ESSAYS ON MACROECONOMICS AND CAPITAL INTERMEDIATION NETWORKS

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Abstract

The topic of aggregate fluctuations in the economic activity is a long-standing question in the study of business cycles, not only the identification of sources of volatility is necessary to forecast the future pace of the economy, but also to understand how the variance of aggregate activity could be reduced. Regarding economic policy, the analysis of economics stability has taken more relevance in the aftermath of the recent financial crisis. The crisis highlighted the need to think of the economy as a complex network where an idiosyncratic shock may precede aggregate consequences, such as the recent problem in arising from the financial sector and its effect on the economy.

Chapter 1 introduces a two-period multi-sector economy with Input-Output linkages and a banking sector. This model is useful to assess the relevance of the economic structure on aggregate volatility. The main finding is that single financial shocks to banks do not average out and could lead to aggregate fluctuations. In particular, aggregate volatility does not go in a single direction when we increase the number of links bank-to-sector (concentration), enhance the number of links shared by two or more banks (integration) or redistribute the links (diversi-

fication).

In Chapter 2, I use a detailed benchmark data of the U.K. input-output accounts spanning from 1997 to 2010, I apply the model of intersectoral linkages by Acemoglu et al. (2012) to identify if the U.K. network structure is prone to the propagation of shocks.

In Chapter 3, I present a model of a multi-sector input-output economy based on Long and Plosser (1983) and Acemoglu et al. (2012) to analyse the effects of capital risk sharing between firms, productivity shocks correlations between firms of the same ownership organisations, and collateral restrictions between firms of separate groups, on aggregate fluctuations.

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Chapter 1

The network effects of capital intermediation on aggregate volatility

1.1 Introduction

Financial contagion has been a recurrent theme in the aftermath of the 2008 great recession. Most of the economic literature has focused on contagion within the financial sector, e.g. Elliot et al (2014) and Acemoglu et al. (2015). However, the real sector plays a role not only as the ultimate economic bloc affected by these shocks but also as an intermediate step in the diffusion of financial shocks. In fact, the network of linkages generated by financial institutions and the input-output network of linkages across firms are not separated but rather blended. In this

This chapter is co-authored with Professor Christian Ghiglino from the University of Essex.

paper, I focus on how shocks affecting the financial sector propagate to the real sector and ultimately generate aggregate fluctuations and how this is affected by the location of the “contacts points” between the two networks.

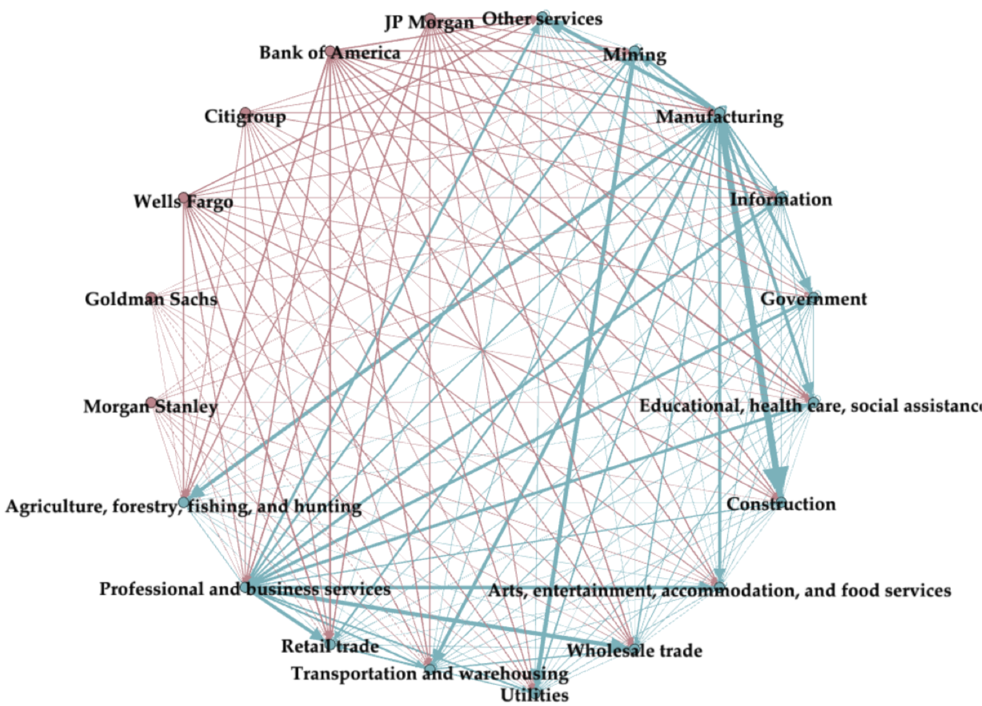
To understand the mechanism of the propagation of shocks we need to have an idea of the network of connections in the economy. It is useful to represent the non-financial sectors and the banks as nodes, and the links as input-output linkages or financial (credit) flow. Indeed, banks provide the intermediation between investors and firms and the way each bank finances production in a given sector is characterised by a link. Below is a graphical representation of this network for the U.S. (Figure 1.1).

The blue arrows represent the input-output linkages: a sector’s technology requires intermediate good from another sector to produce a given good or service. The red arrows represent the financial connections where a sector and a bank have a contract that allows the former to get capital from the latter, and each sector needs capital to produce output.

A remarkable regularity is highlighted by the graph: the pervasive asymmetries of the network, which results in connections with highly dispersed weights. Although the great recession has highlighted the general role of the financial sector as a factor of contagion, I focus more specifically on the contagion from the financial sector to the real sector.

The following diagram (Figure 1.2) illustrates the extent of the intermediation role of financial businesses in providing credit to the real economy, with only a small share of credit coming from non-financial institutions.

(filter: 6 biggest lenders, 14 industries)



Source: BEA, Thomson One.

Figure 1.1: Bank-sector and I-O structure of the U.S. in 2010

The aim of this paper is to study the effect of financial shocks on the real economy, without requiring the type of cascade needed to create a global financial shock. I use the tools recently developed in network theory and applied to the propagation of financial shocks but rather look at contagion from the financial sector to the real sector.

From a modelling point of view, the propagation of financial shocks to the real economy has been widely studied in macroeconomics. Influential examples are Friedman and Schwartz (1963) and Bernanke and Gertler (1989) who show how

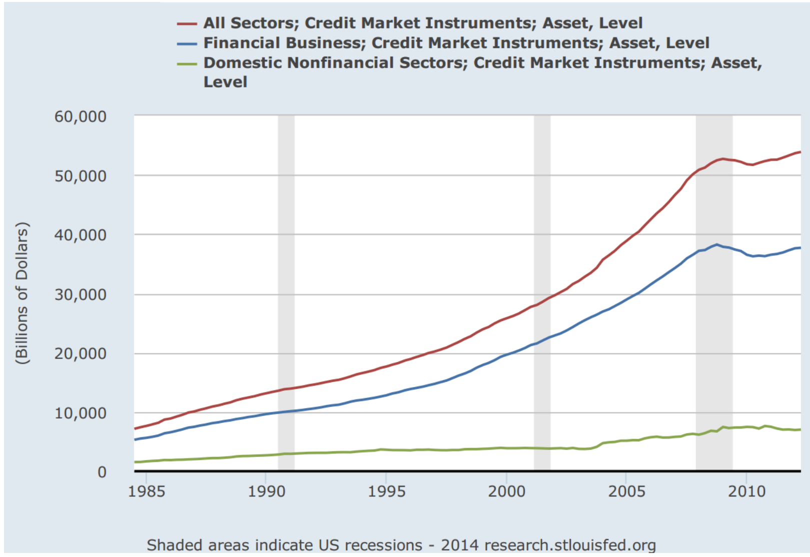


Figure 1.2: Credit to the real economy

credit constraints in the finance of production may generate binding constraints on the possible output and employment. Even if the risks taken by financial institutions increase, as is believed to have happened in the great recession, individual shocks will not lead to aggregate fluctuations because sector-level disturbances would average out and would not have significant aggregate effects. However, the diversification argument made by Dupor (1999) is not relevant when we consider higher order effects that arise from the intersectoral linkages. Explicitly introducing the real and financial interlinkages of the economy may create correlations and finally produce a sizeable aggregate shock via contagion.

The real economy is a multisector economy similar to Long and Plosser (1983) and Acemoglu et al. (2012) the role of interconnections between different firms and sectors serves to propagate idiosyncratic shock through the economy has been recently investigated. In particular, Acemoglu et al. (2012) showed that productivity shocks could propagate through the input-output linkages and produce an aggregate effect. The fundamentals of the model of this chapter are related to both papers.

To analyse the propagation of shocks within the integrated economy, I consider a two-period multisector economy with input-output linkages. In the first period, there is a representative firm producing capital while in the second period there are n non-financial sectors with input-output linkages, m capital intermediaries, banks. There is a representative household, working in the first period and consuming an aggregated consumption good in the second. Households deposit savings (in the form of capital) at the beginning of the second period, D_b, in each bank b, in a competitive deposits market which implies that each bank takes the gross interest rate on deposits, R, as given.

It is important to note that in this present model, each bank is subject to a random friction that limits the amount of deposits that each bank can potentially lend. The friction parameter is realised at the beginning of the second period and takes values between zero and one. We call this an idiosyncratic financial shock because it affects the bank's balance sheet. After the shock is realised, each bank lends x_{ib} capital to each sector i.

Each of the n sectors produces a different good using different quantities of capital, x_{ib}, intermediated by banks b, and intermediate goods, q_{ij}, sold by another sector j. These network of linkages are represented by an IO matrix $\mathbf{W}$. Production is carried out only in the second period, and the output can be used as a consumption good, aggregated by the final good sector, or as an intermediate input in the production of another sector.

The capital intermediated by the banking sector is called a loan, and altogether they form the lending channel from banks to sectors. Each sector i obtains a quantity x_{ib} from each bank in the financial sector, which is priced by r_b. I assume that proportions of x_{ib} in the total capital, x_i, are exogenous and fixed, perfectly know to everybody, and given by ϕ_{ib}. The matrix of lending linkages is represented by a capital intermediation matrix, $\mathbf{\Phi}$.

Firms need to borrow capital from financial institution while these obtain funding from individual depositors and investors. I assume that banks face idiosyncratic shocks in the form of reserve requirements. On the other hand, there are rigidities also in the way firms interact with banks. In fact, the bank-firm relationship is based trust, information or regulation. To summarise these rigidities, I assume that the network of the relationship between firms and banks is exogenous to the model.

Our formulation of the financial sector is related to the canonical model of Gertler and Kiyotaki (2010) of financial intermediation with a random financial friction directly in the balance sheet of the bank that implies a positive spread in the interest rates. Figure 1.3 shows a positive spread between the interest rate on deposits and loans in the US.

The analysis focuses on the role of the structure of the economy, that is the network of connection among physical sectors, W, and between sectors and banks, $\mathbf{\Phi}$, in the propagation of financial shocks to the real economy and its aggregate implications. Analysing different structures of the economy, I find that, in gen-

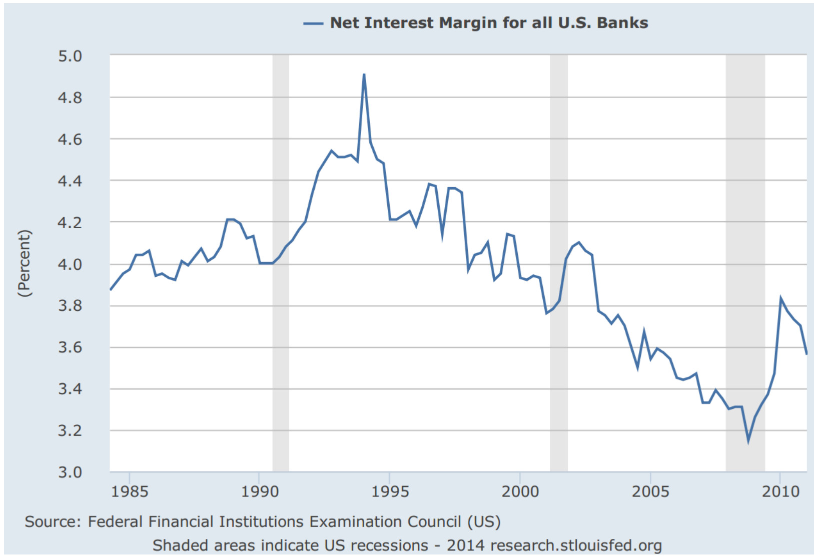


Figure 1.3: Spread between the interest rate on deposits and loans

eral, GDP volatility is decreasing as the variability of the interconnectivity of each bank is reduced, as expressed by the network multiplier.

In other words, the standard deviation of GDP decreases as the interconnectivity coefficient is similar among banks, while it increases when this metric shows a higher variability across banks, mainly because the effects of different individual shocks are propagated with more strength in the presence of asymmetries.

Financial integration can be understood with the analysis of the effect of adding links to the bipartite intermediation network. I first consider the increase in integration provided by adding one link to a specific bank. In this case, the volatility of GDP can decrease or increase depending on the position of the banks and the position of the common sector in the network. On the other hand, when adding m links, one for each bank in the network, the results still depend on the I-O structure, but always reduce volatility for vertical and star structures and have no effect on empty and circle networks.

We also analyse the effect of diversification, obtained as a result of redistributing the links of two or more banks in the network. I find that depending on the position of the bank in the network, reducing diversification can decrease or increase the effect on the volatility of an individual shock even for a particular location. This result is due to the difference in the elements of the influence vector for each sector and the interconnectivity coefficient of the bank-to-sectors network. The latter implies that there is no effect for perfectly balanced networks, as in this case, the influence vector is the same for all sectors.

In general, aggregate fluctuations depends on the distributions of links between banks and sectors and the location of such links. An economy with a uniform distribution of links per bank could be less volatile than an economy with an unequal distribution, provided the bank with less link is not supplying capital to a great influencer, that is, the star sector or the top of the chain in the vertical network. The intuition behind this result is that the multipliers effects of the interactions between the sectors that obtain the funds from the same bank is greater with a non-uniform distribution of the bipartite links under specific locations.

Using input-output data and syndicated loans data to compute the I-O and the bipartite intermediation matrices of the US economy for the period 2000-2010, I find that the structure of the economy is very asymmetric, there are star sectors like manufacturers and professional services, and star banks like Bank of America and JP Morgan. I find that changes in the bipartite structure over time lead to changes in the network multiplier while the I-O network remains relatively steady.

Finally, computing the GDP volatility using the network metrics, I find that individual shocks to banks do not average out and could lead to sizeable fluctuations of GDP. Interestingly, the volatility is considerable higher when taking into account the network multiplier rather than simply using a basic aggregate computation of the financial sector.

This chapter is organised as follows: i) in Section 2, I present a brief literature review; ii) in Section 3, I present the specifics of the model; iii) in Section 4, I describe the equilibrium of the model; iv) in Section 5, I analyse the volatility of the GDP; v) in Section 6, I describe the economy as a network; vi) in section 7, I study the effect of network on the volatility of the GDP; vii) in Section 8, I present an empirical analysis; and viii) finally, I present the conclusions the chapter.

1.2 Related literature

This paper is related to several strands of the literature. As we have pointed out earlier, it takes inspiration from the macroeconomic literature developed in the last two decades, in which frictions in the financial sector cause aggregate economic fluctuations. In their seminal papers, Bernanke and Gertler (1989), Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999) address this issue developing models where imperfections in the credit market influence the business cycle, resulting in the financial accelerator.

Despite being hugely influential, Kocherlakota (2000) argues that these two papers are not quantitatively satisfactory as the observed financial frictions effects are not significant enough to explain the observed aggregate fluctuations. The great recession has shown the role of contagion in the crisis, allowing large aggregate fluctuation to be generated by small individual shocks.

The present paper is related to Gertler and Kiyotaki (2010) as these authors include a financial sector in a dynamic stochastic general equilibrium model (DSGE). In their canonical model, exogenous financial friction arises due to a moral hazard problem in the financial intermediaries. This result generates a spread between the interest rate on deposits and the interest rate on loans, as in our model Gertler and Karadi (2011) expanded that model including nominal rigidities to analyse monetary policy).

More recently different financial frictions specifications have been considered. Jermann and Quadrini (2012) and Christiano, Motto and Rostagno (2014) augmented a DSGE model including a financial accelerator similar to the one developed in Bernanke, Gertler and Gilchrist (1999). The authors found that shocks in risk could influence the business cycle.

In this sense, Liu and Wang (2014) analyse a model with two kinds of firms, productive and unproductive ones, where one kind has credit constraints which generates business cycles. Finally, Brunnermeir and Sanikov (2014) develop a model with two kinds of agents, households and experts, and a financial friction that arises due to a moral hazard problem related to high monitoring costs. They found that the economy could fluctuate around two regimes, an unstable and a stable one because of the propagation non-linear mechanism generated by a shock in the financial sector.

The model of this chapter is also related to the paper of Bigio and La'O (2016). The authors model an economy with financial constraints under in the presence of input-output linkages. However, on top of being static, they do not explicitly model the financial sector. In contrast, I include banks and model the financial constraint as an idiosyncratic and random shock to the balance sheet of each bank while in their model the relation between firms and banks is deterministic.

The diversification argument leading to smooth aggregate behaviour also affects models with individual productivity shocks that cannot result in aggregate volatility. Long and Plosser (1983) present a seminal work with a model that includes a multi-sector economy that is rewritten by Horvath (2000) as a multi-sector economy with input-output trade among sectors. However, as Dupor (1999) argued the law of large number average out any shock on an individual sector, making individual shocks irrelevant for the aggregate fluctuations.

However, that idiosyncratic shocks in a disaggregated production economy can have aggregated effect has been shown in Carvalho (2010). The author analyses the intersectoral trade of inputs by adopting a network perspective on interactions, the result of this model is that independent productivity shocks could lead to aggregate fluctuations in economies that do not have a high degree of diversification of inputs required in production.

Acemoglu et al. (2012) also argue that, in the presence of intersectoral linkages, idiosyncratic shocks to firms or sectors can generate relevant aggregate fluctuations. In particular, they show that the rate at which aggregate volatility decays depends on the structure of the intersectoral intermediate linkages. Finally, Carvalho and Gabaix (2013) develop a two-period multisector model to analyse the individual and aggregate volatility and find that the volatility that emerges from idiosyncratic shocks has the power to explain fluctuation on the aggregate activity.

On the empirical side, Foster et al. (2011) analyse the sectoral and aggregate volatility in the US and find a significant effect of the sectoral shock due to intersectoral linkages. There are also two works that include a theoretical model, and an empirical application of an inter-sectoral economy are Jones (2011) and Fadinger, Ghiglino and Teteryatnikova (2015). Both analyse the effect of a disaggregated economy in the misallocation of resources, the last one using a network formalism.

1.3 Model

In this section, we describe the general framework and the optimisation problems faced by each actor in the economy and consider a two-period multi-sector economy with input-output linkages (Figure 1.4).

In the first period, there is a representative firm producing capital while in the second period there are n non-financial sectors with input-output linkages, m capital intermediaries, banks, each subject to an individual shock. There is a representative household, working in the first period and consuming an aggregate consumption good in the second. The population is constant and of mass one. The following diagram illustrates the actors and the timing in this economy.

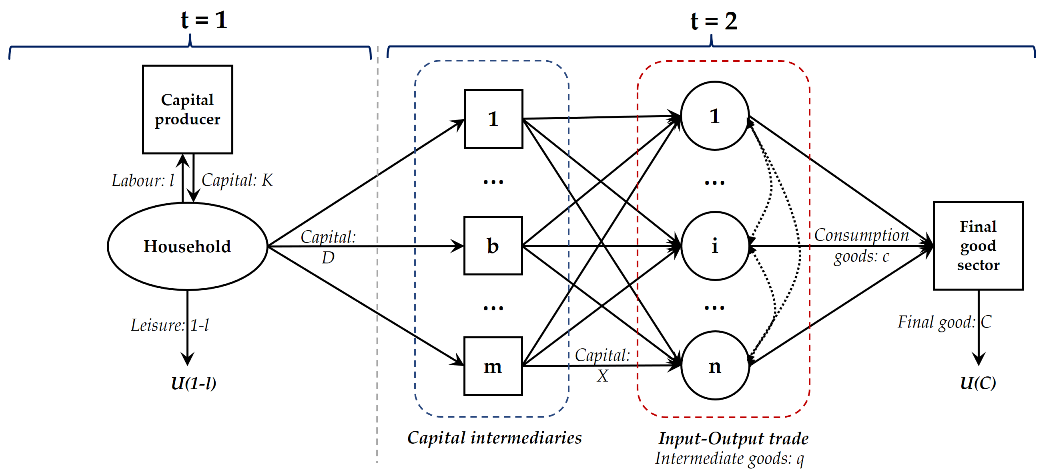


Figure 1.4: Two-period multi-sector economy

The model combines several features present in the literature. The input-output linkages are modelled as in Acemoglu et al. (2012). In my model, these firms need capital, which is provided by the banks. Rather than assuming idiosyncratic productivity shocks to the sectors, we assume that banks are subject to shocks at the beginning of the second period. These shocks to the banks generate a financial friction similarly to the one described by Gertler and Kiyotaki (2010). We now describe the model in greater detail.

1.3.1 Households

The representative household has preferences over leisure, L, and over a final aggregated consumption good, C. The household has separable utility and leisure is consumed in the first period while consumption in the second period. The instantaneous utility for consumption $u'(C)$ satisfies $u'(C) > 0$, $u''(C) < 0$ and $lim_{C\to 0}u'(C) = +\infty$. The same properties hold for the utility of leisure $u'(L)$. Note that leisure is defined as the number of hours minus labour, l, and we normalise to one the total number of hours, $L = 1 - l$.

The household maximises the intertemporal utility, given by the following constrained problem:

$$\max_{L,C} \; u(L) + \beta E[u(C)]$$

$s.t.$

$$wL + D = w \tag{1.1}$$

$$PC = RD \tag{1.2}$$

The intertemporal discount factor $\beta > 0$ captures time preference and the variable D is defined as

$$D = \sum_{b=1}^{m} D_b \tag{1.3}$$

Where D_b is the representative household deposit in bank b. Equation (1) and (2) are the household resource constraints in period 1 and 2, respectively with w being the wage and D are the savings (purchases of capital) that will be called deposits.

Note that deposits are priced at the price of capital in the first period, which is taken as the numeraire. P is the price of the aggregated final consumption good C and caught in the numeraire in the second period, $P = 1$. R is the gross interest rate on deposits, which is the price of deposits in the second period paid by the banks.

Combining the two constraints through the deposits, I obtain the consolidated intertemporal constraint:

$$C = Rw(1 - L) \tag{1.4}$$

To simplify the analysis we assume log-utility on consumption and leisure, $u(C) = \ln C$ and $u(L) = \ln L$. Using the intertemporal constraint to eliminate C from the choice the maximisation problem becomes:

$$\max_{L} \ \ln L + \beta E[\ln(Rw(1 - L))]$$

The first order condition associated with this maximisation problem is given by the following equation

$$L = \frac{1}{1 + \beta} \tag{1.5}$$

Implying that the labour supply is given by

$$l = \frac{\beta}{1 + \beta} \tag{1.6}$$

Using the constraints to find consumption and deposits:

$$C = \frac{Rw\beta}{1 + \beta} \tag{1.7}$$

$$D = \frac{w\beta}{1 + \beta} \tag{1.8}$$

These equations are deterministic because the assumption of log-utility allows the price R to be deterministic and then the demand for labour and leisure.

1.3.2 Firms and sectors

1.3.2.a Capital producers

In the first period, there is a representative firm producing capital in a competitive market. The firm demands labour, l, to produce capital, K, to be sold to the household. The price of the capital is the numeraire in the first period. To produce capital, we assume a linear technology with no productivity shocks. The problem of this sector becomes:

$$\max_{K,l} \; K - wl$$

$s.t.$

$$K = \delta l \tag{1.9}$$

with δ the marginal productivity. When $\delta > w$ there is no solution, $\delta = w$ any K is a solution, while for $\delta < w$, the solution is $K = 0$. The only non-trivial solution is

$$K = wl \tag{1.10}$$

1.3.2.b The multi-sector I-O economy

We assume n sectors, $i = 1, ..., n$, that produce a good, q_i, using different quantities of capital, x_{ib}, intermediated by banks b, and intermediate goods, q_{ij}, sold by another sector j. Production is carried out only in the second period, and the output can be used as a consumption good, aggregated by the final good sector, or as an intermediate input in the production of another sector.

The capital intermediated by the banking sector is called a loan, and altogether they form the lending channel from banks to sectors. Each sector i obtains a quantity x_{ib} from each bank in the financial sector, which is priced by r_b. We also assume that proportions of x_{ib} in the total capital, x_i, are exogenous and fixed, perfectly know to everybody, and given by ϕ_{ib}.

Production is carried out according to a Cobb-Douglas constant returns to scale technology, $f(\cdot)$, which satisfies for each factor $f(0) = 0$, $f'(\cdot) > 0$, $f''(\cdot) < 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. The technology of production is then:

$$q_i = x_i^{\alpha} \prod_{j=1}^{n} q_{ij}{}^{(1-\alpha)w_{ij}} \tag{1.11}$$

where

$$x_i = \sum_{b=1}^{m} x_{ib} \tag{1.12}$$

$$\frac{x_{ib}}{x_i} = \phi_{ib} \tag{1.13}$$

and $\alpha \in (0,1)$ indicates the share of capital in production of i.

The parameters ϕ_{ib} represent the share of capital obtained from each bank b in the total of each sector, which is exogenous and does not depend on the financial situation of each sector. These parameters are required to add up to 1 for each sector

$$\sum_{b=1}^{m} \phi_{ib} = 1 \quad \forall i \tag{1.14}$$

$w_{ij} \geq 0$ represents the share of intermediate good j in the total use in sector i and is also a typical element of the input-output matrix (Leontief matrix) with the following assumption:

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i \tag{1.15}$$

These two set of vectors shape the matrix $\mathbf{W}$ and the matrix of capital intermediation, $\mathbf{\Phi}$. These are important because they define the network of I-O connections and the linkages between banks and sectors.

The price of capital intermediate by bank b is given by r_b. Given these assumptions, each sector has to solve the following maximisation problem subject to the available technology and constraint described above:

$$\max_{x_{ib}\forall b, q_{ij}\forall j} \quad p_i q_i - \sum_{b=1}^{m} r_b x_{ib} - \sum_{j=1}^{n} p_j q_{ij}$$

where the price of capital, r_b, can be interpreted as the gross interest rate paid on loans made by the bank b in the financial sector.

The price of the good produced is given by p_i, and the price of each intermediate input is p_j.

The first order conditions for the capital obtained from each bank and for the intermediate inputs are given by the following equations:

$$x_{ib} = \frac{\alpha p_i q_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}} \tag{1.16}$$

$$q_{ij} = \frac{(1-\alpha) w_{ij} p_i q_i}{p_j} \tag{1.17}$$

Dividing these two equations we can find the ratio of the demand in inputs:

$$\frac{x_{ib}}{q_{ij}} = \frac{\alpha p_j \phi_{ib}}{(1-\alpha) w_{ij} \sum_{b=1}^{m} r_b \phi_{ib}} \tag{1.18}$$

Importantly, the demand for capital depends on a weighted price that takes into account the share in the demand from each bank.

1.3.2.c Final good aggregator sector

To keep tractability, we assume that in the second period the final good is produced by a representative firm in a competitive market that aggregates all consumption goods produced by the I-O sectors into one final good, Y, with P. This sector uses a Cobb-Douglas technology with the same weight for each type of good:

$$Y = \prod_{i=1}^{n} c_i^{1/n} \tag{1.19}$$

The maximisation problem solved by this sector is:

$$\max_{Y, c_i \forall i} \; PY - \sum_{i=1}^{n} p_i c_i$$

The first order conditions are given by the following equations:

$$p_i c_i = \frac{PY}{n} \quad \forall i \tag{1.20}$$

These imply that the ratio between the consumption good i and j is related to the ratio of the prices of these goods

$$\frac{c_i}{c_j} = \frac{p_j}{p_i} \quad \forall i, j \tag{1.21}$$

Substituting the F.O.C. into the production function of the final good sector, we obtain that the price of the final consumption good is a simple function of the individual prices:

$$P = n \prod_{i=1}^{n} p_i^{1/n} \tag{1.22}$$

and recalling that P is the numeraire, the previous equation is the ideal price index:

$$1 = n \prod_{i=1}^{n} p_i^{1/n} \tag{1.23}$$

1.3.3 Banking sector

To introduce the banking sector, we follow a similar specification to the one developed in Gertler and Kiyotaki (2010) and also include this into a multi-sector economy with uncertainty. However, I greatly simplify their model, as we assume only two periods (following the intuition of Cristiano and Ikeda (2011)) and ignore issues related to moral hazard. The friction is, therefore, different, I assume a direct role in the balance sheet and not as an incentive constraint like in the cited papers.

I start by assuming that direct lending from household to I-O sectors is impossible. In other words, members of the banking, or financial sector, have a particular skill that allows them to verify the income declared by the I-O sectors.

We assume m types of banks such that $M = \{\mathbb{N}^+ \leq m\}$, all banks constitute the demand side of a competitive market for deposits. The representative household is the supply side of such market, without any preference over banks. Each type $b \in M$ has a large number of banks with mass 1 that constitute the supply side of a competitive market for loans.

The proportion of loans that each I-O sector can obtain from each type b is exogenous and predefined.[1] The I-O sectors borrowing from type b represents the demand side of a competitive market.[2]

Households deposit savings at the beginning of the second period, D_b, in each

[1] Chodorow-Reich, Gabriel (2014, QJE) suggests that borrowers tend to be stuck with a particular lender and that lender specialised in specific sectors.

[2] A broader implication of this assumption would be having m types of capital. However, we are not considering such case to simplify the analysis

bank b, are in a competitive deposits market which implies that each bank takes the gross interest rate on deposits, R, as given. Each bank is subject to a random friction, θ_b, that limits the amount of deposits that each bank can potentially lend.

The friction parameter is realised at the beginning of the second period and takes values between zero and one. We call this an idiosyncratic financial shock because it affects the bank's balance sheet. After the shock is realised, each bank lends x_{ib} capital to each sector i.

Each bank b has to maximise its profit, Π_b, given by the amount lent to the non-financial institutions priced r_b, less the payment of the household's deposits priced at the interest rate R in the second period.

The bank's decision is given by the following constrained problem:

$$\max_{\{x_{ib}\}_i, D_b} \ \Pi_b = \sum_{i=1}^{n} r_b x_{ib} - RD_b$$

$s.t.$

$$\underbrace{\sum_{i=1}^{n} x_{ib}}_{\text{Loans: capital lent to sectors}} + \underbrace{\theta_b D_b}_{\text{Capital reserved, reduced or injected}} = \underbrace{D_b}_{\text{Deposits: capital from households}} \tag{1.24}$$

$$x_{ib} \geq 0 \ \forall i, \ D_b > 0, \ \theta_b \in (-\infty, 1) \tag{1.25}$$

The first constraint reflects the balance sheet of the representative bank of type b. This equation indicates that the sum of loans to each sector, x_{ib}, plus the part of the deposits that cannot be intermediated due to the rigidity, $\theta_b D_b$, equal to the capital obtained from the representative household as deposits, D_b. There is no exogenous endowment or previous accumulation of capital; hence there is no exogenous capital in the balance sheet.

The financial shock, θ_b, represents the assumption that each bank is subject to an individual shock (not correlated across banks) that can be originated from punitive measures by authorities, idiosyncratic legal capital reserve, exogenous "skin in the game", exogenous injection of capital, and similar events that limit the amount of capital that can be intermediated. To simplify the notation related to the balance sheet of each bank, I define the idiosyncratic financial shock by

$$z_b \equiv 1 - \theta_b \tag{1.26}$$

where

$$z_b \in (0, +\infty) \tag{1.27}$$

Parameter θ_b can only take values below 1, so that the distribution of z_b is log-normal, taking values only positives values. This implies that the distribution of $\ln z_b$ is normal, being its domain the real numbers. I further assume that the mean of z_b is close to one and the parameter of its variance, σ_b^2, is close to zero. Considering our previous assumptions, z_b has the following distribution:

$$z_b \sim \; Lognormal\left(0, \sigma_b^2\right) \tag{1.28}$$

This assumption on the distribution ensures that the shock z_b has a mass near but to the left of one. Intuitively, this hypothesis indicates that the friction is not significant and its occurrence does not destroy the capability of banks to lend. It is important to note that we are assuming independence among shocks.

Substituting the z_b into the bank's problem and the constraint into the objective function:

$$\max_{\{x_{ib}\}_i} \ \Pi_b = \sum_{i=1}^{n} \left[\left(r_b - \frac{R}{z_b} \right) x_{ib} \right]$$

$s.t.$

$$x_{ib} \geq 0 \, \forall i, \ D_b > 0 \tag{1.29}$$

In the maximisation problem above, the solution to the bank problem is demanding any level of deposits that satisfies the profit equation with $x_{ib} \geq 0$, where the interest rate on deposits has to be equal to the interest rate on loans times the shock, $R = z_b r_b$.

There is a financial shock, z_b that affects the amount of capital that each bank can intermediate and will influence the price of loans.

If the interest rates on loans times the financial shock, $z_b r_b$, is lower than the interest rate on deposits, R, this implies negative bank's profits and the bank demand for deposits would be zero, so we can rule out this possibility. If $R < z_B r_b$ then the bank's capital supply and deposits demand would be infinite. However, as we are assuming that there is a significant number of banks with identical cost structure, competition among these banks will drive the price of loans down until there is equality in the price ratio. In consequence, the bank will demand any level of deposit and supply loans according to the following price ratio:

$$\frac{R}{r_b} = z_b \tag{1.30}$$

This result implies that the loans-deposits ratio has to be equal to the idiosyncratic shock, as given by the following equation:

$$\frac{\sum_{i=1}^{n} x_{ib}}{D_b} = z_b \tag{1.31}$$

Finally, the supply of loans is given by:

$$\sum_{i=1}^{n} r_b x_{ib} = R D_b \tag{1.32}$$

1.3.4 The market clearing conditions

The clearing conditions of the economy are given by the constraints described in the previous sections. Additionally, in the first period, savings have to be equal to the capital produced:

$$D = K \tag{1.33}$$

In the second period, besides the constraints of each actor, the production of each good by the I-O sectors has to be equal to the quantity used as the consumption good and the amount used as intermediate inputs, and the final consumption good has to be equal to the aggregated good produced by the final sector:

$$q_i = c_i + \sum_{j=1}^{n} q_{ij} \ \ \forall i \tag{1.34}$$

$$C = Y \tag{1.35}$$

1.4 Equilibrium

In this section, we define the equilibrium of the rigid multi-sector economy under consideration, with financial friction θ. The results for an economy with no financial friction can be obtained by just assuming the parameter θ is deterministic and equal to zero.

Definition 1. *An equilibrium with a random financial shock is defined as the set of allocations* $\{L$, l, D, K, x_{ib}, q_{ij}, q_i, c_i, Y, $C\}$ *and prices* $\{w$, p_i, r_b, $R\}$ *for all* i, j *and* b*, such that:*

(i) *household, non-financial sectors and banks problems are solved,*

(ii) *market clearing conditions are satisfied*

(iii) *price of capital* ($t = 1$) *and price of aggregated final good* ($t = 2$) *are taken as numeraires, and*

(iv) $\{l$, D *and* $C\} > 0$.

In order to obtain the gross domestic product (GDP), we need the demand for intermediate goods and capital at equilibrium:

$$q_{ij} = \frac{v_i(1-\alpha)w_{ij}q_j}{v_j} \tag{1.36}$$

$$x_{ib} = \left(\frac{v_i\phi_{ib}}{\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}}\right)\left(\frac{\delta\beta}{1+\beta}\right) \tag{1.37}$$

These solutions depend on the I-O network, $\mathbf{W}$, given by the so-called influence vector[3] $\mathbf{v}$

$$\mathbf{v} = \frac{\alpha}{n}(\mathbb{I} - (1-\alpha)\mathbf{W}'^{-1}\mathbf{1} \tag{1.38}$$

This is in fact the solution to the ratio output-consumption goods in equilibrium where the matrix of input-output shares, identity matrix and vector of ones are given by:

$$\mathbf{W} \equiv \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}_{nxn} \quad \mathbb{I} \equiv \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn} \quad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

In the same way, such capital solution depends on the network of capital intermediation between banks and I-O sectors implicit by the parameters ϕ_{ib}. This is because such parameter represents a typical element of the matrix, $\mathbf{\Phi}$, that tell

[3]This name was given by Acemoglu et al. (2012) in their seminal paper on networks and aggregate fluctuations.

us the proportion of capital provided by bank b in the production of sector i:

$$\mathbf{\Phi} \equiv \begin{bmatrix} \phi_{11} & \dots & \phi_{1m} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \dots & \phi_{nm} \end{bmatrix}_{nxm}$$

We will focus now on the equilibrium in the second period because this is the time when connections become relevant, and the shock is realised.

Proposition 1. *A unique rational expectations equilibrium exists. In such equilibrium the natural logarithm of the value added in the second period* ($\ln Y$) *is given by:*

$$\ln Y = \Gamma - \sum_{i=1}^{n} \left[v_i \ln \left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) \right] \tag{1.39}$$

where ϕ_{ib} is a typical elements of the matrix of capital intermediation, described previously, and z_b is the idiosyncratic shock. Γ is composed of parameters:

$$\Gamma \equiv \ln \left[\left(\frac{\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}}{n} \right) \left(\frac{\beta\delta}{1+\beta} \right)^{\alpha} \right] + \mathbf{v}' \left(\frac{(1-\alpha)}{\alpha} \left(\mathbf{W} \circ \overline{\mathbf{W}} \right) \mathbf{1} \right) + \left(\mathbf{v}'\mathbf{1} - \frac{1}{n}\mathbf{1}' \right) \overline{\mathbf{v}}$$

where

$$\overline{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{mx1} \qquad \overline{\mathbf{W}} \equiv \begin{bmatrix} \ln w_{11} & \dots & \ln w_{1n} \\ \vdots & \ddots & \vdots \\ \ln w_{n1} & \dots & \ln w_{nn} \end{bmatrix}_{nxn} \qquad \overline{\mathbf{v}} \equiv \begin{bmatrix} \ln v_1 \\ \vdots \\ \ln v_n \end{bmatrix}_{nx1}$$

From the equation of the $\ln Y$ one can see that the real value added in a multi-sector economy with banking capital intermediaries is given by the banking shocks, z_b, escalated by the elements of the influence vector, $\mathbf{v}'$, and the elements of the matrix of capital intermediation, $\mathbf{\Phi}$, that together reflect a network multiplier of connections in the economy, which is the source of co-movement.

1.5 Volatility of GDP

I use as a measure of volatility the standard deviation of the logarithm of GDP. I adopt the following definition:

Definition 2. *The volatility of the aggregate value added is given by:*

$$\sqrt{var(\ln Y)} \equiv \sqrt{var\left(\Gamma - \sum_{i=1}^{n}\left[v_i \ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]\right)} \tag{1.40}$$

Given the assumptions on the distribution of shocks, the variance of the natural logarithm of GDP is:

Proposition 2. *The variance of* $\ln Y$ *is given by:*

$$var[\ln Y] = \sum_{i=1}^{n} v_i^2 \widetilde{\sigma}_i^2 + 2\sum_{i,j:i<j} v_i v_j \widetilde{\sigma}_{ij} \tag{1.41}$$

Where:

$$\widetilde{\sigma}_i^2 \equiv \frac{\sum_{b=1}^{m}\phi_{ib}^2\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)^2} \tag{1.42}$$

$$\widetilde{\sigma}_{ij} \equiv \frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)\left(\sum_{b=1}^{m}\phi_{jb}\sqrt{\left(\sigma_b^2+1\right)}\right)} \tag{1.43}$$

One important insight from this result is that the relevant statistics to characterise the aggregate fluctuations are the variance of the distribution of the idiosyncratic shocks, σ_b^2, the weights, ϕ_{ib} indicating the participation of capital provided by each bank b in the product of i, and the elements of the influence vector, v_i.

The expression for the volatility of GDP can be simplified assuming that the variance of the distribution of shocks is the same for each bank.

Corollary 1. *Assuming that individual variance of the distribution of shocks is the same across banks and economies, $\sigma^2 \forall (b \in M)$, the variance of the natural logarithm of GDP is given by:*

$$var[\ln Y] = \sigma^2 \sum_{b=1}^{m} \eta_b^2 \tag{1.44}$$

where

$$\eta_b \equiv \sum_{i=1}^{n} v_i \phi_{ib} \tag{1.45}$$

which can be expressed as:

$$var[\ln Y] \approx \sigma^2 \sum_{b=1}^{m} \left(\underbrace{\sum_{i=1}^{n} v_i^2 \phi_{ib}^2}_{\textit{variance effect per sector}} + \underbrace{2 \sum_{i,j:i<j} v_i v_j \phi_{ib} \phi_{jb}}_{\textit{covariance effect per sector}} \right)$$

Note that we define η_b as the network multiplier because it includes v_i, which is the ith-element of the influence vector (depending on the I-O network) and ϕ_{ib} is the typical element of the capital intermediation matrix.

This result in the previous Corollary plays a major role in the comparisons of volatilities in different economic structures economies as it highlights the structure parameter η_b in the effect on volatility.

1.6 Economy as a network

Given the expression of volatility derived above, one can see that to compare the volatility of GDP across different economies it is sufficient to know the I-O matrix, $\mathbf{W}$, the matrix of capital intermediation, $\mathbf{\Phi}$, and the parameter α.

Similarly to the recent literature on multi-sector economies, e.g. Acemoglu et al. (2012), there are circumstances in which less information is required, for example, in which volatility of GDP is simply a function of summary network statistics such as degree and connectivity.

In this section, we define the relevant metrics for our model. In the next section, we will show how the standard deviation and the marginal variance depend on this metrics, therefore allowing to relate different levels of volatility to various economic structures. I will demonstrate the effect of the individual shock on aggregate volatility under different topologies of the economy and analyse specific cases.

We start by defining the I-O network and the bipartite network of capital intermediation.

Definition 3. *The I-O directed network, IO, is given by the set of nodes (I-O sectors), $N = \{1, ..., n\}$, the set of directed links among sectors, $E = \left\{\overrightarrow{\langle i, j \rangle}, \forall (i,j) \in N\right\}$, and* *the set of weights that is given by* $w = \left\{ w_{ij}, \forall (i,j) \in N | 0 \leq w_{ij} \leq 1 \wedge \sum_{j=1}^{n} w_{ij} = 1 \right\}$*:*

$$IO = (N, E, w) \tag{1.46}$$

Definition 4. *The capital intermediation network, BN, is a directed bipartite graph given by two disjoint set of nodes, I-O sectors and banks, $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$, respectively; the set of directed links from banks to I-O sectors only, $\overline{E} = \left\{ \overrightarrow{\langle b, i \rangle}, \forall b \in M \wedge \forall i \in N \right\}$, and the set of capital weights, $\phi = \{\phi_{ib}, \forall b \in M \wedge \forall i \in N | 0 \leq \phi_{ib} \leq 1 \wedge \sum_{b=1}^{m}$*

$$BN = (N, M, \overline{E}, \phi) \tag{1.47}$$

where the delimiter $\langle a, b \rangle$ represents a couple of connected nodes and $\rightarrow$ means a directed link from a to b.

The economy can now be defined as networks in the following way.

Definition 5. *The economy, $\mathbb{E}$, in the second period can be represented by the elements of the I-O network, IO, the capital intermediation bipartite network, BN, and the matrix of the variance of financial shocks, $\mathbf{\Sigma}$:*

$$\mathbb{E} = (N, M, E, \overline{E}, w, \phi, \mathbf{\Sigma}) \tag{1.48}$$

We can represent this definition of the economy graphically as networks in the following diagram (Figure 1.5).

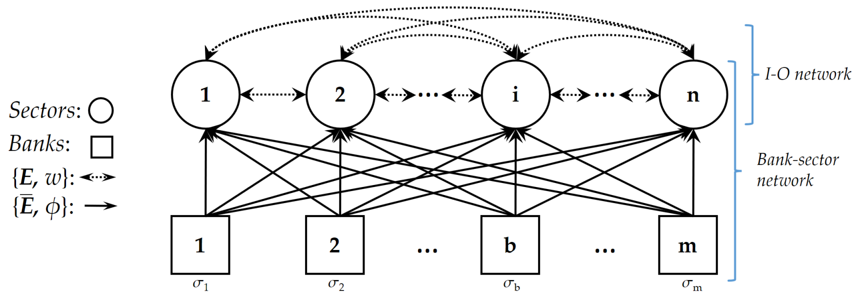


Figure 1.5: Representation of the economy as a network

1.6.1 Network metrics

In this part, we define the network statistics that will be relevant to characterise GDP volatility in the economy. Evidently, these quantities need to capture the connectivity, and more generally, the network importance of each node in the economy. As is now usual, this is obtained under the condition that the network effects are not too large, that is small coupling parameters.

In the appendix, we show how the influence vector of i can be approximated by a function of its out-degree and the out-degrees of the sectors connected to i. For simplicity, we will call it sector degree from now on. The degree of each sector is defined as follows:

Definition 6. *The out-degree of each sector i is given by: $d_i \equiv \sum_{j=1}^{n} w_{ji}$.*

This definition tells us that the degree of a sector i is given by the total number of sectors to whom i is providing inputs. Using the previous definition, we can also obtain the second-order sector out-degree.

Definition 7. *The second-order out-degree of each sector i is given by: $\widehat{d}_i \equiv \sum_{j=1}^{n} d_j w_{ji}$.*

This measure indicates how important is a sector as a provider of inputs to critical sectors. This result shows that the second-order out-degree of sector i will be higher as its sectors customers are providing inputs to a large number of sectors.

Now we define the out-degree of each bank b as the total number of sectors to whom b is providing capital.

Definition 8. *The out-degree of each bank b is given by:* $b_b \equiv \sum_{i=1}^{n} \phi_{ib}$.

As in the case of the sector out-degree, the bank out-degree of b is a measure of the connectivity of b as it is a weighted count of the number of sectors that are obtaining capital from b. The statistics giving the second-order bank out-degree is provided below.

Definition 9. *The bank-sector interconnectivity coefficient is given by:* $B_b \equiv \sum_{i=1}^{n} d_i \phi_{ib}$.

This definition is a measure of the interconnectivity between bank and sectors, and it indicates if each bank is providing capital to sectors with a higher or lower number of connections. The final measure is related to the interconnectivity of banks and sectors, but considering for the latter their relevance as potential hubs of the economy.

Definition 10. *The second-order bank-sector interconnectivity coefficient is given by:* $\widehat{B}_b \equiv \sum_{i=1}^{n} \widehat{d}_i \phi_{ib}$.

The statistics that we just defined can be computed for any given network structure. Such statistics will be relevant as we can express the volatility of GDP as a function of those.

1.6.2 Volatility and network metrics

The variance of GDP depends on the network characteristics, specifically, on the connections and interconnectivity as defined in our network metrics of the previous section.

Corollary 2. *Assuming that individual variance is the same across banks and economies, $\sigma^2 \forall (b \in M)$. The variance of the natural logarithm of GDP is given by:*

$$\sqrt{var(\ln Y)} \approx \sqrt{\left(\frac{\sigma\alpha}{n}\right)^2 \sum_{b=1}^{m} \left(\underbrace{b_b}_{\textit{Bank degree}} + \underbrace{(1-\alpha)B_b}_{\textit{Bank-sector int. coeff.}} + \underbrace{(1-\alpha)^2 \widehat{B_b}}_{\textit{2nd order bank-sector int. coeff.}} \right)^2} \tag{1.49}$$

We can see that the role of the variance of the process underlying the idiosyncratic shocks to any bank b on the total variance of GDP depends on the bank's outdegree, b_b, the bank-sector interconnectivity coefficient, B_b, and the second-order bank-sector interconnectivity coefficient $\widehat{B_b}$. The last two depend on the sector outdegree d_i. This result allows us to obtain the difference of volatilities across economies since it relates volatility with the network metrics that defined previously.

In the next sections, we compare different bipartite structures to analyse the effect on GDP volatility. We will see that depending on the type of comparison it is possible to ignore some metrics that are the same across economies. Additionally, one important result is that the network location might be crucial to evaluate the effect on the volatility of a change in the network topology.

1.6.3 Specific I-O networks

In this section, we define four distinct I-O networks to compare the volatility of different economic structures. We consider the networks n, but with a different structure of connections.

The vertical structure, IO_A, is obtained when connections follow a line from the first sector until the n-sector, node by node, while the connections of the first sector loops (Figure 1.6).

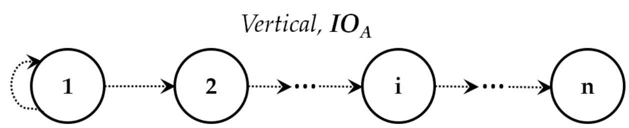


Figure 1.6: Vertical network

Definition 11. *The vertical network, $IO_A = (N, E_A, w_A)$, is given by:*

$$E_A = \left\{ \overrightarrow{\langle 1, 1 \rangle}, \overrightarrow{\langle i, i+1 \rangle}, \forall i \in (N \setminus n) \right\}$$

$$w_A = \{ w_{11} = w_{i+1,i} = 1, \forall i \in (N \setminus n) \}$$

The second typical structure is the star, IO_B, which is defined such that the first sector loops while it sells inputs to the rest of the sectors (Figure 1.7).

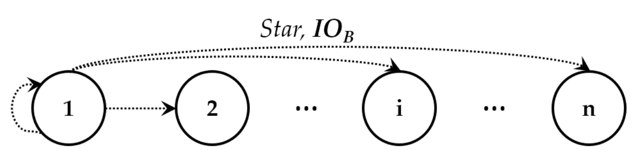


Figure 1.7: Star network

Definition 12. *The star network, $IO_B = (N, E_B, w_B)$, is given by:*

$$E_B = \left\{ \overrightarrow{\langle 1, i \rangle}, \forall i \in N \right\}$$

$$w_B = \{ w_{i1} = 1, \forall i \in N \}$$

The circle structure, IO_D, is also typical. It is obtained when the links form a closed chain from sector 1 to the sector n, sector by sector (Figure 1.8).

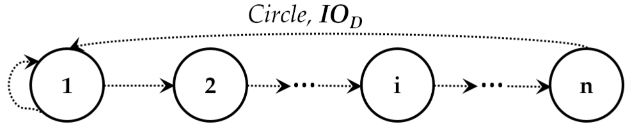


Figure 1.8: Circle network

Definition 13. *The circle network, $IO_D = (N, E_D, w_D)$, is given by:*

$$E_D = \left\{ \overrightarrow{\langle n, 1 \rangle}, \overrightarrow{\langle i, i+1 \rangle}, \forall i \in (N \setminus n) \right\}$$

$$w_D = \{ w_{1n} = w_{i+1,i} = 1, \forall i \in (N \setminus n) \}$$

Finally, the empty structure, IO_F, is given by a structure in which each sector loops, with no other additional linkages (Figure 1.9).

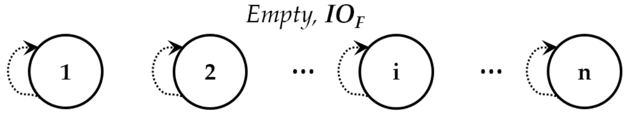


Figure 1.9: Empty network

Definition 14. *The empty network,* $IO_F = (N, E_F, w_F)$*, is given by:*

$$E_F = \left\{ \overrightarrow{\langle i, i \rangle}, \forall i \in N \right\}$$

$$w_F = \{ w_{ii} = 1, \forall i \in N \}$$

We assume, for all of the previous definitions, that all structures share the same N. The fact that all weights are equal to one follows from the assumption of a constant return to scale technology that implies $\sum_{j=1}^{n} w_{ij} = 1$, $\forall i$.

We can see that in the case of the I-O networks IO_A, IO_B and IO_D, there is no isolated node, all sectors are connected to each other indirectly, we call this component-1. The IO_F is different because all nodes are isolated, designated as component-n. To characterise this difference we use the following concept.

Definition 15. *A component-*io*, is a subgraph of the graph* IO*, such that for every couple of nodes,* i*,* j*, there exists a path. Note that a path is a set of edges that connects (directed or undirected) two or more nodes in a network.*

We can categorise the specific networks in two groups, according to the magnitude of the influence vector of directly connected sectors, namely as perfectly balanced and unbalanced networks.

Definition 16. *Any IO network with* $n > 3$ *will be considered perfectly balanced if* $v_i = v_j$*,* $\forall \overrightarrow{\langle i, j \rangle}$.

Definition 17. *Any IO network with* $n > 3$ *and component-*1 *will be considered perfectly unbalanced if* $v_i \geq v_j$*,* $\forall \overrightarrow{\langle i, j \rangle}$.

Where v_i is the i-th element of the influence vector. Applying to this notions to the definitions above, we see that each network belongs to one group only.

Lemma 1. *Circle, IO_D, and empty, IO_F, networks are perfectly balanced.*

Lemma 2. *Vertical, IO_A, and star, IO_B, networks are perfectly unbalanced.*

These results are useful to analyse the effect on GDP volatility of different I-O and capital intermediation structures. This because perfectly unbalanced and balanced networks have different multipliers of propagation over the nodes of the economy.

1.6.4 Specific bipartite networks

As mentioned earlier, we focus on the case in which there are no connections between banks. Although unrealistic, this assumption is useful to isolate the effect of the interaction between banks and sectors and focus on I-O trades as a mechanism of propagation of shocks from the banking sector to the productive side of the economy.

In this section, we define five specific arrangements for the bipartite network, $BN,$ linking banks to sectors. The location of the nodes will matter because we will merge the BN network with the IO network to compare the volatility of the economy, $\mathbb{E}$, under different arrangements.

The first bipartite network to consider is when each bank b lends to only one sector i. In order to allow all sectors to obtain loans, we assume for this configuration, denoted $BN_{1:1}$,that $n = m$.

Definition 18. *The bipartite network, $BN_{1:1} = (N, M, \overline{E}_{1:1}, \phi_{1:1})$, is given by:*

$$N = \left\{\mathbb{N}^{+} \leq n\right\}, M = \left\{\mathbb{N}^{+} \leq m, m = n\right\}, U = \left\{u \in \mathbb{N}^{+} : u \leq n, n \in N\right\}$$

$$\overline{E}_{1:1} = \left\{\overrightarrow{\langle b, i\rangle} : \overrightarrow{\langle u, u\rangle}, \forall u \in U\right\}$$

$$\phi_{1:1} = \left\{\phi_{ib} = 1, \forall \overrightarrow{\langle b, i\rangle} \in \overline{E}_{1:1}\right\}$$

Network $BN_{1:1}$ can be graphically represented as Figure 1.10.

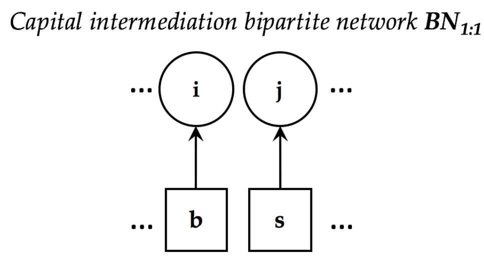


Figure 1.10: Bipartite network where each bank b lends to only one sector i.

This network configuration will allow to compare the volatility generated by different IO networks and therefore analyse the effect of I-O trades one under m links.

The next network is the modification of network $BN_{1:1}$ obtained by adding one link to only one bank. In other words, we consider that bank b is lending to sector i and j, whereas bank s is still lending to bank j, all other banks are lending to one different sector. This assumption implies that now banks b and s are sharing a proportion of the total lending to sector j, , $\phi \in (0, 1)$ and $1 - \phi$, for bank b and bank s, respectively. We call this graph $BN_{1:1,b(1:2)}$.

Definition 19. *The bipartite network, $BN_{1:1,b(1:2)} = (N, M, \overline{E}_{1:1,b(1:2)}, \phi_{1:1,b(1:2)})$, is given by:*

$$N = \left\{\mathbb{N}^+ \leq n\right\}, M = \left\{\mathbb{N}^+ \leq m, m = n\right\}, U = \left\{u \in \mathbb{N}^+ : u \leq n, n \in N\right\}$$

$$\overline{E}_{1:1,b(1:2)} = \left\{\overrightarrow{\langle b, i\rangle} : \overrightarrow{\langle u, u\rangle}, \forall u \in U\right\} \cup \left\{\overrightarrow{\langle b, j\rangle}\right\}$$

$$\phi_{1:1,b(1:2)} = \begin{cases} \phi_{ib} = 1, & \forall \overrightarrow{\langle s, i\rangle} \in \overline{E}_{1:1,b(1:2)} \setminus \left\{\overrightarrow{\langle b, j\rangle}, \overrightarrow{\langle s, j\rangle}\right\} \\ \phi_{jb} = \phi, & if\ \overrightarrow{\langle b, j\rangle} \\ \phi_{js} = 1 - \phi, & if\ \overrightarrow{\langle s, j\rangle} \end{cases}$$

Network $BN_{1:1,b(1:2}$ can be graphically represented as Figure 1.11.

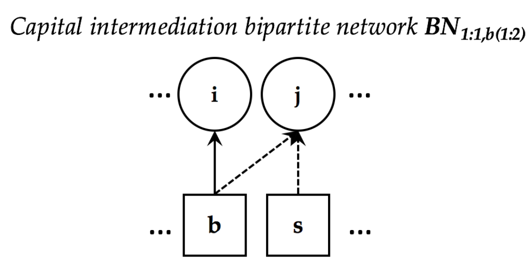


Figure 1.11: Bipartite network where one bank b lends to two sectors.

For the next three specific networks, we will assume that the number of banks, m, is equal to $n/2$, where n is the number of I-O sectors and that n is even, $mod(n, 2) = 0$. This assumption is innocuous as if n is an odd number, the effect of the last sector will always cancel out in the comparisons of volatility.

The third network is also a variation of the first one, $BN_{1:1}$, but now obtained by adding one link per bank. Each bank then provides capital to two sectors, with banks b and s sharing a portion of the lending. More specifically, we are assuming that only two banks share two common sectors and the same weights for the common links, ϕ and $1-\phi$. We call this network $BN_{2:2}$.

Definition 20. *The bipartite network, $BN_{2:2} = (N, M, \overline{E}_{2:2}, \phi_{2:2})$, is given by:*

$$N = \left\{\mathbb{N}^+ \leq n\right\}, M = \left\{\mathbb{N}^+ \leq m, m = n\right\}, U = \left\{u \in \mathbb{N}^+ : u \leq n, n \in N\right\}$$

$$\overline{E}_{2:2} = \left\{\overrightarrow{\langle b, i\rangle} : \overrightarrow{\langle u, u\rangle}, \overrightarrow{\langle u, u+1\rangle} \& mod(u,2) = 1, \overrightarrow{\langle u, u-1\rangle} \& mod(u,2) = 0, \forall u \in U\right\}$$

$$\phi_{1:2} = \left\{\phi_{ib} = 1-\phi, \forall \overrightarrow{\langle b, i = b\rangle} \in \overline{E}_{1:2}\right\} \cup \left\{\phi_{ib} = \phi, \forall \overrightarrow{\langle b, i \neq b\rangle} \in \overline{E}_{1:2}\right\}$$

Network $BN_{2:2}$ can be graphically represented as Figure 1.12.

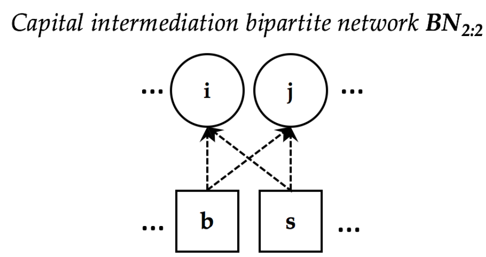


Figure 1.12: Bipartite network where each bank lends to two sectors.

We consider a bipartite network where each bank provides capital to two sectors, and we call it $BN_{1:2}$.

Definition 21. *The bipartite network, $BN_{1:2} = (N, M, \overline{E}_{1:2}, \phi_{1:2})$, is given by:*

$$N = \left\{\mathbb{N}^+ \leq n\right\}, M = \left\{\mathbb{N}^+ \leq m, m = n/2\right\}$$

$$\overline{E}_{1:2} = \left\{ \overrightarrow{\langle b,i \rangle}, \overrightarrow{\langle b,j \rangle} : i \neq j, \nexists \left(\overrightarrow{\langle s,i \rangle} \parallel \overrightarrow{\langle s,j \rangle} \right), \forall (b,s) \in M, \forall (i,j) \in N \right\}$$

$$\phi_{1:2} = \left\{ \phi_{ib} = 1, \forall \overrightarrow{\langle b,i \rangle} \in \overline{E}_{1:2} \right\}$$

Again $BN_{1:2}$ can be graphically represented as Figure 1.13.

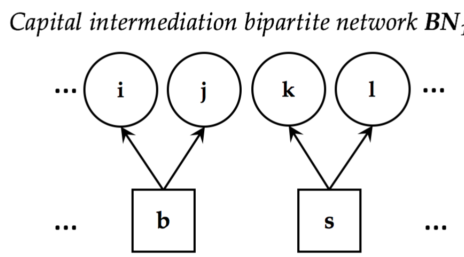


Figure 1.13: Bipartite network where each bank lends to different sectors.

The last network that we study is a variation of the previous one, $BN_{1:2}$. The difference is that now one bank, b, is lending to three sectors while the other bank, s, is lending to only one sector and each of the rest of the banks is lending to two different sectors. This bipartite network is designated as $BN_{1:2,b(1:3),s(1:1)}$.

Definition 22. *The bipartite network* $BN_{1:2,b(1:3),s(1:1)} = (N, M, \overline{E}_{1:2,b(1:3),s(1:1)}, \phi_{1:2,b(1:3),s(1:1)})$, *is given by:*

$$N = \left\{ \mathbb{N}^+ \leq n \right\}, M = \left\{ \mathbb{N}^+ \leq m, m = n/2 \right\}, U = \left\{ u \in \mathbb{N}^+ : 5 \leq u \leq m, m \in M \right\}$$

$$\overline{E}_{1:2,b(1:3),s(1:1)} = \left\{ \overrightarrow{\langle b,i \rangle} : \overrightarrow{\langle u, 2u-1 \rangle}, \overrightarrow{\langle u, 2u \rangle}, \forall u \in U \right\} \cup \left\{ \overrightarrow{\langle b,k \rangle} \right\} \setminus \left\{ \overrightarrow{\langle s,k \rangle} \right\}$$

$$\phi_{1:2,b(1:3),s(1:1)} = \left\{ \phi_{ib} = 1, \forall \overrightarrow{\langle b,i \rangle} \in \overline{E}_{SC1} \right\}$$

Network $BN_{1:2,b(1:3),s(1:1)}$ can be graphically represented as Figure 1.14.

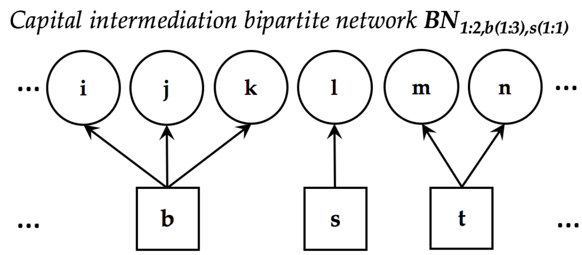


Figure 1.14: Bipartite network where each bank lends to different number sectors.

The specific graphs defined so far will allow analysing the effect of adding one link to specific locations and redistributing the links to different positions in the network. In particular, using $BN_{1:1,b(1:2)}$ and $BN_{2:2}$ we can compare the effect of sharing the provision of capital to a common sector. Finally, using the last two networks, we can examine the effect of diversifying the provision of capital by banks.

1.7 Effect of network on volatility of GDP

To analyse how the network of input-output linkages affect the aggregate volatility and the role played by the connections of banks to sectors, we use the networks structures and metrics defined previously. In this section, we assume that parameters α, n, and m are the same across all economies. Note that the only relevant parameters to characterise the volatility of GDP are σ_b^2, ϕ_{ib}, and v_i, for all $i \in N$ and $b \in M$.

1.7.1 Financial integration

Financial integration is analysed by looking at the effect of adding links between banks and sectors. In particular, the effect of adding one link for only one bank and the effect of adding one link for all banks. In fact, adding one link in a bipartite network in which all sectors are already connected to a bank is equivalent to integrate two banks using common links to one sector. In other words, we define integration as when two or more banks provide capital to one or more common sectors. In this way, we can compare the effect of volatility of integrating banks.

The baseline economies, before adding links are defined as follows.

Definition 23. *We define the following configurations:*

1. *Vertical I-O and Bipartite with one link per bank:* $\mathbb{E}_{A,1:1} = (BN_{1:1} \cup IO_A, \mathbf{\Sigma})$.

2. *Star I-O and Bipartite with one link per bank:* $\mathbb{E}_{B,1:1} = (BN_{1:1} \cup IO_B, \mathbf{\Sigma})$.

3. *Circle I-O and Bipartite with one link per bank:* $\mathbb{E}_{D,1:1} = (BN_{1:1} \cup IO_D, \mathbf{\Sigma})$.

4. *Empty I-O and Bipartite with one link per bank:* $\mathbb{E}_{F,1:1} = (BN_{1:1} \cup IO_F, \mathbf{\Sigma})$.

Where the matrix $\mathbf{\Sigma}$ is the covariance matrix of the shocks, which is assumed to be the same for all economies.

1.7.1.a Integration: Adding one link

In this section, we analyse the effect on the volatility of adding one link for only one bank. The formal analysis of the role of the location of the link on the volatility of GDP requires defining the economies resulting from the addition of one link for bank b.

Definition 24. *We define the following configurations*

1. $\mathbb{E}_{A,1:1,b(1:2)} = (BN_{1:1,b(1:2)} \cup IO_A, \mathbf{\Sigma})$.

2. $\mathbb{E}_{B,1:1,b(1:2)} = (BN_{1:1,b(1:2)} \cup IO_B, \mathbf{\Sigma})$.

3. $\mathbb{E}_{D,1:1,b(1:2)} = (BN_{1:1,b(1:2)} \cup IO_D, \mathbf{\Sigma})$.

4. $\mathbb{E}_{F,1:1,b(1:2)} = (BN_{1:1,b(1:2)} \cup IO_F, \mathbf{\Sigma})$.

We can give the following graphical representation (Figure 1.15).

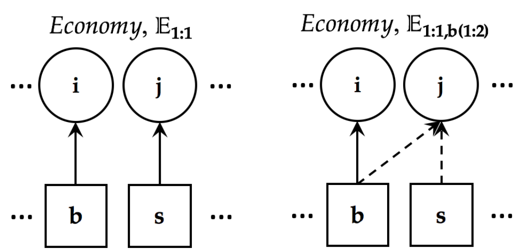


Figure 1.15: Integration

The left diagram is the baseline economy defined at the beginning of the section while the second diagram represents the economy when we add one link. It should be noted that we are not specifying yet the location of the link bank b-sector i in the I-O network as the site will be the variable under consideration. We will focus on four structures, the vertical, star, circle and empty networks.

First, we adopt the vertical I-O structure and assume that n is large.

Proposition 3. *Consider the vertical economies $\mathbb{E}_{A,1:1}$ and $\mathbb{E}_{A,1:1,b(1:2)}$ with $n > 2$ and assume that a uniform idiosyncratic variance, $\sigma_b^2 = \sigma^2$. The difference in volatilities is defined as:*

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \ \textit{if} \ \ (\{i,j\} \neq \{1,n\}) \parallel (i \neq \{1,n\}, j = n) \parallel (i = 1, j \neq 1)$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \ \textit{if} \, [(i \neq 1, j = 1)\&(1-\phi < \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi < \alpha^2)]$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} > [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \ \textit{if} \, [(i \neq 1, j = 1)\&(1-\phi > \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi > \alpha^2)]$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} = [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \ \textit{if} \, [(i \neq 1, j = 1)\&(1-\phi = \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi = \alpha^2)]$$

The above result indicates that the difference in volatility when we add one link, keeping the same vertical structure, depends on the position of the new link. Recalling that the new link represents the integration between two banks that share a common sector. Volatility increases if the integrator sector is other than the top of the vertical chain, or if the common sector of the two integrated banks is at the bottom of the chain.

Additionally, volatility is going to increase if the common sector between integrated banks is the top of the chain or if the non-common sector is the bottom of the chain and a specific condition about the parameter is met ($1-\phi < \alpha^2$). This last situation changes if the status about the parameter changes, yielding a decreasing volatility or no effect at all.

The next comparison is performed using the star network.

Proposition 4. *Consider star economies $\mathbb{E}_{B,1:1}$ and $\mathbb{E}_{B,1:1,b(1:2)}$ with $n > 2$. Assume a* *uniform idiosyncratic variance, $\sigma_b^2 = \sigma^2$. Then the difference in volatilities is:*

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \textit{if} \quad (\{i,j\} \neq 1) \parallel (i = 1, j \neq 1)$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \textit{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) < \alpha\phi$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} > [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \textit{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) > \alpha\phi$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} = [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \textit{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) = \alpha\phi$$

Also, in the case of star economies, the effect of adding one link on the volatility difference depends on the position of the new link. The standard deviation increases if the two integrated banks are lending to satellites sectors, that is, sectors other than the centre of the star. Volatility also increases when the non-common sector is the centre of the star. The difference in volatilities also depends on the location and the value of specific parameters. Volatility increases when the common sector is the centre of the star and $n(1-\phi)(1-\alpha) < \alpha\phi$. On the contrary if $n(1-\phi)(1-\alpha) > \alpha\phi$ volatility is going to decrease after integration. There is no effect if the relationship holds with an equal sign.

Finally, when we compare empty and circle economies, the location does not matter.

Proposition 5. *Consider the empty and the circle economies $\mathbb{E}_{D,1:1}$, $\mathbb{E}_{F,1:1}$, $\mathbb{E}_{D,1:1,b(1:2)}$ and $\mathbb{E}_{F,1:1,b(1:2)}$ with $n > 2$. Assuming a uniform idiosyncratic variance, $\sigma_b^2 = \sigma^2$. Then* *the difference in volatilities is:*

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{D,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{D,1:1,b(1:2)}}$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{F,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{F,1:1,b(1:2)}}$$

The previous result shows that adding one link to integrate two banks will increase output volatility. The effect of adding one link increases volatility in the case of a circle or an empty economy.

To gain some intuition on the previous three propositions, we need to analyse how volatility depends on the network. The comparison in volatility for all the previous configurations can be written as:

$$\left[\underbrace{v_i^2}_{\text{influence bank b}} + \underbrace{v_j^2}_{\text{influence bank s}}\right]_{before} \lesseqgtr \left[\underbrace{(v_i + v_j\phi)^2}_{\text{influence bank b}} + \underbrace{(1-\phi)^2 v_j^2}_{\text{influence bank s}}\right]_{after}$$

After adding one link between bank b and sector j, the network multiplier of bank b increases and the one of bank s decreases, since $\phi \in (0,1)$:

$$0 \lesseqgtr \underbrace{(v_i + v_j\phi)^2 - v_i^2}_{\text{change influence bank b}} + \underbrace{(1-\phi)^2 v_j^2 - v_j^2}_{\text{change influence bank s}}$$

Therefore determine the direction of the change in volatility when one link is added, we need to determine which effect dominates

$$0 < \underbrace{2v_iv_j\phi + v_j^2\phi^2}_{\text{change influence bank b}} + \underbrace{-2v_j^2\phi + v_j^2\phi^2}_{\text{change influence bank s}} \quad \text{true iff} \quad v_j(1-\phi) < v_i$$

Both banks receive the same individual influence from sector j ($v_j^2\phi^2$), but the gain of influence by interacting for bank b ($2v_iv_j\phi$) is greater than the loss of influence of bank s ($-2v_j^2\phi$) if the influence of sector i is equal to or higher than sector j. This result occurs when bank b is lending capital to the most important sector (the centre of the star or the top of the chain), or if the I-O structure is empty or circle.

1.7.1.b Integration: Adding m-links

In this section, we analyse the effect of adding m links, one link per bank. The resulting structure sees each bank providing capital to two sectors, and each sector receiving loans from two banks. We assume that the variance of the idiosyncratic shock σ_b^2 is the same across banks and economies, $\sigma^2 \forall (b \in M)$.

The baseline economies are those defined at the beginning of the section, i.e., $\mathbb{E}_{1:1}$. The economies resulting from the addition of m links are set below where we assume the same vector of shocks $\mathbf{\Sigma}$ for all cases.

Definition 25. *We need the following definition:*

1. $\mathbb{E}_{A,2:2} = (BN_{2:2} \cup IO_A, \mathbf{\Sigma})$.

2. $\mathbb{E}_{B,2:2} = (BN_{2:2} \cup IO_B, \mathbf{\Sigma})$.

3. $\mathbb{E}_{D,2:2} = (BN_{2:2} \cup IO_D, \mathbf{\Sigma})$.

4. $\mathbb{E}_{F,2:2} = (BN_{2:2} \cup IO_F, \mathbf{\Sigma})$.

These configurations can be illustrated as follows (Figure 1.16).

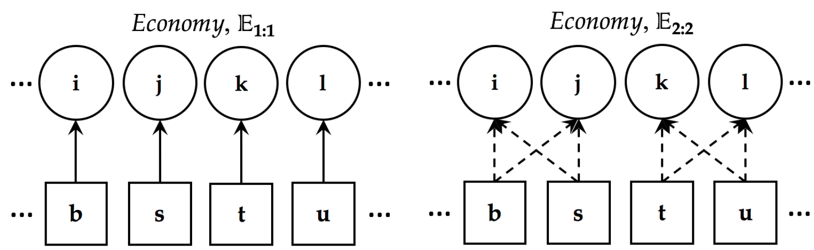


Figure 1.16: Integration: adding m-links

The left diagram represents the baseline economy, for a given I-O structure. The right diagram is the economy resulting from the addition of m links, one per bank.

Given our stated assumptions, we can show the following.

Proposition 6. *Consider economies $\mathbb{E}_{A,1:1}$, $\mathbb{E}_{B,1:1}$, $\mathbb{E}_{A,2:2}$ and $\mathbb{E}_{B,2:2}$, with $n = m$ and $n > 3$. Assume uniform idiosyncratic variance, $\sigma_b^2 = \sigma^2$, for all banks. Then volatilities satisfy*

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:1}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{A,2:2}}$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:1}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,2:2}}$$

This result shows that for economies with perfectly balanced I-O structures, namely $\mathbb{E}_{A,1:1}$ and $\mathbb{E}_{B,1:1}$, there is scope to reduce volatility using integration of two banks, as long as at least one link per bank increases. In fact, as shown in the proof of this proposition, to reduce volatility it is enough to integrate two banks by adding one link per bank and having two common sectors, provided the banks have different network multipliers originated from an asymmetry in the I-O network. On the other hand, there is no possibility to reduce volatility if the I-O structure is perfectly balanced as in the case of the empty and circle economies.

Proposition 7. *Consider economies $\mathbb{E}_{D,1:1}$, $\mathbb{E}_{F,1:1}$, $\mathbb{E}_{D,2:2}$ and $\mathbb{E}_{F,2:2}$, with $n = m$ and $n > 3$. Assume a uniform idiosyncratic variance, $\sigma_b^2 = \sigma^2$, for all banks. Then the difference in volatilities is:*

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:1}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{D,2:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:1}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{F,2:2}}$$

The previous results hold because all sectors have the same influence vector and with bipartite networks, the overall network is the same.

In order to gain more intuition we compare the network multipliers of each structures as this will allow us to understand the process behind the change in volatility when many links are added. The volatility change due to the addition of m links for each typical pair of banks (b and s) is given by the following:

$$\left[\underbrace{v_i^2}_{\text{influence bank b}} + \underbrace{v_j^2}_{\text{influence bank s}}\right]_{before} \lesseqgtr \left[\underbrace{\phi^2 \left(v_i + v_j\right)^2}_{\text{influence bank b}} + \underbrace{\left(1-\phi\right)^2 \left(v_i + v_j\right)^2}_{\text{influence bank s}}\right]_{after}$$

Re-arranging the previous inequality, the change in influence of each pair is given by:

$$0 \lesseqgtr \underbrace{\phi^2 \left(v_i + v_j\right)^2 - v_i^2}_{\text{change influence bank b}} + \underbrace{(1-\phi)^2 \left(v_i + v_j\right)^2 - v_j^2}_{\text{change influence bank s}}$$

Recalling assumption $\phi_{ib} = \phi, \forall \{i, b\}$, the joint change for each pair is:

$$0 \geq \quad \underbrace{-(v_i - v_j)^2}_{\text{joint change bank b and s}}$$

Therefore, the influence of each pair of integrated banks decreases if, and only if, $v_i \neq v_j$. In other words, adding links reduce volatility as long as the influence of the two relevant sectors are different, in this case, new links distribute the impact and reduce the total effect of the shock. This finding applies to the star and vertical I-O networks because the influence of each sector is different, while there is no effect in empty or circle I-O.

1.7.2 Link redistribution

In this section, we analyse the effect of redistributing the existing links in the bipartite network. As when considering links, the location plays a fundamental role in the effect of redistributing links on volatility.

1.7.2.a Diversification

We first analyse the role of the position of the banks within the whole network in the resulting aggregate volatility when at least two links are redistributed. We show that indeed the specific linkages between the sectors to whom each bank provides capital are important for the magnitude of volatility when redistributing at least two connections, everything else equal.

We define as an increase in diversification the increase in the bank-sector interconnectivity coefficient that emerges from changing the positions of the connections of banks. The analysis requires that the interconnectivity importance of each bank be different, so that in the comparisons we use the second order approximation of the influence vector, as discussed in the appendix.

To analyse the effect of diversification on aggregate volatility we compare economies with the same topology of input-output linkages but with different bipartite networks. We decompose the analysis in two cases: 1) the case in which banks provide capital to sectors with the same influence vector, and 2) the case at which banks lend to sectors with different influence vector.

To provide precise results, I focus on two types of I-O networks, the vertical and the circle. Note that this allows us to consider a perfectly unbalanced I-O network and a perfectly balanced I-O network.

Definition 26. *We consider economies with the I-O vertical structure, IO_A, and with the circle network, IO_D. All economies have the same bipartite network, $BN_{1:2}$, but differ in the link's location. They have the same matrix of shocks, $\mathbf{\Sigma}$. The notation is as follows.*

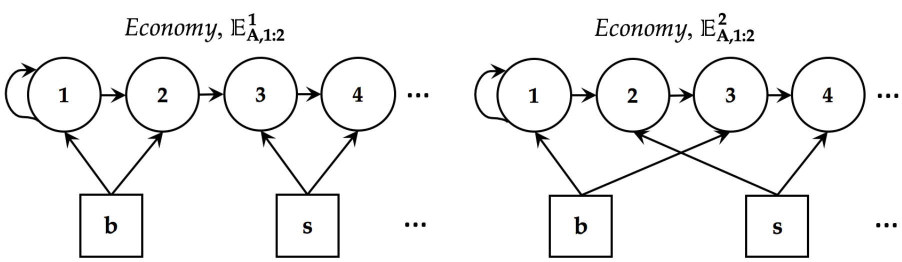
1. *Vertical I-O, bipartite (i.e., two links per bank, one per sector):* $\mathbb{E}_{A,1:2} = (BN_{1:2} \cup IO_A, \mathbf{\Sigma})$

2. *Circle I-O, bipartite (i.e., two links per bank, one per sector):* $\mathbb{E}_{D,1:2} = (BN_{1:2} \cup IO_D, \mathbf{\Sigma})$

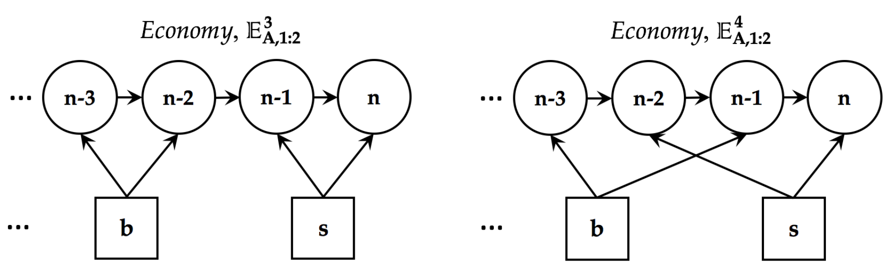
Assuming that the distribution of the bank's shock has the same variance across banks and economies, $\sigma_b^2 = \sigma^2$, for all b, we obtain the following comparison of volatilities for the same IO_A but different bipartite networks.

Proposition 8. *Consider the economy $\mathbb{E}_{A,1:2}$, with $n > 3$ and $m = n/2$, and assume that the variance of idiosyncratic shock is same across economies, $\sigma_b^2 = \sigma^2$. If the links of two banks, b and s, are redistributed as shown in the diagrams below (Figure **??**), the difference in volatilities of GDP is:*

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{A,1:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}^2_{A,1:2}}$$



$$\sqrt{var(\ln Y)}_{\mathbb{E}^3_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}^4_{A,1:2}}$$



The validity of the result rests on the difference in the influence vector affecting the network multipliers of each bank. In the first comparison, only sector 1

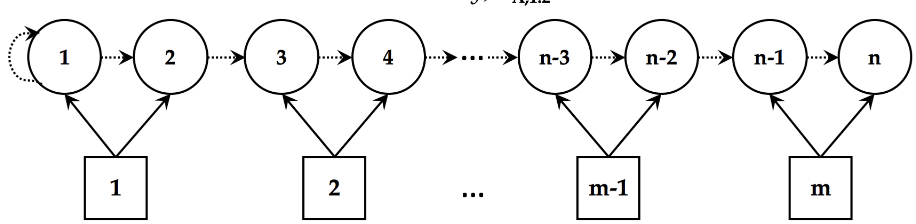
has a different influence vector then the others, so there is no benefit from diversification. In the second comparison, not only the n^{th} sector has different influence but also the $n-1^{th}$ has a different influence in network multipliers. Redistributing the links as illustrated reduces the volatility because the network multiplier has decreased.

For the next comparison, we still use the vertical structure, but the links of all banks are moved horizontally as illustrated by the graph.

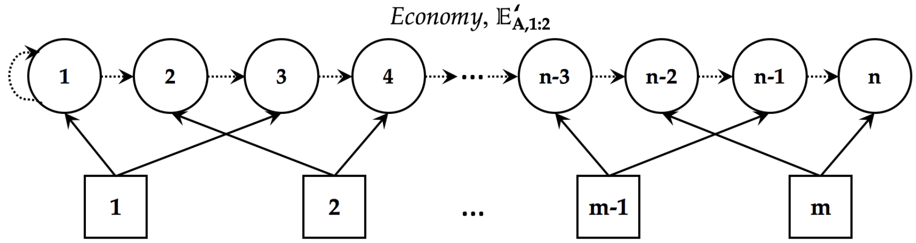
Proposition 9. *Consider the economy $\mathbb{E}_{A,1:2}$, with $n > 3$ and $m = n/2$ and assume that variance of idiosyncratic shock is same across economies, $\sigma_b^2 = \sigma^2$. If the links of all banks are redistributed as shown in the diagrams below, the difference in volatilities of GDP is:*

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}'_{A,1:2}}$$

Economy, $\mathbb{E}^1_{A,1:2}$



Economy, $\mathbb{E}'_{A,1:2}$

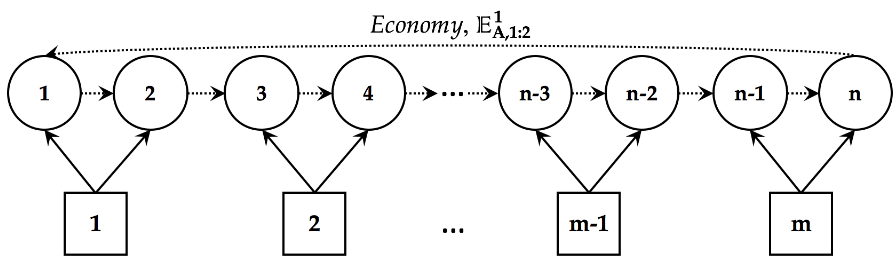


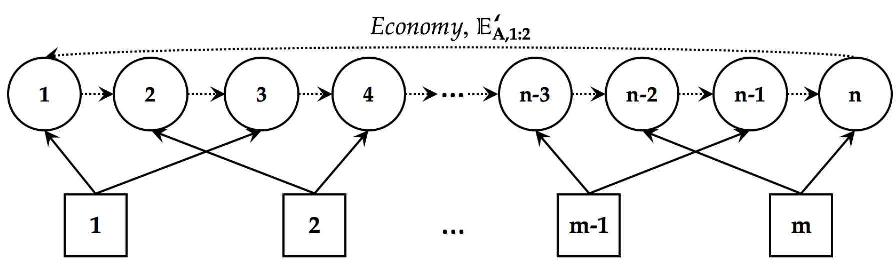
The previous result originates from the fact that at the bottom of the chain there are two sectors that have a different influence than all the remaining sectors, this affects the network multipliers and hence on total volatility.

In the last comparison, we use the I-O circle network. In this structure, the influence of each sector is the same, so that there is no advantage to redistribute the links for the banks.

Proposition 10. *Consider the economy $\mathbb{E}_{D,1:2}$, with $n > 3$ and $m = n/2$, and assume that variance of idiosyncratic shock is same across economies, $\sigma_b^2 = \sigma^2$. If the links of all banks are redistributed as shown in the diagrams below, there is no change in GDP volatility:*

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{D,1:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}'_{D,1:2}}$$





This result tells us that under perfectly balanced I-O structures there is no benefit from diversification, at least in the way we defined diversification. This result rests on the fact that each element of the influence vector of i in any perfectly balanced I-O network is the same for all sectors implying that the interconnectivity importance is the same in any position in the network.

To analyse the three previous propositions, we compare the network multipliers of the structures we considered to evaluate the process behind the change in volatility. In the first economy, we assume that two typical pair of banks, b and s, are lending capital to two sectors each, i and j for b and k and l for s. After diversification, these same banks will lend capital to sectors i and k, in the case of b, and j and l, in the case of s. This ordering represents all the possibilities that we analysed in the previous propositions.

The volatility comparison -before and after moving links- for each typical pair of banks (b and s) is:

$$\left[\underbrace{(v_i + v_j)^2}_{\text{influence bank b}} + \underbrace{(v_k + v_l)^2}_{\text{influence bank s}}\right]_{before} \lesseqgtr \left[\underbrace{(v_i + v_k)^2}_{\text{influence bank b}} + \underbrace{(v_j + v_l)^2}_{\text{influence bank s}}\right]_{after}$$

After re-arranging the previous expression, the change in influence of each pair is given by:

$$0 \lesseqgtr \underbrace{(v_i + v_k)^2 - (v_i + v_j)^2}_{\text{change influence bank b}} + \underbrace{(v_j + v_l)^2 - (v_k + v_l)^2}_{\text{change influence bank s}}$$

After some simplification, the joint change for each pair is

$$0 \lesseqgtr \underbrace{(v_k - v_j)(v_i - v_l)}_{\text{joint change bank b and s}}$$

The change is positive if, and only if, $\{v_k > v_j\}\&\{v_i > v_l\}$ and the change is negative iff $\{v_k > v_j\}\&\{v_i < v_l\}$ or $\{v_k < v_j\}\&\{v_i > v_l\}$. There is no effect if the sectorial influence of the pair is the same: change in volatility requires a difference in sector's influence. More precisely, volatility increases when bank b is lending to two sectors with similar (to s) larger influence. Volatility decreases when b abandons a sector with a more considerable influence and while keeping a sector with low influence. The effect of diversification is obtained considering the distribution of sectoral influence achieved by each bank.

1.7.2.b Concentration

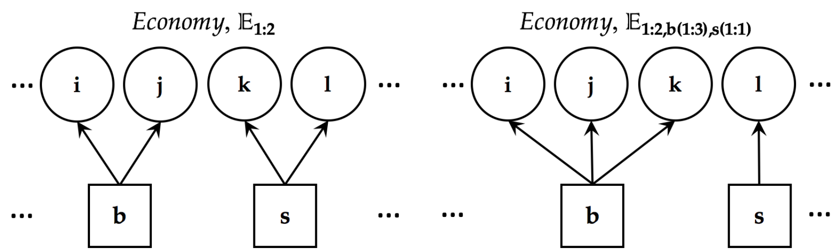
The last comparison of this section is between economies with the same structure of connections among non-financial sectors but that have a different number of connections per bank. The aim of this comparison is to evaluate how the distribution of links between the banks and the sectors is remarkable for the magnitude of the volatility.

We define high concentration as an unequal distribution of links per bank and no concentration as a uniform distribution of connections for each bank. To simplify the comparisons, we consider only perfectly balanced and perfectly unbalanced I-O structures, the explicit I-O structure being specified in the result.

Holding the same I-O network we compare economies that have different bipartite graphs. For the first type of economy, $\mathbb{E}_{1:2}$, we consider the graph $BN_{1:2}$. In the second type, $\mathbb{E}_{1:2,b(1:3),s(1:1)}$ we assume the bipartite network $BN_{1:2,b(1:3),s(1:1)}$, such economies defined as follows.

Definition 27. $\mathbb{E}_{1:2} = (BN_{1:2} \cup IO, \mathbf{\Sigma})$

Definition 28. $\mathbb{E}_{1:2,b(1:3),s(1:1)} = (BN_{1:2,b(1:3),s(1:1)} \cup IO, \mathbf{\Sigma})$



The difference between economies $\mathbb{E}_{1:2}$ and $\mathbb{E}\mathbb{E}_{1:2}$ to economy $\mathbb{E}_{1:2,b(1:3),s(1:1)}$ implies an increase in concentration. In the following comparisons, all banks other than b and s keep the same connections and location in both economies.

Proposition 11. *Consider economies $\mathbb{E}_{1:2}$ and $\mathbb{E}_{1:2,b(1:3),s(1:1)}$ where the I-O structure is either the circle, $I-O_D$, or the empty network, $I-O_F$, with $n > 3$ and $m = n/2$. Assume that the variance of the idiosyncratic shock is uniform, $\sigma_b^2 = \sigma^2$, across banks and economies. Then the difference in volatilities is*

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:2,b(1:3),s(1:1)}}$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:2,b(1:3),s(1:1)}}$$

The previous result reveals that the location plays no role in the effect of concentration on volatility, provided the structure be perfectly balanced. On the other hand, in the case of perfectly balanced structures, it indicates that increasing the degree of concentration yields an increase in the magnitude of aggregate volatility and this independently of the position of the change. Indeed, just modifying the distribution between the links of two banks makes the volatility greater in such structures.

In the next two comparisons, we use two perfectly unbalanced structures.

Proposition 12. *Consider the economies* $\mathbb{E}_{1:2}$ *and* $\mathbb{E}_{1:2,b(1:3),s(1:1)}$*, where the I-O structure is the star network ,*$I - O_B$*, with* $n > 3$ *and* $m = n/2$*. Assume that the variance of the idiosyncratic shock is the same,* $\sigma_b^2 = \sigma^2$*, across banks and economies. The difference in volatilities is*

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$$

if $(\{i = 1 \parallel j = 1\}\&l \neq 1) \parallel \{i, j, l\} \neq 1 \parallel \big(\{i, j\} \neq 1\&l = 1\&n(1-\alpha)(2-\alpha)^2 < 1\big)$

$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$ *if* $\{i, j\} \neq 1\&l = 1\&n(1-\alpha)(2-\alpha)^2 > 1$

$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$ *if* $\{i, j\} \neq 1\&l = 1\&n(1-\alpha)(2-\alpha)^2 = 1$

Proposition 13. *Consider* $\mathbb{E}_{1:2}$ *and* $\mathbb{E}_{1:2,b(1:3),s(1:1)}$*, where the I-O structure is the vertical network ,*$I - O_A$*, with* $n > 3$ *and* $m = n/2$*. Assume that the variance of the idiosyncratic shock is the same,* $\sigma_b^2 = \sigma^2$*, across banks and economies. Then the difference in volatilities is:*

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

$$\textit{if } [l \neq 1] \parallel [l = 1\&\{i, j\} \neq \{n, n-1\}] \parallel \big[l = 1\&\{i, j\} = \{n, n-1\}\&3(1-\alpha)^2 < \alpha\big]$$
$$\parallel \big[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\&\big(\big(\{i \parallel j\} = n\&2(1-\alpha)^2 < \alpha\big) \parallel \big(\{i \parallel j\} = n-1\&2(1-\alpha)^2 < 1\big)\big)\big]$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

$$\textit{if}\left[l=1\&\{i,j\}=\{n,n-1\}\&3(1-\alpha)^2>\alpha\right]$$
$$\parallel\left[l=1\&\{i\parallel j\}\neq\{n,n-1\}\&\left(\left(\{i\parallel j\}=n\&2(1-\alpha)^2>\alpha\right)\parallel\left(\{i\parallel j\}=n-1\&2(1-\alpha)^2>1\right)\right)\right]$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}}=\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

$$\textit{if}\left[l=1\&\{i,j\}=\{n,n-1\}\&3(1-\alpha)^2=\alpha\right]$$
$$\parallel\left[l=1\&\{i\parallel j\}\neq\{n,n-1\}\&\left(\left(\{i\parallel j\}=n\&2(1-\alpha)^2=\alpha\right)\parallel\left(\{i\parallel j\}=n-1\&2(1-\alpha)^2=1\right)\right)\right]$$

The two results above show that the difference in volatilities when there is a redistribution of links between two banks such that the number of links for one bank is increased depends strongly on the location of such banks and their links. We found that the majority of the locations, under some conditions, generate an increase in volatility when the concentration of links on one bank increases.

However, it is important to highlight that it is not always the case that a higher degree of concentration implies a greater volatility when the structure is perfectly unbalanced. In fact, we found that in the majority of cases when the remaining sector of the bank that lost a link is the centre of the star, or the top of the chain, the volatility of such economies is going to decrease. To see why this is happening, we can look at the networks multipliers for a relevant pair of banks (b and s):

$$\left[\underbrace{(v_i+v_j)^2}_{\text{influence bank b}}+\underbrace{(v_k+v_l)^2}_{\text{influence bank s}}\right]_{before}\lesseqgtr\left[\underbrace{(v_i+v_j+v_k)^2}_{\text{influence bank b}}+\underbrace{v_l^2}_{\text{influence bank s}}\right]_{after}$$

After moving one link, sector k and bank b are now connected, and the network multiplier of bank b increases while the one of bank s decreases:

$$0 \lesseqgtr \underbrace{(v_i + v_j + v_k)^2 - (v_i + v_j)^2}_{\text{change influence bank b}} + \underbrace{v_l^2 - (v_k + v_l)^2}_{\text{change influence bank s}}$$

Re-arranging the previous inequality we can see which effect dominates and under what condition:

$$0 < \underbrace{2v_k(v_i + v_j) + v_k^2}_{\text{change influence bank b}} + \underbrace{-2v_k v_l - v_k^2}_{\text{change influence bank s}} \quad \text{true iff:} \quad v_l < v_i + v_j$$

Both banks b and s have the same individual change from sector k (v_k^2) with opposite sign. However, the gain from the interacting effect for bank b ($2v_k(v_i+v_j)$) is greater than the loss for bank s ($-2v_k v_l$) when the joint influence of the original sectors that receive loans from bank b is greater than the influence of the remaining sector that receive loans from bank s ($v_l < v_i + v_j$). This result is always true for the circle and the empty network, and for the vertical and the star network when l is not the most dominant sector.

In light of the previous results, we can conclude that economies that have a bank with a greater number of links is not always going to imply higher volatility, especially when considering asymmetric I-O structures and with different location of banks and links.

1.8 Empirical exercise

In this section, we analyse the structure of the U.S. economy and its changes between 2000 and 2010. In particular, I focus on the Input-Output linkages and the banks-to-sector lending. To I-O linkages are well documented and represented as the $\mathbf{W}$ matrix we have seen before. For the bank-to-sector lending where we use data new data and compute the capital intermediation matrix $\mathbf{\Phi}$.

To construct the I-O matrix, note that a typical element of the matrix $\mathbf{W}$, w_{ij}, in fact, represents the proportion of inputs from the sector j used in the production of sector i. This information for the U.S is provided by the Bureau of Economic Analysis. In the case of bank-to-sector lending, the typical element of the matrix $\mathbf{\Phi}$, ϕ_{ib}, indicates the proportion of lending by bank b in the total borrowing of sector i. This data is not widely available as most of the data that could be used to build such a matrix is proprietary or not disclosed. The strategy of this paper is to create the $\mathbf{\Phi}$ matrix using syndicated loans data, as we are going to describe later.

Once the $\mathbf{W}$ and $\mathbf{\Phi}$ matrices are made available for the different years under consideration, we can analyse the effect of changes in the connections over time have on the aggregated volatility of the economy. As we will see, in fact knowing the elements w_{ij} and ϕ_{ib} for each period and the proportion of input used in the total production $1 - \alpha$, is sufficient to obtain the variation in GDP volatility over the years.

1.8.1 Data

The I-O matrices are constructed using the annual input-output data from the Bureau of Economic Analysis. In particular, I focus on the data about the use of commodities by industry, valued at producer's prices, first for 71 NAICS industries and later for 14 aggregated sectors.[4] The input-output data is originally in dollars, thus, to obtain the **W** matrix we had to compute the proportions of intermediate use for each sector. Note that the financial sector is excluded from this computations as this sector is explicitly model and the relevant connections are in the bank-sector intermediation matrix.

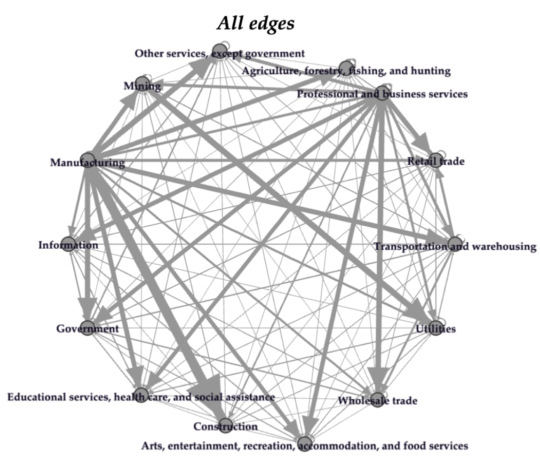


Figure 1.17: I-O Network of the U.S. in 2006

Figure 1.17 shows **W** matrix at the 14 sectors level. The graph representing the full network, including all edges, is a very dense. As many edges are minuscule, imposing a lower threshold of 0.1 makes the graph easier to read and highlights star sectors as manufacturing and professional services.

[4] http://www.bea.gov/industry/io_annual.htm

Another interesting insight is that when considering only edges greater than 0.1, the I-O output structure, or the $\mathbf{W}$ matrix, is nearly constant over time. For example, the right-hand part of the diagram for 2006 and 2010 show a subtle variation in the weights.

The input-output tables also give the average proportion of intermediate goods in the total production, $1-\alpha$, discounting labour as in the model labour does not add value. On average 60% of total production is accounted by the value of inputs implying that the value of α should be 0.4.

The bank-to-sector intermediation matrix, $\mathbf{\Phi}$, is constructed from syndicated loans data. Loans are provided by one or more banks to one specific firm or corporation. We use this data because data on bank intermediation is not publicly available. We think this does not affect our results as the syndicated loans market represent almost half of the commercial and industrial lending in U.S. (see, Chodorow-Reich, Gabriel (2014, QJE)).

Syndicated loans data we use is from Thomson One proprietary database for years 2000 to 2010.[5] This database is worldwide data on syndicated loans that includes the name of the corporation (borrower), own NAICS code, country, amount, maturity, and banks (lenders) name.

To simplify the analysis, we considered the main bank lender as the lead manager on the contract. We examined loans to firms with specific domicile nation as the U.S. The total number of banks represented each year ranged between 273 in 2009 and 444 in 2005. These sums are obtained ignoring the fact that many of

[5] https://www.thomsonone.com

such banks are members of the same holding. In a second part of the analysis, we consider only banks that account for a significant proportion of the lending, the resulting pool of banks being much smaller.

To build each annual $\mathbf{\Phi}$ matrix we had to merge two separate matrices. Indeed, the first set of data contains information about the syndicated loans per firm indicating the respective NAICS code. With this set, we define a matrix with sectors as rows and individual loans as columns, in millions of dollars, designated by $\mathbf{S_1}$. Note that we had to match the Thomson-NAICS codes to the BEA-NAICS classification as in some cases there is a different coding, taking as primary classification the latter.

The second set of data contains information about syndicated loans per bank. We use this set to construct a matrix of zeros and ones with banks as columns and individual loans as rows, $\mathbf{S_2}$[6]. The intermediation matrix with banks as columns and sectors as rows, in millions of dollars, is obtained as the matrix product $\mathbf{S_1'S_2}$. In fact, this data allows deriving the weights, which are represented in the bank-to-sector matrix $\mathbf{\Phi}$.

In Figure 1.18, we represent the input-output network and the bank-to-sector intermediation network, the related links being blue for the I-O network and grey for the bank-to sector network. In the diagram only the 6 biggest lenders are shown and links with weights less than 0.1[7] are ignored.

[6]We assumed proportional participation in the case of loans with more than one lead manager, this due to lack of information.

[7]The biggest six lenders are JP Morgan, Bank of America, Citigroup, Wells Fargo, Goldman Sachs and Morgan Stanley. For these banks, we include all the holding that belong to each major group, and we took into account merges and acquisitions in the period, for example, Bearn Stearns bought by JP Morgan in 2008

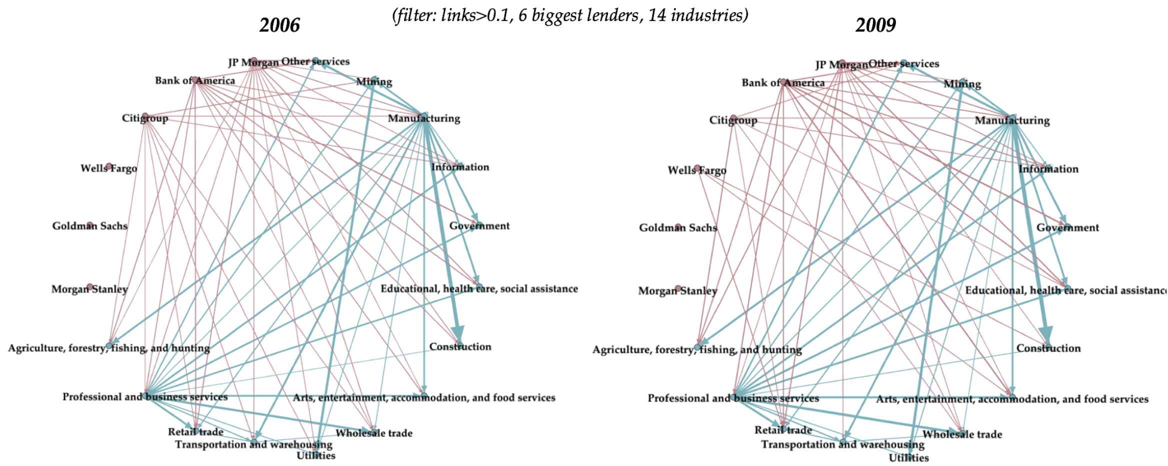


Figure 1.18: Bank-sector and I-O networks in the U.S. in 2006 and 2009

It can be seen that from 2006 to 2009, there are changes in the capital intermediation network but not in the I-O structure. For example, in 2006, Wells Fargo had one link, whereas in 2009 this number increased to 4. In fact, we can see than the number of connection have increased for the majority of banks.

1.8.2 Results

We now compute the empirical variance using the network metrics that defined previously. We use both the matrices of input-output trade, $\mathbf{W}$, and the bank-to-sector intermediation matrix, $\mathbf{\Phi}$, for all nodes but only for the subset of the biggest banks.

The expression for GDP volatility as a function of network metrics is given by:

$$\sqrt{var(\ln Y)} \approx \sqrt{\left(\frac{\sigma\alpha}{n}\right)^2 \sum_{b=1}^{m} \left(\underbrace{b_b}_{\text{Bank degree}} + \underbrace{(1-\alpha)B_b}_{\text{Bank-sector int. coeff.}} + \underbrace{(1-\alpha)^2\widehat{B_b}}_{\text{2nd order bank-sector int. coeff.}} \right)^2}$$

where

$$d_i \equiv \sum_{j=1}^{n} w_{ji}, \quad \widehat{d}_i \equiv \sum_{j=1}^{n} d_j w_{ji}, \quad b_b \equiv \sum_{i=1}^{n} \phi_{ib}, \quad B_b \equiv \sum_{i=1}^{n} d_i \phi_{ib}, \quad \widehat{B}_b \equiv \sum_{i=1}^{n} \widehat{d}_i \phi_{ib}$$

From these matrices, we can compute the relevant metrics. We use the value of α obtained from the data for each year. The multiplier calculated for each year in the period 2000-2010 is given by

$$[\text{Network multiplier}]_{US,t} \equiv \sum_{b=1}^{m} \left(b_{b,t} + \alpha_t B_{b,t} + \alpha_t^2 \widehat{B}_{b,t} \right)^2$$

In Table 1.1 we report the annual change in GDP and take this as the observed volatility.

There are two ways to compute the predicted volatility generated by shocks to the financial sector. The first one is calculated using the share of the total output by the financial sector, multiplied by the industry shock. To model the shock we use the percentage changes in the quantity index for GDP from the financial sector, and this is done to exclude nominal effects. Using this basic measure, we find that the financial sector accounts only for a small part of the total volatility. For example, in 2007 the volatility of GDP generated by the financial sector was only

	% changes in quantity indexes	% of total output	Contribution to GDP volatility (1)	Network multiplier		Network spillover effect (2)-(1)
				Coefficient*	Implied volatility of GDP** (2)	
2001	10.18	7.79	0.79	128.53	2.80	2.01
2002	0.43	7.78	0.03	136.75	0.13	0.09
2003	0.74	7.83	0.06	122.31	0.21	0.15
2004	-0.28	7.78	-0.02	144.08	-0.09	-0.06
2005	8.95	7.75	0.69	138.86	2.62	1.93
2006	4.31	7.84	0.34	103.89	1.09	0.75
2007	-2.62	7.95	-0.21	115.11	-0.69	-0.48
2008	-13.05	7.44	-0.97	130.75	-3.59	-2.62
2009	15.76	7.58	1.19	108.68	4.43	3.23
2010	-0.06	7.29	0.00	110.25	-0.02	-0.01

*Using α computed from NAICS tables, 14 industries.

**Idiosyncratic volatility of financial sector using % changes in quantity indexex for GDP. Implied volatility is calculated as the squared root of the network multiplier scaled by α times idiosyncratic volatility over the number of sectors.

Source: Computed using information from BEA and Thomson One.

Table 1.1: Financial sector and volatility of GDP in the U.S from 2001 to 2010

-0.21% whereas the aggregate fluctuation registered in the national accounts was 1.78%.

The second approach, reported in the second column, is the computed volatility of GDP but using the network multiplier. First, it should be noticed that the bipartite network of bank-to-sectors connections changes over time, leading to a change in the corresponding network multiplier (differently from the I-O network). Using these multipliers, and the expression for the volatility of GDP reported earlier, we compute the fluctuation of GDP generated by the financial sector. The predicted values are greater than the ones obtained using previously with simple weights. For example, in 2008 the fluctuation caused by shocks to the financial sector using basic weights was -0.97%, whereas using network multipliers this value becomes -2.62%. These results re-illustrated in the following diagram (Figure 1.19).

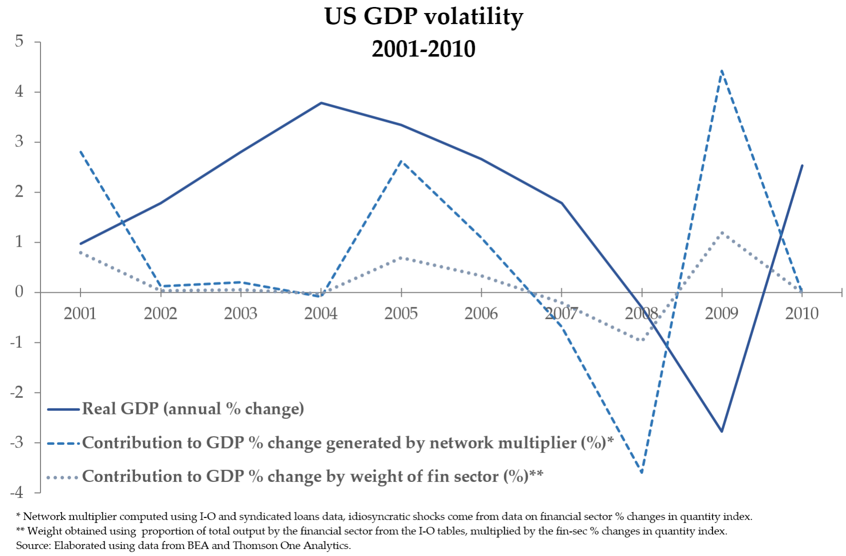


Figure 1.19: U.S. GDP volatility from 2001 to 2010

Finally, Figure 1.20 shows the empirical cumulative distribution of the logarithm of the metrics composing the network multiplier (i.e., the bank-outdegree, the bank-to-sector interconnectivity and the 2^{nd} order bank-to-sector interconnectivity).

The analysis focuses on the years 2002, 2005 and 2010. These are chosen because in 2002 there was no expectation of a recession, in 2005 the fears of a recession start to be widespread, and 2007-2009 are the crisis years, 2010 can be considered in the aftermath.

In 2005 the structure of the economy looked prone to the propagation of idiosyncratic financial shocks. In fact, the network multiplier metrics related to propagation of higher order effects (interconnectivity of 1^{st} and 2^{nd} order) are much higher in 2005 than in the other years; leading to the view that the struc-

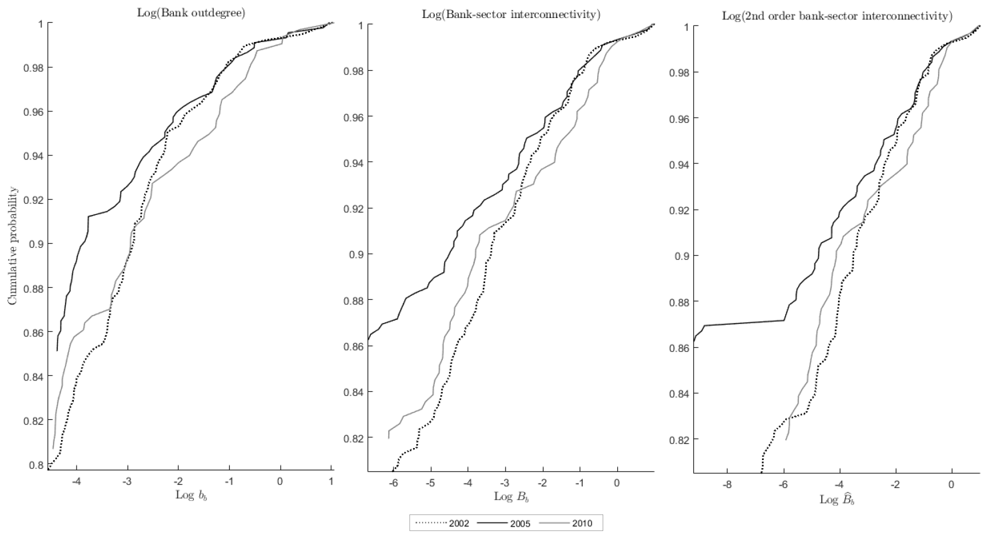


Figure 1.20: Empirical cumulative distribution

ture became highly susceptible to spreading the shock and generate spillovers from the financial institutions to the whole economy.

From these results, we can see that typically individual shocks do not average out and lead to sizeable GDP fluctuations, much larger than those obtained using the aggregate financial sector with basic weights. Although I do not have complete information on the transactions occurring in the credit intermediation market, as in the model aggregate volatility is originated from a small individual financial shock, this result shows the importance of the network in the transmission of the shock.

1.9 Conclusions

In this chapter, we developed a model with a multi-sector production economy with I-O interlinkages and a financial sector to analyse the effect of idiosyncratic shocks to banks on GDP volatility. In the model there is a financial constraint that gives some rigidity to the financial sector, making this sector relevant. The financial shock is characterised by the realisation of a random parameter involved in the financial constraint.

In fact, the random shock represents a financial shock to the bank and could be interpreted as changes in the reserve requirements imposed on the bank or shifts in the "skin in the game" conditions, situations that lead to a reduction of capital that a bank can potentially lend to a firm.

The real economy is modelled following the specification of intersectoral trade via I-O linkages used Acemoglu et al. (2012). These elements are embedded in a two-period model. The resulting multi-sector economy with uncertainty is still tractable thanks to the type of rigidity adopted.

We show that a general equilibrium exists and recover the known result that financial frictions work as a wedge and decrease the level of aggregate output. We then proceed in the main direction of the paper. The analysis focuses on the role of the structure of the economy, that is the network of connection among physical sectors and between sectors and banks, in the propagation of financial shocks to the real economy and its aggregate implications.

Analysing different structures of the economy, we find that, in general, GDP volatility is decreasing as the variability of the interconnectivity of each bank is reduced, as expressed by the network multiplier. In other words, the standard deviation of GDP decreases as the interconnectivity coefficient is similar among banks, while it increases when this metric shows a higher variability across banks. This because the effects of different individual shocks are propagated with more strength in the presence of asymmetries.

We also analysed the effect of adding links to the bipartite intermediation network. We first consider the increase in integration provided by adding one link to a specific bank, in which case after the addition the bank lends to a sector that has in common another bank. In this case, the volatility of GDP can decrease or increase depending on the position of the banks and the location of the common sector in the network. On the other hand, when adding m links, one for each bank in the network, the results still depend on the I-O structure, but always reduces the volatility for vertical and star structures and have no effect on empty and circle networks.

We studied the effect of diversification, obtained as a result of redistributing the links of two or more banks in the network. In this case, we find that depending on the position of the bank in the network, reducing diversification can decrease or increase the effect on the volatility of an individual shock even for a specific location. The latter is due to the difference in the elements of the influence vector for each sector and the interconnectivity coefficient of the bank-to-sectors network implying that there is no effect for perfectly balanced networks, as in this case, the influence vector is the same for all sectors.

In general, aggregate fluctuations depend on the distributions of links between banks and sectors and the location of such links. An economy with a uniform distribution of links per bank could be less volatile than an economy with an unequal distribution, provided the bank with less link is not supplying capital to a great influencer, that is, the star sector or the top of the chain in the vertical network. The intuition behind this result is that the multipliers effects of the interactions between the sectors that obtain the funds from the same bank are greater with a non-uniform distribution of the bipartite links under specific locations.

Finally, we used input-output data and syndicated loans data to compute the I-O and the bipartite intermediation matrices of the US economy for the period 2000-2010. We find that the structure of the economy is highly asymmetric; there are star sectors like manufacturers and professional services, and star banks like Bank of America and JP Morgan. We find that changes in the bipartite structure over time lead to changes in the network multiplier while the I-O network remains relatively steady.

Computing the GDP volatility using the network metrics, we find that individual shocks to banks do not average out and could lead to sizeable fluctuations of GDP. Interestingly, the volatility is considerable higher when taking into account the network multiplier rather than only using a basic aggregate computation of the financial sector.

Chapter 2

Network structure of the sectoral trade and aggregate fluctuations in the United Kingdom

2.1 Introduction

An old but important question in Macroeconomic Theory is whether significant aggregate fluctuations in economic activity can be obtained from independent productivity shocks to individual disaggregated sectors. The Great Financial Crisis (GFC) of 2008 highlighted the need to study further whether a shock in a specific sector or firm could propagate its effect to the whole economy.

The most common view in the Business Cycle Theory has been that idiosyncratic shocks tend to average out in aggregation, the so-called "diversification argument”. This answer discards the possibility that significant aggregate fluctuations may originate from microeconomic shocks to firms or disaggregated sectors.

As argued by Lucas (1981) and Dupor (1999), idiosyncratic shocks cancel out and would only have negligible aggregate effects, in this case, the aggregate output goes to its mean very quickly.

To better understand this argument, assume an economy consisting of N sectors hit by individual shocks, aggregate fluctuations would have a magnitude proportional to $\frac{1}{N}$ because the aggregate volatility of the output, when measured using variance, will have a proportional relation to the independent shock multiplied by such factor: $\sigma^2_{GDP} \propto \frac{\sigma^2}{N}$, where σ^2 is the variance of a specific sector. As N becomes larger, the effect of individual shocks becomes smaller.

This paper applies the original model developed by Acemoglu et al. (2012) to analyse the network structure of the intersectoral trade in the United Kingdom, and specifically, the possibility of aggregated volatility originated by a shock in a specific sector of the economy. At the moment of its elaborating, this was the first attempt to analyse an economy other than the United States and also it is the first application of this model to the United Kingdom data.

A goal is to corroborate whether the results of propagation of idiosyncratic shocks are present in the U.K. data. To do so, I explore different econometric techniques to analyse the results sensitivity. For instance, apart from linear regression I also used a Maximum-likelihood Estimation to address the specific problems of the data.

Additionally, to the empirical application of the Acemoglu et al. (2012) model, I apply a traditional network analysis to characterise the network structure of the disaggregated U.K. economy. Specifically using other network measures to identify (if possible) which sectors of the British economy are the top and bottom players for the economy.

The above serves as a tool to analyse which sectors are "key players" for the economy, being the top providers of inputs for the other sectors. This network context is worth exploring since one would expect areas that have more connections are more likely to be influential because they can directly affect more areas.

Moreover, some sectors could serve as a "core" sector for the economy, with much higher number of connections than many the network. Thus, some sectors could be intermediaries for the rest of the economy.

Finally, one of the main concerns of the analysis is to obtain the rate at which aggregate volatility declines, which depends on the structure of the intersectoral network. It highlights the importance of different sectors as main suppliers to the rest of the economy.

The rest of the chapter is organized as follows: i) Section 2 presents a brief literature review; ii) Section 3 presents the model to estimate, which is the seminal model described in Acemoglu et al. (2012), detailing the multisector economy setup, assumptions and main results; iii) Section 4 describes the main characteristics of the data: the matrix of intersectoral trade in the U.K.; iv) Section 5 presents the analysis of the network structure; and v) finally, I present the conclusions of the chapter.

2.2 Related literature

At the beginning of the Great Financial Crisis of 2008, many of the existing models predicted that individual shocks were not going to have large implications for the aggregate economy (Stiglitz, 2010). The usual answer ignored the presence of interconnections between different firms or sectors functioning as a potential propagation mechanism of individual shocks throughout the economy.

However, the above does not mean there are no papers that tried to model individual shocks in a disaggregated economy (e.g. Long and Plosser (1983, 1987), Norrbin and Schlagenhauf (1990) and Horvath (1998)).

More recently, there are papers where network theory is used to explain aggregate fluctuations. Some of the most known developed in recent year are the ones by Carvalho (2010), Acemoglu et al. (2012). These papers analyse the flow of intermediate inputs across sectors by adopting a network approach.

The general hypothesis of these last papers is that there exists co-movement across sectors as a characteristic of fluctuations, and could act as a shock to the production technology of a sector and could likely propagate to the rest of the economy.

In this chapter, the relevant model is the one by Acemoglu et al. (2012). In this model, they assume an economy with N sectoral goods and each sector is subject to an independent productivity shock of variance σ^2. Whether these sectoral shocks propagate will depend on the network structure, then, we can consider three extreme cases of network structure (incomplete, complete and star) as depicted in the Figure 2.1.

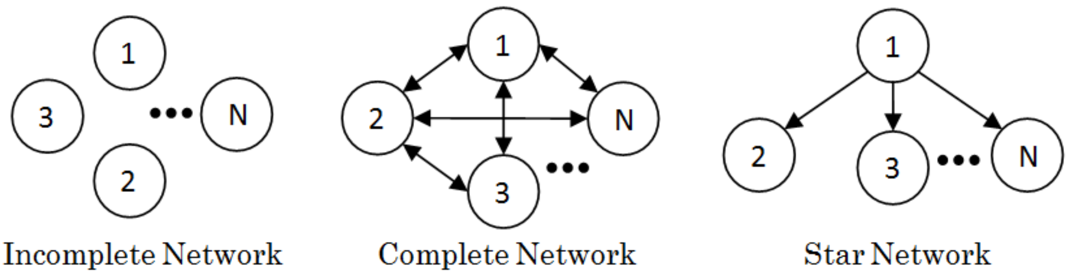


Figure 2.1: Different network structures

Consider the linkages structure, where each directed link indicates the provision of inputs, in the case of incomplete and complete networks we can see that as N increases and the economy becomes more disaggregated; shocks will average out rapidly: $\sigma^2_{GDP} \propto \frac{\sigma^2}{N}$. This situation happens in the case of incomplete networks because there are no links and in the case of complete networks because the level of connectedness is so significant that the disruption in one sector decrease as the number of sectors disaggregates.

An example of this implication, is to consider an independent sectoral volatility in the order of a standard deviation of 2%, and an economy composed by 100 sectors, we will expect an approximate effect in the aggregate volatility of only 0.2%, such GDP volatility is small to account for the empirically measured size of macroeconomic fluctuations of $>= 1\%$, and is increasingly reduced as we additionally disaggregated the economy.

This analysis changes in the case of star networks, if we consider the extreme example in Figure 2.1 where only one sector provides input to the rest of the economy. We can see that if there is a shock in this specific sector, as N increases, sectoral shocks do not average out, shocks to sector 1 propagate strongly to the rest of the economy, generating significant aggregate fluctuations, $\sigma^2_{GDP} \propto \frac{\sigma^2}{M}$.

The problem with this possibility is of how to define M and also which network is present in economies with more realistic patterns of interconnections.

It is worth mentioning that the underlying multi-sector set-up used by Carvalho (2010) and Acemoglu et al. (2010, 2012) is very close to the one in Horvath (1998), Dupor (1999), Shea (2002) and Foerster, Sarte and Watson (2008). All these papers are closely related to the original multisector real business cycle model of Long and Plosser (1983).

2.3 Multisector network model

In this section, I describe the model of intersectoral linkages, which is the model by Acemoglu et al. (2012), to analyse the economy as a network. This analysis is fundamental given that one of the principal objectives of this paper is to explore the assumptions and analytical results of this model using U.K. data.

The model by Acemoglu et al. (2012) is critical for the present analysis because it provides a straightforward and tractable way to test the hypothesis of macroeconomics implications from idiosyncratic shock using data.

The multisector economy section of these models borrows from previous works by Long and Plosser (1983) and Shea (2002). Nevertheless, the principal difference and contribution are that it incorporates tools from the Network Theory that allows them to express the structure of the economy as a network and to link such structure to the volatility of GDP.

2.3.1 Multisector economy

This section describes the model used in U.K. data analysis. The Acemoglu et al. (2012) model consists of a static multisector model for a closed economy without government.

As mentioned before, this model is a version of the seminal work by Long and Plosser (1983) and the static variation by Shea (2002). The model includes a representative household in the consumption side and a multisector economy in the productivity dimension.

Starting with the consumption side, this model considers a representative household that consumes N goods and supplies L labour hours inelastically. Each good is consumed in the same proportions. Given this, and assuming log-utility over the consumption of N goods:

$$U(\{C_i\}_{j=1}^{N}) = \sum_{i=1}^{N} \frac{1}{n} \ln C_i \tag{2.1}$$

$$\text{with } \sum_{i} L_i \leq L \tag{2.2}$$

Consider the production side of the economy, with N sectors producing different kinds of goods, where each sector is in charge of the production of only one good, and all sectors follow a Cobb-Douglas technology with constant returns to scale. Each produced good can be consumed by the representative household, or it can be used as an input in the production of other goods (intermediate good). Given this, the following function represents the production technology:

$$Y_i = z_i L_i^{\alpha} \prod_{j=1}^{N} Y_{ij}^{(1-\alpha)w_{ij}} \tag{2.3}$$

$$\text{subject to } \sum_{j=1}^{N} w_{ij} = 1, \text{ with } \alpha \in (0,1), \text{ where } i = 1, ..., N \tag{2.4}$$

where α is the share of labour hired by the sector i, w_{ij} designates the share of good j in the total intermediate input use of sector i and corresponds to the entries of input-output tables, measuring the value of spending on input j per pound of production of good i. In the equation (4) we can see that an idiosyncratic productivity shock is included, represented by the term z_i, this is a Hicks-neutral shock to the good i, and assuming:

$$z_i = \varepsilon_i, \text{ where } \varepsilon_i \sim N(0, \sigma_i^2) \tag{2.5}$$

This productivity shock can represent changes such as a process of inventories and shipments, a new line of products or R&D, variations in the level of capacity utilisation, or idiosyncratic events like strikes or natural disasters.

The usual market clearing condition implies the following equation:

$$Y_i = C_i + \sum_{j=1}^{N} Y_{ij}, \text{ where } i = 1, ..., N \tag{2.6}$$

Acemoglu et al. (2012) apply the standard method to obtain the competitive equilibrium in this model, this is, substituting the equilibrium input choices into the production function and simplifying. Expressing such equilibrium in logarithms they have:

$$\mathbf{y} = \mathbf{v}'\varepsilon + \mu \tag{2.7}$$

where

$$\mathbf{v} = \frac{\alpha}{N}(\mathbb{I} - (1-\alpha)\mathbf{W}'^{-1})\mathbf{1} \tag{2.8}$$

The N-vector $\mathbf{y}$ gives the logarithm of output in equilibrium and the N-vector μ is a set of constants defined by the parameters of the model. $\mathbb{I}$ is an identity $N \times N$ matrix, and $\mathbf{W}$ is a $N \times N$ I-O matrix that summarises the structure of intersectoral trade because of each element, w_{ij}, corresponds to the share of intermediate goods used in production. The N-vector ε is the logarithm of the productivity shock z_i for each sector.

Therefore, using this model, we can see that considering more than one sector in the economy and independent productivity shocks, z_i, the I-O matrix, $\mathbf{W}$, captures how sectoral productivity shocks potentially affect aggregate volatility. Thus, the next step of this model, described in the next section, is the introduction of the network set-up for the I-O economy to understand the relation between the I-O structure and its potential mechanism of propagation of idiosyncratic volatility.

2.3.2 Matrix W as a network economy

The first important concept is the definition of network, which can also be called a graph. Intuitively, a network is a set of linked nodes. The definitions of this section are obtained from Newman (2010).

Assume N sectors that correspond to the set of nodes $V = \{v_1, ..., v_N\}$. Let the links set, E, be a subset of the collection of all ordered pairs of vertices $\{v_i, v_j\} \epsilon V$, where this set indicates the direction of links among sectors. When one link goes out from a sector that is an input supplier to another sector (where the link goes in), this is an adjacency relation $v_i \rightarrow v_j$ between all the elements of the set V.

We can now define the concept of a network (or graph), given that a network is the set of linked nodes. The definition goes as follows:

Definition 29. *$G = (V, E)$. G is a directed sectoral linkages graph with node set V and edge set E where each element of E is a directed arc from element i to j.*

The adjacency matrix indicates only the presence of a directed link, not its raw value. The concept of an adjacency matrix is the following:

Definition 30. *For a directed sectoral linkages graph $G(V, E)$ define the adjacency matrix $A(G)$ to be an $N \times N$ matrix. If G is a directed graph, define the a_{ij} element of $A(G)$ to be 1 if there is a directed edge from sector i to sector j and $zero$ otherwise.*

We can understand the matrix **W** as a graph, and decompose it as the product of an adjacency matrix and a matrix that includes the shares of inputs traded:

$$\mathbf{W}(G) = A(G)D_{\mathbf{W}} \tag{2.9}$$

where $A(G)$ is an adjacency matrix that represents the structure of the intersectoral trade because a typical element, a_{ij}, would be equal to one if the sector i provides input to the sector j, and zero otherwise.

Matrix $D_{\mathbf{W}}$ is a diagonal matrix with a typical element $D_{kk} = \frac{\gamma_k}{d_k^{in}}$, where is the w_k share of input and d_k^{in} is the indegree of sector k.

Before describing the analytic results as in Acemoglu et al. (2012), we need one definition that is a key assumption in the distribution of the network structure, this is the concept of a power law. According to Clauset et al (2009):

Definition 31. *Quantity k obeys a power law if it is drawn from a probability distribution $p_k \propto ck^{-\zeta}$, where ζ is a constant parameter of the distribution known as the exponent or scaling parameter.*

It is important to note that the scaling parameter, ζ, will be the key to estimating in the application of this model.

We can now describe in a proposition the principal analytical results to be applied. Intuitively, this result assumes that the distribution of the network structure, expressed in the network matrices $\mathbf{W}(G)$ and the corresponding $A(G)$, follows a power law, and that the exponent of this power, ζ, will indicate us if the volatility escalates with N^{-1}, as in the law of large numbers, or with N^{-v} with $v < 1$, giving the possibility of a volatility decline arbitrarily slow. The following proposition comes mainly from Acemoglu et al. (2012):

Proposition 14. *Consider the equilibrium of a static multisector economy, where the input-output intersectoral trade $\mathbf{W}$ is given by a graph $\mathbf{W}(G^{PL})$ for any corresponding* *adjacency matrix $A(G^{PL})$, distributed as a power law with scaling parameter $\zeta \geq 2$, then, whenever this is the case, $\sigma^2_{GDP}(\mathbf{W}(G^{PL}))$ is bounded below by:*

$$O\left[\sigma^2\left(\frac{1}{N}\right)\right] \quad \textit{if} \quad \zeta \geq 2, \textit{ or} \tag{2.10}$$

$$O\left[\sigma^2\left(\frac{1}{N}\right)^{\frac{\zeta-1}{\zeta}}\right] \quad \textit{if} \quad \zeta \in (1,2) \tag{2.11}$$

Where σ^2_{GDP} is the aggregate volatility, and σ^2 is the idiosyncratic sectoral volatility caused by the shock z.

This result shows that the decay rate of the volatility, N^{-v}, where v could be 1 or $\frac{\zeta-1}{\zeta}$, depends on the value that the scaling parameter ζ assumes, and this value will indicate whether the tail of the distribution is thin or fat, which regarding the network intersectoral model shows the degree of symmetry of the linkages in the economy.

If we consider the case of thin-tailed distributions of sectoral outdegrees, $\zeta \geq 2$, this implies that the aggregate volatility will decay as in the particular case of networks assumed by the law of large numbers, near-fully complete and symmetric networks, where the aggregate volatility is proportional to N^{-1}:

$$\sigma^2_{GDP} \propto \frac{\sigma^2}{N} \quad where \quad \zeta \geq 2 \tag{2.12}$$

The above means, intuitively, that in economies with a large number of sectors and a very symmetric structure, aggregate volatility will be negligible.

However, if we consider the case of fat-tailed region for $\zeta \in (1,2)$, aggregate volatility will decline with N^{-v}, where $v = \frac{\zeta-1}{\zeta}$, that means a rate that is lowered significantly:

$$\sigma^2_{GDP} \propto \sigma^2 \left(\frac{1}{N}\right)^{\frac{\zeta-1}{\zeta}} \qquad where \quad \zeta \in (1,2) \tag{2.13}$$

We observe that for the tail parameter, ζ, as it approaches its lower bound, aggregate volatility will go to zero slower. In contrast, if it approaches its upper bound, then volatility will vanish.

In conclusion, we can simplify the analysis to the estimation of the tail parameter, ζ, as it indicates if a particular structure is prone to propagation. Thus, the tail parameter will tell us the implication for the aggregate volatility of a particular network structure obtained from the data.

2.3.3 Traditional networks analysis

This section details the traditional network theory concepts used to characterise the network's structure of the economy modelled in the previous section.

In addition to the major network concepts that Acemoglu et al. (2012) use to characterise the I-O structure; I include different network measures that help to compare the specific qualities and characteristics of the network economy in question.

Now, if one wants to characterise the structure of a graph concerning how well connected, how complete or how symmetric this one is, one needs extra network measures that could provide information about the distribution of the linkages, the centrality of nodes and the possibility of clusters.

The first of these measures, which are used by Acemoglu et al. (2012) are the indegree and outdegree for each node. These core measures are used to identify the possibility of homogeneity or heterogeneity in the I-O network. However, it is illustrative to have the concept of degree, as, in an undirected network, this measure for a particular node is the number of links of this node with any other:

Definition 32. *The degree d_i of a node $v_i \epsilon V$ is given by the cardinality of the set $\{v_j : v_i \leftrightarrow v_i\}$. The degree sequence of a graph $G(V, E)$ is given by $\{d_i, ..., d_N\}$*

It is easier to understand that if we are considering a directed graph, where the direction matters, we have two types of degrees: indegree and outdegree. The first one is related to the number of links that enters the node; the second one is the number of links that go out from a node. It could also be weighted or unweighted.

Saying this, the number of different inputs a sector demands in order to produce, measured by the columns sums of $A(G)$, is the indegree:

Definition 33. *The indegree d_i^{in} of a node $v_i \epsilon V$ is given by the cardinality of the set $\{v_j : v_j \to v_i\}$. The indegree sequence of a graph $G(V, E)$ is given by $\{d_i^{in}, ..., d_N^{in}\}$.*

And, the number of different sectors a sector supplies inputs to, measured by the row sums of $A(G)$, is the outdegree:

Definition 34. *The outdegree d_i^{out} of a node $v_i \epsilon V$ is given by the cardinality of the set $\{v_j : v_i \to v_j\}$. The outdegree sequence of a graph $G(V, E)$ is given by $\{d_i^{out}, ..., d_N^{out}\}$.*

These network concepts are the only ones used by Acemoglu et al. (2012) to characterise the matrix of the I-O structure; however, to compare this empirical application, it is useful to have additional network metrics of connectedness as defined in the following concepts taken from Newman (2010).

Definition 35. *The average in-degree c_{in} and the average out-degree c_{out} of every directed network are equal to:* $c_{in} = \frac{1}{N}\sum_{i=1}^{N} k_i^{in} = \frac{1}{N}\sum_{i=1}^{N} k_i^{out} = c_{out}$

To have a better understanding of the distributions of the input-supply and input-demand of a network, following Newman (2010), we can use the centrality concept.

Definition 36. *The betweenness centrality metric measures the extent to which a node lies on paths between other vertices. Mathematically, let* N_{st}^{i} *be* 1 *if node* i *lies on the geodesic path from* s *to* t *and* 0 *if it does not or if there is no such path. The betweenness centrality* x_i *is given by:* $x_i = \sum_{st} N_{st}^{i}$

Definition 37. *Bonacich (1987) introduced another centrality measure, were considering an adjacency matrix A, the centrality of node* i *(denoted* c_i*), is given by:* $c_i = \sum_j A_{ij}(\alpha + \beta c_j)$*,where* α *and* β *are parameters. The value of* α *is used to normalise the measure, the value of* β *is an attenuation factor which gives the amount of dependence of each node's centrality on the centralities of the vertices it is adjacent to. The centrality of each node is therefore determined by the centrality of vertices it is connected to.*

A different centrality measure that indicates the degree of influence of a specific sector is the clustering coefficient, that identifies possible groups at which the inputs trades concentrate.

Definition 38. *The clustering coefficient for a node* i *represents the average probability that a pair of* i*'s nodes are neighbours of one another and is defined as:*

$$C_i = \frac{(\textit{number of pair of neighbours of } i \textit{ that are connected})}{(\textit{number of pairs of neighbours of } i)}$$

This clustering coefficient measures how influential a node is and whether two neighbours of a node are not connected directly.

2.4 Data

The focus of this chapter is to analyse the intersectoral network structure of the U.K. economy and study its implications for aggregate fluctuations as predicted by Acemoglu et al. (2012).

The data is obtained from the Office for National Statistics (ONS) for the period 1997–2010. According to the Office for National Statistics in the United Kingdom, there are three different approaches to the estimation of Gross Domestic Product (GDP)[1]: production (output), expenditure, and income.

The Supply and Use framework [2] is the part of the National Accounts system which focuses on the production and reflects the production of sectors in which intermediate goods and primary inputs are used.

The supply table shows the output of domestic sectors. The Use Table is composed of intermediate demand and the final expenditure.

The latest set of Supply and Use Tables have been produced based on 109 sectors and constitute the finest level of disaggregation available for the U.K. inter-

[1] http://www.ons.gov.U.K./ons/guide-method/method-quality/specific/economy/national-accounts/a-guide-to-supply-and-use-process/index.html

[2] Ibid.

sectoral trade data [3].

2.5 Analysis of the network structure in The United Kingdom

In this section, using the Supply-Use data for the United Kingdom described previously, I analyse the network of intersectoral trade. After this, and most importantly, using the network measures of indegree and outdegree along with different econometric approaches, I explore the results implied by Acemoglu et al. (2012).

The first part includes the network characterization of the U.K. input-use data for the period 1997-2010. Using graphs and metrics from network theory, I explore not only the centrality and clustering of the network structure but also the degree of symmetry in the distribution of connections among sectors.

The second part considers the empirical evaluation of the assumption that distribution of outdegree follows a power law. I estimate the tail parameter ζ described in the previous section using different methods available and commonly used in the literature of networks and power laws.

Finally, I calculate the decay rate implied by this tail parameter and I described the implications for the output volatility in the U.K.

[3] Ibid.

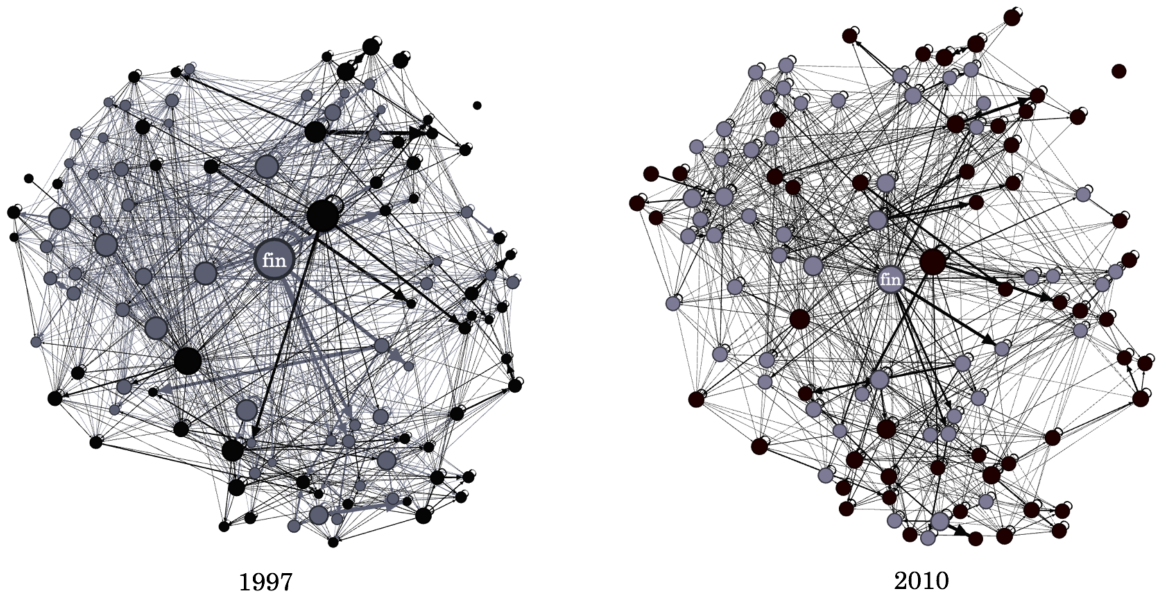
2.5.1 Network Structure

In this section, as a first approximation to the data, I apply networks graphs. In this way, the following graphs show the pattern of connections among sectors.

The empirical analysis (excluding Figure 2.3) considers only links from transactions above 1% of the total purchases of each sector. This decision comes from the fact that I want to discard trivial transactions between sectors and to focus on the main components of total goods necessary to the production of any given sector, as applied in Acemoglu et al. (2012). Following this rule, I gather 85% of the total value of intermediate input trade in the U.K. economy in 2010 and a similar number for all other years considered.

We can see in Figure 2.2 the network structure of input transactions in the U.K. for 2010. To draw this graph (and Figure 2.3), I used a method of force direct graph drawing; specifically, an algorithm called Spring-Embedding with 100 iterations calibrated by the geodesic distance, which consists of applying the repulsion force proportional to the geodesic distance between node (Newman, 2009).

At first sight, it is hard to identify the degree of symmetry of this network structure. Someone may argue that this network is relatively connected and highly symmetric, as in the argument for the irrelevance of idiosyncratic shocks. However, we can see that some sectors are more important than other considering the weight and links, in particular, the financial sector dominates all of the others.

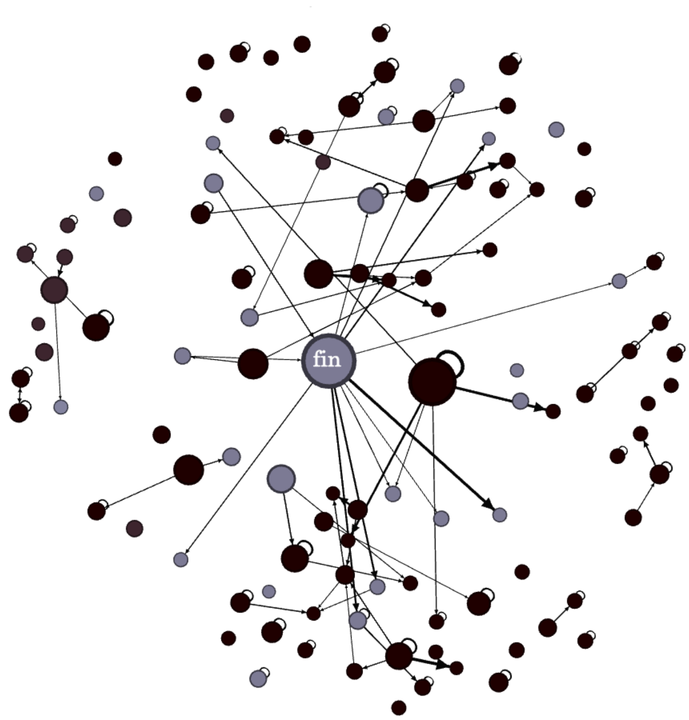


For every input transaction above 1% of the total input purchases of the destination sector, a link between two vertices is drawn. Size of node is proportional to weighted out-degree. Data: ONS.

Figure 2.2: Input-Output trade structure in the U.K. (1997 y 2010)

The problem becomes more complex if instead of the 1% threshold, I had drawn without restrictions, as we would only have a visual mass. However, if we look closely at the graph and filtering even more at 20%, we can see that there are specific sectors that concentrate a big number of links, mostly in the centre of the graph (Figure 2.3).

In Figure 2.3 we can clearly see a certain degree of asymmetry in the input structure of this economy, and identify sectors that concentrate a large number or links and sectors with only a few connections with the rest of the nodes. However, this visual approximation is not able to provide information about whether asymmetry is in the input-demand dimension or the input-supply side, once we consider the direction of the edges.



For every input transaction above 20% of the total input purchases of the destination sector, a link between two vertices is drawn. Size of node is proportional to weighted out-degree. Data: ONS.

Figure 2.3: Intermediate input flows between sectors in the U.K. (2010)

In Table 2.1 we can see different network metrics for the U.K. data in the period where we have observations and the average. These metrics, as defined in the previous section, allow us to identify the degree of connectedness in the network.

	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	Average
Average weighted outdegree	0.990	0.990	0.991	0.991	0.990	0.990	0.990	0.990	0.990	0.990	0.990	0.991	0.990	0.990	**0.990**
Average outdegree	16.90	17.08	17.06	16.66	16.61	16.61	16.41	16.57	16.66	16.69	16.55	16.69	16.77	16.83	**16.72**
Average distance	2.21	2.19	2.22	2.24	2.23	2.28	2.29	2.33	2.32	2.32	2.33	2.33	2.33	2.31	**2.28**

Table 2.1: Input-Output trade structure in the U.K. (2010)

We observe that the average number of edges on each sector is 16.72, without considering the direction, and considering that we have 109 sectors, which implies a small degree of completeness. Same result if we consider the weights, the average weighted outdegree is 0.9900.

From the mean distance from a node to other vertices, we can see that the value is small 2.28 on average, meaning that the average shortest path between two nodes is small. One explanation would be a large asymmetry in the network, this is, we have many sectors with few link, but we also have many sectors with a large number of connections. The latter is related to the assumption of heterogeneity in the input-supply side discussed in the papers by and Acemoglu et al. (2012).

Table 2.2 shows the network metrics of (weighted) outdegree and (weighted) indegreee used in the model described in the previous section, these measures are presented only for the top 5 and bottom five sectors in this economy, ordered by their weighted outdegree.

The values presented in the Table 2.2 indicate the presence of considerable heterogeneity in the input-supply dimension. Considering the outdegree measure we can see that the top 5 sectors have values that go from 50 to 96, but in the bottom, we have isolated sectors and sectors with a unitary degree.

	Weighted outdegree	Outdegree	Weighted Indegree**	Indegree	Weighted betweeness	Bonacich weighted outdegree
Top 5 sectors*						
Financial services, except insurance and pension funding	3.92	96.14	0.41	82.43	81.57	20933
Employment services	2.82	57.36	0.38	85.14	15.56	12039
Land transport services and via pipelines, excluding rail	2.78	66.43	0.50	82.07	45.64	19799
Products of agriculture, hunting and related services	2.58	13.29	0.58	76.14	6.56	3521
Fabricated metal products, excl. machinery and equipmen	2.56	41.93	0.53	64.93	23.63	17504
Bottom 5 sectors*						
Tobacco products	0.02	1.00	0.47	57.79	22.50	17378
Wholesale trade services, except motor vehicles	0.00	0.00	0.51	85.57	49.48	19514
Retail trade services, except of motor vehicles	0.00	0.00	0.40	92.00	103.41	19274
Owner-Occupiers' Housing Services	0.00	0.00	0.15	1.00	0.01	18681
Services of households as employers of domestic personnel	0.00	0.00	0.00	0.00	0.00	0

*Ordered by weighted outdegree.

**Weighted as the proportion of total use taing into account total output.

Table 2.2: Network structure metrics in the U.K. (average 1997-2010)

These upper and lower sectors are according to the intuition, at the top we have the financial sector, and employment services, that are general selling sectors and provide inputs to a large number of sectors in the economy. In contrast, at the bottom, we have the sectors at the end of the economic chain such as retail and wholesale, but also we have very specialised sectors selling inputs to a tiny number of sectors, such as the tobacco products.

We can confirm this feature if we look at the betweenness measure, which measures the extent to which a node lies on paths between other nodes, in this case, the top sectors are significant providers of inputs, in contrast to the bottom sectors. The Bonacich outdegree, which counts the amount of dependence of each node's centrality on the centralities of the nodes it is adjacent to, confirms the high degree of heterogeneity in this dimension.

Also in the Table 2.2 we can identify that if well we have heterogeneity in the outdegree, the values for the indegree of the top and bottom sector are similar in some case; we will later explore in more detail this key assumption of the model.

Table 2.3 comprises another useful centrality measure, the clustering coefficient for the 5 top and bottom sectors.

Top 5 sectors	
Tobacco products	0.909
Products of forestry, logging and related services	0.891
Repair and maintenance of ships and boats	0.881
Remediation services and other waste management services	0.880
Repair and maintenance of aircraft and spacecraft	0.870
Bottom 5 sectors	
Warehousing and support services for transportation	0.662
Telecommunications services	0.662
Financial services, except insurance and pension funding	0.655
Owner-Occupiers' Housing Services	0.000
Services of households as employers of domestic personnel	0.000

Table 2.3: Clustering coefficients per sector in the U.K. (2010)

We can see that the clustering coefficient is inversely correlated with the outdegree type measures presented in Table 2.2; however, this measure allows us to identify how important a sector is regarding the connections between the sectors at which it provides inputs. Lower values of the clustering coefficient mean a more important sector.

In the following part, we will test, using density and distribution estimates for the indegree and outdegrees, the assumption of homogeneity in the demand side of the network and the heterogeneity in the supply side. We can tell that when these assumptions hold it is a network indicator of sectoral volatility by Acemoglu et al. (2012).

In the case of the sector as input demanders, we define the indegree of a sector i as the proportion of input use as a percentage of total output; calculating this value for all the sectors in all the period, we can estimate the density of sectoral indegree. Figures 2.4 and 2.5 show this empirical density for different years.

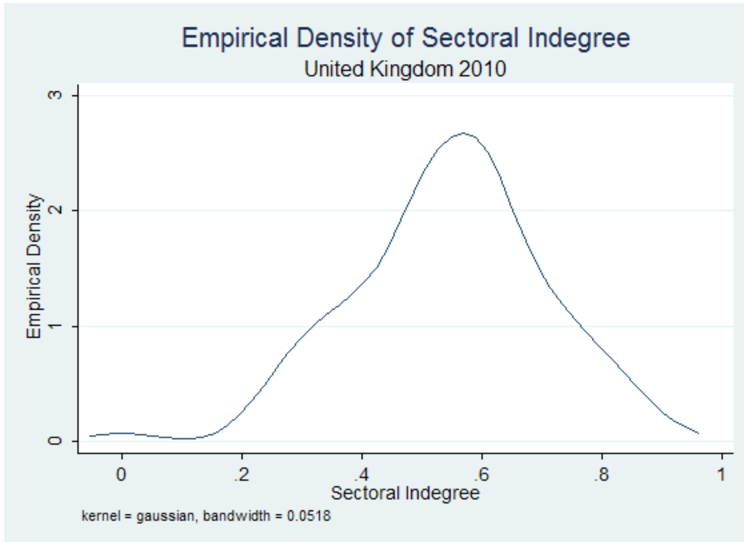


Figure 2.4: Empirical density of sectoral Indegree (2010)

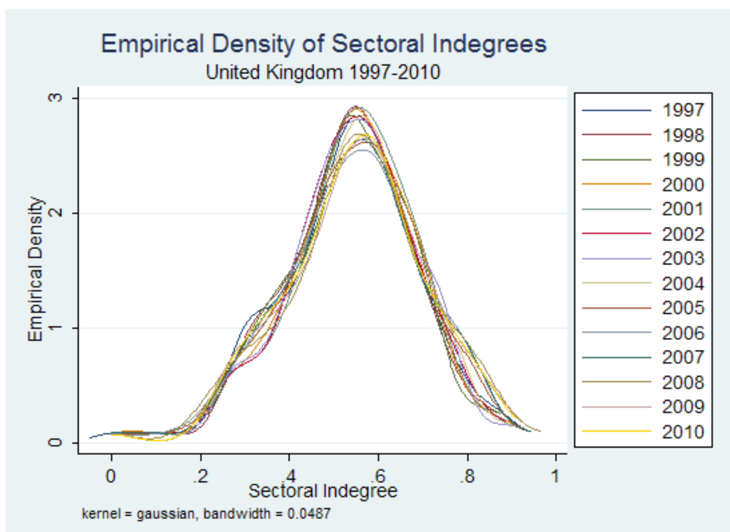


Figure 2.5: Empirical density of sectoral indegrees (1997-2010)

From these densities, we can see that the average sector in the U.K. procures a significant amount of inputs from only a small number of sectors (~ 17 on average across the years). In other words, the average indegree is small compared to the total number of sectors (ratio 0.16), and most sectors have an indegree that is close to the mean. Following Acemoglu et al. (2012), we find homogeneity along the sectoral demand in the data of the U.K.

The next step is to analyse the sectors in their role as input suppliers; we had a first approximation to this assumption with the metrics showed in the previous tables, identifying the possibility of heterogeneity in this dimension. Now we define the outdegree of a sector i as the number of distinct supply transactions. The log-log plot empirical counter-cumulative distribution of the outdegrees, or the probability, $P(k)$, that a randomly selected sector supplies inputs to k or more sectors (Figures 2.6 and 2.7).

I estimate the counter-cumulative outdegree distribution by ranking all the sectors, for each year, according to their outdegree and we assigned an ordinal series of numbers to this ranking, being 1 the sector with the highest outdegree. The plots are in logarithm scale.

From Figures 2.6 and 2.7 we can see that the distribution does not follow a normal behaviour, potentially exhibits fat tails. This observation means a significant heterogeneity across sectors. In other words, we find in the data some specialised suppliers, such Tobacco, but also general purpose sellers, such as the financial sector.

The above confirms the critical assumption of heterogeneity along the supply side in the model detailed in the previous section, at least for the U.K. in for this

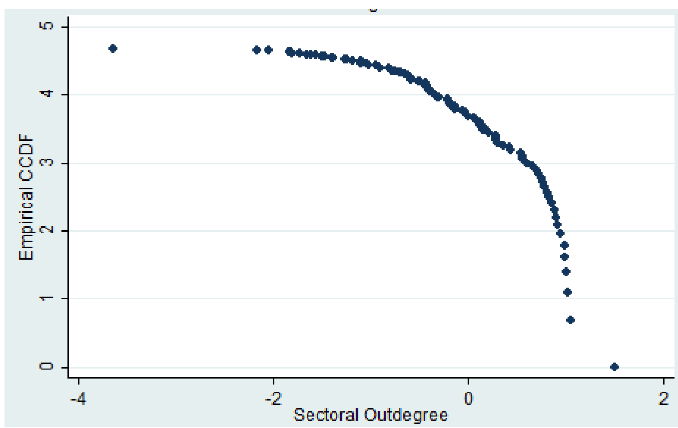


Figure 2.6: Counter-cumulative outdegree distribution in the U.K. (2010)

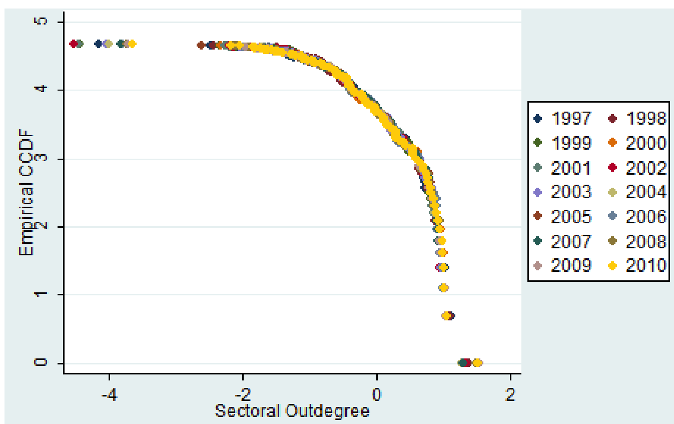


Figure 2.7: Counter-cumulative outdegree distribution in the U.K. (1997-2010)

period.

An important feature we can identify in the past two figures is the apparent linearity in the tail of the distribution, which is usually associated with a power law (Newman, 2010). Given the later, we need to test whether the distributions of the outdegrees follows a power law, if this is true, we can continue with the estimation of the tail parameter, ζ, and the implied decay rate, N^{-v} where $v = \frac{\zeta-1}{\zeta}$.

We can test whether a distribution follows a power law using the method suggested by Clauset et al. (2009) which consists of sampling synthetic data sets derived from a true power-law distribution. This approach is useful to assess how far they fluctuate from the power-law form, and compare the results with similar measurements on the empirical data, using a Kolmogorov-Smirnov statistic.

In case the empirical data set is considerably far from the power-law, then this would not be a good fit to the data. Applying this method, I found on average a $p - value$ of approximately 0.6, which implies that the difference between the empirical data and the model can only be attributed to statistical fluctuations and does not reject the hypothesis that the data follows a power law.

2.5.2 Tail Parameter and volatility decline

In this section, I present the results of the estimation of the tail parameter, under the assumption that the distribution of outdegrees follows a power law.

The estimation of this parameter is the main result of this chapter since the value will indicate if idiosyncratic productivity shock to a specific sector might have a significative aggregate effect in the output volatility of the United Kingdom.

Let be the counter-cumulative distribution of outdegrees, $P(k) = \sum_{k'=k}^{N} p_{k'}$, this is, the probability that a random sector selected from the network economy to k or more sectors, the number of sectors supplied, k, follows a power law distribution where the p.d.f. p_k is given by (where c is a positive constant and ζ is the tail index):

$$p_k = ck^{-\zeta} \quad for \quad \zeta > 1 \tag{2.14}$$

In practice, the power law applies only to values greater than a threshold, $k_{\min}$. We can roughly see in Figures 2.6 and 2.7 that the tail of the distribution follows a power law. An estimate on the value of ζ can be obtained by running an OLS regression. Taking the logarithm of both sides of the previous equation:

$$\ln(p_k) = \zeta \ln(k) + c \tag{2.15}$$

where these terms correspond to the empirical log-counter cumulative distribution function and the log-outdegree sequence, respectively, plus a constant.

Unfortunately, Clauset, Shalizi and Newman (2009) show that OLS methods can produce inaccurate estimates of parameters for power-law distributions. Given this problem, there are two options to correct the estimation. In the first one, as proposed by Gabaix and Ibragimov (2009), we estimate an OLS in the counter-cumulative distribution, taking an exogenous threshold, $k_{\min}$, set at 20% of the number of sectors. The problem with this method, as indicated by Clauset et al. (2009), is that we can rule out important observations that constitute the power law behaviour. The second option is a Maximum Likelihood estimator proposed by Clauset et al (2009) taking an endogenous k_{min}.

We focus the attention in the endogenous threshold. Thus, we need to choose a value that makes the probability distributions of the measured data and the best-fit power-law model. The most common measure for quantifying the distance between two distributions is the Kolmogorov-Smirnov statistic, which is the maximum distance between the cumulative distribution function of the data and the fitted model.

Given this cut-off, the method implements an estimator of ζ for the tail of the distribution, equivalent to the well know Hill-ML estimator, which is asymptotically normal and consistent:

$$\widehat{\zeta} = 1 + N \left[\sum_{i=1}^{N} \ln \frac{k_i}{k_{\min}} \right]^{-1} \tag{2.16}$$

where $k_i, i = 1...N$ are the observed values of k such that $k_i \geq k_{min}$

We assume $\zeta > 1$ since distributions with $\zeta \leq 1$ are not normalisable and hence cannot occur in nature.

Table 2.4 shows the estimated results for the value of the tail parameter, ζ where: i) in the first estimation, I show the OLS over the whole distribution; ii) in the next two estimations, I use an exogenous k_{min}, the first one determined visually at about $ln(k_{min}) \sim 3$ and the second one following the rule of 20%, finally; and iii) I use the estimates obtained by the endogenous Clauset et al. (2009) estimator.

As I discussed earlier, the OLS estimation of the entire distribution presents severe problems of accuracy, even the values for the tail parameter are below 1.

The second estimates, with the exogenous threshold, using the visual rule gives an average estimate of the tail parameter which is near to the lower bound of 2.

Analysing the following estimates with only 20 percent of sectors where I dis-

		OLS		Exogenous Cutoff: OLS-Visual cutoff ln(outdegree)>=0			Exogenous Cutoff: OLS-Gabaix (2009) Only 20% sectors at the tail			Clauset et al (2009) MLE with Endogenous cutoff*		
	ζ	s.e.	Obs.	ζ	s.e.	Obs.	ζ	s.e.	Obs.	ζ	s.e.	Obs.
1997	0.867	0.054	106	2.371	0.147	41	4.667	0.426	8	1.222	0.145	71
1998	0.862	0.056	106	2.378	0.143	41	4.180	0.547	8	1.092	0.118	85
1999	0.859	0.056	106	2.404	0.159	40	4.894	0.841	8	1.233	0.147	72
2000	0.883	0.055	106	2.280	0.161	41	5.307	0.919	8	1.242	0.145	73
2001	0.860	0.056	106	2.382	0.164	40	5.090	0.744	8	1.265	0.147	74
2002	0.856	0.057	106	2.301	0.163	41	4.728	0.723	8	1.061	0.117	82
2003	0.887	0.056	106	2.287	0.166	41	5.236	0.769	8	1.256	0.144	76
2004	0.882	0.055	106	2.323	0.168	41	5.448	0.676	8	1.449	0.180	65
2005	0.881	0.055	106	2.264	0.170	41	5.319	0.697	8	1.387	0.168	68
2006	0.892	0.055	106	2.296	0.167	41	5.146	0.663	8	1.434	0.178	65
2007	0.886	0.054	106	2.236	0.151	43	5.183	0.672	8	1.415	0.174	66
2008	0.903	0.053	106	2.203	0.137	42	3.380	0.736	8	1.309	0.155	71
2009	0.902	0.053	106	2.297	0.144	39	3.091	0.691	8	1.221	0.140	76
2010	0.904	0.053	106	2.254	0.151	39	3.282	0.727	8	1.259	0.147	73
Average	0.880	0.055	106	2.305	0.156	41	4.639	0.702	8	1.275	0.151	73

*Estimated using Clauset et al MATLAB routines: http://www.santafe.edu/~aaronc/powerlaws/

Table 2.4: Estimation of tail parameter (1997-2010)

card many observations on the right side of the distribution, these are, on average, well above the upper bound of 2.

Finally, focusing on the last estimates with endogenous cut-off, we observe the estimated value of the tail parameter, $\widehat{\zeta}$, is near (1) for the period in question.

In fact, the tail parameter (estimated by MLE), $\widehat{\zeta}$, is within $\zeta \in (1, 2)$, implying that aggregate volatility now declines with N at a slower rate than the argument of the irrelevance of individual shocks. I show the decay rates in Table 2.5.

In Table 2.5 we observe the implied decay rates by the tail parameters estimated, $\widehat{\zeta}$, implying a significant fat-tailed behaviour, congruently with the finding that there is substantial heterogeneity in the outdegree distribution of the network structure of the U.K. economy.

Decay Rates using MLE- ζ estimates

	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	Average
Tail parameter ζ	1.222	1.092	1.233	1.242	1.265	1.061	1.256	1.449	1.387	1.434	1.415	1.309	1.221	1.259	1.275
Decay Rate															
$v = (\zeta-1)/\zeta$	0.182	0.085	0.189	0.195	0.210	0.058	0.204	0.310	0.279	0.302	0.293	0.236	0.181	0.206	0.215
n^{-v}	0.428	0.674	0.415	0.403	0.376	0.763	0.387	0.236	0.273	0.244	0.255	0.333	0.429	0.383	0.366

Table 2.5: Decay rates using MLE estimate (1997-2010)

This decay rate indicates that the variance is diverging with the number of sectors, as we disaggregate the economy into finer definitions of sectoral technologies, large input-supplying sectors do not vanish.

As a final example, we can consider that if we move from a 1 sector economy to a 100 sector economy, we expect to find only an approximately one-fold decrease (1.2775 as implicated by the average of $\widehat{\zeta}$) in aggregate volatility, in contrast to a 100-fold decrease implied by the law of large numbers (0.009).

2.6 Conclusions

Using a detailed benchmark of the U.K. input-output accounts spanning from 1997 to 2010, we applied the model of intersectoral linkages by Acemoglu et al. (2012).

It is important to note that in this chapter, I provide a more detailed empirical approach to the evaluation of these assumptions, using additional network measures, testing the power law distribution assumption and comparing different methods to estimate the tail parameter.

Regarding the assumptions of asymmetries in the indegree and outdegree, we found symmetry along the sectoral demand: these are sparse matrices reflecting specialisation occurring at the level of narrowly defined technologies. Estimating of the outdegree distribution, we found asymmetry across sectors in their role as input suppliers. In the data, we have highly specialised input suppliers and general purpose inputs.

On the assumption that input-use network structure follows a power law distribution, implementing MLE estimates of ζ for the tail of the distribution, suggested by Clauset et al. (2009), we found that distribution of outdegrees of the U.K. (1997-2010) follow a power law from an endogenously determined minimum degree, with an average tail parameter $\widehat{\zeta} = 1.2$.

Concerning the question whether aggregate fluctuations in economic activity can be obtained from independent productivity shocks, the estimated tail parameter for the U.K., $\widehat{\zeta}$, implies a decay rate on average of $N^{-v} = 0.381$. In this particular case, will converge to zero slower, in contrast to the irrelevance of individual shocks argued.

Further empirical work would be the application to another available set of data (and big enough) for different economies, addressing the use of a weighted network with an appropriate methodology, and testing further sensitivity analysis exploring medium and extreme samples of the network's structure given by the selected percentage of transactions to be considered.

Chapter 3

The network effects of correlated shocks and capital risk sharing on aggregate fluctuations

3.1 Introduction

The topic of aggregate fluctuations in the economic activity is a long-standing question in the study of business cycles, not only the identification of the sources of volatility is necessary to forecast the future pace of the economy, but also to understand how the variance of the aggregate activity could be reduced. Regarding economic policy, the analysis of the stability of the economy has taken more relevance in the aftermath of the recent GFC.

Considering the existence of exogenous shocks to the endogenous variables that determine the equilibrium of an economy, I may think of these as the source of uncertainty in the economic activity. However, a further inspection of this idea involves the study of which type of shocks are the most relevant, their nature and their relationship with the principal actors in the economy.

In a reductive way, I can think of two type of shocks, according to the number of actors affected at first: aggregate or idiosyncratic. Aggregated shocks have a first order effect or an immediate one, on all the relevant factors or variables in the economy at the same time. Idiosyncratic shock has an initial and individual effect on one sector/firm of the economy, and their far-reaching effect is heterogeneous across the economy.

One major issue is to identify the correspondence of such types of shocks to the reality, and their relevance for the aggregate activity. It is hard to think of an event that represents a proper aggregate shock since even traditional examples such as a supply oil shock could be considered as an idiosyncratic productivity shock in the oil sector. However, at the same time, there are several examples of idiosyncratic shocks that precede aggregate consequences, such as the recent problem in the housing sector and its effect on the recent GFC.

The next non-trivial question is to understand how such individual shocks could have aggregate consequences. One way to think about this problem is to assume propagation from co-movement. Intuitively, I can consider two sources of co-movement: i) explicit correlation between idiosyncratic shocks emanating from implicit characteristics embedded in the nature of the shock; and ii) explicit relationships between the actors affected by individual shocks, working as a transmission mechanism.

In this paper, I consider that modelling a multi-sectoral economy with a network structure of input-output relationships is an adequate approach to study different aspects of the structure of the economy that could be relevant for the propagation of different kind of individual shocks.

The objective of this model is to analyse not only the relevance for aggregate fluctuations but also the difference in aggregate volatility under different structures of the economy and different correlation magnitudes. One of the main interests of this chapter is to contribute to the relevant literature by studying two distinct kinds of idiosyncratic shocks that have not already been analysed under different network structures of the economy. The first type of individual shock is an idiosyncratic productivity shock where, unlike previous models, the shocks are considered correlated, while the second one is an idiosyncratic shock to the capital endowments of the firms.

Shocks to the capital endowments rented by the households to the firms, the second kind of individual shock, assume that the representative agent only rents his capital endowment to a subset of firms, implying joint ownership of firms within a subset. The intuition behind this is the potential aggregate consequences of capital risk sharing between firms under different network structures.

The motivation of this set-up is to identify whether the effect of an individual source of capital is relevant for aggregate fluctuations, identifying whether this shock to the wealth of the capital owner washes out in the aggregate or could indeed propagate under an input-output structure.

The model presented in this chapter could be seen as an analogy to the Bernanke, Gertler and Gilchrist (1999) financial accelerator where the changes to the wealth propagates to the whole economy thanks to a financial friction, but in this case, I assume a shock to the wealth that propagates through the input-output structure without the need of a friction.

Regarding the capital risk sharing assumption, I allow for the possibility of idiosyncratic productivity shocks correlated only in the case of firms within the same subset and independent between firms of different groups.

The intuition of this last assumption is that even if it is true that we observe a correlation of individual shocks, it is small, as documented in Foerster, Sarte and Watson (2011) and Carvalho and Gabaix (2013). Thus, such correlation may only be present between some firms sharing common characteristic not modelled.

In this sense, I follow the finding of Lamont (1997) that firms participating in the same capital market can imply correlation. Additionally, I augment this last model including financial frictions.

Specifically, I model a financial friction as a collateral constraint, similar to the intuition of Kiyotaki and Moore (1997). However, the difference is that I assume that due to a moral hazard problem each firm is not able to completely pledge its corresponding income in the case of intermediate inputs bought from firms of a different subset, a problem not existent in the case of input trading between firms of the same subgroup.

Using the assumptions above, I analyse the aggregate volatility of the GDP implied by each shock under different network structures and assumptions of the distribution parameters. I find that the correlation does not wash out in the aggregate. Specifically, under correlation values different to one, network structures where there are dominant suppliers are more volatile than more homogeneous structures.

The present model implies that under the same network structure, the volatility is greater in economies with greater correlation. In contrast, I also find that an economy with heterogeneous levels of correlation implies a lower volatility than the same structure with common correlation values at least equal to the greater correlation in the first economy.

In the model of capital risk sharing, I find that shocks to the capital endowment do not wash out either, and assuming the same level of specific variance, structures with dominant firms are more volatile.

Moreover, by analysing the same structure of the economy, with different arrangements of groups and assuming the same shock to the group with greater average network centrality, I find that an economy with groups involving directly linked firms is more volatile than one with lower average network centrality.

Additionally, using this last model, I observe that comparing two economies with the same network structure but one with groups of linked firms and another with groups of not linked firms, the first economy presents greater volatility when the correlation value is greater than zero. I also find that economies, where there are dominant firms, are more volatile only under specific conditions of the value of the individual variances and correlations. Finally, I find that the parameter or the financial friction is not relevant for the aggregate fluctuations, but only at the level of the GDP and the sales vector.

The present chapter is organised as follows: i) in Section 2, I introduce a brief literature review of related models; ii) in Section 3, I present the model with correlations of idiosyncratic productivity shocks; iii) in Section 4, I show the model with subsets of capital risk sharing; iv) in Section 5, I present the integrated model with financial friction; and v) I exhibit the conclusions.

3.2 Related literature

Our modelling choices are indebted to the seminal papers that analyse the aggregate implications of input-output relationships under general equilibrium, finding relevance of idiosyncratic shock in the aggregate as Long and Plosser (1983), Jovanovic (1987), Long and Plosser (1983), Bak et al (1992) and Horvath (1998).

The seminal work of Long and Plosser (1983) constitutes an explanation of how I can model relationships between sectors that imply relevant idiosyncratic independent productivity shocks. The authors assume that each sector obtains and sells intermediate inputs from and to the rest of sectors and that such relationships are the cause of aggregate fluctuations.

The focus on idiosyncratic shocks has gained renovated interest following the papers by Gabaix (2011) and Acemoglu et al. (2012). One of the main results of both papers is that individual shocks could have aggregate consequences; however, each offers a different explanation, the first one argues that the size of the effect will depend on the distributions of the size of firms, and in the second one will be depending on the structure of the network of input-outputs relationships.

My analysis of the aggregate fluctuations of the GDP is highly related to Carvalho (2010) and Acemoglu et al. (2012), they model the economy explicitly as a network, finding that the level of aggregate volatility from idiosyncratic shocks depends on the network structure. The difference in this work is that I analyse a different kind of shocks and additional assumptions about the structure of the economy including subsets of firms. I share some similar findings such as the result that network structures with dominant sectors are more volatile, I found additional conditions depending on correlation levels.

The present work is also related to the granular model of Carvalho and Gabaix (2013), first because of the common finding that idiosyncratic shocks could have aggregate consequences, and second because of his alternative specification of correlated shocks. The authors find that this inclusion is empirically relevant. I conclude that under different correlations assumptions and structures, aggregate volatility could differ. Considering this correlation specification, this is similar to the assumption that Foerster, Sarte and Watson (2011) analyse in their model, the difference is that they have a dynamic structural model and do not model specifically the network of the economy.

The model of subsets of firms is close to the network model of ownership of Burlon (2015). The differences, apart from his model being dynamic, is that the author does not model the input-output network, but the network of ownership that emerges from the purchase of shares by the households. In this model, the ownership emerges from the renting of capital by the representative agent to only one subset of firms. Hence the type of shock that I analyse is also different. The common result is the relevance for aggregate fluctuations of a non-trivial specification of ownership groups.

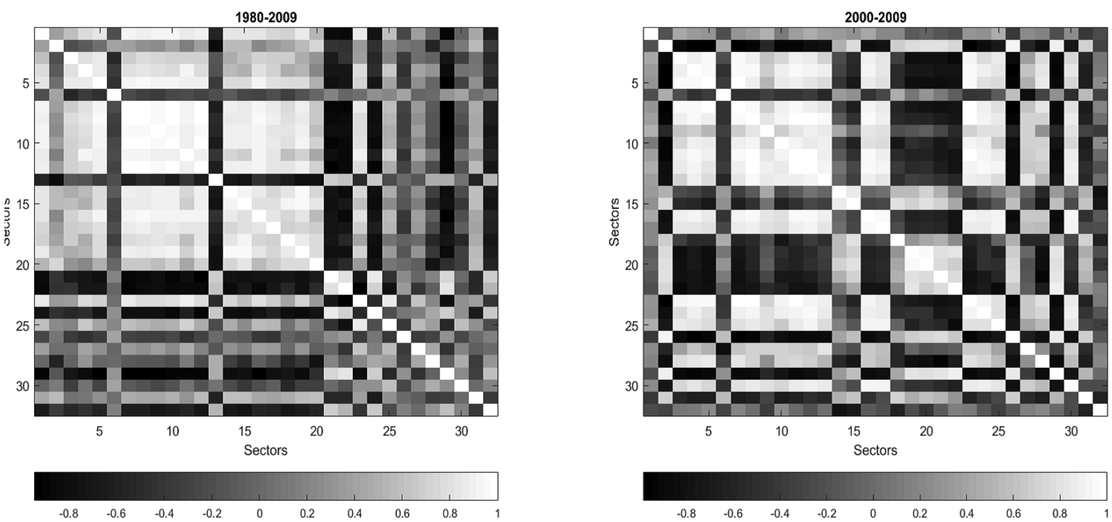
The assumption of financial friction, though I find it not relevant for aggregate fluctuations, is close to the model of Bigio and La'O (2013) since I also assume that the revenue of the firms is not perfectly pledgeable implying a collateral constraint. The difference is that in my model the only transaction subject to a collateral constraint is the purchase of intermediate goods from firms outside the group of ownerships, whereas in their model all input purchases are constrained.

Carvalho and Gabaix (2013) consider idiosyncratic productivity correlated shocks in an input-output model that could potentially be redundant as the co-movement from linkages implies already a correlation. In this sense, I consider that analysing the behaviour of volatility under correlated shocks for different network structures is important for this final purpose of incorporating the correlations under more natural and restrictive assumptions in this last set-up, as I will describe later.

The idea of including financial friction in a model of input-output linkages is shared by the work of Kalemli-Ozcan et al (2014) and Su (2014), though the specifications differ considerably. In the first model, the authors analyse partial equilibrium chains of production and financial shocks. The second model is a dynamic model with a financial friction specification due to an agency problem similar to Christiano, Motto and Rostagno (2014), however, he does not analyse different network structures.

Finally, the result of the relevance of the financial friction for the aggregate level of production, that could be seen as a wedge, is related to the literature of misallocation in a network economy as the papers by Jones (2011) and Fadinger, Ghiglino and Teteryatnikova (2015).

Besides this emerging line of study, there has been also empirical studies that corroborate the importance of idiosyncratic shocks. Foerster, Sarte and Watson (2011), Mizuno, Souma and Watanabe (2014), di Giovanni, Levchenko and Mejean (2014) and Barrot and Sauvagnat (2014), among others, have documented the non-trivial importance of firm and sectoral level volatility for aggregate activity.



Source: Estimated using EU.K.LEMS database

Figure 3.1: U.K. sectoral TFP correlation matrix

Empirically, with a simple analysis of the total factor productivity (TFP) of each sector (Figure 3.1), I find that in the case of the U.K., the correlation matrix is not the identity matrix. For this economy, using sectoral data from the EUKLEMPS database, each sector is positive or negatively correlated with the others. In fact, considering 32 sectors, and calculating the correlations matrix using the TFP index data, I found that the average correlations (excluding autocorrelation) are 0.1280, 0.1576 and 0.0774, in the periods 1980-2009, 1990-2009 and 2000-2009, respectively. The following matrices show the correlation structure in two periods.

3.3 Multi-sector economy with idiosyncratic correlated productivity shocks

3.3.1 Overview

In this model, I will describe the problem of the actors in this economy, the solutions equations and the assumptions of the model. The two most important characteristics of this model are that I specify a disaggregated production economy and that the idiosyncratic productivity shocks are correlated across firms.

One important part of the possible correlation of shocks across firms is given by the input-output connections between firms. However, as I specify such relationships, I assume that the correlation could be present due to another factor not explicitly specified, such as common ownership, common embedded technology or same managerial practices.

3.3.2 Households

I consider a continuum unitary-mass of households with preferences over n-goods. I assume a constant relative risk aversion utility function, with $u'(c_i) > 0$, $u''(c_i) < 0$ and $lim_{c_i \to 0} u'(c_i) = +\infty$:

$$u(c_1, ..., c_n) = \ln \left[\prod_{i=1}^{n} (c_i)^{1/n} \right] = \frac{1}{n} \sum_{i=1}^{n} \ln c_i \tag{3.1}$$

where n indicates the total number of goods in the economy and i indicates one specific good produced by one sector or firm. The household maximises the utility over the consumption of n-goods, given by the following constrained problem:

$$\max_{c_i \forall i} \ \frac{1}{n} \sum_{i=1}^{n} \ln c_i \tag{3.2}$$

s.t.

$$\sum_{i=1}^{n} p_i c_i = r \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i \tag{3.3}$$

c_i is the consumption of the final good i, p_i is the real price of each good, k_i represents capital rented to each sector i at a price r. π_i is the profit of each firm, which in equilibrium will be equal to zero.

3.3.3 Firms

I assume a n number of sectors or firms that produce, in a competitive market, a good i using capital and intermediate goods. The output could be used as a final good consumed by the households or as an intermediate input in the production of another sector.

In order to produce q_i, each sector i uses capital, k_i, rented from the households at a price r, and intermediate inputs, q_{ij}, purchased from sectors j at a price p_j. Production is carried out according to a Cobb-Douglas constant returns to scale technology, $f(\cdot)$, that satisfies for each factor $f(0) = 0$, $f'(\cdot) > 0$, $f''(\cdot) < 0$, $f'(0) = \infty$, $f'(\infty) = 0$. The technology of production is subject to a random productivity shock, z_i, identically distributed and correlated across sector, that takes the form of a Solow augmented capital shock realised at the beginning of the period. Each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_i q_i - r k_i - \sum_{j=1}^{n} p_j q_{ij} \tag{3.4}$$

s.t.

$$q_i = (z_i k_i)^{\alpha} \prod_{j=1}^{n} q_{ij}^{(1-\alpha) w_{ij}} \tag{3.5}$$

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i \tag{3.6}$$

$$z_i = exp(\mu_i), \quad \mu_i \sim N(0, \sigma_i^2, \rho_{ij} \in [-1, 1] \forall j) \;\; \forall i \tag{3.7}$$

Where $\alpha \in (0, 1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents the proportion of intermediate good j in the total use in sector i and is also a typical element of the input-output matrix a-la-Leontief, where the sum of the shares from each sector j in the production of sector i have to add 1 according to the assumption of constant returns to scale. The parameter ρ_{ij} is the variance of the logarithm of the idiosyncratic shock to sector i, and the parameter ρ_{ij} is the correlation between sector i and j that can take values between -1 and 1.

3.3.4 Market clearing

The market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations, where k is the exogenous endowment of capital:

$$c_i + \sum_{j=1}^{n} q_{ij} = q_i \ \ \forall i \tag{3.8}$$

$$\sum_{i=1}^{n} k_i = k \tag{3.9}$$

3.3.5 Equilibrium

Definition 39. *I define a competitive equilibrium in this set-up as the set of allocations* (c_i, k_i, q_{ij}, q_i) *and prices* (r, p_i) *for* $(i, j) = 1, ..., n$*, such that:*

(i) *household and firm problems are solved,*

(ii) *market clearing conditions are satisfied.*

The following are the solutions for intermediate goods and capital in equilibrium:

$$q_{ij} = \frac{\gamma_i(1-\alpha)w_{ij}q_j}{\gamma_j} \tag{3.10}$$

$$k_i = \frac{k\gamma_i}{\sum_{i=1}^{n} \gamma_i} \tag{3.11}$$

These solutions depend on the network given by γ:

$$\gamma = (\mathbb{I} - (1-\alpha)\mathbf{W}')^{-1}\mathbf{1} \tag{3.12}$$

This is in fact the solution to the inverse of the ratio consumption/output in equilibrium; where the matrix of input-output shares, identity matrix and vector of ones are given by:

$$\mathbf{W} \equiv \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}_{nxn} \qquad \mathbb{I} \equiv \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn} \qquad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

3.3.6 GDP

Definition 40. *I define the vector of influence or centrality vector as:*

$$\mathbf{v}' \equiv \left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I} - (1-\alpha)\mathbf{W})^{-1}\right] \tag{3.13}$$

Proposition 15. *I can express the equation for the natural logarithm of the GDP, Y, in the following way:*

$$Y = \mathbf{v}'\mu + \Lambda \tag{3.14}$$

Where μ is the vector of the natural logarithm of idiosyncratic correlated shocks

z_i and Λ is a variable of parameters given by:

$$\Lambda \equiv \ln k + \ln \alpha + \frac{(1-\alpha)}{\alpha}\left(\ln(1-\alpha) + \mathbf{v}'\mathbf{W}\mathbf{w}\right) + \left(\mathbf{v}' - \frac{1}{n}\mathbf{1}'\right)\overline{\gamma} \tag{3.15}$$

$$\mathbf{z} \equiv \begin{bmatrix} \ln z_1 \\ \vdots \\ \ln z_n \end{bmatrix}_{nx1} = \mu \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}_{nx1} \quad \mathbf{w} \equiv \begin{bmatrix} \ln w_{1j} \\ \vdots \\ \ln w_{nj} \end{bmatrix}_{nx1} \quad \overline{\gamma} \equiv \begin{bmatrix} \ln \gamma_1 \\ \vdots \\ \ln \gamma_n \end{bmatrix}_{nx1}$$

3.3.7 Volatility of GDP

I define the aggregate fluctuation in the economic activity as the standard deviation of the logarithm of the GDP. I will call this measure of volatility as the standard deviation or the volatility of GDP indistinctly (omitting the word logarithm). Using the equation found in the previous section I define the volatility as:

Definition 41. *The volatility of the logarithm of the GDP is given by:*

$$[var(Y)]^{1/2} \equiv [var(\mathbf{v}'\mu)]^{1/2} \tag{3.16}$$

Proposition 16. *I can express the standard deviation of the natural logarithm of the GDP in the following way:*

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n} \sigma_i^2 v_i^2 + \sum_{i \neq j} \rho_{ij}\sigma_i\sigma_j v_i v_j\right]^{1/2} \tag{3.17}$$

or

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j v_i v_j\right]^{1/2} \tag{3.18}$$

Where $\sigma_i^2 \in (0, \infty)$ is the variance of each idiosyncratic shock to the sector i, $\rho_{ij} \in [-1, 1]$ is the correlation between the shocks to the sectors i and j, being $\rho_{ij} = 1$ when $i = j$, and v_i is the i-th element of the vector of influence, $\mathbf{v}'$.

Proof Proposition 16*.* To obtain the previous variance equation, and according to the definition of volatility, I need to know the variance of the vector of random correlated shocks, μ, as the vector of influence $\mathbf{v}'$ is compose of parameters only. I know that the distribution of the random shocks is the same, but I also know they are correlated; thus, I need to consider the matrix of variance-covariance. The multivariate distribution of the vector μ is given by:

$$\mu \sim N\left(\mathbf{0}, \mathbf{\Sigma}\right) \tag{3.19}$$

where the parameters of the multivariate normal distribution of the vector of random shocks are given by:

$$\mathbf{0} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{nx1} \qquad \mathbf{\Sigma} \equiv \begin{bmatrix} \sigma_1^2 & \dots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \dots & \sigma_n^2 \end{bmatrix}_{nxn}$$

The matrix $\mathbf{\Sigma}$ is the matrix of variance-covariance where the elements on the diagonal are the individual variances of the shocks and the elements outside the diagonal are the covariances between each pair of shocks. I can obtain the covariance of each pair i, j concerning the correlation and variance parameters in the following way:

$$\Sigma_{ij} = E[\mu_i \mu_j] - E[\mu_i]E[\mu_j] \tag{3.20}$$

Recalling that $E[\mu_i] = 0$ from the distribution of each shock, and that correlation between i and j is given by $\rho_{ij} = E[\mu_i \mu_j]/\sigma_i \sigma_j$, the covariance of each pair is:

$$\Sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \tag{3.21}$$

Then the volatility of GDP is given by:

$$[var(Y)]^{1/2} \equiv [\mathbf{v}' \mathbf{\Sigma} \mathbf{v}]^{1/2} \tag{3.22}$$

where the diagonal of the matrix $\mathbf{\Sigma}$ is composed of the parameters σ_i^2 for each shock, and the rest of elements is given by $\rho_{ij} \sigma_i \sigma_j$. ■

Corollary 3. *Let $\sigma_i^2 = \sigma^2$ and $\rho_{ij} = \rho$ for all i, j, the volatility of the logarithm of GDP is given by:*

$$[var(Y)]^{1/2} = \sqrt{\sigma^2 \left(\rho + (1 - \rho) \|\mathbf{v}\|_2^2\right)} \tag{3.23}$$

Where $\| \cdot \|_2$ is the Euclidean norm of the influence vector, this is, $\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2}$.

This particular result will be crucial later in this paper when I compare volatilities between economies under the assumptions of same individual variance and correlation. Also, in this result, one can already see that even when the Euclidean norm of the influence vector is zero if there is correlation different that zero, the individual volatility will not vanish in the aggregate.

3.3.7.a Volatility of GDP as a function of out-degrees

In order to express the volatility of GDP as a function of the out-degrees, I need to define first such concept.

Definition 42. *The out-degree of a firm i is given by number of firms to which i sells inputs. In this case, given that the input-output matrix is compose of the weights w_{ij}, I will consider the weighted out-degree, d_i, as sum of the shares of the input i in the production of all the firms:*

$$d_i = \sum_{j=1}^{n} w_{ji} \tag{3.24}$$

Proposition 17. *I can express the volatility of GDP as a function of the out-degrees:*

$$[var(Y)]^{1/2} = \frac{\alpha}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_i\sigma_j \left(1 + \Delta_{ij} + \Xi_{ij} + \Psi_{ij}\right)} \tag{3.25}$$

Where:

$$\Delta_{ij} \equiv (1-\alpha)(d_i + d_j) + (1-\alpha)^2 d_i d_j \tag{3.26}$$

$$\Xi_{ij} \equiv (1-\alpha)^2 \left(\sum_{j=1}^{n} d_j w_{ji} + \sum_{i=1}^{n} d_i w_{ij}\right) \tag{3.27}$$

$$\Psi_{ij} \equiv (1-\alpha)^3 \left(d_i \sum_{j=1}^{n} d_j w_{ji} + d_j \sum_{i=1}^{n} d_i w_{ij}\right) + (1-\alpha)^4 \left(\sum_{j=1}^{n} d_j w_{ji} \sum_{i=1}^{n} d_i w_{ij}\right) \tag{3.28}$$

From this equation, the standard deviation of the GDP depends on the degree of connectivity of each node measured by its outdegree, d_i, but also depends on the centrality of each sector measured by the outdegrees of the sector to which this sector is connected, $d_j w_{ji}$. The question that arises from this result is which of the former or the latter effects is more relevant. In the comparison section, I will analyse this issue.

3.3.7.b Asymptotic volatility of GDP

To find the asymptotic volatility of GDP I need to assume that the number of firms goes to a vast quantity, $n \to +\infty$. Additionally, I assume the variance of the shock to be bounded below and above, $\sigma_i^2 \in (\overline{\sigma}_i^2, \underline{\sigma}_i^2)$, and that the variance and the correlation do not depend on n.

Proposition 18. *As n goes to infinity, the volatility of GDP can be expressed in terms of the following lower bound:*

$$[var(Y)]^{1/2} = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i + \sum_{i=1}^{n} d_i w_{ij}\right)\left(d_j + \sum_{j=1}^{n} d_j w_{ji}\right)}\right) \tag{3.29}$$

Where Ω represents the asymptotic notation when $f_n = \Omega(g_n)$ if $\liminf_{n\to\infty} f_n/g_n > 0$. Using that bound I can compare the volatility of GDP and the relevance of individual shocks.

Corollary 4. *The volatility of GDP, in the presence of idiosyncratic correlated productivity shocks, vanishes at an equal or lower speed than the bound implied by the law of large* *numbers:*

$$\Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i+\sum_{i=1}^{n}d_iw_{ij}\right)\left(d_j+\sum_{j=1}^{n}d_jw_{ji}\right)}\right)\geq\Omega\left(\frac{1}{\sqrt{n}}\right) \tag{3.30}$$

In this case, due to the network multipliers and the additional effects of the correlation terms, the volatility decreases slower as n becomes large.

3.3.7.c Variance decomposition of GDP

The elements that are not in the diagonal of the covariance matrix, $\mathbf{\Sigma}$, are composed of the correlation coefficient, ρ_{ij}, multiplied by the square root of the individual variance of each pair, σ_i. I apply the Cholesky decomposition to disentangle the variance and the correlation parameters.

In this way, I can express the covariance matrix of the correlated idiosyncratic shocks in the following way:

$$\mathbf{\Sigma}=\mathbf{C}\mathbf{C}' \tag{3.31}$$

In this specific case, as I described in the previous section, given that the errors have mean zero, each covariance term is given by $\rho_{ij}\sigma_i\sigma_j$. Then, applying the Cholesky algorithm, the lower triangular matrix $\mathbf{C}$ is given by the Hadamard product of the following lower triangular matrices:

$$\mathbf{C} \equiv \left(\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ \sigma_2 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n & \sigma_n & \dots & \sigma_n \end{bmatrix} \circ \begin{bmatrix} f(\rho)_{11} & 0 & \dots & 0 \\ f(\rho)_{21} & f(\rho)_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f(\rho)_{n1} & f(\rho)_{n2} & \dots & f(\rho)_{nn} \end{bmatrix} \right)_{nxn}$$

Where each element of the second matrix, $f(\rho)_{ij}$, is a specific function of the correlation parameters alone. Which in turn introduces the complication of finding each element of the second matrix because, according to the Cholesky algorithm, as I go farther from the first item, the next elements are functions of the correlation parameters of the previous ones. However, having this decomposition for n-small, I can express the volatility of the economy as a sum of terms composed of the influence vector, the individual variance and the function of the correlation parameters.

Proposition 19. *Applying the Cholesky decomposition to the covariance matrix of the correlated idiosyncratic shocks, I can express the volatility of the GDP in the following way:*

$$[var(Y)]^{1/2} = \left[\sum_{j=1}^{n} \left(\sum_{i \geq j} v_i \sigma_i f(\rho)_{ij} \right)^2 \right]^{1/2} \tag{3.32}$$

This expression follows from substituting the matrix $\mathbf{\Sigma}$ by the Cholesky matrices $\mathbf{C}$ and post and pre-multiplying them by the influence vector $\mathbf{v}$. I can use this expression to analyse the data the effect of the correlation parameter and the network position in the presence of an idiosyncratic shock to one specific firm.

This decomposition will be useful to determine the relevance of the correlation parameters in any possible empirical application of the model.

3.3.8 Economy as a network

To analyse whether the aggregate fluctuations depend on the structure of the economy, I analyse specifically related cases. In the next sections, I consider specific examples, in particular, five different kinds of economies. To analyse such cases, I need to define first the economy as a set of nodes and edges.

I consider five economies, A, B, C, D and F, with the same number of sectors, n, but with a different structure of input-output connections represented by a network. I will call the economy A vertical because the input-output downstream follows a direct line of links among sectors. I will call the economy B star because there is only one sector, star, providing inputs to the rest of the economy. I call the economy C tree because the first sector provides inputs to the second sector and this one provides inputs to the remainder of the economy.

I call the economy D circle because the first sector provides input for the second one, this to the third one and so on until the sector n provides inputs to the first sector, closing the circle of inputs-outputs. Finally, I call the economy F empty because each sector is providing inputs to itself only, there are no connections between sectors.

For each of these economies, I have the following definition of a network.

Definition 43. *Each economy represented by a directed network is given by the set of nodes (sectors), $N = \{1, ..., n\}$, set of directed links among sectors, $E = \left\{ \overrightarrow{\langle i, j \rangle}, \forall (i, j) \in N \right\}$,* *and set of weights,* $w = \left\{ w_{ij}, \forall (i,j) \in N | 0 < w_{ij} < 1 \wedge \sum_{j=1}^{n} w_{ij} = 1 \right\}$.

Thus, the vertical economy, A, the star economy, B, the tree economy, C, the circle economy, D, and the empty economy, F, are given by the following networks:

$$A = (N, E_A, w_A) \tag{3.33}$$

$$B = (N, E_B, w_B) \tag{3.34}$$

$$C = (N, E_C, w_C) \tag{3.35}$$

$$D = (N, E_D, w_D) \tag{3.36}$$

$$F = (N, E_F, w_F) \tag{3.37}$$

In each case, according to the assumptions, I can define the set of nodes and weight for each economy as follows, where the symbol $\rightarrow$ means a directed link and the delimiter $\langle ., . \rangle$ represents a tuple of connected nodes.

In the case of the vertical economy, A, the set of edges and the respective weights, are given by the links that include the first sector loop and the connections in a line from this sector to the n-sector, one by one, as described in the following definition.

Definition 44. *Given the set of firms (nodes), N, the vertical economy, A, is given by the following set of edges and weights:*

$$E_A = \left\{ \overrightarrow{\langle 1, 1 \rangle}, \overrightarrow{\langle i, i+1 \rangle}, \forall i \in N \setminus n \right\} \tag{3.38}$$

$$w_A = \{w_{11} = w_{i+1,i} = 1, \forall i \in N \setminus n\} \tag{3.39}$$

The star economy, B, is defined by the set of edges and the respective weights given by the first firm with a loop, this firm providing inputs to the rest of the firms in the economy.

Definition 45. *Given the set of firms (nodes), N, the star economy, B, is given by the following set of edges and weights:*

$$E_B = \left\{ \overrightarrow{\langle 1, i \rangle}, \forall i \in N \right\} \tag{3.40}$$

$$w_B = \{w_{i1} = 1, \forall i \in N\} \tag{3.41}$$

The tree economy, C, is given by the edges and weights that include the loop of the first sector, the connection between this sector and the second one, and the provision of inputs by later to the rest of the economy, as follows.

Definition 46. *Given the set of firms (nodes), N, the tree economy, C, is given by the following set of edges and weights:*

$$E_C = \left\{ \overrightarrow{\langle 1, 1 \rangle}, \overrightarrow{\langle 1, 2 \rangle}, \overrightarrow{\langle 2, i \rangle}, \forall i \in N \setminus (1, 2) \right\} \tag{3.42}$$

$$w_C = \{w_{11} = w_{21} = w_{i2} = 1, \forall i \in N \setminus (1, 2)\} \tag{3.43}$$

In the case of the circle economy, D, the sets of edges and weights are given by the connections in a closed chain from sector 1 to the n-sector, one by one, described in the following definition.

Definition 47. *Given the set of firms (nodes), N, the circle economy, D, is given by the following set of edges and weights:*

$$E_D = \left\{ \overrightarrow{\langle n, 1 \rangle}, \overrightarrow{\langle i, i+1 \rangle}, \forall i \in N \setminus n \right\} \tag{3.44}$$

$$w_D = \{ w_{1n} = w_{i+1,i} = 1, \forall i \in N \setminus n \} \tag{3.45}$$

The empty economy, F, is given by the edges and weights that includes only the loops of each sector, there are no more connections, as follows.

Definition 48. *Given the set of firms (nodes), N, the tree economy, F, is given by the following set of edges and weights:*

$$E_F = \left\{ \overrightarrow{\langle i, i \rangle}, \forall i \in N \right\} \tag{3.46}$$

$$w_F = \{ w_{ii} = 1, \forall i \in N \} \tag{3.47}$$

All the economies share the same set of nodes, N, but they also share the assumption that all the weights are equal to one, $\sum_{i=1}^{n} w_{ij} = 1$. I can represent each of those networks with the following graph (Figure 3.2):

In the following sections, the explicit modelling of the economy as a network will have consequences for aggregate volatility. In fact, considering the existence of an input-output structure implies different levels of volatility, for example, considering the economies described above, and the model parameters. I know that when the share of capital, α, is very close to one, the relevance of the intermediate trade in the product will be close to zero, in this particular case, the only source of propagation will be the correlation parameter.

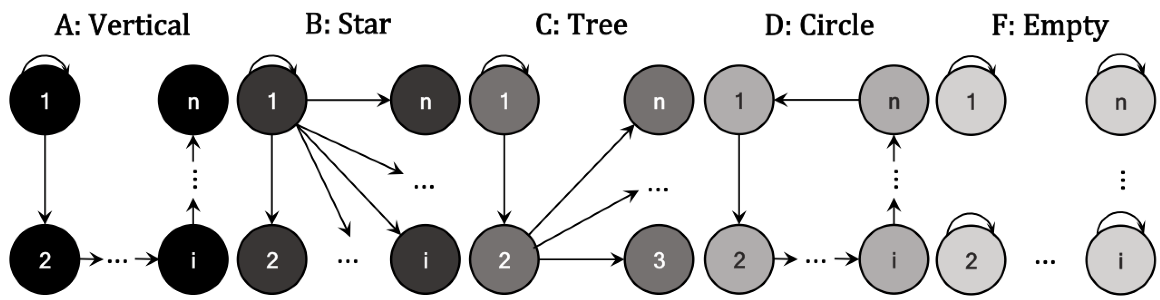


Figure 3.2: Typical network structures

This implication is presented in the following diagram (Figure 3.3), where I plot the five economies described before with given parameters σ^2, ρ, for 100 sectors.

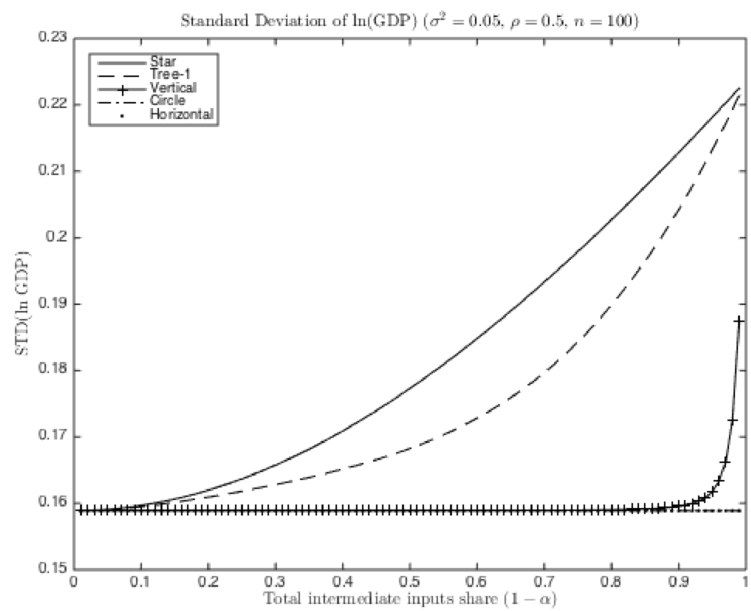


Figure 3.3: Volatility of ln(GDP) when the intermediate inputs share increases

As the relevance of the intermediate inputs become larger, this is, $(1-\alpha) \to 1$, the standard deviation of the GDP will diverge according to the network structure of each economy, scaled by the correlation parameter. In fact, there is a lower bound of the volatility of GDP in the case of some economies. I will describe with more detail this result in the following sections.

3.3.9 Comparison of volatilities

To identify the difference of volatilities given by the network structure, I can analyse the specific examples of the previous economies.

In this section, I compare different economies, under different assumptions. I find that the network structure implies a different level of volatility and that even under a large degree of disaggregation, the volatility never goes to zero as long as the correlation is different that this value.

3.3.9.a Same idiosyncratic variances and correlations, different network structure

To compare the volatility of GDP among the various networks, the main assumption of this section is that the idiosyncratic variance and correlation parameter of each sector are the same within and between economies, this is, the covariance matrix, Σ, is the same for all economies.

Additionally, using the definition of an economy as a network, I will consider the same set of nodes, N, for all economies, but with a different set of nodes, E, and weights, w, as depicted below (Figure 3.4).

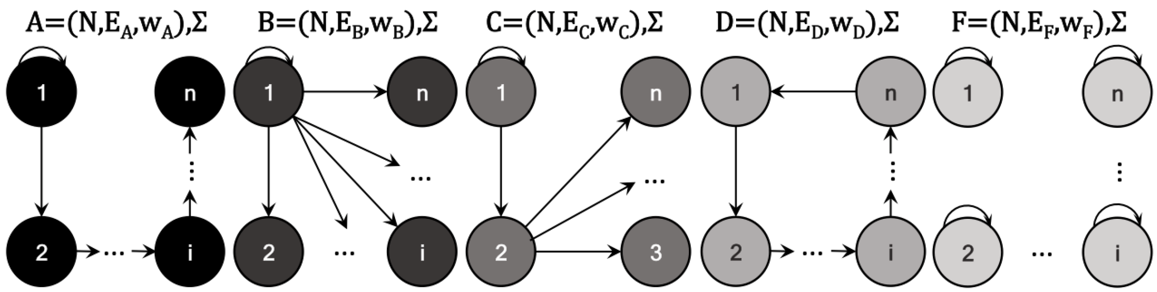


Figure 3.4: Different economies

Proposition 20. *Considering economies A, B, C, D and F, assuming idiosyncratic variance $\sigma_i^2 = \sigma^2$, and correlation $\rho_{ij} = \rho \in [-1, 1)$, same for all sectors and economies, and number of sectors, n, non-trivial but not very large, this is, $3 < n << +\infty$, the difference in volatilities is:*

$$\sqrt{var(Y)}_B > \sqrt{var(Y)}_C > \sqrt{var(Y)}_A > \sqrt{var(Y)}_D = \sqrt{var(Y)}_F$$

The previous proposition indicates that as long as the economy becomes less regular or symmetric, as in the case of the economies B and C, the volatility of GDP will become larger. Whenever economies are more regular and disaggregated (Figure 3.5), the standard deviation of the economy will decrease.

The previous result agrees with the findings of Carvalho (2010) and Acemoglu et al. (2012) for star and regular economies when $\rho = 0$. However, the difference that I found is that such result also holds under the assumption that $\rho \in [-1, 1)$.

In the following proposition I show the consequence when $\rho = 1$ giving place to a new result.

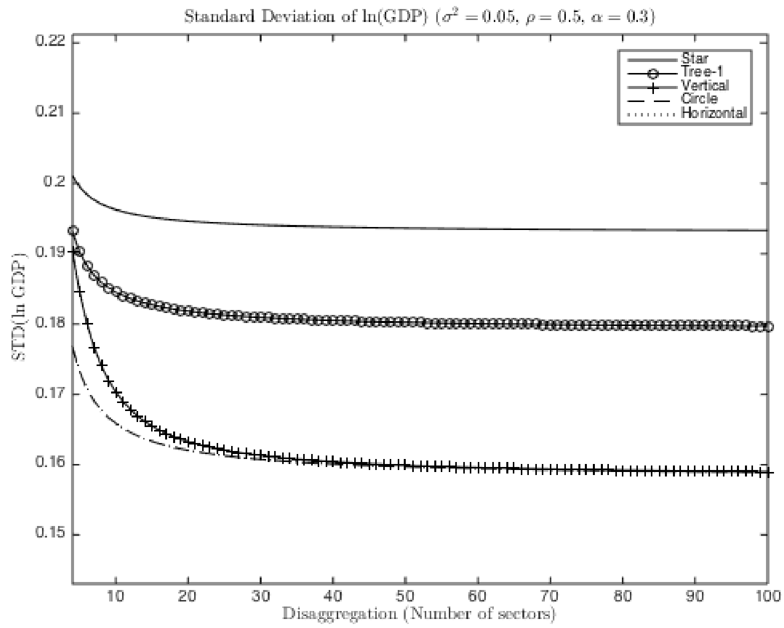


Figure 3.5: Volatility of ln(GDP) as the number of sector increases

Proposition 21. *Considering economies A, B, C, D and F, assuming idiosyncratic variance $\sigma_i^2 = \sigma^2$, perfect positive correlation $\rho_{ij} = \rho = 1$, same for all sectors and economies, and number of sectors, n, non-trivial but not very large, this is, $3 < n << +\infty$, the volatility of GDP is the same for all economies:*

$$\sqrt{var(Y)}_B = \sqrt{var(Y)}_C = \sqrt{var(Y)}_A = \sqrt{var(Y)}_D = \sqrt{var(Y)}_F$$

The network structure becomes irrelevant for the aggregate volatility assuming a perfect and positive correlation. The idiosyncratic shock will be propagated perfectly to each node, mimicking an aggregate shock.

One can see those two results in the following diagram (Figure 3.6). I plot the standard deviation of GDP for different economies as a function of the correlation parameter. I can see that when $\rho = 1$ the volatility of the economy is equal to $\sigma = \sqrt{0.05}$, as an aggregated shock originated from the perfect correlation.

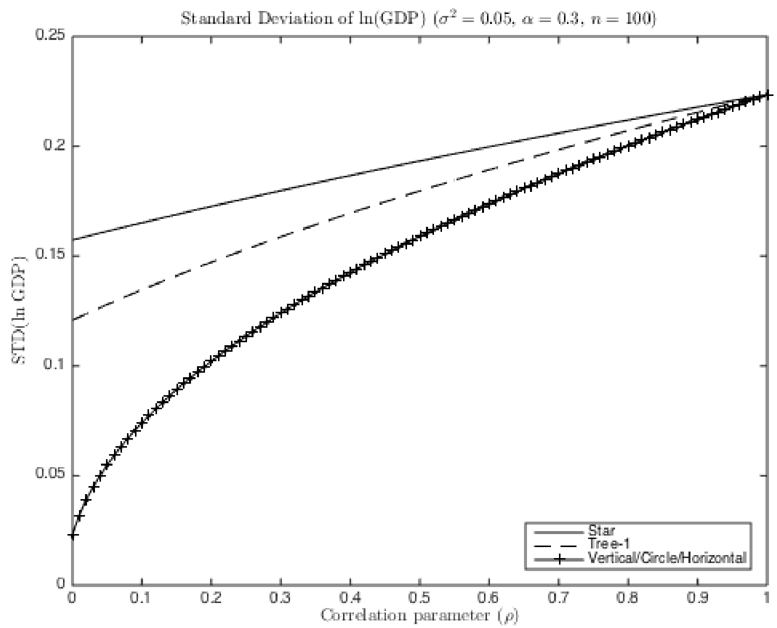


Figure 3.6: Volatility of ln(GDP) as the correlation parameter increases

As long as the correlation increases, the volatility of the GDP will also increase, for all the economies.

In the previous plot of the volatility of GDP as a function of the number of sectors (Figure 3.5), I observe that as the level of disaggregation, the standard deviation decreases, and in the case of the vertical economy, it will converge to the level of the horizontal and circle economies. I analyse this limiting result in the following proposition.

Proposition 22. *Considering economies A, B, C, D and F, assuming idiosyncratic variance $\sigma_i^2 = \sigma^2$, and correlation $\rho_{ij} = \rho \in [-1, 1)$, same for all sectors and economies, and the number of sectors goes to infinity, $n \to \infty$, the difference in volatilities is:*

$$\lim_{n\to\infty} \sqrt{var(Y)}_B > \lim_{n\to\infty} \sqrt{var(Y)}_C > \lim_{n\to\infty} \sqrt{var(Y)}_A = \lim_{n\to\infty} \sqrt{var(Y)}_D = \lim_{n\to\infty} \sqrt{var(Y)}_F$$

When the number of sectors, n, goes to infinity also has implications to analyse if the individual volatility will disappear or not. According to the law of large numbers, when there are no connections and no correlation, in the economies with lower variance, the volatility of GDP generated from idiosyncratic shocks becomes zero as n goes to infinity (Acemoglu at al., 2012).

However, in the presence of correlated shocks with correlation parameter different than zero, the volatility of GDP never goes to zero, only reaches a lower bound determined by the network structure. In particular, I found that the economies with lower volatility (A, D and F) reach the lower bound as a function of the individual variance scaled by the correlation parameter.

Proposition 23. *Considering economies A, B, C, D and F, assuming idiosyncratic variance $\sigma_i^2 = \sigma^2$, non-zero correlation $\rho_{ij} = \rho \neq 0$, as the number of sectors goes to infinity, $n \to \infty$, the volatility of Y never goes to zero and reaches a minimum value given by the economies A, D and F:*

$$\lim_{n\to\infty} \sqrt{var(Y)}_{A,B,C,D,F} > 0$$

$$\inf\left\{\sqrt{var(Y)}_{A,B,C,D,F} : n \to \infty\right\} = \sqrt{\sigma^2\rho}$$

In the first panel of Figure 3.7, one can see that when the correlation is zero, $\rho = 0$, as the level of disaggregation increases, the volatility of GDP goes to zero in the case of vertical, circle and empty economies. However, if the correlation parameter is different from zero, as in the second panel, $\rho = 0.5$, the standard deviation of the GDP never goes to zero for all the economies, reaching a lower bound in the case of vertical, circle and empty economies, equal to $\sqrt{\sigma^2\rho} = 0.1581$.

This result implies that even at greater levels of disaggregation, a shock to an individual sector will have aggregate consequences from the interplaying of two effects, the network effect that depends on the input-output structure, and the correlation effect that depends on the covariance matrix.

As one can see in Figure 3.7, the idiosyncratic shock is propagated differently according to the network structure, but it is scaled by the correlation parameter, implying an increasing level of volatility as the correlation goes to one, value at which the individual shock mimics an aggregated effect.

Now, I analyse if the important effect of the network structure in the presence of an idiosyncratic shock is the outdegree (connectedness) or the higher-order connectivity (centrality) of each node captured by, the higher orders of the influence vector.

In particular, I know that the higher-order connectivity is a broader measure than the outdegree because its recursivity depends not only on the outdegree of each sector but also on the outdegree of the sectors to which this node is connected.

One way to analyse this two effects is comparing the volatility generated in

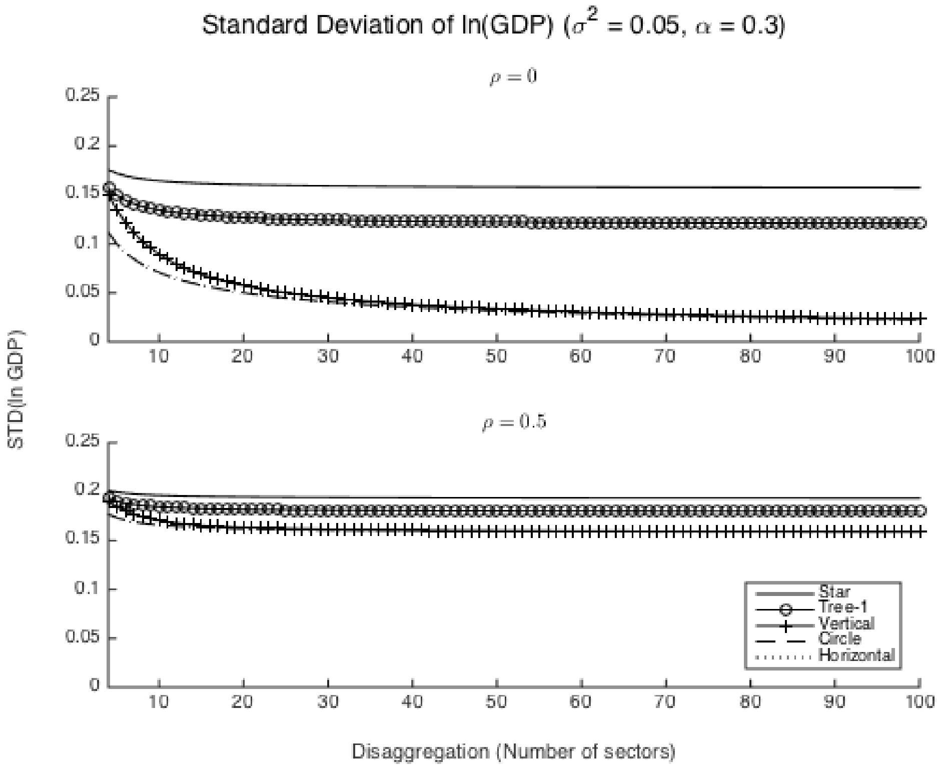


Figure 3.7: Volatility of ln(GDP) scaled by correlation parameter

two economies from a sector with the same outdegree but different centrality between economies.

In the following Figure (3.8), I present the vertical and the tree economy, analysing only the volatility generated by the sector 1. We can observe that the outdegree of this node is 2 (loop and link to sector 2) in both economies, but the centrality is different because in the vertical economy the outdegree of sector 2 is 1 and in the tree economy is $n-1$. The latter implies a different influence vector taking second or higher orders.

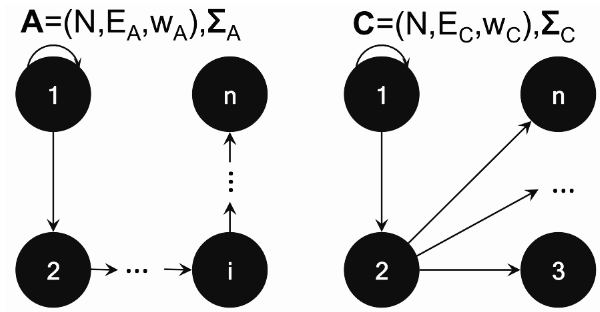


Figure 3.8: Volatility of vertical and tree economies

Proposition 24. *Considering the economies A and C, assuming that the variance of the idiosyncratic shock to the sector 1 is $\sigma_1^2 = \sigma^2$, and $\sigma_i^2 = 0$ for all $i \neq 1$, for both economies and for all $n > 3$, the difference in volatilities of the GDP is the following:*

$$\sqrt{var(Y|\sigma_i^2 = 0, \forall i \neq 1)}_A < \sqrt{var(Y|\sigma_i^2 = 0, \forall i \neq 1)}_C$$

The result of this assumption is that, concerning the volatility of the economy, the relevant characteristic of the network structure is not only the outdegree but mostly the recursivity of the centrality of each node. Thus, it does not only matter the sector to which each node is connected but the connections of its buyers and the links of the buyers to its buyers, and so on.

3.3.9.b Same network structure, same idiosyncratic variances, different correlations

In this section, I depart from the assumption that the correlation parameter, ρ, is the same within and between economies. Specifically, I am interested in analysing if the volatility of the economy depends not only on the network structure but also in the correlation differences.

Moreover, I am also interested to know whether, under the same network architecture, the presence of correlation between sectors that are directly connected implies a lower level of volatility that the correlation between sectors that are not directly connected.

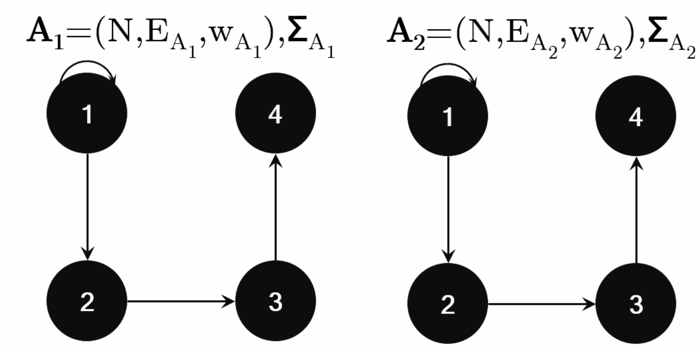


Figure 3.9: Vertical economies with different correlation structure

To make these comparisons, I analyse two vertical economies, A_1 and A_2, with the same set of nodes, links and weights, but with different correlation structure, as depicted in the following graph (Figure 3.9).

To answer the first question, I compare the previous two economies assuming the same correlation within economies but different between networks, as described in the following proposition.

Proposition 25. *Considering two economies with the same vertical network structure, A_1 and A_2; assuming that the variance of each idiosyncratic shock, $\sigma_i^2 = \sigma$, is the same across sectors and also the same in the two economies; different correlation structure be-*

tween economies, but same across sectors of the same economy; $\rho_{ij} = \rho$, *the difference in volatilities of the GDP is given by the following relationships:*

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) > (\rho \in \mathbf{\Sigma}_{A_2})$$

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) < (\rho \in \mathbf{\Sigma}_{A_2})$$

$$[var(Y)]_{A_1}^{1/2} = [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) = (\rho \in \mathbf{\Sigma}_{A_2})$$

I present the result of this comparison in plot (Figure 3.10), where the volatility of the GDP, in the case of the vertical economy, is an increasing function of the correlation parameter.

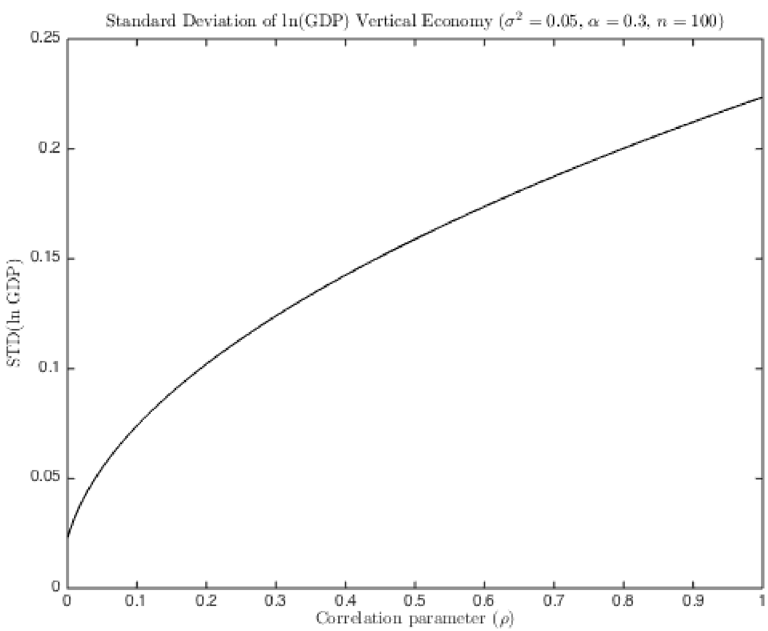


Figure 3.10: Volatility of vertical economy

Comparing to networks with the same structure and parameters but with different correlation, the one with higher correlation will dominate regarding the magnitude of the standard deviation of GDP.

To answer the second question, I compare the same vertical economies again, but now I assume that in the first economy there is correlation only between sectors that are connected and in the second economy there is a correlation between non-directly connected sector, correlation is zero otherwise, as described in the following proposition.

Proposition 26. *Considering two economies with the same vertical network structure, A_1 and A_2; assume that the variance of each idiosyncratic shock, $\sigma_i^2 = \sigma$, is the same across sectors and also the same in the two economies; assume that in the economy A_1 there is correlation only between connected sectors, $\neg\overrightarrow{\langle i,j \rangle}$, and in the economy A_2 there is only correlation between unconnected sectors, $\overrightarrow{\langle i,j \rangle}$, and the correlation parameter, is the same across economies, $\rho_{ij} = \rho$; the difference in volatilities of the GDP is given by the following relationships:*

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \ \ if \ \rho \in (0,1]$$

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \ \ if \ \rho \in [-1,0)$$

In the next plot, I illustrate this result (Figure 3.11). The dashed function corresponds to the economy with correlated connected nodes, which is always greater in volatility of GDP than the solid line, the economy with correlated unconnected nodes, when $\rho \in (0,1]$. Both functions are strictly increasing on the correlation parameter as I previously found.

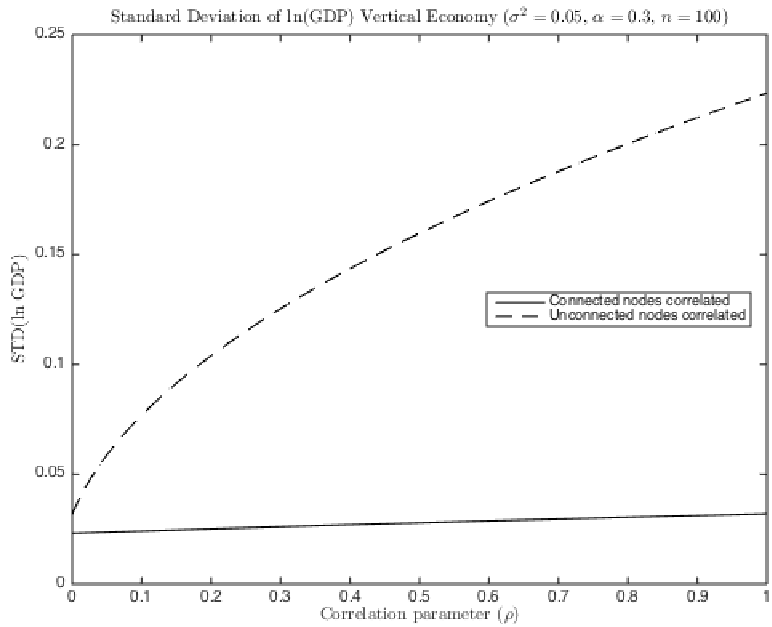


Figure 3.11: Volatility of vertical economy with correlated and uncorrelated nodes

This result tells us that when there is greater diversification regarding the correlation structure that runs over the same network, this is, nodes non-directly linked are more correlated, the volatility of GDP will be greater when there is a positive correlation. The latter is because when there is an idiosyncratic productivity shock hitting any sector, this will propagate downstream through the network structure, but it will also be transmitted immediately to the nodes to which such sector is connected, reaching faster and with more power the whole economy.

Finally, I can analyse specific cases of different correlation structures over the same vertical networks as described in the following proposition.

Proposition 27. *Considering two economies with the same vertical network structure,* A_1 *and* A_2*; assume that the variance of each idiosyncratic shock,* $\sigma_i^2 = \sigma$*, is the same across sectors and also the same in the two economies; assume different correlation structure between economies, but same across sectors of the* A_2 *economy,* $[\rho_{ij} = \rho_{A_2}, \forall (i \neq j) \in \mathbf{\Sigma}_{A_2}]$*; the difference in volatilities of the GDP is given by the following relationships:*

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \ \ if \ [\rho_{A_2} = (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}) > (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}), \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \ \ if \ [\rho_{A_2} = (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}) < (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}), \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

$$[var(Y)]_{A_1}^{1/2} = [var(Y)]_{A_2}^{1/2} \ \ if \ [\rho_{A_2} = (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}) = (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}), \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

The first result indicates that when I compare an economy with homogeneous correlation and one with a greater correlation of connected sectors, the volatility of the homogeneous economy will be larger when its correlation is equal to the higher level of the heterogeneous economy.

The second inequality tells us when the opposite occurs when the economy with homogeneous correlation has a magnitude equal to the lower correlation of the economy with heterogeneous correlation.

These results indicate that to reduce the aggregate volatility, one needs to decrease the correlation level, and to lessen the heterogeneity of the correlation structure over the network.

3.4 Multi-sector economy with idiosyncratic capital risk sharing

3.4.1 Overview

As I did in the previous model, I will start describing the problem of the actors in this economy, the main assumptions and the solutions equations of the model. Besides the specification of a disaggregated production economy, the important characteristic of this model is that instead of assuming idiosyncratic productivity shocks, I assume individual shocks to the capital provided to each firm.

To characterise the relevance of this effect concerning the network structure, I assume that the representative agent only rents capital to a subset of firms, this is, to only one partition of the network economy.

I can interpret each subset of firms or sectors as the partition of the network that receives funding, in the form of capital rented, from a specific agent, analogous to the provision of capital from financial institutions to only some specific firms.

One significant result of this set-up is that the network structure of the input-output economy and the partition arrangements of capital shared interplay implying a different level of volatility of GDP. Under this economy, there will capital risk sharing effects that depend not only on the distribution parameters of the shock but also on the network's arrangements of the economy.

3.4.2 Households

I consider a representative household described in section (2), the difference is that the representative agent is now renting his capital endowment to only a subset of firms. I have the following problem:

$$\max_{c_i \forall i} \ \frac{1}{n}\sum_{i=1}^{n} \ln c_i \tag{3.48}$$

s.t.

$$\sum_{i=1}^{n} p_i c_i = r \sum_{i \in M_s} k_i + \sum_{i \in M_s} \pi_i \tag{3.49}$$

The subset of firms, M_s, is described in the following definition:

Definition 49. *If the set of firms is given by $N = \{i \in \mathbb{N}^+ | i \leq n, n \in \mathbb{N}^+\}$ where i is one firm or sector, I define a family of subsets $\mathbb{M} = \{M_s \in \mathbb{N}^+ | M_s \subseteq N; s \in S\}$ where S is the index set defined by $S = \{s \in \mathbb{N}^+ | s \leq m, m \in \mathbb{N}^+\}$, such that all the subsets are pairwise disjoint, $\cap_{s \in S} M_s = \varnothing$, and the union of all subsets constitute the set of firms, $\cup_{s \in S} M_s = N$.*

This definition will be crucial to compare different economies with a different distribution of the capital shared between firms.

3.4.3 Firms

I assume almost the same specification for the firms as in section (2) with the difference that there is no idiosyncratic Solow capital augmented shock. Each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_i q_i - r k_i - \sum_{j=1}^{n} p_j q_{ij} \tag{3.50}$$

s.t.

$$q_i = (k_i)^\alpha \prod_{j=1}^{n} q_{ij}^{(1-\alpha) w_{ij}} \tag{3.51}$$

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i \tag{3.52}$$

where $\alpha \in (0,1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents capital the use in sector i.

3.4.4 Market clearing

In this section, I specify the different type of shocks, in contrast to the idiosyncratic correlated shock in the model of the previous section.

The household can rent its capital endowment, $\overline{K}_s$, at a price r, to only one subset of firms, M_s, denoted by the index s. I assume that this endowment, $\overline{K}_s$, is subject to an i.i.d. random shock, z_s. Each firm has a total capital demand equal to k_i.

In this way, the market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations:

$$c_i + \sum_{j=1}^{n} q_{ij} = q_i \;\; \forall i \tag{3.53}$$

$$\sum_{i \in M_s} k_i = z_s \overline{K}_s \ \ \forall s \tag{3.54}$$

Where

$$z_s = exp(\mu_s), \ \ \mu_s \sim N(0, \sigma_s^2) \ \ i.i.d. \, \forall s \tag{3.55}$$

3.4.5 Equilibrium

Definition 50. *I define a competitive equilibrium as the set of allocations (c_i, k_i, q_{ij}, q_i) and prices (r, p_i) for $i, j = 1, ..., n$, such that:*

(i) *household and firm problems are solved,*

(ii) *market clearing conditions are satisfied.*

The following are the solutions for intermediate goods and capital in equilibrium:

$$q_{ij} = \frac{\gamma_i (1-\alpha) w_{ij} q_j}{\gamma_j} \tag{3.56}$$

$$k_i = \frac{z_s \overline{K}_s \gamma_i}{\sum\limits_{i \in M_s} \gamma_i} \tag{3.57}$$

These solutions depend on the network given by γ, which is the same vector provided in the previous model.

3.4.6 GDP

Recalling the definition of the influence vector given in the previous section:

$$\mathbf{v}' \equiv \left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right]$$

After using the solutions of the equilibrium and the definition of the GDP as detailed in Appendix 3.8, I can obtain an expression of the natural logarithm of the GDP as a function of the parameters and the random shocks.

Proposition 28. *I can express the equation for the natural logarithm of the GDP in the following way:*

$$Y = \mathbf{v}'\mathbf{I_M}\mu + \Lambda \tag{3.58}$$

where μ is the vector idiosyncratic shocks, z_s, to each capital endowment for each subset of firms, M_s, and Λ is a variable of parameters given by:

$$\Lambda \equiv \mathbf{v}'\left(\mathbf{I_M}(\mathbf{k}-\check{\gamma}) + \frac{(1-\alpha)}{\alpha}\mathbf{W}\mathbf{w} + \overline{\gamma}\right) + \frac{(1-\alpha)}{\alpha}\ln(1-\alpha) + \ln n - \frac{1}{n}\mathbf{1}'\overline{\gamma} \tag{3.59}$$

where

$$\mathbf{z} \equiv \begin{bmatrix} \ln z_1 \\ \vdots \\ \ln z_m \end{bmatrix}_{mx1} = \mu \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}_{mx1} \quad \mathbf{k} \equiv \begin{bmatrix} \ln \overline{K}_1 \\ \vdots \\ \ln \overline{K}_m \end{bmatrix}_{mx1} \quad \mathbf{w} \equiv \begin{bmatrix} \ln w_{1j} \\ \vdots \\ \ln w_{nj} \end{bmatrix}_{nx1}$$

$$\overline{\gamma} \equiv \begin{bmatrix} \ln \gamma_1 \\ \vdots \\ \ln \gamma_n \end{bmatrix}_{nx1} \quad \check{\gamma} \equiv \begin{bmatrix} \ln \sum\limits_{i\in M_1} \gamma_i \\ \vdots \\ \ln \sum\limits_{i\in M_m} \gamma_i \end{bmatrix}_{mx1} \quad \mathbf{I_M} \equiv \begin{bmatrix} 1_{M_1}(1) & \dots & 1_{M_m}(1) \\ \vdots & \ddots & \vdots \\ 1_{M_1}(n) & \dots & 1_{M_m}(n) \end{bmatrix}_{nxm}$$

and the i-th element of the matrix $\mathbf{I_M}$ is an indicator function of the membership of a firm in the subset M_s of N, taking the value 1 for all i in M_s and the value 0 for all elements of N not in M_s:

$$1_{M_s}(i) \equiv \begin{cases} 1 & \text{if } i \in M_s \\ 0 & \text{if } i \notin M_s \end{cases} \tag{3.60}$$

where according to the definition of the subset M_s, $\sum_{s=1}^{m} 1_s(i) = 1$, which means that each i belongs to only one subset.

3.4.7 Volatility of GDP

I define the aggregate fluctuations in the economic activity as the standard deviation of the logarithm of the GDP, using the equation found in the previous section. The definition of volatility is as follows:

Definition 51. *The volatility of the logarithm of the GDP is given by:*

$$[var(Y)]^{1/2} \equiv [var(\mathbf{v}'\mathbf{I_M}\mu)]^{1/2} \tag{3.61}$$

Proposition 29. *I can express the standard deviation of the natural logarithm of the GDP in the following way:*

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n} v_i^2 \left(\sum_{s=1}^{m} (1_s(i))^2 \sigma_s^2\right)\right]^{1/2} \tag{3.62}$$

This result tells us that the structure of the variance matrix of the economy will not be trivial and that it will depend on the arrangements of the partitions of capital shared among sectors.

3.4.8 Economy as a network

To analyse whether the aggregate fluctuations depend on the structure of the economy, I investigate specific appropriate cases as I did in the previous model, but now I incorporate an additional structure given by the subset of firms.

Considering two economies, $A_{n,m}$ and $B_{n,m}$, with the same number of sectors, n, and the same number of subset of sectors, m, but with different input-output structure and various elements in each subset of sectors.

I call the economy $A_{n,m}$ vertical because the input-output downstream follows a direct line of links among sectors. I call the economy $B_{n,m}$ star because there is only one sector, a star, providing inputs to the rest of the economy. Thus, I can represent each of these economies by a network composed of nodes, partitions, links and weights using the following definitions.

Definition 52. *An economy is represented by a directed network given by the set of nodes (sectors), $N = \{i \in \mathbb{N}^{+} | i \leq n, n \in \mathbb{N}^{+}\}$, sets of directed links among sectors, $E = \left\{\overrightarrow{\langle i,j \rangle}, \forall (i,j) \in N\right\}$, and the set of weights, $w = \left\{w_{ij}, \forall (i,j) \in N | 0 < w_{ij} < 1 \wedge \sum_{j=1}^{n} w_{ij} = 1, \forall i \in N\right\}$.*

The realised particular sets of partitions within the network, as I will suppose that each partition (or subset) is receiving capital from the same actor. Thus, the shock to the capital distributed to each partition will have different effects on aggregate volatility according to the structure of the input-output economy and the structure of partitions in the network. I will consider a partition following the next definition.

Definition 53. *Considering an economy represented as a directed network, each of the nodes belongs to a family of subsets, $\mathbb{M} = \{M_s \in \mathbb{N}^+ | M_s \subseteq N, s \in S\}$, where S is the index set defined by $S = \{s \in \mathbb{N}^+ | s \leq m, m \in \mathbb{N}^+\}$ such that all the partitions are pairwise disjoint, $\cap_{s \in S} M_s = \varnothing$, and the union of all partitions constitute the set of nodes, $\cup_{s \in S} M_s = N$.*

Thus, the vertical economy, $A_{n,m}$, and the star economy, $B_{n,m}$, are given by the following networks:

$$A_{n,m} = (N, \mathbb{M}_{A_{n,m}}, E_{A_{n,m}}, w_{A_{n,m}}) \tag{3.63}$$

$$B_{n,m} = (N, \mathbb{M}_{B_{n,m}}, E_{B_{n,m}}, w_{B_{n,m}}) \tag{3.64}$$

In each case, according to the assumptions of each economy, I can define the set of nodes and weight for each economy as follows, where the symbol $\rightarrow$ means a directed link and the delimiter $\langle ., . \rangle$ represents a tuple of connected nodes.

As in the previous section, in the vertical economy, A, the set of edges and the respective weights are given by the links that include the first sector loop and the connections in a line from this sector to the n-sector, one by one.

Definition 54. *Given the set of sectors (nodes), N, the vertical economy, A, is given by the following set of edges, weights and possible partitions:*

$$E_{A_{n,m}} = \left\{ \overrightarrow{\langle 1, 1 \rangle}, \overrightarrow{\langle i, i+1 \rangle}, \forall i \in N \setminus n \right\} \tag{3.65}$$

$$w_{A_{n,m}} = \{w_{11} = w_{i+1,i} = 1, \forall i \in N \setminus n\} \tag{3.66}$$

$$\mathbb{M}_{A_{n,m}} = \left\{ M_{s,A} \in \mathbb{N}^+ | M_{s,A} \subseteq N, s \in S, \cap_{s \in S} M_{s,A} = \varnothing, \cup_{s \in S} M_{s,A} = N \right\} \tag{3.67}$$

The star economy, B, is again defined by the set of edges and the respective weights given by the first sector with loop providing inputs to the rest of the economy.

Definition 55. *Given the set of firms (nodes), N, the star economy, B, is given by the following set of edges, weights and partitions:*

$$E_B = \left\{ \overrightarrow{\langle 1, i \rangle}, \forall i \in N \right\} \tag{3.68}$$

$$w_B = \{ w_{i1} = 1, \forall i \in N \} \tag{3.69}$$

$$\mathbb{M}_{B_{n,m}} = \left\{ M_{s,B} \in \mathbb{N}^+ | M_{s,B} \subseteq N, s \in S, \cap_{s \in S} M_{s,B} = \varnothing, \cup_{s \in S} M_{s,B} = N \right\} \tag{3.70}$$

Considering only four sectors, $n = 4$, which is the minimum non-trivial number, and partitions compose of only two sectors, $m = 2$, the possible combinations of partitions are the following:

$$\mathbb{M}_{4,2} = \{\{1, 2\}, \{3, 4\}\} \tag{3.71}$$

$$\mathbb{M}_{4,2} = \{\{1, 3\}, \{2, 4\}\} \tag{3.72}$$

$$\mathbb{M}_{4,2} = \{\{1, 4\}, \{2, 3\}\} \tag{3.73}$$

For the purpose of analysis, I will consider only the first two possible combinations, as I will show in the next section. In particular, I will use the first two combinations in the case of the vertical economy, having $A^1_{4,2}$ for the first combination, and $A^2_{4,2}$ for the second combination. In the event of the star economy,

$B_{4,2}$, I will use only the first mix. I illustrate these assumptions in the following diagram (Figure 3.12).

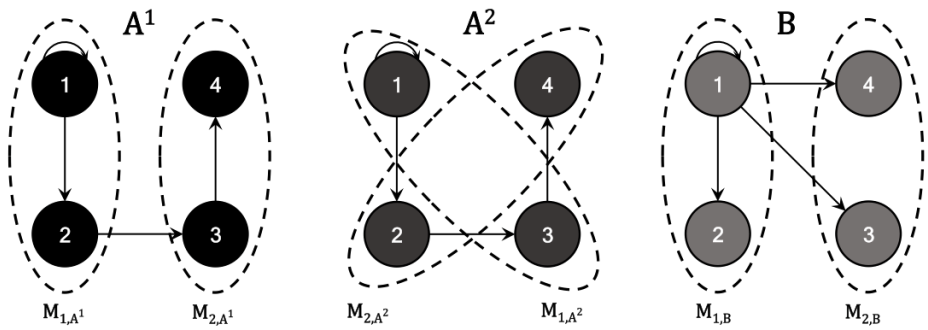


Figure 3.12: Four-sector economies

3.4.9 Comparison of volatilities

In this section, I analyse whether the arrangement of the capital shared by different sectors plays a role in aggregate volatility. To do this, I compare different network structures with different partitions of capital sharing arrangements described in the previous subsection.

I find that the diversification of the subsets of nodes, concerning the connections among firms, has a direct role in the level of aggregate volatility.

3.4.9.a Same vertical input-output structure, different subsets of sectors and variances

In this comparison, I analyse the same input-output structure of two economies, specifically, vertical economies with four sectors, but with a different arrangement of partitions.

I assume that there are two partitions in each economy, but I suppose that in the first one each partition is composed of directly connected sectors whereas, in the second one, I assume that each subset is formed by sectors that are not directly connected, separated by one node.

The idea behind those assumptions is to compare the effect of more diversification in the capital risk sharing of the economy. In Figure 3.13, one can observe that the second economy is more diversified concerning the direct connection between the nodes with each partition.

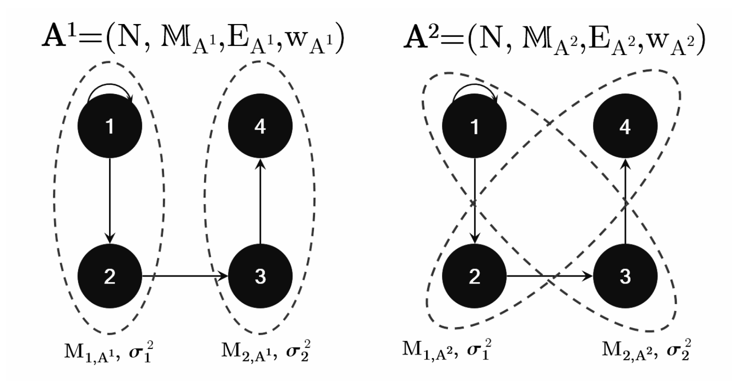


Figure 3.13: Diversification of a four-sector economy

I can suppose that the second economy is composed of representative agents that take the role of financial institutions providing capital to sectors that are not close traders, for example, agriculture and electronic devices.

Proposition 30. *Considering the economies $A^1_{4,2}$ and $A^2_{4,2}$ and assuming that the variance of each idiosyncratic shock, σ_s^2, is the same in the two economies but different across for* *each subset of sectors, the difference in volatilities of the GDP is the following:*

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 = \sigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 > \sigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 < \sigma_2^2$$

The result of this comparison tells us that when the idiosyncratic shocks are the same between and within economies, the partition arrangement is irrelevant. However, when the shock to the first partition of any economy is greater than the shock to the second partition, the economy is less diversified and more volatile.

The opposite occurs when the shock to the capital provided to the second partition is greater than the shock to the first one, for both economies, the economy that is more diversified has a higher level of volatility. This result indicates that the arrangement of the capital shared by each sector plays a first-order role in the propagation of the idiosyncratic shock.

I illustrate the previous result is the following graph (Figure 3.14), fixing the variance of the shock to the second partition, I plot the difference in volatilities of GDP between economies as a function of the difference of the individual variance to the capital shared by each partition.

Whenever the difference between the individual shocks is zero, the difference in volatility of GDP is also zero. The difference of volatilities of GDP is positive when the difference of the variance of the shock to each partition is positive, even when I assume that the variance of each partition is the same between economies.

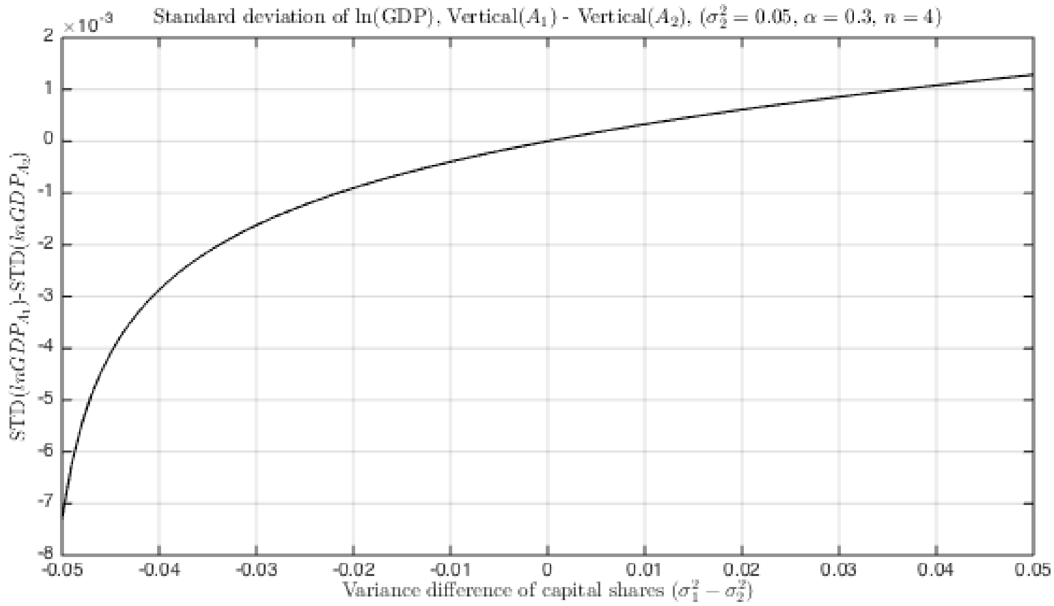


Figure 3.14: Difference in volatilities of GDP

3.4.9.b Different input-output structure, same subsets of sectors and variances

In this section, I analyse the role of the input-output structure under the same partition arrangements. In particular, I will compare the vertical economy and the star economy as depicted in the following graph (Figure 3.15).

Proposition 31. *Considering two economies, $A^1_{4,2}$ and $B_{4,2}$, assuming that the variance of each idiosyncratic shock, $\sigma_s^2 = \sigma_1^2 = \sigma_2^2$, is the same in the two economies and the same across subsets of sectors, the difference in volatilities of the GDP between the two economies is given by:*

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{B_{4,2}}$$

One first finding is that the economy with greater overall centrality and more

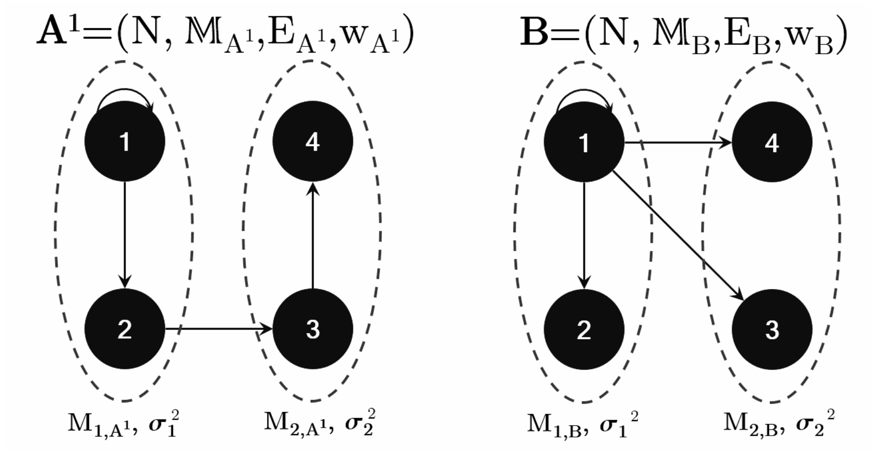


Figure 3.15: Vertical and star economies

asymmetric, the star economy, implies a higher level of volatility even when I suppose that the variance of the shocks to the capital to each partition is the same, as I can see in the following graph (Figure 3.16).

In (Figure 3.16), I know that when the individual variance is the same, the difference in volatilities of GDP is not zero, but negative; this implies that the volatility of the star economy is greater, originated from a higher overall centrality propagation of the shocks. The opposite occurs when the difference of individual variance is negative, this is when the variance of the shock to the firm partition is smaller than the variance of the shock to the second partition.

These results imply that as when the variance of the shock to the capital increases provided to the first partition, holding the variance of the other shock constant, the difference of the volatility of the vertical and the star economy decreases. In contrast to the model without partitions, the input-output structure interacts with the arrangements of the capital risk sharing.

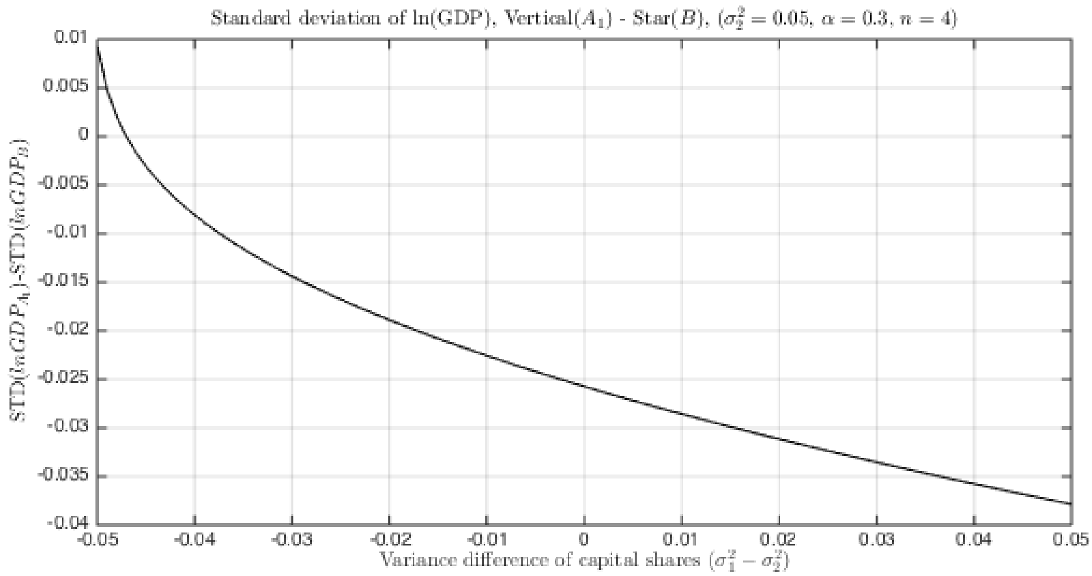


Figure 3.16: Vertical and star economies when the variance is the same

3.5 Multi-sector economy with capital risk sharing, intra-group correlations and collateral constraints

In this section, I integrate the previous two models, the multi-sector model with idiosyncratic correlated productivity shocks and the model with partitions of capital risk sharing.

To have an intuitive integration, I will assume that there is the correlation between the idiosyncratic productivity shock of the sectors that are within the same group of capital sharing.

The idea behind this assumption is that each partition of capital risk sharing can also represent the ownership of the representative agent that is providing capital to the firms within such subset. The correlation assumption has the intuition that firms that are under the same ownership are likely to share some non-explicit common characteristics, such as traditional corporate management.

One additional feature that I include in this model is the presence of financial frictions between the sectors that do not belong to the same partition of capital. The intuition of this assumption is that, for a given firm, it is harder to obtain working capital (inputs financed) frictionless from firms that do not belong to the same group. This assumption implies that each firm cannot fully pledge the income share that otherwise would correspond to pay for the inputs purchased from firms outside its partition, generating a financial friction that I will describe in more detail in next sections.

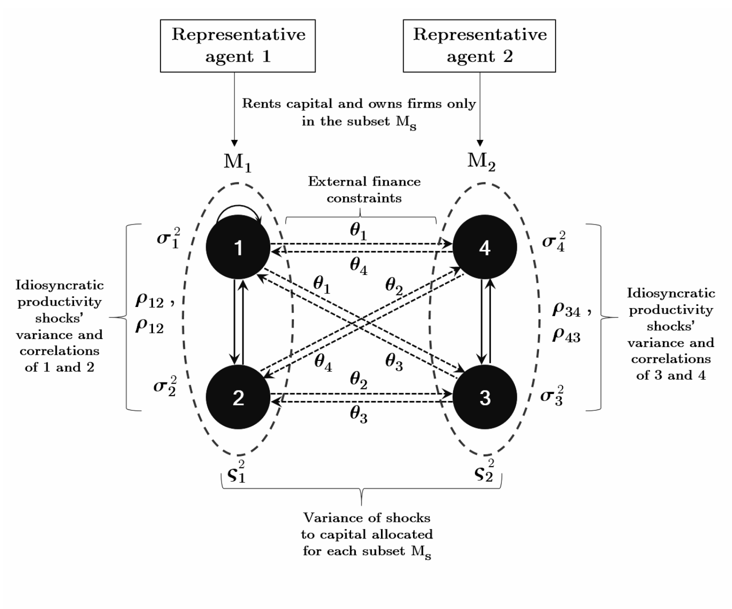


Figure 3.17: Diagram of integrated model

Diagram 3.17 illustrates a simplification of the model of this section considering only four firms, two subsets and specific elements within each group. In the following subsections, I will define more formally the interactions of this economy.

3.5.1 Households

I consider the same representative household described in section (3), the representative agent owns a subset of firms and receives the rent of his capital endowment from only such subset, in fact, owning the firms of this subset.

3.5.2 Firms

I assume n number of firms that produce, in a competitive market, a good i using capital and intermediate goods. The output could be utilised as a final good consumed by the households or as an intermediate input in the production of another sector.

As I defined in the previous section, each firm belongs to a subset, M_s, and each member of this subset shares the characteristics of renting capital from the same agent, then, I can assume that these firms belong to the same group of ownerships.

In this way, to produce q_i, each firm i in the same subset, M_s, uses capital, k_i, rented from the same agent at a price r, and intermediate inputs, q_{ij}, purchased from firms j at a price p_j. I assume that all of these inputs are paid at the end of the period after production is carried out and the revenue is obtained.

The intermediate inputs, q_{ij}, could be purchased from firms of the same subset or from firms belonging to a different subgroup. This distinction allows me to introduce the credit constraint under the assumption that to obtain intermediate inputs from firms of a different subset, and hence of a different ownership, each firm needs a collateral. The later does not happen when the firm gets inputs from firms of the same subset as the potential moral hazard problem does not exist between firms of the same ownership.

Production is carried out according to a Cobb-Douglas constant returns to scale technology, $f(\cdot)$, that satisfies for each factor $f(0) = 0$, $f'(\cdot) > 0$, $f''(\cdot) < 0$, $f'(0) = \infty$, $f'(\infty) = 0$. The technology of production is subject to a random correlated shock, z_i, identically distributed that takes the form of a Solow augmented capital realised at the beginning of the period. This shock is independent to the firms in other subsets but correlated to the firms of the same subset given by the parameter ρ_{ij}.

Each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_i q_i - r k_i - \sum_{j \in M_s} p_j q_{ij} - \sum_{j \notin M_s} p_j q_{ij} \tag{3.74}$$

s.t.

$$q_i = (z_i k_i)^{\alpha} \prod_{j \in M_s} q_{ij}^{(1-\alpha) w_{ij}} \prod_{j \notin M_s} q_{ij}^{(1-\alpha) w_{ij}} \tag{3.75}$$

$$\sum_{j \notin M_s} p_j q_{ij} \leq \theta_i \left(p_i q_i - r k_i - \sum_{j \in M_s} p_j q_{ij} \right) \tag{3.76}$$

Where

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i \tag{3.77}$$

$$z_i = exp(\mu_i), \ \ \mu_i \sim N(0, \sigma_i^2, \rho_{ij} \in [-1, 1] \forall j \in M_s) \ \forall i \tag{3.78}$$

where $\alpha \in (0, 1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents the share of intermediate good j in the total use in sector i.

The second restriction represents the external working capital constraint, by external I mean the inputs obtained from firms of a different subset.

Following the intuition of Kiyotaki and Moore (1997), the constraint indicates that a firm needs to have a fraction θ_i of its revenue as collateral in order to get the intermediate inputs from firms of another subset, this because of the possibility of default, which is not relevant in the case of firms within the same subsets as they are under the same ownership.

3.5.3 Market clearing

The household rents his capital endowment, $\overline{K}_s$, at a price r, to only one subset of firms, M_s, denoted by the index s. I assume that this endowment, $\overline{K}_s$, is subject to an i.i.d. random shock, u_s. Each firm has a total capital demand equal to k_i.

In this way, the market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations:

$$c_i + \sum_{j \in M_s} q_{ij} + \sum_{j \notin M_s} q_{ij} = q_i \ \forall \ (i \in M_s) \tag{3.79}$$

$$\sum_{i \in M_s} k_i = u_s \overline{K}_s \ \forall s \tag{3.80}$$

where

$$u_s = exp(\nu_s), \;\; \nu_s \sim N(0, \varsigma_s^2) \;\; i.i.d. \, \forall s \tag{3.81}$$

3.5.4 Equilibrium

Definition 56. *I define a competitive equilibrium in this set-up as the set of allocations* (c_i, k_i, q_{ij}, q_i) *and prices* (r, p_i) *for* $i, j = 1, ..., n$*, such that:*

(i) *household and firm problems are solved,*

(ii) *market clearing conditions are satisfied,*

(iii) *collateral constraint parameter* $\theta_i \in [0, 1] \; \forall i$*, in order to guarantee a unique equilibrium.*

The following are the solutions for intermediate goods and capital in equilibrium:

$$q_{ij} = \begin{cases} (1-\alpha) w_{ij} q_j \gamma_i \gamma_j^{-1} & \text{if } (i,j) \in M_s \\ (1-\alpha) w_{ij} \theta_i q_j \gamma_i \gamma_j^{-1} & \text{if } (i,j) \notin M_s \end{cases} \tag{3.82}$$

$$k_i = \frac{u_s \overline{K}_s \gamma_i}{\sum\limits_{i \in M_s} \gamma_i} \tag{3.83}$$

These solutions depend on the network given by γ and the collateral parameter θ_i. The vector γ is different to the one found in the previous section and is given by:

$$\gamma = \left[\mathbb{I} - (1-\alpha)\left[\mathbf{W}' \circ \mathbf{I_s}' + \mathbf{W}' \circ (\mathbb{I} - \mathbf{I_s}') \circ \mathbf{\Theta} \mathbf{1}'\right]\right]^{-1} \mathbf{1} \tag{3.84}$$

This is the solution to the consumption/output in equilibrium. The symbol $\circ$ indicates the Hadamard product. The matrix of input-output shares, $\mathbf{W}$, identity matrix, $\mathbb{I}$, indicator matrix, $\mathbf{I_s}$, vector of collateral constraints, $\mathbf{\Theta}$, and vector of ones, $\mathbf{1}$, are given by:

$$\mathbf{W} \equiv \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}_{nxn} \qquad \mathbb{I} \equiv \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn} \qquad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

$$\mathbf{\Theta} \equiv \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}_{nx1} \qquad \mathbf{I_s} \equiv \begin{bmatrix} 1_1(1) & \dots & 1_1(n) \\ \vdots & \ddots & \vdots \\ 1_n(1) & \dots & 1_n(n) \end{bmatrix}_{nxn}$$

where the i-th element of the matrix $\mathbf{I_s}$ is an indicator function of the membership of both firm i and j in the same subset M_s of N, taking the value 1 for all i and j couple in M_s and the value 0 for all couples not in the same group:

$$1_i(j) \equiv \begin{cases} 1 & \text{if } (i,j) \in M_s \\ 0 & \text{if } (i,j) \notin M_s \end{cases} \tag{3.85}$$

3.5.5 GDP

Proposition 32. *I can express the equation for the natural logarithm of the GDP in the following way:*

$$Y = \mathbf{v}'(\mathbf{I_M}\nu + \mu) + \Lambda \tag{3.86}$$

where ν is the vector idiosyncratic shocks, u_s, to each capital endowment for each subset of firms, M_s; and μ is the vector idiosyncratic capital augmenting shocks, z_i, independent from firms outside subset and correlated to the firms of the same group. Λ is a variable of parameters given by:

$$\begin{aligned}\Lambda \equiv \mathbf{v}' \left[\frac{(1-\alpha)}{\alpha} \Big((\mathbf{W} \circ (\mathbb{I} - \mathbf{I_s}) \circ \mathbf{1}\hat{\mathbf{\Theta}}')\mathbf{1} + \mathbf{W}\mathbf{w} \Big) + \mathbf{I_M}(\mathbf{k} - \check{\gamma}) - \overline{\gamma} \right] \\ + \frac{(1-\alpha)}{\alpha} \ln(1-\alpha) - \frac{1}{n}\mathbf{1}'\overline{\gamma} + \ln n \end{aligned} \tag{3.87}$$

Where

$$\mathbf{z} \equiv \begin{bmatrix} \ln z_1 \\ \vdots \\ \ln z_n \end{bmatrix}_{nx1} = \mu \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}_{nx1} \quad \mathbf{u} \equiv \begin{bmatrix} \ln u_1 \\ \vdots \\ \ln u_m \end{bmatrix}_{mx1} = \nu \equiv \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_m \end{bmatrix}_{mx1}$$

$$\mathbf{k} \equiv \begin{bmatrix} \ln \overline{K}_1 \\ \vdots \\ \ln \overline{K}_m \end{bmatrix}_{mx1} \quad \mathbf{w} \equiv \begin{bmatrix} \ln w_{1j} \\ \vdots \\ \ln w_{nj} \end{bmatrix}_{nx1} \quad \hat{\mathbf{\Theta}} \equiv \begin{bmatrix} \ln \theta_1 \\ \vdots \\ \ln \theta_n \end{bmatrix}_{nx1}$$

$$\overline{\gamma} \equiv \begin{bmatrix} \ln \gamma_1 \\ \vdots \\ \ln \gamma_n \end{bmatrix}_{nx1} \quad \check{\gamma} \equiv \begin{bmatrix} \ln \sum\limits_{i \in M_1} \gamma_i \\ \vdots \\ \ln \sum\limits_{i \in M_m} \gamma_i \end{bmatrix}_{mx1} \quad \mathbf{I_M} \equiv \begin{bmatrix} 1_{M_1}(1) & \dots & 1_{M_m}(1) \\ \vdots & \ddots & \vdots \\ 1_{M_1}(n) & \dots & 1_{M_m}(n) \end{bmatrix}_{nxm}$$

The i-th element of the matrix $\mathbf{I_M}$ is an indicator function of the membership of a firm in the subset M_s of N, as defined in the previous section.

From the equation of the natural logarithm of the GDP I can see that the collateral constraints, included in the vector $\hat{\mathbf{\Theta}}$, interact with the influence vector, but also are embedded in the vectors a contain the solution to the output-consumption ratio. However, being an exogenous parameter, it will not have implications for aggregate volatility.

3.5.6 Volatility of GDP

As in the two previous models, I define the aggregate fluctuations in the economic activity as subgroup GDP, using the equation found in the previous section I have the following volatility definition:

Definition 57. *The volatility of the logarithm of the GDP is given by:*

$$[var(Y)]^{1/2} \equiv [var(\mathbf{v}'(\mathbf{I_M}\nu + \mu))]^{1/2} \tag{3.88}$$

Proposition 33. *Under the assumption that the shocks ν and μ are independent, I can express the previous equation in sums of the products of the elements in the following way:*

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n} v_i^2 \left(\sum_{s=1}^{m} (1_s(i))^2 \varsigma_s^2\right) + \sum_{i=1}^{n} \sum_{j \in M_s} \rho_{ij}\sigma_i\sigma_j v_i v_j\right]^{1/2} \tag{3.89}$$

where σ_i^2 is the variance of each idiosyncratic shock to the sector i, $\rho_{ij} \in [-1, 1]$ is the correlation between the shocks to the sectors i and j that belong to the same subset M_s, being $\rho_{ij} = 1$ when $i = j$. The variance of each idiosyncratic shock to the capital endowments of each subset if given by ς_s^2 and v_i is the i-th element of the vector of influence, $\mathbf{v}'$.

3.5.7 Economy as a network

To analyse whether the aggregate fluctuations depend on the structure of the economy, I examine specific relevant cases. I consider the same type of networks defined in the previous model for the case where $n = 4$ (sectors) and $m = 2$ (partitions), that could be represented in the following graph (Figure 3.18), where the dotted lines indicate the subsets of sectors.

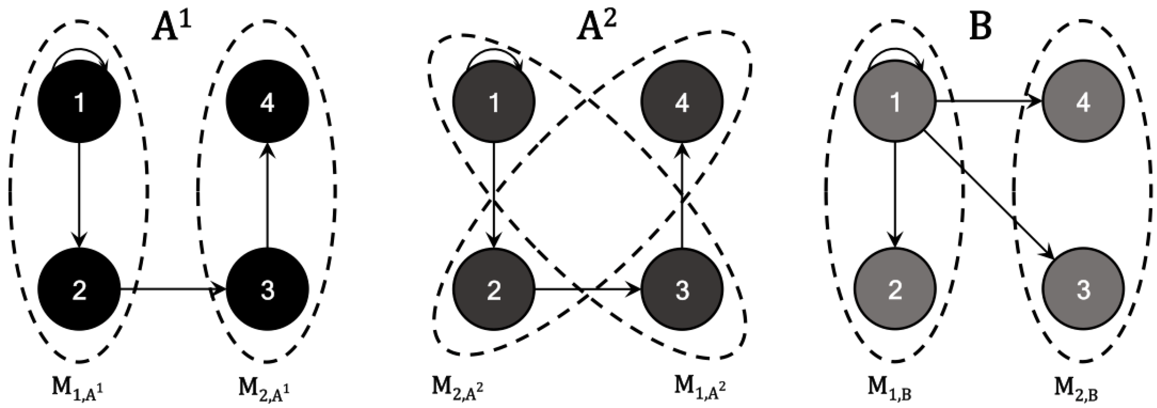


Figure 3.18: Four-sector economies

3.5.8 Comparison of volatilities

In this section, I first compare economies with the same input-output structure, in particular, vertical economies, but with different partition arrangements and parameters. In the next subsection, I compare economies, vertical and star, with the same partition arrangement but with different network structure.

I find that, as in the previous models, the correlation structure, the partition arrangements and the network structure play a significant and interrelated role to determine the magnitude of the volatility and the propagation of individual shocks.

3.5.8.a Same vertical input-output structure, different subsets of firms and variances

For these comparisons, I have the same input-output vertical economy, but in the first case I have partitions of nodes that are directly linked, and in the second case, I have partitions of sectors that are not directly connected, as I did in the previous model of capital risk sharing.

The difference is that I have to type of shocks, idiosyncratic correlated (between sectors of the same group) productivity shocks and individual shocks to the capital rented to each partition, as I illustrate in the following graph (Figure 3.19).

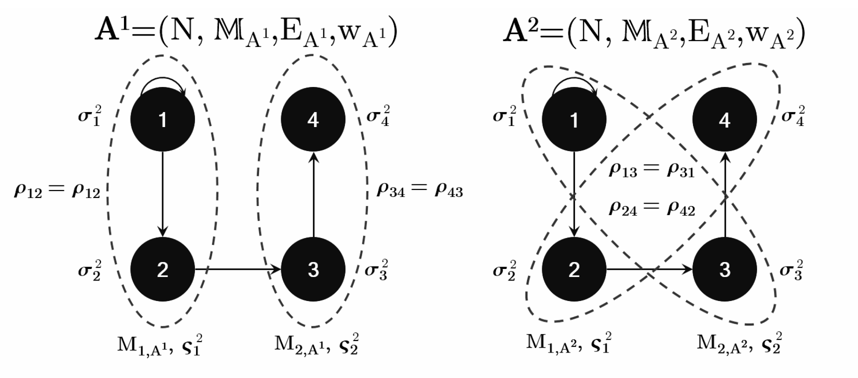


Figure 3.19: Correlated productivity shocks and individual shocks to the capital

Proposition 34. *Considering economies $A^1_{4,2}$ and $A^2_{4,2}$ and assuming that the variance of each idiosyncratic shock, ς_s^2, σ_i^2, is the same in the two economies, $\varsigma_1^2 = \varsigma_2^2$, and across firms, $\sigma_i^2 = \sigma_j^2$. There is correlation, ρ_{ij}, only between firms of the same subset but equal across economies and firms, $[\rho_{ij} \in \mathbf{\Sigma}\, \forall (i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) = (\rho \in A^2_{4,2})]$. The difference in volatilities of the GDP is the following:*

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \;\; if \; \rho = 0$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \;\; if \; \rho \in (0,1]$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho \in [-1, 0)$$

The previous result implies that under the same variance of the shock to the capital shares when the correlation is zero, the partition arrangement is irrelevant. However, when the correlation is positive, the economy that is less diversified is more volatile. (more diversified means sectors not directly connected inside each group). The opposite occurs when the correlation is negative.

This result is different to the corresponding one of the capital risk sharing model. In the latter, the difference of volatility of GDP of economies with the same network structure and different partition arrangement is zero when the variance of the shock to the capital is the same between and within economies, whereas in the former it could be negative or positive depending on the correlation.

Proposition 35. *Consider the economies $A^1_{4,2}$ and $A^2_{4,2}$ and assuming that the variance of each idiosyncratic shock, ς^2_s, σ^2_i, is the same in the two economies, $\varsigma^2_1 = \varsigma^2_2$, and across firms, $\sigma^2_i = \sigma^2_j$. There is correlation, ρ_{ij}, only between firms of the same subset, equal within economies but different across them, $[\rho_{ij} \in \mathbf{\Sigma}\ \forall (i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) \neq (\rho \in A^2_{4,2})]$. The difference in volatilities of the GDP is the following:*

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho \in (0, 1] \wedge [(\rho \in A^1_{4,2}) > (\rho \in A^2_{4,2})]$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho \in [-1, 0) \wedge [(\rho \in A^1_{4,2}) < (\rho \in A^2_{4,2})]$$

This proposition tells us that when the correlation of the first economy is greater than the one of the second economy and positive, the economy with less diversified partitions implies a higher level of volatility of GDP. The opposite occurs when the correlation is negative, and the correlation is the second economy is greater.

Proposition 36. *Considering the economies $A^1_{4,2}$ and $A^2_{4,2}$ and assuming that the variance of each idiosyncratic shock to the capital endowments, ς^2_s, is the same in the two economies but different across for each subset of firms, there is no correlation, $\rho = 0$, and the idiosyncratic shock, σ^2_i, is the same between economies and firms; the difference in volatilities of the GDP is the following:*

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma^2_1 = \varsigma^2_2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma^2_1 > \varsigma^2_2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma^2_1 < \varsigma^2_2$$

This last proposition indicates that, when there is no correlation, the difference between the variance of the shock to the capital shares shows the sign of the difference of volatilities of GDP between the two economies, the same result that I had in the previous model.

3.5.8.b Different input-output structure, same subsets of sectors and variances

Finally, in this section, I compare two different input-output structures with the same parameters (Figure 3.20).

Proposition 37. *Consider the economies $A^1_{4,2}$ and $B_{4,2}$ and assuming that the variance*

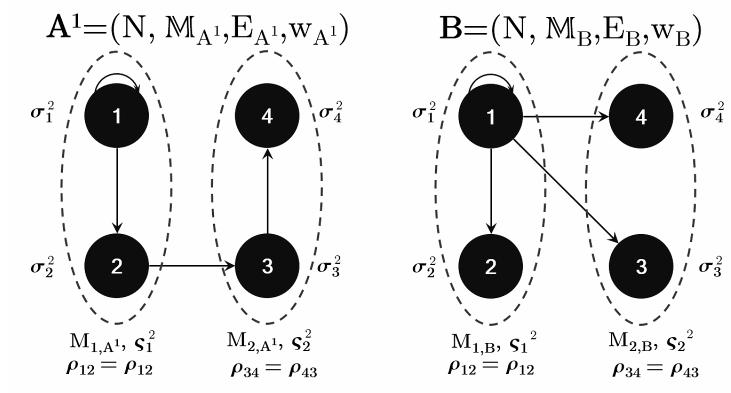


Figure 3.20: Different input-output structures with the same parameters

of each idiosyncratic shock, ς_s^2, σ_i^2, is the same in the two economies, $\varsigma_1^2 = \varsigma_2^2$, and across firms, $\sigma_i^2 = \sigma_j^2$. There is correlation, ρ_{ij}, only between firms of the same subset but equal across economies and firms, $[\rho_{ij} \in \mathbf{\Sigma}\, \forall (i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) = (\rho \in A^2_{4,2})]$. The difference in volatilities of the GDP is the following:

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{B_{4,2}}$$

As in the previous model, I find that the economy with greater overall centrality has larger volatility; thus, the star economy always has higher volatility than the vertical economy. The difference of this result, compared to the first model with correlations, is that even with perfect positive correlation, the star economy implies greater volatility.

3.6 Conclusions

In this chapter, I modelled a disaggregated economy with an input-output structure of connections based on Long and Plosser (1983) and Acemoglu et al. (2012). I analysed the effect of two specific individual shocks. The first one an idiosyncratic productivity correlated shock, and the second one an individual shock to the capital endowments rented from one agent to a specific subset of firms. In a third integrated model, I introduce both idiosyncratic shocks together.

I used this specification to analyse the relevance capital risk sharing between firms, and productivity shocks correlations between firms of the same subset and collateral constraints between firms trading of different subsets. Comparing different networks structures, assumptions of the variances and arrangements of the subsets, I find that the aggregate volatility strongly depends on the structure of the economy.

Having these three different models, I analyse the aggregate volatility of the GDP implied by each shock under different network structures. Mainly, I find that, as the correlation increases, the volatility of the GDP will also increase, for all the economies compared.

As the level of disaggregation increases, the standard deviation decreases, and in the case of the vertical economy, and it will converge to the level of the horizontal and circle economies, the economies with the lower level of volatility.

The first model implies that, under the correlation values different to one, for all the firms, network structures where there is greater overall centrality or more asymmetric structure, are more volatile than homogeneous or regular structures with lower overall centrality.

The difference of this finding to the model of Acemoglu et al. (2012), besides the ordering of volatilities of GDP for additional structures, is that such result does not hold under perfect positive correlation, where the network becomes irrelevant. Indeed, the idiosyncratic shock will be propagated entirely to each node, mimicking an aggregate shock.

I also find that as the share of intermediate inputs in the production becomes larger, the volatility of GDP will increase depending on the network structure, scaled by the correlation parameter.

Concerning limiting results, when the number of sectors goes to infinity, I found that with correlation parameter different than zero, the volatility of GDP never goes to zero, and only reaches a lower bound in the case of vertical, circle and empty economies, equal to $\sqrt{\sigma^2\rho}$. This result implies that even at greater levels of disaggregation, a shock to an individual sector will have aggregate consequences.

I also found that the relevant characteristic of the network structure is not only the outdegree but mostly the centrality of each node, this is, the connections of its buyers. Another interesting result is that when nodes non-directly linked are more correlated, over the same network, the volatility of GDP will be greater when there is a positive correlation.

The final result of this first model is that to reduce the aggregate volatility, and correlation level needs to decrease and also the heterogeneity of the correlation structure.

Using the second model of capital risk sharing between firms of the same subset, I found that assuming the same level of individual variance, structures with greater overall centrality are more volatile.

Under the same structure of the economy, but with different arrangements of groups of capital sharing, I found that an economy with groups involving directly linked firms are more volatile than the same structure of the economy but with groups including not directly connected firms only if the shock to the group with greater average network centrality is the greatest within the same economy.

This result indicates that the arrangement of the capital shared by each sector plays a first-order role in the propagation of the idiosyncratic shock.

In the case of the integrated model, the third model, I confirmed the previous results, but I found additional insights.

When one compares two economies with the same network structure with one with groups of linked firms only and other with groups of firms not connected, and assuming that variance of individual shocks and correlation is the same across groups and economies, I found zero difference in volatility if the correlation value is zero, greater volatility for the first economy when the correlation value is higher than zero and lower when the correlation is lower than zero.

This result is different to the corresponding one of capital risk sharing model. In the latter, the difference of volatility of GDP under same network structure and different partitions is zero when the variance of the shock to the capital is the same across economies, in the former is different from zero.

I find that on the same network if the correlation of one economy is larger and positive, the less diversified economy implies a greater level of volatility

As in the previous model, I found that the economy with higher overall centrality has larger volatility. The additional result, compared to a model with correlations, is that even with perfect positive correlation, the star economy has greater volatility.

Also, I found that the non-stochastic parameter or the financial friction is not relevant for the aggregate fluctuations, but only for the level of the GDP as it represents an intermediate input wedge. This result is in line with the findings of the effect of wedges in Jones (2011), Bigio and La'O (2013) and Fadinger, Ghiglino and Teteryatnikova (2015).

however, when one deviates of this deterministic specification of the friction parameter, assuming it is subject to random shocks, as in Kiyotaki and Moore (2012) or Jermann and Quadrani (2012), this could imply different results for the volatility of GDP under different networks structures taking into account the non-linearity of the friction.

Conclusions

An old but important question in Macroeconomic Theory is whether significant aggregate fluctuations in economic activity can be obtained from independent productivity shocks to individual disaggregated sectors. The Great Financial Crisis (GFC) of 2008 highlighted the need to study further whether a shock in a particular sector or firm could propagate its effect to the whole economy.

The most common view in the Business Cycle Theory has been that idiosyncratic shocks tend to average out in aggregation and discards the possibility that significant aggregate fluctuations may originate from microeconomic shocks to firms or disaggregated sectors.

In Chapter 1, we developed a model with a multi-sector production economy with I-O interlinkages and a financial sector to analyse the effect of idiosyncratic shocks to banks on GDP volatility. In the model, there is a financial constraint that gives some rigidity to the financial sector, making this a relevant sector. The financial shock is characterised by the realisation of a random parameter involved in the financial constraint.

The analysis of Chapter 1 focuses on the role of the structure of the economy, that is the network of connection among physical sectors and between sectors and banks, in the spread of financial shocks to the real economy and its aggregate implications.

We show that a general equilibrium exists and recover the known result that financial frictions work as a wedge and decrease the level of aggregate output. Analysing different structures of the economy, we find that the standard deviation of GDP decreases as the interconnectivity coefficients are similar among banks, while it increases when this metric shows a higher variability across banks.

We studied the role of integration and diversification also analysed each effect separately. We first consider the growth in integration provided by adding one link to a particular bank, in which case after the addition the bank lends to a sector that has in common another bank. When adding m ties, one for each bank in the network, the results depend on the I-O structure, but always reduces the volatility for vertical and star structures and have no effect on empty and circle networks.

Considering the impact of diversification, obtained as a result of redistributing the links of two or more banks in the network, we find that depending on the position of the bank in the network, reducing diversification can decrease or increase the effect on the volatility of an individual shock even for a special location.

In general, aggregate fluctuations depend on the distributions of links between banks and sectors and the place of such links. An economy with a uniform distribution of links per bank could be less volatile than an economy with an unequal distribution, provided the bank with less link is not supplying capital to a great influencer, that is, the star sector or the top of the chain in the vertical network.

Finally, we studied the structure of the U.S. economy and found that it is highly asymmetric; there are star sectors like manufacturers and professional services, and star banks like Bank of America and JP Morgan. We conclude that changes in the bipartite structure over time lead to changes in the network multiplier while the I-O network remains relatively steady. Computing the GDP volatility using the network metrics, we find that individual shocks to banks do not average out and could lead to sizeable fluctuations of GDP.

Chapter 2 relies on the model developed by Acemoglu et al. (2012) to analyse the network structure of the sectoral trade in the United Kingdom and the possibility of an aggregated shock originated by a shock in a distinct sector of the economy.

Estimating the indegree density, we found symmetry along the sectoral demand: these are indicating specialisation. Considering the outdegree distribution, we found asymmetry across sectors in their role as input sellers. In the data, we found specialised input sellers and general purpose providers.

Concerning the assumption that input-use network structure follows a power law distribution, implementing MLE estimates of ζ for the tail of the distribution, suggested by Clauset et al. (2009), we found that distribution of outdegrees of the U.K. (1997-2010) follow a power law from an endogenously determined the minimum degree, with an average tail parameter $\widehat{\zeta} = 1.275$.

In Chapter 3, I presented a model of a disaggregated economy with an input-output structure of connections based on Long and Plosser (1983) and Acemoglu et al. (2012). I analysed the effect of two specific individual shocks. The first one an idiosyncratic productivity correlated shock, and the second one an individual shock to the capital endowments rented from one agent to a specific subset of firms. In a third integrated shock, I introduced together both idiosyncratic shock.

Using these three different models, I analysed the aggregate volatility of the GDP implied by each shock under different network structures. Mainly, I find that, as the correlation increases, the volatility of the GDP will also increase, for all the economies compared.

In the first model, as the level of disaggregation increases, the standard deviation decreases, and in the case of the vertical economy, and it will converge to the level of the horizontal and circle economies, the economies with the lower level of volatility. I find that as the share of intermediate inputs in the production becomes larger, the volatility of GDP will increase depending on the network structure, scaled by the correlation parameter.

Also, I found that with correlation parameter different than zero, the volatility of GDP never goes to zero, and only reaches a lower bound, implying that even in the presence of greater levels of disaggregation, a shock to an individual sector will have aggregate consequences.

Using the second model of capital risk sharing between firms of the same subset, I observed that assuming the same level of individual variance, structures with greater overall centrality are more volatile. This result indicates that the arrangement of the capital shared by each sector plays a first-order role in the propagation of the idiosyncratic shock.

In the case of the integrated model, I confirmed the previous results, but with additional interesting insights. When one compares two economies with the same network structure, one with groups of linked firms only and other with groups of firms not linked, and assuming that variance of individual shocks and correlation is the same across groups and economies, I found zero difference in volatility if the correlation value is zero, greater volatility for the first economy when the correlation value is larger than zero and lower when the correlation is lower than zero.

Interestingly, I discovered that the economy with greater overall centrality has larger volatility. This additional result, compared to the model with correlations, is that even with perfect positive correlation, the star economy has greater volatility.

Appendix

3.7 Chapter 1

3.7.1 Influence vector results

3.7.1.a For $n > 3$, any I-O structure

We define the influence vector, following the same notation as Acemoglu et al. (2012):

$$\mathbf{v}' \equiv \frac{\alpha}{n}\mathbf{1}'\left(\mathbb{I} - (1-\alpha)\,\mathbf{W}\right)^{-1}$$

Where:

$$\mathbf{W} \equiv \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}_{nxn} \qquad \mathbb{I} \equiv \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn} \qquad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

$\mathbf{W}$ represents the input-output structure of the economy, $\mathbf{1}$ is a vector of ones and $\mathbb{I}$ is the identity matrix.

Lemma 3. *Post multiplying the influence vector by the vector of ones is equal to* 1*:*

$$\mathbf{v}'\mathbf{1} = 1$$

Proof Lemma 9*.* Recalling the Perron-Frobenius Theorem for positive matrices, applied $\mathbf{W}$, given that all the eigenvalues of $\mathbf{W}$ are inside the unit circle, we can express the influence vector as a Neumann series:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}'\sum_{k=0}^{\infty}(1-\alpha)^k\mathbf{W}^k$$

Post-multiplying the previous equation by a vector of ones:

$$\mathbf{v}'\mathbf{1} = \frac{\alpha}{n}\mathbf{1}'\sum_{k=0}^{\infty}(1-\alpha)^k\,\mathbf{W}^k\mathbf{1}$$

Simplifying we can see that $\mathbf{v}'\mathbf{1} = 1$ because:

$$\mathbf{v}'\mathbf{1} = \alpha\sum_{k=0}^{\infty}(1-\alpha)^k = \alpha\left(\frac{1}{\alpha}\right) = 1$$

The series converges because $|1-\alpha| < 1$. ∎

Lemma 4. *Each element,* v_i*, of the influence vector,* $\mathbf{v}$*, can be approximated by:*

$$v_i \approx \frac{\alpha}{n}\left(1 + (1-\alpha)d_i + (1-\alpha)^2\sum_{j=1}^{n} d_j w_{ji}\right)$$

Where d_i *is the outdegree of sector* i *defined as* $d_i \equiv \sum_{j=1}^{n} w_{ji}$*.*

Proof Lemma 18. We express the influence vector as a Neumann series as we did before:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}'\sum_{k=0}^{\infty}(1-\alpha)^k\mathbf{W}^k$$

Expanding the series:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}'(\mathbb{I} + (1-\alpha)\mathbf{W} + (1-\alpha)^2\mathbf{W}^2 + ... + \infty)$$

Multiplying the sum by the vector of ones:

$$\mathbf{v}' = \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{1}'\mathbf{W} + (1-\alpha)^2\mathbf{1}'\mathbf{W}\mathbf{W} + ... + \infty)$$

We can substitute the vector of outdegrees, $\mathbf{d}' = [d_1, ..., d_n]$, in the previous equation because $\mathbf{1}'\mathbf{W} = \mathbf{d}'$:

$$\mathbf{v}' = \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W} + ... + \infty)$$

Grouping over the out-degrees vector and taking a second-order approximation, the influence vector can be expressed:

$$\mathbf{v}' \approx \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W})$$

From the equation of the volatility of GDP we can see that we need to express the i-th element of the influence vector, v_i, as a function of the degrees, using the previous results we have the following:

$$v_i \approx \frac{\alpha}{n}\left(1 + (1-\alpha)d_i + (1-\alpha)^2 \sum_{j=1}^{n} d_j w_{ji}\right)$$

$\blacksquare$

3.7.1.b For $n > 3$, structures A, B, D and F

In this section we will show the components of the influence vector for economies with n sectors. Recalling the influence vector in terms of its approximation:

$$\mathbf{v} \approx \frac{\alpha}{n}(\mathbf{1} + (1-\alpha)\mathbf{d} + (1-\alpha)^2\mathbf{W}'\mathbf{d})$$

We can see that in order to obtain the influence vector for the economies A(Vertical), B(Star), D(Circle) and F(Empty), we need to know the input-output matrix, $\mathbf{W}$, and the degree vector, $\mathbf{d}$, in each case. The input-output matrices in each case are the following:

$$\mathbf{W_A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{nxn} \qquad \mathbf{W_B} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}_{nxn}$$

$$
\mathbf{W_D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{nxn} \qquad \mathbf{W_F} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn}
$$

The degree vector in each case is obtained as the column sums of the previous matrices:

$$
\mathbf{d_A} = \begin{bmatrix} 2 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}_{nx1} \qquad \mathbf{d_B} = \begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{nx1} \qquad \mathbf{d_D} = \mathbf{d_F} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}
$$

3.7.2 Perfectly balanced and unbalanced I-O networks

Proof Lemma 7. Circle, IO_D, and empty, IO_F, networks are perfectly balanced. According to the definition provided, any IO network will be considered perfectly balanced if $v_i = v_j$, $\forall \overrightarrow{\langle i,j \rangle}$. In order to verify this claim we recall the approximation of the influence vector:

$$
\mathbf{v} \approx \frac{\alpha}{n}(\mathbf{1} + (1-\alpha)\mathbf{d} + (1-\alpha)^2 \mathbf{W}'\mathbf{d})
$$

Substituting the degree sequence, $\mathbf{d}$, and the I-O matrix, $\mathbf{W}$, of the IO_D and IO_F networks, we obtain the following vectors of influence:

$$\mathbf{v_D} = \mathbf{v_F} \approx \left(\frac{\alpha}{n}\right)(1 + (1-\alpha) + (1-\alpha)^2)\mathbf{1}$$

We can see that each element of the influence vector is the same for all i. ■

Proof Lemma 8. Vertical, IO_A, and star, IO_B, networks are perfectly unbalanced. According to the definition provided, any IO network with component-1 will be considered perfectly unbalanced if $v_i \geq v_j$, $\forall \overrightarrow{\langle i, j \rangle}$.In order to prove this result we recall the approximation of the influence vector:

$$\mathbf{v} \approx \frac{\alpha}{n}(\mathbf{1} + (1-\alpha)\mathbf{d} + (1-\alpha)^2\mathbf{W}'\mathbf{d})$$

Substituting the degree sequence, $\mathbf{d}$, and the I-O matrix, $\mathbf{W}$, of the IO_A and IO_B networks, we obtain the following vectors of influence:

$$\mathbf{v_A} \approx \frac{\alpha}{n}\begin{bmatrix} 1 + 2(1-\alpha) + 3(1-\alpha)^2 \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ \vdots \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ 1 + (1-\alpha) \\ 1 \end{bmatrix}_{nx1} \qquad \mathbf{v_B} \approx \frac{\alpha}{n}\begin{bmatrix} 1 + n(1-\alpha) + n(1-\alpha)^2 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{nx1}$$

Given the assumption $\alpha \in (0, 1)$, we can see that the networks IO_A and IO_B, are perfectly unbalanced. In the case of $\mathbf{v_A}$ we can see straightforward that each element $v_i \geq v_j$ for all $i < j$ because:

$$1 + 2(1-\alpha) + 3(1-\alpha)^2 > 1 + (1-\alpha) + (1-\alpha)^2 > 1$$

For $\mathbf{v_B}$ we can see also that each element $v_i \geq v_j$ for all $i < j$ because:

$$1 + n(1-\alpha) + n(1-\alpha)^2 > 1$$

■

3.7.3 Model

3.7.3.a Households

$$\max_{L,C} \ln L + \beta E[\ln C]$$

$s.t.$

$$t = 1 : wL + D = w$$

$$t = 2 : PC = RD$$

Substituting the first constraint into the second one, through D, we get the intertemporal constraint:

$$PC = R(w - wL)$$

Recalling P is the numeraire at time 2:

$$C = R(w - wL)$$

We formulate the Lagrangian problem using the previous constraint and the intertemporal utility:

$$\mathcal{L} = \ln L + \beta \ln C - \lambda(C - R(w - wL))$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial L} = \frac{1}{L} - \lambda Rw = 0$$

$$\frac{\partial \mathcal{L}}{\partial C} = \frac{\beta}{C} - \lambda = 0$$

Expressing each condition for λ:

$$\Rightarrow \lambda = \frac{1}{LRw}$$

$$\Rightarrow \lambda = \frac{\beta}{C}$$

Equating through λ and simplifying:

$$\Rightarrow \frac{1}{L} - \frac{\beta R w}{C} = 0$$

$$\Rightarrow \frac{1}{wL} = \frac{\beta R}{C}$$

Substituting the intertemporal constraint into C and simplifying:

$$\frac{1}{L} = \frac{\beta}{1-L}$$

Recalling $L = 1 - l$, substituting:

$$\frac{1}{1-l} = \frac{\beta}{l}$$

Simplifying:

$$l = \frac{\beta}{1+\beta}$$

From constraint in the first period we know $D = wl$, then:

$$D = \frac{w\beta}{1+\beta}$$

And substituting into second period constraint:

$$C = \frac{Rw\beta}{1+\beta}$$

3.7.3.b Banks

$$\max_{\{x_{ib}\}_i, D_b} \Pi_b = \sum_{i=1}^{n} r_b x_{ib} - RD_b$$

s.t.

$$\sum_{i=1}^{n} x_{ib} = z_b D_b$$

$$D_b > 0$$

$$x_{ib} \geq 0 \; \forall i$$

$$r_b > 0$$

$$R > 0$$

$$\Pi_b \geq 0$$

$$z_b \sim \; Lognormal\left(0, \sigma_b^2\right)$$

Solution of problem:

$$\sum_{i=1}^{n} r_b x_{ib} - RD_b \geq 0$$
$$\Rightarrow \sum_{i=1}^{n} r_b x_{ib} - \frac{R\sum_{i=1}^{n} x_{ib}}{z_b} \geq 0$$
$$\Rightarrow \sum_{i=1}^{n} x_{ib}\left(r_b - \frac{R}{z_b}\right) \geq 0$$

Bank will not posit quantities such that $R > r_b z_b$ because in this case the supply of deposits by the household would be zero. Additionally, to meet all conditions, we consider that the equation has to bind and the solution condition becomes:

$$r_b = R/z_b$$

Which implies:

$$\sum_{i=1}^{n} r_b x_{ib} = RD_b$$

3.7.3.c Capital producers

$$\max_{K,l} \; K - wl$$

$s.t.$

$$K = \delta l$$

Solution:

$$K = wl$$

and

$$w = \delta$$

3.7.3.d Intermediate goods producers

$$\max_{x_i, q_{ij} \forall j} \quad p_i q_i - \sum_{b=1}^{m} r_b x_{ib} - \sum_{j=1}^{n} p_j q_{ij}$$

s.t.

$$q_i = x_i^{\alpha} \prod_{j=i}^{n} q_{ij}^{(1-\alpha) w_{ij}}$$

$$\frac{x_{ib}}{x_i} = \phi_{ib}$$

Unrestricted problem:

$$\max_{x_i, q_{ij} \forall j} \quad p_i x_i^{\alpha} \prod_{j=i}^{n} q_{ij}^{(1-\alpha) w_{ij}} - x_i \sum_{b=1}^{m} r_b \phi_{ib} - \sum_{j=1}^{n} p_j q_{ij}$$

First order conditions:

$$\frac{\alpha p_i q_i}{x_i} - \sum_{b=1}^{m} r_b \phi_{ib} = 0$$
$$\frac{(1-\alpha) w_{ij} p_i q_i}{q_{ij}} - p_j = 0$$

Rearranging:

$$x_i = \frac{\alpha p_i q_i}{\sum_{b=1}^{m} r_b \phi_{ib}}$$

$$q_{ij} = \frac{(1-\alpha) w_{ij} p_i q_i}{p_j}$$

Recalling constraint, input demand of capital from each bank is given by:

$$x_{ib} = \frac{\alpha p_i q_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}$$

Input ratio:

$$\frac{x_i}{q_{ij}} = \frac{\alpha p_j \phi_{ib}}{(1-\alpha) w_{ij} \sum_{b=1}^{m} r_b \phi_{ib}}$$

3.7.3.e Final goods aggregator sector

$$\max_{c_i \forall i} \Pi_o = PY - \sum_{i=1}^{n} p_i c_i$$

s.t.

$$Y = \prod_{i=1}^{n} c_i^{1/n}$$

First order conditions:

$$\frac{\partial \Pi_o}{\partial c_i} = \frac{PY}{nc_i} - p_i = 0$$

$$\frac{\partial \Pi_o}{\partial c_j} = \frac{PY}{nc_j} - p_j = 0$$

Rearranging for prices:

$$\Rightarrow p_i = \frac{PY}{nc_i}$$

$$\Rightarrow p_j = \frac{PY}{nc_j}$$

Price ratio:

$$p_i c_i = p_j c_j$$

The price of the aggregated final good, P, could be expressed as a function of individual prices:

$$Y = \prod_{i=1}^{n} c_i^{1/n}$$

$$Y = \prod_{i=1}^{n} \left(\frac{Y}{np_i} \right)^{1/n}$$

$$\Rightarrow Y = \frac{Y}{n} \prod_{i=1}^{n} \left(\frac{P}{p_i} \right)^{1/n}$$

$$\Rightarrow n = \prod_{i=1}^{n} \left(\frac{P}{p_i} \right)^{1/n}$$

$$\Rightarrow n \prod_{i=1}^{n} p_i^{1/n} = P$$

Recalling our assumption that P is the numeraire in the second period, previous equation becomes the ideal price index:

$$\Rightarrow n \prod_{i=1}^{n} p_i^{1/n} = 1$$

3.7.3.f Equilibrium

Proof Proposition 15. Recalling the clearing condition for consumption and intermediate goods:

$$c_i + \sum_{j=i}^{n} q_{ij} = q_i \quad \forall i$$

Which holds for good j:

$$\Rightarrow c_j + \sum_{i=i}^{n} q_{ij} = q_j$$

Substituting the first order condition from the intermediate goods into q_{ij}:

$$\Rightarrow c_j + \sum_{i=1}^{n} \frac{(1-\alpha) w_{ij} p_i q_i}{p_j} = q_j$$

Multiplying whole equation by p_j:

$$\Rightarrow p_j c_j + \sum_{i=1}^{n} (1-\alpha) w_{ij} p_i q_i = p_j q_j$$

Substituting first order condition of final goods aggregator sector into p_j and p_i:

$$\Rightarrow \left(\frac{Y}{nc_j}\right) c_j + \sum_{i=1}^{n} (1-\alpha) w_{ij} q_i \left(\frac{Y}{nc_i}\right) = \left(\frac{Y}{nc_j}\right) q_j$$

Simplifying:

$$\Rightarrow 1 + \sum_{i=1}^{n} (1-\alpha) \frac{w_{ij} q_i}{c_i} = \frac{q_j}{c_j}$$

Defining $\gamma_j \equiv q_j/c_j$ and $\gamma_i \equiv q_i/c_i$:

$$\Rightarrow 1 + \sum_{i=1}^{n} (1-\alpha) w_{ij} \gamma_i = \gamma_j$$

Expressing in vectors, stacking over sectors:

$$\Rightarrow \mathbf{1} + (1-\alpha)\mathbf{W}'\gamma = \gamma$$

Solving for γ:

$$\Rightarrow \gamma - (1-\alpha)\mathbf{W}'\gamma = \mathbf{1}$$

$$\Rightarrow (\mathbb{I} - (1-\alpha)\mathbf{W}')\gamma = \mathbf{1}$$

$$\Rightarrow \gamma = (\mathbb{I} - (1-\alpha)\mathbf{W}')^{-1}\mathbf{1}$$

Recalling the price ratio of the final sector:

$$\frac{p_i}{p_j} = \frac{c_j}{c_i}$$

Re-expressing γ_i and γ_j:

$$c_i = \frac{q_i}{\gamma_i}$$

$$c_j = \frac{q_j}{\gamma_j}$$

And substituting into the price ratio of the final sector:

$$\Rightarrow \frac{p_i}{p_j} = \frac{q_j\gamma_i}{q_i\gamma_j}$$

Substituting this relationship into the first order condition of the intermediate sectors for q_{ij}:

$$q_{ij} = (1-\alpha) w_{ij} q_i \left(\frac{q_j \gamma_i}{q_i \gamma_j} \right)$$
$$q_{ij} = (1-\alpha) w_{ij} q_j \left(\frac{\gamma_i}{\gamma_j} \right)$$

Multiplying and dividing by α/n:

$$q_{ij} = (1-\alpha) w_{ij} q_j \left(\frac{(\alpha/n)\gamma_i}{(\alpha/n)\gamma_j} \right)$$

Recalling that influence vector is given by:

$$\mathbf{v} = \frac{\alpha}{n}\gamma = \frac{\alpha}{n}(\mathbb{I} - (1-\alpha)\mathbf{W}')^{-1}\mathbf{1}$$

Previous input equation becomes:

$$q_{ij} = (1-\alpha) w_{ij} q_j \left(\frac{v_i}{v_j} \right)$$

This is the solution for the intermediate goods in equilibrium. Now to find the capital demand from each bank we recall that $p_i q_i = p_j c_j \gamma_i$, substituting into capital demand from I-O sector solution:

$$x_{ib} = \frac{\alpha c_j p_j \gamma_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}$$

Adding over sectors:

$$\sum_{i=1}^{n} x_{ib} = \alpha c_j p_j \sum_{i=1}^{n} \frac{\gamma_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}$$

Dividing previous two equations:

$$x_{ib} = \left(\frac{\gamma_i \phi_{ib}}{\left(\sum_{b=1}^{m} r_b \phi_{ib}\right) \left(\sum_{i=1}^{n} \frac{\gamma_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}} \right)} \right) \sum_{i=1}^{n} x_{ib}$$

Multiplying and dividing by α/n, recalling that $v_i = (\alpha/n)\gamma_i$:

$$x_{ib} = \left(\frac{v_i \phi_{ib}}{\left(\sum_{b=1}^{m} r_b \phi_{ib}\right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}} \right)} \right) \sum_{i=1}^{n} x_{ib}$$

Substituting the interest rate ratio, $r_b = R/z_b$:

$$x_{ib} = \left(\frac{v_i \phi_{ib}}{\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right)} \right) \sum_{i=1}^{n} x_{ib}$$

We need to find the supply of capital by banks, $\sum_{i=1}^{n} x_{ib}$. Recalling the balance sheet constraint from banks, $\sum_{i=1}^{n} x_{ib} = z_b D_b$, and substituting into previous equation:

$$x_{ib} = \left(\frac{v_i \phi_{ib} z_b}{\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right)} \right) D_b$$

To find the equilibrium capital per bank and sector we need to find the deposits in equilibrium per bank, D_b. To do this we recall the bank's solution:

$$\sum_{i=1}^{n} r_b x_{ib} = R D_b$$

Adding over banks:

$$\sum_{b=1}^{m}\sum_{i=1}^{n} r_b x_{ib} = R\sum_{b=1}^{m} D_b$$

Dividing previous two equations:

$$\frac{\sum_{i=1}^{n} r_b x_{ib}}{\sum_{b=1}^{m}\sum_{i=1}^{n} r_b x_{ib}} = \frac{D_b}{\sum_{b=1}^{m} D_b}$$

Recalling the total supply of deposits from households:

$$D = \sum_{b=1}^{m} D_b = \frac{w\beta}{1+\beta}$$

Substituting this equation into the ratio that we found before and recalling from capital producers that $w = \delta$:

$$D_b = \left(\frac{\sum_{i=1}^{n} r_b x_{ib}}{\sum_{b=1}^{m}\sum_{i=1}^{n} r_b x_{ib}}\right)\left(\frac{\delta\beta}{1+\beta}\right)$$

Substituting the capital demand condition from I-O firms $x_{ib} = \alpha c_j p_j \gamma_i \phi_{ib} / \sum_{b=1}^{m} r_b \phi_{ib}$:

$$D_b = \left(\frac{\sum_{i=1}^{n} \frac{r_b \alpha c_j p_j \gamma_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}}{\sum_{b=1}^{m}\sum_{i=1}^{n} \frac{r_b \alpha c_j p_j \gamma_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}}\right)\left(\frac{\delta\beta}{1+\beta}\right)$$

Multiplying and dividing by α/n, recalling that $v_i = (\alpha/n)\gamma_i$, after simplifying:

$$D_b = \left(\frac{\sum_{i=1}^{n} \frac{r_b v_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}}{\sum_{b=1}^{m}\sum_{i=1}^{n} \frac{r_b v_i \phi_{ib}}{\sum_{b=1}^{m} r_b \phi_{ib}}}\right)\left(\frac{\delta\beta}{1+\beta}\right)$$

To further simplify we need to substitute the interest rate ratio, $r_b = R/z_b$. D_b becomes:

$$D_b = \left(\frac{\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{z_b \sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}}}{\sum_{b=1}^{m} \sum_{i=1}^{n} \frac{v_i \phi_{ib}}{z_b \sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}}} \right) \left(\frac{\delta\beta}{1+\beta} \right)$$

The denominator of the previous expression simplifies to one:

$$D_b = \left(\frac{1}{z_b} \right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right) \left(\frac{\delta\beta}{1+\beta} \right)$$

This is deposits per bank in equilibrium which we can substitute now into the equation of capital in equilibrium that we found previously:

$$x_{ib} = \left(\frac{v_i \phi_{ib} z_b}{\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right)} \right) \left(\frac{1}{z_b} \right) \left(\sum_{i=1}^{n} \frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right) \left(\frac{\delta\beta}{1+\beta} \right)$$

Simplifying:

$$x_{ib} = \left(\frac{v_i \phi_{ib}}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right) \left(\frac{\delta\beta}{1+\beta} \right)$$

This is the solution for the capital per bank and sector in equilibrium. Recalling that $x_{ib} = \phi_{ib} x_i$, the capital per sector in equilibrium is:

$$x_i = \left(\frac{v_i}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right) \left(\frac{\delta\beta}{1+\beta} \right)$$

Now we can find the GDP in equilibriumm. Substituting the solutions of q_{ij}

and x_i into the production function of a typical I-O sector:

$$q_i = \left[\left(\frac{v_i}{\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}} \right) \left(\frac{\delta\beta}{1+\beta} \right) \right]^{\alpha} \prod_{j=1}^{n} \left[(1-\alpha)^{w_{ij}} \left(\frac{w_{ij} q_j v_i}{v_j} \right)^{w_{ij}} \right]^{(1-\alpha)}$$

Taking natural logarithms:

$$\ln q_i = \alpha \left[-\ln \left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) + \ln v_i + \ln \kappa \right] + (1-\alpha) \sum_{j=1}^{n} [w_{ij} (\ln(1-\alpha) + \ln w_{ij} + \ln q_j + \ln v_i - \ln v_j)]$$

where:

$$\kappa = \frac{\beta\delta}{1+\beta}$$

Recalling that the coefficient ϕ_{ib} takes values between zero and one and z_b positive real numbers, this implies that we cannot approximate the logarithm of theirs ratio. Stacking over n sectors:

$$\mathbf{q} = \alpha \left[\overline{\mathbf{v}} - \mathbf{\Theta} + \mathbf{1} \ln \kappa \right] + (1-\alpha) \left[\mathbf{W}\mathbf{q} + (\mathbf{W} \circ \overline{\mathbf{W}})\mathbf{1} + \mathbf{1} \ln(1-\alpha) \right]$$

Where $\mathbf{\Theta}' = \left[\sum_{b=1}^{m} \frac{\phi_{1b}}{z_b}, ..., \sum_{b=1}^{m} \frac{\phi_{nb}}{z_b} \right]_{1xn}$

Solving $\mathbf{q}$:

$$[\mathbb{I} - (1-\alpha)\mathbf{W}]\mathbf{q} = \alpha \left[\overline{\mathbf{v}} - \mathbf{\Theta} + \mathbf{1} \ln \kappa \right] + (1-\alpha)[(\mathbf{W} \circ \overline{\mathbf{W}})\mathbf{1} + \mathbf{1} \ln(1-\alpha)]$$

Recalling the influence vector $\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}'(\mathbb{I} - (1-\alpha)\mathbf{W})^{-1}$, $\mathbf{v}'\mathbf{1} = 1$, multiplying appropriately and substituting:

$$\frac{1}{n}\mathbf{1}'\mathbf{q} = -\mathbf{v}'\boldsymbol{\Theta} + \mathbf{v}'\left[\overline{\mathbf{v}} + \frac{1-\alpha}{\alpha}(\mathbf{W} \circ \overline{\mathbf{W}})\mathbf{1}\right] + \ln\kappa + \frac{1-\alpha}{\alpha}\ln(1-\alpha)$$

Defining $\Gamma_0 = \mathbf{v}'\left[\overline{\mathbf{v}} + \frac{1-\alpha}{\alpha}(\mathbf{W} \circ \overline{\mathbf{W}})\mathbf{1}\right] + \ln\kappa + \frac{1-\alpha}{\alpha}\ln(1-\alpha)$, previous equation becomes:

$$\frac{1}{n}\mathbf{1}'\mathbf{q} = \Gamma_0 - \mathbf{v}'\boldsymbol{\Theta}$$

Now we need to relate previous result with the GDP. First, recalling $\gamma_i = q_i/c_i$, multiplied by α/n becomes $v_i = \left(\frac{\alpha}{n}\right)\left(\frac{q_i}{c_i}\right)$, its natural logarithm expressed in vectorial form is given by:

$$\mathbf{c} = \mathbf{q} - \overline{\mathbf{v}} + \mathbf{1}(\ln\alpha - \ln n)$$

Pre-multiplying by a vector of ones and dividing by n:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \frac{1}{n}\mathbf{1}'\mathbf{q} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{v}} + \ln\alpha - \ln n$$

To introduce the final clearing condition and find the fix point for this economy, we recall $C = Y$:

$$\Rightarrow \ln C = \ln Y$$
$$\Rightarrow \ln C = \frac{1}{n}\sum_{i=1}^{n} \ln c_i$$
$$\Rightarrow \ln C = \frac{1}{n}\mathbf{1}'\mathbf{c}$$
$$\Rightarrow \frac{1}{n}\mathbf{1}'\mathbf{c} = \ln Y$$

Putting previous results together:

$$\ln Y = \frac{1}{n}\mathbf{1}'\mathbf{q} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{v}} + \ln\alpha - \ln n$$

$$\Rightarrow \ln Y = \Gamma_0 - \mathbf{v}'\boldsymbol{\Theta} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{v}} + \ln\alpha - \ln n$$

Collecting terms where the shocks, z_b, are not included:

$$\ln Y = \Gamma - \mathbf{v}'\boldsymbol{\Theta}$$

which is equivalent to:

$$\ln Y = \Gamma - \sum_{i=1}^{n}\left[v_i \ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]$$

where $\Gamma = \mathbf{v}'[\frac{1-\alpha}{\alpha}(\mathbf{W}\circ\overline{\mathbf{W}})\mathbf{1}] + (\mathbf{v}'\mathbf{1} - \frac{1}{n}\mathbf{1}')\overline{\mathbf{v}} + \ln\alpha + \ln\kappa_1 - \ln n + \frac{1-\alpha}{\alpha}\ln(1-\alpha)$

■

3.7.4 Volatility of GDP

Proof Proposition 16. Recalling the GDP equation:

$$\ln Y = \Gamma - \sum_{i=1}^{n} \left[v_i \ln \left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) \right]$$

Taking the variance:

$$var[\ln Y] = var \left[\Gamma - \sum_{i=1}^{n} \left[v_i \ln \left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) \right] \right]$$

Considering the first term is composed of constants:

$$var[\ln Y] = var \left[\sum_{i=1}^{n} \left[v_i \ln \left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \right) \right] \right]$$

Even though we have the assumption that each bank's shock, z_b, is independent of the others, in this case the total variance is not just the sum of individual variances per sector, we need to take into account also the covariances between each sector. The reason behind this is caused by the fact that contribution to the total variance per sector depends on the shocks of all banks:

$$var[\ln Y] = var \left[v_1 \ln \left(\sum_{b=1}^{m} \frac{\phi_{1b}}{z_b} \right) + ... + v_n \ln \left(\sum_{b=1}^{m} \frac{\phi_{nb}}{z_b} \right) \right]$$

To avoid this problem we would have to collect the terms per-bank, instead of per-sector. However, we can not do this as we cannot approximate the logarithm expressions. Then we need to compute the variance of each term taking into account their covariances:

$$var[\ln Y] = \sum_{i=1}^{n} var\left[v_i \ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right] + 2\sum_{i,j:i<j} cov\left[v_i \ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right), v_j \ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]$$

According to the properties of the variance and covariance operators and recalling that parameters v_i are constants, we can express the previous equation in the following way:

$$var[\ln Y] = \sum_{i=1}^{n} v_i^2 \left(var\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]\right) + 2\sum_{i,j:i<j} v_i v_j \left(cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right), \ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\right)$$

We can not apply the variance (or covariance) operator directly to each term inside $\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)$ because of the logarithm. However, we can use the following results to obtain the variance and covariance terms.

Lemma 5.

$$var\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right] \approx \frac{\sum_{b=1}^{m}\phi_{ib}^2\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)^2}$$

Proof Lemma 19. Recalling the assumption about the distribution of each shock:

$$z_b \sim \; Lognormal\left(0, \sigma_b^2\right) \forall b$$

The inverse of such variable distributes:

$$z_b^{-1} \sim \; Lognormal\left(0, \sigma_b^2\right) \forall b$$

Multipliying z_b^{-1} by ϕ_{ib} we have the following distribution:

$$\frac{\phi_{ib}}{z_b} \sim \ Lognormal\left(\ln \phi_{ib}, \sigma_b^2\right) \forall b$$

Now we need to find the distribution of the sum of such random variables. There is no closed form expression for the sum of log-normals, however we can use the Fenton-Wilkinson approximation (Marlow (1967), Crow, E. and Shimizu, K. (1987))). Following that method, we will approximate the sum of independent log-normal random variables using another log-normal distribution with parameters $\hat{\mu}_i$ and $\hat{\sigma}_i^2$, [1]. I do not need to know the composition of such parameters but the distribution:

$$\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b} \sim \ Lognormal\left(\hat{\mu}_i, \hat{\sigma}_i^2\right)$$

We know that taking the natural logarithm of a log-normal random variable changes the distribution to normal keeping the same parameters:

$$\ln\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right) \sim \ Normal\left(\hat{\mu}_i, \hat{\sigma}_i^2\right)$$

These two results imply:

$$E\left[\ln\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right)\right] \approx \hat{\mu}_i$$

$$E\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right] \approx e^{\hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2}}$$

[1]More details of such approximation can be found in Hekmat (2006), and Pirinen (2003).

$$var\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]\approx\hat{\sigma}_i^2$$

$$var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]\approx\left(e^{\hat{\sigma}_i^2}-1\right)e^{2\hat{\mu}_i+\hat{\sigma}_i^2}$$

Now we recall the following fact about the variance:

$$var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]=E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]-E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2$$

Diving and rearranging:

$$\frac{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}=1+\frac{var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}$$

We know that the numerator of the left hand side is equal to the variance plus the expectation squared:

$$\frac{var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]+E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}=1+\frac{var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}$$

Now we substitute into the left hand side the variance and expectations that we obtained before:

$$\frac{\left(e^{\hat{\sigma}_i^2}-1\right)e^{2\hat{\mu}_i+\hat{\sigma}_i^2}+e^{2\hat{\mu}_i+\hat{\sigma}_i^2}}{e^{2\hat{\mu}_i+\hat{\sigma}_i^2}}\approx 1+\frac{var\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]^2}$$

We can clearly simplify the left hand side collecting common terms:

$$e^{\hat{\sigma}_i^2} \approx 1 + \frac{var\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]^2}$$

Taking natural logarithm of the whole expression:

$$\hat{\sigma}_i^2 \approx \ln\left[1 + \frac{var\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]^2}\right]$$

From previous result we know that $var\left[\ln\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right)\right] \approx \hat{\sigma}_i^2$, thus:

$$var\left[\ln\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right)\right] \approx \ln\left[1 + \frac{var\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]}{E\left[\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right]^2}\right]$$

Using this expression we can compute the variance, we just need to use the assumption of independence between shocks and the distribution of ϕ_{ib}/z_b that we found before. In this way:

$$var\left[\ln\left(\sum_{b=1}^{m} \frac{\phi_{ib}}{z_b}\right)\right] \approx \ln\left[1 + \frac{\sum_{b=1}^{m}\left(e^{\sigma_b^2} - 1\right) e^{2\ln\phi_{ib} + \sigma_b^2}}{\left(\sum_{b=1}^{m} e^{\ln\phi_{ib} + \frac{\sigma_b^2}{2}}\right)^2}\right]$$

We can simplify the previous expression recalling shocks are small, then σ_b^2 is close to zero, and we use the approximation $e^x \approx 1 + x$ for x close to zero and simplifying the exponential of logarithms:

$$var\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]\approx \ln\left[1+\frac{\sum_{b=1}^{m}\phi_{ib}^2\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)^2}\right]$$

We now that each σ_b^2 is close to zero, this implies that the second term inside the logarithm in the right hand side is close to zero, thus we can approximate it in the following way:

$$var\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)\right]\approx \frac{\sum_{b=1}^{m}\phi_{ib}^2\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)^2}$$

■

Lemma 6.

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\approx \frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{\left(\sigma_b^2+1\right)}\right)\left(\sum_{b=1}^{m}\phi_{jb}\sqrt{\left(\sigma_b^2+1\right)}\right)}$$

Proof Lemma 20. Analogously to the previous proof, we recall the following fact about the covariance:

$$cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]=E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]-E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]$$

Diving and rearranging:

$$\frac{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}=1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

We know that the numerator of the left hand could be re-expressed in the following way:

$$\frac{E\left[e^{\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)}e^{\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}=1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

$$\Rightarrow\frac{E\left[e^{\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right)+\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}=1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

Now we substitute into the left hand side the expectations that we obtained before (previous proof) but now taking into account the covariance of the logarithm terms:

$$\frac{e^{\hat{\mu}_i+\hat{\mu}_j+\frac{1}{2}\left(\hat{\sigma}_i^2+\hat{\sigma}_j^2+2cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\right)}}{e^{\hat{\mu}_i+\frac{\hat{\sigma}_i^2}{2}}e^{\hat{\mu}_j+\frac{\hat{\sigma}_j^2}{2}}}\approx 1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

Simplifying:

$$\frac{e^{\hat{\mu}_i+\hat{\mu}_j+\frac{\hat{\sigma}_i^2}{2}+\frac{\hat{\sigma}_j^2}{2}+cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]}}{e^{\hat{\mu}_i+\frac{\hat{\sigma}_i^2}{2}+\hat{\mu}_j+\frac{\hat{\sigma}_j^2}{2}}}\approx 1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

$$\Rightarrow e^{cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]} \approx 1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}$$

Taking natural logarithm of the whole expression:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right] \approx \ln\left[1+\frac{cov\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b},\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}{E\left[\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right]E\left[\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right]}\right]$$

Using this expression we can compute the covariance, we just need to recall independence assumption and the distribution of ϕ_{ib}/z_b that we found previously. In this way:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right] \approx \ln\left[1+\frac{\sum_{b=1}^{m}cov\left[\frac{\phi_{ib}}{z_b},\frac{\phi_{jb}}{z_b}\right]}{\left(\sum_{b=1}^{m}e^{\ln\phi_{ib}+\frac{\sigma_b^2}{2}}\right)\left(\sum_{b=1}^{m}e^{\ln\phi_{jb}+\frac{\sigma_b^2}{2}}\right)}\right]$$

We have eliminated the covariance terms between z_b and z_s because of the independence assumption. We know covariance operator is linear in constant and the covariance of a variable with itself is its variance, thus:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right] \approx \ln\left[1+\frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}var\left[\frac{1}{z_b}\right]}{\left(\sum_{b=1}^{m}e^{\ln\phi_{ib}+\frac{\sigma_b^2}{2}}\right)\left(\sum_{b=1}^{m}e^{\ln\phi_{jb}+\frac{\sigma_b^2}{2}}\right)}\right]$$

From previous proof we know the inverse of z_b distributed still log-normal with the same variance, previous expression becomes:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\approx\ln\left[1+\frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}\left(e^{\sigma_b^2}-1\right)e^{\sigma_b^2}}{\left(\sum_{b=1}^{m}e^{\ln\phi_{ib}+\frac{\sigma_b^2}{2}}\right)\left(\sum_{b=1}^{m}e^{\ln\phi_{jb}+\frac{\sigma_b^2}{2}}\right)}\right]$$

As we did before, we can simplify previous expression recalling shocks are small and then using the approximation $e^{\sigma_b^2}\approx 1+\sigma_b^2$ for σ_b^2 close to zero:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\approx\ln\left[1+\frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{(\sigma_b^2+1)}\right)\left(\sum_{b=1}^{m}\phi_{jb}\sqrt{(\sigma_b^2+1)}\right)}\right]$$

As each σ_b^2 is close to zero, we can approximate previous expression in the following way:

$$cov\left[\ln\left(\sum_{b=1}^{m}\frac{\phi_{ib}}{z_b}\right),\ln\left(\sum_{b=1}^{m}\frac{\phi_{jb}}{z_b}\right)\right]\approx\frac{\sum_{b=1}^{m}\phi_{ib}\phi_{jb}\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{(\sigma_b^2+1)}\right)\left(\sum_{b=1}^{m}\phi_{jb}\sqrt{(\sigma_b^2+1)}\right)}$$

■

Taking these two previous results to the total variance expression we have the following:

$$var[\ln Y]=\sum_{i=1}^{n}v_i^2\widetilde{\sigma}_i^2+2\sum_{i,j:i<j}v_iv_j\widetilde{\sigma}_{ij}$$

Where:

$$\widetilde{\sigma}_i^2\equiv\frac{\sum_{b=1}^{m}\phi_{ib}^2\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m}\phi_{ib}\sqrt{(\sigma_b^2+1)}\right)^2}$$

$$\widetilde{\sigma}_{ij} \equiv \frac{\sum_{b=1}^{m} \phi_{ib}\phi_{jb}\sigma_b^2\left(\sigma_b^2+1\right)}{\left(\sum_{b=1}^{m} \phi_{ib}\sqrt{(\sigma_b^2+1)}\right)\left(\sum_{b=1}^{m} \phi_{jb}\sqrt{(\sigma_b^2+1)}\right)}$$

■

Proof Corollary 3. Recalling the variance of GDP that we found previously and substituting the assumption $\sigma_b^2 = \sigma^2$ for all b:

$$var[\ln Y] = \sum_{i=1}^{n} v_i^2 \left(\frac{\sum_{b=1}^{m} \phi_{ib}^2\sigma^2\left(\sigma^2+1\right)}{\left(\sum_{b=1}^{m} \phi_{ib}\sqrt{(\sigma^2+1)}\right)^2} \right) + 2\sum_{i,j:i<j} v_i v_j \left(\frac{\sum_{b=1}^{m} \phi_{ib}\phi_{jb}\sigma^2\left(\sigma^2+1\right)}{\left(\sum_{b=1}^{m} \phi_{ib}\sqrt{(\sigma^2+1)}\right)\left(\sum_{b=1}^{m} \phi_{jb}\sqrt{(\sigma^2+1)}\right)} \right)$$

Simplifying:

$$var[\ln Y] = \sigma^2 \left[\sum_{i=1}^{n} v_i^2 \left(\frac{\sum_{b=1}^{m} \phi_{ib}^2}{\left(\sum_{b=1}^{m} \phi_{ib}\right)^2} \right) + 2\sum_{i,j:i<j} v_i v_j \left(\frac{\sum_{b=1}^{m} \phi_{ib}\phi_{jb}}{\left(\sum_{b=1}^{m} \phi_{ib}\right)\left(\sum_{b=1}^{m} \phi_{jb}\right)} \right) \right]$$

Recalling the assumption $\sum_{b=1}^{m} \phi_{ib} = 1$ for all i:

$$var[\ln Y] = \sigma^2 \left[\sum_{i=1}^{n} v_i^2 \left(\sum_{b=1}^{m} \phi_{ib}^2 \right) + 2\sum_{i,j:i<j} v_i v_j \left(\sum_{b=1}^{m} \phi_{ib}\phi_{jb} \right) \right]$$

This equation could be expressed in a simpler way:

$$var[\ln Y] = \sigma^2 \sum_{b=1}^{m} \left(\sum_{b=1}^{m} v_i \phi_{ib} \right)^2$$

■

3.7.5 Volatility and network metrics

Proof Corollary 4. Using the expression of volatility that we obtain in the previous section, and the assumption that σ is the same for all banks, the volatility of GDP is:

$$\sqrt{var(\ln Y)} = \sqrt{\sigma^2 \left(\sum_{b=1}^{m} \eta_b^2\right)}$$

Using the approximation of the influence vector (detailed in the first part of the appendix), the variance is a function of the networks of banks and sectors as η_b can be expressed as:

$$\eta_b = \sum_{i=1}^{n} v_i \phi_{ib}$$

$$\eta_b \approx \frac{\alpha}{n} \sum_{i=1}^{n} \left(1 + (1-\alpha)d_i + (1-\alpha)^2 \sum_{j=1}^{n} d_j w_{ji}\right) \phi_{ib}$$

$$\eta_b \approx \frac{\alpha}{n} \sum_{i=1}^{n} \left(\phi_{ib} + (1-\alpha)d_i \phi_{ib} + (1-\alpha)^2 \sum_{j=1}^{n} d_j w_{ji} \phi_{ib}\right)$$

After substituting the definitions of bank out-degree, bank-sector interconnectivity coefficient and second-order interconnectivity coefficient, the network multiplier becomes:

$$\eta_b \approx \frac{\alpha}{n}(b_b + (1-\alpha)B_b + (1-\alpha)^2 \widehat{B_b})$$

And then the volatility is given by:

$$\sqrt{var(\ln Y)} \approx \sqrt{\left(\frac{\sigma\alpha}{n}\right)^2 \sum_{b=1}^{m} \left(b_b + (1-\alpha)B_b + (1-\alpha)^2 \widehat{B_b}\right)^2}$$

■

3.7.6 Effect of network on volatility of GDP

3.7.6.a Adding one link

To prove the propositions of this section we will use results in the first section of the appendix. It is useful to recall the influence vector approximation that we found in previous sections, for the vertical (A), star (B), circle (D) and empty (F) structures:

$$\mathbf{v_D} = \mathbf{v_F} \approx \left(\frac{\alpha}{n}\right)(1 + (1-\alpha) + (1-\alpha)^2)\mathbf{1}$$

$$\mathbf{v_A} \approx \frac{\alpha}{n} \begin{bmatrix} 1 + 2(1-\alpha) + 3(1-\alpha)^2 \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ \vdots \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ 1 + (1-\alpha) \\ 1 \end{bmatrix}_{nx1} \qquad \mathbf{v_B} \approx \frac{\alpha}{n} \begin{bmatrix} 1 + n(1-\alpha) + n(1-\alpha)^2 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{nx1}$$

Proof Proposition 19. In this proof we are assuming that n is fairly large and than when i or j are not the top or the bottom of the chain, there are located close to the middle of the structure. We need to prove that the following inequalities hold for the parameters $\phi \in (0,1)$, $\alpha \in (0,1)$, $2 < n$, and $\sigma_b^2 > 0$ and the conditions given:

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \text{ if } (\{i,j\} \neq \{1,n\}) \parallel (i \neq \{1,n\}, j = n) \parallel (i = 1, j \neq 1)$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \text{ if } [(i \neq 1, j = 1)\&(1-\phi < \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi < \alpha^2)]$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} > [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \text{ if } [(i \neq 1, j = 1)\&(1-\phi > \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi > \alpha^2)]$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1}} = [var(\ln Y)]^{1/2}_{\mathbb{E}_{A,1:1,b(1:2)}} \text{ if } [(i \neq 1, j = 1)\&(1-\phi = \alpha)] \parallel [(i = n, j \neq 1)\&(1-\phi = \alpha^2)]$$

In fact, what we are comparing in all cases, after substituting the variance expression that we found before and eliminating common constants is the following:

$$\left[\sum_{b=1}^{m}\left(\sum_{i=1}^{n} v_i\phi_{ib}\right)^2\right]_{\mathbb{E}_{A,1:1}} \lesseqgtr \left[\sum_{b=1}^{m}\left(\sum_{i=1}^{n} v_i\phi_{ib}\right)^2\right]_{\mathbb{E}_{A,1:1,b(1:2)}}$$

Taking into account the structure of the economy and eliminating common terms:

$$v_i^2 + v_j^2 \lesseqgtr (v_i + \phi v_j)^2 + (1-\phi)^2 v_j^2$$

Simplifying:

$$v_j^2 \lesseqgtr 2\phi v_i v_j + \phi^2 v_j^2 + (1-\phi)^2 v_j^2$$

$$v_j \lesseqgtr 2\phi v_i + \phi^2 v_j + (1-\phi)^2 v_j$$

$$0 \lesseqgtr 2\phi v_i + \phi^2 v_j - 2\phi v_j + \phi^2 v_j$$

$$1 - \phi \lesseqgtr \frac{v_i}{v_j}$$

Using this relationship and according to the conditions given, substituting the corresponding influence vectors, the first inequality of the proposition becomes:

$$-\phi < 0 \parallel -\phi < (1-\alpha)+(1-\alpha)^2 \parallel -\phi < \frac{1+2(1-\alpha)+3(1-\alpha)^2}{1+(1-\alpha)+(1-\alpha)^2}-1 \parallel -\phi < 2(1-\alpha)+3(1-\alpha)^2$$

We can see that all of these relationships are true because of the assumptions about α and ϕ. For the rest of the inequalities, after simplifying in the same way as before, we have the following:

$$\left(1-\phi < \frac{1+(1-\alpha)+(1-\alpha)^2}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi < \alpha\right) \parallel \left(1-\phi < \frac{1}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi < \alpha^2\right)$$

$$\left(1-\phi > \frac{1+(1-\alpha)+(1-\alpha)^2}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi > \alpha\right) \parallel \left(1-\phi > \frac{1}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi > \alpha^2\right)$$

$$\left(1-\phi = \frac{1+(1-\alpha)+(1-\alpha)^2}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi = \alpha\right) \parallel \left(1-\phi = \frac{1}{1+2(1-\alpha)+3(1-\alpha)^2} \text{ if } 1-\phi = \alpha^2\right)$$

Recalling that the polynomials that contain α in the previous inequalities where obtained from a second order approximation for the elements of the influence vector, the infinite series converge because $|1-\alpha| < 1$ and then $\sum_{k=0}^{\infty}(1-\alpha)^k = 1/\alpha$ and $\sum_{k=0}^{\infty}(k+1)(1-\alpha)^k = 1/\alpha^2$. Substituting these facts into the previous inequalities:

$$1-\phi < \alpha \parallel 1-\phi < \alpha^2$$

$$1-\phi > \alpha \parallel 1-\phi > \alpha^2$$

$$1-\phi = \alpha \parallel 1-\phi = \alpha^2$$

Which are exactly the conditions that we provided. ■

Proof Proposition 20. In the same way that the previous proof, we need to prove that the following inequalities hold for the parameters $\phi \in (0,1)$, $\alpha \in (0,1)$, $2 < n$, and $\sigma_b^2 > 0$ and the conditions given:

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \text{if} \quad (\{i,j\} \neq 1) \parallel (i = 1, j \neq 1)$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \text{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) < \alpha\phi$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} > [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \text{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) > \alpha\phi$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1}} = [var(\ln Y)]^{1/2}_{\mathbb{E}_{B,1:1,b(1:2)}} \quad \text{if} \quad i \neq 1, j = 1, n(1-\phi)(1-\alpha) = \alpha\phi$$

As we did in the previous proof, substituting the variance expression, eliminating common constants and terms taking into account the structure of the economy, and substituting the corresponding influence vectors, the first inequality of the proposition becomes:

$$-\phi < 0 \parallel -\phi < n(1-\alpha) + n(1-\alpha)^2$$

These relationships are true taking into account the assumptions about α and ϕ. For the last three inequalities of the proposition, after simplifying we have the following:

$$(1-\alpha) + (1-\alpha)^2 < \frac{\phi}{n(1-\phi)} \quad \text{if} \quad n(1-\phi)(1-\alpha) < \alpha\phi$$

$$(1-\alpha)+(1-\alpha)^2 > \frac{\phi}{n(1-\phi)} \;\text{ if }\; n(1-\phi)(1-\alpha) > \alpha\phi$$

$$(1-\alpha)+(1-\alpha)^2 = \frac{\phi}{n(1-\phi)} \;\text{ if }\; n(1-\phi)(1-\alpha) = \alpha\phi$$

As we did in the previous proof, we substitute the infinite convergent series $\sum_{k=1}^{\infty}(1-\alpha)^k = (1-\alpha)/\alpha$ into the left hand side of the previous inequalities:

$$n(1-\phi)(1-\alpha) < \alpha\phi$$

$$n(1-\phi)(1-\alpha) > \alpha\phi$$

$$n(1-\phi)(1-\alpha) = \alpha\phi$$

Which are exactly the conditions that we provided. ■

Proof Proposition 21. We need to prove that the following inequalities hold for the parameters $\phi \in (0,1)$, $\alpha \in (0,1)$, $2 < n$, and $\sigma_b^2 > 0$:

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{D,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{D,1:1,b(1:2)}}$$

$$[var(\ln Y)]^{1/2}_{\mathbb{E}_{F,1:1}} < [var(\ln Y)]^{1/2}_{\mathbb{E}_{F,1:1,b(1:2)}}$$

As we did before, substituting the variance expression, eliminating common constants and terms taking into account that the structure of the economy implies the same influence vectors for all sector, both inequalities become:

$$1-\phi < 1$$

Which we know is true because $\phi \in (0, 1)$. ■

3.7.6.b Adding m-links

To prove the propositions of this section we will use results in the first section of the appendix.

Proof Proposition 22. We need to prove that the inequalities hold for the parameters $\alpha \in (0, 1)$, $3 < n$, and $\sigma_b^2 = \sigma^2 > 0$:

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:1}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{A,2:2}}$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:1}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,2:2}}$$

Recalling the definition of volatility, the structure $(1 : 1)$ that implies $\phi_{ib} = 1$ $\forall \overrightarrow{\langle b, i \rangle} \in BN_{1:1}$, and the structure $(2 : 2)$, after simplifying both inequalities become:

$$\sum_{i=1}^{n} v_i^2 > \sum_{\forall i \in M : mod(i,2)=1} ((1-\phi)v_i + \phi v_{i+1})^2 + \sum_{\forall i \in M : mod(i,2)=0} (\phi v_{i-1} + (1-\phi)v_i)^2$$

Expanding the squared terms:

$$\begin{aligned}\sum_{i=1}^{n} v_i^2 > &\sum_{\forall i \in M : mod(i,2)=1} ((1-\phi)^2 v_i^2 + \phi^2 v_{i+1}^2 + 2(1-\phi)\phi v_i v_{i+1}) \\ &+ \sum_{\forall i \in M : mod(i,2)=0} (\phi^2 v_{i-1}^2 + (1-\phi)^2 v_i^2 + 2(1-\phi)\phi v_{i-1} v_i)\end{aligned}$$

Collecting common terms and simplifying:

$$\sum_{i=1}^{n} v_i^2 > \sum_{\forall i \in M: mod(i,2)=1} [(v_i^2 + v_{i+1}^2)((1-\phi)^2 + \phi^2) + 4(1-\phi)\phi v_i v_{i+1}]$$

Taking all terms to the left hand side:

$$\sum_{\forall i \in M: mod(i,2)=1} [(v_i^2 + v_{i+1}^2)(1 - (1-\phi)^2 - \phi^2) - 4(1-\phi)\phi v_i v_{i+1}] > 0$$

Simplifying the terms that contain ϕ:

$$\sum_{\forall i \in M: mod(i,2)=1} [(v_i^2 + v_{i+1}^2)2(1-\phi)\phi - 4(1-\phi)\phi v_i v_{i+1}] > 0$$

Dividing the whole inequality by $2(1-\phi)\phi$ as we know that this term is positive because $\phi \in (0, 1)$:

$$\sum_{\forall i \in M: mod(i,2)=1} [v_i^2 + v_{i+1}^2 - 2v_i v_{i+1}] > 0$$

Factorising:

$$\sum_{\forall i \in M: mod(i,2)=1} (v_i - v_{i+1})^2 > 0$$

Which is true because from previous results we know that for vertical and star structures there is at least one pair $v_i \neq v_j$. ■

Proof Proposition 23*.* We need to prove that the inequality hold for the parameters $\alpha \in (0,1)$, $3 < n$, and $\sigma_b^2 = \sigma^2 > 0$:

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:1}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{D,2:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:1}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{F,2:2}}$$

Following the same intuition that the previous proof, recalling the definition of volatility, after simplifying the first two sides of the inequality become (last two sides are equal to the following):

$$\sum_{i=1}^{n} v_i^2 = \sum_{\forall i \in M: mod(i,2)=1} ((1-\phi)v_i + \phi v_{i+1})^2 + \sum_{\forall i \in M: mod(i,2)=0} (\phi v_{i-1} + (1-\phi)v_i)^2$$

Recalling from previous results that for circle and empty structures v_i is the same for all sectors:

$$nv_i^2 = \frac{n}{2}(v_i^2) + \frac{n}{2}(v_i^2)$$

Which implies:

$$v_i^2 = v_i^2$$

■

3.7.6.c Diversification

To prove the propositions of this section we will use again the results of the first section of the appendix. In particular, we recall the influence vector approximation for the vertical (A) and circle structures:

$$\mathbf{v_D} \approx \left(\frac{\alpha}{n}\right)(1 + (1-\alpha) + (1-\alpha)^2)\mathbf{1}$$

$$\mathbf{v_A} \approx \frac{\alpha}{n} \begin{bmatrix} 1 + 2(1-\alpha) + 3(1-\alpha)^2 \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ \vdots \\ 1 + (1-\alpha) + (1-\alpha)^2 \\ 1 + (1-\alpha) \\ 1 \end{bmatrix}_{nx1}$$

Proof Proposition 24. Recalling the inequalities that we need to prove:

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{A,1:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}^2_{A,1:2}}$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}^3_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}^4_{A,1:2}}$$

Recalling the volatility expression and according to the bipartite structure of each economy, after eliminating common terms, inequalities become:

$$(v_1 + v_2)^2 + (v_3 + v_4)^2 = (v_1 + v_3)^2 + (v_2 + v_4)^2$$

$$(v_{n-3} + v_{n-2})^2 + (v_{n-1} + v_n)^2 > (v_{n-3} + v_{n-1})^2 + (v_{n-2} + v_n)^2$$

Expanding the squared terms, eliminating common terms and diving whole inequalities by 2:

$$v_1 v_2 + v_3 v_4 = v_1 v_3 + v_2 v_4$$

$$v_{n-3} v_{n-2} + v_{n-1} v_n > v_{n-3} v_{n-1} + v_{n-2} v_n$$

Re-arranging:

$$(v_1 - v_4)(v_2 - v_3) = 0$$

$$(v_{n-3} - v_n)(v_{n-2} - v_{n-1}) > 0$$

The inequalities are true because we know that for the vertical structure, $v_2 = v_3$ and $v_{n-3} = v_{n-2} > v_{n-1} > v_n$. ■

Proof Proposition 25. The inequality that we want to prove is:

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}'_{A,1:2}}$$

Recalling the definition of volatility, taking squares and eliminating constants, the comparison becomes:

$$(v_1 + v_2)^2 + ... + (v_{n-1} + v_n)^2 > (v_1 + v_3)^2 + ... + (v_{n-2} + v_n)^2$$

Expanding squared terms:

$$\|\mathbf{v}\|_2^2 + 2(v_1 v_2 + ...v_{n-1}v_n) > \|\mathbf{v}\|_2^2 + 2(v_1 v_3 + ...v_{n-2}v_n)$$

Simplifying:

$$v_1 v_2 + ...v_{n-1}v_n > v_1 v_3 + ...v_{n-2}v_n$$

Rearranging:

$$v_1(v_2 - v_3) - v_4(v_2 - v_3) + ... + v_{n-3}(v_{n-2} - v_{n-1}) - v_n(v_{n-2} - v_{n-1}) > 0$$

Collecting common terms:

$$(v_1 - v_4)(v_2 - v_3) + ... + (v_i - v_{i+3})(v_{i+1} - v_{i+2}) + ... + (v_{n-3} - v_n)(v_{n-2} - v_{n-1}) > 0$$

The above is true because we know that for the vertical structure, $v_i \geq v_j \; \forall (i < j) \in N$, in particular $v_{n-3} > v_n$ and $v_{n-2} > v_{n-1}$. ■

Proof Proposition 26. The equality that we need to prove is:

$$\sqrt{var(\ln Y)}_{\mathbb{E}^1_{D,1:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}'_{D,1:2}}$$

Analogous to the previous proof, following the same steps and after simplifying we have the following equality:

$$(v_1 - v_4)(v_2 - v_3) + ... + (v_i - v_{i+3})(v_{i+1} - v_{i+2}) + ... + (v_{n-3} - v_n)(v_{n-2} - v_{n-1}) = 0$$

The above is true because we know that for the circle structure, $v_i = v_j \; \forall (i, j) \in N$. ■

3.7.6.d Concentration

As in the previous cases, to prove the propositions of this section we will use results in the first section of the appendix:

$$\mathbf{v_D} = \mathbf{v_F} \approx \left(\frac{\alpha}{n}\right)(1 + (1 - \alpha) + (1 - \alpha)^2)\mathbf{1}$$

$$\mathbf{v_A} \approx \frac{\alpha}{n} \begin{bmatrix} 1+2(1-\alpha)+3(1-\alpha)^2 \\ 1+(1-\alpha)+(1-\alpha)^2 \\ \vdots \\ 1+(1-\alpha)+(1-\alpha)^2 \\ 1+(1-\alpha) \\ 1 \end{bmatrix}_{nx1} \quad \mathbf{v_B} \approx \frac{\alpha}{n} \begin{bmatrix} 1+n(1-\alpha)+n(1-\alpha)^2 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{nx1}$$

Proof Proposition 27. The inequalities that we want to prove are:

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{D,1:2,b(1:3),s(1:1)}}$$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{F,1:2,b(1:3),s(1:1)}}$$

Recalling the definition of volatility and the assumption that $\sigma_b^2 = \sigma$, $\forall b \in M$, together with the given bipartite structure that implies $\phi_{ib} = 1 \ \forall \overrightarrow{\langle b, i \rangle} \in BN$ for both economies, after taking squares and eliminating constants and common terms, the comparison becomes (for both inequalities):

$$(v_i + v_j)^2 + (v_k + v_l)^2 < (v_i + v_j + v_k)^2 + v_l^2$$

Expanding squared terms, simplifying and eliminating common terms:

$$v_l < v_i + v_j$$

Which we know is true because for all perfectly balanced I-O networks, like circle and empty, $v_i = v_j = v_l \; \forall (i,j,l) \in N$. ■

Proof Proposition 28. In this case the inequalities that we want to prove are:

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$$

if $(\{i = 1 \parallel j = 1\} \& l \neq 1) \parallel \{i,j,l\} \neq 1 \parallel \big(\{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 < 1\big)$

$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$ if $\{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 > 1$

$\sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{B,1:2,b(1:3),s(1:1)}}$ if $\{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 = 1$

Analogous to our previous proof, after taking squares, eliminating constants and common terms, and simplifying, comparisons become:

$$v_l < v_i + v_j \text{ if } (\{i = 1 \parallel j = 1\} \& l \neq 1) \parallel \{i,j,l\} \neq 1 \parallel \big(\{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 < 1\big)$$

$$v_l > v_i + v_j \text{ if } \{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 > 1$$

$$v_l = v_i + v_j \text{ if } \{i,j\} \neq 1 \& l = 1 \& n(1-\alpha)(2-\alpha)^2 = 1$$

Taking into account the conditions and the structure of the star network, inequalities become:

$$-1 < n(1-\alpha) + n(1-\alpha)^2 \parallel 1 < 2 \parallel n(1-\alpha)(2-\alpha)^2 < 1$$

$$n(1-\alpha)(2-\alpha)^2 > 1$$

$$n(1-\alpha)(2-\alpha)^2 = 1$$

The first two inequalities are true because of the assumptions about n and α and the rest are exactly the conditions that we provided. ■

Proof Proposition 29. The inequalities that we want to prove are:

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}} < \sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

if $[l \neq 1] \parallel [l = 1\&\{i,j\} \neq \{n, n-1\}] \parallel \left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 < \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\& \left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 < \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 < 1\right)\right)\right]$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}} > \sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

if $\left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 > \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\& \left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 > \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 > 1\right)\right)\right]$

$$\sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2}} = \sqrt{var(\ln Y)}_{\mathbb{E}_{A,1:2,b(1:3),s(1:1)}}$$

if $\left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 = \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\& \left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 = \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 = 1\right)\right)\right]$

As we did in our previous proof, inequalities become:

$$v_l < v_i + v_j$$

if $[l \neq 1] \parallel [l = 1\&\{i,j\} \neq \{n, n-1\}] \parallel \left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 < \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\&\left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 < \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 < 1\right)\right)\right]$

$$v_l > v_i + v_j$$

if $\left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 > \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\&\left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 > \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 > 1\right)\right)\right]$

$$v_l = v_i + v_j$$

if $\left[l = 1\&\{i,j\} = \{n, n-1\}\&3(1-\alpha)^2 = \alpha\right]$

$\parallel \left[l = 1\&\{i \parallel j\} \neq \{n, n-1\}\&\left(\left(\{i \parallel j\} = n\&2(1-\alpha)^2 = \alpha\right) \parallel \left(\{i \parallel j\} = n-1\&2(1-\alpha)^2 = 1\right)\right)\right]$

Taking into account the conditions and the structure of the vertical network, inequalities become:

$$-1 < 3(1-\alpha)+3(1-\alpha)^2 \parallel -1 < 2(1-\alpha)+3(1-\alpha)^2 \parallel -1 < 2(1-\alpha)+2(1-\alpha)^2 \parallel$$
$$-1 < (1-\alpha)+2(1-\alpha)^2 \parallel -1 < (1-\alpha)+(1-\alpha)^2 -1 < (1-\alpha)^2 \parallel -1 < (1-\alpha) \parallel -1 < 2$$
$$(1-\alpha)^2 < 1 \parallel 3(1-\alpha)^2 < \alpha \parallel 2(1-\alpha)^2 < \alpha \parallel 2(1-\alpha)^2 < 1$$

$$3(1-\alpha)^2 > \alpha \parallel 2(1-\alpha)^2 > \alpha \parallel 2(1-\alpha)^2 > 1$$

$$3(1-\alpha)^2 = \alpha \parallel 2(1-\alpha)^2 = \alpha \parallel 2(1-\alpha)^2 = 1$$

The first eight parts of the first set of inequalities are true because of the assumption α. All the rest of inequalities a are exactly the conditions that we provided. ■

3.8 Chapter 3

3.8.1 Influence vector results

3.8.1.a Results for $n > 3$, any economy

Recalling the definition of the influence vector, and following Carvalho (2010) and Acemoglu et al. (2012), I define the centrality vector of the intersectoral trade as the following:

$$\mathbf{v}' \equiv \frac{\alpha}{n}\mathbf{1}'\left(\mathbb{I} - (1-\alpha)\,\mathbf{W}\right)^{-1}$$

Where:

$$\mathbf{W} \equiv \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix}_{nxn} \qquad \mathbb{I} \equiv \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn} \qquad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

where the matrix $\mathbf{W}$ represents the input-output table of the economy, in other words, the network of trading among sectors, $\mathbf{1}$ is a vector of ones and $\mathbb{I}$ is the identity matrix.

Lemma 7. *Post multiplying the influence vector by the vector of ones I have the following result:*

$$\mathbf{v}'\mathbf{1} = 1$$

Proof Lemma 7. Recalling the Perron-Frobenius Theorem for positive matrices, applied to the input-output matrix, $\mathbf{W}$, which implies that all the eigenvalues of $\mathbf{W}$ are inside the unit circle, and hence, I can express the influence vector as a Neumann series:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}' \sum_{k=0}^{\infty} (1-\alpha)^k \mathbf{W}^k$$

Post-multiplying the previous equation by a vector of ones:

$$\mathbf{v}'\mathbf{1} = \frac{\alpha}{n}\mathbf{1}' \sum_{sk=0}^{\infty} (1-\alpha)^k \, \mathbf{W}^k \mathbf{1}$$

Simplifying I can see that $\mathbf{v}'\mathbf{1} = 1$ because:

$$\mathbf{v}'\mathbf{1} = \alpha \sum_{k=0}^{\infty} (1-\alpha)^k = \alpha \left(\frac{1}{\alpha}\right) = 1$$

The series converges because $|1-\alpha| < 1$. ■

Lemma 8. *The natural logarithm of the $L_1 - norm$ of the vector that gives us the solution for the ratio output-consumption, when there are no wedges or frictions is given by:*

$$\ln \|\gamma\|_1 = \ln n - \ln \alpha$$

Proof Lemma 8. Taking the exponential of both sides:

$$\|\gamma\|_1 = n/\alpha$$

Expressing this equation as a product of vectors:

$$\gamma'\mathbf{1} = n/\alpha$$

Substituting the definition of the vector γ', and multiplying the equation by α/n:

$$\frac{\alpha}{n}\mathbf{1}'\left(\mathbb{I} - (1-\alpha)\,\mathbf{W}\right)^{-1}\mathbf{1} = 1$$

Using the definition of the influence vector, I can expressed the previous equation in the following way, which was proven to be true en the previous result:

$$\mathbf{v}'\mathbf{1} = 1$$

■

Lemma 9. *Subtracting the vector of ones transposed, scaled by* $1/n$*, from the influence vector and multiplied by the vector of ones I have the following result:*

$$\left(\mathbf{v}' - \frac{1}{n}\mathbf{1}'\right)\mathbf{1} = 0$$

Proof Lemma 9. Recalling the result:

$$\mathbf{v}'\mathbf{1} = 1$$

Subtracting from both sides:

$$\mathbf{v}'\mathbf{1} - \frac{1}{n}\mathbf{1}'\mathbf{1} = 1 - \frac{1}{n}\mathbf{1}'\mathbf{1}$$

Rearranging and using the fact that $\mathbf{1}'\mathbf{1}/n = 1$:

$$\left(\mathbf{v}' - \frac{1}{n}\mathbf{1}'\right)\mathbf{1} = 0$$

■

Lemma 10. *Each element, v_i, of the influence vector, $\mathbf{v}$, can be approximated by:*

$$v_i \approx \frac{\alpha}{n}\left(1 + (1-\alpha)d_i + (1-\alpha)^2 \sum_{j=1}^{n} d_j w_{ji}\right)$$

Where d_i is the outdegree of sector i defined as $d_i \equiv \sum_{j=1}^{n} w_{ji}$.

Proof Lemma 10*.* We express the influence vector as a Neumann series as I did before:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}' \sum_{k=0}^{\infty} (1-\alpha)^k \mathbf{W}^k$$

Expanding the series:

$$\mathbf{v}' = \frac{\alpha}{n}\mathbf{1}'(\mathbb{I} + (1-\alpha)\mathbf{W} + (1-\alpha)^2\mathbf{W}^2 + ... + \infty)$$

Multiplying the sum by the vector of ones:

$$\mathbf{v}' = \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{1}'\mathbf{W} + (1-\alpha)^2\mathbf{1}'\mathbf{W}\mathbf{W} + ... + \infty)$$

I can substitute the vector of outdegrees, $\mathbf{d}' = [d_1, ..., d_n]$, in the previous equation because $\mathbf{1}'\mathbf{W} = \mathbf{d}'$:

$$\mathbf{v}' = \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W} + ... + \infty)$$

Grouping over the out-degrees vector and taking a second-order approximation, the influence vector can be expressed:

$$\mathbf{v}' \approx \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W})$$

From the equation of the volatility of GDP I can see that I need to express the i-th element of the influence vector, v_i, as a function of the degrees, using the previous results I have the following:

$$v_i \approx \frac{\alpha}{n}\left(1 + (1-\alpha)d_i + (1-\alpha)^2\sum_{j=1}^{n} d_j w_{ji}\right)$$

■

Lemma 11. *The inner product of the influence vector,* $\mathbf{v}$*, can be approximated by:*

$$\|\mathbf{v}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2) + (1-\alpha)^2\mathbf{d}'\mathbf{d} + (1-\alpha)^3\mathbf{d}'(\mathbf{W}'+\mathbf{W})\mathbf{d} + (1-\alpha)^4\mathbf{d}'\mathbf{W}'\mathbf{W}\mathbf{d}\right]$$

Where $\mathbf{d}$ *is the vector of outdegrees,* $\mathbf{d}' = [d_1, ..., d_n]$*, and* W *is the input-output matrix.*

Proof Lemma 11. Recalling the approximation of the influence vector from the previous result:

$$\mathbf{v}' \approx \frac{\alpha}{n}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W})$$

Multiplying this equation by its transpose:

$$\|\mathbf{v}\|_2^2 \approx \frac{\alpha^2}{n^2}(\mathbf{1}' + (1-\alpha)\mathbf{d}' + (1-\alpha)^2\mathbf{d}'\mathbf{W})(\mathbf{1} + (1-\alpha)\mathbf{d} + (1-\alpha)^2\mathbf{W}'\mathbf{d})$$

Expanding the product and using the definition $\mathbf{d} \equiv \mathbf{W}'\mathbf{1}$, assumptions $\mathbf{W}\mathbf{1} = \mathbf{1}$,

$\mathbf{d}'\mathbf{1} = n$, and fact $\mathbf{1}'\mathbf{1} = n$, the previous equation simplifies to:

$$\|\mathbf{v}\|_2^2 \approx \frac{\alpha^2}{n^2}[\mathbf{1}'\mathbf{1}+(1-\alpha)\mathbf{1}'\mathbf{d}+(1-\alpha)^2\mathbf{1}'\mathbf{W}'\mathbf{d}+(1-\alpha)\mathbf{d}'\mathbf{1}+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'\mathbf{W}'\mathbf{d}$$
$$+ (1-\alpha)^2\mathbf{d}'\mathbf{W}\mathbf{1} + (1-\alpha)^3\mathbf{d}'\mathbf{W}\mathbf{d} + (1-\alpha)^4\mathbf{d}'\mathbf{W}\mathbf{W}'\mathbf{d}]$$

$$= \frac{\alpha^2}{n^2}[n+(1-\alpha)n+(1-\alpha)^2\mathbf{1}'\mathbf{d}+(1-\alpha)n+(1-\alpha)^2\mathbf{1}'\mathbf{W}\mathbf{W}'\mathbf{1}+(1-\alpha)^3\mathbf{1}'\mathbf{W}\mathbf{W}'\mathbf{W}'\mathbf{1}$$
$$+ (1-\alpha)^2\mathbf{1}'\mathbf{W}\mathbf{W}\mathbf{1} + (1-\alpha)^3\mathbf{1}'\mathbf{W}\mathbf{W}\mathbf{W}'\mathbf{1} + (1-\alpha)^4\mathbf{1}'\mathbf{W}\mathbf{W}\mathbf{W}'\mathbf{W}'\mathbf{1}]$$

$$= \frac{\alpha^2}{n^2}[n+(1-\alpha)n+(1-\alpha)^2n+(1-\alpha)n+(1-\alpha)^2||\mathbf{d}\|_2^2+(1-\alpha)^3\mathbf{d}'\mathbf{W}'\mathbf{d}+(1-\alpha)^2\mathbf{d}'\mathbf{1}$$
$$+ (1-\alpha)^3\mathbf{d}'\mathbf{W}\mathbf{d} + (1-\alpha)^4\mathbf{d}'\mathbf{W}\mathbf{W}'\mathbf{d}]$$

$$= \frac{\alpha^2}{n^2}[n(1+(1-\alpha)+(1-\alpha)^2+(1-\alpha))+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'\mathbf{W}'\mathbf{d}+(1-\alpha)^2n$$
$$+ (1-\alpha)^3\mathbf{d}'\mathbf{W}\mathbf{d} + (1-\alpha)^4\mathbf{d}'\mathbf{W}\mathbf{W}'\mathbf{d}]$$

$$= \frac{\alpha^2}{n^2}[n(1+2(1-\alpha)+2(1-\alpha)^2)+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'\mathbf{W}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'\mathbf{W}\mathbf{d}+(1-\alpha)^4\mathbf{d}'\mathbf{W}\mathbf{W}'\mathbf{d}]$$

$$= \frac{\alpha^2}{n^2}[n(1+2(1-\alpha)(2-\alpha))+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'(\mathbf{W}'+\mathbf{W})\mathbf{d}+(1-\alpha)^4\mathbf{d}'\mathbf{W}\mathbf{W}'\mathbf{d}]$$

Finally, expanding the term that is multiplying n in the first product inside the sum:

$$\|\mathbf{v}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'(\mathbf{W}'+\mathbf{W})\mathbf{d}+(1-\alpha)^4\mathbf{d}'\mathbf{W}'\mathbf{W}\mathbf{d}\right]$$

■

3.8.1.b Results for $n > 3$, economies A, B, C, D and F

In this section, I will use the second order approximation of the influence vector to analyse the difference between economies with n sectors. Recalling the inner product of the influence vector in terms of its approximation:

$$\|\mathbf{v}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(1-\alpha)^2\mathbf{d}'\mathbf{d}+(1-\alpha)^3\mathbf{d}'(\mathbf{W}'+\mathbf{W})\mathbf{d}+(1-\alpha)^4\mathbf{d}'\mathbf{W}'\mathbf{W}\mathbf{d}\right]$$

I can see that in order to obtain the inner product of the influence vector for the economies A(Vertical), B(Star), C(Tree), D(Circle) and F(Empty), I need to know the input-output matrix, $\mathbf{W}$, and the degree vector, $\mathbf{d}$, in each case. The input-output matrices in each case are the following:

$$\mathbf{W_A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{nxn} \quad \mathbf{W_B} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}_{nxn} \quad \mathbf{W_C} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}_{nxn}$$

$$\mathbf{W_D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{nxn} \quad \mathbf{W_F} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{nxn}$$

The degree vector in each case is obtained as the column sums of the previous matrices:

$$\mathbf{d_A} = \begin{bmatrix} 2 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}_{nxn} \quad \mathbf{d_B} = \begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{nxn} \quad \mathbf{d_C} = \begin{bmatrix} 2 \\ n-2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{nxn} \quad \mathbf{d_D} = \mathbf{d_F} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{nxn}$$

With these matrices and vector I can find the inner product of the influence vector using the approximation above. For each economy, after simplifying, the euclidean norms squared of the influence vectors are the following:

$$\|\mathbf{v_A}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4\right]$$

$$\|\mathbf{v_B}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n^2((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right]$$

$$\|\mathbf{v_C}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(8+n(n-2))(1-\alpha)^2+4n(1-\alpha)^3+n^2(1-\alpha)^4\right]$$

$$\|\mathbf{v_D}\|_2^2 = \|\mathbf{v_F}\|_2^2 \approx \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right]$$

Using these equations I can proceed to prove the following comparisons.

Lemma 12. $\|\mathbf{v_B}\|_2^2 > \|\mathbf{v_C}\|_2^2$, *for* $3 < n << \infty$

Proof Lemma 12. Substituting the approximations, the inequality becomes:

$$\begin{aligned}&\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n^2((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right]\\&\quad>\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(8+n(n-2))(1-\alpha)^2+4n(1-\alpha)^3+n^2(1-\alpha)^4\right]\end{aligned}$$

Multiplying by $\frac{n^2}{\alpha^2}$ and eliminating the common term:

$$n^2((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4) > (1-\alpha)^2(8+n(n+2))+2(1-\alpha)^3 2n+(1-\alpha)^4 n^2$$

Dividing by n^2:

$$(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4 > (1-\alpha)^2\left(\frac{8}{n^2}+\frac{n-2}{n}\right)+2(1-\alpha)^3\left(\frac{2}{n}\right)+(1-\alpha)^4$$

Rearranging:

$$(1-\alpha)^2\left(1-\left(\frac{8}{n^2}+\frac{n-2}{n}\right)\right)+2(1-\alpha)^3\left(1-\frac{2}{n}\right)>0$$

Recalling the assumptions $\alpha \in (0,1)$ and $3 < n << \infty$, this inequality will be always true. First term is non negative because $(1-\alpha)^2 > 0$ and:

$$1-\frac{8}{n^2}-\frac{n-2}{n}\geq 0$$

Simplifying:

$$\frac{n^2-8}{n^2}-\frac{n-2}{n}\geq 0$$

Using n^2 as common denominator:

$$\frac{n^2-8-n(n-2)}{n^2}\geq 0$$

Multiplying everything by n^2 and expanding:

$$n^2-8-n^2+2n\geq 0$$

Simplifying:

$$n \geq 4$$

Second term is always positive because $2(1-\alpha)^3 > 0$ and

$$1 - \frac{2}{n} > 0$$

Simplifying:

$$n > 2$$

Which is true according to this assumption. ■

Lemma 13. $\|\mathbf{v_C}\|_2^2 > \|\mathbf{v_A}\|_2^2$, *for* $3 < n << \infty$

Proof Lemma 13. We substitute the approximation to the inner product of the influence vector:

$$\begin{aligned}&\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(8+n(n-2))(1-\alpha)^2+4n(1-\alpha)^3+n^2(1-\alpha)^4\right]\\&>\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4\right]\end{aligned}$$

After simplifying:

$$(1-\alpha)^2(8+n(n-2))+2(1-\alpha)^3 2n+(1-\alpha)^4 n^2 > (1-\alpha)^2(2+n)+2(1-\alpha)^3(3+n)+(1-\alpha)^4(6+n)$$

Rearranging:

$$(1-\alpha)^2(8+n(n-2)-2-n)+2(1-\alpha)^3(2n-3-n)+(1-\alpha)^4(n^2-6-n)>0$$

By assumption $\alpha \in (0,1)$ and $3 < n << \infty$. Thus, first term is always positive because $(1-\alpha)^2 > 0$ and:

$$8 + n(n-2) - 2 - n > 0$$

Expanding:

$$8 + n^2 - 2n - 2 - n > 0$$

Simplifying:

$$6 + n^2 - 3n > 0$$

Factorising:

$$6 + n(n-3) > 0$$

Second term is always positive because $2(1-\alpha)^3 > 0$ and:

$$2n - 3 - n > 0$$

Simplifying:

$$n > 3$$

Finally, third term is also positive because $(1-\alpha)^4 > 0$ and:

$$n^2 - 6 - n > 0$$

Simplifying:

$$n^2 - n > 6$$

Factorising:

$$n(n-1) > 6$$

$\blacksquare$

Lemma 14. $\|\mathbf{v_A}\|_2^2 > \|\mathbf{v_D}\|_2^2 = \|\mathbf{v_F}\|_2^2$, *for* $3 < n << \infty$

Proof Lemma 14. Substituting the approximations to the inner product of the influence vector for each economy, the inequality becomes:

$$\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4\right]$$
$$> \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right]$$

Eliminating common terms and rearranging:

$$(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4-n((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4) > 0$$

Factorising the terms that contain α:

$$2(1-\alpha)^2+6(1-\alpha)^3+6(1-\alpha)^4 > 0$$

This polynomial inequality is always positive because $\alpha \in (0,1)$. $\blacksquare$

I can compare also the inner products as n goes to infinity:

Lemma 15. $\lim_{n\to\infty} \|\mathbf{v_B}\|_2^2 > \lim_{n\to\infty} \|\mathbf{v_C}\|_2^2$

Proof Lemma 15. We start with the inequality of the inner products:

$$\|\mathbf{v_B}\|_2^2 > \|\mathbf{v_C}\|_2^2 \tag{3.90}$$

Substituting the approximations:

$$\begin{aligned}
&\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n^2((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right] \\
&\quad > \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(8+n(n-2))(1-\alpha)^2+4n(1-\alpha)^3+n^2(1-\alpha)^4\right]
\end{aligned}$$

Taking limits to infinity with respect to n of both sides, eliminating α^2 and rearranging:

$$\begin{aligned}
&\lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2}{n}+(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4\right] \\
&\quad > \lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2+(n-2)(1-\alpha)^2+4(1-\alpha)^3}{n}+\frac{8(1-\alpha)^2}{n^2}+(1-\alpha)^4\right]
\end{aligned}$$

Calculating the limits the inequality becomes:

$$(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4 > (1-\alpha)^4$$

Simplifying:

$$(1-\alpha)^2+2(1-\alpha)^3 > 0$$

Which is true because $\alpha \in (0,1)$. ■

Lemma 16. $\lim_{n\to\infty} \|\mathbf{v_C}\|_2^2 > \lim_{n\to\infty} \|\mathbf{v_A}\|_2^2$

Proof Lemma 16. As before, I start with the inequality of the inner products:

$$\|\mathbf{v_C}\|_2^2 > \|\mathbf{v_A}\|_2^2 \tag{3.91}$$

We substitute the approximations:

$$\begin{aligned}
&\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(8+n(n-2))(1-\alpha)^2+4n(1-\alpha)^3+n^2(1-\alpha)^4\right]\\
&> \frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4\right]
\end{aligned}$$

Taking limits to infinity with respect to n of both sides, eliminating α^2 and rearranging:

$$\begin{aligned}
&\lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2+(n-2)(1-\alpha)^2+4(1-\alpha)^3}{n}+\frac{8(1-\alpha)^2}{n^2}+(1-\alpha)^4\right]\\
&> \lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2+(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4}{n}+\frac{2(1-\alpha)^2+6(1-\alpha)^3+6(1-\alpha)^4}{n^2}\right]
\end{aligned}$$

Calculating the limits the inequality becomes:

$$(1-\alpha)^4 > 0$$

Which is true because $\alpha \in (0,1)$. ■

Lemma 17. $\lim_{n\to\infty} \|\mathbf{v_A}\|_2^2 = \lim_{n\to\infty} \|\mathbf{v_D}\|_2^2 = \lim_{n\to\infty} \|\mathbf{v_F}\|_2^2$

Proof Lemma 17. Substituting the approximation of the inner product in each side of the inequality:

$$\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+(2+n)(1-\alpha)^2+2(3+n)(1-\alpha)^3+(6+n)(1-\alpha)^4\right]$$
$$=\frac{\alpha^2}{n^2}\left[n(5-6\alpha+2\alpha^2)+n((1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4)\right]$$

Taking the limit when n goes to infinity of each side, rearranging the n, and eliminating the common α^2:

$$\lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2+(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4}{n}+\frac{2(1-\alpha)^2+6(1-\alpha)^3+6(1-\alpha)^4}{n^2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{5-6\alpha+2\alpha^2+(1-\alpha)^2+2(1-\alpha)^3+(1-\alpha)^4}{n}\right]$$

Calculating the limits as n goes to infinity I find that both sides of this equation are equal to zero. ∎

3.8.1.c Results for $n = 4$

In this section I will use the closed form solution in the case of 4-sectors(firms) for the star and vertical economies. Recalling the definition of the influence vector, $\mathbf{v}'$, and the input-output structure implied by such economies, in particular, $w_{ij} = 1$ for all $\overrightarrow{\langle i,j \rangle}$, after inverting ($Mathematica$ code at the end of the Appendix 3.8) the Leontief matrix, the influence vector elements for each economy are the following:

$$Star : \mathbf{v'_S} \equiv [v_{1S}, v_{2S}, v_{3S}, v_{4S}]$$

$$Vertical : \mathbf{v'_V} \equiv [v_{1V}, v_{2V}, v_{3V}, v_{4V}]$$

Where

$$v_{iS} = \begin{cases} \frac{4-3\alpha}{4} & \text{if } i = 1 \\ \\ \frac{\alpha}{4} & \text{if } i \in (2,4) \end{cases}$$

$$v_{iV} = \begin{cases} \frac{(2-\alpha)(2+\alpha(\alpha-2))}{4} & \text{if } i = 1 \\ \\ \frac{\alpha(3+\alpha(\alpha-3))}{4} & \text{if } i = 2 \\ \\ \frac{\alpha(2-\alpha)}{4} & \text{if } i = 3 \\ \\ \frac{\alpha}{4} & \text{if } i = 4 \end{cases}$$

Using these elements I can obtain the following results that I will use in the proofs of the volatilities comparisons.

Lemma 18. $\|\mathbf{v_S}\|_2^2 > \|\mathbf{v_V}\|_2^2$, *for* $n = 4$

Proof Lemma 18. Substituting the elements of each influence vector obtained previously, expanding and simplifying collecting α's I have the following inequality:

$$12\alpha - 35\alpha^2 + 39\alpha^3 - 22\alpha^4 + 7\alpha^5 - \alpha^6 > 0$$

The solution to this inequality is given by the intervals $\alpha \in (0,1)$ and $\alpha \in (1, 2.5260)$. Recalling the assumption about the values that the parameter α can take, $\alpha \in (0,1)$, I can see that the range of the inequality is always positive over the domain specified by α. ∎

Lemma 19. $\sum_{i \neq j}^{n} v_{iV} v_{jV} > \sum_{i \neq j}^{n} v_{iS} v_{jS}$, *for* $n = 4$

Proof Lemma 19. Following the same steps as before, substituting the elements of each influence vector and simplifying:

$$12\alpha - 35\alpha^2 + 39\alpha^3 - 22\alpha^4 + 7\alpha^5 - \alpha^6 > 0$$

Which is the same polynomial inequality that the one in the previous lemma, thus I know that the range of the inequality for the domain of α is positive. ■

Lemma 20. $\sum_{i \neq j}^{n} v_{iV} v_{jV} - \sum_{i \neq j}^{n} v_{iS} v_{jS} = \|\mathbf{v_S}\|_2^2 - \|\mathbf{v_V}\|_2^2$, *for* $n = 4$

Proof Lemma 20. This equation is true because of the two previous inequalities above. ■

Lemma 21. $\sum_{i \neq j}^{n} v_{iV} v_{jV} > 0$, *for* $n = 4$

Proof Lemma 21. Following the procedure, substituting the elements of the influence vector:

$$3\alpha - \frac{41\alpha^2}{8} + \frac{39\alpha^3}{8} - \frac{11\alpha^4}{4} + \frac{7\alpha^5}{8} - \frac{\alpha^6}{8} > 0$$

The solution to this polynomial inequality is given by the interval $\alpha \in (0, 2.2859)$. Since the values of α are given by the assumption of $\alpha \in (0, 1)$, hence the range of the inequality is positive. ■

3.8.2 Ideal Price Index

Definition 58. *The ideal price index is given by:*

$$(p_1 \cdot ... \cdot p_n)^{1/n} = 1$$

I can express the ideal price index in terms of vectors, taking logarithm and defining $\mathbf{p} \equiv [\ln p_1, ..., \ln p_n]$:

$$\frac{1}{n}\mathbf{1}'\mathbf{p} = 0$$

3.8.3 Multi-sectoral economy with idiosyncratic correlated productivity shocks

3.8.3.a Equilibrium

The household maximises the utility over the consumption of n-goods, given by the following constrained problem:

$$\max_{c_i \forall i} \ \frac{1}{n}\sum_{i=1}^{n} \ln c_i$$

s.t.

$$\sum_{i=1}^{n} p_i c_i = r\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i$$

c_i is the consumption of the final good i, p_i is the real price of each good, k_i represents capital rented to each sector i at a price r. π_i is the profit of each firm, which in equilibrium will be equal to zero.

The first order conditions of this maximisation problem are found solving the following Lagrangian problem:

$$\mathcal{L} = \frac{1}{n}\sum_{i=1}^{n} \ln c_i - \lambda\left(\sum_{i=1}^{n} p_i c_i - r\sum_{i=1}^{n} k_i - \sum_{i=1}^{n} \pi_i\right)$$

The F.O.C. are given by:

$$\frac{1}{nc_i} - \lambda p_i = 0 \implies \lambda = \frac{1}{np_ic_i}$$

$$\sum_{i=1}^{n} p_ic_i - r\sum_{i=1}^{n} k_i - \sum_{i=1}^{n} \pi_i = 0$$

The first condition implies:

$$p_ic_i = p_jc_j \ \ \forall i, j$$

Substituting this relationship into the budget constraint, I get the optimal consumption of the household:

$$c_i = \frac{r\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i}{np_i}$$

Each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_iq_i - rk_i - \sum_{j=1}^{n} p_jq_{ij}$$

s.t.

$$q_i = (z_ik_i)^{\alpha} \prod_{j=1}^{n} q_{ij}^{(1-\alpha)w_{ij}}$$

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i$$

$$z_i = exp(\mu_i), \ \ \mu_i \sim N(0, \sigma_i^2, \rho_{ij} \forall j) \ \ \forall i$$

Where $\alpha \in (0,1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents the share of intermediate good j in the total use in sector i and is also a typical element of the input-output matrix a-la-Leontief, where the sum of the shares from each sector j in the production of sector i have to add 1.

The first order conditions of each firm are given by:

$$\frac{\partial \pi_i}{\partial k_i} = \frac{p_i \alpha q_i}{k_i} - r = 0 \implies k_i = \frac{p_i \alpha q_i}{r}$$

$$\frac{\partial \pi_i}{\partial q_{ij}} = \frac{p_i (1-\alpha) w_{ij} q_i}{q_{ij}} - p_j = 0 \implies q_{ij} = \frac{p_i (1-\alpha) w_{ij} q_i}{p_j}$$

The market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations, where k is the exogenous endowment of capital:

$$c_i + \sum_{j=1}^{n} q_{ij} = q_i \ \ \forall i$$

$$\sum_{i=1}^{n} k_i = k$$

To find the quantities in equilibrium I start substituting firm's F.O.C. for q_{ij} into the clearing condition of goods market for j:

$$c_j + \sum_{i=1}^{n} \frac{p_i (1-\alpha) w_{ij} q_i}{p_j} = q_j$$

Multiplying by p_j:

$$p_j c_j + \sum_{i=1}^{n} \frac{p_j p_i (1-\alpha) w_{ij} q_i}{p_j} = p_j q_j$$

Substituting equation of optimal consumption from households through p_j and p_i into the previous equation:

$$\frac{r \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i}{n c_j} c_j + \sum_{i=1}^{n} (1-\alpha) w_{ij} q_i \frac{r \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i}{n c_i} = \frac{r \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \pi_i}{n c_j} q_j$$

Simplifying:

$$1 + \sum_{i=1}^{n} \frac{(1-\alpha) w_{ij} q_i}{c_i} = \frac{q_j}{c_j}$$

Defining:

$$\gamma_j \equiv q_j / c_j$$

and

$$\gamma_i \equiv q_i / c_i$$

Substituting into previous equation:

$$1 + \sum_{i=1}^{n} (1-\alpha) w_{ij} \gamma_i = \gamma_j$$

Stacking over sectors and solving for γ:

$$\gamma = (\mathbb{I} - (1-\alpha)\mathbf{W}')^{-1}\mathbf{1}$$

Substituting the household's F.O.C. into the firm's F.O.C. for q_{ij}:

$$q_{ij} = \frac{c_j(1-\alpha)w_{ij}q_i}{c_i}$$

Substituting γ_i and γ_j into the previous equation:

$$q_{ij} = \frac{\gamma_i(1-\alpha)w_{ij}q_j}{\gamma_j}$$

This is the solution for intermediate goods in equilibrium, it depends on the network given by γ.

Substituting the household's F.O.C. into the firm's F.O.C. for k_i:

$$k_i = \frac{c_j p_j \alpha q_i}{c_i r}$$

Substituting γ_i into the previous equation:

$$k_i = \frac{c_j p_j \alpha \gamma_i}{r}$$

Adding over sectors and substituting the clearing condition of the capital market:

$$k = \frac{c_j p_j \alpha \sum_{i=1}^{n} \gamma_i}{r}$$

Dividing the previous two equations:

$$k_i = \frac{k\gamma_i}{\sum_{i=1}^{n} \gamma_i}$$

This is the solution for the capital in equilibrium, it depends on the network given by γ.

Taking into account that each element of gamma is positive because $\alpha \in (0,1)$ and $w_{ij} \in [0,1]$, I can express the previous equation in the following way:

$$k_i = \frac{k\gamma_i}{\|\gamma\|_1}$$

3.8.3.b GDP

Proof Proposition 15. Substituting the solutions of q_{ij} and k_i into the production function:

$$q_i = \left(z_i \frac{k\gamma_i}{\|\gamma\|_1}\right)^{\alpha} \prod_{j=1}^{n} \left(\frac{\gamma_i(1-\alpha)w_{ij}q_j}{\gamma_j}\right)^{(1-\alpha)w_{ij}}$$

Taking logs of the previous equation:

$$\begin{aligned} \ln q_i = \alpha\left(\ln z_i + \ln k + \ln \gamma_i - \ln \|\gamma\|_1\right) \\ + (1-\alpha) \sum_{j=1}^{n} w_{ij}\left(\ln(1-\alpha) + \ln w_{ij} + \ln \gamma_i - \ln \gamma_j + \ln q_j\right) \end{aligned}$$

I define the following vectors:

$$\mathbf{q} \equiv \begin{bmatrix} \ln q_1 \\ \vdots \\ \ln q_n \end{bmatrix}_{nx1} \quad \mathbf{z} \equiv \begin{bmatrix} \ln z_1 \\ \vdots \\ \ln z_n \end{bmatrix}_{nx1} = \mu \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}_{nx1} \quad \mathbf{w} \equiv \begin{bmatrix} \ln w_{1j} \\ \vdots \\ \ln w_{nj} \end{bmatrix}_{nx1} \quad \overline{\gamma} \equiv \begin{bmatrix} \ln \gamma_1 \\ \vdots \\ \ln \gamma_n \end{bmatrix}_{nx1}$$

Simplifying, stacking over sectors and substituting the result $\ln \|\gamma\|_1 = \ln n - \ln \alpha$:

$$\mathbf{q} = \alpha \left(\mu + \overline{\gamma} - \mathbf{1}(\ln n - \ln \alpha) + \mathbf{1} \ln k \right) + \mathbf{1}(1-\alpha)\ln(1-\alpha) + (1-\alpha)\mathbf{W}\left(\mathbf{w} + \mathbf{q}\right)$$

Solving for $\mathbf{q}$:

$$\mathbf{q} = \left(\mathbb{I} - (1-\alpha)\mathbf{W}\right)^{-1} \left[\alpha \left(\mu + \overline{\gamma} - \mathbf{1}(\ln n - \ln \alpha) + \mathbf{1} \ln k \right) + \mathbf{1}(1-\alpha)\ln(1-\alpha) + (1-\alpha)\mathbf{W}\mathbf{w} \right]$$

Recalling $\gamma_i = q_i / c_i$, taking logs, stacking over sectors, rearranging for $\mathbf{c}$, dividing by n and pre-multiplying by $\mathbf{1}'$:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \frac{1}{n}\mathbf{1}'\mathbf{q} - \frac{1}{n}\mathbf{1}'\overline{\gamma}$$

Where $\mathbf{c}' = [\ln c_1, ..., \ln c_n]$. Substituting the equation that I found for $\mathbf{q}$ into the

previous one:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right](\mu+\overline{\gamma}-\mathbf{1}(\ln n-\ln\alpha)+\mathbf{1}\ln k)$$
$$+\left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right]\left(\mathbf{1}\frac{(1-\alpha)}{\alpha}\ln(1-\alpha)+\frac{(1-\alpha)}{\alpha}\mathbf{W}\mathbf{w}\right)-\frac{1}{n}\mathbf{1}'\overline{\gamma}$$

Substituting the definition of the influence vector:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \mathbf{v}'(\mu+\overline{\gamma}-\mathbf{1}(\ln n-\ln\alpha)+\mathbf{1}\ln k)+\mathbf{v}'\left(\mathbf{1}\frac{(1-\alpha)}{\alpha}\ln(1-\alpha)+\frac{(1-\alpha)}{\alpha}\mathbf{W}\mathbf{w}\right)-\frac{1}{n}\mathbf{1}'\overline{\gamma}$$

Substituting the result $\mathbf{v}'\mathbf{1}=1$ and grouping common terms:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \mathbf{v}'\mu+\ln k+\ln\alpha+\frac{(1-\alpha)}{\alpha}\ln(1-\alpha)+\frac{(1-\alpha)}{\alpha}\mathbf{v}'\mathbf{W}\mathbf{w}-\ln n+\left(\mathbf{v}'-\frac{1}{n}\mathbf{1}'\right)\overline{\gamma}$$

To find the logarithm of the GDP, from the budget constraint of the household I know that:

$$GDP = \sum_{i=1}^{n} p_i c_i$$

Substituting the optimal consumption and taking logarithms, this equation becomes:

$$Y = \ln c_i + \ln p_i + \ln n$$

Stacking over $i's$, diving by n and pre-multiplying by $\mathbf{1}'$:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \frac{1}{n}\mathbf{1}'\mathbf{p} + \ln n$$

Substituting the definition of the ideal price index, $\frac{1}{n}\mathbf{1}'\mathbf{p} = 0$, the previous equation implies:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \ln n$$

Substituting this previous result, I can express the equation for the GDP in the following way:

$$Y = \mathbf{v}'\mu + \Lambda$$

Where μ is logarithm of idiosyncratic correlated shocks z_i and Λ is a variable of parameters given by:

$$\Lambda \equiv \ln k + \ln \alpha + \frac{(1-\alpha)}{\alpha}\left(\ln(1-\alpha) + \mathbf{v}'\mathbf{W}\mathbf{w}\right) + \left(\mathbf{v}' - \frac{1}{n}\mathbf{1}'\right)\overline{\gamma}$$

■

3.8.3.c Volatility of GDP under same variance and correlation parameters

Proof Corollary 3. From the expression of the variance of the GDP, I know that in terms of the influence vector, $\mathbf{v}$, and covariance matrix, $\mathbf{\Sigma}$, I have:

$$[var(Y)]^{1/2} = [\mathbf{v}'\mathbf{\Sigma}\mathbf{v}]^{1/2}$$

Using the assumptions that $\sigma_i^2 = \sigma^2$ and $\rho_{ij} = \rho$ for all i, j, I can express the covariance matrix in the following way:

$$\mathbf{\Sigma} \equiv \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \dots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \dots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \dots & \sigma_n^2 \end{bmatrix}_{nxn} = \sigma^2 \begin{bmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \dots & 1 \end{bmatrix}_{nxn} = \sigma^2\rho \begin{bmatrix} \frac{1}{\rho} & 1 & \dots & 1 \\ 1 & \frac{1}{\rho} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \frac{1}{\rho} \end{bmatrix}$$

I can decompose further the last matrix using the fact that $1 + (1-\rho)/\rho = 1/\rho$:

$$\mathbf{\Sigma} = \sigma^2\rho \left(\begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} + \frac{1-\rho}{\rho} \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} \right)_{nxn}$$

Where the second matrix is the identity, $\mathbb{I}_{nxn}$, and the first matrix is compose of ones and can be expressed as the outer product of a vector of ones, $\mathbf{1}_{nxn}$:

$$\mathbf{\Sigma} = \sigma^2 \left(\rho \mathbf{1}\mathbf{1}' + (1-\rho)\mathbb{I}\right)$$

Now, I substitute this special case of the covariance matrix into the variance of Y above:

$$[var(Y)]^{1/2} = \left[\mathbf{v}'\sigma^2 \left(\rho\mathbf{1}\mathbf{1}' + (1-\rho)\mathbb{I}\right)\mathbf{v}\right]^{1/2}$$

Multiplying the elements of the sum by the outside vectors:

$$[var(Y)]^{1/2} = \left[\sigma^2 \left(\rho\mathbf{v}'\mathbf{1}\mathbf{1}'\mathbf{v} + (1-\rho)\mathbf{v}'\mathbf{v}\right)\right]^{1/2}$$

Recalling the result about the influence vector, $\mathbf{v}'\mathbf{1} = 1$, and hence, $\mathbf{1}'\mathbf{v} = 1$, the previous equation becomes:

$$[var(Y)]^{1/2} = [\sigma^2 (\rho + (1-\rho)\mathbf{v}'\mathbf{v})]^{1/2}$$

And finally, the inner product of the influence vector can be expressed as the squared of its euclidean norm:

$$[var(Y)]^{1/2} = \sqrt{\sigma^2 (\rho + (1-\rho)\|\mathbf{v}\|_2^2)}$$

■

3.8.3.d Volatility of GDP as a function of out-degrees

Proof Proposition 17. We need to express the influence vector as a function of the out-degrees. In order to do this, I recall the approximation:

$$v_i \approx \frac{\alpha}{n}\left(1 + (1-\alpha)d_i + (1-\alpha)^2 \sum_{j=1}^{n} d_j w_{ji}\right)$$

Substituting this result into the variance of GDP, I have the following expression as a function of the out-degrees:

$$var(Y) =$$
$$\frac{\alpha^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j\left(1 + (1-\alpha)d_i + (1-\alpha)^2\sum_{j=1}^{n} d_j w_{ji}\right)\left(1 + (1-\alpha)d_j + (1-\alpha)^2\sum_{i=1}^{n} d_i w_{ij}\right)$$

Expanding the product of the influence vector v_i and v_j, the previous equation becomes:

$$var(Y) = \frac{\alpha^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j + \frac{\alpha^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j\left((1-\alpha)(d_i+d_j)+(1-\alpha)^2 d_i d_j\right)$$
$$+\frac{\alpha^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j\left((1-\alpha)^2\left(\sum_{j=1}^{n}d_j w_{ji}+\sum_{i=1}^{n}d_i w_{ij}\right)\right)$$
$$+\frac{\alpha^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j\left((1-\alpha)^3\left(d_i\sum_{j=1}^{n}d_j w_{ji}+d_j\sum_{i=1}^{n}d_i w_{ij}\right)+(1-\alpha)^4\left(\sum_{j=1}^{n}d_j w_{ji}\sum_{i=1}^{n}d_i w_{ij}\right)\right)$$

I can simplify the previous expression in the following:

$$var(Y)^{1/2} = \frac{\alpha}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j\left(1+\Delta_{ij}+\Xi_{ij}+\Psi_{ij}\right)}$$

Where:

$$\Delta_{ij} \equiv (1-\alpha)(d_i+d_j)+(1-\alpha)^2 d_i d_j$$
$$\Xi_{ij} \equiv (1-\alpha)^2\left(\sum_{j=1}^{n}d_j w_{ji}+\sum_{i=1}^{n}d_i w_{ij}\right)$$
$$\Psi_{ij} \equiv (1-\alpha)^3\left(d_i\sum_{j=1}^{n}d_j w_{ji}+d_j\sum_{i=1}^{n}d_i w_{ij}\right)+(1-\alpha)^4\left(\sum_{j=1}^{n}d_j w_{ji}\sum_{i=1}^{n}d_i w_{ij}\right)$$

■

3.8.3.e Asymptotic volatility of GDP

Proof Proposition 18. To find the lower bound of the volatility of GDP as n goes to infinity, I can use the expression of the volatility in terms of the degrees and input shares that I found previously. But first I need to find the tighter bounds, given the approximation of the influence vector as a function of outdegrees and the input-output matrix, I know the following:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_i v_j \geq \Theta\left(\frac{1}{n}\right) + \Theta\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i + \sum_{i=1}^{n} d_i w_{ij}\right)\left(d_j + \sum_{j=1}^{n} d_j w_{ji}\right)\right)$$

Where Θ represents the asymptotic notation when the lower and the upper bounds are meet at the same time, $f_n = \Omega(g_n)$ if $\liminf_{n\to\infty} f_n/g_n > 0$ and $f_n = O(g_n)$ if $\limsup_{n\to\infty} f_n/g_n < \infty$. The second term of the previous inequality is greater or equal than the first term, because of the following:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i + \sum_{i=1}^{n} d_i w_{ij}\right)\left(d_j + \sum_{j=1}^{n} d_j w_{ji}\right) \geq n$$

Recalling that $\sum_{i=1}^{n} d_i = n$, substituting it into the right side and expanding the left side:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_i d_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i \sum_{j=1}^{n} d_j w_{ji} + d_j \sum_{i=1}^{n} d_i w_{ij} + \sum_{j=1}^{n} d_j w_{ji} \sum_{i=1}^{n} d_i w_{ij}\right) \geq \sum_{i=1}^{n} d_i$$

I can express the first term of the left hand side as the sum of two terms using the fact that when $i = j$ is the sum of the squares of the degrees:

$$\sum_{i=1}^{n} d_i^2 + \sum_{j\neq i} d_i d_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i \sum_{j=1}^{n} d_j w_{ji} + d_j \sum_{i=1}^{n} d_i w_{ij} + \sum_{j=1}^{n} d_j w_{ji} \sum_{i=1}^{n} d_i w_{ij}\right) \geq \sum_{i=1}^{n} d_i$$

We know that the second and third terms of the left hand side of the inequality are equal or greater than zero because the degrees and the inputs shares take values over this range. Hence, in order to be true such inequality I need to prove that the first term of the left side is greater or equal than the right hand side:

$$\sum_{i=1}^{n} d_i^2 \geq \sum_{i=1}^{n} d_i$$

Expressing this inequality using the euclidean and the L_1 norms:

$$\|\mathbf{d}\|_2^2 \geq \|\mathbf{d}\|_1$$

Taking squared root of both sides and dividing by $\sqrt{n}$:

$$\frac{\|\mathbf{d}\|_2}{\sqrt{n}} \geq 1 = \frac{\|\mathbf{d}\|_1}{n}$$

Which is true because it has the form of the QM-AM (Quadratic Mean - Arithmetic Mean) for n real numbers. Having establish this relationship, I can express the volatility of GDP in terms of the following lower bound:

$$[var(Y)]^{1/2} = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i + \sum_{i=1}^{n} d_i w_{ij}\right)\left(d_j + \sum_{j=1}^{n} d_j w_{ji}\right)}\right)$$

$\blacksquare$

Proof Corollary 4. Following the same reasoning as before, this lower bound is greater than that one implied by the law of large numbers:

$$\Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_i+\sum_{i=1}^{n}d_iw_{ij}\right)\left(d_j+\sum_{j=1}^{n}d_jw_{ji}\right)}\right)\geq\Omega\left(\frac{1}{\sqrt{n}}\right)$$

This because after applying the same reasoning as in the previous proof, I only need to compare:

$$\frac{1}{n}\sqrt{\sum_{i=1}^{n}d_i^2}\geq\frac{1}{\sqrt{n}}$$

Knowing that $\|\mathbf{d}\|_1 = n$, taking squares and multiplying by n:

$$\frac{\|\mathbf{d}\|_2}{\sqrt{n}}\geq 1=\frac{\|\mathbf{d}\|_1}{n}$$

Which again is true because it has the form of the QM-AM for n real numbers. ■

3.8.3.f Variance decomposition of GDP

Proof Proposition 19. To express the volatility of GDP in terms of the Cholesky matrices, I need to apply the Cholesky algorithm to the covariance matrix, $\mathbf{\Sigma}$. In particular, I use the algorithm implementation in Mathematica, which according to the Cholesky–Banachiewicz or Cholesky–Crout algorithms, the decomposition of the covariance matrix will yield a triangular matrix, $\mathbf{C}$, such that:

$$\mathbf{\Sigma}=\mathbf{C}\mathbf{C}'$$

Where the matrix $\mathbf{C}$, in the $4x4$ case, is composed of:

$$\mathbf{C} = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

Where:

$$C_{ij} = \begin{cases} \sqrt{\Sigma_{j,j} - \sum_{k=1}^{j-1} C_{j,k}^2} & \text{if } i = j \\ \frac{1}{C_{j,j}} \left(\Sigma_{i,j} - \sum_{k=1}^{j-1} C_{i,k} C_{j,k} \right) & \text{if } i > j \end{cases}$$

After implementing this algorithm, I found the Cholesky matrix:

$$\mathbf{C} \equiv \left(\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ \sigma_2 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n & \sigma_n & \dots & \sigma_n \end{bmatrix} \circ \begin{bmatrix} f(\rho)_{11} & 0 & \dots & 0 \\ f(\rho)_{21} & f(\rho)_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f(\rho)_{n1} & f(\rho)_{n2} & \dots & f(\rho)_{nn} \end{bmatrix} \right)_{nxn}$$

Where in the $4x4$ case is given by:

$$f(\rho)_{11} = 1$$

$$f(\rho)_{21} = \rho_{12}$$

$$f(\rho)_{31} = \rho_{13}$$

$$f(\rho)_{41} = \rho_{14}$$

$$f(\rho)_{22} = \sqrt{1-\rho_{12}^2}$$

$$f(\rho)_{32} = \frac{(\rho_{23}-\rho_{12}\rho_{13})}{\sqrt{1-\rho_{12}^2}}$$

$$f(\rho)_{42} = \frac{(\rho_{24}-\rho_{12}\rho_{14})}{\sqrt{1-\rho_{12}^2}}$$

$$f(\rho)_{33} = \sqrt{\frac{\rho_{12}^2-2\rho_{13}\rho_{23}\rho_{12}+\rho_{13}^2+\rho_{23}^2-1}{\rho_{12}^2-1}}$$

$$f(\rho)_{43} = \frac{((\rho_{24}-\rho_{12}\rho_{14})\,\rho_{23}+\rho_{13}\,(\rho_{14}-\rho_{12}\rho_{24})+(\rho_{12}^2-1)\,\rho_{34})}{(\rho_{12}^2-1)\sqrt{\frac{\rho_{12}^2-2\rho_{13}\rho_{23}\rho_{12}+\rho_{13}^2+\rho_{23}^2-1}{\rho_{12}^2-1}}}$$

$$f(\rho)_{44}^2 = (\left(1-\rho_{34}^2\right)\rho_{12}^2+2\left(\rho_{13}\left(\rho_{24}\rho_{34}-\rho_{23}\right)+\rho_{14}\left(\rho_{23}\rho_{34}-\rho_{24}\right)\right)\rho_{12}+\rho_{23}^2+\rho_{24}^2+\rho_{34}^2-\rho_{14}^2\left(\rho_{23}^2-1\right)$$
$$-\rho_{13}^2\left(\rho_{24}^2-1\right)+2\rho_{13}\rho_{14}\left(\rho_{23}\rho_{24}-\rho_{34}\right)-2\rho_{23}\rho_{24}\rho_{34}-1)\left(\frac{1}{\rho_{12}^2-2\rho_{13}\rho_{23}\rho_{12}+\rho_{13}^2+\rho_{23}^2-1}\right)$$

■

3.8.3.g Comparison of volatilities, specific cases

In order to prove the propositions of this subsection I will use the influence vector results detailed in the first section of Appendix 3.8.

Proof Proposition 20. According to the assumptions, I need to verify that the following inequality holds for the parameters $\alpha \in (0,1)$, $3 < n << \infty$, $\sigma_i^2 = \sigma^2 > 0$ and $\rho_{ij} = \rho \in [-1,1)$, that are the same between and within economies:

$$\sqrt{var(Y)}_B > \sqrt{var(Y)}_C > \sqrt{var(Y)}_A > \sqrt{var(Y)}_D = \sqrt{var(Y)}_F$$

From previous result I found that:

$$[var(Y)]^{1/2} = \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v}\|_2^2\right)}$$

Substituting into the inequality:

$$\sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v_B}\|_2^2\right)} > \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v_C}\|_2^2\right)} > \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v_A}\|_2^2\right)}$$
$$> \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v_D}\|_2^2\right)} = \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v_F}\|_2^2\right)}$$

Taking squares, diving by σ^2 and eliminating the common ρ:

$$(1-\rho)\|\mathbf{v_B}\|_2^2 > (1-\rho)\|\mathbf{v_C}\|_2^2 > (1-\rho)\|\mathbf{v_A}\|_2^2 > (1-\rho)\|\mathbf{v_D}\|_2^2 = (1-\rho)\|\mathbf{v_F}\|_2^2$$

I can divide each term by $(1-\rho)$ since is positive because of the assumption $\rho \in [-1, 1)$, then previous inequality becomes:

$$\|\mathbf{v_B}\|_2^2 > \|\mathbf{v_C}\|_2^2 > \|\mathbf{v_A}\|_2^2 > \|\mathbf{v_D}\|_2^2 = \|\mathbf{v_F}\|_2^2$$

I know that these relationships are true because of the results that I found in the first section. ■

Proof Proposition 21. In this case I need to prove the following inequality for the assumptions $\alpha \in (0, 1)$, $3 < n << \infty$, $\sigma_i^2 = \sigma^2 > 0$ and $\rho_{ij} = \rho = 1$:

$$\sqrt{var(Y)}_B = \sqrt{var(Y)}_C = \sqrt{var(Y)}_A = \sqrt{var(Y)}_D = \sqrt{var(Y)}_F$$

Substituting the expression for the volatility under the assumptions that I found previously:

$$\sqrt{\sigma^2\left(\rho+(1-\rho)\|\mathbf{v_B}\|_2^2\right)}=\sqrt{\sigma^2\left(\rho+(1-\rho)\|\mathbf{v_C}\|_2^2\right)}=\sqrt{\sigma^2\left(\rho+(1-\rho)\|\mathbf{v_A}\|_2^2\right)}$$
$$=\sqrt{\sigma^2\left(\rho+(1-\rho)\|\mathbf{v_D}\|_2^2\right)}=\sqrt{\sigma^2\left(\rho+(1-\rho)\|\mathbf{v_F}\|_2^2\right)}$$

Taking squares of both sides, diving by σ^2 and subtracting ρ:

$$(1-\rho)\|\mathbf{v_B}\|_2^2=(1-\rho)\|\mathbf{v_C}\|_2^2=(1-\rho)\|\mathbf{v_A}\|_2^2=(1-\rho)\|\mathbf{v_D}\|_2^2=(1-\rho)\|\mathbf{v_F}\|_2^2$$

This inequality is true because of the assumption $\rho=1$. ■

Proof Proposition 22. In this case I have the assumptions $\alpha\in(0,1)$, $\sigma_i^2=\sigma^2>0$ and $\rho_{ij}=\rho\in[-1,1)$, so I need to prove the following inequality:

$$\lim_{n\to\infty}\sqrt{var(Y)}_B>\lim_{n\to\infty}\sqrt{var(Y)}_C>\lim_{n\to\infty}\sqrt{var(Y)}_A=\lim_{n\to\infty}\sqrt{var(Y)}_D=\lim_{n\to\infty}\sqrt{var(Y)}_F$$

As I did in the previous proofs, substituting the equation of the volatility of GDP under the assumptions given, after simplifying I have the following relationship:

$$\lim_{n\to\infty}(1-\rho)\|\mathbf{v_B}\|_2^2>\lim_{n\to\infty}(1-\rho)\|\mathbf{v_C}\|_2^2>\lim_{n\to\infty}(1-\rho)\|\mathbf{v_A}\|_2^2=\lim_{n\to\infty}(1-\rho)\|\mathbf{v_D}\|_2^2=\lim_{n\to\infty}(1-\rho)\|\mathbf{v_F}\|_2^2$$

I can divide everything by $(1-\rho)$ because the assumption $\rho\in[-1,1)$ implies this term is positive, then previous inequality becomes:

$$\lim_{n\to\infty} \|\mathbf{v_B}\|_2^2 > \lim_{n\to\infty} \|\mathbf{v_C}\|_2^2 > \lim_{n\to\infty} \|\mathbf{v_A}\|_2^2 = \lim_{n\to\infty} \|\mathbf{v_D}\|_2^2 = \lim_{n\to\infty} \|\mathbf{v_F}\|_2^2$$

And, as I found in the previous section, I know that all these relationships are true. ■

Proof Proposition 23. Let $\sigma_i^2 = \sigma^2$ and $\rho_{ij} = \rho \neq 0$ between and within economies for all i, j. I need to prove for the economies A, B, C, D and F:

$$\lim_{n\to\infty} \sqrt{var(Y)} = \lim_{n\to\infty} \sqrt{\sigma^2 \left(\rho + (1-\rho)\|\mathbf{v}\|_2^2\right)} > 0$$

From the previous proposition I know that the economies with the minimum volatility are A, D and F, and in those cases the previous equation becomes:

$$\sqrt{\sigma^2 \rho} > 0$$

This follows from the result, in the first section, $\lim_{n\to\infty} \|\mathbf{v_{A,D,F}}\|_2^2 = 0$. ■

Proof Proposition 24. Considering the economies A(Vertical) and C(Tree), let ($\sigma_1^2 = \sigma^2$), and ($\sigma_i^2 = 0$) for all $i \neq 1$, for both economies and for all $n > 3$. Recalling the definition of volatility and the assumptions about the variance of the idiosyncratic shocks, the inequality of volatilities could be simplified to:

$$\sum_{i=1}^{n} \sigma^2 v_{1A}^2 < \sum_{i=1}^{n} \sigma^2 v_{1C}^2$$

Dividing by n and by σ^2:

$$v_{1A}^2 < v_{1C}^2$$

Taking the square root of each side and recalling that each element of the influence vector is always positive:

$$v_{1A} < v_{1C}$$

Substituting the expression of the influence vector for the first elements as a function of the degrees using the second order approximation described previously, the inequality becomes:

$$\frac{\alpha}{n}\left(1 + (1-\alpha)d_{1A} + (1-\alpha)^2 \sum_{j=1}^{n} d_{jA} w_{j1A}\right) < \frac{\alpha}{n}\left(1 + (1-\alpha)d_{1C} + (1-\alpha)^2 \sum_{j=1}^{n} d_{jC} w_{j1C}\right)$$

Simplifying the previous inequality and rearranging:

$$0 < (1-\alpha)(d_{1C} - d_{1A}) + (1-\alpha)^2 \left(\sum_{j=1}^{n} d_{jC} w_{j1C} - \sum_{j=1}^{n} d_{jA} w_{j1A}\right)$$

The input-output structure of each of these economies is given by the following matrices, when n is unknown:

$$\mathbf{W_A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{nxn} \qquad \mathbf{W_C} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}_{nxn}$$

Hence the degree sequence for each economy is the following:

$$\mathbf{d'_A} = [2, 1, ..., 1, 0]_{1xn}$$

$$\mathbf{d'_C} = [2, n-2, 0, ..., 0]_{1xn}$$

Having the vector of degrees and the input-output matrices in these cases, I can simplify the inequality using the fact that the degree of 1 is the same in the two economies and that the other terms are given by the inner product of the vector of degrees ($\mathbf{d'}$) and the first column of the input-output matrix ($col_1(\mathbf{W})$):

$$0 < (1-\alpha)^2 \left(\mathbf{d'_C} col_1(\mathbf{W_C}) - \mathbf{d'_A} col_1(\mathbf{W_A})\right)$$

After simplifying the inner products given the matrices described above:

$$0 < (1-\alpha)^2 (n-3)$$

Using the fact that $1-\alpha$ is positive, the inequality becomes:

$$3 < n$$

Which is the assumption under I consider the inequality to be true. ■

***Proof Proposition 25**.* Considering the vertical economies A_1 and A_2, let $(\sigma_i^2 = \sigma)$ for all firms and $(\rho_{ij} = \rho)$ for all firms within the same economy:

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) > (\rho \in \mathbf{\Sigma}_{A_2})$$

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) < (\rho \in \mathbf{\Sigma}_{A_2})$$

$$[var(Y)]_{A_1}^{1/2} = [var(Y)]_{A_2}^{1/2} \;\; if \; (\rho \in \mathbf{\Sigma}_{A_1}) = (\rho \in \mathbf{\Sigma}_{A_2})$$

Recalling the definition of volatility and the assumption that both A_1 and A_2 have the same vertical structure, start assuming that the difference of volatilities is equal to zero. After taking squares and calling ρ_{A_1} and ρ_{A_2} the correlation of economy 1 and 2, respectively:

$$\sigma^2 \sum_{i=1}^{n} v_{iV}^2 + \sigma^2 \rho_{A_1} \sum_{i \neq j} v_{iV} v_{jV} = \sigma^2 \sum_{i=1}^{n} v_{iV}^2 + \sigma^2 \rho_{A_2} \sum_{i \neq j} v_{iV} v_{jV}$$

Dividing by σ^2, and simplifying:

$$\sum_{i \neq j} v_{iV} v_{jV} \left(\rho_{A_1} - \rho_{A_2}\right) = 0$$

From previous results I know that the term that is multiplying the difference of correlations is positive. Then, if the correlations are equal, $\rho_{A_1} = \rho_{A_2}$, this equality holds. However, if $\rho_{A_1} > \rho_{A_2}$ the previous equation becomes:

$$\sum_{i \neq j} v_{iV} v_{jV} \left(\rho_{A_1} - \rho_{A_2}\right) > 0$$

If the difference of correlations is $\rho_{A_1} < \rho_{A_2}$, the inequality is:

$$\sum_{i \neq j} v_{iV} v_{jV} \left(\rho_{A_1} - \rho_{A_2}\right) < 0$$

Which in both cases the inequalities comply with the assumptions. This completes the proof. ■

Proof Proposition 26*.* Consider the vertical economies A_1 and A_2, let ($\sigma_i^2 = \sigma$) for all firms, and ($\rho_{ij} = \rho$) for all connected sectors, $\neg\overrightarrow{\langle i, j \rangle}$, in the economy A_1 and the unconnected sectors, $\overrightarrow{\langle i, j \rangle}$, in the economy A_2, otherwise ($\rho_{ij} = 0$):

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \;\; if \; \rho \in (0, 1]$$

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \;\; if \; \rho \in [-1, 0)$$

Substituting the equation of volatility, the assumption that both A_1 and A_2 have the same vertical structure, and the assumption about the parameters, the first inequality becomes:

$$\sigma^2\sum_{i=1}^{n}v_{iV}^2+\sigma^2\rho(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V})<\sigma^2\sum_{i=1}^{n}v_{iV}^2+\sigma^2\rho(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V})$$

Simplifying the inequality eliminating the first terms of both sides and dividing by σ^2, after re-arranging:

$$0<\rho(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V}-(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V}))$$

This inequality depends on the value of ρ and the difference of the elements of the influence vector. First, I analyse if the difference of the terms that are multiplying ρ is always positive, I substitute each element and after simplifying:

$$v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V}-(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V})=3\alpha^2-9\alpha^3+10\alpha^4-5\alpha^5+\alpha^6>0$$

The solution to this polynomial inequality is given by the intervals $\alpha\in(-\infty,0)$, $\alpha\in(0,1)$ and $\alpha\in(1,+\infty)$. The second interval is the assumption about the parameter α, hence, the range of this polynomial inequality in this case is always positive. Thus, the original inequality is true if and only if the correlation parameter is positive, which in the first case is the assumption given $\rho\in(0,1]$. Following the same steps, the second inequality becomes:

$$0>\rho(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V}-(v_{1V}v_{2V}+v_{2V}v_{3V}+v_{3V}v_{4V}))$$

Which I know is true under the assumption that the correlation parameter takes negatives values $\rho \in [-1, 0)$, because the term that depends on the difference of the elements of the influence vector is always positive. ■

Proof Proposition 27. Consider the vertical economies A_1 and A_2, let $(\sigma_i^2 = \sigma)$ for all firms, and $[\rho_{ij} = \rho_{A_2}, \forall (i \neq j) \in \mathbf{\Sigma}_{A_2}]$:

$$[var(Y)]_{A_1}^{1/2} < [var(Y)]_{A_2}^{1/2} \ if \ [\rho_{A_2} = (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}) > \rho_{A_1} = (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}), \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

$$[var(Y)]_{A_1}^{1/2} > [var(Y)]_{A_2}^{1/2} \ if \ [\rho_{A_2} = (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}) < \rho_{A_1} = (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}), \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

$$[var(Y)]_{A_1}^{1/2} = [var(Y)]_{A_2}^{1/2} \ if \ [\rho_{A_2} = (\rho_{ij} \forall \overrightarrow{\langle i, j \rangle}) = (\rho_{ij} \forall \neg \overrightarrow{\langle i, j \rangle}) = \rho_{A_1}, \forall (i \neq j) \in \mathbf{\Sigma}_{A_1}]$$

Substituting the equation of volatility, the assumption that both A_1 and A_2 have the same vertical structure, and the assumption about the parameters, I start proving the difference of volatilities equal to zero:

$$\sigma^2 \sum_{i=1}^{n} v_{iV}^2 + \sigma^2 \rho_{A_2} \sum_{i \neq j} v_{iV} v_{jV} = \sigma^2 \sum_{i=1}^{n} v_{iV}^2 + \sigma^2 \rho_{A_2} \sum_{i \neq j} v_{iV} v_{jV}$$

This relationship is true if all the correlations are the same as is the assumption.

Now, to prove the first inequality where the volatility of A_2 is greater than the one of A_1, I take into account the assumption about the correlations for this case, and dividing by σ^2, after simplifying:

$$2\rho_{A_2}(v_{1V}v_{2V} + v_{2V}v_{3V} + v_{3V}v_{4V}) + 2\rho_{A_1}(v_{1V}v_{3V} + v_{1V}v_{4V} + v_{2V}v_{4V}) < \rho_{A_2} \sum_{i \neq j} v_{iV} v_{jV}$$

The sum in the right hand side contains the same products of the elements of the influence vector as in the left had side, but with the difference that are multiplied all by ρ_{A_2}, then I can simplify the inequality in the following way:

$$0 < (\rho_{A_2} - \rho_{A_1})(v_{1V}v_{3V} + v_{1V}v_{4V} + v_{2V}v_{4V})$$

For this inequality, by assumption, I know that the difference of correlations is positive, then to prove the inequality I need to prove that the other term is positive. After substituting the elements of the influence vector and simplifying, I obtain:

$$v_{1V}v_{3V} + v_{1V}v_{4V} + v_{2V}v_{4V} = \frac{3\alpha}{2} - \frac{19\alpha^2}{8} + \frac{15\alpha^3}{8} - \frac{3\alpha^4}{4} + \frac{\alpha^5}{8} > 0$$

This polynomial is greater than zero for the domain given by α because the solution to this inequality is given by $\alpha \in (0, +\infty)$. This completes the proof for the first difference of volatilities.

To prove the second inequality of the difference of volatilities, I follow the same steps as above, taking into account the assumptions:

$$2\rho_{A_1}(v_{1V}v_{2V} + v_{2V}v_{3V} + v_{3V}v_{4V}) + 2\rho_{A_2}(v_{1V}v_{3V} + v_{1V}v_{4V} + v_{2V}v_{4V}) > \rho_{A_2}\sum_{i \neq j} v_{iV}v_{jV}$$

After simplifying:

$$0 < (\rho_{A_1} - \rho_{A_2})(v_{1V}v_{2V} + v_{2V}v_{3V} + v_{3V}v_{4V})$$

By assumption in this case the difference of correlations is positive, then in order to corroborate the inequality I need to know if the second term is positive, after substituting the elements of the influence vector and simplifying:

$$v_{1V}v_{2V} + v_{2V}v_{3V} + v_{3V}v_{4V} = \frac{3\alpha}{2} - \frac{11\alpha^2}{4} + 3\alpha^3 - 2\alpha^4 + \frac{3\alpha^5}{4} - \frac{\alpha^6}{8} > 0$$

Which is true because the solution to this polynomial inequality is $\alpha \in (0, 2)$. This completes the proof. ■

3.8.4 Multi-sector economy with idiosyncratic capital risk sharing

3.8.4.a Equilibrium

The representative household solves the following problem:

$$\max_{c_i \forall i} \; \frac{1}{n}\sum_{i=1}^{n} \ln c_i$$

s.t.

$$\sum_{i=1}^{n} p_i c_i = r \sum_{i \in M_s} k_i + \sum_{i \in M_s} \pi_i$$

The first order conditions of this maximisation problem are found solving:

$$\mathcal{L} = \frac{1}{n}\sum_{i=1}^{n} \ln c_i - \lambda \left(\sum_{i=1}^{n} p_i c_i - r \sum_{i \in M_s} k_i - \sum_{i \in M_s} \pi_i \right)$$

The F.O.C. are given by:

$$p_i c_i = p_j c_j \ \ \forall i, j$$

Optimal consumption of the household:

$$p_i c_i = \frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{n}$$

Then each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_i q_i - r k_i - \sum_{j=1}^{n} p_j q_{ij}$$

s.t.

$$q_i = (k_i)^\alpha \prod_{j=1}^{n} q_{ij}^{(1-\alpha) w_{ij}}$$

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i$$

Where $\alpha \in (0, 1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents the share of intermediate good j in the total use in sector i. The first order conditions of each firm are given by:

$$\frac{\partial \pi_i}{\partial k_i} = \frac{p_i \alpha q_i}{k_i} - r = 0 \implies k_i = \frac{p_i \alpha q_i}{r}$$

$$\frac{\partial \pi_i}{\partial q_{ij}} = \frac{p_i (1-\alpha) w_{ij} q_i}{q_{ij}} - p_j = 0 \implies q_{ij} = \frac{p_i (1-\alpha) w_{ij} q_i}{p_j}$$

The market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations:

$$c_i + \sum_{j=1}^{n} q_{ij} = q_i \ \ \forall i$$

$$\sum_{i \in M_s} k_i = z_s \overline{K}_s \ \ \forall s$$

Where

$$z_s = exp(\mu_s), \ \ \mu_s \sim N(0, \sigma_s^2) \ \ i.i.d. \ \forall s$$

In the same way that section (2), I start substituting firm's F.O.C. for q_{ij} into the clearing condition of goods market for j:

$$c_j + \sum_{i=1}^{n} \frac{p_i(1-\alpha)w_{ij}q_i}{p_j} = q_j$$

Multiplying by p_j:

$$p_j c_j + \sum_{i=1}^{n} \frac{p_j p_i(1-\alpha)w_{ij}q_i}{p_j} = p_j q_j$$

Substituting equation of optimal consumption from households through p_j and p_i into the previous equation:

$$\frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{nc_j} c_j + \sum_{i=1}^{n} (1-\alpha) w_{ij} q_i \frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{nc_i} = \frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{nc_j} q_j$$

Simplifying:

$$1 + \sum_{i=1}^{n} \frac{(1-\alpha) w_{ij} q_i}{c_i} = \frac{q_j}{c_j}$$

Defining:

$$\gamma_j \equiv q_j / c_j$$

and

$$\gamma_i \equiv q_i / c_i$$

Substituting into previous equation:

$$1 + \sum_{i=1}^{n} (1-\alpha) w_{ij} \gamma_i = \gamma_j$$

Stacking over sectors and solving for γ:

$$\gamma = (\mathbb{I} - (1-\alpha)\mathbf{W}')^{-1}\mathbf{1}$$

Substituting the household's F.O.C. into the firm's F.O.C. for q_{ij}:

$$q_{ij} = \frac{c_j (1-\alpha) w_{ij} q_i}{c_i}$$

Substituting γ_i and γ_j into the previous equation:

$$q_{ij} = \frac{\gamma_i (1-\alpha) w_{ij} q_j}{\gamma_j}$$

This is the solution for intermediate goods in equilibrium, it depends on the network given by γ.

Substituting the household's F.O.C. into the firm's F.O.C. for k_i:

$$k_i = \frac{c_j p_j \alpha q_i}{c_i r}$$

Substituting γ_i into the previous equation:

$$k_i = \frac{c_j p_j \alpha \gamma_i}{r}$$

Adding over the subset of sectors M_s and substituting the clearing condition of the capital market:

$$z_s \overline{K}_s = \frac{c_j p_j \alpha \sum\limits_{i \in M_s} \gamma_i}{r}$$

Dividing the previous two equations:

$$k_i = \frac{z_s \overline{K}_s \gamma_i}{\sum\limits_{i \in M_s} \gamma_i}$$

This is the solution for the capital in equilibrium, it depends on the network given by γ.

3.8.4.b GDP

Proof Proposition 28. Substituting the solutions of q_{ij} and k_i into the production function:

$$q_i = \left(\frac{z_s \overline{K}_s \gamma_i}{\sum\limits_{i \in M_s} \gamma_i}\right)^{\alpha} \prod_{j=1}^{n} \left(\frac{\gamma_i (1-\alpha) w_{ij} q_j}{\gamma_j}\right)^{(1-\alpha) w_{ij}}$$

Taking logs of the previous equation:

$$\ln q_i = \alpha \left(\ln z_s + \ln \overline{K}_s + \ln \gamma_i - \ln \sum_{i \in M_s} \gamma_i\right) + (1-\alpha) \sum_{j=1}^{n} w_{ij} \left(\ln(1-\alpha) + \ln w_{ij} + \ln \gamma_i - \ln \gamma_j + \ln q_j\right)$$

I define the following vectors:

$$\mathbf{z} \equiv \begin{bmatrix} \ln z_1 \\ \vdots \\ \ln z_m \end{bmatrix}_{mx1} = \mu \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}_{mx1} \quad \mathbf{k} \equiv \begin{bmatrix} \ln \overline{K}_1 \\ \vdots \\ \ln \overline{K}_m \end{bmatrix}_{mx1} \quad \mathbf{q} \equiv \begin{bmatrix} \ln q_1 \\ \vdots \\ \ln q_n \end{bmatrix}_{nx1}$$

$$\mathbf{w} \equiv \begin{bmatrix} \ln w_{1j} \\ \vdots \\ \ln w_{nj} \end{bmatrix}_{nx1} \quad \overline{\gamma} \equiv \begin{bmatrix} \ln \gamma_1 \\ \vdots \\ \ln \gamma_n \end{bmatrix}_{nx1} \quad \check{\gamma} \equiv \begin{bmatrix} \ln \sum\limits_{i \in M_1} \gamma_i \\ \vdots \\ \ln \sum\limits_{i \in M_m} \gamma_i \end{bmatrix}_{mx1} \quad \mathbf{I_M} \equiv \begin{bmatrix} 1_{M_1}(1) & \dots & 1_{M_m}(1) \\ \vdots & \ddots & \vdots \\ 1_{M_1}(n) & \dots & 1_{M_m}(n) \end{bmatrix}_{nxm}$$

Where the i-th element of the matrix $\mathbf{I_M}$ is an indicator function of the membership of a firm in the subset M_s of N, taking the value 1 for all i in M_s and the value 0 for all elements of N not in M_s:

$$1_{M_s}(i) \equiv \begin{cases} 1 & \text{if } i \in M_s \\ \\ 0 & \text{if } i \notin M_s \end{cases}$$

where according to the definition of the subset M_s, $\sum_{s=1}^{m} 1_s(i) = 1$, which means that each i belongs to only one subset.

Stacking over sectors:

$$\mathbf{q} = \alpha \left(\mathbf{I_M}(\mu + \mathbf{k} - \check{\gamma}) + \overline{\gamma} \right) + (1-\alpha)\mathbf{W} \left(\mathbf{1} \ln(1-\alpha) + \mathbf{w} + \mathbf{q} \right)$$

Solving for $\mathbf{q}$:

$$\mathbf{q} = \left(\mathbb{I} - (1-\alpha)\mathbf{W} \right)^{-1} \left[\alpha \left(\mathbf{I_M}(\mu + \mathbf{k} - \check{\gamma}) + \overline{\gamma} \right) + (1-\alpha)\mathbf{W} \left(\mathbf{1} \ln(1-\alpha) + \mathbf{w} \right) \right]$$

Recalling $\gamma_i = q_i/c_i$, taking logs, stacking over sectors, rearranging for $\mathbf{c}$, dividing by n and pre-multiplying by $\mathbf{1}'$:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \frac{1}{n}\mathbf{1}'\mathbf{q} - \frac{1}{n}\mathbf{1}'\overline{\gamma}$$

Where $\mathbf{c}' = [\ln c_1, ..., \ln c_n]$. Substituting the equation that I found for $\mathbf{q}$ into the

previous one:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right](\mathbf{I_M}(\mu+\mathbf{k}-\check{\gamma})+\overline{\gamma}) \\ +\left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right]\left(\frac{(1-\alpha)}{\alpha}\mathbf{W}\left(\mathbf{1}\ln(1-\alpha)+\mathbf{w}\right)\right)-\frac{1}{n}\mathbf{1}'\overline{\gamma} \quad (3.92)$$

From the budget constraint of the household I know that:

$$\sum_{i=1}^{n} p_i c_i = GDP$$

Substituting the optimal consumption and taking logarithms this equation becomes:

$$Y = \ln c_i + \ln p_i + \ln n$$

Stacking over $i's$, diving by n and pre-multiplying by $\mathbf{1}'$:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \frac{1}{n}\mathbf{1}'\mathbf{p} + \ln n$$

Substituting the ideal price index $(p_1 \cdot ... \cdot p_n)^{1/n} = 1$, the previous equation implies:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \ln n$$

Substituting this previous results and the vector of influence into the previous equation, I can express the equation for the GDP in the following way:

$$Y = \mathbf{v}'\mathbf{I_M}\mu + \Lambda$$

Where μ is the vector idiosyncratic shocks, z_s, to each capital endowment for each subset of firms, M_s, and Λ is a variable of parameters given by:

$$\Lambda \equiv \mathbf{v}'\left(\mathbf{I_M}(\mathbf{k} - \breve{\gamma}) + \frac{(1-\alpha)}{\alpha}\mathbf{W}\mathbf{w} + \overline{\gamma}\right) + \frac{(1-\alpha)}{\alpha}\ln(1-\alpha) + \ln n - \frac{1}{n}\mathbf{1}'\overline{\gamma}$$

■

3.8.4.c Volatility of GDP

***Proof Proposition 29**.* The volatility of the logarithm of the GDP was defined as:

$$[var(Y)]^{1/2} \equiv [var(\mathbf{v}'\mathbf{I_M}\mu)]^{1/2}$$

I can express the previous equation in sums of the products of the elements in the following way:

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n} v_i^2 \left(var \sum_{s=1}^{m} 1_s(i)\mu_s\right)\right]^{1/2}$$

I can see from the previous equation that in order to know the volatility of the logarithm of the GDP I need to know the variance of the sum, over the subsets of firms, of the logarithm of the idiosyncratic shock times the indicator function. I can use the assumption that the idiosyncratic shock are i.i.d to express the variance in the following way:

$$[var(Y)]^{1/2} = \left[\sum_{i=1}^{n} v_i^2 \left(\sum_{s=1}^{m} (1_s(i))^2 \sigma_s^2\right)\right]^{1/2}$$

■

3.8.4.d Comparison of volatilities, specific cases

Proof Proposition 30. Considering $A^1_{4,2}$ and $A^2_{4,2}$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 = \sigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 > \sigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \sigma_1^2 < \sigma_2^2$$

We start by finding the variance of the logarithm of the GDP for each case in terms of the elements of the influence vector and the idiosyncratic variance of each capital endowment, substituting in the first inequality:

$$[v_1^2\sigma_1^2 + v_2^2\sigma_1^2 + v_3^2\sigma_2^2 + v_4^2\sigma_2^2]^{1/2} = [v_1^2\sigma_1^2 + v_2^2\sigma_2^2 + v_3^2\sigma_1^2 + v_4^2\sigma_2^2]^{1/2}$$

Collecting the σ_s^2:

$$[\sigma_1^2(v_1^2 + v_2^2) + \sigma_2^2(v_3^2 + v_4^2)]^{1/2} = [\sigma_1^2(v_1^2 + v_3^2) + \sigma_2^2(v_2^2 + v_4^2)]^{1/2}$$

Squaring both sides and rearranging:

$$\sigma_1^2(v_1^2 + v_2^2) - \sigma_1^2(v_1^2 + v_3^2) = \sigma_2^2(v_2^2 + v_4^2) - \sigma_2^2(v_3^2 + v_4^2)$$

Simplifying:

$$\sigma_1^2(v_2^2 - v_3^2) = \sigma_2^2(v_2^2 - v_3^2)$$

In the case of the first equation this is true because of the assumption $\sigma_1^2 = \sigma_2^2$. For the second inequality, following the same steps, I have:

$$\sigma_1^2(v_2^2 - v_3^2) > \sigma_2^2(v_2^2 - v_3^2)$$

Rearranging:

$$(v_2^2 - v_3^2)(\sigma_1^2 - \sigma_2^2) > 0$$

We know that the second term is positive in the case of the second inequality because of the assumption $\sigma_1^2 > \sigma_2^2$. The first term is also positive because $v_2^2 > v_3^2$ as I can see substituting the elements of the influence vector:

$$\left(\frac{\alpha(2-\alpha)}{4}\right)^2 > \left(\frac{\alpha}{4}\right)^2$$

Simplifying:

$$1 > \alpha$$

Which I know is true according to this assumption about α. For the last inequality, applying an analogous reasoning I find:

$$(v_2^2 - v_3^2)(\sigma_1^2 - \sigma_2^2) < 0$$

We know that this last inequality is true because the first term is always positive and the second term is negative because of the assumption given in the third inequality, $\sigma_1^2 < \sigma_2^2$. ■

Proof Proposition 31. Considering $A^1_{4,2}$ and $B_{4,2}$, let $\sigma_s^2 = \sigma^2$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{B_{4,2}} \quad if\ \sigma_1^2 = \sigma_2^2$$

Substituting the elements of the influence vector and the idiosyncratic variance of each capital endowment into the previous inequality:

$$[v_{1V}^2\sigma_1^2 + v_{2V}^2\sigma_1^2 + v_{3V}^2\sigma_2^2 + v_{4V}^2\sigma_2^2]^{1/2} < [v_{1S}^2\sigma_1^2 + v_{2S}^2\sigma_2^2 + v_{3S}^2\sigma_1^2 + v_{4S}^2\sigma_2^2]^{1/2}$$

Where the sub-index V indicates an element from the vertical economy and S and elements from the star economy, as described in the previous sections. Taking squares of both sides and simplifying using the assumption that $\sigma_s^2 = \sigma^2$:

$$\|\mathbf{v_V}\|_2^2 < \|\mathbf{v_S}\|_2^2$$

Which I know is true as was proved in the previous subsections. ■

3.8.5 Multi-sector economy with capital risk sharing, intra-group correlations and collateral constraints

3.8.5.a Equilibrium

We consider a representative household solving the following problem:

$$\max_{c_i \forall i} \ \frac{1}{n}\sum_{i=1}^{n} \ln c_i$$

s.t.

$$\sum_{i=1}^{n} p_i c_i = r \sum_{i \in M_s} k_i + \sum_{i \in M_s} \pi_i$$

The first order conditions of this maximisation problem are found solving:

$$\mathcal{L} = \frac{1}{n}\sum_{i=1}^{n} \ln c_i - \lambda \left(\sum_{i=1}^{n} p_i c_i - r \sum_{i \in M_s} k_i - \sum_{i \in M_s} \pi_i \right)$$

The F.O.C. are given by:

$$p_i c_i = p_j c_j \ \ \forall i, j$$

Optimal consumption of the household:

$$p_i c_i = \frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{n}$$

Each firm maximises its profits according to the following problem:

$$\max_{k_i, q_{ij}} \pi_i = p_i q_i - r k_i - \sum_{j \in M_s} p_j q_{ij} - \sum_{j \notin M_s} p_j q_{ij}$$

s.t.

$$q_i = (z_i k_i)^{\alpha} \prod_{j \in M_s} q_{ij}^{(1-\alpha) w_{ij}} \prod_{j \notin M_s} q_{ij}^{(1-\alpha) w_{ij}}$$

$$\sum_{j \notin M_s} p_j q_{ij} \leq \theta_i \left(p_i q_i - r k_i - \sum_{j \in M_s} p_j q_{ij} \right)$$

Where

$$\sum_{j=1}^{n} w_{ij} = 1 \quad \forall i$$

$$z_i = exp(\mu_i), \;\; \mu_i \sim N(0, \sigma_i^2, \rho_{ij} \forall j \in M_s) \;\; non - i., i.d.$$

Where $\alpha \in (0,1)$ indicates the share of capital in production, $w_{ij} \geq 0$ represents the share of intermediate good j in the total use in sector i.

The second restriction represents the external working capital constraint, by external I mean the inputs obtained from firms of a different subset. We need to solve the following lagrangian problem to find the quantity of inputs demanded by the firm i:

$$\mathcal{L} = p_i (z_i k_i)^{\alpha} \prod_{j=1}^{n} q_{ij}^{(1-\alpha) w_{ij}} - r k_i - \sum_{j \in M_s} p_j q_{ij} + \lambda \left(\theta_i \left(p_i q_i - r k_i - \sum_{j \in M_s} p_j q_{ij} \right) - \sum_{j \notin M_s} p_j q_{ij} \right)$$

The solution to this problem is given by the following equations:

$$q_{ij} = \begin{cases} (1-\alpha) w_{ij} p_i q_i p_j^{-1} & \text{if } (i,j) \in M_s \\ (1-\alpha) w_{ij} \theta_i p_i q_i p_j^{-1} & \text{if } (i,j) \notin M_s \end{cases}$$

$$k_i = \frac{\alpha p_i q_i}{r}$$

Compared to the previous case, I can see that with credit constraints for the external working capital, as the collateral becomes more relevant $(\theta \to 0)$, the demand of each external intermediate input decreases.

The market clearing conditions for the final and intermediate goods, and for the capital are given by the following equations:

$$c_i + \sum_{j \in M_s} q_{ij} + \sum_{j \notin M_s} q_{ij} = q_i \;\; \forall \, (i \in M_s)$$

$$\sum_{i \in M_s} k_i = u_s \overline{K}_s \;\; \forall s$$

Where

$$u_s = exp(\nu_s), \;\; \nu_s \sim N(0, \varsigma_s^2) \;\; i.i.d. \, \forall s$$

In the same way that previous sections, I start substituting firm's F.O.C. for q_{ij} into the clearing condition of goods market for good j:

$$c_j + \sum_{i \in M_s} \frac{(1-\alpha) w_{ij} p_i q_i}{p_j} + \sum_{i \notin M_s} \frac{(1-\alpha) w_{ij} p_i q_i \theta_i}{p_j} = q_j$$

Multiplying by p_j:

$$p_j c_j + \sum_{i \in M_s} (1-\alpha) w_{ij} p_i q_i + \sum_{i \notin M_s} (1-\alpha) w_{ij} p_i q_i \theta_i = p_j q_j$$

Substituting equation of optimal consumption from households through p_j and p_i into the previous equation:

$$\frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{n} \left(1 + \sum_{i \in M_s} \frac{(1-\alpha) w_{ij} q_i}{c_i} + \sum_{i \notin M_s} \frac{(1-\alpha) w_{ij} q_i \theta_i}{c_i} \right) = \frac{r \sum\limits_{i \in M_s} k_i + \sum\limits_{i \in M_s} \pi_i}{n c_j} q_j$$

Simplifying:

$$1 + \sum_{i \in M_s} \frac{(1-\alpha) w_{ij} q_i}{c_i} + \sum_{i \notin M_s} \frac{(1-\alpha) w_{ij} q_i \theta_i}{c_i} = \frac{q_j}{c_j}$$

Defining:

$$\gamma_j \equiv q_j / c_j$$

and

$$\gamma_i \equiv q_i / c_i$$

Substituting into previous equation:

$$1 + \sum_{i \in M_s} (1-\alpha) w_{ij} \gamma_i + \sum_{i \notin M_s} (1-\alpha) w_{ij} \gamma_i \theta_i = \frac{q_j}{c_j}$$

Stacking over firms and solving for γ:

$$\gamma = \left[\mathbb{I} - (1-\alpha)\left[\mathbf{W}' \circ \mathbf{I_s}' + \mathbf{W}' \circ (\mathbb{I} - \mathbf{I_s}') \circ \mathbf{\Theta 1}'\right]\right]^{-1} \mathbf{1}$$

This is in the solution to the consumption/output in equilibrium.

Now I need to find the quantities in equilibrium. Substituting the household's F.O.C. into the firm's F.O.C. for q_{ij}:

$$q_{ij} = \begin{cases} (1-\alpha) w_{ij} c_j q_i c_i^{-1} & \text{if } (i,j) \in M_s \\ (1-\alpha) w_{ij} \theta_i c_j q_i c_i^{-1} & \text{if } (i,j) \notin M_s \end{cases}$$

Substituting γ_i and γ_j into the previous equation:

$$q_{ij} = \begin{cases} (1-\alpha) w_{ij} q_j \gamma_i \gamma_j^{-1} & \text{if } (i,j) \in M_s \\ (1-\alpha) w_{ij} \theta_i q_j \gamma_i \gamma_j^{-1} & \text{if } (i,j) \notin M_s \end{cases}$$

This is the solution for intermediate goods in equilibrium, it depends on the network given by γ and the collateral parameter θ_i.

Substituting the household's F.O.C. into the firm's F.O.C. for k_i:

$$k_i = \frac{\alpha c_j p_j q_i}{c_i r}$$

Substituting γ_i into the previous equation:

$$k_i = \frac{\alpha c_j p_j \gamma_i}{r}$$

Adding over the subset of sectors M_s and substituting the clearing condition of the capital market:

$$u_s\overline{K}_s = \frac{\alpha c_j p_j \sum\limits_{i\in M_s}\gamma_i}{r}$$

Dividing the previous two equations:

$$k_i = \frac{u_s\overline{K}_s\gamma_i}{\sum\limits_{i\in M_s}\gamma_i}$$

This is the solution for the capital in equilibrium, it depends on the network given by γ.

3.8.5.b GDP

Proof Proposition 32. Substituting the solutions of q_{ij} and k_i into the production function:

$$q_i = \left(z_i\frac{u_s\overline{K}_s\gamma_i}{\sum\limits_{i\in M_s}\gamma_i}\right)^{\alpha}\prod_{j\in M_s}\left(\frac{\gamma_i(1-\alpha)w_{ij}q_j}{\gamma_j}\right)^{(1-\alpha)w_{ij}}\prod_{j\notin M_s}\left(\frac{\gamma_i(1-\alpha)w_{ij}\theta_i q_j}{\gamma_j}\right)^{(1-\alpha)w_{ij}}$$

Taking logs of the previous equation:

$$\begin{aligned}\ln q_i = \alpha&\left(\ln z_i + \ln u_s + \ln\overline{K}_s + \ln\gamma_i - \ln\sum_{i\in M_s}\gamma_i\right)\\ &+ (1-\alpha)\sum_{j\in M_s} w_{ij}\left(\ln(1-\alpha) + \ln w_{ij} + \ln\gamma_i - \ln\gamma_j + \ln q_j\right)\\ &\quad + (1-\alpha)\sum_{j\notin M_s} w_{ij}\left(\ln(1-\alpha) + \ln w_{ij} + \ln\gamma_i - \ln\gamma_j + \ln q_j + \ln\theta_i\right)\end{aligned}$$

Simplifying after collapsing common terms:

$$\ln q_i = \alpha \left(\ln z_i + \ln u_s + \ln \overline{K}_s + \ln \gamma_i - \ln \sum_{i \in M_s} \gamma_i \right)$$
$$+ (1-\alpha) \left(\sum_{j=1}^{n} w_{ij} \left(\ln(1-\alpha) + \ln w_{ij} + \ln \gamma_i - \ln \gamma_j + \ln q_j \right) + \sum_{j \notin M_s} w_{ij} \ln \theta_i \right)$$

Stacking over firms:

$$\mathbf{q} = \alpha \left[\mathbf{I_M}(\nu + \mathbf{k} - \check{\gamma}) + \mu + \overline{\gamma} \right] + (1-\alpha) \left[\mathbf{W} \left(\mathbf{1} \ln(1-\alpha) + \mathbf{w} + \mathbf{q} \right) + (\mathbf{W} \circ (\mathbb{I} - \mathbf{I_s}) \circ \mathbf{1}\hat{\mathbf{\Theta}}')\mathbf{1} \right]$$

Solving for $\mathbf{q}$:

$$\mathbf{q} = (\mathbb{I} - (1-\alpha)\mathbf{W})^{-1} \alpha \left[\mathbf{I_M}(\nu + \mathbf{k} - \check{\gamma}) + \mu + \overline{\gamma} \right]$$
$$+ (\mathbb{I} - (1-\alpha)\mathbf{W})^{-1} (1-\alpha) \left[\mathbf{W} \left(\mathbf{1} \ln(1-\alpha) + \mathbf{w} \right) + (\mathbf{W} \circ (\mathbb{I} - \mathbf{I_s}) \circ \mathbf{1}\hat{\mathbf{\Theta}}')\mathbf{1} \right]$$

Recalling $\gamma_i = q_i/c_i$, taking logs, stacking over sectors, rearranging for $\mathbf{c}$, dividing by n and pre-multiplying by $\mathbf{1}'$:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \frac{1}{n}\mathbf{1}'\mathbf{q} - \frac{1}{n}\mathbf{1}'\overline{\gamma}$$

Where $\mathbf{c}' = [\ln c_1, ..., \ln c_n]$. Substituting the equation that I found for $\mathbf{q}$ into the

previous one:

$$\frac{1}{n}\mathbf{1}'\mathbf{c} = \left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right]\left[\mathbf{I_M}(\nu+\mathbf{k}-\breve{\gamma})+\mu+\overline{\gamma}\right]$$
$$+\left[\frac{(1-\alpha)}{\alpha}\right]\left[\frac{\alpha}{n}\mathbf{1}'(\mathbb{I}-(1-\alpha)\mathbf{W})^{-1}\right]\left[\mathbf{W}\left(\mathbf{1}\ln(1-\alpha)+\mathbf{w}\right)+(\mathbf{W}\circ(\mathbb{I}-\mathbf{I_s})\circ\mathbf{1}\hat{\mathbf{\Theta}}')\mathbf{1}\right]-\frac{1}{n}\mathbf{1}'\overline{\gamma}$$

From the budget constraint of the household I know that:

$$\sum_{i=1}^{n} p_i c_i = GDP$$

Substituting the optimal consumption and taking logarithms this equation becomes:

$$Y = \ln c_i + \ln p_i + \ln n$$

Stacking over $i's$, diving by n and pre-multiplying by $\mathbf{1}'$:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \frac{1}{n}\mathbf{1}'\mathbf{p} + \ln n$$

Substituting the ideal price index $(p_1 \cdot ... \cdot p_n)^{1/n} = 1$, the previous equation implies:

$$Y = \frac{1}{n}\mathbf{1}'\mathbf{c} + \ln n$$

Substituting this previous results and the vector of influence into the previous equation, I can express the equation for the GDP in the following way:

$$Y = \mathbf{v}'(\mathbf{I_M}\nu + \mu) + \Lambda$$

Where ν is the vector idiosyncratic shocks, z_s, to each capital endowment for each subset of firms, M_s, μ is the idiosyncratic productivity shocks only correlated between firms of the same group and Λ is a variable of parameters given by:

$$\Lambda \equiv \mathbf{v}'\left[\frac{(1-\alpha)}{\alpha}\Big((\mathbf{W}\circ(\mathbb{I}-\mathbf{I_s})\circ \mathbf{1}\hat{\mathbf{\Theta}}')\mathbf{1} + \mathbf{W}\mathbf{w}\Big) + \mathbf{I_M}(\mathbf{k}-\check{\gamma}) - \overline{\gamma}\right] + \frac{(1-\alpha)}{\alpha}\ln(1-\alpha) - \frac{1}{n}\mathbf{1}'\overline{\gamma} + \ln n \quad (3.93)$$

■

3.8.5.c Volatility of GDP

Proof Proposition 33. Follows the same steps that I use in the proof of the corresponding proposition of the previous two models. ■

3.8.5.d Comparison of volatilities, specific cases

Proof Proposition 34. Consider $A^1_{4,2}$ and $A^2_{4,2}$, let $\varsigma^2_1 = \varsigma^2_2 = \varsigma^2$ and $\sigma^2_i = \sigma^2_j = \sigma^2$. Let the correlation to be given by $[\rho_{ij} \in \mathbf{\Sigma}\ \forall(i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) = (\rho \in A^2_{4,2})]$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho = 0$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho \in (0,1]$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if \ \rho \in [-1, 0)$$

Considering the first equation, after substituting the elements of the influence vector and the idiosyncratic variance of each capital endowment and each idiosyncratic shock:

$$\left[v_1^2 \varsigma_1^2 + v_2^2 \varsigma_1^2 + v_3^2 \varsigma_2^2 + v_4^2 \varsigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 v_1 v_2 + 2\rho_{34}\sigma_3\sigma_4 v_3 v_4 + \sum_{i=1}^{4} \sigma_i^2 v_i^2 \right]^{1/2} =$$
$$\left[v_1^2 \varsigma_1^2 + v_2^2 \varsigma_2^2 + v_3^2 \varsigma_1^2 + v_4^2 \varsigma_2^2 + 2\rho_{13}\sigma_1\sigma_3 v_1 v_3 + 2\rho_{24}\sigma_2\sigma_4 v_2 v_4 + \sum_{i=1}^{4} \sigma_i^2 v_i^2 \right]^{1/2}$$

Taking squares and simplifying using the assumptions about the variance and correlation parameters:

$$\varsigma^2(v_1^2 + v_2^2 + v_3^2 + v_4^2) + 2\rho\sigma^2(v_1 v_2 + v_3 v_4) = \varsigma^2(v_1^2 + v_2^2 + v_3^2 + v_4^2) + 2\rho\sigma^2(v_1 v_3 + v_2 v_4)$$

Simplifying:

$$\rho(v_1 v_2 + v_3 v_4 - v_1 v_3 - v_2 v_4) = 0$$

Rearranging:

$$\rho(v_1 - v_4)(v_2 - v_3) = 0$$

Which in the case of the first equation of the proposition is true because of the assumption that $\rho = 0$. Following the same steps for the second inequality of the proposition, I arrive at:

$$\rho(v_1 - v_4)(v_2 - v_3) > 0$$

The second of the left hand side is always positive because, after substituting the elements of the influence vector, I have $-\alpha^3+4\alpha^2-7\alpha+4$, which is always positive in the interval $\alpha \in (-\infty, 1)$. Hence I can express the previous inequality as:

$$\rho(v_2 - v_3) > 0$$

After substituting the elements of the influence vector for the second term of the inequality I have:

$$\rho(\alpha - 1)(\alpha - 1) > 0$$

As $\alpha \in (0, 1)$ I can further simplify:

$$\rho > 0$$

Which is true given the assumption of the second inequality of the proposition. Following the same steps in the case of the third inequality, I arrive at the following condition:

$$\rho < 0$$

Which is true given the assumption of the inequality that I am analysing. ■

Proof Proposition 35. Consider $A^1_{4,2}$ and $A^2_{4,2}$, let $\varsigma_1^2 = \varsigma_2^2 = \varsigma^2$ and $\sigma_i^2 = \sigma_j^2 = \sigma^2$. Let the correlation to be given by $[\rho_{ij} \in \mathbf{\Sigma}\ \forall (i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) \neq (\rho \in A^2_{4,2})]$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \ \ if\ \rho \in (0, 1] \wedge [(\rho \in A^1_{4,2}) > (\rho \in A^2_{4,2})]$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \;\; if \; \rho \in [-1,0) \wedge [(\rho \in A^1_{4,2}) < (\rho \in A^2_{4,2})]$$

Following the same steps of the previous proof, I consider the first equation, after substituting the elements of the influence vector and the idiosyncratic variance of each capital endowment and each idiosyncratic shock:

$$\left[v_1^2\varsigma_1^2 + v_2^2\varsigma_1^2 + v_3^2\varsigma_2^2 + v_4^2\varsigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 v_1 v_2 + 2\rho_{34}\sigma_3\sigma_4 v_3 v_4 + \sum_{i=1}^{4}\sigma_i^2 v_i^2\right]^{1/2} >$$
$$\left[v_1^2\varsigma_1^2 + v_2^2\varsigma_2^2 + v_3^2\varsigma_1^2 + v_4^2\varsigma_2^2 + 2\rho_{13}\sigma_1\sigma_3 v_1 v_3 + 2\rho_{24}\sigma_2\sigma_4 v_2 v_4 + \sum_{i=1}^{4}\sigma_i^2 v_i^2\right]^{1/2}$$

Taking squares and simplifying using the assumptions about the variance and correlation parameters:

$$\varsigma^2(v_1^2+v_2^2+v_3^2+v_4^2)+2\rho_{A_1}\sigma^2(v_1v_2+v_3v_4) > \varsigma^2(v_1^2+v_2^2+v_3^2+v_4^2)+2\rho_{A_2}\sigma^2(v_1v_3+v_2v_4)$$

Where ρ_{A_1} denotes the correlation of the first economy and ρ_{A_2} the correlation of the second one. After simplifying:

$$\rho_{A_1}(v_1v_2 + v_3v_4) - \rho_{A_2}(v_1v_3 + v_2v_4) > 0$$

Substituting the elements of the influence vector and simplifying:

$$-\alpha^6\rho_{A_1} - \alpha^5(\rho_{A_2} - 7\rho_{A_1}) - \alpha^4(21\rho_{A_1} - 5\rho_{A_2}) - \alpha^3(11\rho_{A_2} - 33\rho_{A_1})$$
$$- \alpha^2(28\rho_{A_1} - 13\rho_{A_2}) - \alpha(8\rho_{A_2} - 12\rho_{A_1}) > 0$$

For values of $\alpha \in (0,1)$, the solution to this polynomial inequality is given by the conditions ($\rho_{A_2} < 0 \wedge 3\rho_{A_1} \geq 2\rho_{A_2}$), ($\rho_{A_2} = 0 \wedge \rho_{A_1} > 0$) and ($\rho_{A_2} > 0 \wedge \rho_{A_1} \geq \rho_{A_2}$). All cases do not violate the assumption for this inequality, $\rho_{A_1} > \rho_{A_2}$. Following the same steps in the case of the second inequality I arrive at the following:

$$-\alpha^6\rho_{A_1} - \alpha^5(\rho_{A_2} - 7\rho_{A_1}) - \alpha^4(21\rho_{A_1} - 5\rho_{A_2}) - \alpha^3(11\rho_{A_2} - 33\rho_{A_1}) \\ - \alpha^2(28\rho_{A_1} - 13\rho_{A_2}) - \alpha(8\rho_{A_2} - 12\rho_{A_1}) < 0$$

Given the assumption of $\alpha \in (0,1)$, the solution to this polynomial inequality is given by the conditions ($\rho_{A_2} > 0 \wedge 3\rho_{A_1} \leq 2\rho_{A_2}$), ($\rho_{A_2} = 0 \wedge \rho_{A_1} < 0$) and ($\rho_{A_2} < 0 \wedge \rho_{A_1} \leq \rho_{A_2}$). Which in all cases the assumption for this inequality, $\rho_{A_1} < \rho_{A_2}$, is not violated. ■

Proof Proposition 36. Considering $A^1_{4,2}$ and $A^2_{4,2}$, let $\rho_{ij} = 0$ and $\sigma_i^2 = \sigma^2$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} = [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma_1^2 = \varsigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} > [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma_1^2 > \varsigma_2^2$$

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{A^2_{4,2}} \quad if\ \varsigma_1^2 < \varsigma_2^2$$

For the first inequality of the proposition, substituting into the definition of variance I have:

$$\left[v_1^2\varsigma_1^2 + v_2^2\varsigma_1^2 + v_3^2\varsigma_2^2 + v_4^2\varsigma_2^2 + \sum_{i=1}^{4}\sigma_i^2 v_i^2\right]^{1/2} = \left[v_1^2\varsigma_1^2 + v_2^2\varsigma_2^2 + v_3^2\varsigma_1^2 + v_4^2\varsigma_2^2 + \sum_{i=1}^{4}\sigma_i^2 v_i^2\right]^{1/2}$$

Taking squares of both sides, collecting the ς_s^2, and simplifying:

$$\varsigma_1^2(v_1^2+v_2^2)+\varsigma_2^2(v_3^2+v_4^2)=\varsigma_1^2(v_1^2+v_3^2)+\varsigma_2^2(v_2^2+v_4^2)$$

This is true for the first equation of the proposition because the assumption given in this case, $\varsigma_1^2=\varsigma_2^2$. Following the same steps in the case of the second inequality of the proposition, I arrive at the following:

$$\varsigma_1^2(v_1^2+v_2^2)+\varsigma_2^2(v_3^2+v_4^2)>\varsigma_1^2(v_1^2+v_3^2)+\varsigma_2^2(v_2^2+v_4^2)$$

After simplifying and rearranging:

$$(\varsigma_1^2-\varsigma_2^2)(v_2^2-v_3^2)>0$$

Substituting the elements of the influence vector, the previous expression becomes:

$$(\varsigma_1^2-\varsigma_2^2)(\alpha-1)^2((\alpha-2)^2+1)>0$$

As the second and the third terms between parenthesis of the left hand side are always positive because $\alpha\in(0,1)$, the inequality is given by:

$$\varsigma_1^2>\varsigma_2^2$$

Which is true because this is the assumption given in the proposition for this case. In the case of the last inequality of the proposition, following the same steps as before, I arrive at the following condition:

$$\varsigma_1^2 < \varsigma_2^2$$

Which completes the proof because is the same that the assumption given for the inequality analysed. ■

Proof Proposition 37. Consider $A^1_{4,2}$ and $B_{4,2}$, let $\varsigma_1^2 = \varsigma_2^2$ and $\sigma_i^2 = \sigma_j^2$. Let the correlation to be given by $[\rho_{ij} \in \mathbf{\Sigma}\, \forall (i,j) \in N | \rho_{ij} = \rho, i \neq j, (\rho \in A^1_{4,2}) = (\rho \in A^2_{4,2})]$:

$$[var(Y)]^{1/2}_{A^1_{4,2}} < [var(Y)]^{1/2}_{B_{4,2}}$$

Considering the first equation, using the definition of variance and simplifying terms using the assumptions given:

$$\left[v_1^2\varsigma_1^2 + v_2^2\varsigma_1^2 + v_3^2\varsigma_2^2 + v_4^2\varsigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 v_1 v_2 + 2\rho_{34}\sigma_3\sigma_4 v_3 v_4 + \sum_{i=1}^{4} \sigma_i^2 v_i^2 \right]^{1/2}_{A^1_{4,2}} <$$
$$\left[v_1^2\varsigma_1^2 + v_2^2\varsigma_1^2 + v_3^2\varsigma_2^2 + v_4^2\varsigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 v_1 v_2 + 2\rho_{34}\sigma_3\sigma_4 v_3 v_4 + \sum_{i=1}^{4} \sigma_i^2 v_i^2 \right]^{1/2}_{B_{4,2}}$$

Taking squares and simplifying using the assumptions about the variance and correlation parameters, denoting the elements of the vertical economy with the sub-index V and the ones of the star economy with the sub-index S:

$$\left(\sigma^2 + \varsigma^2\right) \left(\sum_{i=1}^{4} v_{iV}^2 - \sum_{i=1}^{4} v_{iS}^2 \right) - \rho\sigma^2 (v_{1S}v_{2S} + v_{3S}v_{4S} - v_{1V}v_{2V} - v_{3V}v_{4V}) < 0$$

After substitute the elements of the influence vector and simplifying, I have the following inequality:

$$\alpha^2(35(\sigma^2+\varsigma^2)-26\sigma^2\rho)+\alpha^4(22(\sigma^2+\varsigma^2)-21\sigma^2\rho)+\alpha^6(\sigma^2+\varsigma^2-\sigma^2\rho)$$
$$+\alpha^5(-7(\sigma^2+\varsigma^2)+7\sigma^2\rho)+\alpha(-12(\sigma^2+\varsigma^2)+8\sigma^2\rho)+\alpha^3(-39(\sigma^2+\varsigma^2)+33\sigma^2\rho)<0$$

Under the assumptions of the parameters $\alpha \in (0,1)$, $\sigma^2 > 0$ and $\varsigma^2 > 0$, the solution to this polynomial inequality is given by $\rho \leq< 3/2$, which completes the proof because I know that the correlation parameter can takes values $\rho \in [-1,1]$.

■

3.8.6 Mathematica code

3.8.6.a Influence vector star and vertical economies

ClearAll["Global`*"]

ClearAll[W1, W2, W3, V, S, T, α, n]

$n = 4;$

$$\mathrm{W1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\text{W2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

$$\text{W3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

Text[“Vertical n=4”]

$$V = \text{FullSimplify}\left[(\alpha/n) * \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} . \left(\text{Inverse}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - ((1-\alpha) * \text{W1})\right]\right)\right]$$

Text[“Vertical squared n=4”]

$V.(\text{Transpose}[V])$

Text[“Star n=4”]

$$S = \text{FullSimplify}\left[(\alpha/n) * \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} . \left(\text{Inverse}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - ((1-\alpha) * \text{W2})\right]\right)\right]$$

Text[“Star squared n=4”]

$S.(\text{Transpose}[S])$

Text[“Tree n=4”]

$$T = \text{FullSimplify}\left[(\alpha/n)*\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.\left(\text{Inverse}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - ((1-\alpha)*\text{W3})\right]\right)\right]$$

Text[“Tree squared n=4”]

T.(Transpose[T])

Quit[]

3.8.6.b Simplification of influence vectors and inequalities

ClearAll[“Global`*”]

ClearAll[vs, vv, vt, a, parameter1, diff1, diff4, diff7, diff8, diff9, diff10, diff11, diff12, vvij, vsij, vvijnc,

elementsv1234, elementsv1324, elementsv1234V, elementsv1234S, X, Y, Z]

vs[a_] $= ((4-3*a)/4)\hat{}2 + 3*(a/4)\hat{}2;$

vv[a_] $= (((2-a)*(2+(-2+a)*a))/4)\hat{}2$

$+((a*(3+(-3+a)*a))/4)\hat{}2 + ((a*(1-(a-1)))/4)\hat{}2 + (a/4)\hat{}2;$

vsij[a_] $= 2*(((4-3*a)/4)*(a/4) + ((4-3*a)/4)*(a/4)$

$+((4-3*a)/4)*(a/4) + (a/4)*(a/4) + (a/4)*(a/4) + (a/4)*(a/4));$

vvij[a_] =

$2*((((2-a)*(2+(-2+a)*a))/4)*((a*(3+(-3+a)*a))/4)$

$+(((2-a)*(2+(-2+a)*a))/4)*((a*(1-(a-1)))/4)+$

$(((2 - a) * (2 + (-2 + a) * a))/4) * (a/4) + ((a * (3 + (-3 + a) * a))/4) * ((a * (1 - (a - 1)))/4)$

$+((a * (3 + (-3 + a) * a))/4) * (a/4)+$

$((a * (1 - (a - 1)))/4) * (a/4));$

vvijnc[a_] $=$

$2 * ((((2 - a) * (2 + (-2 + a) * a))/4) * ((a * (1 - (a - 1)))/4) + (((2 - a) * (2 + (-2 + a) * a))/4) * (a$

$+((a * (3 + (-3 + a) * a))/4) * (a/4));$

vvijcc[a_] $=$

$2 * ((((2 - a) * (2 + (-2 + a) * a))/4) * ((a * (3 + (-3 + a) * a))/4)$

$+((a * (3 + (-3 + a) * a))/4) * ((a * (1 - (a - 1)))/4)+$

$((a * (1 - (a - 1)))/4) * (a/4));$

parameter1 $= 1 > a > 0;$

Text["Comparing Star vs Vertical"]

Resolve[ForAll[$\{a\}$, parameter1, vs[a] $>$ vv[a]]]

diff1[a_] $=$ Collect[FullSimplify[vs[a] $-$ vv[a]], a];

Collect[FullSimplify[vs[a] $-$ vv[a] > 0], a]

Reduce[diff1[a] $> 0, a$]

Text["Comparing Starij vs Verticalij"]

Resolve[ForAll[$\{a\}$, parameter1, vsij[a] $<$ vvij[a]]]

diff4[a_] $=$ Collect[FullSimplify[vvij[a] $-$ vsij[a]], a];

Collect[FullSimplify[vvij[a] $-$ vsij[a] > 0], a]

Reduce[diff4[a] $> 0, a$]

Text["Comparing Star-Vertical=Verticalij-Starij"]

Resolve[ForAll[$\{a\}$, parameter1, vs[a] $-$ vv[a]==vvij[a] $-$ vsij[a]]]

diff7[a_] $=$ Collect[FullSimplify[vs[a] $-$ vv[a] $-$ (vvij[a] $-$ vsij[a])], a]

Text["Analysing if verticalij is always positive"]

diff8[a_] $=$ Collect[FullSimplify[vvij[a]], a];

Collect[FullSimplify[vvij[a] > 0], a]

Reduce[diff8[a] $> 0, a$]

Text["Analysing if the non-connected terms of the ij product of vertical are positive"]

diff9[a_] $=$ Collect[FullSimplify[vvijnc[a]], a];

Collect[FullSimplify[vvijnc[a] > 0], a]

Reduce[diff9[a] $> 0, a$]

Text["Analysing if the connected terms of the ij product of vertical are positive"]

diff10[a_] $=$ Collect[FullSimplify[vvijcc[a]], a];

Collect[FullSimplify[vvijcc[a] > 0], a]

Reduce[diff10[a] $> 0, a$]

Text["Comparing the connected and unconnected terms of the ij product of vertical"]

Resolve[ForAll[$\{a\}$, parameter1, vvijcc[a] $<$ vvijnc[a]]]

diff11[a_] $=$ Collect[FullSimplify[vvijnc[a] $-$ vvijcc[a]], a];

Collect[FullSimplify[vvijnc[a] $-$ vvijcc[a] > 0], a]

Reduce[diff11[a] $> 0, a$]

Text["Substituting the elements into p1(v1v2+v3v4)-p2(v1v3+v2v4)>0 (vertical)"]

elementsv1234[a_, p1_] = p1 $*((((2-a)*(2+(-2+a)*a))/4)*((a*(3$

$+(-3+a)*a))/4)+((a*(1-(a-1)))/4)*(a/4));$

elementsv1324[a_, p2_] = p2 $*((((2-a)*(2+(-2+a)*a))/4)*((a*(1-(a-1)))/4)$

$+((a*(3+(-3+a)*a))/4)*(a/4));$

diff11[a_, p1_, p2_] = Collect[FullSimplify[elementsv1234[a, p1] $-$ elementsv1324[a, p2]], a];

Collect[FullSimplify[diff11[a, p1, p2] > 0], a]

Resolve[ForAll[{a}, parameter1, diff11[a, p1, p2] > 0]]

Reduce[diff11[a, p1, p2] $> 0, a$];

Text["Substituting the elements into p1(v1v2+v3v4)-p2(v1v3+v2v4)<0 (vertical)"]

Collect[FullSimplify[diff11[a, p1, p2] < 0], a]

Resolve[ForAll[{a}, parameter1, diff11[a, p1, p2] < 0]]

Reduce[diff11[a, p1, p2] $> 0, a$];

Text["Substituting the elements into X(vv-vs)-Y((v1v2+v3v4(star))-((v1v2+v3v4(vertical)))=0"]

elementsv1234V[a_] = $(((2-a)*(2+(-2+a)*a))/4)*((a*(3+(-3+a)*a))/4)$

$+((a*(1-(a-1)))/4)*(a/4);$

elementsv1234S[a_] = $((4-3*a)/4)*(a/4)+(a/4)*(a/4);$

diff12[a_, X_, Y_, Z_] = Collect[FullSimplify[$(X+Y)*$(vv[a]

```
-vs[a]) - 2 * Z * X * (elementsv1234S[a] - elementsv1234V[a])], a];
Collect[FullSimplify[diff12[a, X, Y, Z]==0], a]
Resolve[Exists[{a, X, Y}, diff12[a, X, Y, Z]==0&&X > 0&&Y > 0&&parameter1], Reals]
Resolve[ForAll[{a, Y, X}, parameter1&&X > 0&&Y > 0, diff12[a, X, Y, Z]==0], Reals]
Reduce[diff12[a, X, Y, Z]==0, a, Reals];

Text["Substituting the elements into X(vv-vs)-Y((v1v2+v3v4(star))-((v1v2+v3v4(vertical)))<0"]
Collect[FullSimplify[diff12[a, X, Y, Z] < 0], a]
Resolve[Exists[{a, X, Y}, diff12[a, X, Y, Z] < 0&&X > 0&&Y > 0&&parameter1], Reals]
Resolve[ForAll[{a, Y, X}, parameter1&&X > 0&&Y > 0, diff12[a, X, Y, Z] < 0], Reals]
Reduce[diff12[a, X, Y, Z] < 0, {a, X, Y, Z}, Reals];

Quit[]
```

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