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The aspherical Cavicchioli–Hegenbarth–Repovš generalized Fibonacci groups

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Abstract. The Cavicchioli–Hegenbarth–Repovš generalized Fibonacci groups are defined by the presentations $G_n(m,k) = \langle x_1, \ldots, x_n \mid x_i x_{i+m} = x_{i+k} \ (1 \leqslant i \leqslant n) \rangle$. These cyclically presented groups generalize Conway's Fibonacci groups and the Sieradski groups. Building on a theorem of Bardakov and Vesnin we classify the aspherical presentations $G_n(m,k)$. We determine when $G_n(m,k)$ has infinite abelianization and provide sufficient conditions for $G_n(m,k)$ to be perfect. We conjecture that these are also necessary conditions. Combined with our asphericity theorem, a proof of this conjecture would imply a classification of the finite Cavicchioli–Hegenbarth–Repovš groups.

1 Introduction

A group Γ is said to be *cyclically presented* if it has a presentation of the form

$$\langle x_1, \ldots, x_n \mid w, \eta(w), \ldots, \eta^{n-1}(w) \rangle$$

where w is a word in $X = \{x_1, \ldots, x_n\}$ and η is an automorphism of the free group $F(X)$ whose action on the generators is given by $\eta(x_i) = x_{i+1}$ (subscripts taken modulo n). Cyclically presented groups have been studied both for algebraic and for geometric reasons.

If w takes the form $w = x_i x_{i+m} x_{i+k}^{-1}$ then we obtain a class of groups introduced in [4]. Specifically, the *Cavicchioli–Hegenbarth–Repovš generalized Fibonacci groups* are the groups defined by the presentations

$$G_n(m,k) = \langle x_1, \ldots, x_n \mid x_i x_{i+m} = x_{i+k} \ (1 \leqslant i \leqslant n) \rangle \tag{1.1}$$

where all indices are taken modulo n and take their values from the set $\{1, \ldots, n\}$. We shall sometimes refer to $G_n(m,k)$ as groups, when we mean the groups defined by the presentations.

The groups $G_n(1,2)$ are the Fibonacci groups $F(2,n)$ introduced by Conway [7] (see [18] for a comprehensive survey of such groups); the groups $G_n(2,1)$ are the

Sieradski groups $S(n)$ considered in [15], [19]. The groups $G_n(t,1)$ are the Gilbert–Howie groups $H(n,t)$ studied in [11].

The asphericity of cyclically presented groups and generalizations of Fibonacci groups have been studied in [1], [6], [11], [14], [16]. In this paper we consider the asphericity of presentations $G_n(m,k)$. In [11], Gilbert and Howie give (with certain excluded cases) necessary and sufficient conditions for $G_n(t,1) \cong H(n,t)$ to be aspherical. A presentation $G = A * B$ is aspherical if and only if at least one of the presentations A, B is aspherical (from [1, Lemma 2.1]), and for this reason it is enough to consider only cases where $G_n(m,k)$ does not factorize as a free product. Moreover, since the asphericity of presentations $G_n(t,1)$ was considered in [11], we also do not need to consider these cases.

The presentation $G_n(m,k)$ is said to be *irreducible* if n, m, k satisfy

$$0 < m < k < n, \quad (n,m,k) = 1, \tag{1.2}$$

and is *strongly irreducible* if it is irreducible and additionally

$$(n,k) > 1, \quad (n,k-m) > 1. \tag{1.3}$$

If $G_n(m,k)$ is not irreducible then it is either trivial, cyclic, or factorizable into a free product by [1, Lemma 1.2]. If $G_n(m,k)$ is irreducible but not strongly irreducible, then it is isomorphic to some Gilbert–Howie group $H(n,t)$ by [1, Lemma 1.3]. (We remark that Edjvet [8] defines irreducibility for an arbitrary cyclically presented group. According to his definition, $G_n(m,k)$ is irreducible if and only if $(n,m,k) = 1$, and so this is a slightly weaker property than the one used here.)

In [1], Bardakov and Vesnin give sufficient conditions for a strongly irreducible presentation $G_n(m,k)$ to be aspherical. In Section 2 we build on this to determine precisely when such presentations are aspherical (Theorem 2). In Section 3 we determine when $G_n(m,k)$ has infinite abelianization (Theorem 4) and provide sufficient conditions for $G_n(m,k)$ to be perfect (Lemma 5). We conjecture (Conjecture 6) that these conditions are also necessary; using Magma we have verified this for $n \leqslant 200$. Bardakov and Vesnin [1] have asked for a classification of the finite Cavicchioli–Hegenbarth–Repovš groups. We address this question in Section 4 and show that the classification of the finite, strongly irreducible groups $G_n(m,k)$ would follow from a proof of Conjecture 6; therefore, we have obtained such a classification for $n \leqslant 200$.

Many of the results in this paper were formulated after performing computational experiments in Magma [2].

2 Asphericity

A presentation P is said to be *aspherical* if $\pi_2(P) = 0$; the group defined by an aspherical presentation is torsion-free, and hence either trivial or infinite.

For $(n,t) \notin \{(8,3),(9,4),(9,7)\}$ Gilbert and Howie have determined precisely when the presentation $H(n,t)$ is aspherical ([11, Theorem 3.2]). If $(n,t) = (8,3)$ then a calculation in Magma shows that $H(n,t)$ defines a finite group of order 295245, and hence $H(8,3)$ is not aspherical.

We shall consider when a strongly irreducible presentation $G_n(m,k)$ is aspherical. Bardakov and Vesnin [1] have provided the following sufficient condition.

Theorem 1 ([1]). *Let $G_n(m,k)$ be strongly irreducible. Then $G_n(m,k)$ is aspherical if none of the following conditions are satisfied*:

(1) *there exists an integer $l \geqslant 1$ such that n divides $l(2k-m)$ and also $1/l + (n,k)/n + (n,k-m)/n > 1$;*

(2) $n = k + m$;

(3) $n = 2(k-m)$ *and* $(n,k) \leqslant n/2$;

(4) $n = 2k$ *and* $(n,k-m) < n/2$.

We build on this result to provide necessary and sufficient conditions for $G_n(m,k)$ to be aspherical.

Theorem 2. *Suppose that $G_n(m,k)$ is strongly irreducible. If $(m,k) = 1$ and either $n = 2k$ or $n = 2(k-m)$ then $G_n(m,k)$ is not aspherical. In all other cases $G_n(m,k)$ is aspherical.*

Before embarking on the proof, we first identify classes of finite cyclic groups among the groups $G_n(m,k)$.

Lemma 3. *Suppose that $(m,k) = 1$ and either* (i) $n = 2k$, *or* (ii) $n = 2(k-m)$. *Then $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

Proof. Consider first case (i). Let P be the presentation of $G_n(m,k)$ defined as in (1.1). That is, P has generators $x_1, \ldots, x_{2k}$ and relations

$$x_i x_{i+m} = x_{i+k} \quad (1 \leqslant i \leqslant 2k) \tag{2.1}$$

(with subscripts taken modulo $2k$). Setting $i = q-m-k$ in (2.1) we obtain $x_{q-m} = x_{q-m-k}x_{q-k}$; on the other hand, setting $i = q-m$ gives $x_{q-m} = x_{q-m+k}x_q^{-1}$. Hence $x_{q-k} = x_q^{-1}$ and so we may add the relations

$$x_{i-k} = x_i^{-1} \quad (1 \leqslant i \leqslant 2k) \tag{2.2}$$

to P without changing the group that it defines. Relations (2.1) and (2.2) imply $x_i x_{i+m} = x_i^{-1}$ and so we may add relations

$$x_{i+m} = x_i^{-2} \quad (1 \leqslant i \leqslant 2k). \tag{2.3}$$

The relations (2.1) are a consequence of (2.2) and (2.3) and so can be removed.

Suppose that m is even; then k is odd. By (2.2) we may remove all generators x_i where i is odd (and hence also the relations (2.2)). Thus P has generators x_{2j} $(1 \leqslant j \leqslant m)$ and relations

$$x_{2j+m} = x_{2j}^{-2} \quad (1 \leqslant j \leqslant k), \tag{2.4}$$

(from (2.3)). Since $(m,k) = 1$ and m is even, $(m/2,k) = 1$, and so for each j with $1 \leqslant j \leqslant k$ there exists a unique J with $1 \leqslant J \leqslant k$ such that $J(m/2) = j \bmod k$ and hence $Jm = 2j \bmod 2k$. Thus we can write the generators of P as x_{Jm} $(1 \leqslant J \leqslant k)$ and the relations (2.4) as

$$x_{(J+1)m} = x_{Jm}^{-2} \quad (1 \leqslant J \leqslant k). \tag{2.5}$$

Then for each J we have

$$x_{Jm} = x_{(J-1)m}^{-2} = x_{(J-2)m}^{(-2)^2} = \cdots = x_m^{(-2)^{J-1}} = x_{km}^{(-2)^J}$$

and so we may add relations

$$x_{Jm} = x_{km}^{(-2)^J} \quad (1 \leqslant J \leqslant k). \tag{2.6}$$

The relations (2.5) are a consequence of (2.6) and so we may remove them. Using (2.6) we may remove generators x_{Jm} for $1 \leqslant J \leqslant k-1$ together with the corresponding relations, leaving the presentation

$$P = \langle x_{km} \mid x_{km} = x_{km}^{(-2)^k} \rangle = \langle x_{km} \mid x_{km}^{(-2)^k - 1} = 1 \rangle,$$

and so (since m is even and k is odd) $G_{2k}(m,k) \cong \mathbb{Z}_s$, as required.

Suppose then that m is odd. Then $(m,2k) = 1$ and so for each i with $1 \leqslant i \leqslant 2k$ there exists a unique J with $1 \leqslant J \leqslant 2k$ such that $i = Jm$. Thus the generators of P can be written as x_{Jm} $(1 \leqslant J \leqslant 2k)$, the relations (2.2) as

$$x_{(J+k)m} = x_{Jm}^{-1} \quad (1 \leqslant J \leqslant 2k), \tag{2.7}$$

and (2.3) as

$$x_{(J+1)m} = x_{Jm}^{-2} \quad (1 \leqslant J \leqslant k-1), \tag{2.8}$$

$$x_{(k+1)m} = x_{km}^{-2}, \tag{2.9}$$

$$x_{(J+k+1)m} = x_{(J+k)m}^{-2} \quad (1 \leqslant J \leqslant k-1), \tag{2.10}$$

$$x_m = x_{2km}^{-2}. \tag{2.11}$$

Using (2.7) we can remove generators $x_{(J+k)m}$ $(1 \leqslant J \leqslant k)$ and the relations (2.7). Since $x_{(J+k)m} = x_{Jm}^{-1}$ for each J the relations (2.10) become equivalent to (2.8) and so can be removed; relations (2.9) and (2.11) both become

$$x_m = x_{km}^2. \tag{2.12}$$

Thus P has generators x_{Jm} $(1 \leqslant J \leqslant k)$ and relations (2.8) and (2.12). For each J we therefore obtain

$$x_{Jm} = x_{(J-1)m}^{-2} = x_{(J-2)m}^{(-2)^2} = \cdots = x_{(J-(J-1))m}^{(-2)^{J-1}} = (x_{km}^2)^{(-2)^{J-1}}.$$

We can therefore add the relations

$$x_{Jm} = x_{km}^{-(-2)^J} \quad (1 \leqslant J \leqslant k). \tag{2.13}$$

The relations (2.8) are a consequence of these, so we may remove them. Using (2.13) we may remove generators x_{Jm} for $1 \leqslant J \leqslant k-1$ together with the corresponding relations, leaving the presentation

$$P = \langle x_{km} \,|\, x_{km} = x_{km}^{-(-2)^k} \rangle.$$

Since m is odd we have $G_{2k}(m,k) \cong \mathbb{Z}_s$, as required.

Consider then case (ii). We shall use [1, Theorem 1.1] to show that if a number K is coprime to m then the group $G_{2K}(m, m+K)$ is isomorphic to $G_{2K}(m,K)$. To this end, let $K' = m+K$, $m' = m$, $r = (2K, K-m)$. Then $r = (m,K) = 1$, and so there exists J with $1 \leqslant J \leqslant 2K$ such that $J(K-m) = 1 \bmod 2K$. Let $i = 1$, $j = -mJ \bmod 2K$; then $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant 2K/r$, and

$$i + j(K-m) = 1 - m \bmod 2K, \quad i + jK' = 1 + m' \bmod 2K.$$

By [1, Theorem 1.1] we have $G_{2K}(m, m+K) \cong G_{2K}(m,K)$ as claimed; since also $(m,K) = 1$ part (i) of this lemma shows that this is isomorphic to $\mathbb{Z}_s$ where $s = 2^K - (-1)^{K+m}$. Now if $n = 2(k-m)$, then setting $K = k-m$ gives that $G_n(m,k) = G_{2K}(m, m+K) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$, as required. □

Proof of Theorem 2. If $(m,k) = 1$ and either $n = 2k$ or $n = 2(k-m)$ then by Lemma 3 the group $G_n(m,k)$ is finite and non-trivial, and so is not aspherical. Suppose then that neither of these possibilities occurs. We shall show that none of the conditions (1), (2), (3), (4) of Theorem 1 holds, and hence $G_n(m,k)$ is aspherical.

Condition (1). If $n/(n,k) = 2$ then (since $0 < k < n$) we have $n = 2k$, and so by our assumption, $(m,k) > 1$. On the other hand, $(n,m,k) = (2k,m,k) = (m,k) > 1$, a contradiction. If $n/(n,k-m) = 2$ then $n = 2(k-m)$ and so by our assumption, $(m,k) > 1$. On the other hand, $(n,m,k) = (2(k-m),m,k) = (m,k) > 1$, a contradiction.

Thus we may assume that $n/(n,k), n/(n,k-m) \geqslant 3$, and so the inequality $1/l + (n,k)/n + (n,k-m)/n > 1$ implies that either (i) $l = 1$ or (ii) $l = 2$ and $n/(n,k) = p$, $n/(n,k-m) = q$ where $\{p,q\} = \{3,3\}, \{3,4\}, \{3,5\}$.

Suppose that $l = 1$. Then n must divide $(2k - m)$. But $k < n$ implies that $2k - m < 2n$ and hence $n = 2k - m$. Then $1 = (n, m, k) = (2k - m, m, k) = (m, k)$ and $1 < (n, k) = (2k - m, k) = (m, k)$, a contradiction.

Suppose then that $l = 2$. Then $n/(n, k) = p$, $n/(n, k - m) = q$ imply $\alpha n = pk$, $\beta n = q(k - m)$ for some α, β with $(\alpha, p) = 1$, $(\beta, q) = 1$. Thus $\beta pk = \alpha q(k - m)$ and hence $\alpha qm = (\alpha q - \beta p)k$. But $m > 0$ and $k > 0$, so that $\alpha q > \beta p$.

If $p = q = 3$ then $\alpha, \beta \in \{1, 2\}$ and $3\alpha > 3\beta$ so $\alpha = 2$, $\beta = 1$. That is, $2n = 3k$ and $n = 3(k - m)$ and so $k = 2m$, $n = 3m$. Then $(n, m, k) = m = (n, k - m)$. But $(n, m, k) = 1$, $(n, k - m) > 1$, and we have a contradiction. If $p = 3$ and $q = 4$ then $\alpha \in \{1, 2\}$, $\beta \in \{1, 3\}$ and $4\alpha > 3\beta$, so that $(\alpha, \beta) = (1, 1)$ or $(2, 1)$. If $(\alpha, \beta) = (1, 1)$ then $n = 3k$, $n = 4(k - m)$, so that $k = 4m$, $n = 12m$, and then $(n, k, m) = 1$ implies $m = 1$, and so $k = 4$, $n = 12$. But then $n \nmid l(2k - m)$, and so condition (1) does not hold. If $(\alpha, \beta) = (2, 1)$ then $2n = 3k$, $n = 4(k - m)$. Then $5k = 8m$ so $5|m$; let $m = 5M$, say. Then $k = 8M$, $n = 12M$, and $(n, m, k) = 1$ implies $M = 1$. Hence $m = 5$, $k = 8$, $n = 12$ and so $n \nmid l(2k - m)$.

Similar arguments show that if $(p, q) = (4, 3)$ then the tuple (n, m, k) is $(12, 5, 9)$ or $(12, 1, 9)$; if $(p, q) = (3, 5)$ then (n, m, k) is one of $(15, 2, 5), (15, 7, 10), (15, 4, 10)$ or $(15, 1, 10)$; if $(p, q) = (5, 3)$ then (n, m, k) is one of $(15, 1, 6), (15, 4, 9), (15, 7, 12)$ or $(15, 2, 12)$. In each case $n \nmid l(2k - m)$.

Condition (2). Strong irreducibility implies

$$(n, m, k) = 1, \quad (n, k) > 1.$$

But $n = k + m$ implies $(n, m, k) = (n, k)$, a contradiction.

Condition (3). Under this condition $n = 2(k - m)$, and so by our initial assumption $(m, k) > 1$. Then $(n, m, k) = (2(k - m), m, k) = (m, k) > 1$; but $(n, m, k) = 1$ and so we have a contradiction.

Condition (4). Under this condition $n = 2k$, and so by our initial assumption $(m, k) > 1$. Then $(n, m, k) = (2k, m, k) = (m, k) > 1$, but $(n, m, k) = 1$ and so we have a contradiction. □

3 Abelianizations

In this section we consider the abelianizations of groups $G_n(m, k)$. Using [13, Lemma 1.1] we have the following criterion that determines when $G_n(m, k)$ has infinite abelianization or when $G_n(m, k)$ is perfect. Let $f(t) = t^m - t^k + 1$ and

$$R_n(f) = \prod_{\theta^n = 1} f(\theta).$$

Then $G_n(m, k)^{\mathrm{ab}}$ is infinite if and only if $R_n(f) = 0$, and $G_n(m, k)$ is perfect if and only if $R_n(f) = \pm 1$.

Theorem 4. *Suppose that* $(n, m, k) = 1$. *Then* $G_n(m, k)$ *has infinite abelianization if and only if* $n = 0 \bmod 6$ *and* $(m \bmod 6, k \bmod 6) = (2, 1)$ *or* $(4, 5)$.

Proof. Observe that $R_n(f) = 0$ if and only if $f(\lambda) = 0$ for some $\lambda^n = 1$.

Suppose first that $n = 0 \bmod 6$ and $(m \bmod 6, k \bmod 6) = (2, 1)$ or $(4, 5)$, and that $(n, m, k) \geqslant 1$. Let $\lambda^3 = -1$, $\lambda \neq -1$; then $\lambda^6 = 1$, and therefore $\lambda^n = 1$. If $m = 2 \bmod 6$ and $k = 1 \bmod 6$ then $f(\lambda) = \lambda^2 - \lambda + 1$. If $m = 4 \bmod 6$ and $k = 5 \bmod 6$ then $f(\lambda) = \lambda^4 - \lambda^5 + 1 = -\lambda + \lambda^2 + 1$. Thus $(\lambda + 1)f(\lambda) = \lambda^3 + 1 = 0$ and since $(\lambda + 1) \neq 0$ we have $f(\lambda) = 0$.

For the converse, suppose that $(n, m, k) = 1$ and that $f(\lambda) = 0$ for some $\lambda^n = 1$. Thus $|\lambda| = 1$ so $\bar{\lambda} = \lambda^{-1}$ and then $f(\lambda) = 0$, $f(\lambda^{-1}) = f(\bar{\lambda}) = 0$ imply

$$\lambda^m = \lambda^k - 1, \tag{3.1}$$

$$\lambda^{-m} = \lambda^{-k} - 1. \tag{3.2}$$

Hence $1 = \lambda^m \cdot \lambda^{-m} = 2 - \lambda^k - \lambda^{-k}$, and so

$$\lambda^{2k} - \lambda^k + 1 = 0. \tag{3.3}$$

Thus $\lambda^k = (1 \pm i\sqrt{3})/2$, and so λ is a $(6k)$th root of unity. Since $\lambda^n = 1$, $6k$ divides n and so, in particular,

$$n = 0 \bmod 6. \tag{3.4}$$

By (3.1) and (3.3) we have $\lambda^m = \lambda^{2k}$, so that $\lambda^{m-2k} = 1$ and therefore

$$m - 2k = 0 \bmod 6. \tag{3.5}$$

Since $(n, m, k) = 1$, it follows from (3.4) and (3.5) that $(m \bmod 6, k \bmod 6) = (2, 1)$ or $(4, 5)$, as required. □

We remark that necessary and sufficient conditions for $G_n(m, k)$ to have infinite abelianization were incorrectly asserted in [17, Example 3(3)].

We now consider when $G_n(m, k)$ is perfect. By [1, Lemma 1.1], if $k = 0$ or $m \bmod n$ then $G_n(m, k)$ is trivial, and hence perfect. Here is another sufficient condition:

Lemma 5. *If* $m = 2k \bmod n$ *and* $(n/(n, m, k), 6) = 1$ *then* $G_n(m, k)$ *is perfect.*

Proof. Suppose first that $g(t)$ is any polynomial, and $d = (k, n)$. Then

$$\prod_{\theta^n=1} g(\theta^k) = \prod_{q=0}^{n-1} g(e^{2\pi i q k/n}) = \prod_{q=0}^{n-1} g(e^{2\pi i q (k/d)/(n/d)})$$

$$= \left(\prod_{q=0}^{n/d-1} g(e^{2\pi i q (k/d)/(n/d)}) \right)^d.$$

Since $(k/d, n/d) = 1$, for each $q \in \{0, \ldots, (n/d - 1)\}$ there exists a unique $Q \in \{0, \ldots, (n/d - 1)\}$ such that $q(k/d) = Q \bmod n/d$. Hence

$$\prod_{q=0}^{n/d-1} g(e^{2\pi i q(k/d)/(n/d)}) = \prod_{Q=0}^{n/d-1} g(e^{2\pi i Q/(n/d)}) = \prod_{\phi^{n/d}=1} g(\phi)$$

and so

$$\prod_{\theta^n=1} g(\theta^k) = \Bigg(\prod_{\phi^{n/(n,k)}=1} g(\phi)\Bigg)^{(n,k)}.$$

If $m = 2k \bmod n$ then setting $g(t) = t^2 - t + 1$ we have

$$R_n(f) = \prod_{\theta^n=1} g(\theta^k) = \Bigg(\prod_{\phi^{n/(n,k)}=1} g(\phi)\Bigg)^{(n,k)}.$$

Now

$$\begin{aligned}\prod_{\phi^{n/(n,k)}=1} g(\phi) &= \prod_{\phi^{n/(n,k)}=1} (\phi - e^{2\pi i/6}) \prod_{\phi^{n/(n,k)}=1} (\phi - e^{-2\pi i/6}) \\ &= ((e^{2\pi i/6})^{n/(n,k)} - 1)((e^{-2\pi i/6})^{n/(n,k)} - 1) \\ &= 2 - 2\cos\left(\frac{2\pi n/(n,k)}{6}\right) \\ &= 1\end{aligned}$$

since $(n/(n,k), 6) = (n/(n,m,k), 6) = 1$. Hence $R_n(f) = 1$ and $G_n(m,k)$ is perfect, as required. □

Since this paper was written a complete description of the abelianization of $G_n(2k,k)$ has been provided in [3, Lemma 7]. We conjecture that the sufficient conditions that we have given for $G_n(m,k)$ to be perfect are also necessary; we have verified this (using MAGMA) for $n \leqslant 200$.

Conjecture 6. If $G_n(m,k)$ is perfect then either $m = 2k \bmod n$ and $(n/(n,m,k), 6) = 1$ or $k = 0$ or $m \bmod n$.

Lemma 5 and Conjecture 6 form a natural generalization of [13, Theorem 2(ii), (iii)] (see also [11, Theorem 2.3(b)]). It seems likely that Odoni's methods [13] can be applied to prove Conjecture 6 in the general case.

4 Finiteness

In [1, Question 1], Bardakov and Vesnin posed the following question:

Question. For which values of the defining parameters n, m, k subject to the natural restrictions (1.2) are groups $G_n(m,k)$ finite?

If $(n,k) = 1$ or $(n, m-k) = 1$ then by [1, Lemma 1.3] the group $G_n(m,k)$ is isomorphic to some Gilbert–Howie group $H(n,t)$. More precisely, we have

Lemma 7. (i) *If* $(n,k) = 1$ *then*

$$G_n(m,k) \cong G_n(t,1) = H(n,t)$$

where $tk = m \bmod n$.
(ii) *If* $(n, k-m) = 1$ *then*

$$G_n(m,k) \cong G_n(t,1) = H(t,1)$$

where $t(k-m) = n - m \bmod n$.

Proof. (i) See the proof of [1, Lemma 1.3].
(ii) By [1, Lemma 1.1(3)] we have

$$G_n(m,k) \cong G_n(n-m, n-m+k) = G_n(K-k, K)$$

where $K = n + (k-m)$. Then $(n,K) = (n,k-m) = 1$, and so by (i) we have $G_n(K-k,K) \cong G_n(t,1)$, where $tK = K - k \bmod n$, i.e. where

$$t(k-m) = n - m \bmod n,$$

as required. □

Now the finite Gilbert–Howie groups $H(n,t)$ have almost been classified. In [11] the following theorem is proved:

Theorem 8 ([11]). *Suppose that* $(n,t) \neq (8,3), (9,3), (9,4), (9,6), (9,7)$. *Then* $H(n,t)$ *is finite if and only if* $t = 0, 1$ *or* $(n,t) = (2k, k+1)$ *where* $k \geqslant 1$, *or* $(n,t) \in \{(3,2), (4,2), (5,2), (5,3), (5,4), (6,3), (7,4), (7,6)\}$.

As mentioned in Section 2, a calculation in Magma shows that $H(8,3)$ is finite of order 295245. Cavicchioli, O'Brien and Spaggiari [3, Lemma 16] have recently proved that the (isomorphic) groups $H(9,3)$ and $H(9,6)$ are infinite. It remains unknown whether $H(9,4)$ and $H(9,7)$ define finite or infinite groups.

Suppose that $(n,k) > 1$ and $(n, m-k) > 1$, i.e. $G_n(m,k)$ is strongly irreducible.

Since any non-trivial group with an aspherical presentation is infinite, the following is an immediate corollary of Theorem 2 and Lemma 3.

Corollary 9. *Let $G = G_n(m,k)$ be strongly irreducible and assume that $G \neq 1$. Then G is finite if and only if $(m,k) = 1$ and $n = 2k$ or $n = 2(k-m)$, in which case $G \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

Thus, to give a complete classification of the finite, strongly irreducible groups $G_n(m,k)$ it suffices to prove that every such (strongly irreducible) group is non-trivial. The problem as to which cyclically presented groups are trivial has been of interest recently; see for example [8], [9], [10], [12].

If Conjecture 6 holds, it follows that every strongly irreducible group $G_n(m,k)$ is not perfect, and hence non-trivial. To see this, observe that Conjecture 6 implies that if $G_n(m,k)$ is perfect then $m = 2k \bmod n$ and so $(n,m,k) = (n,k)$. But if $G_n(m,k)$ is strongly irreducible then $(n,m,k) = 1$, $(n,k) > 1$, a contradiction. Our Magma calculations supporting Conjecture 6 therefore imply a classification of the finite, strongly irreducible groups $G_n(m,k)$ for $n \leqslant 200$.

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