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THE TITS ALTERNATIVE FOR NON-SPHERICAL PRIDE GROUPS

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Abstract

Pride groups, or "groups given by presentations in which each defining relator involves at most two types of generators", include Coxeter groups, Artin groups, triangles of groups, and Vinberg's groups defined by periodic paired relations. We show that every non-spherical Pride group that is not a triangle of groups satisfies the Tits alternative.

1. *Introduction*

Pride groups, or "groups given by presentations in which each defining relator involves at most two types of generators" [**11**], include Coxeter groups, Artin groups, triangles of groups, and Vinberg's groups defined by periodic paired relations. The cohomology of Pride groups was considered in [**11**], geometric invariants were considered in [**8**], and a Freiheitssatz was proven in [**3**].

In this paper we consider the Tits alternative for the class of Pride groups. Recall that a class of groups $\mathcal{C}$ satisfies the *Tits alternative* if each group in $\mathcal{C}$ contains a non-abelian free subgroup or has a soluble subgroup of finite index. This property is named after Tits who established that it is satisfied by the class of linear groups [**14**]; in particular, it holds for Coxeter groups.

The Tits alternative has been considered, for example, for the classes of mapping class groups of compact surfaces [**7**, **9**], outer automorphism groups of free groups of finite rank [**1**, **2**], subgroups of Gromov hyperbolic groups [**5**], groups acting on CAT(0) cubical complexes [**12**], triangles of groups [**6**], and groups defined by periodic paired relations [**15**, **16**].

In this paper we prove the following

Theorem 1. *Every non-spherical Pride group G based on a graph with at least 4 vertices contains a non-abelian free subgroup, unless it is based on the graph shown in Figure 1, in which case G is virtually abelian and has presentation*

$$\langle x_1, x_2, x_3, x_4 \,|\, x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_2x_3)^2, (x_3x_4)^2, (x_4x_1)^2\rangle.$$

It is interesting to note that the "negatively curved" property of containing a non-abelian free subgroup is found in this non-positively curved class of groups.

We now give our formal definitions. Let $\mathcal{G}$ be a finite simplicial graph with vertex set $I(\mathcal{G})$, and edge set $E(\mathcal{G})$. Further, let there be non-trivial groups G_i (with fixed finite presentations) associated to each vertex $i \in I(\mathcal{G})$ and, in addition, for each edge $\{i,j\} \in E(\mathcal{G})$ let R_{ij} be a (possibly empty) finite collection of cyclically reduced words. We assume each word in R_{ij} is of free product length greater than or equal to 2 in $G_i * G_j$. The *Pride group* based on the graph $\mathcal{G}$ with groups G_i assigned to the vertices and with edge relations $R = \cup_{\{i,j\}\in E(\mathcal{G})} R_{ij}$ is the group $G = *_{i\in I(\mathcal{G})} G_i/N$, where N is the normal closure of R in $*_{i\in I(\mathcal{G})} G_i$.

We refer to the groups G_i as *vertex groups*, and we define the *edge groups* to be $G_{ij} = \{G_i * G_j\}/N_{ij}$, where $\{i,j\} \in E(\mathcal{G})$ and where N_{ij} is the normal closure of R_{ij} in $G_i * G_j$.

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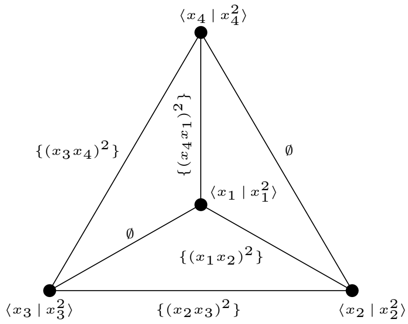

FIGURE 1.

More generally, if $\mathcal{F}$ is any full subgraph of $\mathcal{G}$ with vertex set $I(\mathcal{F}) \subseteq I(\mathcal{G})$, then the *subgraph group* $G_{\mathcal{F}}$ is $\{*_{i\in I(\mathcal{F})}G_i\}/\{N_{ij}|\{i,j\} \in E(\mathcal{F})\}$. In particular, $G_{\mathcal{G}} = G$.

For each $i, j \in I(\mathcal{G})$, the natural homomorphisms $G_i \to G_{ij}$, $G_j \to G_{ij}$ determine a homomorphism $G_i * G_j \to G_{ij}$. Let m_{ij} denote the length of a shortest non-trivial element in its kernel (in the usual length function on the free product), or put $m_{ij} = \infty$ if the kernel is trivial. Note that either $m_{ij} = 1$ (in which case one of the natural maps $G_i \to G_{ij}$, $G_j \to G_{ij}$ is not injective), or m_{ij} is even or infinite. The *Gersten-Stallings angle* $(G_{ij}; G_i, G_j)$ between the groups G_i and G_j in the group G_{ij} is defined to be $2\pi/m_{ij}$ for $m_{ij} > 1$, and 0 for $m_{ij} = \infty$ [**13**].

In [**11**] Pride formulated the following asphericity condition. A Pride group G based on a graph $\mathcal{G}$ (with $|I(\mathcal{G})| \geq 3$) is said to be *non-spherical* if

(i) $(G_{ij}; G_i, G_j) \leq \pi/2$ for all $i, j \in I(\mathcal{G})$; and
(ii) for any triangle $\{i, j, k\}$ in $\mathcal{G}$

$$(G_{ij}; G_i, G_j) + (G_{jk}; G_j, G_k) + (G_{ik}; G_i, G_k) \leq \pi.$$

In the non-spherical case we can assume that the graph $\mathcal{G}$ is complete. To see this, observe that if $i, j \in I(\mathcal{G})$ and $\{i, j\} \notin E(\mathcal{G})$ then we can add the edge $\{i, j\}$ and set $R_{ij} = \emptyset$ without changing the group G.

If $|I(\mathcal{G})| = 3$ then the Pride group G is the colimit of a triangle of groups. In [**6**], it was proved that if the angle sum of the triangle is strictly less than π then G contains a non-abelian free subgroup. In the same paper the Tits alternative was proved for a particular class of non-spherical triangles of groups, namely, for non-spherical generalized tetrahedron groups. In general, it is unknown if this property holds for non-spherical triangles of groups.

We also remark that every Pride group in which $m_{ij} > 1$ for all i, j can be represented in terms of a 2-complex of groups. Moreover, if the Pride group is non-spherical then the corresponding complex can be chosen to be non-spherical.

2. *Proof of Theorem 1*

Our method of proof has evolved from that developed in [**4**] and [**6**].

Let $G = G_{\mathcal{G}}$ be a non-spherical Pride group, where $\mathcal{G}$ is complete. First suppose that $\mathcal{G}$ has four vertices. Let $I(\mathcal{G}) = \{1, 2, 3, 4\}$ and let $X = G_1$, $Y = G_2$, $Z = G_3$ and $T = G_4$. We shall sometimes write G_{XY} for G_{12}, G_{XZ} for G_{13} and so on. Label the vertices of $\mathcal{G}$ by the vertex groups and each edge $\{i, j\}$ by $(G_{ij}; G_i, G_j)$.

If $(G_{ij}; G_i, G_j) + (G_{jk}; G_j, G_k) + (G_{ik}; G_i, G_k) < \pi$ for some $\{i, j, k\} \subset I(\mathcal{G})$ then, by [**6**], G_{ijk} contains a non-abelian free subgroup. By [**3**], every subgraph group embeds, so G also contains a non-abelian free subgroup. Hence, we may assume that for all $i, j, k \in I(\mathcal{G})$ the angle sum is exactly π.

Suppose that the edges incident to T are labelled by θ, α, and β. Since the angle sum is π for each triangle it follows that the edges that do not share any vertices have the same labels and all triangles in $\mathcal{G}$ are labelled by one of $\{\theta,\alpha,\beta\} = \{\pi/2,\pi/2,0\}$, $\{\pi/2,\pi/3,\pi/6\}$, $\{\pi/2,\pi/4,\pi/4\}$, $\{\pi/3,\pi/3,\pi/3\}$. Without loss of generality we may assume that $\theta \geq \alpha \geq \beta$ and that

$$\begin{aligned}(G_{XZ};G_X,G_Z) &= (G_{YT};G_Y,G_T) = \theta,\\ (G_{XY};G_X,G_Y) &= (G_{ZT};G_Z,G_T) = \alpha,\\ (G_{YZ};G_Y,G_Z) &= (G_{XT};G_X,G_T) = \beta.\end{aligned}$$

Suppose $(\theta,\alpha,\beta) \neq (\pi/2,\pi/2,0)$ and consider a presentation $\mathcal{P}$ for G. Since all the vertex groups are non-trivial, we may choose non-trivial elements $x \in X$, $y \in Y$, $z \in Z$ and $t \in T$ such that x, y, z, t are all generators of $\mathcal{P}$. We shall show that $u = xyztxyz$ has infinite order in G and that t and u generate a free product.

Let $w(t,u) = t^{p_1}u^{q_1}\dots t^{p_m}u^{q_m}$ or $w(t,u) = u^{q_1}$, where $m \geq 1$ and each $p_i, q_i \neq 0$, and assume that $w(t,u) = 1$ in G. Consider a van Kampen diagram K over $\mathcal{P}$ whose boundary label is $w(t,u)$. Let D be an extremal disk of K. We divide D into G_{ij}-regions. If two G_{ij}-regions intersect at least at one edge, then we can amalgamate them into a single region. We continue in this way as often as possible, and so get a division of D into *maximal G_{ij}-regions*. (Note that the resulting division of D is not necessarily unique.)

By [**3**], the edge groups embed, so it can be assumed that the maximal regions are simply connected. Let $\widehat{D}$ be the resulting diagram. On the boundary of $\widehat{D}$ an edge of $\widehat{D}$ is defined to be a longest path whose edges are labelled by elements of the same vertex group. In the interior an edge is defined to be the intersection of two adjacent maximal G_{ij}- and G_{ik}-regions. Note that it is a path labelled by elements of G_i.

Now place $\widehat{D}$ on the sphere and take its dual D^*. Let v_0 be the vertex corresponding to $\mathbb{S}^2\backslash\widehat{D}$. We call a region of D^* *exterior* if it involves v_0 and *interior* otherwise. We give each corner at a vertex of D^* of degree δ the angle $2\pi/\delta$. The curvature $c(\Delta)$ of a region Δ of degree q whose vertices have degrees $\delta_1, \delta_2, \dots, \delta_q$ is then defined by

$$c(\Delta) = (2-q)\pi + \sum_{i=1}^{q} \frac{2\pi}{\delta_i}.$$

Then

$$\sum_{\Delta\subset D^*} c(\Delta) = 2\pi\chi(\mathbb{S}^2) = 4\pi.$$

We remark that one can use the Gersten-Stallings angles to estimate the curvature as follows. Suppose that a vertex $v \neq v_0$ of D^* has degree δ and comes from a maximal G_{ij}-region of $\widehat{D}$. Since the Gersten-Stallings angle $(G_{ij};G_i,G_j)$ is defined to be $2\pi/m_{ij}$, where m_{ij} is the length of a shortest non-trivial element in the kernel of $G_i * G_j \to G_{ij}$, we have that $\delta \geq m_{ij}$ so $2\pi/\delta \leq (G_{ij};G_i,G_j)$. Moreover, the non-spherical condition implies $(G_{ij};G_i,G_j) \leq \pi/2$ so $\delta \geq 4$.

Let Δ be an interior region of D^* of degree q. Observe that $q \geq 3$. If $q \geq 4$ then

$$c(\Delta) \leq (2-q)\pi + \sum_{i=1}^{q} \frac{\pi}{2} \leq 0$$

and if $q = 3$ then (for some distinct $i, j, k \in I(\mathcal{G})$)

$$c(\Delta) \leq -\pi + (G_{ij};G_i,G_j) + (G_{jk};G_j,G_k) + (G_{ik};G_i,G_k) = 0.$$

Thus, the sum of the curvatures of interior regions is non-positive.

Consider exterior regions. Observe that an exterior region can be a 2-gon. It is convenient to define $d(\Delta) = c(\Delta) - 2\pi/N$, where $N = \deg(v_0)$ is the number of exterior regions. Then

$$\sum_{\text{ext.}\Delta} c(\Delta) = \sum_{\text{ext.}\Delta} d(\Delta) + 2\pi.$$

We aim to show $\sum_{\text{ext.}\Delta} d(\Delta) \le 0$ and obtain a contradiction.

Split the boundary of $\widehat{D}$ into t^{p_i} and $u^\varepsilon = (xyztxyz)^\varepsilon$ pieces, where $\varepsilon = \pm 1$. We now consider the sum of the curvatures of the exterior regions of D^* arising from each u^ε piece.

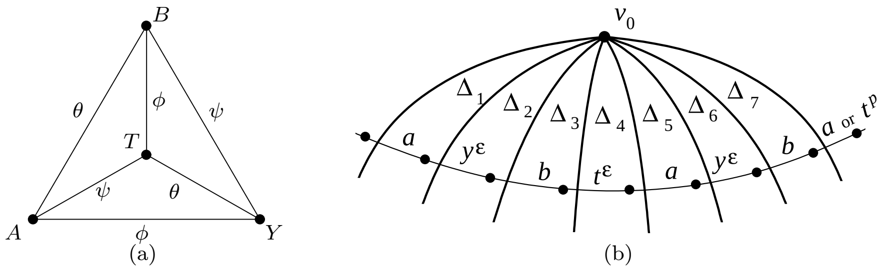

FIGURE 2.

The analysis of a u^ε piece is essentially the same for both $\varepsilon = +1$ and $\varepsilon = -1$, so we introduce the following notation. Let $(a, b) = (x, z)$, $(A, B) = (X, Z)$, $(\phi, \psi) = (\alpha, \beta)$ if $\varepsilon = +1$ and let $(a, b) = (z^{-1}, x^{-1})$, $(A, B) = (Z, X)$, $(\phi, \psi) = (\beta, \alpha)$ if $\varepsilon = -1$. Note that with this convention $\theta \ge \phi$ and $\theta \ge \psi$. Figure 2(a) indicates the Gersten-Stallings angles between the vertex groups and Figure 2(b) shows the form of a u^ε piece, where the exterior regions of D^* are labelled by Δ_i $(1 \le i \le 7)$ and each Δ_i is a q_i-gon.

Since any three consecutive edges on the boundary of $\widehat{D}$ are labelled by elements of three different vertex groups, no two exterior 2-gons of D^* can be adjacent. Therefore, at most four of the Δ_i can be 2-gons. Denote the chain $\Delta_1\Delta_2 \ldots \Delta_7$ by S and write $d(S) = \sum_{i=1}^{7} d(\Delta_i)$. Denote by v_1 the vertex of $\Delta_1 \backslash \Delta_2$ adjacent to v_0.

We shall make frequent use of the following observations. Let Δ be an exterior q-gon. If $q = 2$ then $d(\Delta) = 2\pi/\delta \le \pi/2$. If $q = 3$ then no two adjacent vertices of Δ arise from maximal G_{AB}- or G_{YT}-regions, and so $d(\Delta) \le -\pi + \pi/2 + \pi/3 = -\pi/6$. Similarly, if $q \ge 4$ then $d(\Delta) \le -2\pi/3$.

CLAIM 1. *If v_1 does not arise from any maximal G_{AT}-region then $d(S) \le 0$.*

Proof. Note that if $d(\Delta_i) > \pi/3$ then $i = 7$. Hence, if $|\{i \,|\, q_i = 2\}| \le 2$ then $d(S) \le \pi/2 + \pi/3 - 5\pi/6 = 0$.

Suppose that $|\{i \,|\, q_i = 2\}| = 3$. If $q_i \ge 4$ for some i then $d(S) \le 2\pi/3 + \pi/2 + 3(-\pi/6) + (-2\pi/3) = 0$. Hence, we may assume that $q_i \le 3$ for $1 \le i \le 7$. However, if $q_i = q_{i+2} = 2$ for any $i \le 4$ then $q_{i+1} \ge 4$; moreover, if $q_7 \ne 2$ then this condition holds for some i. Thus $\{i \,|\, q_i = 2\} = \{1, 4, 7\}$, $\{1, 5, 7\}$, $\{2, 5, 7\}$. Label consecutive vertices of S by $v_2, \ldots, v_5$ so that v_2 is adjacent to v_1.

$\{1, 4, 7\}$. The vertices v_1 and v_3 arise from maximal G_{AY}- and G_{AT}-regions, respectively. Moreover, at most two of the five angles in Δ_5, Δ_6, and Δ_7 can be greater than $\pi/3$. Then

$$d(S) \le \phi + (\phi + \theta - \pi) + (\theta + \psi - \pi) + \psi + 3\frac{\pi}{3} + 2\frac{\pi}{2} - 2\pi = 2\phi + 2\psi + 2\theta - 2\pi = 0.$$

$\{1, 5, 7\}$. Since v_1 and v_4 arise from maximal G_{AY}-regions and $q_2 = q_3 = q_4 = 3$, either v_2 arises from a maximal G_{AB}-region and v_3 arises from a maximal G_{AT}-region, or v_2 arises from a maximal G_{YB}-region and v_3 arises from a maximal G_{YT}-region, see Figure 3(a). In both cases

$$d(S) \leq 5\phi + 2\psi + 4\theta - 4\pi = 3\phi + 2\theta - 2\pi \leq 0.$$

$\{2, 5, 7\}$. Since v_2 and v_4 arise from maximal G_{BY}- and G_{AY}-regions, we immediately get

$$d(S) \leq 3\phi + 3\psi + 5\theta - 4\pi = 2\theta - \pi \leq 0.$$

Finally, suppose that $|\{i \,|\, q_i = 2\}| = 4$. Then $\{i \,|\, q_i = 2\} = \{1, 3, 5, 7\}$ and hence $q_2 \geq 4$ and $q_4 \geq 4$. Since $d(\Delta_i) \leq \phi$ for $i = 1, 3, 5$ and $d(\Delta_7) \leq \pi/2$, we get $d(S) \leq 3\phi + \pi/2 - 4\pi/3 - \pi/6 \leq 0$. □

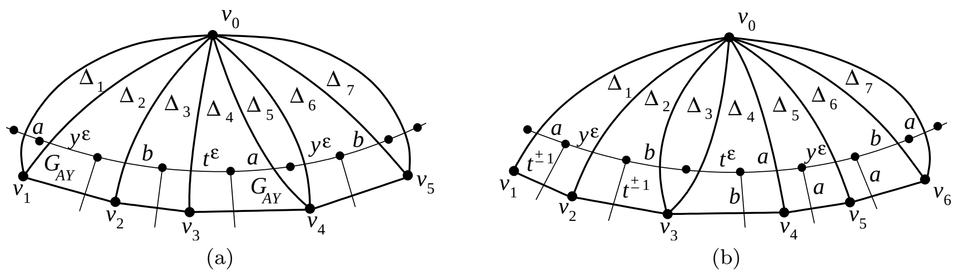

FIGURE 3.

CLAIM 2. *If v_1 arises from a maximal G_{AT}–region then $d(S) \leq -\pi/3$.*

Proof. Since v_1 arises from a maximal G_{AT}-region, Δ_1 is not a 2-gon and, therefore, $|\{i \,|\, q_i = 2\}| \leq 3$. If $|\{i \,|\, q_i = 2\}| \leq 1$ then $d(S) \leq \pi/2 + 6(-\pi/6) = -\pi/2$.

Suppose that $|\{i \,|\, q_i = 2\}| = 2$. If $q_i \geq 4$ for some i then $d(S) \leq -\pi/2$. Hence, we may assume that $q_i \leq 3$ for $1 \leq i \leq 7$. However, if $q_i = q_{i+2} = 2$ for any $i \leq 4$ then $q_{i+1} \geq 4$; moreover, if $q_2 = 2$ then $q_1 \geq 4$. This reduces us to the four cases: $\{i \,|\, q_i = 2\} = \{3, 6\}$, $\{3, 7\}$, $\{4, 7\}$, $\{5, 7\}$. Label consecutive vertices of S by $v_2, \ldots, v_6$ so that v_2 is adjacent to v_1.

$\{3, 6\}$. Since v_3 arises from a maximal G_{BT}-region and v_5 arises from a maximal G_{YB}-region, we have $d(S) \leq 4\psi + 3\phi + 5\theta - 5\pi \leq -2\pi/3$.

$\{3, 7\}$. If u^ε is followed by t^{p_i}, then both v_3 and v_6 arise from maximal G_{BT}-regions and thus

$$\begin{aligned} d(S) &= d(\Delta_1\Delta_2\Delta_3) + d(\Delta_4\Delta_5\Delta_6) + d(\Delta_7) \\ &\leq (\psi + 2\phi + 2\theta - 2\pi) - 3\frac{\pi}{6} + \phi \leq -\frac{\pi}{3}. \end{aligned}$$

So suppose that u^ε is followed by a (see Figure 3(b)). Then v_6 arises from a maximal G_{AB}-region. If v_4 arises from a maximal G_{AT}-region or v_5 arises from a maximal G_{BY}-region then $d(S) \leq 3\psi + 3\phi + 6\theta - 5\pi \leq -\pi/2$. Hence, we may assume that v_4 arises from a maximal G_{AB}-region and v_5 arises from a maximal G_{AY}-region. It follows that $\deg(v_4) \geq 5$, that is, the angle at v_4 is at most $2\pi/5$, so we have

$$\begin{aligned} d(S) &= d(\Delta_1\Delta_2\Delta_3) + d(\Delta_4\Delta_5) + d(\Delta_6\Delta_7) \\ &\leq (\psi + 2\phi + 2\theta - 2\pi) + \left(2\phi + 2\frac{2\pi}{5} - 2\pi\right) + \left(-\frac{\pi}{6} + \frac{\pi}{2}\right) \leq -\frac{11\pi}{30}. \end{aligned}$$

$\{4, 7\}$. Since both v_1 and v_4 arise from maximal G_{AT}-regions, $d(\Delta_1\Delta_2\Delta_3) \leq 2\psi + 2\phi + 2\theta - 3\pi = -\pi$. Then $d(S) = d(\Delta_1\Delta_2\Delta_3) + d(\Delta_4\Delta_5\Delta_6\Delta_7) \leq -\pi + \psi - 2\pi/6 + \pi/2 \leq -\pi/2$.

{5, 7}. If u^ε is followed by t^{p_i}, then $q_5 \geq 4$, a contradiction. Suppose u^ε is followed by a. The vertex v_5 arises from a maximal G_{AY}-region and v_6 arises from a maximal G_{AB}-region. If v_3 arises from a maximal G_{BY}-region or v_4 arises from a maximal G_{AT}-region then $d(S) \leq 3\psi+3\phi+6\theta-5\pi \leq -\pi/2$. Therefore, we may assume that v_4 arises from a maximal G_{YT}-region. It follows that v_3 arises from a maximal G_{BT}-region and hence $\deg(v_4) \geq 5$. Thus,

$$d(S) \leq \psi + 5\phi + 4\theta + 2\frac{2\pi}{5} - 5\pi \leq -\frac{11\pi}{30}.$$

Now suppose that $|\{i \mid q_i = 2\}| = 3$. Then $\{i \mid q_i = 2\}$ is one of the following: $\{2,4,6\}$, $\{2,4,7\}$, $\{2,5,7\}$, $\{3,5,7\}$. However, if $q_2 = q_4 = 2$ then $q_1 \geq 4$ and $q_3 \geq 4$, and so $d(S) \leq -\pi/2$.

{2, 5, 7}. Since $q_1 \geq 4$, we may assume that $q_i \leq 3$ for all $1 < i \leq 7$. In particular, $q_6 = 3$, so the 2-gon Δ_7 comes from a maximal G_{AB}-region. Since Δ_2 and Δ_5 come from maximal G_{BY}- and G_{AY}-regions, respectively, we have

$$d(S) \leq 3\phi + 4\psi + 5\theta - 5\pi = \psi + 2\theta - 2\pi \leq -\frac{2\pi}{3}.$$

{3, 5, 7}. Then $q_4 \geq 4$ and again we may assume that $q_i \leq 3$ for all $i \neq 4$ and therefore the 2-gon Δ_7 comes from a maximal G_{AB}-region. Since Δ_3 and Δ_5 come from maximal G_{BT}- and G_{AY}-regions, respectively, we have

$$d(S) \leq 6\phi + \psi + 5\theta - 5\pi = 5\phi + 4\theta - 4\pi \leq -\frac{\pi}{3}.$$

□

If $w(t,u) = u^{q_1}$ then Claims 1 and 2 imply the required contradiction that $\sum_{\text{ext.}\Delta} d(\Delta) \leq 0$. Hence, u has infinite order in G.

Now suppose $w(t,u) = t^{p_1}u^{q_1} \ldots t^{p_m}u^{q_m}$. For each syllable $t^{p_i}u^{q_i} = t^{p_i}u^\varepsilon u^{q_i-\varepsilon}$, consider the part of the boundary corresponding to $t^{p_i}u^\varepsilon$. Label the first exterior region Δ_0 and, as before, label the remaining regions $\Delta_1, \ldots, \Delta_7$.

If $q_0 \neq 2$ then $d(\Delta_0) \leq -\pi/6$ and, therefore, $d(\Delta_0 S) = d(\Delta_0) + d(S) < 0$. If Δ_0 is a 2-gon then v_1 arises from a G_{AT}-region. But then, by Claim 2, $d(S) \leq -\pi/3$ and, hence, $d(\Delta_0 S) \leq \psi - \pi/3 \leq 0$. It follows that the sum of the d-values of all exterior regions arising from any syllable $t^{p_i}u^{q_i}$ is non-positive, and the required contradiction follows. Hence, t and u generate a free product in G and, since u has infinite order, G contains a non-abelian free subgroup.

Now suppose $(\theta, \alpha, \beta) = (\pi/2, \pi/2, 0)$. Then G is isomorphic to an amalgamated free product $L *_K M$, where $L = G_{14} *_{G_4} G_{34}$, $M = G_{12} *_{G_2} G_{23}$ and $K = G_1 * G_3$. We may assume that $|G_1| = |G_3| = 2$ for otherwise K (and hence G) contains a non-abelian free subgroup. Similarly we may assume that $|G_2| = |G_4| = 2$, so $G_i = \langle x_i \mid x_i^2 \rangle$ for all i. Further, each $m_{i,i+1} = 4$ so $G_{i,i+1} = \langle x_i, x_{i+1} \mid x_i^2, x_{i+1}^2, (x_i x_{i+1})^2 \rangle \cong D_4$. Therefore, G has presentation

$$\langle x_1, x_2, x_3, x_4 \mid x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_2x_3)^2, (x_3x_4)^2, (x_4x_1)^2 \rangle.$$

Since G is a group of isometries of the Euclidean plane, it is virtually abelian.

Thus the theorem is proved when G is based on a graph with four vertices. To complete the proof in the general case, it remains to note that when $\mathcal{G}$ has five or more vertices it is impossible to label the edges of $\mathcal{G}$ so that all four vertex subgraphs give rise to the virtually abelian group. Therefore, one of the four vertex subgraph groups contains a non-abelian free subgroup. Since by [**3**] subgraph groups embed, Theorem 1 is proved.

3. *Application*

We consider the following class of groups which generalizes the groups defined by periodic paired relations [**15**]. Let $n \geq 3$, $1 \leq i, j \leq n$, $n_{ij} \geq 1$ and $1 \leq t \leq n_{ij}$. For each such i, j, t let $2 \leq q_i, q_{i,j;t} \leq \infty$ and suppose $w_{i,j;t}(x_i, x_j)$ is a cyclically reduced word in x_i and x_j. Define

$$\Gamma = \langle x_1, \ldots, x_n \,|\, x_i^{q_i}, w_{i,j;t}(x_i, x_j)^{q_{i,j;t}} (1 \leq i, j \leq n, 1 \leq t \leq n_{ij}) \rangle.$$

Each group Γ can be realized as a Pride group by setting $G_i = \langle x_i \,|\, x_i^{q_i} \rangle$ and $R_{ij} = \{w_{i,j;t}(x_i, x_j)^{q_{i,j;t}} \,|\, 1 \leq t \leq n_{ij}\}$. For each i, j define $r_{ij} = \min\{\ell_{i,j;t} q_{i,j;t} \,|\, 1 \leq t \leq n_{ij}\}$, where $\ell_{i,j;t}$ denotes the free product length of $w_{i,j;t}(x_i, x_j)$. If $1/r_{ij} + 1/r_{jk} + 1/r_{ik} \leq 1/2$ for all distinct $1 \leq i, j, k \leq n$ then by the Spelling Theorem for generalized triangle groups [**6**], the Pride group Γ is non-spherical.

Corollary. *Let Γ be as defined above with $n \geq 4$. If $1/r_{ij} + 1/r_{jk} + 1/r_{ik} \leq 1/2$ for all distinct i, j, k, then Γ contains a non-abelian free subgroup unless it has presentation*

$$\langle x_1, x_2, x_3, x_4 \,|\, x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_2x_3)^2, (x_3x_4)^2, (x_4x_1)^2 \rangle,$$

in which case Γ is virtually abelian.

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