_{\alpha_1 \ldots \alpha_k}$ ($k = 1, 2, \ldots$) denote the k-th order partial derivatives of the function w with respect to the variables x^1 and x^2, i.e.,

$$w_{\alpha_1 \alpha_2 \ldots \alpha_k} = \frac{\partial^k w}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_k}}, \qquad k = 1, 2, \ldots$$

The mean curvature H of the surface $\mathcal{S}$ and its Gaussian curvature K are given as follows

$$H = \frac{1}{2} g^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} g^{-3/2} \left(\delta^{\alpha\beta} w_{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu \right) \qquad (6)$$

and

$$K = \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}b_{\alpha\beta}b_{\mu\nu} = \frac{1}{2}g^{-2}e^{\alpha\mu}e^{\beta\nu}w_{\alpha\beta}w_{\mu\nu}. \tag{7}$$

The rotationally–invariant solutions of the Willmore equation are sought in the form

$$w = w(r), \qquad r = \sqrt{(x^1)^2 + (x^2)^2}.$$

On account of equations (3) – (7), upon such a symmetry reduction equation (2) reads

$$\begin{aligned}
\mathcal{R} &\equiv (2\,r^3 + 4\,r^3\,w_r^2 + 2\,r^3\,w_r^4)w_{rrrr} \\
&\quad +(4\,r^2 + 8\,r^2\,w_r^2 + 4\,r^2\,w_r^4 - 20\,r^3\,w_r w_{rr} - 20\,r^3\,w_r^3 w_{rr})w_{rrr} \\
&\quad -5r^2\,(3\,w_r + 3\,w_r^3 + r\,w_{rr} - 6\,r\,w_r^2 w_{rr})w_{rr}^2 \\
&\quad +(r\,w_r^6 - 2\,r - 3\,r\,w_r^2)w_{rr} + 2\,w_r + 7w_r^3 + 9\,w_r^5 + 5\,w_r^7 + w_r^9 = 0
\end{aligned} \tag{8}$$

where

$$w_r = \frac{\mathrm{d}w}{\mathrm{d}r}, \qquad w_{rr} = \frac{\mathrm{d}^2 w}{\mathrm{d}r^2}, \qquad w_{rrr} = \frac{\mathrm{d}^3 w}{\mathrm{d}r^3}, \qquad w_{rrrr} = \frac{\mathrm{d}^4 w}{\mathrm{d}r^4}.$$

Simultaneously, the mean and Gaussian curvatures take the forms

$$H = \frac{1}{2r}\frac{rw_{rr} + w_r^3 + w_r}{\left(1 + w_r^2\right)^{3/2}}, \qquad K = \frac{1}{r}\frac{w_{rr}w_r}{\left(1 + w_r^2\right)^2}. \tag{9}$$

3. A Special Class of Axially Symmetric Willmore Surfaces

Consider the following normal system of two ordinary differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}r} = v, \qquad \frac{\mathrm{d}v}{\mathrm{d}r} = \pm\frac{1}{r}\left(v^2 + 1\right)\sqrt{v^2 + 2\,a\sqrt{v^2 + 1}} \tag{10}$$

(where a is a real constant), which is equivalent to the single second-order equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} = \pm\frac{1}{r}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + 1\right]\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + 2\,a\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + 1}}. \tag{11}$$

Substituting equation (11) into expression $\mathcal{R}$ one obtains $\mathcal{R} = 0$ and thus shows that each solution of system (10) or equation (11) is a solution of the reduced Willmore equation $\mathcal{R} = 0$. In this way one determines a special class of axially symmetric Willmore surfaces. It is worth nothing that system (10) and equation (11) turn out to be invariant under the translations of the variable w and the scaling transformations

$$w \to w\eta, \qquad r \to r\eta, \qquad \eta \in \mathbb{R}.$$

Substituting equations (10) into expressions (9) one can see that the mean and Gaussian curvatures of a surface belonging to the forgoing special class are given as follows

$$H = \frac{v \pm \sqrt{v^2 + 2\,a\sqrt{v^2 + 1}}}{2r\sqrt{v^2 + 1}}, \qquad K = \pm\frac{v\sqrt{v^2 + 2\,a\sqrt{v^2 + 1}}}{r^2\,(v^2 + 1)} \tag{12}$$

where v is any solution of the second equation of system (10).

The substitutions

$$u = \sqrt{v^2 + 1}, \qquad \rho = \ln r \tag{13}$$

transform the system (10) to the following one

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} = \mathrm{e}^{\rho}\sqrt{u^2 - 1} \tag{14}$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)^2 = u^2\,(u^2 - 1)\,(u^2 + 2au - 1). \tag{15}$$

In terms of a new variable t such that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{u} \tag{16}$$

equation (15) may be written in the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = P\,(u), \qquad P\,(u) = (u^2 - 1)\,(u^2 + 2au - 1) \tag{17}$$

and equation (14) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \mathrm{e}^{\rho}\frac{1}{u}\sqrt{u^2 - 1}. \tag{18}$$

Using the standard approach (see [7, §20.6] and [1, pp. 649–652]), one can express a class of solutions of the equation (17) corresponding to the root $u = 1$ of the polynomial $P(u)$ as follows

$$u\,(t) = \frac{2\sqrt{a^2 + 1} - \left(\sqrt{a^2 + 1} - a + 1\right)\mathrm{sn}^2\,(\lambda t, k)}{2\sqrt{a^2 + 1} - \left(\sqrt{a^2 + 1} + a + 1\right)\mathrm{sn}^2\,(\lambda t, k)} \tag{19}$$

where

$$\lambda = \sqrt[4]{a^2 + 1}, \qquad k = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{1}{\sqrt{a^2 + 1}}}$$

and $\mathrm{sn}(\cdot, \cdot)$ denotes the sine Jacobi elliptic function. Then, using expression (19), one can write down the solution $\rho\,(t)$ of equation (16) in the form

$$\rho\,(t) = \left(\lambda^2 + a\right)t - \frac{\lambda^2 + a - 1}{\lambda}\Pi\left(\frac{\lambda^2 - a + 1}{2\lambda^2}, \mathrm{am}\,(\lambda t, k), k\right) \tag{20}$$

where $\Pi(\cdot,\cdot,\cdot)$ denotes the incomplete elliptic integral of the third kind.

Finally, using the second one of equations (13) and equation (14) we arrive at the following analytic representation of the parametric equations for the profile curves of the axially symmetric Willmore surfaces determined by equation (11)

$$r(t) = e^{\rho(t)}, \qquad w(t) = \int e^{\rho(t)} \frac{1}{u(t)} \sqrt{u(t)^2 - 1}\, dt + b \qquad (21)$$

where b is an arbitrary real constant.

Unfortunately, the parametric equations (21) are too complicated to be used for displaying the respective surfaces directly. This, however, can be done by solving numerically system (10) taking as initial values at $r = 1$ an arbitrary real number for w (because of the invariance of system (10) under the translation of this variable) and $v = 0$. Indeed, for each surface of the considered class equations (19) and (20) imply $u(0) = 1$ and $\rho(0) = 0$ and hence, in view of equations (13), $v = 0$ at $r = 1$. Two Willmore surfaces obtained in this way are depicted in Figure 1.

Figure 1. Willmore surface constructed by the profile curves $\Gamma_- \cup \Gamma_+$ (left) and $\hat{\Gamma}_- \cup \hat{\Gamma}_+$ (right).

First of them (Figure 1, left) is constructed by joining two profile curves Γ_- and Γ_+ (see Figure 2, left), which are generated by solving numerically system (10), choosing respectively sign "-" and sign "+" of the right-hand side of the second equation in this system, setting $a = 0.2$ and taking $v = 0$ as initial condition at $r = 1$. The second one is constructed by joining another couple of profile curves $\hat{\Gamma}_-$ and $\hat{\Gamma}_+$ (see Figure 2, right) obtained in the same manner, but now $a = 1$. The Gaussian curvatures corresponding to both profile curves Γ_- and Γ_+ are identical, while the respective mean curvatures are symmetric with respect to the r-axis. The same holds true for the Gaussian and mean curvatures of the curves $\hat{\Gamma}_-$ and $\hat{\Gamma}_+$.

4. Concluding Remarks

In this work, we have determined analytically only one class of axially symmetric Willmore surfaces – those that arises from the solutions of equation (17) corresponding to the root $u = 1$ of the polynomial $P(u)$. There are however other
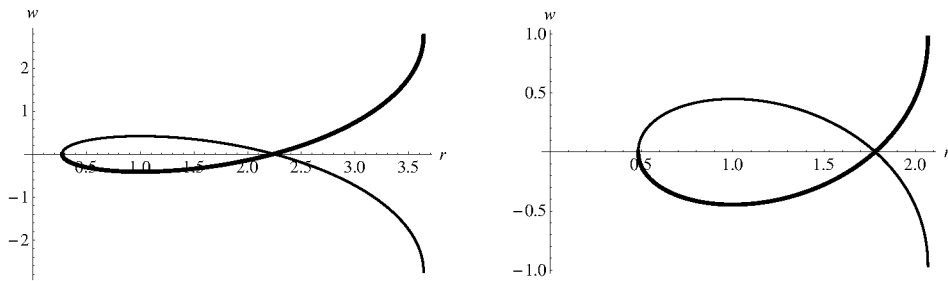
Figure 2. The curves Γ_- (left, thick), Γ_+ (left, thin), $\hat{\Gamma}_-$ (right, thick) and $\hat{\Gamma}_+$ (right, thin).

possibilities, which will be analysed elsewhere. It is worth noting as well that spheres and catenoids belong to the class of axially symmetric Willmore surfaces determined by equation (11). Indeed, it is easy to verify that the functions $w = \pm\sqrt{R^2 - r^2}$ and $w = R\ln\left(r \pm \sqrt{r^2 - R^2}\right)$, where R is an arbitrary real constant, determining the corresponding profile curves satisfy equation (11) in the case $a = 0$.

References

[1] Abramowitz M. and Stegun I., *Handbook of Mathematical Functions*, Dover, New York 1972.

[2] Blaschke W., *Vorlesungen über Differentialgeometrie*, vol. III, Springer, Berlin 1929.

[3] Nitsche J., *Lectures on Minimal Surfaces*, vol. **1**. *Introduction, Fundamentals, Geometry and Basic Boundary Value Problems*, Cambridge University Press, Cambridge 1989.

[4] Nitsche J., *Boundary Value Problems for Variational Integrals Involving Surface Curvatures*, Q. Appl. Math. **60** (1993) 363–387.

[5] Thomsen G., *Grundlagen der konformen Flächentheorie*, Abh. Math. Sem. Univ. Hamburg **3** (1923) 31–56.

[6] White J., *A Global Invariant of Conformal Mappings in Space*, Proc. Amer. Math. Soc. **38** (1973) 162–164.

[7] Whittaker E. and Watson G., *A Course of Modern Analysis*, Cambridge University Press, Cambridge 1927.

[8] Willmore T., *Note on Embedded Surfaces*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Se çt. Ia Mat. **11** (1965) 493-496.

[9] Willmore T., *Riemannian Geometry*, Oxford University Press, Oxford 1993.

KILLING FORMS ON KERR-NUT-(A)dS SPACES OF EINSTEIN-SASAKI TYPE[*]

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Abstract. In certain scaling limits the higher-dimensional Euclideanized Kerr-NUT-(A)dS metrics are related to the Einstein-Sasaki ones. The complete set of Killing forms of the Einstein-Sasaki spaces are presented. It is pointed out the existence of two additional Killing forms on these spaces associated with the complex volume form of the Calabi-Yau cone manifold. As a concrete example we present the complete set of Killing-Yano tensors on the five-dimensional Einstein-Sasaki $Y(p,q)$ spaces.

1. Introduction

In the last time the properties of higher-dimensional black holes have become of large interest. The most general known higher-dimensional metrics describing rotating black holes with NUT parameters in an asymptotically AdS spacetimes were described in [6]. The general Kerr-NUT-AdS metrics have $(2n-1)$ non-trivial parameters where the spacetime dimension is $(2n+1)$ in the odd-dimensional case and $(2n)$ in the even dimensional case.

In certain scaling limits [12, 13] these metrics are related to the Einstein-Sasaki ones. On the other hand the Einstein-Sasaki geometries have been the object of much attention in connection with the supersymmetric backgrounds relevant to the AdS/CFT correspondence.

The Kerr-NUT-(A)dS metrics possess isometries and hidden symmetries encoded in a series of Killing vectors and Stäckel-Killing tensors [6]. These symmetries are connected with a set of conserved quantities which are functionally independent,

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in involution, and guarantee the complete integrability of the geodesic motions [11, 15, 18].

In the case of Sasaki spaces the hidden symmetries are derived from the characteristic Sasakian one-form and a tower of Killing-Yano and conformal Killing-Yano tensors can be constructed [12].

The main purposes of this paper is to point out the special case of the higher dimensional Kerr-NUT-(A)dS metrics which are related to the Einstein-Sasaki ones. In this instance there are two additional Killing-Yano tensors taking into account that the metric cone is Calabi-Yau [5, 17]. These two exceptional Killing forms can be also described using the Killing spinors of an Einstein-Sasaki manifold [2].

In Section 2 we review some basic facts about the Stäckel-Killing and Killing-Yano tensors. In Section 3 we present the close connection between Einstein-Sasaki and Einstein-Kähler geometries. In the next Section we discuss the Killing forms on Einstein-Sasaki spaces which proceed from Euclideanized Kerr-NUT-(A)dS metrics in certain scaling limits. We identity two new Killing forms associated with the complex volume form of the cone manifold. In Section 5 we restrict to the five-dimensional $Y(p, q)$ manifolds and present the complete set of Killing forms. Finally we give our conclusions in Section 6.

2. Killing Tensors

Let (M, g) be a n-dimensional differentiable manifold equipped with a (pseudo)-Riemannian metric

$$\mathrm{d}s^2 = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j. \tag{1}$$

Definition 1. *A vector field X on M is said to be a Killing vector if the Lie derivative with respect to X of the metric g vanishes*

$$\mathcal{L}_X g = 0. \tag{2}$$

In coordinates this means that

$$X_{(i;j)} = 0 \tag{3}$$

where a semicolon precedes an index i of covariant differentiation associated with the Levi-Civita connection and a round bracket denotes symmetrization over the indices within.

A symmetric generalization of the Killing vectors is that of Stäckel-Killing tensors.

Definition 2. *A symmetric tensor $K_{(i_1\cdots i_r;j)}$ of rank $r > 1$ satisfying the generalized Killing equation*

$$K_{(i_1\cdots i_r;j)} = 0 \tag{4}$$

is called a Stäckel-Killing tensor.

From the generalized Killing equation (4) we get that for any geodesic γ with tangent vector $\dot{\gamma}^i$

$$Q_K = K_{i_1 \cdots i_r} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_r} \tag{5}$$

is constant along γ.

Antisymmetric Killing-Yano tensors [21] are a different generalization of the Killing vectors.

Definition 3. *A Killing-Yano tensor is a p-form $f (p \leq n)$ which satisfies*

$$\nabla_X f = \frac{1}{p+1} X \lrcorner \, \mathrm{d}\omega \tag{6}$$

for any vector field X, where 'hook' operator $\lrcorner$ is dual to the wedge product.

This definition is equivalent with the property that $\omega_{i_1 \cdots i_p; j}$ is totally antisymmetric or, in components

$$\omega_{i_1 \cdots i_{p-1}(i_p; j)} = 0. \tag{7}$$

It was observed that Killing-Yano tensors generate *non-standard supersymmetries* in the dynamics of pseudo-classical spinning particles being the natural objects to be coupled with the fermionic degrees of freedom. At the quantum level, Killing-Yano tensors generate conserved *non-standard Dirac operators* which commute with the standard one.

These two generalizations of the Killing vectors could be related. Given two Killing-Yano tensors $\omega^{i_1, \ldots, i_k}$ and $\sigma^{i_1, \ldots, i_k}$ it is possible to associate with them a Stäckel-Killing tensor of rank two

$$K_{ij}^{(\omega, \sigma)} = \omega_{i i_2 \ldots i_k} \sigma_j^{\ i_2 \ldots i_k} + \sigma_{i i_2 \ldots i_k} \omega_j^{\ i_2 \ldots i_k}. \tag{8}$$

Therefore a method to generate higher order integrals of motion is to identify the complete set of Killing-Yano tensors. The existence of enough integrals of motion leads to complete integrability or even superintegrability of the mechanical system when the number of functionally independent constants of motion is larger than its number of degrees of freedom.

The conformal extension of the Killing vectors is given by the conformal Stäckel-Killing and conformal Killing-Yano tensors.

Definition 4. *A conformal Killing-Yano tensor of rank p is a p-form ω which satisfies*

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner \, \mathrm{d}\omega - \frac{1}{n-p+1} X^* \wedge \mathrm{d}^* \omega \tag{9}$$

for any vector field X on M, where X^* is the one-form dual to the vector field X with respect to the metric g, and d^* is the adjoint of the exterior differential d.

If ω is co-closed in (9), then we obtain the definition of a Killing-Yano tensor [21]. We mention that Killing-Yano tensors are also called Yano tensors or Killing forms, and conformal Killing-Yano tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms.

Definition 5. *A Killing form ω is said to be a special Killing form if it satisfies for some constant c the additional equation*

$$\nabla_X(\mathrm{d}\omega) = cX^* \wedge \omega \tag{10}$$

for any vector field X on M.

3. Kähler and Sasakian Manifolds

The Sasakian geometry, defined on an odd dimensional manifold, is the closest possible analogue of the Kähler geometry of even dimension.

There are several equivalent definitions of the Sasakian structure and for describing the problems of interest here it is more convenient to use the following definition:

Definition 6. *A compact Riemannian manifold (S, g) is Sasakian if and only if the metric cone*

$$C(S) = \mathbb{R}_{>0} \times S, \qquad \bar{g} = \mathrm{d}r^2 + r^2 g \tag{11}$$

is Kähler.

A Sasakian manifold inherits a number of geometrical structures from the Kähler structure of its cone. Let us note that if the odd dimension of the Sasaki space is $(2n+1)$, the Kähler cone has the complex dimension $(n+1)$. In local holomorphic coordinates $(z^1, ..., z^{n+1})$ the associated Kähler form Ω can be written as

$$\Omega = \mathrm{i}g_{j\bar{k}}\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k = \sum X_j^* \wedge Y_j^* = \frac{\mathrm{i}}{2}\sum Z_j^* \wedge \bar{Z}_j^* \tag{12}$$

where $(X_1, Y_1, ..., X_{n+1}, Y_{n+1})$ is an adapted local orthonormal field (i.e., such that $Y_j = JX_j$), and $(Z_j, \bar{Z}_j)$ is the associated complex frame given by

$$Z_j = \frac{1}{2}(X_j - \mathrm{i}Y_j), \qquad \bar{Z}_j = \frac{1}{2}(X_j + \mathrm{i}Y_j). \tag{13}$$

There is an intimate connection between its Kähler form and the volume form (which is just the Riemannian volume form determined by the metric) as follows

$$\mathrm{d}\mathcal{V} = \frac{1}{(n+1)!}\,\Omega^{n+1} \tag{14}$$

where $\mathrm{d}\mathcal{V}$ denotes the volume form of $C(S)$, Ω^{n+1} is the wedge product of Ω with itself $n + 1$ times [1]. Hence the volume form is a real $(n + 1, n + 1)$-form on

$C(S)$. On the other hand, if the volume of a Kähler manifold is written as

$$d\mathcal{V} = \frac{i^{n+1}}{2^{n+1}}(-1)^{n(n+1)/2}\, dV \wedge \overline{dV} \tag{15}$$

then dV is the complex volume holomorphic $(n+1, 0)$ form of $C(S)$.

An Einstein-Sasaki manifold is a Riemannian manifold (S, g) that is both Sasaki and Einstein, i.e., a Sasakian manifold satisfying the Einstein condition

$$\mathrm{Ric}_g = \lambda g \tag{16}$$

for some real constant λ, where Ric_g denotes the Ricci tensor of g. Einstein manifolds with $\lambda = 0$ are called Ricci-flat manifolds. Similarly, an Einstein-Kähler manifold is a Riemannian manifold that is both Kähler and Einstein. An important subclass of Einstein-Kähler manifolds are the Calabi-Yau manifolds which are Kähler and Ricci-flat.

A simple calculation shows that we have

Corollary 7. *A Sasakian metric g is Einstein with*

$$\mathrm{Ric}_g = 2ng \tag{17}$$

if and only if the cone metric $\bar{g}$ is Ricci flat, i.e., Calabi-Yau.

Suppose we have an Einstein-Sasaki metric g_{ES} on a manifold S_{2n+1} of odd dimension $2n + 1$. An Einstein-Sasaki manifold can always be written as a fibration over an Einstein-Kähler manifold M_{2n} with the metric g_{EK} twisted by the overall U(1) part of the connection [10]

$$ds^2_{ES} = (d\psi_n + 2A)^2 + ds^2_{EK} \tag{18}$$

where dA is given as the Kähler form of the Einstein-Kähler base. This can be easily seen when we write the metric of the cone manifold $M_{2n+2} = C(S_{2n+1})$ as

$$ds^2_{\mathrm{cone}} = dr^2 + r^2\, ds^2_{ES} = dr^2 + r^2\left((d\psi_n + 2A)^2 + ds^2_{EK}\right). \tag{19}$$

The cone manifold is Calabi-Yau and its Kähler form can be written as

$$\Omega_{\mathrm{cone}} = r\, dr \wedge (d\psi_n + 2A) + r^2\Omega_{EK} \tag{20}$$

and the Kähler condition $d\Omega_{\mathrm{cone}} = 0$ implies

$$dA = \Omega_{EK} \tag{21}$$

where Ω_{EK} is Kähler form of the Einstein-Kähler base manifold M_{2n}.

The Sasakian one-form of the Einstein-Sasaki metric is

$$\eta = 2A + d\psi_n \tag{22}$$

which is a special unit-norm Killing one-form obeying for all vector fields X [17]

$$\nabla_X \eta = \frac{1}{2}X \lrcorner\, d\eta, \qquad \nabla_X(d\eta) = -2X^* \wedge \eta. \tag{23}$$

4. Killing Forms on Kerr-NUT-(A)dS Space in a Certain Scaling Limit

In recent time new Einstein-Sasaki spaces have been constructed by taking certain BPS [7] or scaling limits [12, 13] of the Euclideanized Kerr-de Sitter metrics.

In even dimensions, performing the scaling limit on the Euclideanized Kerr-NUT-(A)dS spaces, the Einstein-Kähler metric g_{EK} and the Kähler potential A are [12]

$$g_{EK} = \frac{\Delta_\mu \, \mathrm{d}x_\mu^2}{X_\mu} + \frac{X_\mu}{\Delta_\mu} \left(\sum_{j=0}^{n-1} \sigma_\mu^{(j)} \, \mathrm{d}\psi_j \right)^2 \tag{24}$$

with

$$X_\mu = -4 \prod_{i=1}^{n+1} (\alpha_i - x_\mu) - 2b_\mu, \qquad A = \sum_{k=0}^{n-1} \sigma^{(k+1)} \mathrm{d}\psi_k \tag{25}$$

and

$$\Delta_\mu = \prod_{\nu \neq \mu} (x_\nu - x_\mu), \quad \sigma_\mu^{(k)} = \sum_{\substack{\nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1} \ldots x_{\nu_k}, \quad \sigma^{(k)} = \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1} \ldots x_{\nu_k}. \tag{26}$$

Here, coordinates $x_\mu \, (\mu = 1, \ldots, n)$ stands for the Wick rotated radial coordinate and longitudinal angles and the Killing coordinates $\psi_k \, (k = 0, \ldots, n-1)$ denote time and azimuthal angles with Killing vectors $\xi^{(k)} = \partial_{\psi_k}$. Also α_i, $i = 1, \ldots, n+1$ and b_μ are constants related to the cosmological constant, angular momenta, mass and NUT parameters [6].

We mention that in the case of odd-dimensional Kerr-NUT-(A)dS spaces the appropriate scaling limit leads to the same Einstein-Sasaki metric (18).

The hidden symmetries of the Sasaki manifold M_{2n+1} are described by the special Killing $(2k+1)-$forms [17]

$$\Psi_k = \eta \wedge (\mathrm{d}\eta)^k, \qquad k = 0, 1, \ldots, n-1. \tag{27}$$

In [17] Semmelmann has proved that special Killing forms on a Riemannian manifold M are exactly those forms which translate into parallel forms on the metric cone $C(M)$. Therefore, the metric cone being either flat or irreducible, the problem of finding all special Killing forms is reduced to a holonomy problem [4]. In the case of holonomy U(n+1), i.e., the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler, or equivalently M_{2n+1} is Sasaki, it follows that all special Killing forms are spanned by the forms Ψ_k (27). Besides these Killing forms, there are n closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (\mathrm{d}\eta)^k, \qquad k = 1, \ldots, n. \tag{28}$$

In the case of holonomy SU(n+1), i.e., the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler and Ricci-flat, or equivalently M_{2n+1} is Einstein-Sasaki, there are *two additional* Killing forms of degree n on the manifold M_{2n+1}.

In order to write explicitly these additional Killing forms, we introduce the complex vierbeins on the Einstein-Kähler manifold M_{2n}. First of all we shall write the metric g_{EK} in the form

$$g_{EK} = o^{\hat{\mu}} o^{\hat{\mu}} + \tilde{o}^{\hat{\mu}} \tilde{o}^{\hat{\mu}} \tag{29}$$

and the Kähler two-form

$$\Omega = \mathrm{d}A = o^{\hat{\mu}} \wedge \tilde{o}^{\hat{\mu}} \tag{30}$$

where

$$o^{\hat{\mu}} = \sqrt{\frac{\Delta_\mu}{X_\mu(x_\mu)}}\, \mathrm{d}x_\mu, \qquad \tilde{o}^{\hat{\mu}} = \sqrt{\frac{X_\mu(x_\mu)}{\Delta_\mu}} \sum_{j=0}^{n-1} \sigma_\mu^{(j)}\, \mathrm{d}\psi_j. \tag{31}$$

We introduce the following complex vierbeins on Einstein-Kähler manifold M_{2n} [20]

$$\zeta_\mu = o^{\hat{\mu}} + \mathrm{i}\tilde{o}^{\hat{\mu}}, \qquad \mu = 1, \dots, n. \tag{32}$$

On the Calabi-Yau cone manifold M_{2n+2} we take $\Lambda_\mu = r\zeta_\mu$ for $\mu = 1, \dots, n$ and

$$\Lambda_{n+1} = \mathrm{d}r + \mathrm{i}r\eta. \tag{33}$$

The standard complex volume form of the Calabi-Yau cone manifold M_{2n+2} is [20]

$$\mathrm{d}V = \Lambda_1 \wedge \Lambda_2 \wedge \dots \wedge \Lambda_{n+1}. \tag{34}$$

The real Killing forms are given the real respectively the imaginary part of the complex volume form.

The additional Killing forms on the Einstein-Sasaki spaces are connected with the parallel forms on the metric cone. For this purpose we make use of the fact that for any p-form ω^M on the space M_{2n+1} we can define an associated $(p+1)$-form ω^C on the cone $C(M_{2n+1})$

$$\omega^C := r^p\, \mathrm{d}r \wedge \omega^M + \frac{r^{p+1}}{p+1}\, \mathrm{d}\omega^M. \tag{35}$$

Moreover ω^C is parallel if and only if ω^M is a special Killing form (10) with constant $c = -(p+1)$ [17]. The one-to-one-correspondence between special Killing p-forms on M_{2n+1} and parallel $(p+1)$-forms on the metric cone $C(M_{2n+1})$ allows us to describe the additional Killing forms on Einstein-Sasaki spaces.

Therefore in order to find the additional Killing forms on the manifold M_{2n+1} we must identify the ω^M form in the complex volume form of the Calabi-Yau cone. An explicit example is presented in the next Section.

5. $Y(p,q)$ **Manifolds**

Recently an infinite family $Y(p,q)$ of Einstein-Sasaki metrics on $S^2 \times S^3$ have been discovered[7–9, 13]. Such manifolds provide supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p,q)$ of an S^1-fibration over $S^2 \times S^2$ with relative prime winding numbers p and q is topologically $S^2 \times S^3$. The starting point is the explicit local metric of the five-dimensional $Y(p,q)$ manifold given by the line element [8, 9, 16]

$$\mathrm{d}s_{ES}^2 = \frac{1-y}{6}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{1}{p(y)}\,\mathrm{d}y^2 + \frac{1}{36}p(y)(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)^2$$

$$+ \frac{1}{9}[\mathrm{d}\psi' - \cos\theta\,\mathrm{d}\phi + y(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)]^2 \tag{36}$$

with

$$p(y) = \frac{2(a - 3y^2 + 2y^3)}{1-y} \tag{37}$$

and a is a constant.

From (22) in the case of the $Y(p,q)$ space the Sasakian one-form is

$$\eta = \frac{1}{3}\,\mathrm{d}\psi' + 2A \tag{38}$$

with

$$A = \frac{1}{6}\left(-\cos\theta\,\mathrm{d}\phi + y(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)\right) \tag{39}$$

connected with local Kähler form Ω_{EK} as in (21).

The form of the metric (36) with the one-form (38) is the standard one for a locally Einstein-Sasaki metric with $\frac{\partial}{\partial\psi'}$ the Reeb vector field. Note also that the holomorphic $(2,0)$-form of the Einstein-Kähler base manifold is

$$\mathrm{d}V_{EK} = \sqrt{\frac{1-y}{6p(y)}}(\mathrm{d}\theta + \mathrm{i}\sin\theta\,\mathrm{d}\phi) \wedge \left(\mathrm{d}y + \mathrm{i}\frac{p(y)}{6}(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)\right). \tag{40}$$

From the isometries $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ the momenta P_ϕ, P_ψ, P_α and the Hamiltonian describing the geodesic motions are conserved [3, 16]. P_ϕ is the third component of the SU(2) angular momentum, while P_ψ and P_α are associated with the U(1) factors. Additionally, the total SU(2) angular momentum given by

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\Psi^2 \tag{41}$$

is also conserved.

In what follows we are looking for further conserved quantities specific for motions in Einstein-Sasaki spaces. First of all, according to (27), the Killing one-form η (38) together with the third rank form

$$
\begin{aligned}
\Psi &= \eta \wedge \mathrm{d}\eta \\
&= \frac{1}{9}\left((1-y)\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\psi' + \mathrm{d}y \wedge \mathrm{d}\beta \wedge \mathrm{d}\psi' + \cos\theta\,\mathrm{d}y \wedge \mathrm{d}\phi \wedge \mathrm{d}\psi'\right. \\
&\quad \left. - \cos\theta\,\mathrm{d}y \wedge \mathrm{d}\beta \wedge \mathrm{d}\phi + (1-y)y\sin\theta\,\mathrm{d}\beta \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi\right)
\end{aligned}
\tag{42}
$$

are special Killing forms (10) with constants $c = -2$ and $c = -4$ respectively. Let us note also that Φ_k (28) with $k = 1, 2$ are closed conformal Killing forms.

On the Calabi-Yau manifold $C(M_{2n+1})$ the Kähler form (20) with the Sasakian one-form (38) is

$$
\begin{aligned}
\Omega_{\mathrm{cone}} &= r^2 \frac{1-y}{6} \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi + \frac{r^2}{6}\mathrm{d}y \wedge (\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi) \\
&\quad + \frac{1}{3} r\,\mathrm{d}r \wedge \left(y\,\mathrm{d}\beta + \mathrm{d}\psi' - (1-y)\cos\theta\,\mathrm{d}\phi\right).
\end{aligned}
\tag{43}
$$

The complex volume holomorphic $(3,0)$ form on the metric cone is [14]

$$
\begin{aligned}
\mathrm{d}V_{\mathrm{cone}} &= \mathrm{e}^{\mathrm{i}\psi'} r^2\,\mathrm{d}V_{EK} \wedge (\mathrm{d}r + \mathrm{i}r\eta) \\
&= \mathrm{e}^{\mathrm{i}\psi'} r^2 \sqrt{\frac{1-y}{6p(y)}} (\mathrm{d}\theta + \mathrm{i}\sin\theta\,\mathrm{d}\phi) \\
&\quad \wedge \left(\mathrm{d}y + \mathrm{i}\frac{p(y)}{6}(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)\right) \\
&\quad \wedge \left(\mathrm{d}r + \mathrm{i}\frac{r}{3}(y\,\mathrm{d}\beta + \mathrm{d}\psi' - (1-y)\cos\theta\,\mathrm{d}\phi)\right).
\end{aligned}
\tag{44}
$$

Extracting from the complex volume (44) the form ω^M on the Einstein-Sasaki space according to (35) for $p = 2$ we get the following additional Killing 2-forms

of the $Y(p, q)$ spaces written as real forms [19]

$$
\begin{aligned}
\Xi = \Re\, \omega^M = {} & \sqrt{\frac{1-y}{6\,p(y)}} \\
& \times \left(\cos\psi' \left(-\mathrm{d}y \wedge \mathrm{d}\theta + \frac{p(y)}{6} \sin\theta\, \mathrm{d}\beta \wedge \mathrm{d}\phi \right) \right. \\
& \left. - \sin\psi' \left(-\sin\theta\, \mathrm{d}y \wedge \mathrm{d}\phi - \frac{p(y)}{6}\, \mathrm{d}\beta \wedge \mathrm{d}\theta + \frac{p(y)}{6} \cos\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi \right) \right) \\
\Upsilon = \Im\, \omega^M = {} & \sqrt{\frac{1-y}{6\,p(y)}} \\
& \times \left(\cos\psi' \left(-\sin\theta\, \mathrm{d}y \wedge \mathrm{d}\phi - \frac{p(y)}{6}\, \mathrm{d}\beta \wedge \mathrm{d}\theta + \frac{p(y)}{6} \cos\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi \right) \right. \\
& \left. + \sin\psi' \left(-\mathrm{d}y \wedge \mathrm{d}\theta + \frac{p(y)}{6} \sin\theta\, \mathrm{d}\beta \wedge \mathrm{d}\phi \right) \right).
\end{aligned}
\tag{45}
$$

The Stäckel-Killing tensors associated with the Killing forms Ψ, Ξ, Υ are constructed as in (8). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors P_ϕ, P_ψ, P_α and the total angular momentum J^2 (41) these Stäckel-Killing tensors provide the superintegrability of the $Y(p, q)$ geometries.

6. Concluding Remarks

In general it is a hard task to find solutions of the Killing-Yano equation (6) or conformal Killing-Yano equation (9). However in the case of spaces endowed with special geometrical structures, the existence of Killing forms and their explicit construction is granted.

In this paper we presented the complete set of Killing forms on Einstein-Sasaki spaces associated with Euclideanized Kerr-NUT-(A)dS spaces in a certain scaling limit. The multitude of Killing-Yano and Stäckel-Killing tensors makes possible a complete integrability of geodesic equations.

As an exemplification of the general framework we have presented the complete set of Killing forms on five-dimensional Einstein-Sasaki $Y(p, q)$ spaces. The multitude of Stäckel-Killing tensors associated with these Killing forms implies the superintegrability of the geodesic motions.

These remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon and Dirac equations on Einstein-Sasaki spaces.

Acknowledgements

References

[1] Ballmann W., *Lectures on Kähler Manifolds*, ESI Lectures in Mathematics and Physics, Eur. Math. Soc., Zürich 2006.

[2] Bär C., *Real Killing Spinors and Holonomy*, Comm. Math. Phys. **154** (1993) 509–521.

[3] Benvenuti S. and Kruczenski M., *Semiclassical Strings in Sasaki-Einstein Manifolds and Long Operators in $\mathcal{N} = 1$ Gauge Theories*, J. High Energy Phys. **06** (2006) 051.

[4] Besse A., *Einstein Manifolds*, Springer, Berlin 1987.

[5] Boyer C. and Galicki K., *Sasakian Geometry, Holonomy, and Supersymmetry*, Handbook of Pseudo-Riemannian Geometry and Supersymmetry (IRMA Lect. Math. Theor. Phys.) vol. 16, Eur. Math. Soc., Zürich 2010.

[6] Chen W., Lü H. and Pope C., *General Kerr-NUT-AdS Metrics in All Dimensions*, Class. Quantum Grav. **23** (2006) 5323–5340.

[7] Cvetič M., Lü H., Page D. and Pope C., *New Einstein-Sasaki and Einstein Spaces from Kerr-de Sitter*, J. High Energy Phys. **07** (2009) 082.

[8] Gauntlett J., Martelli D., Sparks J. and Waldram D., *Sasaki-Einstein Metrics on $S^2 \times S^3$*, Adv. Theor. Math. Phys. **8** (2004) 711–734.

[9] Gauntlett J., Martelli D., Sparks J. and Waldram D., *Supersymmetric AdS_5 Solutions of M-Theory*, Class. Quant. Grav. **21** (2004) 4335–4366.

[10] Gibbons G., Hartnoll S. and Pope C., *Bohm and Einstein-Sasaki Metrics, Black Holes and Cosmological Event Horizons*, Phys. Rev. D **67** (2003) 084024.

[11] Krtouš P., Kubizňák D., Page D. and Varsudevan M., *Constants of Geodesic Motion in Higher-Dimensional Black Hole Spacetimes*, Phys. Rev. D **76** (2007) 084034.

[12] Kubizňák D., *On the Supersymmetric Limit of Kerr-NUT-AdS Metrics*, Phys. Lett. B **675** (2009) 110–115.

[13] Martelli D. and Sparks J., *Toric Sasaki-Einstein Metrics on $S^2 \times S^3$*, Phys. Lett. B **621** (2005) 208–212.

[14] Martelli D. and Sparks J., *Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals*, Commun. Math. Phys. **262** (2006) 51–89.

[15] Page D., Kubizňák D., Varsudevan M. and Krtouš P., *Complete Integrability of Geodesic Motion in General Kerr-NUT-AdS Spacetimes*, Phys. Rev. Lett. **98** (2007) 061102.

[16] Rubín de Celis E. and Santillán O., *Massless Geodesics in $AdS_5 \times Y(p, q)$ as a Superintegrable System*, J. High Energy Phys. **09** (2012) 032.

[17] Semmelmann U., *Conformal Killing Forms on Riemannian Manifolds*, Math. Z. **245** (2003) 503–527.

[18] Sergyeyev A. and Krtouš P., *Complete Set of Commuting Symmetry Operators for the Klein-Gordon Equation in Generalized Higher-Dimensional Kerr-NUT-(A)dS Spacetimes*, Phys. Rev. D **77** (2008) 044033.

[19] Visinescu M., *Killing Forms on the Five-Dimensional Einstein-Sasaki Y(p,q) Spaces*, Mod. Phys. Lett. A **27** (2012) 1250217.

[20] Visinescu M. and Vîlcu G., *Hidden Symmetries of Euclideanised Kerr-NUT-(A)dS Metrics in Certain Scaling Limit*, SIGMA **8** (2012) 058.

[21] Yano K., *Some Remarks on Tensor Fields and Curvature*, Ann. of Math. **55** (1952) 328–347.

RECURSION OPERATORS AND EXPANSIONS OVER ADJOINT SOLUTIONS FOR THE CAUDREY-BEALS-COIFMAN SYSTEM WITH $\mathbb{Z}_p$ REDUCTIONS OF MIKHAILOV TYPE

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Abstract. We consider the Caudrey-Beals-Coifman linear problem and the theory of the Recursion Operators (Generating Operators) related to it in the presence of $\mathbb{Z}_p$ reduction of Mikhailov type.

1. Introduction

1.1. The Generalized Zakharov-Shabat and Caudrey-Beals-Coifman Systems

As it is well known nonlinear evolution equations (NLEEs) of soliton type are equations (systems) that can be written into the form $[L, A] = 0$ (Lax representation) where L, A are linear operators on ∂_x, ∂_t depending also on some functions $q_\alpha(x, t), 1 \leq \alpha \leq s$ (called 'potentials') and the spectral parameter λ. The corresponding system is of a course system of partial differential equations on $q_\alpha(x, t)$. Usually the equation is a part of a hierarchy of NLEEs related to $L\psi = 0$ (auxiliary linear problem) which consists of the equations that can be obtained by changing A and fixing L, [7, 15]. The soliton equations possess many interesting properties but for our purposes we shall mention only that they can be solved explicitly through various schemes, most of which share the property that the Lax representation permits to pass from the original evolution to the evolution of some spectral data related to the problem $L\psi = 0$. The Caudrey-Beals-Coifman (CBC) system, called the Generalized Zakharov-Shabat (GZS) system in the case when the

---

element J is real, is one of the best known auxiliary linear problems

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = 0. \tag{1}$$

The system has a long history of study and generalizations see [2–6,29,30], finally it has been realized that one can assume that $q(x)$ and J belong to a fixed simple Lie algebra $\mathfrak{g}$ in some finite dimensional irreducible representation, [17]. Then the element J should be regular, that is $\ker(\mathrm{ad}_J)$ $(\mathrm{ad}_J(X) \equiv [J, X]$, $X \in \mathfrak{g})$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $q(x)$ should belong to the orthogonal complement $\mathfrak{h}^\perp \equiv \bar{\mathfrak{g}}$ of $\mathfrak{h}$ with respect to the Killing form: $\langle X, Y \rangle = \mathrm{tr}(\mathrm{ad}_X \,\mathrm{ad}_Y)$, $X, Y \in \mathfrak{g}$. Thus $q(x) = \sum_{\alpha \in \Delta} q_\alpha(x) E_\alpha$ where E_α are the root vectors, Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. (We use notation and normalizations as in [20].) The scalar functions $q_\alpha(x)$ are defined on $\mathbb{R}$, are complex valued, smooth and tend to zero as $x \to \pm\infty$. We shall assume that they are Schwartz-type functions. Classical Zakharov-Shabat system is obtained for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $J = \mathrm{diag}(1, -1)$.

1.2. The AKNS Approach to the Soliton Equations

Let us construct the so-called adjoint solutions of the system L, that is functions of the type $w = mXm^{-1}$ where $X = \mathrm{const}$, $X \in \mathfrak{g}$ and m is fundamental solution of $Lm = 0$. They satisfy the equation

$$[L, w] = (\mathrm{i}\partial_x w + [q(x) - \lambda J, w]) = 0.$$

Let $w^{\mathrm{a}} = \pi_0$, $w^{\mathrm{d}} = (\mathrm{id} - \pi_0)w$ where π_0 is the orthogonal projector (with respect to the Killing form) of w over $\mathfrak{h}^\perp$ and $\mathfrak{h}$ respectively. We cannot go in detail into the AKNS approach, its history and generalizations, we just mention the seminal work [1] according to which the approach has been named and refer to [15] for all the details. Very roughly speaking the main facts are the following

- If a suitable set of adjoint solutions $(w_i(x, \lambda))_i$ is taken, for λ on the spectrum of L the functions $w_i^{\mathrm{a}}(x, \lambda)$ form a complete set in the space of potentials $q(x)$.

- If one expands the potential over $(w_i(x, \lambda))_i$ as coefficients one gets the minimal scattering data for L.

2. Recursion Operators

Relation to the expansions over adjoint solutions. From the above follows that passing from the potentials to the scattering data can be considered as Generalized Fourier Transform. For it the functions $w_i^{\mathrm{a}}(x, \lambda)$ play the role the exponents play in the Fourier transform. The Recursion Operators (Generating Operators, Λ-operators) are the operators for which the adjoint solutions $w_i^{\mathrm{a}}(x, \lambda)$ introduced

above are eigenfunctions and therefore for the Generalized Fourier transform they play the same role as the differentiation operator in the Fourier transform method.

For the above reason recursion operators are theoretical tools containing most of the information about the NLEEs associated with L. Through them can be obtained

i) The hierarchies of the nonlinear evolution equations solvable through L.

ii) The conservation laws for these NLEEs.

iii) The hierarchies of Hamiltonian structures for these NLEEs.

It is not hard to get that the recursion operators related to L have the form

$$\Lambda_\pm(X(x)) = \mathrm{ad}_J^{-1}\left(\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{i}\,\mathrm{ad}_q \int_{\pm\infty}^x (\mathrm{id} - \pi_0)[q(y), X(y)]\mathrm{d}y\right) \quad (2)$$

where of course $\mathrm{ad}_q(X) = [q, X]$ and X is a smooth, fast decreasing function with values in $\mathfrak{h}^\perp$.

Relation to recursion identities. The name recursion operators has the following origin. If for the NLEEs such that $[L, A] = 0$ the operator A is of the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^n \lambda^k A_k, \qquad A_n \in \mathfrak{h}, \qquad A_n = \mathrm{const}, \qquad A_{n-1} \in \mathfrak{h}^\perp$$

then first $A_{n-1} = \mathrm{ad}_J^{-1}[q, A]$ and for $0 < k < n - 1$ one gets the recursion relations

$$\pi_0 A_{k-1} = \Lambda_\pm(\pi_0 A_k), \qquad (\mathrm{id} - \pi_0)A_k = \mathrm{i}(\mathrm{id} - \pi_0)\int_{\pm\infty}^x [q, \pi_0 A_k](y)\mathrm{d}y. \quad (3)$$

Moreover, the NLEEs related to L can be written into one of the two forms

$$\mathrm{iad}_J^{-1}q_t + \Lambda_\pm^n\left(\mathrm{ad}_J^{-1}[A_n, q]\right) = 0. \quad (4)$$

Thus the recursion operators could be introduced also algebraically as the operators solving the above recursion relations.

Geometric Interpretation. The recursion operators have interesting geometric interpretation as dual objects to a Nijenhuis tensors N on the manifold of potentials on which it is defined a special geometric structure, Poisson-Nijenhuis structure [15, 22]. The corresponding NLEEs are fundamental fields of that structure.

Summarizing, the recursion operators have three important aspects

- They appear naturally by considering the recursion relations arising from the Lax representations of the NLEEs related with L.
- In the generalized Fourier expansions they play the role similar of the role of differentiation in the Fourier expansions.

- Their adjoint operators are Nijenhuis tensors for some special geometric structure on the manifold of potentials - Poisson-Nijenhuis structures.

In this work we shall discuss the implications of the Mikhailov-type reductions on the theory of recursion operators. The topic has been considered recently in several papers, for example [12–14, 25–27]. The case treated in these papers is of the CBC system in pole gauge. The CBC system in canonical gauge (the one we discuss now) subject to reductions has been considered earlier. For example, in [18, 19] were investigated the implications to the scattering data. In [16] the recursion operators in the presence of reductions has been considered from spectral theory viewpoint. General result about the geometry of the recursion operators for L in canonical gauge is presented in [28]. From the other side, though there are number of papers treating what happens with the spectral expansions related with the recursion operators in concrete situations with $\mathbb{Z}_p$ reductions, there has been no general treatment and we shall try to fill this gap.

3. Fundamental Solutions to the CBC System

If $q(x) = \sum_{\alpha \in \Delta} q_\alpha(x) E_\alpha$ we define: $\|q\|_1 = \sum_{\alpha \in \Delta} \int_{-\infty}^{+\infty} |q_\alpha(x)| \mathrm{d}x$. Potentials for which $\|q\|_1 < \infty$ form a Banach space $L^1(\bar{\mathfrak{g}}, \mathbb{R})$. Some important facts about the solutions of (1) with $q \in L^1(\bar{\mathfrak{g}})$ in some irreducible matrix representation defined on a space V are obtained in [17]. We remind them in this and the next section. Let $m(x, \lambda) = \psi(x, \lambda) \exp \mathrm{i}\lambda J x$ where ψ satisfies CBC system. Then

$$\mathrm{i}\partial_x m + q(x)m - \lambda Jm + \lambda mJ = 0, \qquad \lim_{x \to -\infty} m = \mathbf{1}_V. \qquad (5)$$

Theorem 1. *Suppose that for a fixed λ the bounded fundamental solution $m(x, \lambda)$, satisfying the equation (5) exists. Suppose that λ does not belong to the bunch of straight lines $\Sigma = \cup_{\alpha \in \Delta} l_\alpha$ where*

$$l_\alpha = \{\lambda; \; \mathrm{Im}(\lambda\alpha(J)) = 0\}. \qquad (6)$$

Then the solution $m(x, \lambda)$ is unique. (In the above Im *denotes the imaginary part.)*

The connected components of $\mathbb{C} \setminus \Sigma$ are open sectors in the λ-plain. In every such sector either $\mathrm{Im}[\lambda\alpha(J)]$, $\alpha \in \Delta$ is identically zero or it has the same sign. We denote these sectors by Ω_ν and order them anti-clockwise. Clearly ν takes values from one to some even number $2M$. The boundary of the sector Ω_ν consists of two rays - L_ν and $L_{\nu+1}$ (L_ν comes before $L_{\nu+1}$ when we turn anti-clockwise) so that $\bar{\Omega}_\nu \cap \bar{\Omega}_{\nu-1} = L_\nu$. Of course, we understand the number ν modulo $2M$.

For small potentials ($\|q\|_1 < 1$) in any representation of $\mathfrak{g}$ there is no discrete spectrum and in each sector Ω_ν there exists unique fundamental solution $m_\nu(x, \lambda)$

of (5), analytic in λ. The solution admits extension by continuity to the boundary of Ω_ν, that is to the rays L_ν and $L_{\nu+1}$. For potentials that are not small the typical approach is to consider potentials on compact support and then to pass to Lebesgue integrable potentials. The situation is complicated, there is discrete spectrum etc., [17]. For our purposes however we shall limit ourselves to the situation when there is no discrete spectrum.

4. Expansions Over Adjoint Solutions

We first define in each Ω_ν analytic solutions $\chi_\nu(x, \lambda)$ of equation (1)

$$m_\nu(x, \lambda) = \chi_\nu(x, \lambda)e^{i\lambda J x} \tag{7}$$

and then we set

$$e_\alpha^\nu(x, \lambda) = \pi_0(\chi_\nu(x, \lambda)E_\alpha\chi_\nu^{-1}(x, \lambda)), \qquad \lambda \in \bar\Omega_\nu. \tag{8}$$

This notation is better to be changed because for $\lambda \in L_\nu$ it will be good to retain the index ν to refer to the ray L_ν. Then it becomes necessary to distinguish from which sector the solution is extended. So for $\lambda \in L_\nu$ we shall write $e_\alpha^{(+;\nu)}(x, \lambda)$ if the solution is extended from the sector $\Omega_{\nu-1}$ and $e_\alpha^{(-;\nu)}(x, \lambda)$ if the solution is extended from the sector Ω_ν. In other words, for $\lambda \in L_\nu$

$$e_\alpha^{\nu;+}(x, \lambda) = \pi_0(\chi_{\nu-1}(x, \lambda)E_\alpha\chi_{\nu-1}^{-1}(x, \lambda))$$

$$e_\alpha^{\nu;-}(x, \lambda) = \pi_0(\chi_\nu(x, \lambda)E_\alpha\chi_\nu^{-1}(x, \lambda)). \tag{9}$$

In order to write the completeness relations, let is denote

$$\Pi_0 = \sum_{\gamma \in \Delta} \frac{|\gamma\rangle\langle\gamma|}{\gamma(J)}, \qquad \delta_\nu^\pm = \Delta_\nu^\pm \cap \delta_\nu \tag{10}$$

$$\delta_\nu = \{\alpha \in \Delta \; ; \; \text{Im}(\lambda\alpha(J)) = 0 \text{ for } \lambda \in L_\nu\}. \tag{11}$$

Let us also assume that the rays L_ν are oriented from 0 to ∞. Then the completeness relations (no discrete spectrum) amount to the formula

$$\Pi_0\delta(x - y)$$

$$= \frac{1}{2\pi}\sum_{\nu=1}^{2M}\int_{L_\nu} d\lambda\{\sum_{\alpha\in\delta_\nu^+}[e_\alpha^{(-;\nu)}(x) \otimes e_{-\alpha}^{(-;\nu)}(y) - e_{-\alpha}^{(+;\nu)}(x) \otimes e_\alpha^{(+;\nu)}(y)]\} \tag{12}$$

where we have omitted the dependence on λ in order to be able to write the relation (12) more nicely. The above formula should be understood in the following way: first, it is assumed that $\mathfrak{g}^*$ is identified with $\mathfrak{g}$, assuming that the pairing is given by the Killing form. So for example, for $X, Y, Z \in \mathfrak{g}$ making a contraction of $X \otimes Y$ with Z on the right we obtain $X\langle Y, Z\rangle$ and making contraction on the left we get

$\langle Z, X \rangle Y$. Next, the formula for Π_0 implies that making a contraction with Π_0 on the right we get $\Pi_0 X = \operatorname{ad}_J^{-1} \pi_0 X$ and similarly on the left $X \Pi_0 = -\operatorname{ad}_J^{-1} \pi_0 X$. (On the space $\bar{\mathfrak{g}}$ the operator ad_J is invertible.) Finally, if we have a L^1-integrable function $h : \mathbb{R} \mapsto \bar{\mathfrak{g}}$ then making a contraction of $\operatorname{ad}_J h = [J, h]$ with (12) from the right (left) and integrating over y from $-\infty$ to $+\infty$ we get

$$h(x) = \frac{\epsilon}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \{ \sum_{\alpha \in \delta_\nu^+} [e_{\epsilon\alpha}^{(-;\nu)}(x) \langle\langle e_{-\epsilon\alpha}^{(-;\nu)}, [J, h] \rangle\rangle$$

$$- e_{\epsilon\alpha}^{(+;\nu)}(x) \langle\langle e_{-\epsilon\alpha}^{(+;\nu)}, [J, h] \rangle\rangle]\} \mathrm{d}\lambda. \quad (13)$$

We have two expansions here for $\epsilon = +1$ and $\epsilon = -1$ and we adopted the notation

$$\langle\langle e_\alpha^{(\pm;\nu)}, [J, h] \rangle\rangle = \int_{-\infty}^{+\infty} \langle e_\alpha^{(\pm;\nu)}(x), [J, h(x)] \rangle \mathrm{d}x. \quad (14)$$

We must make some comments here

1. It can be shown that the expansion (13) converges in the same sense as the Fourier expansions for $h(x)$. These are the so-called Generalized Fourier Expansions and the functions $e_\alpha^{\pm;\nu}(x, \lambda)$ are the Generalized Exponents. When one expands over the Generalized Exponents the potential $q(x)$ one gets as coefficients the minimal scattering data.

2. One can prove that

$$(\Lambda_- - \lambda) e_\alpha^{(-;\nu)} = 0, \qquad (\Lambda_- - \lambda) e_{-\alpha}^{(+;\nu)} = 0, \qquad \alpha \in \delta_\nu^+ \quad (15)$$

$$(\Lambda_+ - \lambda) e_{-\alpha}^{(-;\nu)} = 0, \qquad (\Lambda_+ - \lambda) e_\alpha^{(+;\nu)} = 0, \qquad \alpha \in \delta_\nu^+ \quad (16)$$

and therefore the expansions (13) are in fact the spectral decompositions for the operators Λ_- and Λ_+, that is they play for these expansions the role that $\mathrm{i}\partial_x$ plays for the Fourier expansion.

5. $\mathbb{Z}_p$ Reductions in the CBC System Defined by an Automorphism

We shall consider now special type of linear problems of the type (1) in which the potential function $q(x)$ and the element J obey some special requirements resulting from Mikhailov-type reductions, [21, 23, 24]. We shall consider the case when the Mikhailov reduction group G_0 is generated by one element, which we denote by H. It acts on the fundamental solutions in the following way

$$H(\psi(x, \lambda)) = \mathcal{K}(\psi(x, \omega^{-1}\lambda)) \quad (17)$$

where $\omega = \exp \dfrac{2\pi\mathrm{i}}{p}$ and $\mathcal{K}$ is automorphism of order p of the Lie group corresponding to the algebra $\mathfrak{g}$. $\mathcal{K}$ generates an automorphism of $\mathfrak{g}$ which we shall denote by

the same letter $\mathcal{K}$. We shall require in the above situation that the automorphism leaves invariant the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to which the element J in the CBC system belongs. We proceed with some general remarks and technical results.

1. Suppose $\mathcal{K}$ is an automorphism of $\mathfrak{g}$ and $\mathcal{K}^p = \mathrm{id}$, $\mathcal{K}\mathfrak{h} \subset \mathfrak{h}$. (In case of Coxeter automorphisms p is the Coxeter number.) The Coxeter automorphisms are internal that is each $\mathcal{K}$ is internal and can be represented as $\mathcal{K} = \mathrm{Ad}(K)$, K belonging to the corresponding group G with algebra $\mathfrak{g}$.

2. The automorphisms leave the Killing form invariant, a fact that we shall use constantly.

3. The algebra $\mathfrak{g}$ splits into a direct sum of eigenspaces of $\mathcal{K}$, that is

$$\mathfrak{g} = \oplus_{s=0}^{p-1} \mathfrak{g}^{[s]} \tag{18}$$

where for each $X \in \mathfrak{g}^{[s]}$ we have $\mathcal{K}X = \omega^s X$ and the spaces $\mathfrak{g}^{[s]}$, $\mathfrak{g}^{[k]}$ for $k \neq s$ are orthogonal with respect to the Killing form.

4. Because $\mathcal{K}$ leaves $\mathfrak{h}$ invariant, it leaves invariant also the orthogonal complement $\bar{\mathfrak{g}}$ of $\mathfrak{h}$. Thus each $\mathfrak{g}^{[s]}$ splits into $\bar{\mathfrak{g}}^{[s]} \oplus \mathfrak{h}^{[s]}$ and

$$\bar{\mathfrak{g}} = \oplus_{s=0}^{p-1} \bar{\mathfrak{g}}^{[s]}, \qquad \mathfrak{h} = \oplus_{s=0}^{p-1} \mathfrak{h}^{[s]}. \tag{19}$$

The spaces $\bar{\mathfrak{g}}^{[k]}$ and $\mathfrak{h}^{[s]}$ are orthogonal for arbitrary k and s. We shall denote the projectors over the space $\bar{\mathfrak{g}}^{[k]}$ by $\pi_0^{[s]}$.

After the above preliminaries, let us assume that the set of fundamental solutions for the spectral problem (1) is invariant under G_0. Then as it is easy to see that we must have

$$\mathcal{K}(J) = \omega J, \qquad \mathcal{K}q = q \tag{20}$$

that is, $J \in \mathfrak{g}^{[1]}$, $q(x) \in \mathfrak{g}^{[0]}$. In fact, suppose we have a Lax representation $[L, A] = 0$ where A has the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \qquad A_n \in \mathfrak{h}, \qquad A_n = \mathrm{const}, \qquad A_{n-1} \in \bar{\mathfrak{g}}.$$

If the common fundamental solutions for $L\psi = 0$, $A\psi = 0$ are invariant under G_0 then we also have

$$\mathcal{K}(A_s) = \omega^s A_s, \qquad s = 0, 1, 2, \ldots n. \tag{21}$$

The above reductions are compatible with the evolution in the sense that if at the moment $t = 0$ we have (20), (21) we have the same relations at arbitrary moment t. The invariance of the set of the fundamental solutions can be additionally specified if we take the solutions $m_\nu(x, \lambda)$ defined in the sectors Ω_ν, $\nu = 1, 2, \ldots 2M$ defined by the straight lines $l_\alpha = \{\lambda ; \mathrm{Im}(\lambda\alpha(J)) = 0, \alpha \in \Delta\}$. (Of course, one obtains the same line for α and $-\alpha$ but it can happen that $\alpha \neq \beta$ and $l_\alpha = l_\beta$.)

Taking into account the uniqueness of the solutions $m(x, \lambda)$ we get that $\mathcal{K}(m(x, \lambda))$ is equal to $m(x, \omega\lambda)$. Consequently, we obtain that

$$\mathcal{K}(\chi(x, \lambda)) = \mathcal{K}(m(x, \lambda)e^{-iJx\lambda}) = m(x, \omega\lambda)e^{-iJx\omega\lambda} = \chi(x, \omega\lambda) \qquad (22)$$

is analytic in $\omega\Omega_\nu$. If l_α, l_β form the boundary of Ω_ν then $\omega l_\alpha, \omega l_\beta$ are the straight lines defining the boundary of $\omega\Omega_\nu$.

Let us define $\hat{\mathcal{K}} : \mathfrak{h} \mapsto \mathfrak{h}$ by $\hat{\mathcal{K}} = (\mathcal{K}^*)^{-1}$. The map $\hat{\mathcal{K}}$ defines the coadjoint action of $\mathcal{K}$ on $\mathfrak{h}^*$. Naturally $\hat{\mathcal{K}}^p = \mathrm{id}$ and

$$\langle \hat{\mathcal{K}}\xi, \mathcal{K}H \rangle = \langle \xi, H \rangle, \qquad \xi \in \mathfrak{h}^*, H \in \mathfrak{h}. \qquad (23)$$

It is a general fact from the theory of the automorphisms is that for all roots we have $\mathcal{K}E_\alpha = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$, $q(\alpha)q(\beta) = q(\alpha + \beta)$ if $\alpha + \beta \in \Delta$. *One easily gets that* $\omega l_\alpha = l_{\hat{\mathcal{K}}^{-1}\alpha}$. *Thus we have an action of the automorphism $\mathcal{K}$ (the group $\mathbb{Z}_p$) on the bunch of lines $\{l_\alpha\}_{\alpha \in \Delta}$ defined by $\hat{\mathcal{K}}^{-1}$ and similarly the action on the set of sectors Ω_ν, $\nu = 1, 2, \ldots, 2M$. We have*

Proposition 2. *The representatives from the different orbits of $\mathbb{Z}_p$ on the set of sectors Ω_ν, $\nu = 1, 2 \ldots, a$ can be taken to be adjacent, which we shall always assume.*

6. Expansions in Presence of Reductions Defined by Automorphisms

6.1. $\mathbb{Z}_p$ Reductions of General Type

Consider the general case of automorphism $\mathcal{K}$ of order p, let $\Omega_1, \Omega_2, ..., \Omega_a$ be the fundamental sectors (moving anticlockwise when we go from Ω_1 to Ω_a) and let us label the rays that form the boundaries of the sectors in such a way that Ω_ν is locked between the rays L_ν and $L_{\nu+1}$ that are oriented from zero to infinity. Since multiplication by ω^p is identity (turning by angle 2π) the number of sectors is $2M = pa$. Multiplying by ω we get from the sector Ω_ν the sector $\Omega_{a+\nu}$ and multiplying by ω^{2M} we get again Ω_ν so we shall understand the labels modulo $2M$. Naturally, $L_{a+\nu} = \omega L_\nu$. For each $\alpha \in \Delta$ we have $\mathcal{K}(E_\alpha) = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha)$ are numbers, such that $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$ and $q(\alpha)q(\beta) = q(\alpha+\beta)$ if $\alpha + \beta \in \Delta$. It is not hard to obtain that

$$[\mathcal{K} \circ \pi_0](\chi_\nu(x, \lambda)E_\alpha\chi_\nu^{-1}(x, \lambda)) = \pi_0(\chi_{\nu+a}(x, \omega\lambda)\mathcal{K}(E_\alpha)\chi_{\nu+a}^{-1}(x, \omega\lambda))$$
$$= q(\alpha)\pi_0(\chi_{\nu+a}(x, \omega\lambda)E_{\hat{\mathcal{K}}\alpha}\chi_{\nu+a}^{-1}(x, \omega\lambda))$$

and as a consequence

$$\mathcal{K}(e_\alpha^\nu(x, \lambda)) = q(\alpha)e_{\hat{\mathcal{K}}\alpha}^{\nu+a}(x, \omega\lambda). \qquad (24)$$

Changing the variables for the integrals over the rays that do not belong to the set $\{L_1, L_2, \ldots L_a\}$ we transform expansion (12) into

$$\Pi_0\, \delta(x - y) = \frac{1}{2\pi} \sum_{\nu=1}^{a} \sum_{k=1}^{p} \int_{L_\nu} \{ \sum_{\alpha \in \delta_\nu^+} [\omega^k \mathcal{K}^k \otimes \mathcal{K}^k(e_\alpha^{(-;\nu)}(x) \otimes e_{-\alpha}^{(-;\nu)}(y))$$

$$- \sum \omega^k \mathcal{K}^k \otimes \mathcal{K}^k(e_{-\alpha}^{(+;\nu)}(x) \otimes e_\alpha^{(+;\nu)}(y))]\} d\lambda \tag{25}$$

where $(\mathcal{K} \otimes \mathcal{K})(X \otimes Y) = \mathcal{K}(X) \otimes \mathcal{K}(Y)$. Note that the numbers $q(\alpha)$ do not appear any more, this occurs because we apply $\mathcal{K}$ always on products of the type $E_\alpha \otimes E_{-\alpha}$. The rays L_ν are orientated from 0 to ∞ and the index ν is understood modulo a.

The expansions of a function $h(x)$ over the adjoint solutions can be simplified further, if for arbitrary x the value $h(x) \in \mathfrak{g}^{[s]}$, where $\mathfrak{g}^{[s]}$ is the eigenspace corresponding to the eigenvalue ω^s. As the Killing form is invariant with respect to the action of the automorphism, we get

$$\langle \mathcal{K}^k(e_\alpha^\nu(x, \lambda)), [J, h(x)] \rangle = \langle e_\alpha^\nu(x, \lambda), \mathcal{K}^{-k}([J, h(x)]) \rangle$$

$$= \omega^{-k(s+1)} \langle e_\alpha^\nu(x, \lambda), [J, h(x)] \rangle.$$

The expansions over the adjoint solutions run as follows

$$h(x) = \frac{\epsilon}{2\pi} \sum_{\nu=1}^{a} \int_{L_\nu} \{ \sum_{\alpha \in \delta_\nu^+} [\sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(e_{\epsilon\alpha}^{(-;\nu)}(x, \lambda)) \langle\langle e_{-\epsilon\alpha}^{(-;\nu)}, [J, h] \rangle\rangle$$

$$- \sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(e_{-\epsilon\alpha}^{(+;\nu)}(x, \lambda)) \langle\langle e_{\epsilon\alpha}^{(+;\nu)}, [J, h] \rangle\rangle] \} d\lambda. \tag{26}$$

Actually here we have two expansions, one for $\epsilon = +1$ and the other for $\epsilon = -1$ and the index ν is understood modulo a. Thus we see that $h(x)$ is expanded over the functions

$$e_\alpha^{(\pm;\nu;s)}(x, \lambda) = \sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(e^{(\pm;\nu)}(x, \lambda)) \in \mathfrak{g}^{[s]}, \qquad \nu = 1, 2, \ldots, a \tag{27}$$

since for arbitrary $X \in \mathfrak{g}$ we have $\sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(X) \in \mathfrak{g}^{[s]}$. We shall denote by $e_\alpha^{(\nu;s)}(x, \lambda)$ the expressions

$$e_\alpha^{(\nu;s)}(x, \lambda) = \sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(e_\alpha^\nu(x, \lambda)), \qquad \lambda \in \Omega_\nu. \tag{28}$$

Clearly, $e_\alpha^{(\pm;\nu;s)}(x, \lambda)$ are just the limits of $e_\alpha^{(\nu-1;s)}(x, \lambda)$ and $e_\alpha^{(\nu;s)}(x, \lambda)$ when λ approaches one of the rays L_ν from one or the other side. If as before $h(x) \in \mathfrak{g}^{[s]}$,

we get $\langle e_\alpha^{(\nu;s+1)}(x,\lambda), [J, h(x)]\rangle = p\langle e_\alpha^\nu, [J, h(x)]\rangle$ and the expansions (26) can be cast into the form

$$
h(x) = \frac{\epsilon}{2\pi p} \sum_{\nu=1}^a \int_{L_\nu} \{ \sum_{\alpha\in\delta_\nu^+} [e_{\epsilon\alpha}^{(-;\nu;s)}(x,\lambda)\langle\langle e_{-\epsilon\alpha}^{(-;\nu;s+1)}, [J, h]\rangle\rangle
$$

$$
- e_{-\epsilon\alpha}^{(+;\nu;s)}(x,\lambda))\langle\langle e_{+\epsilon\alpha}^{(+;\nu;s+1)}, [J, h]\rangle\rangle]\} d\lambda. \tag{29}
$$

(As before we have two expansions, for $\epsilon = +1$ and for $\epsilon = -1$.)

6.2. Coxeter Automorphisms Reductions

Coxeter automorphisms are the automorphisms for which

$$
\hat{\mathcal{K}} = S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_r}, \qquad \mathcal{K}^p = \mathrm{id}, \qquad p - \text{the Coexter number}
$$

and S_{α_i} are the Weyl reflections corresponding to the simple roots $\alpha_1, \alpha_2, \ldots \alpha_r$ of $\mathfrak{g}$. We are able to prove the following

Theorem 3. *Assume we have the* CBC *problem for the classical series of simple Lie algebras and the* $\mathbb{Z}_p$ *reduction is defined as in the above using the Coxeter automorphism* $\mathcal{K}$. *Then we have two adjacent fundamental sectors of analyticity for the fundamental analytic solutions* $m_\nu(x, \lambda)$ *and they can be chosen to be*

$$
\Omega_0 = \{\lambda ; \frac{\pi}{2} < \arg(\lambda) < \frac{\pi}{2} + \frac{\pi}{p}\}
$$

$$
\Omega_1 = \{\lambda ; \frac{\pi}{2} + \frac{\pi}{p} < \arg(\lambda) < \frac{\pi}{2} + \frac{2\pi}{p}\}. \tag{30}
$$

For a reduction defined by Coxeter automorphism of order p on some fixed algebra from the classical series of simple Lie algebras the expansion we considered specify even further. First, for the sake of symmetry we label the fundamental sectors by 0 and 1, that is they are Ω_0 and Ω_1 (as in the above). Their boundaries are formed by the rays L_0, L_1, L_2. Next, if $\alpha \in \delta_\nu^+$ then

- $\nu = 2k$ leads to $\hat{\mathcal{K}}^{-k}\alpha \in \delta_0^+ = \delta_{2p}^+$
- $\nu = 2k + 1$ leads to $\hat{\mathcal{K}}^{-k}\alpha \in \delta_1^+$.

The completeness relations we have considered, namely the general formula (25) and the expansions (26),(29) can be written easily for the case of Coxeter automorphism reduction. The only thing one needs to do is not to sum over ν instead from 1 to a but from 0 to 1. Of course, p is then the Coxeter number.

7. Recursion Operators and $\mathbb{Z}_p$ Reductions Related to Automorphisms

7.1. Algebraic Aspects

When we have $\mathbb{Z}_p$ reductions of the type we consider the algebra splits in a direct sum, see (18) and $q \in \mathfrak{g}^{[0]}$ while $J \in \mathfrak{h}^{[1]}$. In particular, this means that

$$\mathrm{ad}_J(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s+1]}, \qquad \mathrm{ad}_J^{-1}(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s-1]} \tag{31}$$

(the superscripts are understood modulo p.) Also, if $X \in \bar{\mathfrak{g}}^{[s]}$ then $\partial_x X \in \bar{\mathfrak{g}}^{[s]}$, $\partial_x^{-1} X \in \bar{\mathfrak{g}}^{[s]}$, $[q, X] \in \bar{\mathfrak{g}}^{[s]}$ and

$$\Lambda_\pm X = \mathrm{ad}_J^{-1}\{\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{ad}_q\,\partial_x^{-1}(1 - \pi_0)[q, X]\} \in \bar{\mathfrak{g}}^{[s-1]}. \tag{32}$$

If we use the projectors $\pi_0^{[s]}$ introduced earlier the above expression can also be written as

$$\Lambda_\pm X = \mathrm{ad}_J^{-1}\{\mathrm{i}\partial_x + \pi_0\,\mathrm{ad}_q + \mathrm{ad}_q\,\partial_x^{-1}(1 - \pi_0)\,\mathrm{ad}_q\}\pi_0^{[s]}X. \tag{33}$$

Further on we will denote

- by $\mathfrak{F}(\bar{\mathfrak{g}})$ the space of smooth, rapidly decreasing $\bar{\mathfrak{g}}$-valued functions.
- by $\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$ the space of smooth, rapidly decreasing $\bar{\mathfrak{g}}^{[s]}$-valued functions.
- by $\Lambda_{\pm;s}$ the operator $\Lambda_\pm\pi_0^{[s]}$, that is $\Lambda_{\pm;s}X = \Lambda_\pm X$ if $X \in \mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$.

The spaces $\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$ are mapped by $\Lambda_\pm$ and are invariant under the action of $\Lambda_\pm^p$

$$\Lambda_\pm|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})} = \Lambda_{\pm;s}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})}, \qquad \Lambda_{\pm;s}\mathfrak{F}(\bar{\mathfrak{g}}^{[s]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}^{[s-1]}). \tag{34}$$

Also

$$\Lambda_\pm^p|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})} = \Lambda_{\pm;s-p+1} \ldots \Lambda_{\pm;s-1}\Lambda_{\pm;s} \tag{35}$$

and the indexes are understood modulo p. In particular

$$\Lambda_\pm^p|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[0]})} = \Lambda_{\pm;1} \ldots \Lambda_{\pm;p-2}\Lambda_{\pm;p-1}\Lambda_{\pm;p}. \tag{36}$$

Recall that the recursion operators arise naturally when looking for the NLEEs that have Lax representation $[L, A] = 0$ with L being the CBC system operator and A is the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^n \lambda^k A_k, \qquad \mathfrak{h} \ni A_n = \mathrm{const}, \qquad A_{n-1} \in \bar{\mathfrak{g}}. \tag{37}$$

Then from $[L, A] = 0$ we first obtain $A_{n-1} = \mathrm{ad}_J^{-1}[q, A]$ and next for $0 < k < n - 1$ the recursion relations $\pi_0 A_{k-1} = \Lambda_\pm(\pi_0 A_k)$ and the NLEEs (4).

Assume that we have $\mathbb{Z}_p$ reduction. Then $q \in \bar{\mathfrak{g}}^{[0]}$, $J \in \mathfrak{h}^{[0]}$ and we must have $\mathcal{K}(A_s) = \omega^s A_s$. Assume that $A_n \in \mathfrak{h}^{[n]}$. Then $A_{n-1} \in \bar{\mathfrak{g}}^{[n-1]}$ and we see that $A_s \in \mathfrak{g}^{[s]}$. Therefore the reduction requirements will be satisfied automatically

when we choose $A_n \in \mathfrak{h}^{[n]}$. Since n is a natural number let us write it into the form $n = kp + m$ where k, p, m are natural numbers and $0 \leq m < p$. Then

$$\Lambda_\pm^n \operatorname{ad}_J^{-1}[A_n, q] = \Lambda_\pm^{kp} \Lambda_\pm^m \operatorname{ad}_J^{-1}[A_n, q]$$

$$= \left(\Lambda_{\pm;0} \ldots \Lambda_{\pm;p-2} \Lambda_{\pm;p-1}\right)^k \Lambda_{\pm;0} \ldots \Lambda_{\pm;m-2} \Lambda_{\pm;m-1} \operatorname{ad}_J^{-1}[A_n, q].$$

Starting from the works [8,9] it is frequently said that when reductions are present the recursion operator becomes of higher order in the derivative ∂_x and factorizes into a product of first order operators with respect to ∂_x. The above has been used by some authors to justify the claim that the recursion operators $R_\pm$ in the presence of $\mathbb{Z}_p$ reduction factorize to become

$$R_\pm = \Lambda_{\pm;0} \ldots \Lambda_{\pm;p-2} \Lambda_{\pm;p-1}. \tag{38}$$

To our opinion more accurate would be simply to say that they are restrictions of the recursion operator in general position on some subspaces

$$\begin{array}{ccccccc}
& \Lambda_{\pm;0} & & \Lambda_{\pm;p-1} & & \Lambda_{\pm;1} & \\
\mathfrak{F}(\bar{\mathfrak{g}}^{[p]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) & \to & \mathfrak{F}(\bar{\mathfrak{g}}^{[p-1]}) & \to & \ldots & \to & \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[p]}).
\end{array} \tag{39}$$

The above shows that the role of the recursion operators in case of $\mathbb{Z}_p$ reductions is taken now by $\Lambda_\pm^p$. This view is supported also by the geometric picture, [28], since the operators $(\Lambda_\pm^p)^*$ are also Nijenhuis tensors.

7.2. Expansions Over Adjoint Solutions

Let us see how the operators we introduced act on the set of functions (27), (28) over which the expansions (26) are written. Using the properties of the automorphism $\mathcal{K}$ (the fact that it commutes with the projection π_0 on $\mathfrak{h}$) and the facts that $\mathcal{K}q = q$ and $\mathcal{K}J = \omega J$ we easily get

Lemma 4. *If $\mathcal{K}$ is an automorphism of order p defining the $\mathbb{Z}_p$ reduction then*

$$\Lambda_\pm \circ \mathcal{K} = \omega \mathcal{K} \circ \Lambda_\pm. \tag{40}$$

As a consequence

$$\Lambda_\pm^p \circ \mathcal{K} = \mathcal{K} \circ \Lambda_\pm^p. \tag{41}$$

Then for $\lambda \in \Omega_\nu$ we immediately obtain

$$\Lambda_\pm e_\alpha^{(\nu;s)}(x, \lambda) = \lambda \sum_{k=1}^{p} \omega^{-k(s-1)} \mathcal{K}^k \Lambda_\pm(e_\alpha^\nu(x, \lambda)) = \lambda e_\alpha^{(\nu;s-1)}(x, \lambda). \tag{42}$$

After some calculations we get that

$$\Lambda_- e_\alpha^{(-;\nu;s)} = \lambda e_\alpha^{(-;\nu,s-1)}, \qquad \Lambda_- e_\alpha^{(+;\nu,s)} = \lambda e_\alpha^{(+;\nu.s-1)}, \quad \alpha \in \delta_\nu^+ \tag{43}$$

$$\Lambda_+ e_{-\alpha}^{(-;\nu,s)} = \lambda e_{-\alpha}^{(-;\nu,s-1)}, \qquad \Lambda_+ e_\alpha^{(+;\nu,s)} = \lambda e_\alpha^{(+;\nu,s-1)}, \quad \alpha \in \delta_\nu^+. \tag{44}$$

As a corollary

$$\Lambda_-^p e_\alpha^{(-;\nu;s)} = \lambda^p e_\alpha^{(-;\nu,s)}, \qquad \Lambda_-^p e_{-\alpha}^{(+;\nu,s)} = \lambda^p e_{-\alpha}^{(+;\nu.s)}, \quad \alpha \in \delta_\nu^+ \qquad (45)$$

$$\Lambda_+^p e_{-\alpha}^{(-;\nu,s)} = \lambda^p e_{-\alpha}^{(-;\nu,s)}, \qquad \Lambda_+^p e_\alpha^{(+;\nu,s)} = \lambda^p e_\alpha^{(+;\nu,s)}, \quad \alpha \in \delta_\nu^+ \qquad (46)$$

and we have

Theorem 5. *For the expansions* (26) *the role of the recursion operators are played by the p-th powers of the operators* $\Lambda_\pm$.

8. Conclusions

- The above considerations show that both from recursion relations viewpoint and expansion over adjoint solutions viewpoint the role of the recursion operators in case of $\mathbb{Z}_p$ reductions is played by the operators $\Lambda_\pm^p$.
- The same conclusion is drawn from the geometric considerations [28] so the theory now is complete in all aspects - algebraic, spectral and geometric.

References

[1] Ablowitz M., Kaup D., Newell A. and Segur H, *The Inverse Scattering Method - Fourier Analysis for Nonlinear Problems*, Studies in Appl. Math. **53** (1974) 249–315.

[2] Beals R. and Coifman R., *Scattering and Inverse Scattering for First Order Systems*, Comm. Pure & Apppl. Math. **37** (1984) 39–90.

[3] Beals R. and Coifman R., *Inverse Scattering and Evolution Equations*, Commun. Pure & Appl. Math. **38** (1985) 29–42.

[4] Beals R. and Coifman R., *Scattering and Inverse Scattering for First Order Systems II*, Inverse Problems **3** (1987) 577–593.

[5] Beals R. and Sattinger D., *On the Complete Integrability of Completely Integrable Systems*, Comm. Math. Phys. **138** (1991) 409–436.

[6] Caudrey P., *The Inverse Problem for a General $N \times N$ Spectral Equation*, Physica D **6** (1982) 51–56.

[7] Faddeev L. and Takhtadjan L., *Hamiltonian Method in the Theory of Solitons*, Springer, Berlin 1987.

[8] Fordy A. and Gibbons J., *Factorization of Operators I. Miura Transformations*, J. Math. Phys. **21** (1980) 2508–2510.

[9] Fordy A. and Gibbons J., *Factorization of Operators II*, J. Math. Phys. **22** (1981) 1170–1175.

[10] Gerdjikov V. and Kulish P., *The Generating Operator for $n \times n$ Linear System*, Physica D **3** (1981) 549–562.

[11] Gerdjikov V., *Generalized Fourier Transforms for the Soliton Equations. Gauge-covariant Formulation*, Inverse Problems **2** (1986) 51–74.

[12] Gerdjikov V., Mikhailov A. and Valchev T., *Reductions of Integrable Equations on A.III-Symmetric Spaces*, J. Phys. A: Math & Theor. **43** (2010) 434015.

[13] Gerdjikov V., Mikhailov A. and Valchev T., *Recursion Operators and Reductions of Integrable Equations on Symmetric Spaces*, J. Geom. Symmetry Phys. **20** (2010) 1–34.

[14] Gerdjikov V., Grahovski G., Mikhailov A. and Valchev T., *Polynomial Bundles and Generalized Fourier Transforms for Integrable Equations on A. III-type Symmetric Spaces*, SIGMA **7** (2011) 096.

[15] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies - Spectral and Geometric Methods*, Springer, Heidelberg 2008.

[16] Gerdjikov V., Kostov N. and Valchev T., *Generalized Zakharov-Shabat Systems and Nonlinear Evolution Equations with Deep Reductions*, In: BGSIAM'09, S. Margenov, S. Dimova and A. Slavova (Eds), Demetra, Sofia 2010, pp 51–57.

[17] Gerdjikov V. and Yanovski A., *Completeness of the Eigenfunctions for the Caudrey-Beals-Coifman System*, J. Math. Phys. **35** (1994) 3687–3721.

[18] Grahovski G., *On the Reductions and Scattering Data for the CBC System*, Geometry, Integrability and Quantization **3** (2002) 262–277.

[19] Grahovski G., *On the Reductions and Scattering Data for the Generalized Zakharov-Shabat Systems*, In: Nonlinear Physics: Theory and Experiment II, M. Ablowitz, M. Boiti, F. Pempinelli and B. Prinari (Eds), World Scientific, Singapore 2003, pp 71–78.

[20] Goto M. and Grosshans F., *Semisimple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics **38**, M. Dekker, New York 1978.

[21] Lombardo S. and Mikhailov A., *Reductions of Integrable Equations. Dihedral Group*, J. Phys. A **37** (2004) 7727–7742.

[22] Magri F., *A Simple Model of the Integrable Hamiltonian Equations*, J. Math. Phys. **19** (1978) 1156–1162.

[23] Mikhailov A., *Reduction in the Integrable Systems. Reduction Groups* (in Russian), Lett. JETF (Letters to Soviet Journal of Experimental and Theoretical Physics) **32** (1979) 187–192.

[24] Mikhailov A., *The Reduction Problem and Inverse Scattering Method*, Physica D **3** (1981) 73–117.

[25] Valchev T., *On Certain Reduction of Integrable Equations on Symmetric Spaces*, In: AIP Conference Proceedings 1340, K. Sekigawa, V. Gerdjikov, Y. Matsushita and I. Mladenov (Eds), Melville, New York 2011, pp 154–163.

[26] Yanovski A., *Geometric Interpretation of the Recursion Operators for the Generalized Zakharov-Shabat System in Pole Gauge on the Lie Algebra A_2*, J. Geom. Symmetry Phys. **23** (2011) 97–11.

[27] Yanovski A., *On the Recursion Operators for the Gerdjikov, Mikhailov and Valchev System*, J. Math. Phys. **52** (2011) 082703.

[28] Yanovski A., *Geometry of the Recursion Operators for Caudrey-Beals-Coifman System in the Presence of Mikhailov $\mathbb{Z}_p$ Reductions*, J. Geom. Symmetry Phys. **25** (2012) 77–97.

[29] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons*: *The Inverse Scattering Method*, Plenum, New York, 1984.

[30] Zhou X., *Direct and Inverse Scattering Transforms with Arbitrary Spectral Singularities*, Comm. Pure & Appl. Math. **42** (1989) 895–938.

Selected list of publications

Monographs and Editorials

[1] R. Balakrishnan, R. Dandoloff, V. Gerdjikov, D. Pushkarov, A. Saxena. *Editorial: Topical issue on geometry, integrability and nonlinearity in condensed matter physics.* European J. Phys. **29B**, 155 (2002).

[2] V. Gerdjikov, M. Tsvetkov and P. Fiziev. Foreword. In: Prof. G. Manev's Legacy in Contemporary Aspects of Astronomy, Theoretical and Gravitational Physics, Eds.: V. Gerdjikov and M. Tsvetkov, Heron Press Ltd, Sofia, 2005. pp. 5-6.

[3] V. S. Gerdjikov, G. Vilasi, A. B. Yanovski. *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods* Lecture Notes in Physics **748**, Springer Verlag, Berlin, Heidelberg, New York (2008). ISBN: 978-3-540-77054-1.

[4] Editors: Kouei Sekigawa, Vladimir S. Gerdjikov and Stancho Dimiev. *Trends in Differential Geometry, Complex Analysis and Mathematical Physics.* World Scientific (2009).

[5] Complex Structures, Integrability and Vector Fields. Editors: Kouei Sekigawa, Vladimir S. Gerdjikov, Yasuo Matsushita, Ivaïlo M. Mladenov AIP Conference Proceeding, vol. **1340**, (2011).

[6] Symmetries and Integrability of Difference Equations (SIDE-9). Special issue of SIGMA. Editors: Boyka Aneva, Vladimir Gerdjikov, Plamen Iliev, Vassilis Papageorgiou, vol. **7**, (2011).

Articles in Journals

The Inverse scattering method as Generalized Fourier Transform

[1] V. S. Gerdjikov, E. Kh. Khristov. *On the expansions over the products of solutions of two one–dimensional Dirac systems.* Mathematical Notes **28**, No 4, 710 - 716 (1980).

[2] V. S. Gerdjikov, E. Kh. Khristov. *On the evolution equations solvable with the inverse scattering problem. I. The spectral theory.* Bulgarian J. Phys. **7**, No.1, 28–41, (1980). (In Russian).

[3] V. S. Gerdjikov, E. Kh. Khristov. *On the evolution equations solvable with the inverse scattering problem. II. Hamiltonian structures and Backlund transformations.* Bulgarian J. Phys. **7**, No.2, 119–133, (1980) (In Russian).

[4] V. S. Gerdjikov, M. I. Ivanov, P. P. Kulish. *The quadratic bundle and the nonlinear equations.* Theoretical and Mathematical Physics **44**, 784-795, (1980).

[5] V. S. Gerdjikov, M. I. Ivanov. *The quadratic bundle of general form and the nonlinear evolution equations. I. Expansions over the "squared" solutions are generalized Fourier transforms.* Bulgarian J. Phys. **10,** No.1, 13–26, (1983). (In Russian). JINR communication E2–81–811, (1981), Dubna.

[6] V. S. Gerdjikov, M. I. Ivanov. *The quadratic bundle of general form and the nonlinear evolution equations. II. Hierarchies of Hamiltonian structures.* Bulgarian J. Phys. **10,** No.2, 130–143, (1983). (In Russian). JINR communication E2–81–812, (1981), Dubna.

[7] V. S. Gerdjikov, P. P. Kulish. *The generating operator for the $n \times n$ linear system.* Physica **3D,** n. 3, 549–564, (1981).

[8] V. S. Gerdjikov, M. I. Ivanov, P. P. Kulish. *Complete integrability of the difference evolution equations.* JINR preprint E2–80–882, (1980), Dubna, USSR.

[9] V. S. Gerdjikov, M. I. Ivanov. *The Hamiltonian structure of the difference multicomponent nonlinear Schrödinger equations.* Hamiltonian-stucture of multicomponent non-linear Schrodinger-equations in difference form Theor. Math. Phys. **52** no. 1, 676-685 (1982).

[10] V. S. Gerdjikov. *On the spectral theory of the integro–differential operator Λ, generating nonlinear evolution equations.* Lett. Math. Phys. **6,** n. 6, 315–324, (1982).

[11] V. S. Gerdjikov, M. I. Ivanov, P. P. Kulish. *Expansions over the "squared" solutions and difference evolution equations.* Journ. Math. Phys. **25,** n. 1, 25–34, (1984).

[12] V. S. Gerdjikov, A. B. Yanovsky. *Gauge covariant formulation of the generating operator. I. The Zakharov–Shabat system.* Phys. Lett. A, **103A,** n. 5, 232–236, (1984).

[13] V. S. Gerdjikov, A. B. Yanovsky. *Gauge covariant formulation of the generating operator. II. Systems on homogeneous spaces.* Phys. Lett. A, **110A,** n. 2, 53–58, (1985).

[14] V. Gerdjikov. *Generalised Fourier transforms for the soliton equations. Gauge covariant formulation.* Inverse Problems **2,** 51–74, (1986).

[15] M. Boiti, V. S. Gerdjikov, F. Pempinelli. *The WKIS system: Backlund transformations, generalised Fourier transforms and all that.* Prog. Theor. Phys. **75,** n. 5, 1111–1141, (1986).

[16] V. S. Gerdjikov, A. B. Yanovsky. *Gauge covariant theory of the generating operator. I. The Zakharov–Shabat system.* Commun. Math. Phys. **103,** n. 4, 549–568, (1986).

[17] V. S. Gerdjikov. *The Zakharov–Shabat dressing method and the representation theory of the semisimple Lie algebras.* Phys. Lett. A, **126A,**

n. 3, 184–188, (1987).

[18] V. S. Gerdjikov. Complete integrability, gauge equivalence and Lax representations of the inhomogeneous nonlinear evolution equations. Theor. Math. Phys. **92**, 374–386 (1992).

[19] V. S. Gerdjikov, A. B. Yanovski. *Completeness of the eigenfunctions for the Caudrey–Beals–Coifman system.* J. Math. Phys. **35**, no. 7, 3687–3725 (1994).

[20] V. S. Gerdjikov. *Algebraic and Analytic Aspects of N-wave Type Equations.* Contemporary Mathematics **301**, 35-68 (2002).

[21] Adrian Constantin, Vladimir S. Gerdjikov, Rossen I. Ivanov. *Inverse Scattering Transform for the Camassa-Holm equation.* Inverse Problems **22**, 2197-2207 (2006).

[22] Adrian Constantin, Vladimir S. Gerdjikov, Rossen I. Ivanov. *Generalized Fourier Transform for the Camassa-Holm equation.* Inverse Problems **23**, 1565-1597 (2007).

Soliton Interactions

[1] V. S. Gerdjikov, I. M. Uzunov. *Self–Frequency Shift of Dark Solitons in Optical Fibers.* Phys. Rev. A, **47A**, 1582–1585 (1993)

[2] V. S. Gerdjikov, N. A. Kostov. *Inverse Scattering Transform Analysis of Stokes–anti–Stokes Stimulated Raman Scattering.* Phys. Rev. A **54**, No. 5, 4339-4350 (1996).

[3] V. S. Gerdjikov, I. M. Uzunov, E. G. Evstatiev, G. L. Diankov. *Nonlinear Schrödinger Equation and N–soliton Interactions: Generalized Karpman–Soloviev Approach and the Complex Toda Chain.* Phys. Rev. E **55**, No 5, 6039-6060 (1997).

[4] V. S. Gerdjikov, D. J. Kaup, I. M. Uzunov, E. G. Evstatiev. *Asymptotic behavior of N-Soliton Trains of the Nonlinear Schrödinger Equation.* Phys. Rev. Lett. **77**, 3943-3946 (1996).

[5] V. S. Gerdjikov, E. G. Evstatiev, D. J. Kaup, G. L. Diankov, I. M. Uzunov. *Stability and Quasi-Equidistant Propagation of NLS Soliton Trains.* Phys. Lett. A **241**, 323-328 (1998).

[6] V. S. Gerdjikov, E. G. Evstatiev, R. I. Ivanov. *The complex Toda chains and the simple Lie algebras – solutions and large time asymptotics.* J. Phys. A: Math & Gen. **31**, 8221-8232 (1998).

[7] V. S. Gerdjikov, E. G. Evstatiev, R. I. Ivanov. *The complex Toda Chains and the Simple Lie Algebras – Solutions and Large Time Asymptotics – II.* J. Phys. A: Math & Gen. **33**, 975-1006 (2000).

[8] V. S. Gerdjikov, I. M. Uzunov. *Adiabatic and Non-adiabatic Soliton Interactions in Nonlinear Optics.* Physica D **152-153**, 355-362 (2001).

[9] V. S. Gerdjikov, E. V. Doktorov, J. Yang. *Adiabatic Interaction of N Ultrashort Solitons: Universality of the Complex Toda Chain Model.* Phys. Rev. E **64**, 056617 (2001).

[10] V.S. Gerdjikov, E.V. Doktorov, N. P. Matsuka. *N-soliton Train and Generalized Complex Toda Chain for Manakov System.* Teor. Mat. Fiz. **151**, No 3, 391-404 (2007) (In Russian).

[11] V. S. Gerdjikov, N.A. Kostov, E.V. Doktorov, N.P. Matsuka. *Generalized Perturbed Complex Toda Chain for Manakov System and Exact Solutions of the Bose-Einstein Mixtures.* Mathematics and Computers in Simulation **80**, 112-119 (2009).

[12] V. S. Gerdjikov, R. I. Ivanov and A. V. Kyuldjiev. On the N-wave equations and soliton interactions in two and three dimensions. Wave Motion **48**, 791–804 (2011).

[13] V. S. Gerdjikov. Modeling Soliton Interactions of the perturbed vector nonlinear Schrödinger equation Bulgarian J. Phys. **38** 274–283 (2011).

[14] V. S. Gerdjikov, M. D. Todorov. N-soliton interactions for the Manakov system: effects of external potentials. Nonlinear Systems and Complexity **7**, 147–169 (2014).

[15] V. S. Gerdjikov, M. D. Todorov. On the Effects of sech-like Potentials on Manakov Solitons. AIP Conf. proc. **1561**, pp. 75–83, (2013), Melville, NY.

Multicomponent Soliton Equations and their Reductions

[1] V. S. Gerdjikov, P. P. Kulish. *Multicomponent nonlinear Schrödinger equation in the case of nonzero boundary conditions.* Journal of Mathematical Sciences **30**, No 4, 2261-2269 (1985).

[2] V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov. *Reductions of N-wave interactions related to low–rank simple Lie algebras. I: $\mathbb{Z}_2$- reductions.* J. Phys. A: Math & Gen. **34**, 9425-9461 (2001).

[3] V. S. Gerdjikov, G. G. Grahovski, R. I. Ivanov, N. A. Kostov. *N-wave interactions related to simple Lie algebras. $\mathbb{Z}_2$- reductions and soliton solutions.* Inverse Problems **17**, 999-1015 (2001).

[4] V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov. *On N-wave Type Systems and Their Gauge Equivalent.* European J. Phys. **29B**, 243-248 (2002).

[5] V.S. Gerdjikov, N.A. Kostov, T.I. Valchev. *N*-Wave Equations with Orthogonal Algebras: $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ Reductions and Soliton Solutions. SIGMA **3**, paper 039 (2007); 19 pages.

[6] Nikolay Kostov, Vladimir Gerdjikov. *Reductions of multicomponent mKdV equations on symmetric spaces of* **DIII***-type.* SIGMA **4** (2008), paper 029, 30 pages.

[7] V. S. Gerdjikov, D. J. Kaup, N. A. Kostov, T. I. Valchev. *On clas-sification of soliton solutions of multicomponent nonlinear evolution equations.* J. Phys. A: Math. Theor. **41** (2008) 315213 (36pp).

[8] V. S. Gerdjikov, G. G. Grahovski. Multi-component NLS Models on Symmetric Spaces: Spectral Properties versus Representations The-ory. SIGMA **6** (2010), 044, 29 pages.

[9] V. S. Gerdjikov, N. A. Kostov. *Multi-component nonlinear Schrödinger equation on symmetric spaces with constant boundary conditions. I* J. Geometry and Symmetry in Physics. **19**, 1–28 (2010).

[10] V S Gerdjikov, A V Mikhailov, and T I Valchev. Reductions of inte-grable equations on **A.III**-type symmetric spaces. J. Phys. A: M&T **43** 434015 (2010).

[11] V. S. Gerdjikov, G. G. Grahovski. *Two soliton interactions of* **BD.I** *multicomponent NLS equations and their gauge equivalent* AIP Conf. Proc. **1301**, pp. 561-572 (2010).

[12] V. Gerdjikov, A. Mikhailov, T. Valchev. Recursion Operators and Re-ductions of Integrable Equations on Symmetric Spaces. J. Geometry and Symmetry in Physics **20**, 1–34 (2010).

[13] V. S. Gerdjikov, G. G. Grahovski, A. V. Mikhailov, T. I. Valchev. Rational Bundles and Recursion Operators for Integrable Equations on **A.III**-type Symmetric Spaces. Theor. Math. Phys. **167**, 740-750, (2011).

[14] V. S. Gerdjikov, G. G. Grahovski, A. V. Mikhailov, T. I. Valchev. Re-cursion Operators and Hamiltonian structures for Integrable Equations on **A.III**-type Symmetric Spaces I. The Polynomial Case. SIGMA **7** (2011), 096, 48 pages.

[15] V. Gerdjikov, G. Grahovski, A. Mikhailov, T. Valchev. *On Soliton Interactions for the Hierarchy of a Generalized Heisenberg Ferro-magnetic Model on $SU(3)/S(U(1) \times U(2))$ Symmetric Space.* J. Geom. Symm. Phys. **25**, 23–55 (2012).

Dynamical Systems and their Real Hamiltonian Forms

[1] V. Gerdjikov, A. Kyuldjiev, G. Marmo, G. Vilasi. *Complexifications and Real Forms of Hamiltonian Structures.* European J. Phys. **29B**, 177-182 (2002).

[2] V. S. Gerdjikov, A. Kyuldjiev, G. Marmo, G. Vilasi. *Real Hamilton-ian forms of Hamiltonian systems.* European Phys. J. B. **38** 635-649 (2004).

[3] V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov. *On the multi-component NLS type equations on symmetric spaces and their reduc-tions.* Theor. Math. Phys. **144** No. 2 1147-1156 (2005).

[4] V. S. Gerdjikov, G. G. Grahovski *On Reductions and Real Hamiltonian Forms of Affine Toda Field Theories.* J. Nonlinear Math. Phys. **12**, Supplement 2, 153-168 (2005).

[5] V. S. Gerdjikov, G. G. Grahovski. *Real Hamiltonian Forms of Affine Toda Models Related to Exceptional Lie Algebras.* SIGMA **2**, paper 022 (2006); 11 pages.

[6] N. A. Kostov, V. S. Gerdjikov, V. Mioc. *Exact solutions for a class of integrable Hénon-Heiles-type systems* J. Math. Phys, **51**, 022702 (2010).

Multicomponent NLS Equations and Bose-Einstein Condensates

[1] N. A. Kostov, V. Z. Enol'skii, V. S. Gerdjikov, V. V. Konotop, M. Salerno. *On two-component Bose-Einstein condensates in periodic potential.* Phys. Rev. E. **70**, 056617 (2004).

[2] V. Gerdjikov, B. Baizakov, M. Salerno. *Modelling adiabatic N-soliton interactions and perturbations.* Theor. Math. Phys. **144**, 1138-1146 (2005).

[3] V. S. Gerdjikov, B. B. Baizakov, M. Salerno, N. A. Kostov. *Adiabatic N-soliton interactions of Bose-Einstein condensates in external potentials.* Phys. Rev. E. **73**, 046606 (2006).

[4] N. A. Kostov, V. S. Gerdjikov, T. I. Valchev. *Exact solutions for equations of Bose-Fermi mixtures in one-dimensional optical lattice* SIGMA 3 (2007), paper 071, 14 pages.

[5] V. S. Gerdjikov, N. A. Kostov, T. I. Valchev. Solutions of multi-component NLS models and Spinor Bose-Einstein condensates Physica D **238** 1306-1310 (2009).

[6] V. A. Atanasov, V. S. Gerdjikov, G. G. Grahovski and N. A. Kostov. *Fordy-Kulish models and spinor Bose-Einstein condensates* J. Nonlinear Math. Phys., **15**, No. 3, 291-298 (2008)

[7] V. S. Gerdjikov. *Bose-Einstein Condensates and spectral properties of multicomponent nonlinear Schrödinger equations.* Discrete and Continuous Dynamical Systems B **4**, No. 5, 1181-1197 (2011).

The Riemann-Hilbert Problem and New Integrable Interactions

[1] V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Pliska Stud. Math. Bulgar. **21**, 201–216 (2012).

[2] V. S. Gerdjikov and G. G. Grahovski. On the 3-wave Equations with Constant Boundary Conditions. Pliska Stud. Math. Bulgar. **21**, 217–236 (2012).

[3] V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012).

[4] V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, 573-582 (2013).

[5] V. S. Gerdjikov, A. B. Yanovski *On soliton equations with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions: conservation laws and generating operators*. J. Geom. Symmetry Phys. **31**, 57–92 (2013).

Reports, delivered at the conference and list of participants

Alexander Mikhailov	Darboux transformations and symmetries of partial difference equations
Georgi Grahovski	On the 3-wave Equations with Constant Boundary Conditions
Anca Visinescu	Statistical approach of modulation instability beyond Gaussian approximation
Rossen Dandoloff	Geometry and Quantum Mechanics
Georgi Boyadzhiev	Existence theorems for non - cooperative elliptic systems
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Rossen Ivanov	Geometric Mechanics and Coupled Equations of Camassa-Holm Type.
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Vladimir I. Pulov	Helfrich's Equilibrium Membranes' Shape Model: Lie Group Analysis, Determining System and Symmetries.
Nikolay M. Nikolov	Geometry and topology in renormalization theory.
Jing Ping Wang	Recursion Operator for Narita-Itoh-Bogoyavlensky Lattice.
V. S. Gerdjikov	Riemann-Hilbert Problems with canonical normalization and families of commuting operators.

Stancho Dimiev	Aspects of the multicomplex number systems.
Dimitar Mladenov	Integrable many-body systems related to geodesic flows on $GL(n, R)$ group manifold.
Andrey K. Angelow	Schrödinger Minimum Uncertainty States of Electromagnetic Field in Non-stationary Media with negative conductivity.
Petko Nikolov	Formal integrability and dimensional reduction of differential operators.
Nikolay V. Vitanov	High-fidelity control of quantum dynamics with composite pulse sequences.

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Integrability, Recursion Operators and Soliton Interactions

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