Isomorphisms amongst certain classes of cyclically presented groups

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Dedicated to

My parents' souls, my brother Nuruddin's soul, my wife, and all my family members who continually pray for my fortune.

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Abstract

In this thesis we consider isomorphisms amongst certain classes of cyclically presented groups. We give isomorphism theorems for two families of cyclically presented groups, the groups $G_n(h,k,p,q,r,s,l)$, and the groups $G_n^{\varepsilon}(m,k,h)$, which were introduced by Cavicchioli, Repovs and Spaggiari. These families contain many subfamilies of cyclically presented groups, we have results for two of them, the groups $G_n(m,k)$, which were introduced by Johnson and Mawdesley, and the groups $\Gamma_n(k,l)$, which were introduced by Cavicchioli, Repovs and Spaggiari.

The abelianization of the Fibonacci groups $F(2,n)$ was proved by Lyndon to be finite and its order can be expressed in terms of the Lucas numbers. Bardakov and Vesnin have asked if there is a formula for the order of the abelianization of $G_n(m,k)$ groups that can be expressed in terms of Fibonacci numbers. We produce formulas that compute the order of $G_{pm}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}, G_{pk}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}$ for certain values of p where m,k are coprime, and for the groups $\Gamma_n(1,\frac{n}{2}-1)^{\mathrm{ab}}$ (this formula is given in terms of Lucas numbers).

The values of the number of non-isomorphic $G_n(m,k)$ groups was conjectured by Cavicchioli, O'Brien and Spaggiari for $n=p^l$, where p is prime and l is a positive integer, we show that these values provide an upper bound for the number of non-isomorphic $G_n(m,k)$ groups. We also give lower bounds and upper bounds for the number of non- isomorphic $G_n(m,k)$ and $\Gamma_n(k,l)$ groups for certain values of n. Similar to the investigation of the type of isomorphisms of $G_n(m,k)$ groups for $n \leq 27$ that was carried by Cavicchioli, O'Brien and Spaggiari, we perform a similar investigation for $\Gamma_n(k,l)$ groups for $n \leq 29$.

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Chapter 1

Introduction

1.1 Introduction

In this chapter, we give in Section 1.2 initial definitions and background material about cyclically presented groups. In Section 1.3, we present some families of groups that we have results about such as the family of groups $G_n(h,k,p,q,r,s,\ell)$ which were introduced by Cavicchioli, Repovs, and Spaggiari in [CRS03], and the family of the groups $G_n^{\epsilon}(m,k,h)$, which were introduced by Cavicchioli, Repovs, and Spaggiari in [CRS05]. Both of theses families contain various families of cyclically presented groups, such as the groups $G_n(m,k)$ and $\Gamma_n(k,l)$ which we will consider in Sections 1.5 and 1.6. We give essential definitions of circulant matrices in Section 1.4, which are important to understand the structure of cyclically presented groups. Thesis outline will be given in Section 1.7.

1.2 Introduction to Cyclically Presented Groups

Cyclically presented groups is a rich source of interesting groups. It provides a wide range of study as it is connected to many branches of mathematics. We study isomorphisms amongst particular classes of cyclically presented groups.

Definition 1.2.1. Let G be a group generated by a set $X = \{x_0, x_1, ..., x_{n-1}\}$. Each element of G can be expressed as a product of $x_i^{\pm 1}, 0 \leq i \leq n-1$. such a product is called *a word* $\omega = \omega(x_0, ..., x_{n-1})$.

Definition 1.2.2. A group F is said to be *free* on a subset $X \subseteq F$ if, for any group G and any mapping $\theta : X \to G$, there is a unique homomorphism $\theta' : F \to G$ such that $x_i\theta' = x_i\theta$ for all $x_i \in X$.

Definition 1.2.3. Let $\omega = \omega(x_0, ..., x_{n-1})$ be a cyclically reduced word in the free group F with generators $x_0, ..., x_{n-1}$ and let $\theta(x_i) = x_{i+1}$ for each $0 \leq i \leq n-1$ (subscripts mod n). The presentation

$$\mathcal{G}_n(\omega) = \langle x_0, x_1, ..., x_{n-1} \mid \omega, \theta(\omega), ..., \theta^{n-1}(\omega)\rangle$$

is said to be *a cyclic presentation*, and the group $G_n(\omega)$ that defined by $\mathcal{G}_n(\omega)$ is called *a cyclically presented group.*

1.3 Families of cyclically presented groups

We present here two families and many subfamilies of cyclic presentations of groups which we have results for.

[I] The family $G_n(h, k, p, q, r, s, \ell)$, which was introduced by Cavicchioli, Repovš and Spaggiari in [CRS03].

$$\begin{aligned} G_n(h, k, p, q, r, s, \ell) &= G_n((\prod_{j=0}^{r-1} x_{jp})^{\ell}(\prod_{j=0}^{s-1} x_{h+jq})^{-k}) \\ &= \langle x_0, x_1, ..., x_{n-1} \mid (x_i x_{i+p} ... x_{i+p(r-1)})^{\ell} = \\ &\qquad (x_{i+h} x_{i+h+q} ... x_{i+h+q(s-1)})^{k}, i = 0, ..., n-1\rangle \end{aligned} \tag{1.1}$$

where $r \geq 1, s \geq 1, 0 \leq p, q, h \leq n-1, \ell, k \in \mathbb{Z}$, and all subscripts are taken modulo n.

This family contains many classes of cyclic presented groups, considered before by different authors. These groups were studied in terms of their topological properties in [CRS03], and they illustrated them as follow

(1) The groups $G_n(s, c, 1, 1, r, 1, 1) = G_n(x_0 x_1 ... x_{r-1} x_s^{-c})$, which were introduced in [JO94] and denoted by $F(r, s, c, n)$. This group is a generalization of the following groups

(a) The groups $G_n(r, 1, 1, 1, r, 1, 1) = G_n(x_0 x_1 ... x_{r-1} x_r^{-1})$, which were introduced in [JWW74] and denoted by $F(r,n)$. They are called Fibonacci groups and in terms of isomorphisms they give in [JWW74] a table that is showing in most cases isomorphisms for $n, r \leq 7$. They were also studied by many authors, geometrically (see for example [HKM98]).

(b) The groups $G_n(r+k-1,1,1,1,r,1,1) = G_n(x_0x_1...x_{r-1}x_{r+k-1}^{-1})$, which were introduced in [CR75c] and denoted by $F(r,n,k)$.

(2) The groups $G_n(\ell,-1,k,0,2,1,1) = G_n(x_0x_kx_\ell)$, which the groups we denote $\Gamma_n(k,\ell)$, which were introduced in [CRS05], and studied further in [EW10].

(3) The groups $G_n(k-1,1,q,q,r,s,1) = G_n((\prod_{j=0}^{r-1} x_{jq})(\prod_{j=0}^{s-1} x_{k+jq})^{-1})$, which were introduced in [Pri95] and denoted by $P(r,n,k,s,q)$. They generalize the following groups

(a) The groups $R(r,n,k,h) = G_n((r-1)h+k,1,h,0,r,1,1)$, which were introduced in [CR75a] and denoted by $R(r,n,k,h)$. They are called Fibonacci type groups and studied in terms of isomorphisms by different people (see for example [CR75a]).

(b) The groups $G_n(k,1,m,1,2,1,1) = G_n(x_0x_mx_k^{-1})$, which were were introduced in [JM75]. They are called Fibonacci type groups and subsequently studied in [BV03], [COS08], [Wil09], see [Wil12] for survey of these groups $G_n(x_0x_mx_k^{-1})$. They are generalizations of the Gilbert-Howie groups defined in [GH95] as

$$H(n,m) = G_n(x_0x_mx_1^{-1}) = G_n(m,1).$$

(c) The groups $G_n(r,1,1,1,r,k,1) = G_n((x_0x_1...x_{r-1})(x_rx_{r+1}...x_{r+k})^{-1}$, which were introduced in [CR75b] and denoted by $H(r,n,k)$. These groups $F(r,n,k), H(r,n,k)$ are present generalizations of the Fibonacci groups $F(r,n)$, and also have been studied topologically and geometrically (see [BV03], [Odo99] and [SV00]).

[II] The family $G_n^\varepsilon(m,k,h)$, which was introduced by Cavicchioli, Repovš and Spaggiari in [CRS05]

$$G_n^\varepsilon(m,k,h) = \langle x_0, x_1, ..., x_{n-1} \mid x_i^a x_{i+k}^b x_{i+h+m}^a = (x_{i+h}^r x_{i+m}^r)^s, i = 0, ..., n-1\rangle \quad (1.2)$$

where $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4, n \geq 2, m, k$ and h are taken modulo n, and the integer parameters m, k and h are taken modulo n.

This class of groups contains also well-known groups considered by different people, most of these groups are illustrated in [CRS05], including the following

(1) If $a = b = s = 1, r = 2$ and $h = 0$, then the groups $G_n^{\varepsilon}(m,k,h)$ have defining relations $x_i x_{i+m} = x_{i+k}$ of the groups $G_n(m,k)$, as described above.

(2) If $a = s = 1, b = -1, r = 2, m = k, k = \ell$ and $h = 0$, then the groups $G_n^{\varepsilon}(m,k,h)$ have defining positive relators $x_i x_{i+k} x_{i+\ell}$ of the groups $\Gamma_n(k,l)$, which described above.

Results about these two families of groups will be given in Chapter 2.

Definition 1.3.1. *The abelianization* of $G_n(\omega)$ group can be defined by

$$G_n(\omega)^{\mathrm{ab}} = \langle x_0, x_1, ..., x_{n-1} \mid \omega, \theta(\omega), ..., \theta^{n-1}(\omega), x_i x_j = x_j x_i, 0 \le i, j \le n-1 \rangle. \tag{1.3}$$

1.4 Circulant matrices

Circulant matrices play a role in understanding the structure of cyclically presented groups, we give essential definitions of circulant matrices (see [Dav12])

Definition 1.4.1. The polynomial $f(t) = f_{n,\omega}(t)$ associated with the cyclically presented group $G = G_n(\omega)$ is given by

$$f(t) = \sum_{i=0}^{n-1} a_i t^i \tag{1.4}$$

where a_i is the exponent sum of x_i in $\omega, 0 \le i \le n-1$.

Since the n permutants of ω under powers of θ comprise a set of defining relators for $G_n(\omega)$, it follows that the matrix

$$C = \begin{pmatrix} a_0 & a_1 & . & . & . & a_{n-1} \\ a_{n-1} & a_0 & . & . & . & a_{n-2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_1 & a_2 & . & . & . & a_0 \end{pmatrix} \tag{1.5}$$

is a relation matrix for $G_n(\omega)^{ab}$. By [Dav12, Equation (3.2.14)], [Joh80, Page 77-Theorem 1], its determinant is known.

Theorem 1.4.2. *[Joh80, Page 77-Theorem 1] With the notation of (1.4) and (1.5)*

$$det(C) = \prod_{i=0}^{n-1} f(\omega_i), \tag{1.6}$$

where ω_i ranges over the set of complex nth roots of unity.

Accordance with the theory of §6 of [Joh80] we can write $\prod_{\theta^n=1} f(\theta) = \prod_{i=0}^{n-1} f(\omega_i)$, and we have

Theorem 1.4.3. *[Joh80, Page 77-Theorem 2] If f is the polynomial associated with ω, then*

$$|G_n(\omega)^{\text{ab}}| = |\prod_{\theta^n=1} f(\theta)|. \tag{1.7}$$

Therefore by (1.6),(1.7) we have.

$$|G_n(\omega)^{\text{ab}}| = |det(C)|. \tag{1.8}$$

det(C) = 0 (C is singular) is interpreted as $G_n(\omega)^{\text{ab}}$ is infinite, otherwise $G_n(\omega)^{\text{ab}}$ is finite.

Put

$$R_n(f) = \prod_{\theta^n=1} f(\theta). \tag{1.9}$$

Now the following lemma is in [Dav12, Page (76)] and [Odo99, Lemma 2.1], which will be used in Chapter 3 in finding the order of the abelianization.

Lemma 1.4.4. *[Dav12, Page (76)], [Odo99, Lemma 2.1] Let $f(x) \in \mathbb{Z}[x]$, $degf = k \geq 1$, and suppose that $f(x) = c\prod_{j=1}^{k}(x - \beta_j) \in \mathbb{C}[x]$, where $0 \neq c \in \mathbb{Z}$. Then*

$$R_n(f) = \left((-1)^k c\right)^n \prod_{j=1}^{k} (\beta_j^n - 1). \tag{1.10}$$

Definition 1.4.5. *Any finitely generated abelian group is isomorphic to a group of the form $G_0 \oplus \mathbb{Z}^\beta$ where G_0 is a finite abelian group and β is called the Betti number (or torsion-free rank of $G_n(\omega)$) (see for example [Fra03, Theorem 2.11]). Therefore $G_n(\omega)$ is infinite if and only if $\beta(G_n(\omega)) \geq 1$.*

The following method for calculating Betti number for cyclically presented groups was observed in [Wil17].

Let $g(t) = t^n - 1$, it is shown in the following [Ing56], [New83, Theorem 1] that the rank of C can be expressed in terms of the polynomials f, g.

Theorem 1.4.6. *[Ing56], [New83, Theorem 1] The rank of C is given by the formula*

$$r(C) = n - deg(gcd(f(t), g(t))). \tag{1.11}$$

and so

$$\beta(G_n(\omega)^{\mathrm{ab}}) = deg(gcd(f(t), g(t))). \tag{1.12}$$

In the following two sections we will pay attention to the groups $G_n(m,k), \Gamma_n(k,l)$.

1.5 Generalized Fibonacci Type Groups $G_n(m,k)$.

This class of cyclically presented groups $G_n(m,k)$ was introduced by Johnson and Mawdesley in [JM75], and studied by Bardakov in [BV03], then by Cavicchioli, O'Brien, and Spaggiari in [COS08], and Williams [Wil09], and revisited by Williams in [Wil12], and can be defined as

$$G_n(m,k) = \langle x_0, x_1, ..., x_{n-1} \mid x_i x_{i+m} = x_{i+k} (0 \leq i \leq n-1) \rangle \tag{1.13}$$

The groups generalize various groups that have previously been studied such as: Gilbert and Howie groups $H(n,m) = G_n(m,1)$, see [GH95], Conway's Fibonacci groups $F(2,n) = G_n(1,2)$ [CWLF67], and the Sieradski groups $S(2,n) = G_n(2,1)$ [Sie86]. As described in Section 1.3 the groups $G_n(m,k)$ fit into the wider classes of cyclically presented groups $R(r,n,k,h)$ of [CR75b], $P(r,n,k,s,q)$ of [Pri95], $G_n^{\varepsilon}(m,k,h)$ of [CRS05] and $G_n(h,k,p,q,r,s,\ell)$ of [CRS03].

Definition 1.5.1. The presentation $G_n(m,k)$ is said to be *irreducible* if n, m, k satisfy

$$0 < m < k < n-1, \quad (n,m,k) = 1,$$

and is *strongly irreducible* if it is irreducible and additionally

$$(n,k) > 1, \quad (n,k-m) > 1.$$

Lemma 1.5.2. *[BV03, Lemma1.2] The group $G_n(m,k)$ is isomorphic to the free product of (n,m,k) copies of $G_N(M,K)$ where $N = n/(n,m,k), M = m/(n,m,k), K = k/(n,m,k)$.*

Lemma 1.5.3. *[BV03, Lemma1.2]*

a. $G_n(m,0) = G_n(m,m) = 1$;

b. $G_n(0,k)$ *is isomorphic to the free product of* (n,k) *copies of* $\mathbb{Z}_{2^n-1}$.

By Lemma 1.5.2 and Lemma 1.5.3 we may assume (and often will) that $(n,m,k) = 1$ and $1 \le m \ne k \le n-1$.

1.5.1 Finiteness

Building on [GH95] and [Edj03], a classification of finite groups $G_n(m,k)$ was given in [Wil09] with 2 exceptions. The groups are described in Section 4 of [Wil12] when $n < 9$, and when $n = 9$ we have

Theorem 1.5.4. *[Wil12, Theorem 4.4] Let $n = 9, 1 \le m \ne k \le n-1, (n,m,k) = 1$. Then $G_n(m,k)$ is isomorphic to exactly one of $F(2,9), S(2,9), H(9,3), H(9,4), H(9,7)$. The groups $F(2,9), S(2,9), H(9,3)$ are infinite; it is unknown whether $H(9,4), H(9,7)$ are finite or infinite.*

For $n \ge 10$ the classification of finite groups $G_n(m,k)$ is given by the following theorem.

Theorem 1.5.5. *([GH95], [Wil09]) Let $n \ge 10, 1 \le m \ne k \le n-1$, and $(n,m,k) = 1$. Then $G_n(m,k)$ is finite if and only if $2k \equiv 0$ mod n or $2(k-m) \equiv 0$ mod n, in which case $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{\frac{n}{2}} - (-1)^{m+\frac{n}{2}}$.*

Lemma 1.5.6. *[Wil09, Lemma 3] Suppose that $(m,k) = 1, k \not\equiv 0$ mod n and either $2k \equiv 0$ or $2(k-m) \equiv 0$ mod n. Then $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{\frac{n}{2}} - (-1)^{m+\frac{n}{2}}$.*

1.5.2 Abelianization

We will apply different techniques on abelianization of groups in order to identify isomorphism types.

The survey article [Wil12] includes the following theorems. For the Fibonacci groups $F(2,n)$ the order of $F(2,n)^{\mathrm{ab}}$ is given by the following theorem, where L_n denotes the nth Lucas number where $L_{n+2} = L_{n+1} + L_n,\ L_0 = 2, L_1 = 1$.

Theorem 1.5.7. *(Lyndon, [CWLF67])* $|F(2,n)^{\mathrm{ab}}| = L_n - 1 - (-1)^n$ *. In particular,* $F(2,n)^{\mathrm{ab}}$ *is finite for all* n*.*

Theorem 1.5.8. *(Bumby, [CWLF67])*

$$F(2,n)^{\mathrm{ab}} = \begin{cases} \mathbb{Z}_s & if\ (n,6) = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2s} & if\ (n,6) = 2 \\ \mathbb{Z}_s \oplus \mathbb{Z}_s & if\ (n,6) = 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{5s} & if\ (n,6) = 6 \end{cases}$$

where s *can be found from Theorem 1.5.7*

For the Sieradski groups, the structure of $S(2,n)^{ab}$ is given by the following theorem

Theorem 1.5.9. *([JO94], [COS08])*

$$S(2,n)^{\mathrm{ab}} = \begin{cases} 1 & if\ (n,6) = 1 \\ \mathbb{Z}_3 & if\ (n,6) = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & if\ (n,6) = 3 \\ \mathbb{Z} \oplus \mathbb{Z} & if\ (n,6) = 6 \end{cases}$$

1.5.3 Isomorphisms Problems of $G_n(m,k)$.

For any $n \geq 2$, let $f(n)$ denotes the number of isomorphisms types among the irreducible groups $G_n(m,k)$ and $g(n)$ denotes the number of abelianization isomorphisms (note that $f(n)$ is different from $f(t)$ in (1.4)). The general result on isomorphisms is the following theorem, which is a corrected simplification of [BV03, Theorem 1.1]. Although developed independently, it is in fact a generalization of [GH95, Lemma 2.1] which deals with the case $k = k' = 1$.

Theorem 1.5.10. *[COS08, Theorem 2] Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups and assume that $(n,k') = 1, (n,m-k) = 1$. If $m'(m-k) \equiv mk' \bmod n$ then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.*

Lemma 1.5.11. *[BV03, Lemma 1.3]*

$$G_n(m,k) \cong G_n(n-m, n-m+k).$$

By Lemma 1.5.11 we may also assume $m < k$.

Further isomorphisms are given in the following proposition.

Proposition 1.5.12. *[COS08, Proposition 6.]*

1. $G_n(m,k) \cong G_n(m, n+m-k) \cong G_n(n-m, n-m+k)$.

2. *If* $(n,t) = 1$, *then* $G_n(m,k) \cong G_n(mt,kt)$.

3. $G_{2h}(2h-1, 2h-2) \cong G_{2h}(2h-1, 1) \cong G_{2h}(1, 2h-1) \cong G_{2h}(1,2) \cong F(2,2h)$.

The following tells us that in certain cases $G_n(m,k)$ is isomorphic to a Gilbert-Howie group.

Lemma 1.5.13. *[BV03, Lemma 1.3.] - [Wil09, Lemma 3.4]*

a. *If* $(n,k) = 1$ *then* $G_n(m,k) \cong H(n,t)$ *where* $tk \equiv m \bmod n$;

b. *If* $(n,k-m) = 1$ *then* $G_n(m,k) \cong G_n(t,1) = H(n,t)$ *where* $t(k-m) \equiv n-m \bmod n$.

Similarly, if $(n,m) = 1$ then $G_n(m,k) \cong G_n(1,k')$ where $k' = kt$ where $tm \equiv 1 \bmod n$.

The following conjecture was stated in [COS08]

Conjecture 1.5.14. *[COS08, Conjecture 8] If $n = p^l$ for an odd prime p and positive integer l, then $f(n) = p^l - \frac{(p-1)}{2}p^{(l-1)} - 1$. If $l > 2$ then $f(2^l) = 3(2^{l-2})$.*

We will show in Chapter 6 that the values given in Conjecture 1.5.14 are upper bound for $f(n)$.

The following questions are from [BV03]

Question 1.5.15. *[BV03, Question 2] Is it possible to compute the function $f(n)$ which, given an integer $n \geq 3$, yields the number of pairwise non isomorphic groups $G_n(m,k)$, where $0 < m < k < n$.*

Question 1.5.16. *[BV03, Question 5] Can groups $G_n(m,k)$ and $G'_n(m',k')$ be isomorphic for $n \neq n'$?*

As in Theorem 1.5.7, the abelianization of the Fibonacci groups $F(2,n)$ is finite and its order is equal to $L_n - 1 - (-1)^n$, where L_n is Lucas number and appears in [Joh80] in the form $f_n - 1 - (-1)^n$, where f_n is a Fibonacci number.

Question 1.5.17. *[BV03, Question 6] Does there exist a similar formula which gives the order of an abelianization of $G_n(m,k)$ if the abelianization is finite? Can such a formula be given in terms of numbers generalizing the fibonacci numbers?*

Further results on isomorphisms of cyclically presented groups appear in [BP16]. We give in Chapter 4 lower bounds for $f(n)$ of $G_n(m,k)$ groups and for certain values of n. In Chapter 5 we investigate when $\Gamma_n(k,l) \cong \Gamma'_n(k',l')$ requires $n = n'$ and when it does not. In Chapter 3 we produce formulas that compute the order of $G_{pm}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}, G_{pk}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}$ for certain values of p where $(m,k) = 1$. In Chapter 5 we produce a formula that compute $|\Gamma_n(1, \frac{n}{2} - 1)^{\mathrm{ab}}|$ when $n \equiv 2$ or $4 \bmod 6$, the formula includes Lucas number. Also in Chapter 6 we give results about the number of non-isomorphic $\Gamma_n(k,l)$ groups.

1.5.4 Investigating $G_n(m,k)$ for small values of n

Cavicchioli, O'Brien and Spaggiari in [COS08] investigated isomorphisms among the irreducible groups $G_n(m,k)$. They obtained the value of $f(n)$ for all $n \leq 27$. They did this by using isomorphism results to obtain an upper bound on $f(n)$ and used invariants of groups to obtain a lower bound on $f(n)$. $U(n)$ denotes the least upper bound they were able to obtain, $L(n)$ denotes the greatest lower bound they were able to obtain (so when $L(n) = U(n)$ we have $f(n) = L(n) = U(n)$). For the cases $n = 17, 19, 21, 23$ they showed that

$f(17) = 7$ or 8, $f(19) = 8$ or 9, $f(21) = 15$ or 16 and $f(23) = 10$ or 11, for all other cases they gave the exact value of $f(n)$.

They summarise their results in [COS08, Table 1], which we reproduce as Table 1.1, and listed the unresolved cases in [COS08, Table 2], which we produce as Table 1.2.

Table 1.1: Lower and upper bounds for $f(n)$ for $n \leq 27$ ([COS08, Table 1]).

n	$L(n)$	$U(n)$	Parameters (m,k)
3	1	1	$(1,2)$
4	2	2	$(1,2),(2,3)$
5	2	2	$(1,k)k \in \{2,3\}$
6	5	5	$(1,k)k \in \{2,3\},(2,3),(3,4),(4,5)$
7	3	3	$(1,k)k \in \{2,3,4\}$
8	6	6	$(1,k)k \in \{2,3,4\},(2,3),(2,5),(4,5)$
9	5	5	$(1,k)k \in \{2,...,5\},(3,4)$
10	8	8	$(1,k)k \in \{2,...,5\},(2,k)k \in \{3,5\},(4,7),(5,6)$
11	5	5	$(1,k)k \in \{2,...,6\}$
12	12	12	$(1,k)k \in \{2,...,6\},(2,k)k \in \{3,7\},(3,k)k \in \{4,5\},(4,k)k \in \{5,7\},(6,7)$
13	6	6	$(1,k)k \in \{2,...,7\}$
14	11	11	$(1,k)k \in \{2,...,7\},(2,k)k \in \{3,5,7\},(4,9),(7,8)$
15	12	12	$(1,k)k \in \{2,...,8\},(3,k)k \in \{4,5,7\},(5,6),(5,7)$
16	12	12	$(1,k)k \in \{2,...,8\},(2,k)k \in \{3,5,9\},(4,5),(8,9)$
17	7	8	$(1,k)k \in \{2,...,9\}$
18	17	17	$(1,k)k \in \{2,...,9\},(2,k)k \in \{3,5,7,9\},(3,k)k \in \{4,7\},(4,11),(6,7),(9,10)$
19	8	9	$(1,k)k \in \{2,...,10\}$
20	18	18	$(1,k)k \in \{2,...,10\},(2,k)k \in \{3,5,11\},(4,k)k \in \{5,7,11\},(5,6),(5,8),(10,11)$
21	15	16	$(1,k)k \in \{2,...,11\},(3,k)k \in \{4,5,7,8\},(7,8),(7,9)$
22	17	17	$(1,k)k \in \{2,...,11\},(2,k)k \in \{3,5,7,9,11\},(4,13),(11,12)$
23	10	11	$(1,k)k \in \{2,...,12\}$
24	26	26	$(1,k)k \in \{2,...,12\},(2,k)k \in \{3,5,7,13\},(3,k)k \in \{4,5,8,10\},$ $(4,k)k \in \{5,7\},(6,k)k \in \{7,13\},(8,k)k \in \{9,13\},(12,13)$
25	14	14	$(1,k)k \in \{2,...,13\},(5,6),(5,7)$
26	20	20	$(1,k)k \in \{2,...,13\},(2,k)k \in \{3,5,7,9,11,13\},(4,15),(13,14)$
27	17	17	$(1,k)k \in \{2,...,14\},(3,k)k \in \{4,5,10\},(9,10)$

Table 1.2: Possible isomorphisms amongst $G_n(m,k)$ [COS08, Table 2].

n	Parameters (m,k)
17	(1,3), (1,4)
19	(1,3), (1,6)
21	(1,6), (1,9)
23	(1,3), (1,7)

1.5.5 Initial results

We interpreted [COS08, Table 1] and expressed in Table 1.4 as many of the groups as possible as Gilbert and Howie groups $H(n,m) = G_n(m,1)$ for $n \leq 27$. In order to obtain that table, we applied isomorphism relations of [COS08] and [Wil14], also we used computer program (Maple) to compute the abelianization of the groups. The table contains type of groups whenever we know from previous studies, which group is finite or infinite and the abelianization for each group, in addition to the values of $f(n)$ and $g(n)$. The unsolved cases will appear in Table 1.3. We give here part of the table in Table 1.4 below and the full table will appear in Table A.1 in the appendix. This table will be used in Chapter 4 in counting $G_n(m,k)$ groups.

Table 1.3: Possible isomorphisms (unsolved cases) amongst $G_n(m,1) = H(n,m)$.

n	m
17	6, 11
19	9, 15
21	4,13
23	8,10

Table 1.4: Isomorphisms classes of $G_n(m,k)$ groups for $n \leq 27$.

n	$f(n)$	$g(n)$	Groups	type of group	Details	Abelianization
3	1	1	$H(3,2)$	-	Q_8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
4	2	2	$H(4,2)$	$S(2,4)$	$SL(2,3)$	$\mathbb{Z}_3$
			$H(4,3)$	$F(2,4)$	$\mathbb{Z}_5$	$\mathbb{Z}_5$
5	2	2	$H(5,2)$	$S(2,5)$	$SL(2,5)$	1
			$H(5,3)$	$F(2,5)$	$\mathbb{Z}_{11}$	$\mathbb{Z}_{11}$
6	5	4	$H(6,2)$	$S(2,6)$	infinite	$\mathbb{Z} \oplus \mathbb{Z}$
			$H(6,3)$	-	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$	$\mathbb{Z}_7$
			$H(6,4)$	-	$\mathbb{Z}_9$	$\mathbb{Z}_9$
			$H(6,5)$	$F(2,6)$	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$
			$G_6(1,3)$	-	$\mathbb{Z}_7$	$\mathbb{Z}_7$
7	3	3	$H(7,2)$	$S(2,7)$	infinite	1
			$H(7,3)$	-	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(7,4)$	$F(2,7)$	$\mathbb{Z}_{29}$	$\mathbb{Z}_{29}$
8	6	6	$H(8,2)$	$S(2,8)$	infinite	$\mathbb{Z}_3$
			$H(8,3)$	-	group of order $3^{10}.5$	$\mathbb{Z}_5$
			$H(8,4)$	-	infinite	$\mathbb{Z}_{15}$
			$H(8,5)$	-	$\mathbb{Z}_{17}$	$\mathbb{Z}_{17}$
			$H(8,6)$	-	infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
			$H(8,7)$	$F(2,8)$	infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_{15}$
9	5	5	$H(9,2)$	$S(2,9)$	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(9,3)$	-	infinite	$\mathbb{Z}_7$
			$H(9,4)$	-	Unknown	$\mathbb{Z}_{19}$
			$H(9,5)$	$F(2,9)$	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{38}$
			$H(9,7)$	-	Unknown	$\mathbb{Z}_{37}$
10	8	5	$H(10,2)$	$S(2,10)$	Infinite	$\mathbb{Z}_3$
			$H(10,3)$	-	Infinite	$\mathbb{Z}_{11}$
			$H(10,4)$	-	Infinite	$\mathbb{Z}_{33}$
			$H(10,5)$	-	Infinite	$\mathbb{Z}_{31}$
			$H(10,6)$	-	$\mathbb{Z}_{33}$	$\mathbb{Z}_{33}$
			$H(10,7)$	-	Infinite	$\mathbb{Z}_{11}$
			$H(10,9)$	$F(2,10)$	Infinite	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$
			$G_{10}(1,5)$	-	$\mathbb{Z}_{31}$	$\mathbb{Z}_{31}$

We record here the following propositions which were suggested by Table 1.4, and do not seem to have been explicitly stated before. The main results of $G_n(m,k)$ that we obtained will be shown in Chapters 3, 4 and 5.

Proposition 1.5.18. *Let* n ≥ 10 , n $\equiv 2 \bmod 4$ *then the strongly irreducible group* $G_n(1,\frac{n}{2})$ *is not isomorphic to* $H(n,m)$ *for any* $1 \leq m \leq n-1$.

Proof. Lemma 1.5.6 implies that $G_n(1,\frac{n}{2}) \cong \mathbb{Z}_s$ where $s = 2^{n/2}-1$. Now assume that $G_n(1,n/2) \cong H(n,m) = G_n(m,1)$ therefore $G_n(m,1)$ is finite of the same order. If $G_n(m,1)$ is finite then by Theorem 1.5.5, we have either $2k \equiv 0 \bmod n$ (impossible since $k = 1, n \geq 10$) or $2(m-1) \equiv 0 \mod n$. Since $n \equiv 2 \bmod 4$ we have $(m-1)$ is odd, so m must be even. Therefore by Theorem 1.5.5 we have that $G_n(m,1) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2} = 2^{n/2}+1$, so $G_n(1,n/2) \ncong G_n(m,1)$. □

Proposition 1.5.19. *If* $G_n(m,k)$ *is strongly irreducible and finite then* $n \equiv 2$ *mod* 4 *and* $G_n(m,k) \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$.

Proof. Assume that $G_n(m,k)$ is strongly irreducible and finite then by Theorem 1.5.5 we have that n is even and

$$G_n(m,k) \cong \mathbb{Z}_{2^{\frac{n}{2}}-(-1)^{m+\frac{n}{2}}} \tag{1.14}$$

In here we need to show that $m+\frac{n}{2}$ is always even so that $G_n(m,k) \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$. Now since that n is even there are two cases

Case 1: $n \equiv 0$ *mod* 4

From Theorem 1.5.5, then either $2k \equiv 0$ *mod* n or $2(k-m) \equiv 0$ *mod* n. If $2k \equiv 0$ *mod* n, then $k = \frac{n}{2}$ (since $m \leq k \leq n$) and k is even, therefore m can not be even since if m is even, then $(n,m,k) = (2k,m,k) \geq 2$ this is contradiction with $G_n(m,k)$ irreducible. Now if m is odd then $1 = (n,m,k) = (2k,m,k) = (m,k)$ but $1 < (n,k-m) = (2k,k-m) = (k,k-m) = (k,m) = 1$ which is also contradiction since $G_n(m,k)$ is strongly irreducible.

If $2(k-m) \equiv 0$ *mod* n, $k-m = \frac{n}{2}$ and it is even so either m,k are even or odd. If m,k are even then $(n,m,k) = (2k,m,k) \geq 2$ contradict the assumption. If m,k are odd then $1 < (n,k) = (2(k-m),k) = (2(k-m),(k-m),k) = (n,m,k) = 1$ which also contradicts the assumption. Therefore when $n \equiv 0$ *mod* 4 there are no cases to consider.

Case 2: $n \equiv 2 \; mod \; 4$

If $2k \equiv 0 \; mod \; n$ then $k = \frac{n}{2}$ and it is odd, if m is even, $k - m$ is odd and then by strong irreducibility we have $1 < (n, k-m) = (2k, k-m) = (k, k-m)$, but $(k, k-m) = (k, m) = 1$ this is a contradiction. Thus m is odd, so $m + \frac{n}{2}$ is even and the results follows from equation (1.14).

If $2(k - m) \equiv 0 \; mod \; n$ therefore $k - m = \frac{n}{2}$ and it is odd so either k is even, m is odd or vice versa. If k is odd, m is even then $1 < (n, k) = (2(k - m), k) = (k - m, k) = (m, k)$ but $1 = (n, m, k) = (2(k - m), m, k) = (m, k) > 1$ which contradicts the assumption. If k is even, m is odd then $m + \frac{n}{2}$ is even and $G_n(m, k) \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$. □

Proposition 1.5.20. *Let n be even, then $H(n, \frac{n}{2} + 1) \cong \mathbb{Z}_s$ where $s = 2^{\frac{n}{2}} + 1$.*

Proof. By Lemma 1.5.6, since $(m, k) = 1, k \not\equiv 0$ mod $n, k \not\equiv m$ mod n and $2(k - m) = 2(1 - n/2 - 1) = -n \equiv 0 \; mod \; n$ then $G_n(m, k) = G_n(n/2 + 1, 1) = H(n, n/2 + 1) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$. Now since $m + n/2 = n/2 + 1 + n/2 = n + 1$ (which is odd) then $s = 2^{n/2} + 1$. □

1.6 $\Gamma_n(k, l)$ groups

We also continue an investigation that was carried by Edjvet and Williams in [EW10] into the cyclic presentation

$$P_n(k, l) = \langle x_0, x_1, ..., x_{n-1} \mid x_i x_{i+k} x_{i+l} = 1, \; i = 0, 1, ..., n-1 \rangle$$

and the group $\Gamma_n(k, l)$, where $1 \leq k, l \leq n-1$ and subscripts are taking mod n. They described the groups' structures, and stated their results in terms of the following conditions

(A) (A)$n \equiv 0 \, (\mathrm{mod}\, 3)$ and $k + l \equiv 0 \, (\mathrm{mod}\, 3)$.

(B) $k + l \equiv 0 \, (\mathrm{mod}\, n)$ or $2l - k \equiv 0 \, (\mathrm{mod}\, n)$ or $2k - l \equiv 0 \, (\mathrm{mod}\, n)$.

(C) $3l \equiv 0 \, (\mathrm{mod}\, n)$ or $3k \equiv 0 \, (\mathrm{mod}\, n)$ or $3(l - k) \equiv 0 \, (\mathrm{mod}\, n)$.

(D) $2(k + l) \equiv 0 \, (\mathrm{mod}\, n)$ or $2(2l - k) \equiv 0 \, (\mathrm{mod}\, n)$ or $2(2k - l) \equiv 0 \, (\mathrm{mod}\, n)$.

They summarized their results for $(n,k,l)=1, k\neq l$ in terms of three conditions $(A),(B),(C)$ being true or false in [EW10, Table 1], which we reproduce as Table 1.5, (where $\alpha = 3(2^{n/3}-(-1)^{n/3}), \gamma = (2^{n/3}-(-1)^{n/3})/3)$), and they denoted by ∞ the group of infinite order whose structure is unknown, Metacyclic denotes a metacyclic group of order $s = 2^n - (-1)^n$ (G is called metacyclic if it has a normal subgroup H such that both H and G/H are cyclic), Large denotes a large group (that is, one that has a finite index subgroup that maps homomorphically onto the free group of rank 2).

Table 1.5: *Summary of structures of* $\Gamma_n(k,l)$ *[EW10, Table 1]*

(A)	(B)	(C)		Aspherical	Abelianization	Group
F	F	F		Yes	finite$\neq 1$	∞
F	F	T		No	$\mathbb{Z}_\alpha$	Metacyclic
F	T	F		No	$\mathbb{Z}_3$	$\mathbb{Z}_3$
T	F	F	$n\neq 18$	Yes	∞	Large
T	F	F	$n=18$	No	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_{19}$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_{19}$
T	F	T		No	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$
T	T	F		No	$\mathbb{Z}*\mathbb{Z}$	$\mathbb{Z}*\mathbb{Z}$
T	T	T		No	$\mathbb{Z}*\mathbb{Z}$	$\mathbb{Z}*\mathbb{Z}$

The following lemma was proved in [EW10] and gives isomorphisms between $\Gamma_n(k,l)$ groups

Lemma 1.6.1. *[EW10, Lemma 2.1.] Let* $1\leq k,l\leq n-1$ *then*

1. $\Gamma_n(k,l)\cong\Gamma_n(l-k,-k)$.
2. $\Gamma_n(k,l)\cong\Gamma_n(l,k)$.
3. $\Gamma_n(k,l)\cong\Gamma_n(k-l,-l)$.
4. $\Gamma_n(k,l)\cong\Gamma_n(k,k-l)$.
5. *If* $(k,n)=1$ *then* $\Gamma_n(k,l)\cong\Gamma_n(1,Kl)$, *where* $Kk\equiv 1(\bmod n)$.
6. *If* n *is even and* $(l,n)=1$ *then* $\Gamma_n(k,l)\cong\Gamma_n(1,Lk+1)$, *where* $Ll\equiv -1(\bmod n)$.

Corollary 1.6.2. *[EW10, Corollary 5.2.] Suppose that* $(n,k,l)=1, k\neq l$. *If non of* $(B),(C),(D)$ *hold then* $\Gamma_n(k,l)$ *contains a non-abelian free subgroup.*

1.7 Thesis outline

In Chapter 2, we generalize an isomorphism theorem for the class of groups $G_n(m,k)$, which was proved in [BV03, Theorem 1.1.] and updated in [COS08, Theorem 2]. We generalize this result to the class of groups $G_n(h,k,p,q,r,s,\ell)$ which were introduced in [CRS03]. We also have identified a mistake in the proof of isomorphism theorem was asserted in [CRS05], about the groups $G_n^{\varepsilon}(m,k,h)$, and we provide a new isomorphism theorem for that group.

In Chapter 3, we give an answer for the first part of Question 1.5.17. We produce formulas that compute the order of $G_{pm}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}, G_{pk}(x_0x_mx_k^{\pm 1})^{\mathrm{ab}}$ for certain values of p where $(m,k)=1$. Similar formulas were given in [BW16, Corollary 4.5] for $p \in \{4,6\}$. We give formulas for $p \in \{2,3,4,6,12\}$. We use these formulas in Chapter 4 to determine lower bounds for the number of non-isomorphic $G_n(m,k)$ groups for certain values of n, and in Chapter 5 to compute $|\Gamma_n(1,\frac{n}{2}-1)^{\mathrm{ab}}|$.

In Chapter 4, we give an answer for Question 1.5.15. For certain values of n we calculate lower bounds for the minimum number of generators of $G_n(m,k)^{\mathrm{ab}}$ and we use this with the finiteness classification of $G_n(m,k)$ and the order of $G_n(m,k)^{\mathrm{ab}}$ to give lower bounds for the number of non isomorphic $G_n(m,k)$ groups for certain values of n.

In Chapter 5, we give an answer for Question 1.5.15 when it considers $\Gamma_n(k,l)$ groups instead of $G_n(m,k)$. We count $\Gamma_n(k,l)$ groups, and give lower bounds of the number of non isomorphic $\Gamma_n(k,l)$ groups for certain values of n. These results suggest that the groups $\Gamma_n(1,\frac{n}{2}-1)$ deserve further study. We obtain results concerning their abelianization, and in relation to Question 1.5.17 we provide a formula for the order $|\Gamma_n(k,l)^{\mathrm{ab}}|$ in terms of Lucas numbers (where this abelianization is finite).

In Chapter 6, we prove that the values given in Conjecture 1.5.14 provide an upper bound for $f(n)$ of the groups $G_n(m,k)$ where $n=p^l, p$ is prime. We also give results about the upper bound of $f(n)$ of $\Gamma_n(k,l)$ groups, for $n=p^{\alpha}q^{\beta}$, and $n=p^{\alpha}q^{\beta}r^{\gamma}$, where p,q and r are distinct primes We carry out a similar study of $G_n(m,k)$ groups in [COS08] for $\Gamma_n(k,l)$ groups. We produce a table similar to Table 1.1 for $\Gamma_n(k,l)$ groups for $n \leq 29$.

Chapter 2

Isomorphism Theorems

2.1 Introduction

Our starting point for this chapter is the following isomorphism theorem for the class of groups $G_n(m,k)$, which is Theorem 1.5.10.

Theorem 2.1.1. *[COS08, Theorem 2] Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups and assume that $(n,k')=1$. If $m'(m-k) \equiv mk' \bmod n$ then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.*

A version of this result was initially proved in [BV03, Theorem 1.1.]. In [COS08, Theorem 2] it was observed that the hypothesis $(n,k')=1$ (missing from the original statement) is necessary. Following a comment from the referee of [COS08] this formulation was obtained. In Section 2.2, we generalize this result to the class of groups $G_n(h,k,p,q,r,s,\ell)$ which were introduced in [CRS03]. In Section 2.3, we consider the groups $G_n^{\epsilon}(m,k,h)$ which were considered in [CRS05]. We have identified a mistake in the proof of isomorphism theorem [CRS05, Theorem 2.6.], we show why the proof is wrong, and we provide a corrected version for these groups. More information about $G_n(h,k,p,q,r,s,\ell), G_n^{\epsilon}(m,k,h)$ groups can be seen in Section 1.3.

2.2 $G_n(h,k,p,q,r,s,\ell)$ groups

Recall from the introduction that, let $r \geq 2, s \geq 1, 0 \leq p,q,h \leq n-1, \ell,k \in \mathbb{Z}$, we define the group $G_n(h,k,p,q,r,s,\ell)$ to be the group

$$G_n((\prod_{j=0}^{r-1} x_{jp})^{\ell}(\prod_{j=0}^{s-1} x_{h+jq})^{-k}) = \langle x_0, x_1, ..., x_{n-1} \mid (x_i x_{i+p} ... x_{i+p(r-1)})^{\ell} =$$
$$(x_{i+h} x_{i+h+q} ... x_{i+h+q(s-1)})^k, i = 0, ..., n-1 \rangle$$

Note: unlike in [CRS03], we allow $\ell, k < 0$. We obtain a condition under which the groups $G_n(h,k,p,q,r,s,\ell), G_n(h',k,p',q',r,s,\ell)$ are isomorphic.

It is convenient to express our result in terms of parameters A, B, A', B' where

$$A = h, B = -p(r-1) + A + q(s-1), A' = h', B' = -p'(r-1) + A' + q'(s-1) \tag{2.1}$$

Our proof of the following theorem follows the method of the proof of Theorem 2.1.1 [CRS03, BV03].

Theorem 2.2.1. *If* $(n, A) = 1, (n, B') = 1$, $p'A \equiv -pB'$ *mod* n, $q'A \equiv -qB'$ *mod* n, *then* $G_n(h,k,p,q,r,s,\ell) \cong G_n(h',k,p',q',r,s,\ell)$.

Proof. Since $(n, B') = 1$ there exist integers $\alpha, \beta \in \mathbb{Z}$ such that $\alpha n + \beta B' = 1$ therefore $\beta B' \equiv 1$ mod n. Now by setting $f = \beta p'$, the condition $p'A \equiv -pB'$ mod n implies that

$$\begin{aligned} fA &\equiv \beta p' A \mod n \\ &\equiv -\beta p B' \mod \text{n} \qquad \text{by assumption} \\ fA &\equiv -p \mod \text{n}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} fB' &\equiv \beta p' B' \mod \text{n} \qquad \text{by assumption} \\ fB' &\equiv p' \mod \text{n}. \end{aligned} \tag{2.3}$$

Similarly by setting $g = \beta q'$,the condition $q'A \equiv -qB' \bmod n$ implies that

$$\begin{aligned} gA &\equiv \beta q' A \bmod n \\ &\equiv -\beta q B' \bmod \mathrm{n} \qquad \text{by assumption} \\ gA &\equiv -q \bmod \mathrm{n}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} gB' &\equiv q'B'\beta \bmod \mathrm{n} \qquad \text{by assumption} \\ gB' &\equiv q' \bmod \mathrm{n}. \end{aligned} \tag{2.5}$$

We will use conditions (2.2), (2.3), (2.4), (2.5) to complete the proof. Now let us consider the group

$$G_n(h,k,p,q,r,s,\ell) \cong \langle x_0, x_1, ..., x_{n-1} \mid (x_i x_{i+p} ... x_{i+p(r-1)})^\ell = (x_{i+A} x_{i+A+q} ... x_{i+A+q(s-1)})^k, i = 0, ..., n-1 \rangle$$

Inverting the relations gives

$$\begin{aligned} &\cong \langle x_0, x_1, ..., x_{n-1} \mid (x^{-1}_{i+p(r-1)} ... x^{-1}_{i+p} x^{-1}_i)^\ell = \\ &\qquad (x^{-1}_{i+A+q(s-1)} ... x^{-1}_{i+A+q} x^{-1}_{i+A})^k, i = 0, 1, ..., n-1 \rangle \\ &put\, c_i = x_i^{-1} \\ &\cong \langle c_0, c_1, ..., c_{n-1} \mid (c_{i+p(r-1)} ... c_{i+p} c_i)^\ell = \\ &\qquad (c_{i+A+q(s-1)} ... c_{i+A+q} c_{i+A})^k, i = 0, 1, ..., n-1 \rangle \\ &put\, j = i + p(r-1) \end{aligned}$$

$$\begin{aligned} G_n(h,k,p,q,r,s,\ell) &\cong \langle c_0, c_1, ..., c_{n-1} \mid (c_j ... c_{j-p(r-2)} c_{j-p(r-1)})^\ell = \\ &\qquad (c_{j-p(r-1)+A+q(s-1)} ... c_{j-p(r-1)+A+q} c_{j-p(r-1)+A})^k, j = 0, 1, ..., n-1 \rangle \\ &\cong \langle c_0, c_1, ..., c_{n-1} \mid (c_j ... c_{j-p(r-2)} c_{j-p(r-1)})^\ell = \\ &\qquad (c_{j+B} ... c_{j+B-q(s-2)} c_{j+B-q(s-1)})^k, j = 0, 1, ..., n-1 \rangle \end{aligned}$$

Since $(n, A) = 1$ there exist $\zeta, \delta \in \mathbb{Z}$ such that $\zeta n + \delta A \equiv 1 \bmod n$ therefore $\delta A \equiv 1 \bmod n$, now for each $j = 0, 1, ..., n-1$ set $u \equiv \delta j \bmod n$ so $c_{uA} = c_{\delta jA} = c_{j\delta A} = c_j$. Now we write

$$\begin{aligned} G_n(h,k,p,q,r,s,\ell) &\cong \langle c_0, c_1, ..., c_{n-1} \mid (c_{uA}...c_{uA-p(r-2)}c_{uA-p(r-1)})^{\ell} = \\ &\qquad (c_{uA+B}c_{uA+B-q}...c_{uA+B-q(s-2)}c_{uA+B-q(s-1)})^k, u = 0, 1, ..., n-1\rangle \\ &\cong \langle c_0, c_1, ..., c_{n-1} \mid \Big(\prod_{\gamma=0}^{r-1} c_{uA-\gamma p}\Big)^{\ell} = \Big(\prod_{\gamma=0}^{s-1} c_{uA+B-\gamma q}\Big)^k, u = 0, ...(n-1)\rangle \end{aligned} \tag{2.6}$$

Now let us consider the group

$$\begin{aligned} G_n(h',k,p',q',r,s,\ell) \cong \langle y_0, y_1, ..., y_{n-1} \mid (y_i y_{i+p'} ... y_{i+p'(r-1)})^{\ell} = \\ (y_{i+A'} y_{i+A'+q'} ... y_{i+A'+q'(s-1)})^k,\ i = 0, ..., n-1\rangle \end{aligned}$$

since $(n, B') = 1$ then there exist $\alpha', \beta' \in \mathbb{Z}$ such that $\alpha' n + \beta' B' = 1$ therefore $\beta' B' \equiv 1 \bmod n$. Now for each $i = 0, 1, ..., n-1$ set $v \equiv \beta' i \bmod n$ so $y_{vB'} = y_{\beta' iB'} = y_{i\beta' B'} = y_i$. so we could write

$$\begin{aligned} G_n(h',k,p',q',r,s,\ell) &\cong \langle y_0, y_1, ..., y_{n-1} \mid (y_{vB'} y_{vB'+p'} ... y_{vB'+p'(r-1)})^{\ell} = \\ &\qquad (y_{vB'+A'} y_{vB'+A'+q'} ... y_{vB'+A'+q'(s-1)})^k, i = 0, ..., n-1\rangle \\ &\cong \langle y_0, y_1, ..., y_{n-1} \mid \Big(\prod_{\gamma=0}^{r-1} y_{vB'+p'\gamma}\Big)^{\ell} = \Big(\prod_{\gamma=0}^{s-1} y_{vB'+q'\gamma+A'}\Big)^k, v = 0, ...(n-1)\rangle \end{aligned} \tag{2.7}$$

Now define a map as follow

$$\Phi : \{c_0, ..., c_{n-1}\} \longrightarrow \{y_0, ..., y_{n-1}\}$$

acting on the set of generators $\{c_j | j = 0, ..., n-1\} = \{c_{uA} | u = 0, ..., n-1\}$ by the rule

$$\Phi(c_{uA}) = y_{uB'}$$

For each $\gamma = 0, ..., r-1$

$$\begin{aligned}
\Phi(c_{uA-\gamma p}) &= \Phi(c_{uA+\gamma fA}) \qquad \text{using (2.2)}\\
&= \Phi(c_{(u+\gamma f)A})\\
&= y_{(u+\gamma f)B'}\\
&= y_{uB'+\gamma fB'}\\
&= y_{uB'+\gamma p'} \qquad \text{using (2.3)}
\end{aligned}$$

and for each $\gamma = 0, ..., s-1$

$$\begin{aligned}
\Phi(c_{uA+B-\gamma q}) &= \Phi(c_{uA-p(r-1)+A+q(s-1)+g\gamma A}) \qquad \text{using (2.1), (2.4)}\\
&= \Phi(c_{uA+fA(r-1)+A-gA(s-1)+g\gamma A}) \qquad \text{using (2.2), (2.4)}\\
&= \Phi(c_{(u+f(r-1)+1-g(s-1)+g\gamma)A})\\
&= y_{(u+f(r-1)+1-g(s-1)+g\gamma)B'}\\
&= y_{uB'+fB'(r-1)+B'-gB'(s-1)+gB'\gamma}\\
&= y_{uB'+p'(r-1)-p'(r-1)+A'+q'(s-1)-q'(s-1)+q'\gamma} \qquad \text{using (2.3), (2.5), (2.1)}\\
&= y_{uB'+q'\gamma+A'}
\end{aligned}$$

Comparing presentations (2.6), (2.7) it is clear that for each $0 \le \gamma \le (n-1)$ that Φ is an epimorphism.

Now let us define a map as follow

$$\Theta : \{y_0, ..., y_{n-1}\} \longrightarrow \{c_0, ..., c_{n-1}\}$$

acting on the set of generators $\{y_j | j = 0, ..., n-1\} = \{y_{vB'} | v = 0, ..., n-1\}$ by the rule

$$\Theta(y_{vB'}) = c_{vA}$$

For each $\gamma = 0, ..., r-1$

$$\begin{aligned}
\Theta(y_{vB'+\gamma p'}) &= \Theta(y_{vB'+\gamma fB'}) \qquad \text{using (2.3)} \\
&= \Theta(y_{(v+\gamma f)B'}) \\
&= c_{(v+\gamma f)A} \\
&= c_{vA+\gamma fA} \\
&= c_{vA-p\gamma} \qquad \text{using (2.2)}
\end{aligned}$$

and for each $\gamma = 0, ..., s-1$

$$\begin{aligned}
\Theta(y_{vB'+q'\gamma+A'}) &= \Theta(y_{vB'+gB'\gamma+B'+p'(r-1)-q'(s-1)}) \qquad \text{using (2.5), (2.1)} \\
&= \Theta(y_{vB'+gB'\gamma+B'+fB'(r-1)-gB'(s-1)}) \qquad \text{using (2.3), (2.5)} \\
&= \Theta(y_{(v+g\gamma+1+f(r-1)-g(s-1))B'}) \\
&= c_{(v+g\gamma+1+f(r-1)-g(s-1))A} \\
&= c_{(vA+gA\gamma+A+fA(r-1)-gA(s-1))} \\
&= c_{(vA-q\gamma+A-p(r-1)+q(s-1))} \qquad \text{using (2.2), (2.4)} \\
&= c_{vA+B-q\gamma} \qquad \text{using (2.1)}
\end{aligned}$$

Comparing presentations (2.6), (2.7) it is clear that for each $0 \leq \gamma \leq (n-1)$ that $\Theta : G_n(h',k,p',q',r,s,\ell) \to G_n(h,k,p,q,r,s,\ell)$ is an epimorphism and $\phi(\Theta(y_{uB'})) = \phi(c_{uA}) = y_{uB'}$ and $\Theta(\phi(c_{uA})) = \Theta(y_{uB'}) = c_{uA}$, therefore $\phi^{-1} = \Theta, \Theta^{-1} = \phi$. thus the composition of epimorphisms $G_n(h,k,p,q,r,s,\ell) \underset{\rightarrow}{\Phi} G_n(h',k,p',q',r,s,\ell) \underset{\rightarrow}{\Theta} G_n(h,k,p,q,r,s,\ell)$ shows that

$$G_n(h,k,p,q,r,s,\ell) \cong G_n(h',k,p',q',r,s,\ell)$$ □

Now we will apply Theorem 2.2.1 to classes of cyclic presentations of groups which were considered previously, and have been shown in [CRS03] as special cases of the groups $G_n(h,k,p,q,r,s,\ell)$. see Section 1.2 for definition of the groups in the following corollaries.

Corollary 2.2.2. *Suppose* $(n,s) = 1$, *then* $F(r,s,c,n) \cong F(r, r-s-1, c, n)$.

Proof. $F(r,s,c,n) = G_n(s,c,1,1,r,1,1) = G_n(x_0x_1...x_{r-1}x_s^{-c})$, then from (2.1) we have $A = s, B = s-r+1$. Similarly $F(r, r-s-1, c, n) = G_n(r-s-1, c, 1, 1, r, 1, 1) = G_n(x_0x_1...x_{r-1}x_{r-s-1}^{-c})$, then from

(2.1) we have $A' = r-s-1, B' = (r-s-1)-(r-1) = -s$, and in the notation $G_n(h,k,p,q,r,s,\ell)$ we have $p = p' = q = q' = 1$. since $(n, A) = (n, s) = 1, (n, B') = (n, -s) = 1$ and $p'.A = 1.s = s, -pB' = -1.(-s) = s$, so $p'A \equiv -pB'$ mod n, also $q'.A = 1.s = s, -qB' = -1.(-s) = s$, so $q'A \equiv -qB'$ mod n. Therefore Theorem 2.2.1 implies that $F(r,s,c,n) \cong F(r, r-s-1, c, n)$. □

Corollary 2.2.3. *Suppose* $(n, r+k-1) = 1$, *then* $F(r,n,k) \cong F(r,n,1-r-k)$.

Proof. $F(r,n,k) = G_n(r+k-1, 1, 1, 1, r, 1, 1)$, then from (2.1) we have $A = r+k-1, B = (r+k-1)-(r-1) = k$. Similarly $F(r,n,1-r-k) = G_n(-k,1,1,1,r,1,1)$, then from (2.1) we have $A' = -k, B' = (-k)-(r-1) = 1-r-k$, and in the notation $G_n(h,k,p,q,r,s,\ell)$ we have $p = p' = q = q' = 1$. Since $(n, A) = (n, r+k-1) = 1, (n, B') = (n, -(r+k-1)) = 1$ and $p'.A = 1.(r+k-1) = r+k-1, -pB' = -1.(1-r-k) = r+k-1$, so $p'A \equiv -pB'$ mod n, also $q'.A = 1.(r+k-1) = r+k-1, -qB' = -1.(1-r-k) = r+k-1$, so $q'A \equiv -qB'$ mod n. Theorem 2.2.1 implies that $F(r,n,k) \cong F(r,n,1-r-k)$. □

Corollary 2.2.4. *Suppose* $(n, k-1) = 1$, $(n, k'-1-q'(r-s)) = 1, q'(k-1) \equiv -q(k'-1-q'(r-s))$ *mod* n, *then* $P(r,n,k,s,q) \cong P(r,n,k',s,q')$.

Proof. $P(r,n,k,s,q) = G_n(k-1,1,q,q,r,s,1)$, then from (2.1) we have $A = k-1, B = -q(r-1) + k-1+q(s-1) = k-q(r-s)-1$. Similarly $P(r,n,k',s,q') = G_n(k'-1,1,q',q',r,s,1)$, then from (2.1) we have $A' = k'-1, B' = -q'(r-1)+k'-1+q'(s-1) = k'-1-q'(r-s)$. By assumptions we have $(n, A) = (n, k-1) = 1, (n, B') = (n, k'-1-q'(r-s)) = 1$ and $q'A = q'(k-1), -qB' = -q(k'-1-q'(r-s))$, so $q'A \equiv -qB'$ mod n, and since $p = q, p' = q'$ we have $p'A \equiv -pB'$ mod n, therefore Theorem 2.2.1 implies that $G_n(k-1,1,q,q,r,s,1) \cong G_n(k'-1,1,q',q',r,s,1)$ and then $P(r,n,k,s,q) \cong P(r,n,k',s,q')$. □

Let $K = k-1, K' = k'-1, q = m, q' = m'$ In Corollary 2.2.4, then we have the following corollary

Corollary 2.2.5. *Suppose* $(n, K) = 1$, $(n, K'-m'(r-s)) = 1, m'K \equiv -m(K'-m'(r-s))$ *mod* n, *then*

$$P(r,n,K+1,s,m) \cong P(r,n,K'+1,s,m').$$

Now put $r = 2, s = 1$ we get the following corollary

Corollary 2.2.6. *[COS08, Theorem 2] Suppose $(n,K) = 1$, $(n,K' - m') = 1, m'K \equiv -m(K' - m')$ then $G_n(m,K)$ is isomorphic to $G_n(m',K')$.*

Proof. $G_n(m,k) = P(2,n,K+1,1,m)$, and $G_n(m',k') = P(2,n,K'+1,1,m')$, so the result follows by putting $r = 2, s = 1$ in Corollary 2.2.5. □

Corollary 2.2.7. *[GH95, Lemma 2.1.] Let n,t be non-negative integers with $n > t$ such that $(n,t-1) = 1$. Let s satisfy $0 \le s < n$ and $t \equiv (t-1)s$ mod n then $H(n,t) \cong H(n,s)$.*

Proof. Since $H(n,t) = G_n(t,1)$ the proof follows from proof of Corollary 2.2.6, by setting $m = s, K = 1, m' = t, K' = 1$. □

Corollary 2.2.8. *Suppose $(n,l) = 1, (n,l' - k') = 1$ and $k'l \equiv -k(l' - k')$ mod n, then $\Gamma_n(k,l) \cong \Gamma_n(k',l')$.*

Proof. $\Gamma_n(k,l) = G_n(l,-1,k,0,2,1,1) = \Gamma_n(x_0x_kx_l)$, and by definition (2.1) we have $A = l, B = l-k, p = k, q = 0$, we also have $\Gamma_n(k',l') = G_n(l',-1,k',0,2,1,1) = \Gamma_n(x_0x_{k'}x_{l'})$, and $A' = l', B' = l' - k', p' = k', q' = 0$. By hypotheses we have $(n,A) = (n,l) = 1, (n,B') = (n,l'-k') = 1$, and since $p'A = k'l, -pB' = -k(l'-k')$ therefore $p'A \equiv -pB'$ mod n, also since $q = q' = 0$ therefore $q'A \equiv -qB'$ mod n. Theorem 2.2.1 implies that $\Gamma_n(m,k) \cong \Gamma_n(m',k')$. □

2.3 $G_n^\varepsilon(m,k,h)$ groups

Recall from the introduction, we have the groups $G_n^\varepsilon(m,k,h)$. For $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4, n \ge 2$, m,k and h are modulo n, the group $G_n^\varepsilon(m,k,h)$ defined to be

$$G_n^\varepsilon(m,k,h) = \langle x_0, x_1, ..., x_{n-1} \mid x_i^a x_{i+k}^b x_{i+h+m}^a = (x_{i+h}^r x_{i+m}^r)^s, i = 0, ..., n-1 \rangle$$

The following isomorphism theorem was asserted in [CRS05]

Theorem 2.3.1. *[CRS05, Theorem 2.6.] Suppose that $\rho = gcd(n, k-h-m)$ divides k' and there* *exist positive integers α, β, γ and δ such that*

$$\begin{aligned} \alpha + \beta(k-h-m) &\equiv 1-m \mod n, \\ \gamma + \delta(k-h-m) &\equiv 1-h \mod n, \\ \alpha + \beta k' &\equiv 1+m' \mod n, \\ \gamma + \delta k' &\equiv 1+h' \mod n, \end{aligned}$$

where $1 \le \alpha, \gamma \le \rho$ and $1 \le \beta, \delta \le \frac{n}{\rho}$. Then $G_n^\varepsilon(m, n-(h+m), h) \cong G_n^\varepsilon(m', 2(h'+m'), h')$.

We have identified a mistake in the proof of Theorem 2.3.1 given in [CRS05]. In the next example we show why the proof is wrong, we consider the groups $G_6^\varepsilon(1,3,0), G_6^\varepsilon(3,4,0)$ where $\varepsilon = (1,1,2,1)$. This example was also used in [COS08] to highlight a mistake in [BV03, Theorem1.1.].

Example 2.3.2. Let $n = 6, m = 1, k = 3, h = 0, n = 6, m' = 3, k' = 4, h' = 0$ and $a = b = s = 1, r = 2$ therefore $\varepsilon = (a,b,r,s) = (1,1,2,1), \rho = (n, k-h-m) = 2$ divides k' and the integers $\alpha = 2, \beta = 2, \gamma = 1, \delta = 3$ satisfy $1 \le \alpha, \gamma \le \rho$ and $1 \le \beta, \delta \le \frac{n}{\rho}$, and imply that

$$\begin{aligned} \alpha + \beta(k-h-m) &\equiv 1-m \bmod n \\ \gamma + \delta(k-h-m) &\equiv 1-h \bmod n \\ \alpha + \beta k' &\equiv 1+m' \bmod n \\ \gamma + \delta k' &\equiv 1+h' \bmod n \end{aligned}$$

Then [CRS05, Theorem 2.6.] gives that $G_6^\varepsilon(1,3,0) \cong G_6^\varepsilon(3,4,0)$, but that is wrong since it is known that $\mathbb{Z}_7 \cong G_6^\varepsilon(1,3,0) \not\cong G_6^\varepsilon(3,4,0) \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, see example [COS08, Page 3]. We now explain where the mistake in the proof in [CRS05] occurs.

The proof in [CRS05] starts as follow: The group $G_n^\varepsilon(m,k,h) = G_6^\varepsilon(1,3,0)$ has a finite presentation with generators $y_0, \dots, y_5$, and defining relations $y_i y_{i+5} = y_{i+2}$ for $i = 0, 2, \dots, 5$. we set $\ell = \frac{n}{\rho} = 3$. Then we separate the generators $y_0, \dots, y_5$ into $\rho = 2$ sets A_1, A_2 of $\ell = 3$ elements each, where $A_j = \{y_j, y_{j+k-h-m}, \dots, y_{j+(\ell-1)(k-h-m)}\}$ therefore $A_0 = \{y_0, y_2, y_4\}, A_1 = \{y_1, y_3, y_5\}$. This gives a partition of the relations into $\rho = 2$ sets R_1, R_2 of $\ell = 3$ elements each one, we got $R_0 = \{y_0 y_5 = y_2, y_2 y_1 = y_4, y_4 y_3 = y_0\}$, $R_1 = \{y_1 y_0 = y_3, y_3 y_2 = y_5, y_5 y_4 = y_1\}$

Let us consider $G_n^\varepsilon(m',k',h') = G_6^\varepsilon(3,4,0)$ with generators $z_0, ..., z_5$, and defining relations $z_i z_{i+3} = z_{i+4}$. As we did above we separate the generators $z_0, ..., z_5$ into $\rho = 2$ sets B_1, B_2 of $\ell = 3$ elements each one, where $B_j = \{z_j, z_{j+k'}, ..., z_{j+(\ell-1)(k')}\}$ therefore $B_0 = \{z_0, z_4, z_2\}, B_1 = \{z_1, z_5, z_3\}$, we obtain a partition of the defining relations of $G_n^\varepsilon(m',k',h') = G_6^\varepsilon(3,4,0)$ into $\rho = 2$ sets S_1, S_2 of $\ell = 3$ elements each one, where $S_0 = \{z_0z_3 = z_4, z_4z_1 = z_2, z_2z_5 = z_0\}$, $S_1 = \{z_1z_4 = z_5, z_5z_2 = z_3, z_3z_0 = z_1\}$. They define the correspondence Ψ from $G_n^\varepsilon(m,k,h) = G_6^\varepsilon(1,3,0)$ onto $G_n^\varepsilon(m',k',h') = G_6^\varepsilon(3,4,0)$ by its action on the generators $\Psi(y_{j+\tau(k-h-m)}) = z_{j+\tau k'}$ therefore

$$\Psi(y_{j+2\tau}) = z_{j+4\tau}$$

for $0 \leq j \leq 1$ and $0 \leq \tau \leq 2$, we got $\Psi(y_0) = z_0, \Psi(y_2) = z_4, \Psi(y_4) = z_2, \Psi(y_1) = z_1, \Psi(y_3) = z_5, \Psi(y_5) = z_3$. In the proof given in [COS08, Theorem 1], it is claimed that Ψ maps each defining relation of $G_n(m,k,h) = G_6(1,3,0)$ to a defining relation of $G_n^\varepsilon(m',k',h') = G_6^\varepsilon(3,4,0)$, but this is not the case here, for example Ψ maps the relation $y_1y_0 = y_3$ to the relation $z_1z_0 = z_5$, but this is not a relation of $G_6^\varepsilon(3,4,0)$, so their claim is incorrect.

Now we provide a corrected and improved version of Theorem 2.3.1 for the group $G_n^\varepsilon(m,k,h)$ as follows. Our proof combines methods from [CRS05] and [COS08].

Theorem 2.3.3. *If* $(n, h+m) = 1, (n, h'+m') = 1$, *and*

$$\begin{aligned} m'(h+m) &\equiv m(h'+m') \bmod n, \\ h'(h+m) &\equiv h(h'+m') \bmod n. \end{aligned} \tag{2.8}$$

then $G_n^\varepsilon(m, n-(h+m), h) \cong G_n^\varepsilon(m', 2(h'+m'), h')$, *for any* $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4$.

Proof. Since $(n,(h'+m')) = 1$ there exist integers $\beta, \gamma \in \mathbb{Z}$ such that $\beta n + \gamma(h'+m') = 1$ therefore $\gamma(h'+m') \equiv 1 \bmod n$. Now by setting $f = \gamma m'$, the condition $m'(h+m) \equiv m(h'+m') \bmod n$ implies that

$$\begin{aligned} f(h+m) &\equiv \gamma m'(h+m) \bmod n \\ &\equiv \gamma m(h'+m') \bmod n \qquad \text{by (2.8)} \\ &\equiv m \bmod n \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} f(h'+m') &\equiv \gamma m'(h'+m') \bmod n \qquad \text{by (2.8)} \\ &\equiv m' \bmod n \end{aligned} \tag{2.10}$$

Similarly by setting $g=\gamma h'$,the condition $h'(h+m)\equiv h(h'+m') \bmod n$ implies that

$$\begin{aligned} g(h+m) &\equiv \gamma h'(h+m) \bmod n \\ &\equiv \gamma h(h'+m') \bmod n \qquad \text{by (2.8)} \\ &\equiv h \bmod n \end{aligned} \tag{2.11}$$

$$\begin{aligned} g(h'+m') &\equiv \gamma h'(h'+m') \bmod n \qquad \text{by (2.8)} \\ &\equiv h' \bmod n \end{aligned} \tag{2.12}$$

We will use conditions (2.9), (2.10), (2.11), (2.12) to complete the proof. Now let us consider the group

$$\begin{aligned} G_n^\varepsilon(m,-(h+m),h) &\cong \langle x_0,x_1,\dots,x_{n-1} \mid x_i^a x_{i-(h+m)}^b x_{i+(h+m)}^a = (x_{i+h}^r x_{i+m}^r)^s, i=0,\dots,n-1\rangle \\ &\quad \text{Inverting the relations gives} \\ &\cong \langle x_0,x_1,\dots,x_{n-1} \mid (x_{i+(h+m)}^{-1})^a (x_{i-(h+m)}^{-1})^b (x_i^{-1})^a = (x_{i+m}^{-r} x_{i+h}^{-r})^s, i=0,\dots,n-1\rangle \\ &\cong \langle x_0,x_1,\dots,x_{n-1} \mid (x_{i+(h+m)}^{-1})^a (x_{i-(h+m)}^{-1})^b (x_i^{-1})^a = ((x_{i+m}^{-1})^r (x_{i+h}^{-1})^r)^s, i=0,\dots,n-1\rangle \\ &\quad \mathit{Put}\ c_i = x_i^{-1} \\ &\cong \langle c_0,c_1,\dots,c_{n-1} \mid (c_{i+(h+m)})^a (c_{i-(h+m)})^b (c_i)^a = ((c_{i+m})^r (c_{i+h})^r)^s, i=0,\dots,n-1\rangle \\ &\quad \mathit{Put}\ j=i+(h+m).\ \mathit{Then} \\ G_n^\varepsilon(m,-(h+m),h) &\cong \langle c_0,c_1,\dots,c_{n-1} \mid (c_j)^a (c_{j-(h+m)-(h+m)})^b (c_{j-(h+m)})^a \\ &\qquad = ((c_{j-(h+m)+m})^r (c_{j-(h+m)+h})^r)^s, j=0,\dots,n-1\rangle, \\ &\cong \langle c_0,c_1,\dots,c_{n-1} \mid (c_j)^a (c_{j-2(h+m)})^b (c_{j-(h+m)})^a \\ &\qquad = ((c_{j-h})^r (c_{j-m})^r)^s, j=0,\dots,n-1\rangle \end{aligned}$$

Since $(n,-(h+m))=1$ there exist $\delta,\zeta\in\mathbb{Z}$ such that $\delta n+\zeta.(-(h+m))=1$ mod n therefore $\zeta.(-(h+m))\equiv 1$ mod n, now for each $j=0,1,...,n-1$ set $u\equiv\zeta j$ mod n therefore

$$c_{u(-(h+m))}=c_{\zeta j.(-(h+m))}=c_{j\zeta.(-(h+m))}=c_j,$$

we can write

$$\begin{aligned}G_n^{\varepsilon}(m,-(h+m),h)\cong\langle c_0,c_1,...,c_{n-1}\mid (c_{u.(-(h+m))})^a(c_{u.(-(h+m))-2(h+m)})^b(c_{u.(-(h+m))-(h+m)})^a\\ =((c_{u.(-(h+m))-h})^r(c_{u.(-(h+m))-m})^r)^s,u=0,...,n-1\rangle\end{aligned}\tag{2.13}$$

Now let us consider the group

$$G_n^{\varepsilon}(m',2(h'+m'),h')\cong\langle y_0,y_1,...,y_{n-1}\mid y_i^a y_{i+2(h'+m')}^b y_{i+(h'+m')}^a=((y_{i+h'})^r(y_{i+m'})^r)^s,i=0,...,n-1\rangle$$

since $(n,(h'+m'))=1$ there exist $\delta',\zeta'\in\mathbb{Z}$ such that $\delta' n+\zeta'(h'+m')=1$ mod n therefore $\zeta'(h'+m')\equiv 1$ mod n. Now for each $i=0,1,...,n-1$ set $v\equiv\zeta' i$ mod n therefore

$$y_{v(h'+m')}=y_{\zeta' i(h'+m')}=y_{i\zeta'(h'+m')}=y_i,$$

we can write

$$\begin{aligned}G_n^{\varepsilon}(m',2(h'+m'),h')\cong\langle y_0,y_1,...,y_{n-1}\mid y_{v(h'+m')}^a y_{v(h'+m')+2(h'+m')}^b y_{v(h'+m')+(h'+m')}^a\\ =((y_{v(h'+m')+h'})^r(y_{v(h'+m')+m'})^r)^s,v=0,...,n-1\rangle\end{aligned}\tag{2.14}$$

Now define a map as follow

$$\Phi:\{c_0,...,c_{n-1}\}\longrightarrow\{y_0,...,y_{n-1}\}$$

acting on the set of generators $\{c_j|j=0,...,n-1\}=\{c_{u(h+m)}|u=0,...,n-1\}$ by the rule

$$\Phi(c_{u.(-(h+m))})=y_{u(h'+m')}\tag{2.15}$$

For each $u = 0, ..., n-1$, we will map all generators in (2.13) to generators in (2.14)

For the left hand side of (2.13) we have

$$\Phi(c_{u.(-(h+m))}) = y_{u(h'+m')} \qquad \text{using (2.15)}$$

$$\begin{aligned}\Phi(c_{u.(-(h+m))-2(h+m)}) &= \Phi(c_{(u+2).(-(h+m))}) \\ &= y_{(u+2)(h'+m')} \qquad \text{using (2.15)} \\ &= y_{u(h'+m')+2(h'+m')}\end{aligned}$$

$$\begin{aligned}\Phi(c_{u.(-(h+m))-(h+m)}) &= \Phi(c_{(u+1).(-(h+m))}) \\ &= y_{(u+1)(h'+m')} \qquad \text{using (2.15)} \\ &= y_{u(h'+m')+(h'+m')}\end{aligned}$$

For the right hand side of (2.13) we have

$$\begin{aligned}\Phi(c_{u.(-(h+m))-h}) &= \Phi(c_{u.(-(h+m))+g.(-(h+m))}) \qquad \text{using (2.11)} \\ &= \Phi(c_{(u+g)(-(h+m))}) \\ &= y_{(u+g)(h'+m')} \qquad \text{using (2.15)} \\ &= y_{u(h'+m')+g(h'+m')} \\ &= y_{u(h'+m')+h'} \qquad \text{using (2.12)}\end{aligned}$$

$$\begin{aligned}\Phi(c_{u.(-(h+m))-m}) &= \Phi(c_{u.(-(h+m))+f.(-(h+m))}) \qquad \text{using (2.9)} \\ &= \Phi(c_{(u+f)(-(h+m))}) \\ &= y_{(u+f)(h'+m')} \qquad \text{using (2.15)} \\ &= y_{u(h'+m')+f(h'+m')} \\ &= y_{u(h'+m')+m'} \qquad \text{using (2.10)}\end{aligned}$$

Since Φ maps the u'th relators of (2.13) to the u'th relators of (2.14), it is clear that the map $\Phi : G_n^{\varepsilon}(m,k,h) \to G_n^{\varepsilon}(m',k',h')$ is an epimorphism.

Now define a map as follow

$$\Theta : \{y_0, ..., y_{n-1}\} \longrightarrow \{c_0, ..., c_{n-1}\}$$

acting on the set of generators $\{y_i | i = 0, ..., n-1\} = \{y_{vB'} | v = 0, ..., n-1\}$ by the rule

$$\Theta(y_{v(h'+m')}) = c_{v.(-(h+m))} \tag{2.16}$$

For each $v = 0, ..., n-1$, we will map all generators in (2.14) to generators in (2.13)

For the left hand side of (2.14) we have,

$$\Theta(y_{v(h'+m')}) = c_{v.(-(h+m))} \qquad \text{using (2.16)}$$

$$\begin{aligned}
\Theta(y_{(v(h'+m')+2(h'+m'))}) &= \Theta(y_{(v+2)(h'+m')}) \\
&= c_{(v+2)(-(h+m))} \qquad \text{using (2.16)} \\
&= c_{v.(-(h+m))+2.(-(h+m))} \\
&= c_{v.(-(h+m))-2(h+m)}
\end{aligned}$$

$$\begin{aligned}
\Theta(y_{(v(h'+m')+(h'+m'))}) &= \Theta(y_{(v+1)(h'+m')}) \\
&= c_{(v+1)(-(h+m))} \qquad \text{using (2.16)} \\
&= c_{v.(-(h+m))-(h+m)}
\end{aligned}$$

For the right hand side of (2.14) we have,

$$\begin{aligned}
\Theta(y_{(v(h'+m')+h')}) &= \Theta(y_{(v(h'+m')+g(h'+m'))}) \qquad \text{using (2.12)} \\
&= \Theta(y_{(v+g)(h'+m')}) \\
&= c_{(v+g)(-(h+m))} \qquad \text{using (2.16)} \\
&= c_{v.(-(h+m))+g.(-(h+m))} \\
&= c_{v.(-(h+m))-h} \qquad \text{using (2.11)}
\end{aligned}$$

$$\begin{aligned}
\Theta(y_{(v(h'+m')+m')}) &= \Theta(y_{(v(h'+m')+f(h'+m'))}) \qquad \text{using (2.10)} \\
&= \Theta(y_{(v+f)(h'+m')}) \\
&= c_{(v+f)(-(h+m)} \qquad \text{using (2.16)} \\
&= c_{v.(-(h+m))+f.(-(h+m))} \\
&= c_{v.(-(h+m))-m} \qquad \text{using (2.9)}
\end{aligned}$$

Since Θ maps the u'th relators of (2.14) to the u'th relators of (2.13), it is clear that for each $0 \le \gamma \le (n-1)$ that $\Theta : G_n(h', k, p', q', r, s, \ell) \to G_n(h, k, p, q, r, s, \ell)$ is an epimorphism and $\phi(\Theta(y_{uB'})) = \phi(c_{uA}) = y_{uB'}$ and $\Theta(\phi(c_{uA})) = \Theta(y_{uB'}) = c_{uA}$, therefore $\phi^{-1} = \Theta, \Theta^{-1} = \phi$. thus the composition of epimorphisms $G_n(h,k,p,q,r,s,\ell) \underset{\to}{\Phi} G_n(h',k,p',q',r,s,\ell) \underset{\to}{\Theta} G_n(h,k,p,q,r,s,\ell)$ shows that

$$G_n(h,k,p,q,r,s,\ell) \cong G_n(h',k,p',q',r,s,\ell) \qquad \square$$

According to [CRS05, page 42] we will consider classes of groups that can be defined using definition of $G_n^{\varepsilon}(m,k,h)$ for chosen parameters

For $a = 0, b = r = s = 1, h = 0$ the groups $G_n^{\varepsilon}(m,k,h)$ have defining relations $x_i x_{i+m} = x_{i+k}$ of the group $G_n(m,k)$ which was introduced in [JM75], and subsequently studied in [BV03], [CRS03] (see Section 1.3 for more details).

Corollary 2.3.4. *Suppose* $(n,m)=1,(n,m')=1$ *then* $G_n(m,n-m)\cong G_n(m',2m')\cong G_n(1,2)=F(2,n)$.

Proof. Since $G_n(m,n-m)=G_n^{\varepsilon}(M,K,h)=G_n(x_0x_mx_{(n-m)}^{-1})$, where $\varepsilon=(a,b,r,s)=(0,1,1,1),h=0,M=m,K=n-m$, and $G_n(m',2m')=G_n^{\varepsilon}(M',K',h')=G_n(x_0x_{m'}x_{2m'}^{-1})$, where $\varepsilon=(a,b,r,s)=(0,1,1,1),h'=0,M'=m',K'=2m'$. Since $(n,h+m)=(n,m)=1,(n,h'+m')=(n,m')=1$ and $m'(h+m)=m'm,m(h'+m')=mm'$ therefore $m'(h+m)\equiv m(h'+m') \bmod n$ and since $h=h'=0$ therefore $h'(h+m)\equiv h(h'+m') \bmod n$. Theorem (2.3.3) implies that $G_n(m,n-m)\cong G_n(m',2m')$ and by [BV03, Lemma 1.4.] since $(n,m')=1$ we have $G_n(m',2m')\cong G_n(1,2)$. □

Chapter 3

Order of $G_{pM}(x_0x_M^\delta x_K^\varepsilon)^{\mathrm{ab}}$

3.1 Introduction

In this chapter we produce a technical formula for the order of the abelianization of the group $G_{pM}(x_0x_M^\delta x_K^\varepsilon)$ where $\delta = \pm 1, \varepsilon = \pm 1$ and $(M, K) = 1$. This formula is in terms of the parameters $p, M, K, \delta, \varepsilon$. By restricting to particular values of p we are able to obtain numerical values of $|G_{pM}(x_0x_M^\delta x_K^\varepsilon)^{\mathrm{ab}}|$, we apply it to give precise values for the order of abelianization of $G_{pm}(x_0x_mx_k^{-1})$, $G_{pk}(x_0x_mx_k^{-1})$, and $\Gamma_{pk}(x_0x_kx_l)$ when $p \in \{2, 3, 4, 6, 12\}$. The reason we have chosen these numbers is that we can carry out the relevant manipulations with roots of unity in these cases but they are harder in others. This will be used in counting $G_n(m, k)$ groups (Chapter 4), and in counting $\Gamma_n(k, l)$ groups (Chapter 6). In Section 3.2 we give in Theorem 3.2.3 the general formula for $|G_{pM}(x_0x_M^\delta x_K^\varepsilon)^{ab}|$. In Section 3.3 we calculate $|G_{pM}(x_0x_Mx_K^\varepsilon)^{ab}|$ for $p \in \{2, 3, 4, 6, 12\}$ (Theorem 3.3.1), and apply it to obtain $|G_{pm}(x_0x_mx_k^{-1})^{ab}|$ (Corollary 3.3.2), and to obtain $|G_{pk}(x_0x_kx_l)^{ab}|$ (Corollary 3.3.3). In Section 3.4 we calculate $|G_{pk}(x_0x_mx_k^{-1})^{ab}|$ for $p \in \{2, 3, 4, 6, 12\}$ (Theorem 3.4.1).

Thus the results of this chapter give for $p \in \{2, 3, 4, 6, 12\}$, formulas for the orders of the groups $G_{pm}(x_0x_mx_k^{-1})^{\mathrm{ab}}$, $G_{pk}(x_0x_mx_k^{-1})^{\mathrm{ab}}$, $G_{pk}(x_0x_kx_l)^{\mathrm{ab}}, G_{pl}(x_0x_kx_l)^{\mathrm{ab}}$ (since by Lemma 1.6.1 (2) we have $G_{pk}(x_0x_kx_l)^{\mathrm{ab}} \cong G_{pk}(x_0x_lx_k)^{\mathrm{ab}}$).

3.2 The general formula

The following is equation (1.7)

$$|G_n(\omega)^{\text{ab}}| = |\prod_{\theta^n=1} f(\theta)| \tag{3.1}$$

Let $R_n(f) = \prod_{\theta^n=1} f(\theta) \in \mathbb{Z}$. Recall the following Lemma from introduction Lemma 1.4.4 is.

Lemma 3.2.1. *Let* $f(t) = c\prod_{j=1}^{k}(t-\beta_j)$*. Then* $R_n(f) = \left((-1)^k c\right)^n \prod_{j=1}^{k}(\beta_j^n - 1)$.

From now on we shall use the notation $\zeta_p = e^{\frac{2\pi i}{p}}$ for any $p \geq 1$. (Note that we do not required p to be prime).

Lemma 3.2.2. *Let* $g(x) = x^k - w$*. If* $(n,k) = 1$ *then* $\prod_{j=0}^{n-1} g(\zeta_n^j) = (-1)^n(w^n - 1)$.

Proof. Let $f(x) = x - w$, from Lemma 3.2.1 we have $R_n(f) = \left((-1)^k c\right)^n \prod_{j=1}^{k}(\beta_j^n - 1)$, and by setting $c = 1, k = 1, \beta_j = w$, then

$$R_n(f) = \prod_{j=0}^{n-1} f(\zeta_n^j) = (-1)^n(w^n - 1) \tag{3.2}$$

and since $(n,k) = 1$ we have

$$\begin{aligned}\prod_{j=0}^{n-1} f(\zeta_n^j) &= \prod_{\varphi\in U} f(\varphi), \qquad U = \{\zeta_n^0, \zeta_n^1, ..., \zeta_n^{n-1}\} = \{\zeta_n^j | j = 0, ..., n-1\} = \{\zeta_n^{jk} | j = 0, ..., n-1\} \\ &= \prod_{j=0}^{n-1} f(\zeta_n^{jk}) \\ &= \prod_{j=0}^{n-1} g(\zeta_n^j)\end{aligned}$$

and the result follows from (3.2). □

The relation matrix of the cyclically presented group $G_{pM}(x_0 x_M^\delta x_K^\varepsilon)$ is the $n \times n$ circulant matrix where first row is $(1\ 0\ \ldots\ \delta\ 0\ \ldots\ 0\ \varepsilon\ 0\ \ldots\ 0)$ where δ is the M'th entry, and ε is K'th entry. So

$$f(t) = 1 + \delta t^M + \varepsilon t^K \tag{3.3}$$

Let

$$P^{\delta,\varepsilon}_{i,p}(M,K) = \Big((1+\delta\zeta_p^i)^M - (-\varepsilon)^M(\zeta_p^{iK})\Big) \tag{3.4}$$

Theorem 3.2.3. *Let* $(M,K)=1$ *then* $|G_{pM}(x_0x_M^\delta x_K^\varepsilon)^{ab}| = |\prod_{i=0}^{p-1} P^{\delta,\varepsilon}_{i,p}(M,K)|$

Proof. From (3.1), we have that $|G_{pM}(x_0x_Mx_K^{-1})^{ab}| = |P|$ where

$$P = \prod_{j=0}^{pM-1} f(\zeta_{pM}^j) = \prod_{j\in S} f(\zeta_{pM}^j), \qquad f(t) = 1+\delta t^M + \varepsilon t^K \tag{3.5}$$

Where $S = \{j | j = 0,1,...,(pM-1)\}$. For each $0 \le i \le p-1$ let $S_i = \{pt+i | t = 0,1,\ldots M-1\}$; then $S = S_0 \cup \ldots \cup S_{p-1}$. We write P in the form

$$P = Q^{\delta,\varepsilon}_{0,p}(M,K)Q^{\delta,\varepsilon}_{1,p}(M,K)...Q^{\delta,\varepsilon}_{(p-1),p}(M,K) = \prod_{i=0}^{p-1} Q^{\delta,\varepsilon}_{i,p}(M,K), \text{where } Q^{\delta,\varepsilon}_{i,p}(M,K) = \prod_{j\in S_i} f(\zeta_{pM}^j) \tag{3.6}$$

If $j \in S_i$, $0 \le i \le p-1$ then $j = pt+i$, $t = 0,1,...,(M-1)$ so

$$\begin{aligned}
Q^{\delta,\varepsilon}_{i,p}(M,K) &= \prod_{j\in S_i} f(\zeta_{pM}^j)\\
&= \prod_{t=0}^{M-1} f(\zeta_{pM}^{pt+i})\\
&= \prod_{t=0}^{M-1} \Big(1+\delta(\zeta_{pM}^{pt+i})^M + \varepsilon(\zeta_{pM}^{pt+i})^K\Big)\\
&= \prod_{t=0}^{M-1} \Big(1+\delta(\zeta_{pM}^{pMt})(\zeta_{pM}^{iM}) + \varepsilon(\zeta_{pM}^{pKt})(\zeta_{pM}^{iK})\Big)\\
&= \prod_{t=0}^{M-1} \Big((1+\delta\zeta_p^i) + \varepsilon(\zeta_M^{Kt})(\zeta_{pM}^{iK})\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{t=0}^{M-1}(\varepsilon\zeta_{pM}^{iK})\Big(\varepsilon(1+\delta\zeta_p^i).\zeta_{pM}^{-iK} + (\zeta_M^{Kt})\Big)\\
&= \prod_{t=0}^{M-1}(\varepsilon\zeta_{pM}^{ik})\prod_{t=0}^{M-1}(\zeta_M^{Kt} - (-\varepsilon(1+\delta\zeta_p^i)\zeta_{pM}^{-iK}))\Big)\\
&= (\varepsilon\zeta_{pM}^{ik})^M\prod_{t=0}^{M-1}(\zeta_M^{Kt} - (-\varepsilon(1+\delta\zeta_p^i)\zeta_{pM}^{-iK}))\Big)\\
&= (\varepsilon\zeta_{pM}^{ik})^M\Big[(-1)^M\Big((-\varepsilon(1+\delta\zeta_p^i)\zeta_{pM}^{-iK})^M - 1)\Big] \quad \text{by Lemma 3.2.2}\\
&= \varepsilon^M(\zeta_{pM}^{iKM})\Big[(-1)^M\Big((-\varepsilon)^M(1+\delta\zeta_p^i)^M(\zeta_{pM}^{-iKM}) - 1\Big)\Big]\\
&= \varepsilon^M(\zeta_p^{iK})\Big[(-1)^M\Big((-\varepsilon)^M(1+\delta\zeta_p^i)^M(\zeta_p^{-iK}) - 1\Big)\Big]\\
&= (1+\delta\zeta_p^i)^M - (-\varepsilon)^M(\zeta_p^{iK})\\
&= P_{i,p}^{\delta,\varepsilon}(M,K)
\end{aligned}
$$

□

Corollary 3.2.4.

a. If $(m,k)=1$ *then* $|G_{pm}(x_0x_mx_k^{-1})^{ab}| = |\prod_{i=0}^{p-1} P_{i,p}^{1,-1}(m,k)|$ *where* $P_{i,p}^{1,-1}(m,k) = (1+\zeta_p^i)^m - (\zeta_p^{ik})$.

b. If $(m,k)=1$ *then* $|G_{pk}(x_0x_mx_k^{-1})^{ab}| = |\prod_{i=0}^{p-1} P_{i,p}^{-1,1}(k,m)|$ *where* $P_{i,p}^{-1,1}(k,m) = (-1)^k\Big((\zeta_p^i-1)^k - (\zeta_p^{im})\Big)$.

c. If $(k,l)=1$ *then* $|G_{pk}(x_0x_kx_l)^{ab}| = |\prod_{i=0}^{p-1} P_{i,p}^{1,1}(k,l)|$ *where* $P_{i,p}^{1,1}(k,l) = (1+\zeta_p^i)^k + (-1)^{k+1}(\zeta_p^i)^l$.

Proof. a. It follows from Theorem 3.2.3 by setting $M=m, K=k, \delta=1, \varepsilon=-1$.

b. Since $|G_{pk}(x_0x_mx_k^{-1})^{ab}| = |G_{pk}(x_0x_k^{-1}x_m)^{ab}|$, therefore proof is done by setting $M=k, K=m$, $\delta=-1, \varepsilon=1$ in Theorem 3.2.3.

c. It follows from Theorem 3.2.3 by setting $M=k, K=l, \delta=1, \varepsilon=1$.

□

In the following lemma we calculate $P_{i,p}^{\delta,\varepsilon}(M,K)$ in some important cases

Lemma 3.2.5. *1.* $P_{0,p}^{\delta,\varepsilon}(M,K) = (1+\delta)^M - (-\varepsilon)^M$.

2. $P^{\delta,\varepsilon}_{1,2}(M,K) = (1-\delta)^M - (-\varepsilon)^M(-1)^k$.

3. $P^{\delta,\varepsilon}_{ti,tp}(M,K) = P^{\delta,\varepsilon}_{i,p}(M,K)$ *for any* $t \geq 1$.

4. $P^{\delta,\varepsilon}_{p-i,p}(M,K) = P^{\delta,\varepsilon}_{-i,p}(M,K)$

5. *If* $t \geq 1$ *then* $\prod_{i=0}^{p-1} P^{\delta,\varepsilon}_{i,p}(M,K)$ *divides* $\prod_{i=0}^{tp-1} P^{\delta,\varepsilon}_{i,tp}(M,K)$.

6. $P^{\delta,\varepsilon}_{i,p}(M,K).P^{\delta,\varepsilon}_{-i,p}(M,K) = 2^M\Big(1+\delta\cos\frac{2\pi i}{p}\Big)^M + 1 - h_{i,p}$ *where*

$$h_{i,p} = \Big(\zeta_p^{iK}(1+\zeta_p^{-i})^M + \zeta_p^{-iK}(1+\zeta_p^{i})^M\Big)$$

Proof. By using (3.4) we have $P^{\delta,\varepsilon}_{i,p}(M,K) = (1+\delta\zeta_p^i)^M - (-\varepsilon)^M(\zeta_p^{iK})$, and therefore

1.
$$P^{\delta,\varepsilon}_{0,p}(M,K) = (1+\delta)^M - (-\varepsilon)^M \tag{3.7}$$

2.
$$\begin{aligned} P^{\delta,\varepsilon}_{1,2}(M,K) &= (1+\delta\zeta_2)^M - (-\varepsilon)^M(\zeta_2^K) \\ &= (1-\delta)^M - (-\varepsilon)^M(-1)^k \end{aligned} \tag{3.8}$$

3.
$$\begin{aligned} P^{\delta,\varepsilon}_{ti,tp}(M,K) &= (1+\delta\zeta_{tp}^{ti})^M - (-\varepsilon)^M(\zeta_{tp}^{ti})^K \\ &= (1+\delta\zeta_p^{i})^M - (-\varepsilon)^M(\zeta_p^{i})^K \\ &= P^{\delta,\varepsilon}_{i,p}(M,K) \end{aligned} \tag{3.9}$$

4.

$$P^{\delta,\varepsilon}_{p-i,p}(M,K) = (1+\zeta_p^{p-i})^M - (-\varepsilon)^M(\zeta_p^{p-i})^K$$

$$\text{since} \quad \zeta_p^p = 1 \quad \text{then} \quad \zeta_p^{p-i} = \zeta_p^{-i} \text{ and}$$

$$\begin{aligned} P^{\delta,\varepsilon}_{p-i,p}(M,K) &= (1+\zeta_p^{-i})^M - (\zeta_p^{-i})^K \\ &= P_{-i,p}(M,K) \end{aligned} \tag{3.10}$$

5. Since $\{0,t,2t,...,(p-1)t\} \subseteq \{0,1,2,...,(tp-1)\}$ then

$$\prod_{i\in\{0,t,2t,...,(p-1)t\}} P^{\delta,\varepsilon}_{i,tp}(M,K) \quad \text{divides} \quad \prod_{i=0}^{tp-1} P^{\delta,\varepsilon}_{i,tp}(M,K) = \prod_{i\in\{0,1,2,...,(tp-1)\}} P^{\delta,\varepsilon}_{i,tp}(M,K)$$

and

$$\begin{aligned} \prod_{i\in\{0,t,2t,...,(p-1)t\}} P^{\delta,\varepsilon}_{i,tp}(M,K) &= \prod_{i=t\imath,\imath\in\{0,1,2,...,(p-1)\}} P^{\delta,\varepsilon}_{i,tp}(M,K) \\ &= \prod_{\imath\in\{0,1,2,...,(p-1)\}} P^{\delta,\varepsilon}_{t\imath,tp}(M,K) \\ &= \prod_{\imath\in\{0,1,2,...,(p-1)\}} P^{\delta,\varepsilon}_{\imath,p}(M,K) \end{aligned}$$

6.

$$\begin{aligned} P^{\delta,\varepsilon}_{i,p}(M,K).P^{\delta,\varepsilon}_{-i,p}(M,K) &= [(1+\delta\zeta_p^i)^M - (-\varepsilon)^M(\zeta_p^i)^K][(1+\delta\zeta_p^{-i})^M - (-\varepsilon)^M(\zeta_p^{-i})^K] \\ &= \Big((1+\delta\zeta_p^i)(1+\delta\zeta_p^{-i})\Big)^M - (-\varepsilon)^M\Big(\zeta_p^{iK}(1+\delta\zeta_p^{-i})^M + \zeta_p^{-iK}(1+\delta\zeta_p^i)^M\Big) + 1 \\ &= \Big(2+\delta\zeta_p^i + \delta\zeta_p^{-i}\Big)^M - h_{i,p} + 1, \\ \text{where } h_{i,p} &= (-\varepsilon)^M\Big(\zeta_p^{iK}(1+\delta\zeta_p^{-i})^M + \zeta_p^{-iK}(1+\delta\zeta_p^i)^M\Big) \\ &= \Big(2+2\delta\cos\frac{2\pi i}{p}\Big)^M + 1 - h_{i,p} \\ &= 2^M\Big(1+\delta\cos\frac{2\pi i}{p}\Big)^M + 1 - h_{i,p} \end{aligned} \tag{3.11}$$

□

3.3 Order of $G_{pM}(x_0x_Mx_K^{\varepsilon})^{\mathrm{ab}}$ where $p \in \{2,3,4,6,12\}$

Theorem 3.3.1. *If* $(M,K)=1$ *then*

1. $|G_{2M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |(-\varepsilon)^M(-1)^{K+1}(2^M-(-\varepsilon)^M)|$.

2. $|G_{3M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |\Big(2^M-(-\varepsilon)^M\Big)\Big(2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big)|$.

3. $|G_{4M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |\Big((-\varepsilon)^M(-1)^{K+1}\Big(2^M-(-\varepsilon)^M\Big)\Big((2^M+1)-(-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4}\Big)|$.

4. $$|G_{6M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |(-\varepsilon)^M(-1)^{K+1}\Big(2^M-(-\varepsilon)^M\Big)\Big(3^M+1-(-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}\Big)$$
$$\Big(2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big)|.$$

5. $$\begin{aligned}|G_{12M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |&(-\varepsilon)^M(-1)^{k+1}\Big(2^M-(-\varepsilon)^M\Big)\\ &\Big((3^M+1)-(-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}\Big)\\ &\Big(2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big)\\ &\Big(2^M+1-(-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4}\Big)\\ &\Big(2^M(1+\frac{\sqrt{3}}{2})^M+1-(-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M.2\cos\frac{\pi(2k-M)}{12}\Big)\\ &\Big(2^M(1-\frac{\sqrt{3}}{2})^M+1-(-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M.2\cos\frac{5\pi(2K-M)}{12}\Big)|.\end{aligned}$$

Proof.

1. From Theorem 3.2.3 we get that $|G_{2M}(x_0x_Mx_K^{\varepsilon})^{ab}| = \prod_{i=0}^{1} P_{i,2}^{1,\varepsilon}(M,K) = P_{0,2}^{1,\varepsilon}(M,K).P_{1,2}^{1,\varepsilon}(M,K)$, and since

$$P_{0,2}^{1,\varepsilon}(M,K) = 2^M-(-\varepsilon)^M \quad \text{by (3.7)}$$
$$P_{1,2}^{1,\varepsilon}(M,K) = (-\varepsilon)^M(-1)^{k+1} \quad \text{by (3.8)}$$

therefore

$$|G_{2M}(x_0x_Mx_k^\varepsilon)^{ab}| = |(-\varepsilon)^M(-1)^{K+1}(2^M - (-\varepsilon)^M)|$$

2. From Theorem 3.2.3 we get that $|G_{3m}(x_0x_Mx_K^\varepsilon)^{ab}| = \prod_{i=0}^{2} P_{i,3}^{1,-1}(M,K)$, and since

$$P_{0,3}^{1,\varepsilon}(M,K) = 2^M - (-\varepsilon)^M \quad \text{by (3.7)}$$
$$P_{1,3}^{1,\varepsilon}(M,K) = (1+\zeta_3)^M - (-\varepsilon)^M(\zeta_3^K) \quad \text{by (3.4)}$$
$$P_{2,3}^{1,\varepsilon}(M,K) = (1+\zeta_3^2)^M - (-\varepsilon)^M(\zeta_3^{2K}) \quad \text{by (3.4)}$$

observe

$$1+\zeta_3 = 1 + (\frac{-1}{2} + i(\frac{\sqrt{3}}{2})) = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \zeta_6 \tag{3.12}$$

$$1+\zeta_3^2 = 1 + (\frac{-1}{2} - i(\frac{\sqrt{3}}{2})) = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \zeta_6^{-1} \tag{3.13}$$

Then

$$\begin{aligned}
P_{1,3}^{1,\varepsilon}(M,K).P_{2,3}^{1,\varepsilon}(M,K) &= P_{1,3}^{1,\varepsilon}(M,K).P_{-1,3}^{1,\varepsilon}(M,K) \quad \text{by (3.10)} \\
&= 2^M\Big(1+\cos\frac{2\pi}{3}\Big)^M + 1 - h_{1,3} \quad \text{by (3.11)} \\
&= 2^M\Big(1-\frac{1}{2}\Big)^M + 1 - (-\varepsilon)^M\Big(\zeta_3^K(1+\zeta_3^{-1})^M + \zeta_3^{-K}(1+\zeta_3)^M\Big) \\
&= 2 - (-\varepsilon)^M(\zeta_3^K.\zeta_6^{-M} + \zeta_3^{-K}.\zeta_6^M) \quad \text{by (3.12), (3.13)} \\
&= 2 - (-\varepsilon)^M(\zeta_6^{2K}.\zeta_6^{-M} + \zeta_6^{-2K}.\zeta_6^M) \\
&= 2 - (-\varepsilon)^M(\zeta_6^{(2K-M)} + \zeta_6^{-(2k-M)}) \\
&= 2 - (-\varepsilon)^M 2\cos\frac{(2K-M)2\pi}{6} \\
&= 2 - (-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3} \qquad (3.14)
\end{aligned}$$

Therefore

$$|G_{3M}(x_0x_Mx_K^\varepsilon)^{ab}| = |\Big(2^M - (-\varepsilon)^M\Big)\Big(2 - (-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big)|.$$

3. From Theorem 3.2.3 we get that $|G_{4M}(x_0x_Mx_K^{\varepsilon})^{ab}| = \prod_{i=0}^{3} P_{i,4}^{1,-1}(M,K)$, and since

$$\begin{aligned}
P_{0,4}^{1,\varepsilon}(M,K) &= 2^M - (-\varepsilon)^M \quad \text{by (3.7)}\\
P_{1,4}^{1,\varepsilon}(M,K) &= (1+\zeta_4)^M - (-\varepsilon)^M(\zeta_4^K) \quad \text{by (3.4)}\\
P_{2,4}^{1,\varepsilon}(M,K) &= P_{1,2}^{1,\varepsilon}(M,K) = (-\varepsilon)^M(-1)^{K+1} \quad \text{by (3.9), (3.8)}\\
P_{3,4}^{1,\varepsilon}(M,K) &= (1+\zeta_4^3)^M - (-\varepsilon)^M(\zeta_4^{3K}) \quad \text{by (3.4)}
\end{aligned}$$

observe

$$1+\zeta_4 = 1+(0+i) = 1+i = \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2}e^{\frac{2\pi i}{8}} = \sqrt{2}\zeta_8 \tag{3.15}$$

$$1+\zeta_4^3 = 1+(0-i) = 1-i = \sqrt{2}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) = \sqrt{2}e^{\frac{2\pi i}{8}} = \sqrt{2}\zeta_8^{-1} \tag{3.16}$$

Then

$$\begin{aligned}
P_{1,4}^{1,\varepsilon}(M,K).P_{3,4}^{1,\varepsilon}(M,K) &= P_{1,4}^{1,\varepsilon}(M,K).P_{-1,4}^{1,\varepsilon}(M,K) \quad \text{by (3.10)}\\
&= 2^M\Big(1+\cos\frac{2\pi}{4}\Big)^m + 1 - h_{1,4}\\
&= 2^M + 1 - (-\varepsilon)^M\Big(\zeta_4^K(1+\zeta_4^{-1})^M + \zeta_4^{-K}(1+\zeta_4)^M\Big)\\
&= 2^M + 1 - (-\varepsilon)^M(\zeta_4^K.(\sqrt{2}\zeta_8)^{-M} + \zeta_4^{-K}.(\sqrt{2}\zeta_8)^M) \quad \text{by (3.15), (3.16)}\\
&= 2^M + 1 - (-\varepsilon)^M(\sqrt{2})^M[\zeta_8^{(2K-M)} + \zeta_8^{-(2k-M)}]\\
&= (2^M+1) - (-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)2\pi}{8}\\
&= (2^M+1) - (-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4}
\end{aligned} \tag{3.17}$$

therefore

$$|G_{4M}(x_0x_Mx_K^{\varepsilon})^{ab}| = |(-\varepsilon)^M(-1)^{K+1}\Big(2^M - (-\varepsilon)^M\Big)\Big((2^M+1) - (-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4}\Big)|$$

4. From Theorem 3.2.3 we get that $|G_{6M}(x_0x_Mx_K^{\varepsilon})^{ab}| = \prod_{i=0}^{5} P_{i,6}^{1,-1}(M,K)$, and since

$$\begin{aligned}
P_{0,6}^{1,\varepsilon}(M,K) &= 2^M - (-\varepsilon)^M \quad \text{by (3.7)}\\
P_{1,6}^{1,\varepsilon}(M,K) &= (1+\zeta_6)^M - (-\varepsilon)^M(\zeta_6^K) \quad \text{by (3.4)}\\
P_{2,6}^{1,\varepsilon}(M,K) &= P_{1,3}^{1,-1}(M,K) = (1+\zeta_3)^M - (-\varepsilon)^M(\zeta_3^K) \quad \text{by (3.9)}\\
P_{3,6}^{1,\varepsilon}(M,K) &= P_{1,2}^{1,-1}(M,K) = (-\varepsilon)^M(-1)^{K+1} \quad \text{by (3.9), (3.8)}\\
P_{4,6}^{1,\varepsilon}(M,K) &= P_{2,3}^{1,-1}(M,K) = (1+\zeta_3^2)^M - (-\varepsilon)^M(\zeta_3^{2K}) \quad \text{by (3.9)}\\
P_{5,6}^{1,\varepsilon}(M,K) &= (1+\zeta_6^5)^M - (-\varepsilon)^M(\zeta_6^{5K}) \quad \text{by (3.4)}
\end{aligned}$$

observe

$$1+\zeta_6 = 1+(\frac{1}{2}+i(\frac{\sqrt{3}}{2})) = \frac{3}{2}+i\frac{\sqrt{3}}{2} = \sqrt{3}(\frac{\sqrt{3}}{2}+i\frac{1}{2}) = \sqrt{3}e^{\frac{2\pi i}{12}} = \sqrt{3}\zeta_{12} \tag{3.18}$$

$$1+\zeta_6^5 = 1+(\frac{1}{2}-i(\frac{\sqrt{3}}{2})) = \frac{3}{2}-i\frac{\sqrt{3}}{2} = \sqrt{3}(\frac{\sqrt{3}}{2}-i\frac{1}{2}) = \sqrt{3}e^{\frac{-2\pi i}{12}} = \sqrt{3}\zeta_{12}^{-1} \tag{3.19}$$

Now let us calculate

$$\begin{aligned}
P_{1,6}^{1,\varepsilon}(M,K).P_{5,6}^{1,\varepsilon}(M,K) &= P_{1,6}^{1,\varepsilon}(M,K).P_{-1,6}^{1,\varepsilon}(M,K) \quad \text{by} \quad \text{Lemma 3.2.5}\\
&= 2^M\Big(1+\cos\frac{2\pi}{6}\Big)^M + 1 - h_{1,6}\\
&= 2^M\Big(1+\frac{1}{2}\Big)^M + 1 - (-\varepsilon)^M\Big(\zeta_6^K(1+\zeta_6^{-1})^M + \zeta_6^{-K}(1+\zeta_6)^M\Big)\\
&= 3^M + 1 - (-\varepsilon)^M(\zeta_6^K.\sqrt{3}\zeta_{12}^{-M} + \zeta_6^{-K}.\sqrt{3}\zeta_{12}^{M}) \quad \text{by (3.18), (3.19)}\\
&= 3^M + 1 - (-\varepsilon)^M(\sqrt{3})^M[\zeta_{12}^{(2K-M)} + \zeta_{12}^{-(2K-M)}]\\
&= (3^M+1) - (-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)2\pi}{12}\\
&= (3^M+1) - (-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}.
\end{aligned}$$

Similarly we find

$$P^{1,\varepsilon}_{2,6}(M,K).P^{1,\varepsilon}_{4,6}(M,K) = P^{1,-1}_{1,3}(M,K).P^{1,-1}_{-1,3}(M,K) \quad \text{by Lemma 3.2.5. parts 3 and 4.}$$
$$= 2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3} \qquad \text{by (3.14)}$$

and $P^{1,-1}_{3,6}(M,K) = P^{1,-1}_{1,2}(M,K) = (-\varepsilon)^M(-1)^{k+1}$ by (3.9), (3.8), therefore

$$|G_{6M}(x_0x_Mx_K^{-1})^{ab}| = |P^{1,-1}_{0,6}(M,K)P^{1,-1}_{1,6}(M,K)...P^{1,-1}_{5,6}(M,K)|$$
$$= |(-\varepsilon)^M(-1)^{K+1}\big(2^M-(-\varepsilon)^M\big)\Big((3^M+1)-(-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}\Big)$$
$$\Big(2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big)|. \qquad (3.20)$$

5. From Theorem 3.2.3 we get that $|G_{12M}(x_0x_Mx_K^{\varepsilon})^{ab}| = \prod_{i=0}^{11} P^{1,-1}_{i,12}(M,K)$, and since

$$P^{1,\varepsilon}_{0,12}(M,K) = 2^M-(-\varepsilon)^M \quad \text{by (3.7)}$$
$$P^{1,\varepsilon}_{1,12}(M,K) = (1+\zeta_{12})^M-(-\varepsilon)^M(\zeta_{12}^K) \quad \text{by (3.4)}$$
$$P^{1,\varepsilon}_{2,12}(M,K) = P^{1,-1}_{1,6}(M,K) = (1+\zeta_6)^M-(-\varepsilon)^M(\zeta_6^K) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{3,12}(M,K) = P^{1,-1}_{1,4}(M,K) = (1+\zeta_4)^M-(-\varepsilon)^M(\zeta_4^K) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{4,12}(M,K) = P^{1,-1}_{1,3}(M,K) = (1+\zeta_3)^M-(-\varepsilon)^M(\zeta_3^K) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{5,12}(M,K) = (1+\zeta_{12}^5)^M-(\varepsilon)^M(\zeta_{12}^{5K}) \quad \text{by (3.4)}$$
$$P^{1,\varepsilon}_{6,12}(M,K) = P^{1,-1}_{1,2}(M,K) = (-\varepsilon)^M(-1)^{K+1} \quad \text{by (3.9), (3.8)}$$
$$P^{1,\varepsilon}_{7,12}(M,K) = (1+\zeta_{12}^7)^M-(\zeta_{12}^K)^7 \quad \text{by (3.4)}$$
$$P^{1,\varepsilon}_{8,12}(M,K) = P^{1,-1}_{2,3}(M,K) = (1+\zeta_3^2)^M-(\zeta_3^{2K}) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{9,12}(M,K) = P^{1,-1}_{3,4}(M,K) = (1+\zeta_4^3)^M-(\zeta_4^{3K}) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{10,12}(M,K) = P^{1,-1}_{5,6}(M,K) = (1+\zeta_6^5)^M-(\zeta_6^{5K}) \quad \text{by (3.9)}$$
$$P^{1,\varepsilon}_{11,12}(M,K) = (1+\zeta_{12}^{11})^M-(-\varepsilon)^M(\zeta_{12}^{11K}) \quad \text{by (3.4)}$$

so by Lemma 3.2.5 we have

$$\begin{aligned} P^{1,\varepsilon}_{0,12}(M,K)P^{1,\varepsilon}_{2,12}(M,K)P^{1,\varepsilon}_{4,12}(M,K)P^{1,\varepsilon}_{6,12}(M,K)P^{1,\varepsilon}_{8,12}(M,K)P^{1,\varepsilon}_{10,12}(M,K) &= \prod_{i=0}^{5} P^{1,-1}_{i,6}(M,K) \\ = (-\varepsilon)^M(-1)^{k+1}\Big(2^M - (-\varepsilon)^M\Big)\Big((3^M+1) - (-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}\Big) \\ \Big(2 - (-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} P^{1,\varepsilon}_{3,12}(M,K)P^{1,\varepsilon}_{9,12}(M,K) &= P^{1,\varepsilon}_{1,4}(M,K)P^{1,\varepsilon}_{3,4}(M,k) \\ &= (2^M+1) - (-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4} \quad \text{by (3.17)} \end{aligned} \tag{3.22}$$

Now observe

$$1+\zeta_{12} = \frac{\sqrt{6}+\sqrt{2}}{2}\zeta_{24}, \qquad 1+\zeta_{12}^{-1} = \frac{\sqrt{6}+\sqrt{2}}{2}\zeta_{24}^{-1} \tag{3.23}$$

$$1+\zeta_{12}^{5} = \frac{\sqrt{6}-\sqrt{2}}{2}\zeta_{24}^{5}, \qquad 1+\zeta_{12}^{-5} = \frac{\sqrt{6}-\sqrt{2}}{2}\zeta_{24}^{-5} \tag{3.24}$$

$$\begin{aligned} P^{1,\varepsilon}_{1,12}(M,K)P^{1,\varepsilon}_{11,12}(M,K) &= P^{1,\varepsilon}_{1,12}(M,K)P^{1,\varepsilon}_{-1,12}(M,K) \quad \text{by} \quad \text{Lemma 3.2.5} \\ &= 2^M(1+\cos\frac{2\pi}{12})^M + 1 - h_{1,12} \quad \text{by} \quad \text{Lemma 3.2.5} \\ &= 2^M(1+\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M\Big(\zeta_{12}^{K}(1+\zeta_{12}^{-1})^M + \zeta_{12}^{K}(1+\zeta_{12})^M\Big) \\ &= 2^M(1+\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M\Big(\zeta_{12}^{K}\zeta_{24}^{-M} + \zeta_{12}^{-K}\zeta_{24}^{M}\Big) \quad \text{by (3.23)} \\ &= 2^M(1+\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M\Big(\zeta_{24}^{2K-M} + \zeta_{24}^{-(2K-M)}\Big) \\ &= 2^M(1+\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M.2\cos\frac{2\pi(2K-M)}{24} \\ &= 2^M(1+\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M.2\cos\frac{\pi(2K-M)}{12} \end{aligned} \tag{3.25}$$

We also have

$$\begin{aligned}
P^{1,\varepsilon}_{5,12}(M,K)P^{1,\varepsilon}_{7,12}(M,K) &= P^{1,\varepsilon}_{5,12}(M,K)P^{1,\varepsilon}_{-5,12}(M,K) \quad \text{by} \quad \text{Lemma 3.2.5} \\
&= 2^M(1+\cos\frac{10\pi}{12})^M + 1 - h_{5,12} \\
&= 2^M(1-\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M\Big(\zeta_{12}^{5K}(1+\zeta_{12}^{-5})^M + \zeta_{12}^{-5K}(1+\zeta_{12}^{5})^M\Big) \\
&= 2^M(1-\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M\Big(\zeta_{12}^{5K}\zeta_{24}^{-5M} + \zeta_{12}^{-5K}\zeta_{24}^{5M}\Big) \quad \text{by (3.24)} \\
&= 2^M(1-\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M\Big(\zeta_{24}^{10K-5M} + \zeta_{24}^{-(10K-5M)}\Big) \\
&= 2^M(1-\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M.2\cos\frac{10\pi(2K-M)}{24} \\
&= 2^M(1-\frac{\sqrt{3}}{2})^M + 1 - (-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M.2\cos\frac{5\pi(2K-M)}{12}
\end{aligned} \tag{3.26}$$

By using (3.21), (3.40), (3.43) and (3.44) we get

$$\begin{aligned}
|G_{12M}(x_0x_Mx_K^{-1})^{ab}| = |&(-\varepsilon)^M(-1)^{K+1}\Big(2^M-(-\varepsilon)^M\Big) \\
&\Big((3^M+1)-(-\varepsilon)^M(\sqrt{3})^M.2\cos\frac{(2K-M)\pi}{6}\Big) \\
&\Big(2-(-\varepsilon)^M 2\cos\frac{(2K-M)\pi}{3}\Big) \\
&\Big(2^M+1-(-\varepsilon)^M(\sqrt{2})^M.2\cos\frac{(2K-M)\pi}{4}\Big) \\
&\Big(2^M(1+\frac{\sqrt{3}}{2})^M+1-(-\varepsilon)^M(\frac{\sqrt{6}+\sqrt{2}}{2})^M.2\cos\frac{\pi(2K-M)}{12}\Big) \\
&\Big(2^M(1-\frac{\sqrt{3}}{2})^M+1-(-\varepsilon)^M(\frac{\sqrt{6}-\sqrt{2}}{2})^M.2\cos\frac{5\pi(2K-M)}{12}\Big)|
\end{aligned}$$

□

Corollary 3.3.2. *If* $(m,k)=1$ *then*

1. $|G_{2m}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}(2^m-1)|$
2. $|G_{3m}(x_0x_mx_k^{-1})^{ab}| = |(2^m-1)(2-2\cos(\frac{(2k-m)\pi}{3}))|$
3. $|G_{4m}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}(2^m-1)((2^m+1)-(\sqrt{2})^m.2\cos\frac{(2k-m)\pi}{4})|$

4. $|G_{6m}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}(2^m-1)(3^m+1-2(\sqrt{3})^m\cos(\frac{(2k-m)\pi}{6}))(2-2\cos(\frac{(2k-m)\pi}{3}))|$

5.
$$\begin{aligned}|G_{12m}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^m-1\Big)\Big(3^m+1-2(\sqrt{3})^m\cos(\frac{(2k-m)\pi}{6})\Big)\Big(2-2\cos(\frac{(2k-m)\pi}{3})\Big)\\ \Big(2^m+1-(\sqrt{2})^m.2\cos\frac{(2k-m)\pi}{4}\Big)\\ \Big(2^m(1+\frac{\sqrt{3}}{2})^m+1-(\frac{\sqrt{6}+\sqrt{2}}{2})^m.2\cos\frac{(2k-m)\pi}{12}\Big)\\ \Big(2^m(1-\frac{\sqrt{3}}{2})^m+1-(\frac{\sqrt{6}-\sqrt{2}}{2})^m.2\cos\frac{5(2k-m)\pi}{12}\Big)|\end{aligned}$$

Proof. Proof is done by substituting $M = m, K = k, \varepsilon = -1$ in Theorem 3.3.1 □

Corollary 3.3.3. *If* $(k,l) = 1$ *then*

1. $|\Gamma_{2k}(k,l)^{ab}| = |(-1)^{k+l+1}(2^k-(-1)^k)|$

2. $|\Gamma_{3k}(k,l)^{ab}| = |(2^k-(-1)^k)(2-(-1)^k.2\cos\frac{(2l-k)\pi}{3})|$

3. $|\Gamma_{4k}(k,l)^{ab}| = |(-1)^{k+l+1}(2^k-(-1)^k)((2^k+1)-(-1)^k(\sqrt{2})^k.2\cos\frac{(2l-k)\pi}{4})|$

4. $|\Gamma_{6k}(k,l)^{ab}| = |(-1)^{k+l+1}(2^k-(-1)^k)((3^k+1)-(-1)^k(\sqrt{3})^k.2\cos\frac{(2l-k)\pi}{6})$
$[2-(-1)^k.2\cos\frac{(2l-k)\pi}{3}]|$

5.
$$\begin{aligned}|\Gamma_{12k}(k,l)^{ab}| = |(-1)^{k+l+1}\Big(2^k-(-1)^k\Big)\Big(3^k+1-(-1)^k2(\sqrt{3})^k\cos\frac{(2l-k)\pi}{6}\Big)\\ \Big(2-(-1)^k.2\cos(\frac{2l-k}{3})\pi\Big)\Big(2^k+1-(-1)^k(\sqrt{2})^k.2\cos\frac{(2l-k)\pi}{4}\Big)\\ \Big(2^k(1+\frac{\sqrt{3}}{2})^k+1-(-1)^k(\frac{\sqrt{6}+\sqrt{2}}{2})^k.2\cos\frac{(2l-k)\pi}{12}\Big)\\ \Big(2^k(1-\frac{\sqrt{3}}{2})^k+1-(-1)^k(\frac{\sqrt{6}-\sqrt{2}}{2})^k.2\cos\frac{5(2l-k)\pi}{12}\Big)|\end{aligned}$$

Proof. Proof is done by substituting $M = k, K = l, \varepsilon = 1$ in Theorem 3.3.1 □

3.4 Order of $G_{pk}(x_0x_mx_k^{-1})^{\mathrm{ab}}$ where $p \in \{2,3,4,6,12\}$

Theorem 3.4.1. *If* $(m,k) = 1$ *then*

1. $|G_{2k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)|.$

2. $|G_{3k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)|.$

3. $|G_{4k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)\Big(2^k + 1 - (-\sqrt{2})^k.2\cos\frac{\pi(k+2m)}{4}\Big)|.$

4. $|G_{6k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)\Big(2 - 2.\cos\frac{\pi(2k-m)}{3}\Big)|.$

5.

$$\begin{aligned}|G_{12k}(x_0x_mx_k^{-1})^{ab}| = |((-1)^{2k+m} - (-2)^k)(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6}))\\ (2 - 2.\cos\frac{\pi(2k-m)}{3})(2^k + 1 - (\sqrt{2})^k.2\cos\frac{\pi(k+2m)}{4})\\ 2^k(1 + \frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{2}-\sqrt{6}}{2})^k.2\cos\frac{\pi(5k+2m)}{12}\\ (2^k(1 - \frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{6}+\sqrt{2}}{2})^k.2\cos\frac{\pi(11k-10m)}{12})|.\end{aligned}$$

Proof.

1. From Corollary 3.2.4 (b) we have $|G_{2k}(x_0x_mx_k^{-1})^{ab}| = \prod_{i=0}^{1} P_{i,2}^{-1,1}(k,m) = P_{0,2}^{-1,1}(k,m).P_{1,2}^{-1,1}(k,m)$, and since

$$P_{0,2}^{-1,1}(k,m) = (-1)^{k+1} \quad \text{by (3.7)}$$
$$P_{1,2}^{-1,1}(k,m) = 2^k - (-1)^{k+m} \quad \text{by (3.8)}$$

so

$$|G_{2k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)| \tag{3.27}$$

2. From Corollary 3.2.4 (*b*), we have $|G_{3k}(x_0x_mx_k^{-1})^{ab}| = \prod_{i=0}^{2} P_{i,3}^{-1,1}(m,k)$, and since

$$P_{0,3}^{-1,1}(k,m) = (-1)^{k+1} \quad \text{by (3.7)}$$
$$P_{1,3}^{-1,1}(k,m) = (-1)^{k}[(\zeta_3 - 1)^k - \zeta_3^m)] \quad \text{by (3.4)}$$
$$P_{2,3}^{-1,1}(k,m) = (-1)^{k}[(\zeta_3^2 - 1)^k - \zeta_3^{2m})] \quad \text{by (3.4)}$$

Observe

$$\zeta_3 - 1 = e^{\frac{2\pi i}{3}} - 1 = \frac{-1}{2} + i(\frac{\sqrt{3}}{2}) - 1 = \frac{-3}{2} + i(\frac{\sqrt{3}}{2}) = -\sqrt{3}\zeta_{12}^{-1} \tag{3.28}$$

$$\zeta_3^2 - 1 = e^{-\frac{2\pi i}{3}} - 1 = \frac{-1}{2} - i(\frac{\sqrt{3}}{2}) - 1 = \frac{-3}{2} - i(\frac{\sqrt{3}}{2}) = -\sqrt{3}\zeta_{12} \tag{3.29}$$

Now let us calculate

$$\begin{aligned}
|G_{3k}(x_0x_mx_k^{-1})^{ab}| &= |P_{0,3}^{-1,1}(k,m)P_{1,3}^{-1,1}(m,k)P_{2,3}^{-1,1}(k,m)| \\
&= |(-1)^{k+1}.(-1)^{k}[(\zeta_3 - 1)^k - \zeta_3^m)].(-1)^{k}[(\zeta_3^2 - 1)^k - \zeta_3^{2m})]| \\
&= |(-1)^{k+1}.[(-\sqrt{3}\zeta_{12}^{-1})^k - \zeta_3^m][(-\sqrt{3}\zeta_{12})^k - \zeta_3^{2m}]| \quad \text{by (3.28), (3.29)} \\
&= |(-1)^{k+1}\Big(3^k - (-\sqrt{3})^k(\zeta_{12}^{-k}\zeta_3^{-m} + \zeta_{12}^{k}\zeta_3^{m}) + 1\Big)| \\
&= |(-1)^{k+1}\Big(3^k - (-\sqrt{3})^k(\zeta_{12}^{-(k+4m)} + \zeta_{12}^{(k+4m)}) + 1\Big)| \\
&= |(-1)^{k+1}\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{2\pi(k+4m)}{12})\Big)| \\
&= |(-1)^{k+1}\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)|
\end{aligned} \tag{3.30}$$

3. From Corollary 3.2.4 (*b*), we have $|G_{4k}(x_0x_mx_k^{-1})^{ab}| = \prod_{i=0}^{3} P_{i,4}^{-1,1}(m,k)$ where

$$P_{0,4}^{-1,1}(k,m) = (-1)^{k+1} \quad \text{by (3.7)}$$
$$P_{1,4}^{-1,1}(k,m) = (-1)^k[(\zeta_4-1)^k - \zeta_4^m)] \quad \text{by (3.4)}$$
$$P_{2,4}^{-1,1}(k,m) = P_{1,2}^{-1,1}(m,k) = 2^k - (-1)^{k+m} \quad \text{by (3.8)}$$
$$P_{3,4}^{-1,1}(k,m) = (-1)^k[(\zeta_4^3-1)^k - \zeta_4^{3m})] \quad \text{by (3.4)}$$

Observe

$$\zeta_4 - 1 = (0+i) - 1 = -\sqrt{2}\zeta_8^{-1} \tag{3.31}$$

$$\zeta_4^3 - 1 = (0-i) - 1 = -\sqrt{2}\zeta_8 \tag{3.32}$$

Now let us calculate

$$\begin{aligned}
P_{1,4}^{-1,1}(k,m)P_{3,4}^{-1,1}(k,m) &= P_{1,4}^{-1,1}(k,m)P_{-1,4}^{-1,1}(k,m) \quad \text{by} \quad \text{Lemma 3.2.5}\\
&= 2^k\Big(1-\cos\frac{2\pi}{4}\Big)^k + 1 - \Big(\zeta_4^m(\zeta_4^{-1}-1)^k + \zeta_4^{-m}(\zeta_4-1)^k\Big)\\
&= 2^k + 1 - (-\sqrt{2})^k(\zeta_4^m\zeta_8^k + \zeta_4^{-m}\zeta_8^{-k}) \quad \text{by (3.31), (3.32)}\\
&= 2^k + 1 - (-\sqrt{2})^k(\zeta_8^{k+2m} + \zeta_8^{-(k+2m)})\\
&= 2^k + 1 - (-\sqrt{2})^k.2\cos\frac{2\pi(k+2m)}{8}\\
&= 2^k + 1 - (-\sqrt{2})^k.2\cos\frac{\pi(k+2m)}{4}
\end{aligned} \tag{3.33}$$

and from Lemma 3.2.5 and equation (3.27) we have

$$P_{0,4}^{-1,1}(k,m)P_{2,4}^{-1,1}(m,k) = P_{0,2}^{-1,1}(k,m)P_{1,2}^{-1,1}(k,m) = (-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big) \tag{3.34}$$

so by (3.33),(3.34) we have

$$|G_{4k}(x_0x_mx_k^{-1})^{ab}| = |(-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)\Big(2^k + 1 - (-\sqrt{2})^k.2\cos\frac{\pi(k+2m)}{4}\Big)|$$

4. From b of Corollary 3.2.4, we have $|G_{6k}(x_0x_mx_k^{-1})^{ab}| = \prod_{i=0}^{5} P_{i,6}^{-1,1}(m,k)$ where

$$
\begin{aligned}
P_{0,6}^{-1,1}(k,m) &= (-1)^{k+1} \quad \text{by (3.7)}\\
P_{1,6}^{-1,1}(k,m) &= (-1)^k\Big((\zeta_6-1)^k - \zeta_6^m\Big) \quad \text{by (3.4)}\\
P_{2,6}^{-1,1}(k,m) &= P_{1,3}^{-1,1}(m,k) = (-1)^k\Big((\zeta_3-1)^k - \zeta_3^m\Big)\\
P_{3,6}^{-1,1}(k,m) &= P_{1,2}^{-1,1}(m,k) = 2^k - (-1)^{k+m} \quad \text{by (3.8)}\\
P_{4,6}^{-1,1}(k,m) &= P_{2,3}^{-1,1}(m,k) = (-1)^k\Big((\zeta_3^2-1)^k - \zeta_3^{2m}\Big)\\
P_{5,6}^{-1,1}(k,m) &= (-1)^k\Big((\zeta_6^5-1)^k - \zeta_p^{5m}\Big) \quad \text{by (3.4)}
\end{aligned}
$$

$$
\begin{aligned}
P_{0,6}^{-1,1}(m,k)P_{2,6}^{-1,1}(m,k)P_{4,6}^{-1,1}(m,k) &= P_{0,3}^{-1,1}(m,k)P_{1,3}^{-1,1}(m,k)P_{2,3}^{-1,1}(m,k)\\
&= (-1)^{k+1}\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big) \qquad (3.35)
\end{aligned}
$$

Observe

$$
\zeta_6 - 1 = \zeta_3, \quad \zeta_6^5 - 1 = \zeta_3^{-1} \qquad (3.36)
$$

Now let us calculate

$$
\begin{aligned}
P_{1,6}^{-1,1}(m,k).P_{5,6}^{-1,1}(k,m) &= P_{1,6}^{-1,1}(k,m).P_{-1,6}^{-1,1}(k,m) \quad \text{by} \quad \text{Lemma 3.2.5}\\
&= 2^k\Big(1-\cos\frac{2\pi}{6}\Big)^k + 1 - \Big(\zeta_6^m(\zeta_6^{-1}-1)^k + \zeta_6^{-m}(\zeta_6-1)^k\Big)\\
&= 1 + 1 - (\zeta_6^m\zeta_3^{-k} + \zeta_6^{-m}\zeta_3^k) \quad \text{by (3.36)}\\
&= 2 - (\zeta_6^{2k-m} + \zeta_6^{-(2k-m)})\\
&= 2 - 2.\cos\frac{2\pi(2k-m)}{6}\\
&= 2 - 2.\cos\frac{\pi(2k-m)}{3} \qquad (3.37)
\end{aligned}
$$

$$|G_{6k}(x_0x_mx_k^{-1})^{ab}| = P_{0,6}^{-1,1}(k,m).P_{1,6}^{-1,1}(k,m)...P_{5,6}^{-1,1}(k,m)$$
$$= |(-1)^{k+1}\Big(2^k-(-1)^{k+m}\Big)\Big(3^k+1-(-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)\Big(2-2.\cos\frac{\pi(2k-m)}{3}\Big)|. \tag{3.38}$$

5. From Corollary 3.2.4, we have that $|G_{12m}(x_0x_mx_k^{-1})^{ab}| = \prod_{i=0}^{11} P_{i,12}^{-1,1}(k,m)$ where

$$P_{0,12}^{-1,1}(k,m) = (-1)^{k+1} \quad \text{by (3.7)}$$
$$P_{1,12}^{-1,1}(k,m) = (-1)^k((\zeta_{12}-1)^k-(\zeta_{12})^m) \quad \text{by (3.4)}$$
$$P_{2,12}^{-1,1}(k,m) = P_{1,6}^{-1,1}(m,k) = (-1)^k((\zeta_6-1)^k-\zeta_6^m))$$
$$P_{3,12}^{-1,1}(k,m) = P_{1,4}^{-1,1}(m,k) = (-1)^k((\zeta_4-1)^k-(\zeta_4)^m)$$
$$P_{4,12}^{-1,1}(k,m) = P_{1,3}^{-1,1}(m,k) = (-1)^k((\zeta_3-1)^k-\zeta_3^m))$$
$$P_{5,12}^{-1,1}(k,m) = (-1)^k((\zeta_{12}^5-1)^k-(\zeta_{12}^5)^m) \quad \text{by (3.4)}$$
$$P_{6,12}^{-1,1}(k,m) = P_{1,2}^{-1,1}(m,k) = (-1)^{2k+m}-(-2)^k \quad \text{by (3.8)}$$
$$P_{7,12}^{-1,1}(k,m) = (-1)^k[(\zeta_{12}^7-1)^k-(\zeta_{12}^7)^m] \quad \text{by (3.4)}$$
$$P_{8,12}^{-1,1}(k,m) = P_{2,3}^{-1,1}(m,k) = (-1)^k((\zeta_3^2-1)^k-\zeta_3^{2m}))$$
$$P_{9,12}^{-1,1}(k,m) = P_{3,4}^{-1,1}(m,k) = (-1)^k((\zeta_4^3-1)^k-(\zeta_4^3)^m)$$
$$P_{10,12}^{-1,1}(k,m) = P_{5,6}^{-1,1}(m,k) = (-1)^k((\zeta_6^5-1)^k-\zeta_6^{5m}))$$
$$P_{11,12}^{-1,1}(k,m) = (-1)^k((\zeta_{12}^{11}-1)^k-(\zeta_{12}^{11})^m) \quad \text{by (3.4)}$$

so by Corollary 3.2.5 and equation (3.38) we have

$$P_{0,12}^{-1,1}(k,m).P_{2,12}^{-1,1}(k,m)...P_{10,12}^{-1,1}(k,m) = P_{0,6}^{-1,1}(k,m)P_{1,6}^{-1,1}(k,m)...P_{5,6}^{-1,1}(k,m)$$

$$= (-1)^{k+1}\Big(2^k-(-1)^{k+m}\Big)\Big(3^k+1-(-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)\Big(2-2.\cos\frac{\pi(2k-m)}{3}\Big) \tag{3.39}$$

and

$$\begin{aligned} P_{3,12}^{-1,1}(k,m).P_{9,12}^{-1,1}(k,m) &= P_{1,4}^{-1,1}(k,m).P_{3,4}^{-1,1}(k,m) \qquad \text{by (3.33)} \\ &= 2^k + 1 - (\sqrt{2})^k.2\cos\frac{\pi(2m+k)}{4} \end{aligned} \tag{3.40}$$

Now observe

$$\zeta_{12} - 1 = \frac{\sqrt{2}-\sqrt{6}}{2}\zeta_{24}^{-5}, \zeta_{12}^{-1} - 1 = \frac{\sqrt{2}-\sqrt{6}}{2}\zeta_{24}^{5} \tag{3.41}$$

$$\zeta_{12}^{5} - 1 = \frac{\sqrt{6}+\sqrt{2}}{2}\zeta_{24}^{11}, \zeta_{12}^{-5} - 1 = \frac{\sqrt{6}+\sqrt{2}}{2}\zeta_{24}^{-11} \tag{3.42}$$

$$\begin{aligned} P_{1,12}^{-1,1}(k,m).P_{11,12}^{-1,1}(k,m) &= P_{1,12}^{-1,1}(k,m)P_{-1,12}^{-1,1}(k,m) \quad \text{by Lemma 3.2.5} \\ &= 2^k\Big(1+\cos\frac{2\pi}{12}\Big)^k + 1 - h_{1,12} \quad \text{by Corollary (2.2.)} \\ &= 2^k(1+\frac{\sqrt{3}}{2})^k + 1 - \Big(\zeta_{12}^{m}(\zeta_{12}^{-1}-1)^k + \zeta_{12}^{-m}(\zeta_{12}-1)^k\Big) \\ &= 2^k(1+\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{2}-\sqrt{6}}{2})^k(\zeta_{12}^{m}\zeta_{24}^{5k} + \zeta_{12}^{-m}\zeta_{24}^{-5k}) \\ &= 2^k(1+\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{2}-\sqrt{6}}{2})^k(\zeta_{24}^{5k+2m} + \zeta_{24}^{-(5k+2m)}) \quad \text{by (3.41), (3.42)} \\ &= 2^k(1+\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{2}-\sqrt{6}}{2})^k.2\cos\frac{2\pi(5k+2m)}{24} \end{aligned} \tag{3.43}$$

$$\begin{aligned} P_{5,12}^{-1,1}(k,m)P_{7,12}^{-1,1}(k,m) &= P_{5,12}^{-1,1}(m,k)P_{-5,12}^{-1,1}(k,m) \quad \text{by Lemma 3.2.5} \\ &= 2^k(1+\cos\frac{10\pi}{12})^k + 1 - h_{5,12} \\ &= 2^k(1-\frac{\sqrt{3}}{2})^k + 1 - \Big(\zeta_{12}^{5m}(\zeta_{12}^{-5}-1)^k + \zeta_{12}^{-5m}(\zeta_{12}^{5}-1)^k\Big) \\ &= 2^k(1-\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{6}+\sqrt{2}}{2})^k(\zeta_{12}^{5m}\zeta_{24}^{-11k} + \zeta_{12}^{-5m}\zeta_{24}^{11k}) \\ &= 2^k(1-\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{6}+\sqrt{2}}{2})^k(\zeta_{24}^{(11k-10m)} + \zeta_{24}^{-(11k-10m)}) \\ &= 2^k(1-\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{6}+\sqrt{2}}{2})^k.2\cos\frac{2\pi(11k-10m)}{24} \end{aligned} \tag{3.44}$$

using (3.39),(3.40), (3.43), (3.44) we get

$$
\begin{aligned}
|G_{12k}(x_0x_mx_k^{-1})^{ab}| &= |P^{-1,1}_{0,12}(k,m)P^{-1,1}_{1,12}(k,m)...P^{-1,1}_{11,12}(k,m) \\
= (-1)^{k+1}\Big(2^k - (-1)^{k+m}\Big)&\Big(3^k + 1 - (-\sqrt{3})^k(2.\cos\frac{\pi(k+4m)}{6})\Big)\Big(2 - 2.\cos\frac{\pi(2k-m)}{3}\Big) \\
&\Big((2 - 2.\cos\frac{\pi(2k-m)}{3})(2^k + 1 - (\sqrt{2})^k.2\cos\frac{\pi(k+2m)}{4})\Big) \\
&\Big(2^k(1+\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{2}-\sqrt{6}}{2})^k.2\cos\frac{\pi(5k+2m)}{12})\Big) \\
&\Big(2^k(1-\frac{\sqrt{3}}{2})^k + 1 - (\frac{\sqrt{6}+\sqrt{2}}{2})^k.2\cos\frac{\pi(11k-10m)}{12}\Big)|
\end{aligned}
$$

□

Chapter 4

Counting $G_n(m,k)$ groups.

4.1 Introduction

In this chapter we count $G_n(m,k)$ groups up to isomorphisms, in order to do this we use classification of finite $G_n(m,k)$ groups in [Wil12], the number of generators of $G_n(m,k)^{\mathrm{ab}}$ groups, and the order of $G_n(m,k)^{\mathrm{ab}}$. In Section 4.2, we determine a lower bound for the number of generators of $G_n(m,k)^{\mathrm{ab}}$ groups for certain values of n, by giving homomorphisms from the groups whose number of generators we do not know to groups whose minimum number of generators we do know (identified by Table A.1). In Section 4.3 we use methods we mentioned above to give lower bounds for $f(n)$ of $G_n(m,k)$ groups for certain values of n.

4.2 Lower bound for number of generators of $G_n(m,k)^{ab}$ groups

Definition 4.2.1. For a group G let $d(G)$ denotes *the minimum number of generators of* G^{ab}.

Lemma 4.2.2. If $n \equiv 0 \bmod q$ then $G_n(m,k)$ maps onto $G_q(m,k)$.

Proof. Let $\twoheadrightarrow$ denotes the sujective homomorphism between two groups, then we have

$$\begin{aligned} G_{\alpha q}(m,k) &\cong \langle x_0, x_1, ..., x_{\alpha q-1} \mid x_i x_{i+m} = x_{i+k},\ i = 0,1,...,\alpha q - 1\rangle \\ &\twoheadrightarrow \langle x_0, x_1, ..., x_{\alpha q-1} \mid x_i x_{i+m} = x_{i+k}, x_i = x_{i+q},\ i = 0,1,...,\alpha q - 1\rangle \\ &\twoheadrightarrow \langle x_0, x_1, ..., x_{q-1} \mid x_i x_{i+m} = x_{i+k},\ i = 0,1,...,q - 1\rangle = G_q(m,k) \end{aligned}$$

□

In certain cases, we can use this to obtain lower bounds for $d(G_n(m,k))$.

Example 4.2.3. Let $n \equiv 0 \bmod 7$ then $d\Big(H(n,3)^{ab}\Big) \geq 3$.

Proof. Suppose that $n = 7q$. By Lemma 4.2.2 we have that $H(n,3) = G_n(3,1)$ maps onto $H(7,3) = G_7(3,1)$. Now from Table A.1 we have $H(7,3)^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $d\Big(H(7,3)^{ab}\Big) \geq 3$ hence $d\Big(H(n,3)^{ab}\Big) \geq 3$. □

Corollary 4.2.4.

(a) If $n \equiv 0$ mod 7 then $d(H(n,3)) \geq 3$.

(b) If $n \equiv 0$ mod 12 then $d(H(n,8)) \geq 3$.

(c) If $n \equiv 0$ mod 13 then $d(G_n(1,3)) \geq 3$.

(d) If $n \equiv 0$ mod 15 then $d(H(n,4)) \geq 4$.

(e) If $n \equiv 0$ mod 18 then $d(H(n,8)) \geq 3$.

(f) If $n \equiv 0$ mod 24 then $d(H(n,3)) \geq 3$.

(g) If $n \equiv 0$ mod 24 then $d(H(n,8)) \geq 3$.

(h) If $n \equiv 0$ mod 30 then $d(H(n,4)) \geq 4$.

(i) If $n \equiv 0$ mod 30 then $d(H(n,8)) \geq 3$.

Proof. From Lemma 4.2.2 and Table A.1 we have

(a) $H(n,3) \twoheadrightarrow H(7,3) \twoheadrightarrow H(7,3)^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, therefore $d(H(7,3)^{ab}) = 3$ and $d(H(n,3)^{ab}) \geq 3$.

(b) $H(n,8) \twoheadrightarrow H(12,8) \twoheadrightarrow H(12,8)^{ab} = \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$, therefore $d(H(12,8)^{ab}) = 3$ and $d(H(n,8)^{ab}) \geq 3$.

(c) $G_n(1,3) \twoheadrightarrow G_{13}(1,3) \twoheadrightarrow G_{13}(1,3)^{ab} = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, therefore $d(G_{13}(1,3)^{ab}) = 3$ and $d(G_n(1,3)^{ab}) \geq 3$.

(d) $H(n,4) \twoheadrightarrow H(15,4) \twoheadrightarrow H(15,4)^{ab} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{22}$, therefore $d(H(15,4)^{ab}) = 4$ and $d(H(n,4)^{ab}) \geq 4$.

(e) $H(n,8) \twoheadrightarrow H(18,8) \twoheadrightarrow H(18,8)^{ab} = \mathbb{Z}_{19} \oplus \mathbb{Z} \oplus \mathbb{Z}$, therefore $d(H(18,8)^{ab}) = 3$ and $H(n,8)^{ab}) \geq 3$.

(f) $H(n,3) \twoheadrightarrow H(24,3) \twoheadrightarrow H(24,3)^{ab} = \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{35}$, therefore $d(H(24,3)^{ab}) = 3$ and $d(H(n,3)^{ab}) \geq 3$.

(g) $H(n,8) \twoheadrightarrow H(24,8) \twoheadrightarrow H(24,8)^{ab} = \mathbb{Z}_{85} \oplus \mathbb{Z} \oplus \mathbb{Z}$, therefore $d(H(24,8)^{ab}) = 3$ and $d(H(n,8)^{ab}) \geq 3$.

(h) $H(n,4) \twoheadrightarrow H(30,4) \twoheadrightarrow H(30,4)^{ab} = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{396}$, then $d(H(30,4)^{ab}) = 4$ and $d(H(n,4)^{ab}) \geq 4$.

(i) $H(n,8) \twoheadrightarrow H(30,8) \twoheadrightarrow H(30,8)^{ab} = \mathbb{Z}_{341} \oplus \mathbb{Z} \oplus \mathbb{Z}$, therefore $d(H(30,8)^{ab}) = 3$ and $d(H(30,8)^{ab}) \geq 3$.

□

4.3 Lower bounds on $f(n)$

In here we give answer for Question 1.5.15, we give lower bounds for the number of non isomorphic $G_n(m,k)$ groups for certain values of n. In order to do this we use lower bounds for the minimum number of generators of $G_n(m,k)^{\mathrm{ab}}$, the finiteness classification of $G_n(m,k)$ and the order of $G_{pm}(m,k)^{\mathrm{ab}}, G_{pk}(m,k)^{\mathrm{ab}}$ when $p=3$.

From Corollary 3.3.2 and Theorem 3.4.1, we have

$$|G_{3m}(x_0x_mx_k^{-1})^{ab}| = (2^m-1)(2-2\cos\frac{(2k-m)\pi}{3}) \quad \text{and}$$

$$|G_{3k}(x_0x_mx_k^{-1})^{ab}| = 3^k+1-(-\sqrt{3})^k(2.\cos\frac{(k+4m)\pi}{6}).$$

The following corollaries follow from this

Corollary 4.3.1.

$$| G_{3k}(x_0x_1x_k^{-1})^{ab} |= \begin{cases} 3^k+3^{\frac{k}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 0 \text{ or } 4 \bmod 12 \\ 3^k+1 & \text{when} \quad \mathrm{k} \equiv 5 \bmod 12 \\ 3^k-3^{\frac{k}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 6 \text{ or } 10 \bmod 12 \\ 3^k-3^{\frac{k+1}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 1 \text{ or } 3 \bmod 12 \\ 3^k-2.3^{\frac{k}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 8 \bmod 12 \\ 3^k+2.3^{\frac{k}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 2 \bmod 12 \\ 3^k+3^{\frac{k+1}{2}}+1 & \text{when} \quad \mathrm{k} \equiv 7 \text{ or } 9 \bmod 12 \end{cases}$$

Proof. From the fact that

$$-(-\sqrt{3})^k.2\cos\frac{(4+k)\pi}{6} = \begin{cases} 3^{\frac{k}{2}} & \text{when} \quad \mathrm{k} \equiv 0 \text{ or } 4 \bmod 12 \\ 0 & \text{when} \quad \mathrm{k} \equiv 5 \bmod 12 \\ -3^{\frac{k}{2}} & \text{when} \quad \mathrm{k} \equiv 6 \text{ or } 10 \bmod 12 \\ -3^{\frac{k+1}{2}} & \text{when} \quad \mathrm{k} \equiv 1 \text{ or } 3 \bmod 12 \\ -2.3^{\frac{k}{2}} & \text{when} \quad \mathrm{k} \equiv 8 \bmod 12 \\ 2.3^{\frac{k}{2}} & \text{when} \quad \mathrm{k} \equiv 2 \bmod 12 \\ 3^{\frac{k+1}{2}} & \text{when} \quad \mathrm{k} \equiv 7 \text{ or } 9 \bmod 12 \end{cases}$$

□

Corollary 4.3.2.

$$|G_{3m}(x_0 x_m x_1^{-1})^{ab}| = \begin{cases} 3(2^m - 1) & \text{when} \quad \text{m} \equiv 0 \text{ or } 4 \bmod 6 \\ 2^m - 1 & \text{when} \quad \text{m} \equiv 1 \text{ or } 3 \bmod 6 \\ 0 & \text{when} \quad \text{m} \equiv 2 \bmod 6 \\ 4(2^m - 1) & \text{when} \quad \text{m} \equiv 5 \bmod 6 \end{cases}$$

Proof. From the fact that

$$2 - 2\cos\frac{(m-2)\pi}{3} = \begin{cases} 3 & \text{when} \quad \text{m} \equiv 0 \text{ or } 4 \bmod 6 \\ 1 & \text{when} \quad \text{m} \equiv 1 \text{ or } 3 \bmod 6 \\ 0 & \text{when} \quad \text{m} \equiv 2 \bmod 6 \\ 4 & \text{when} \quad \text{m} \equiv 5 \bmod 6 \end{cases}$$

□

Definition 4.3.3. See for example [NZM08, pages 199 – 200] *The Lucas sequence* is defined by

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, L_1 = 1, L_2 = 3, n \geq 2. \tag{4.1}$$

The first few Lucas number are $2, 1, 3, 4, 7, 11, 18, 29, ...$ L_n is determined by the relation

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{4.2}$$

Recall from the Introduction Theorems 4.3.4 and 4.3.5 below, which will be used in counting $G_n(m,k)^{ab}$ groups.

Theorem 4.3.4. *[CWLF67, Lyndon]* $|F(2,n)^{\mathrm{ab}}| = L_n - 1 - (-1)^n$ *. In particular,* $F(2,n)^{\mathrm{ab}}$ *is finite for all n. Let* $m = n - 1$ *then*

$$F(2,n)^{\mathrm{ab}} = \begin{cases} \mathbb{Z}_s & if\ (n,6) = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2s} & if\ (n,6) = 2 \\ \mathbb{Z}_s \oplus \mathbb{Z}_s & if\ (n,6) = 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{5s} & if\ (n,6) = 6 \end{cases}$$

For the Sieradski groups, the structure of $S(2,n)^{ab}$ is given by the following theorem

Theorem 4.3.5. *[JO94, COS08]*

$$S(2,n)^{\text{ab}} = \begin{cases} 1 & if\ (n,6) = 1 \\ \mathbb{Z}_3 & if\ (n,6) = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & if\ (n,6) = 3 \\ \mathbb{Z} \oplus \mathbb{Z} & if\ (n,6) = 6 \end{cases}$$

Lemma 4.3.6. *If $n \equiv 0$ mod 3 then $\mid F(2,n)^{ab} \mid$ is even.*

Proof. From Theorem 4.3.4, we have that $\mid F(2,n)^{ab} \mid = L_n - 1 - (-1)^n$, to prove our claim we need to prove that L_n is even for $n \equiv 0$ mod 3. Now let $n = 3j$ therefore we will prove that L_{3j} is even for any $j \geq 1$, ie $L_{3j} \equiv 0$ mod 2 for any $j \geq 1$. Now since $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ therefore

$$L_{j+3} = L_{j+2} + L_{j+1} \tag{4.3}$$

$$= L_j + 2L_{j+1} \tag{4.4}$$

therefore

$L_{j+3} \equiv L_j$ mod 2, so $L_{3j} \equiv L_{3(j-1)} \equiv L_{3(j-2)} ... \equiv L_6 \equiv L_3 \equiv L_0 \equiv 2 \equiv 0$ mod 2

□

Theorem 4.3.7. For certain values of n lower bounds of isomorphisms classes will be as follow

(a) If $n \equiv 0$ mod 4, $n \geq 8$ then $f(n) \geq 4$.

(b) If $n \equiv 2$ mod 4, $n \geq 10$ then $f(n) \geq 5$.

(c) If $n \equiv 3$ mod 6, $n \geq 9$ then $f(n) \geq 4$.

(d) If $n \equiv 0$ mod 7 then $f(n) \geq 3$.

(e) If $n \equiv 0$ mod 12 then $f(n) \geq 6$.

(f) If $n \equiv 0$ mod 13 then $f(n) \geq 3$.

(g) If $n \equiv 0$ mod 15 then $f(n) \geq 4$.

(h) If $n \equiv 0 \bmod 18$ then $f(n) \geq 8$.

(i) If $n \equiv 0 \bmod 24$ then $f(n) \geq 6$.

(j) If $n \equiv 0 \bmod 30$ then $f(n) \geq 7$.

Proof. In Table 4.1 we bound below the number of isomorphism classes of groups by exhibiting various isomorphism classes of groups and showing none of them are isomorphic. We identify groups by showing whether or not the group is finite, groups have different values of d or different $|G_n(m,k)^{ab}|$). Lower bounds of d come from Corollary 4.2.4, and the orders of $|G_n(m,k)^{ab}|$ are from Corollaries 4.3.1, 4.3.2 and Theorems 4.3.4, 4.3.5.

In here I will explain one case of the table and the argument will be similar for the others. When $n \equiv 0 \bmod 4, n \geq 8$ we have 4 groups $F(2,n), S(2,n), H(n, \frac{n}{2})$ and $H(n, \frac{n}{2}+1)$, the group $H(n, \frac{n}{2}+1)$ is the only group in this case which is finite while the others are infinite, therefore it is not isomorphic to any of them. For the group $F(2,n)$ we have that $d = 2, 3 < |F(2,n)^{ab}| < \infty$, so it is not isomorphic to the group $S(2,n)$ since $S(2,n)^{ab} \cong \mathbb{Z}_3$ or $\mathbb{Z} \oplus \mathbb{Z}$, and not isomorphic to $H(n, \frac{n}{2})$ since $H(n, \frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$ (d=1). Also $S(2,n) \ncong H(n, \frac{n}{2})$ since they have different abelianization.

□

Table 4.1: *The lower bound of f(n) for certain values of n.*

n	Groups	Finite	Informations
(a) $n \equiv 0 \bmod 4$, $n \geq 8$	$G_n(1,2) = F(2,n)$	No	$d = 2, 3 < \lvert F(2,n)^{ab}\rvert < \infty$
	$G_n(2,1) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z}_3$ or $\mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
(b) $n \equiv 2 \bmod 4$, $n \geq 10$	$G_n(1,2) = F(2,n)$	No	$d = 2, 3 < \lvert F(2,n)^{ab}\rvert < \infty$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z}_3$ or $\mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
	$G_n(1,\frac{n}{2})$	$\mathbb{Z}_{2^{\frac{n}{2}}-1}$	
(c) $n \equiv 3 \bmod 6$, $n \geq 9$	$G_n(1,2) = F(2,n)$	No	$d = 2, \lvert F(2,n)^{ab}\rvert > 4$ $4 \mid \lvert F(2,n)^{ab}\rvert$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$
	$G_n(x_0x_1x_{\frac{n}{3}}^{-1})$	No	$\mid G_n(x_0x_1x_{\frac{n}{3}}^{-1})^{ab} \mid = 3^{\frac{n}{3}} \pm 3^{\frac{\frac{n}{3}+1}{2}} + 1$
	$G_n(x_0x_{\frac{n}{3}}x_1^{-1})$	No	$\lvert G_n(x_0x_{\frac{n}{3}}x_1^{-1})^{ab}\rvert = 2^{\frac{n}{3}} - 1$
(d) $n \equiv 0 \bmod 7$	$G_n(1,2) = F(2,n)$	No	$d = 1, F(2,n)^{ab} > 4$
	$G_n(1,2) = S(2,n)$	No	$d = 1, S(2,n)^{ab} \cong 1$ or $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$
	$G_n(3,1) = H(n,3)$	No	$d \geq 3$

n	Groups	Finite	Informations
(e) $n \equiv 0 \bmod 12$	$G_n(1,2) = F(2,n)$	No	$d = 2, 0 < \lvert F(2,n)^{ab}\rvert < \infty, 5 \mid \lvert F(2,n)^{ab}\rvert$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
	$G_n(8,1) = H(n,8)$	No	$d \geq 3, \lvert H(n,8)^{ab} \rvert = \infty$
	$G_n(x_0x_1x_{\frac{n}{3}}^{-1})$	No	$5 \nmid \lvert G_n(x_0x_1x_{\frac{n}{3}}^{-1})^{ab}\rvert = 3^{\frac{n}{3}} - 2.3^{\frac{n}{6}} + 1$ or $3^{\frac{n}{3}} + 3^{\frac{n}{6}} + 1 (odd)$
(f) $n \equiv 0 \bmod 13$	$G_n(1,2) = F(2,n)$	No	$d = 1, \lvert F(2,n)^{ab}\rvert > 4$
	$G_n(1,2) = S(2,n)$	No	$d = 1, \lvert S(2,n)^{ab}\rvert \cong 1$ or $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$
	$G_n(1,3)$	No	$d \geq 3$
(g) $n \equiv 0 \bmod 15$	$G_n(1,2) = F(2,n)$	No	$d = 2, \lvert F(2,n)^{ab}\rvert = \mathbb{Z}_2 \oplus \mathbb{Z}_{2s}, s > 1$
	$S(2,n)$	No	$d = 2, \lvert S(2,n)^{ab}\rvert \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$
	$G_n(4,1) = H(n,4)$	No	$d \geq 4, 2 \mid \lvert H(n,4)^{ab}\rvert < \infty$
	$G_n(x_0x_{\frac{n}{3}}x_1^{-1})$	No	$\lvert G_n(x_0x_{\frac{n}{3}}x_1^{-1})^{ab}\rvert = 3(2^{\frac{n}{3}} - 1)$ or ∞
(h) $n \equiv 0 \bmod 18$	$G_n(1,2) = F(2,n)$	No	$d = 2, 0 < \lvert F(2,n)^{ab}\rvert < \infty (even)$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
	$G_n(1,\frac{n}{2})$	$\mathbb{Z}_{2^{\frac{n}{2}}-1}$	
	$G_n(8,1) = H(n,8)$	No	$d \geq 3, \lvert H(n,8)^{ab}\rvert = \infty$
	$G_n(x_0x_{\frac{n}{3}}x_1^{-1})$	No	$\lvert G_n(x_0x_{\frac{n}{3}}x_1^{-1})^{ab}\rvert = 3(2^{\frac{n}{3}} - 1)$
	$G_n(x_0x_1x_{\frac{n}{3}}^{-1})$	No	$\lvert G_n(x_0x_1x_{\frac{n}{3}}^{-1})^{ab}\rvert = 3^{\frac{n}{3}} \pm 3^{\frac{n}{6}} + 1$

n	Groups	Finite	Informations
(i) $n \equiv 0 \bmod 24$	$G_n(1,2) = F(2,n)$	No	$d = 2, 5 \mid \lvert F(2,n)^{ab} \rvert < \infty$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
	$G_n(8,1) = H(n,8)$	No	$d \geq 3, \mid H(n,8)^{ab} \mid = \infty$
	$G_n(x_0x_1x_{\frac{n}{3}}^{-1})$	No	$5 \nmid \lvert G_n(x_0x_1x_{\frac{n}{3}}^{-1})^{ab} \rvert = 3^{\frac{n}{3}} - 2.3^{\frac{n}{6}} + 1$ or $3^{\frac{n}{3}} + 3^{\frac{n}{6}} + 1 (odd)$
(j) $n \equiv 0 \bmod 30$	$G_n(1,2) = F(2,n)$	No	$d = 2, \lvert F(2,n)^{ab} \rvert < \infty$
	$G_n(1,2) = S(2,n)$	No	$S(2,n)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$
	$G_n(\frac{n}{2},1) = H(n,\frac{n}{2})$	No	$H(n,\frac{n}{2})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$
	$G_n(\frac{n}{2}+1,1) = H(n,\frac{n}{2}+1)$	$\mathbb{Z}_{2^{\frac{n}{2}}+1}$	
	$G_n(1,\frac{n}{2})$	$\mathbb{Z}_{2^{\frac{n}{2}}-1}$	
	$G_n(8,1) = H(n,8)$	No	$d \geq 3$ $\lvert H(n,8)^{ab} \rvert = \infty$
	$G_n(4,1) = H(n,4)$	No	$d \geq 4,$ $4 \mid \lvert H(n,4)^{ab} \rvert < \infty$

Chapter 5

Counting $\Gamma_n(k,l)$ groups.

In this chapter we count $\Gamma_n(k,l)$ groups up to isomorphism, our results are based on results of Edjvet and Williams in [EW10]. Their results were stated in terms of three conditions $(A),(B),(C)$ being true or false, and have been summarised in [EW10, Table 1], which we reproduce as Table 5.1 (this is Table 1.5). We study the groups in terms of the four conditions $(A),(B),(C),(D)$ being true or false, this gives 16 combinations.

Table 5.1: *Summary of structures of* $\Gamma_n(k,l)$ *[EW10, Table 1]*

(A)	(B)	(C)		Aspherical	Abelianization	Group
F	F	F		Yes	finite$\neq 1$	∞
F	F	T		No	$\mathbb{Z}_\alpha$	Metacyclic
F	T	F		No	$\mathbb{Z}_3$	$\mathbb{Z}_3$
T	F	F	$n \neq 18$	Yes	∞	Large
T	F	F	$n = 18$	No	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_{19}$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_{19}$
T	F	T		No	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_\gamma$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$
T	T	F		No	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}*\mathbb{Z}$
T	T	T		No	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}*\mathbb{Z}$

In Section 5.1, we show that the first 3 conditions are preserved under isomorphisms, however we are unable to do so for the fourth condition (D).

In Section 5.2 we show that six out of the 16 combinations are impossible namely $FTTF, FTTT$, $TTTF, TTFF, FTFF$, $FFTT$. Furthermore the case $TTTT$ occurs when $n = 3$ or $n = 6$ and the case $TFTT$ occurs when $n = 12$. These two cases will be studied in Chapter 6, where we give information about $\Gamma_n(k,l)$ for $n \leq 29$, for simplicity we consider $n > 12$ (the cases when $n < 12$ are well understood).

Now eight combinations are left to be studied *FFFF, FFTF, FTFT, TFFF, TFTF, TTFT, FFFT, TFFT*.

In Section 5.3 we consider the six cases *FFTF, FTFT, TFTF, TTFT, FFFT, TFFT*. We show that in each case there is exactly one group or no groups, we determine values of n for which we get 1 group, and show that for other values of n the number of groups is 0. We express our results by using the following definition

Definition 5.0.8. *Let* $(n,k,l) = 1, k \neq l, 1 \leq k, l \leq n-1$. *We define* $f^{(abcd)}(n)$ *to be the number of* $\Gamma_n(k,l)$ *groups up to isomorphism,* where $a,b,c,d \in \{T,F,-\}$ and a,b,c,d denote to conditions $(A),(B),(C),(D)$

$$a = T \text{ means (A) holds}$$
$$a = F \text{ means (A) does not hold}$$
$$a = - \text{ means there is no restriction on A}$$

Similarly for b,c,d.

In the cases *FFFF, TFFF* we are unable to find the precise number of the groups, but we are able to obtain lower bounds of $\Gamma_n(k,l)$ groups for certain values of n, we do this in Section 5.4.

It turns out (see Lemma 5.2.4) that in the cases *FFFT* and *TFFT* the groups $\Gamma_n(k,l)$ are isomorphic to $\Gamma_n(1, \frac{n}{2}-1)$, in [EW10, page 774] it was observed without proof that in the case *FFFT* we have $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2}-1)$. In Lemma 5.2.4 we prove that observation and extend it to include the case *TFFT*. We study this group in more detail in Section 5.5. We show in Theorem 5.5.2 that in the case *FFFT*, the group has finite abelianization of order $|\Gamma_n(1, \frac{n}{2}-1)^{ab}| = 3\left(L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}}\right)$, where L_n is Lucas number of order n. In the case *TFFT* the group is known to have infinite abelianization and we show in Theorem 5.5.1 that it has torsion free rank 2. In Section 5.6, we investigate a question similar to question 1.5.16 which was about $G_n(m,k)$ groups. Our question is when does $\Gamma_n(k,l) \cong \Gamma_{n'}(k',l')$ imply $n = n'$?

We determine here $\Gamma_n^{\mathrm{ab}}(1, \frac{n}{2})$, in order to identify groups later in this chapter.

Proposition 5.0.9. *Let n be even then*

$$\Gamma_n(1,\frac{n}{2})^{ab} = \begin{cases} \mathbb{Z}_{2^{\frac{n}{2}}-1} & \text{If} \quad n \equiv 0 \bmod 4 \\ \mathbb{Z}_{2^{\frac{n}{2}}+1} & \text{If} \quad n \equiv 2 \bmod 4 \end{cases}$$

Proof. Now by definition

$$\Gamma_n(1,\frac{n}{2})^{ab} = \langle x_0, x_1, ..., x_{n-1} \mid x_i x_{i+1} x_{i+\frac{n}{2}} = 1, x_i x_j = x_j x_i, 0 \le i,j \le n-1 \rangle^{ab}$$

We could now add to the previous presentation the relation $x_i = x_{i+\frac{n}{2}}$ which come from adding $\frac{n}{2}$ to the index i in the relation

$$x_i x_{i+1} x_{i+\frac{n}{2}} = 1 \tag{5.1}$$

we get

$$x_{i+\frac{n}{2}} x_{i+\frac{n}{2}+1} x_i = 1 \tag{5.2}$$

then by using abelianization in relations (5.1), (5.2) we get $x_{i+1} = x_{i+\frac{n}{2}+1}$ and by subtracting 1 from the index in relation we get $x_i = x_{i+\frac{n}{2}}$.

$$\Gamma_n(1,\frac{n}{2})^{ab} = \langle x_0, x_1, ..., x_{n-1} \mid x_i x_{i+1} x_{i+\frac{n}{2}} = 1, x_i = x_{i+\frac{n}{2}}, x_i x_j = x_j x_i, 0 \le i,j \le n-1 \rangle$$

This allow us to write the presentation in this form

$$= \langle x_0, x_1, ..., x_{\frac{n}{2}-1} \mid x_i^2 x_{i+1} = 1, i = 0, 1, ..., \frac{n}{2} - 1 \rangle$$

$$= \langle x_0, x_1, ..., x_{\frac{n}{2}-1} \mid x_0^2 x_1 = 1, x_1^2 x_2 = 1, ..., x_{\frac{n}{2}-2}^2 x_{\frac{n}{2}-1} = 1, x_{\frac{n}{2}-1}^2 x_0 = 1 \rangle$$

By eliminating $x_0 = x_{\frac{n}{2}-1}^{-2}$ we get

$$\Gamma_n(1,\frac{n}{2}) = \langle x_1, ..., x_{\frac{n}{2}-1} \mid x_{\frac{n}{2}-1}^{-4} x_1 = 1, x_1^2 x_2 = 1, ..., x_{\frac{n}{2}-2}^2 x_{\frac{n}{2}-1} = 1 \rangle$$

By eliminating $x_{\frac{n}{2}-1} = x_{\frac{n}{2}-2}^{-2}$ we get

$$\Gamma_n(1,\frac{n}{2}) = \langle x_0, x_1, ..., x_{\frac{n}{2}-2} \mid x_{\frac{n}{2}-1}^{8} x_1 = 1, x_1^2 x_2 = 1, ..., x_{\frac{n}{2}-3}^2 x_{\frac{n}{2}-2} = 1\rangle$$

when $n \equiv 0 \ mod\ 4$ therefore $\frac{n}{2}$ is even and by doing $\frac{n}{2} - 1$ eliminations we get

$$\Gamma_n(1,\frac{n}{2})^{\mathrm{ab}} = \langle x_1 \mid x_1^{2^{\frac{-n}{2}}} x_1 = 1\rangle = \langle x_1 \mid x_1^{2^{\frac{n}{2}}-1} = 1\rangle = \mathbb{Z}_{2^{\frac{n}{2}}-1}$$

when $n \equiv 2 \ mod\ 4$ therefore $\frac{n}{2}$ is odd and by doing $\frac{n}{2} - 1$ eliminations we get

$$\begin{aligned}\Gamma_n(1,\frac{n}{2})^{\mathrm{ab}} &= \langle x_1 \mid x_1^{2^{\frac{n}{2}}} x_1 = 1\rangle \\ &= \langle x_1 \mid x_1^{2^{\frac{n}{2}}+1} = 1\rangle \\ &= \mathbb{Z}_{2^{\frac{n}{2}}+1}\end{aligned}$$

□

We summarise our results in Table 5.2, and we assume that $(n,k,l) = 1, k \neq l$. We record here that [EW10, Theorem B (i)] stated that if $k = l$, in which case $\Gamma \cong \mathbb{Z}_s$ where $s = 2^n - (-1)^n$. In Table 5.2 for the 6 cases *FFTF, FTFT, TFTF, TTFT, FFFT, TFFT*, $f^{(abcd)}(n) = 1$ for values of n that appear in the table, and $f^{(abcd)}(n) = 0$ for values of n that do not appear in the table. $f^{(abcd)}(n) \geq 1$ for the cases *FFFF, TFFF*.

Table 5.2: $(n,k,l) = 1, k \neq l, k \neq 0, l \neq 0, n > 12, \alpha = 3(2^{n/3} - (-1)^{n/3}), \gamma = (2^{n/3} - (-1)^{n/3})/3)$

Values of n	(A)	(B)	(C)	(D)	Aspherical	Abelianization	Group	$\Gamma_n(k,l) \cong$	$f^{(abcd)}(n)$
	F	F	F	F	Yes	finite$\neq 1$	∞	Unknown	≥ 1
$n \equiv 2$ or 4 mod 6	F	F	F	T	Yes	$\lvert\Gamma_n(k,l)^{ab}\rvert = 3\left(L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}}\right)$	∞	$\Gamma_n(1,\frac{n}{2}-1)$	1
$n \equiv 0$ mod $3, n \geq 6$	F	F	T	F	No	$\mathbb{Z}_\alpha$	Metacyclic	$\Gamma_n(\frac{n}{3},\frac{1+2n}{3})$	1
$n \equiv 1$ or 2 mod 3	F	T	F	T	No	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\Gamma_n(1,2)$	1
	T	F	F	F	Yes	∞	Large	Unknown	≥ 1
$n \equiv 0$ mod 6, $n > 18$	T	F	F	T	Yes	∞ has torsion − free rank 2	∞	$\Gamma_n(1,\frac{n}{2}-1)$	1
$n \equiv 0$ mod 6, $n = 18$	T	F	F	T	No	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_{19}$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_{19}$	$\Gamma_n(1,\frac{n}{2}-1)$	1
$n \equiv 3$ or 6 mod 9	T	F	T	F	No	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_\gamma$	$\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$	$\Gamma_n(1,\frac{n}{3})$	1
$n \equiv 0$ mod $3, n \geq 9$	T	T	F	T	No	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}*\mathbb{Z}$	$\Gamma_n(1,2)$	1

We recall from introduction the following lemma (Lemma 1.6.1), which gives isomorphisms between $\Gamma_n(k,l)$ groups

Lemma 5.0.10. *[EW10, Lemma 2.1.] Let $1 \leq k, l \leq n-1$ then*

1. $\Gamma_n(k,l) \cong \Gamma_n(l-k,-k)$.

2. $\Gamma_n(k,l) \cong \Gamma_n(l,k)$.

3. $\Gamma_n(k,l) \cong \Gamma_n(k-l,-l)$.

4. $\Gamma_n(k,l) \cong \Gamma_n(k,k-l)$.

5. *If* $(k,n)=1$ *then* $\Gamma_n(k,l) \cong \Gamma_n(1,Kl)$, *where* $Kk \equiv 1(\bmod n)$.

6. *If n is even and* $(l,n)=1$ *then* $\Gamma_n(k,l) \cong \Gamma_n(1,Lk+1)$, *where* $Ll \equiv -1(\bmod n)$.

5.1 Preservation of conditions $(A),(B),(C),(D)$ under isomorphisms

Here we consider the four conditions $(A),(B),(C),(D)$, we prove the following theorem

Theorem 5.1.1. *Suppose* $(n,k_1,l_1)=1, 1 \leq k_1 \leq n-1, 1 \leq l_1 \leq n-1, k_1 \neq l_1$ *and* $(n,k_2,l_2)=1, 1 \leq k_2 \leq n-1, 1 \leq l_2 \leq n-1, k_2 \neq l_2$ *and suppose* $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2)$ *then*

(a) *If n,k_1,l_1 satisfy (A) then n,k_2,l_2 satisfy (A).*

(b) *If n,k_1,l_1 satisfy (B) then n,k_2,l_2 satisfy (B).*

(c) *If n,k_1,l_1 satisfy (C) then n,k_2,l_2 satisfy (C).*

(d) *If n,k_1,l_1 satisfy (D) then one of the following holds*

- **(i)** *n,k_2,l_2 satisfy (D).*
- **(ii)** *n,k_1,l_1 satisfy FFFT and n,k_2,l_2 satisfy FFFF.*
- **(iii)** *n,k_1,l_1 satisfy TFFT and n,k_2,l_2 satisfy TFFF.*

Proof.

(a) Suppose n, k_1, l_1 satisfy (A), then [EW10, Lemma2.2.] implies that $|\Gamma_n(k_1,l_1)^{ab}| = \infty$. If n, k_2, l_2 does not satisfy (A), then $|\Gamma_n(k_2,l_2)^{ab}| < \infty$, this contradicts $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2)$, therefore n, k_2, l_2 satisfy (A).

(b) Suppose (n, k_1, l_1) satisfy (B), then from Table 5.1, $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z}$ or $\mathbb{Z}_3$. Suppose for contradiction that (n, k_2, l_2) does not satisfy (B), then one of the following hold

From line 1 in Table 5.1, $|\Gamma_n(k_2,l_2)^{ab}| < \infty, |\Gamma_n(k_2,l_2)| = \infty$. Therefore $\Gamma_n(k_2,l_2) \ncong \mathbb{Z}_3$, and since $|(\mathbb{Z} * \mathbb{Z})^{ab}| = \infty$, we get $\Gamma_n(k_2,l_2) \ncong \mathbb{Z} * \mathbb{Z}$, a contradiction.

From line 2 in Table 5.1, $|\Gamma_n(k_2,l_2)| = 2^n - (-1)^n$. Therefore $\Gamma_n(k_2,l_2) \ncong \mathbb{Z} * \mathbb{Z}$, and $\Gamma_n(k_2,l_2) \cong \mathbb{Z}_3$, only when $n = 1$ or $n = 2$, a contradiction.

From line 4 in Table 5.1, $\Gamma_n(k_2,l_2)$ is large, so $\Gamma_n(k_2,l_2) \ncong \mathbb{Z}_3$. Suppose for contradiction that $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z}$, then the presentation $P_n(k_2,l_2)$ is an aspherical presentation of $\mathbb{Z} * \mathbb{Z}$ of deficiency zero, a contradiction.

From line 5 in Table 5.1, $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19} \ncong \mathbb{Z} * \mathbb{Z}$ or $\mathbb{Z}_3$, a contradiction.

From line 6 in Table 5.1, $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma, \gamma = \left(2^{\frac{n}{3}} - (-1)^{\frac{n}{3}}\right)/3$. Therefore $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma \ncong \mathbb{Z}_3$, $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma \cong \mathbb{Z} * \mathbb{Z}$ only when $n = 3$ or $n = 6$, a contradiction.

(c) Suppose (n, k_1, l_1) satisfy (C), then from Table 5.1, either $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2) \cong \Gamma_n(\frac{n}{3}, \frac{1+2n}{3})$ (this was conjectured in [EW10, Conjectur 3.4.] and proved in [BW17, Lemma 23 and Corollary D.]), which is metacyclic of order $2^n - (-1)^n$ or $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma, \gamma = 2^{\frac{n}{3}} - (-1)^{\frac{n}{3}}/3$, (note that it was stated in [EW10, page 761] that line 8 only occurs when $n = 3$ or 6, and we will give full proof for it in Lemma 5.2.3). Suppose for contradiction that (n, k_2, l_2) does not satisfy (C), then one of the following hold

From line 1 in Table 5.1, $|\Gamma_n(k_2,l_2)^{ab}| < \infty, |\Gamma_n(k_2,l_2)| = \infty$. Therefore $\Gamma_n(k_2,l_2)$ is not isomorphic to a Metacyclic group, and since $|(\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma)^{ab}| = \infty$, $|(\mathbb{Z} * \mathbb{Z})^{ab}| = \infty$, we get $\Gamma_n(k_2,l_2) \ncong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ and $\Gamma_n(k_2,l_2) \ncong \mathbb{Z} * \mathbb{Z}$, a contradiction.

From line 3 in Table 5.1, $|\Gamma_n(k_2,l_2)| \cong \mathbb{Z}_3$. Therefore $\Gamma_n(k_2,l_2) \ncong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$, and $\Gamma_n(k_2,l_2)$ isomorphic to a metacyclic group only when $n = 1$ or $n = 2$, a contradiction.

From line 4 in Table 5.1, $\Gamma_n(k_2,l_2)$ is large, so $\Gamma_n(k_2,l_2)$is not isomorphic to a metacyclic group. Suppose for contradiction that $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$, then in both cases the presentation $P_n(k_2,l_2)$ is an aspherical presentation of $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ of deficiency zero, a contradiction (since $def(\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma) = 2$).

From line 5 in Table 5.1, $\Gamma_n(k_2,l_2)$ is not isomorphic to a metacyclic group, $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19} \ncong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ (since $\gamma \neq 19$) , a contradiction.

From line 7 in Table 5.1, $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z}$ is not isomorphic to a metacyclic group ,$\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} \ncong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ (since $\gamma = 1$ when $n = 3$) , a contradiction.

(d) Suppose (n,k_1,l_1) satisfy (D), then from Table 5.2 the group $\Gamma_n(k_1,l_1)$ can be one of the following

1. If $n \equiv 2$ or $4 \bmod 6$, then $|\Gamma_n(k_1,l_1)^{ab}| < \infty$ where $|\Gamma_n(k_1,l_1)| = \infty$ in line 2 (the case *FFFT*).
2. If $n \equiv 1$ or $2 \bmod 3$, then $\Gamma_n(k_1,l_1) \cong \mathbb{Z}_3$, in line 4 (the case *FTFT*).
3. If $n \equiv 0 \bmod 6, n \neq 18$, then $|\Gamma_n(k_1,l_1)| = \infty, |\Gamma_n(k_1,l_1)^{ab}| = \infty$, in line 6 (the case *TFFT*).
4. If $n = 18$, then $\Gamma_n(k_1,l_1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$, in line 7 (the case *TTFT*).
5. If $n \equiv 0 \bmod 3, n \geq 9$, then $\Gamma_n(k_1,l_1) \cong \mathbb{Z} * \mathbb{Z}$, in line 9 (the case *TTFT*).

Suppose for contradiction that (n,k_2,l_2) does not satisfy (D), then one of the following hold

In line 1 in Table 5.2 (the case *FFFF*), then $|\Gamma_n(k_2,l_2)^{ab}| < \infty, |\Gamma_n(k_2,l_2)| = \infty$. We do not know if there any parameters n,k_1,l_1,k_2,l_2 such that (n,k_1,l_1) satisfies *FFFT* (case 1) and (n,k_2,l_2) satisfies *FFFF* and $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2)$. But $\Gamma_n(k_2,l_2) \ncong \mathbb{Z}_3$ in case (2), a contradiction. Also $|\Gamma_n(k_2,l_2)^{ab}| \neq |\Gamma_n(k_1,l_1)^{ab}|$ in case (3), a contradiction. $|\Gamma_n(k_2,l_2)^{ab}| \neq |\Gamma_n(k_1,l_1)^{ab}| = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{19})^{ab}$ in case (4), a contradiction. Similarly $|\Gamma_n(k_2,l_2)^{ab}| \neq |\Gamma_n(k_1,l_1)^{ab}| = (\mathbb{Z} \times \mathbb{Z})^{ab}$ in (5), a contradiction.

In line 3 in Table 5.2, when $n \equiv 0 \bmod 3, n \geq 6$ (the case *FFTF*), then $|\Gamma_n(k_2,l_2)| = 2^n - (-1)^n$. We only need to consider the groups in cases (3), (4), (5), as the value of n is different in cases (1), (2). Now $|\Gamma_n(k_1,l_1)| = \infty$ in (3), (4), (5), and $|\Gamma_n(k_2,l_2)| = 2^n - (-1)^n$. $\Gamma_n(k_2,l_2) \ncong \Gamma_n(k_1,l_1)$, a contradiction.

In line 5 in Table 5.2, this is *TFFF*, then $\Gamma_n(k_2,l_2)$ is large and has infinite abelianization. We do not know are there any parameters n,k_2,l_1,k_2,l_2 such that (n,k_1,l_1) satisfies *TFFT* in case (3) and (n,k_2,l_2) satisfies *TFFF* and $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2)$. But $|\Gamma_n(k_2,l_2)^{ab}| \neq |\Gamma_n(k_1,l_1)^{ab}|$ in (1), a contradiction. Also $\Gamma_n(k_2,l_2) \ncong \mathbb{Z}_3$ in case (2), a contradiction. Now suppose for contradiction that $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$, then the presentation $P_n(k_2,l_2)$ is an aspherical of deficiency zero, and since $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$ has a presentation of deficiency 2 (namely the presentation $\langle x,y,z \mid z^{19}=1\rangle$), a contradiction. Suppose for contradiction that $\Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z}$, then the presentation $P_n(k_2,l_2)$ is an aspherical presentation of $\mathbb{Z} * \mathbb{Z}$ of deficiency zero, and since $\mathbb{Z} * \mathbb{Z}$ has a presentation of deficiency 2 (namely the presentation $\langle x,y \mid \rangle$), a contradiction.

In line 8 in Table 5.2, when $n \equiv 3$ or $6 \bmod 9$ (the case *TFTF*), then $\Gamma_n(k_2,l_2) \cong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$, $\gamma = \left(2^{\frac{n}{3}} - (-1)^{\frac{n}{3}}\right)/3$. we only consider cases (3), (4), (5), since the value of n is different in cases (1), (2). $\Gamma_n(k_2,l_2)$ does not imply cases (4) above, since $P_n(k_1,l_1)$ is aspherical presentation of deficiency zero, now suppose for contradiction that $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$, then the presentation $P_n(k_2,l_2)$ is an aspherical presentation of $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$ of deficiency zero, and since $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$ has a presentation of deficiency 2 (namely the presentation $\langle x,y,z \mid z^{19}=1\rangle$), a contradiction. Also $\Gamma_n(k_2,l_2) = \mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma \ncong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}_{19}$ in (4) since $\gamma \neq 19$. And $\Gamma_n(k_2,l_2) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma \cong \mathbb{Z} * \mathbb{Z}$ only when $n = 6$, a contradiction. □

In most of cases we showed that the condition (D) is preserved under isomorphisms, but in case (*ii*) we do not know if there any parameters n, k_2, l_1, k_2, l_2 such that (n,k_1,l_1) satisfies *FFFT* and (n,k_2,l_2) satisfies *FFFF* and $\Gamma_n(k_1,l_1) \cong \Gamma_n(k_2,l_2)$. Since that both of the groups $\Gamma_n(k_1,l_1), \Gamma_n(k_2,l_2)$ have infinite order and $\Gamma_n(k_1,l_1)^{ab}, \Gamma_n(k_2,l_2)^{ab}$ have finite order. Also both of presentations $P_n(k_1,l_1), P_n(k_2,l_2)$ are aspherical. Similarly for part (*iii*).

5.2 Combinations of $(A),(B),(C),(D)$ that are not possible for $n > 12$

Here we shall show why the eight combinations *FTTF*, *FTTT*, *TTTF*, *TTFF*, *FTFF*, *FFTT*, *TTTT*, *TFTT* are not possible when $n > 12$.

Lemma 5.2.1. *Suppose that $(n,k,l) = 1, k \neq l$. Then the combinations FTTF, TTTF, TTFF, FTFF are not possible.*

Proof. Since that when (B) holds then (D) holds, therefore proof is done. □

Lemma 5.2.2. *Let $n > 12$, if $(n,k,l) = 1, k \neq l$, and* (C), (D) *hold. Then* (A) *holds, and therefore the combinations FFTT, FTTT are not possible.*

Proof. We shall show that when $(C),(D)$ are true, then (A) is true. This is enough to show the combination $FFTT, FTTT$ are not possible. Now $(C),(D)$ are T, this means that $n \equiv 0 \bmod 6$ (first part of A holds). Assume for contradiction that second part of (A) does not hold ($k + l \not\equiv 0 \bmod 3$). Now let $n = 6m, m > 2$. If C part one ($3l \equiv 0 \bmod 6m$) holds, therefore $l = 2m$ or $l = 4m$. Now by considering (D) is T, we have three cases for each value of l

- $l = 2m$

  1. If $2(k + l) \equiv 0 \bmod 6m$, therefore $(k + l) \equiv 0 \bmod 3m$, so $(k + l) \equiv 0 \bmod 3$, a contradiction.
  2. If $2(2l - k) \equiv 0 \bmod 6m$, therefore $(2l - k) \equiv 0 \bmod 3m$, so $k \equiv 2l \bmod 3m$ then $k + l \equiv 3l \equiv 0 \bmod 3$, a contradiction.
  3. If $2(2k - l) \equiv 0 \bmod 6m$, therefore $(2k - l) \equiv 0 \bmod 3m$, so $2k \equiv l \bmod 3m$, therefore $k + l \equiv 3k \equiv 0 \bmod 3$, a contradiction.

- $l = 4m$

  1. If $2(k + l) \equiv 0 \bmod 6m$, therefore $k + l \equiv 0 \bmod 3m$, so $k + l \equiv 0 \bmod 3$, a contradiction.
  2. If $2(2l - k) \equiv 0 \bmod 6m$, therefore $(2l - k) \equiv 0 \bmod 3m$, so $k \equiv 2l \bmod 3m$,therefore $k + l \equiv 3l \equiv 0 \bmod 3m$, and then $k + l \equiv 0 \bmod 3$, a contradiction.
  3. If $2(2k - l) \equiv 0 \bmod 6m$, therefore $(2k - l) \equiv 0 \bmod 3m$, so $2k \equiv l \bmod 3m$, therefore $k + l \equiv 3k \equiv 0 \bmod 3$, and then $k + l \equiv 0 \bmod 3$, a contradiction.

If (C) part two ($3k \equiv 0 \bmod 6m$) holds, therefore $k = 2m$ or $k = 4m$, by similar argument we used above we will get that (A) is T. If (C) part three ($3(l - k) \equiv 0 \bmod 6m$) holds, then $l - k \equiv 2m \bmod 2m$, therefore

$$l - k = 2\alpha m \quad (\alpha \in \mathbb{Z}). \tag{5.3}$$

If (D) is T we have

1. If $2(k + l) \equiv 0 \bmod 6m$, then $(k + l) \equiv 0 \bmod 3m$, so $(k + l) \equiv 0 \bmod 3$, a contradiction.

2. If $2(2l-k) \equiv 0 \mod 6m$, then $2l-k \equiv 0 \mod 3m$, therefore

$$2l - k = 3\beta m \quad (\beta \in Z). \tag{5.4}$$

By (5.3), (5.4), we have $k = (3\beta - 4\alpha)m, l = (3\beta - 2\alpha)m$, therefore

$$\begin{aligned} k + l &= 6(\beta - \alpha)m \\ \text{therefore} \quad k + l &\equiv 0 \mod 6m \\ k + l &\equiv 0 \mod 3. \end{aligned}$$

A contradiction.

3. If $2(2k-l) \equiv 0 \mod 6m$, then $2k-l \equiv 0 \mod 3m$, therefore

$$2k - l = 3\gamma m \quad (\gamma \in Z). \tag{5.5}$$

By (5.3), (5.5), we have $k = (3\gamma + 2\alpha)m, l = (3\gamma + 4\alpha)m$, therefore

$$\begin{aligned} &k + l = 6(\gamma + \alpha)m \\ \Rightarrow\ &k + l \equiv 0 \mod 6m \\ \Rightarrow\ &k + l \equiv 0 \mod 3 \qquad \text{a contradiction.} \end{aligned}$$

□

Now we show that the case $TTTT$ only occurs when $n = 3$ or $n = 6$, and the case $TFTT$ only occurs when $n = 12$. We study these cases in more detail in Chapter 6.

The following lemma is stated without proof in [EW10, page 761, line 12]

Lemma 5.2.3. *For $n > 6$ there are no values of $k, l, k \neq l, (n, k, l) = 1, 1 \leq k, l \leq n-1$ such that $(A)(B)(C)(D)$ are $TTTT$.*

Proof. (A) is T gives that $n \equiv 0 \mod 3$, and $l \equiv -k \mod 3$. Assume for contradiction that $n = 3m, m > 2$. If $3|k$ then $3|l$ so $3|(n,k,l) = 1$ contradiction, therefore $3 \not| k$ and $3 \not| l$. If (C) part one ($3l \equiv 0 \mod 3m$) hold, therefore $l = m$ or $l = 2m$. Now by considering (B) is T we have three cases for each value of l

When $l = m$ we have,

1. If $k + l \equiv 0 \mod 3m$, then $k \equiv 2m \mod 3m$. Therefore $1 = (n, k, l) = (3m, 2m, m) = m$, a contradiction.

2. If $2l - k \equiv 0 \mod 3m$, then $k \equiv 2m \mod 3m$. Therefore $1 = (n, k, l) = (3m, 2m, m) = m$, a contradiction.

3. If $2k - l \equiv 0 \mod 3m$, then $2k \equiv m \mod 3m$. Therefore $k \equiv 2m \mod 3m$, so $1 = (n, k, l) = (3m, 2m, m) = m$, a contradiction.

When $l = 2m$ we have

1. If $k + l \equiv 0 \mod 3m$, then $k \equiv m \mod 3m$. Then $1 = (n, k, l) = (3m, m, 2m) = m$, a contradiction.

2. If $2l - k \equiv 0 \mod 3m$, then $k \equiv m \mod 3m$. Then $1 = (n, k, l) = (3m, m, 2m) = m$, a contradiction.

3. If $2k - l \equiv 0 \mod 3m$, then $k \equiv m \mod 3m$. Then $1 = (3m, m, 2m) = m$, a contradiction.

If (C) part two ($3k \equiv 0 \mod 3m$) hold, therefore $k = m$ or $k = 2m$, by considering B and using similar argument we used above we will get similar contradictions. If (C) part three ($3(l - k) \equiv 0 \mod 3m$) hold, therefore $l - k \equiv m \mod 3m$, by considering B we have

1. If $k + l \equiv 0 \mod 3m$, and $l - k \equiv m \mod 3m$, therefore $l = 2m, k = m$, so $1 = (n, k, l) = (3m, m, 2m) = m$ contradiction $m > 2$.

2. If $2l - k \equiv 0 \mod 3m$, and $l - k \equiv m \mod 3m$, therefore $l = 2m, k = m$, so $1 = (n, k, l) = (3m, m, 2m) = m$ contradiction $m > 2$.

3. If $2k - l \equiv 0 \mod 3m$, and $l - k \equiv m \mod 3m$, therefore $l = 2m, k = m$, so $1 = (n, k, l) = (3m, m, 2m) = m$ contradiction $m > 2$.

□

In [EW10, page 774] it was observed without proof that in the case *FFFT* we have $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$. The following Lemma gives a proof of that observation, and shows that the observation also holds in particular for the case *TFFT* and in general for all cases $-F-T$. This result will also be needed for the proof of Lemma 5.2.5

Lemma 5.2.4. *If $(n,k,l) = 1, k \neq l$, and (D) hold and (B) does not hold then $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$*

Proof. If (D) holds then n is even and we have three cases to consider

1. If $2(2k-l) \equiv 0 \bmod n$ then $(2k-l) \equiv 0$ or $\frac{n}{2} \bmod n$ but (B) is F therefore $(2k-l) \equiv \frac{n}{2} \bmod n$ so $l \equiv \frac{n}{2} + 2k \bmod n$. Using part 4 of Lemma 5.0.10, we have $\Gamma_n(k,l) = \Gamma_n(k, \frac{n}{2} + 2k) \cong \Gamma_n(k, \frac{n}{2} - k)$.

2. If $2(2l-k) \equiv 0 \bmod n$ then $(2l-k) \equiv 0$ or $\frac{n}{2} \bmod n$ but (B) is F therefore $(2l-k) \equiv \frac{n}{2} \bmod n$ so $k \equiv \frac{n}{2} + 2\ell \bmod n$, by using number $(2),(4)$ in Lemma 5.0.10 we can see $\Gamma_n(k,l) = \Gamma_n(\frac{n}{2} + 2l, l) \cong \Gamma_n(l, \frac{n}{2} + 2l) \cong \Gamma_n(l, \frac{n}{2} - l) = \Gamma_n(k_1, \frac{n}{2} - k_1)$ where $k_1 = l$.

3. If $2(k+l) \equiv 0 \bmod n$ then $(k+l) \equiv 0$ or $\frac{n}{2} \bmod n$ but (B) is F therefore $(k+l) \equiv \frac{n}{2} \bmod n$ $\Rightarrow l \equiv \frac{n}{2} - k \bmod n$, so we can write $\Gamma_n(k,l) = \Gamma_n(k, \frac{n}{2} - k)$.

It suffices to consider $\Gamma_n(k,l) = \Gamma_n(k, \frac{n}{2} - k)$. Now since $1 = (n,k,l) = (n,k,\frac{n}{2} - k)$ then $(n,k,\frac{n}{2} - k) = 1$, and thus that $(\frac{n}{2}, k) = 1$. Let $d = (n,k,\frac{n}{2} - k)$ therefore $d | k + (\frac{n}{2} - k) = \frac{n}{2}$ which means $d | (\frac{n}{2}, k)$ so $(n,k,\frac{n}{2} - k) | (\frac{n}{2}, k)$ so $(\frac{n}{2}, k) = (n,k,\frac{n}{2} - k) = 1$.

Suppose that k is even then since $(\frac{n}{2}, k) = 1$ we have that $\frac{n}{2}$ is odd, let $k_2 = \frac{n}{2} - k$. Then k_2 is odd, and by using number (2) in Lemma 5.0.10 we have $\Gamma_n(k, \frac{n}{2} - k) \cong \Gamma_n(\frac{n}{2} - k, k) = \Gamma_n(k_2, \frac{n}{2} - k_2)$. But $(k_2, n) = 1$, so $\Gamma_n(k_2, \frac{n}{2} - k_2) \cong \Gamma_n(1, K(\frac{n}{2} - 1))$ by number (5) in Lemma 5.0.10 where $Kk_2 \equiv 1 \bmod n$, therefore $\Gamma_n(k_2, \frac{n}{2} - k_2) \cong \Gamma_n(1, \frac{n}{2} - 1)$.

So we may assume k is odd so $(\frac{n}{2}, k) = 1 \Rightarrow (n,k) = 1$, and by using Lemma 5.0.10 we may assume that $k = 1$. Then $\Gamma_n(k,l) = \Gamma_n(k, \frac{n}{2} - k) \cong \Gamma_n(\alpha k, \alpha(\frac{n}{2} - k)) \cong \Gamma_n(1, \alpha(\frac{n}{2} - k))$ where $\alpha k \equiv 1 \bmod n$, $(n,\alpha) = 1$ which implies that α is odd, therefore $\Gamma_n(1, \alpha(\frac{n}{2} - k)) \cong \Gamma_n(1, \alpha.\frac{n}{2} - 1) = \Gamma_n(1, \frac{n}{2} - 1)$. That is $\Gamma_n(k,l) \cong \Gamma_n(k, \frac{n}{2} - k) \cong \Gamma_n(1, \frac{n}{2} - 1)$ □

Lemma 5.2.5. *Suppose that $(n,k,l) = 1, k \neq l, k \not\equiv 0 \bmod n$, $l \not\equiv 0 \bmod n$. If $n \neq 12$, then the combination TFTT is not possible.*

Proof. $(A), (D)$ are T, T give that $n \equiv 0 \mod 6$, and $l \equiv -k \mod 3$. Lemma 5.2.4 implies that $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$. If $n = 6$, the only cases imply that (A) is T, and $(n,k,l) = 1$ are $\Gamma_6(1,2), \Gamma_6(1,5), \Gamma_6(2,1)$, $\Gamma_6(5,1)$, this contradicts (B) is F. When $n = 12$, we have $\Gamma_{12}(1,5)$ implies that $ABCD$ are $TFTT$. Now when $n > 12$ assume for contradiction that $n = 6m, m > 2$. If $3|k$ then $3|l$ so $3|(n,k,l) = 1$ contradiction, therefore $3 \not| k$ and $3 \not| l$. If (C) part one ($3l \equiv 0 \mod n$) hold, therefore $l = 2m$ or $l = 4m$. Now by considering (D) is T, (B) is F we have three cases for each value of l

When $l = 2m$ we have,

1. If $2(k + l) \equiv 0 \mod 6m$, therefore $(k + l) \equiv 0 \mod 3m$, so $k \equiv m \mod 3m$. Then $1 = (n,k,l) = (6m, m, 2m) = m$, a contradiction.

2. If $2(2l - k) \equiv 0 \mod 6m$, therefore $(2l - k) \equiv 0 \mod 3m$, so $k \equiv m \mod 3m$. Then $1 = (n,k,l) = (6m, m, 2m) = m$, a contradiction.

3. If $2(2k - l) \equiv 0 \mod 6m$, therefore $(2k - l) \equiv 0 \mod 3m$, so $k \equiv m \mod 3m$. Then $1 = (n,k,l) = (6m, m, 2m) = m$, a contradiction.

When $l = 4m$ we have,

1. If $2(k + l) \equiv 0 \mod 6m$, therefore $(k + l) \equiv 0 \mod 3m$, so $k \equiv 2m \mod 3m$. Then $1 = (n,k,l) = (6m, 2m, 4m) = 2m$, a contradiction.

2. If $2(2l - k) \equiv 0 \mod 6m$, therefore $(2l - k) \equiv 0 \mod 3m$, so $k \equiv 2m \mod 3m$. Then $1 = (n,k,l) = (6m, 2m, 4m) = 2m$, a contradiction.

3. If $2(2k - l) \equiv 0 \mod 6m$, therefore $(2k - l) \equiv 0 \mod 3m$, so $k \equiv 2m \mod 3m$. Then $1 = (n,k,l) = (6m, 2m, 4m) = 2m$, a contradiction.

If (C) part two ($3k \equiv 0 \mod 6m$) hold, therefore $k = m$ or $k = 2m$, by considering (B) and using similar argument we used above we will get similar contradictions. If (C) part three hold, we have

$$\begin{aligned} &3(l - k) \equiv 0 \mod 6m \\ &\Rightarrow l - k \equiv 0 \mod 2m \\ &\Rightarrow l - k = 2m\alpha \quad (\alpha \in \mathbb{Z}) \end{aligned} \tag{5.6}$$

Now by considering (D) is T we have three cases

1. If

$$2(k+l) \equiv 0 \mod 6m$$
$$\Rightarrow k+l \equiv 0 \mod 3m$$
$$\Rightarrow k+l = 3m\beta \quad (\beta \in \mathbb{Z}) \tag{5.7}$$

   Thus, substituting (5.6) from (5.7) gives $k = \frac{(3\beta-2\alpha)m}{2}$, and we know that k is an integer. Also adding (5.7) and (5.6) gives $l = \frac{(3\beta+2\alpha)m}{2}$, and we know that l is an integer. Therefore $m|(n,k,l) = (6m, \frac{(3\alpha-2\beta)m}{2}, \frac{(3\alpha+2\beta)m}{2}) = 1$, a contradiction.

2. If

$$2(2l-k) \equiv 0 \mod 6m$$
$$\Rightarrow 2l-k \equiv 0 \mod 3m$$
$$\Rightarrow 2l-k = 3m\beta' \quad (\beta' \in \mathbb{Z}) \tag{5.8}$$

   Then (5.8) − (5.6) gives $l = (3\beta - 2\alpha)m$, (5.8) − 2.(5.6) gives $k = (3\beta' - 4\alpha)m$, therefore $m|(n,k,l) = (6m, (3\beta-4\alpha)m, (3\beta' - 2\alpha')m) = 1$, a contradiction.

3. If

$$2(2k-l) \equiv 0 \mod 6m$$
$$\Rightarrow 2k-l \equiv 0 \mod 3m$$
$$\Rightarrow 2k-l = 3m\beta'' \quad (\beta'' \in Z) \tag{5.9}$$

   Then (5.9) + (5.6) gives $k = (3\beta'' + 2\alpha)m$, (5.9) + 2.(5.6) gives $l = (3\beta'' + 4\alpha)m$, therefore $m|(n,k,l) = (6m, (3\beta + 2\alpha'')m, (3\beta + 4\alpha'')m) = 1$, a contradiction.

□

Theorem 5.2.6. *Suppose that $(n,k,l) = 1, k \neq l$. If $n > 12$ then the following 8 combinations FTTF, TTTF, TTFF, FTFF, TTTT, FFTT, FTTT, TFTT of $(A)(B)(C)(D)$ being true or false, are not* *possible.*

Proof. Lemma 5.2.1 shows that the combinations $FTTF, TTTF, TTFF, FTFF$ are not possible. Lemma 5.2.3 shows that the combination $TTTT$ is not possible. Lemma 5.2.2 shows that the combinations $FFTT, FTTT$ are not possible. Lemma 5.2.5 shows that the combination $TFTT$ is not possible. □

5.3 $f^{(abcd)}(n)$ for the six cases $FFTF, FTFT, TFTF, TTFT, FFFT, TFFT$

The five cases $FFTF, FTFT, TFTF, TTFT, FFFT$ have been studied in [EW10] (Information can be seen in Table 5.1). The combination ($FFTF$) in Table 5.2 represents (FFT) in Table 5.1 (since $FFTT$ is impossible see Lemma 5.2.2), we have the group $\Gamma_n(k,l)$ is metacyclic, and $\Gamma_n(k,l)^{ab} \cong \mathbb{Z}_\alpha, \alpha = 3(2^{n/3} - (-1)^{n/3})$, and the presentation $P_n(k,l)$ is not aspherical. The combination ($FTFT$) represents (FTF) (if B true D true), and we have the group $\Gamma_n(k,l) \cong \mathbb{Z}_3$, and $\Gamma_n(k,l)^{ab} \cong \mathbb{Z}_3$, and the presentation $P_n(k,l)$ is not aspherical. The combination ($TFTF$) represents (TFT) (since $TFTT$ is impossible see Lemma 5.2.5), we have the group $\Gamma_n(k,l) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$, and $\Gamma_n(k,l)^{ab} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_\gamma$ where $\gamma = (2^{n/3} - (-1)^{n/3})/3)$, and the presentation $P_n(k,l)$ is not aspherical. The combination ($TTFT$) represents (TTF) (since $TTFF$ is impossible see Lemma5.2.2), and we have the group $\Gamma_n(k,l) \cong \mathbb{Z}*\mathbb{Z}$, and $\Gamma_n(k,l)^{ab} \cong \mathbb{Z}\times\mathbb{Z}$, and the presentation $P_n(k,l)$ is not aspherical. The combination $TFFT$ has not been studied in [EW10], but they studied a special case of it when $n = 18$, they showed that the group is $\Gamma_{18}(k,l) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$, and $\Gamma_{18}(k,l)^{ab} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{19}$, and the presentation $P_{18}(k,l)$ is not aspherical. We study the group $\Gamma_n(1, \frac{n}{2} - 1)$ in more detail in Section 5.5.

Lemma 5.3.1 below comes from Lemma 2.4. of [EW10].

Lemma 5.3.1. *If* $(n,k,l) = 1, k \neq l$ *and* B *holds then* $\Gamma_n(k,l) \cong \Gamma_n(1,2)$. *If, in addition,* (A) *holds then* $\Gamma_n(1,2) \cong \mathbb{Z} * \mathbb{Z}$; *otherwise* $\Gamma_n(1,2) \cong \mathbb{Z}_3$.

Here we record our results about $f^{(abcd)}(n)$ of the six groups we mentioned above where $k \neq l$

Lemma 5.3.2.

$$f^{(TTFT)}(n) = \begin{cases} 1 & \text{If n} \equiv 0 \text{ mod } 3,\ n \geq 9 \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $n \equiv 0 \bmod 3, n \geq 9$. Let $k = 1, l = 2$, then $(A), (B), (C), (D)$ are $TTFT$, so $f^{(TTFT)}(n) \geq 1$, but Lemma 5.3.1 implies that $\Gamma_n(k,l) \cong \Gamma_n(1,2) \cong \mathbb{Z}*\mathbb{Z}$, so $f^{(TTFT)}(n) \leq 1$,therefore $f^{(TTFT)}(n) = 1$. The reason we consider $n \geq 9$ is that $\Gamma_n(k,l) \cong \Gamma_n(1,2)$ contradicts (C) is false when $n = 3$ or 6. If $n \equiv 1, 2 \bmod 3$ this is a contradiction with A is T, therefore $f^{(TTFT)}(n) = 0$. □

Lemma 5.3.3.

$$f^{(FTFT)}(n) = \begin{cases} 1 & \text{If} \quad \text{n} \equiv 1,2 \bmod 3 \\ 0 & \text{If} \quad \text{n} \equiv \quad 0 \bmod 3 \end{cases}$$

Proof. If $n \equiv 1, 2 \bmod 3$ and let $k = 1, l = 2$, then $(A), (B), (C), (D)$ are $FTFT$, so $f^{(FTFT)}(n) \geq 1$, but Lemma 5.3.1 implies that $\Gamma_n(k,l) \cong \Gamma_n(1,2) \cong \mathbb{Z}_3$, so $f^{(FTFT)}(n) \leq 1$, therefore $f^{(FTFT)}(n) = 1$. Suppose $n \equiv 0 \bmod 3$. By Lemma 5.3.1 we have $\Gamma_n(k,l) \cong \Gamma_n(1,2)$, since $\Gamma_n(1,2)$ implies A is T, therefore by Theorem 5.1.1 $\Gamma_n(k,l)$ implies A is T, this is a contradiction and $f^{(FTFT)}(n) = 0$. □

Lemma 5.3.4.

$$f^{(FFTF)}(n) = \begin{cases} 1 & \text{If} \quad \text{n} = 0 \bmod 3 \text{ ,n} \geq 6 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $n = 0 \bmod 3$. Then let $n = 3m, k = 1, l = m$ and $m \equiv 0$ or $1 \bmod 3$ then $(A), (B), (C), (D)$ are $FFTF$, therefore $f(n)^{(FFTF)} \geq 1$. But [EW10, Lemma 3.4] implies that when (C) holds and (A) does not then $\Gamma_n(k,l) = B((2^n - (-1)^n)/3, 3, 2^{2n/3}, 1)$ of [BW17], so $f(n) \leq 1$ therefore $f^{(FFTF)}(n) = 1$. If $n \equiv 1, 2 \bmod 3$, then this contradicts C is T, therefore $f(n)^{(FFTF)} = 0$ □

Lemma 5.3.5.

$$f^{(TFTF)}(n) = \begin{cases} 1 & \text{If} \quad \text{n} \equiv 3 \text{ or } 6 \bmod 9 \\ 0 & \text{If} \quad \text{n} \not\equiv 3 \text{ or } 6 \bmod 9 \end{cases}$$

Proof. Let n $\equiv 3 \bmod 9$. Let $k = 1, l = \frac{n}{3} + 1$, then $(A), (B), (C), (D)$ are $TFTF$, therefore $f(n)^{(TFTF)} \geq 1$. But [EW10, Lemma 2.5] implies that, when (B) does not hold and $(A), (C)$ both hold then $\Gamma_n(k,l) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ where $\gamma = (2^{\frac{n}{3}} - (-1)^{\frac{n}{3}})/3$, so $f^{(TFTF)}(n) \leq 1$, therefore $f^{(TFTF)}(n) = 1$. Similarly when $n \equiv 6 \;\; mod \; 9$, let $k = 1, l = \frac{n}{3}$, then $(A), (B), (C), (D)$ are $TFTF$, therefore $f(n)^{(TFTF)} \geq 1$. But [EW10, Lemma 2.5] implies that $\Gamma_n(k,l) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$ where $\gamma = (2^{\frac{n}{3}} - (-1)^{\frac{n}{3}})/3$, so $f^{(TFTF)}(n) \leq 1$, therefore $f^{(TFTF)}(n) = 1$. If $n \equiv 1, 2, 4, 5, 7$ or $8 \bmod 9$ that contradicts (A) is T therefore $f^{(TFTF)}(n) = 0$. □

Lemma 5.3.6.

$$f^{(FFFT)}(n) = \begin{cases} 1 & \text{If} \quad \text{n} \equiv 2,4 \bmod 6 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $n \equiv 2,4 \bmod 6$, if $k = 1, l = \frac{n}{2} - 1$ then $(A),(B),(C),(D)$ are $FFFT$, so $f^{(FFFT)}(n) \geq 1$. But Lemma 5.2.4 implies that when D holds and B does not then $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$ so $f^{(FFFT)}(n) \leq 1$ therefore $f^{(FFFT)}(n) = 1$.

Suppose $n \equiv 0 \bmod 6$ then $\frac{n}{2} \equiv 0 \bmod 3$, if $2(k+l) \equiv 0 \bmod n$, then $(k+l) \equiv 0 \bmod n/2$ contradicts A is F. If $2(2k-l) \equiv 0 \bmod n$ then

$$\begin{aligned} 2k - l &\equiv 0 \bmod n/2 \\ \Rightarrow \ l &\equiv \ 2k \bmod n/2 \\ \Rightarrow \ k + l &\equiv \ 3k \bmod n/2 \\ \Rightarrow \ k + l &\equiv \ 3k \bmod 3 \\ \Rightarrow \ k + l &\equiv \ 0 \bmod 3 \quad \text{contradicts (A) is F} \end{aligned}$$

Similarly if $2(2l-k) \equiv 0 \bmod n$. Then $k + l \equiv 0 \bmod 3$ which contradicts (A) is F, therefore $f^{(FFFT)}(n) = 0$. Then $n \equiv 1,3,5 \bmod 6$, contradicts with D is T, therefore $f^{(FFFT)}(n) = 0$. □

Lemma 5.3.7.

$$f^{(TFFT)}(n) = \begin{cases} 1 & \text{If n} \equiv 0 \bmod 6,\ n \geq 18 \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $n \equiv 0 \bmod 6$,$n \geq 18$. Then let $k = 1, l = \frac{n}{2} - 1$. Then $(A),(B),(C),(D)$ are $TFFT$, so $f(n)^{(TFFT)} \geq 1$. But Lemma 5.2.4 implies that where $ABCD$ are $TFFT$ then $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$ so $f^{(TFFT)}(n) \leq 1$ therefore $f^{(TFFT)}(n) = 1$. The reason we consider $n \geq 18$ is that $\Gamma_n(k,l) \cong \Gamma_n(1, \frac{n}{2} - 1)$ contradicts C is false for $n = 6$ or 12. If $n \equiv 1,3,5 \bmod 6$, then this gives a contradiction to D is T. If $n \equiv 2,4 \bmod 6$ (this contradicts A is T), therefore $f^{(TFFT)}(n) = 0$. □

We combine these results in Table 5.2, where we can easily determine the number of $\Gamma_n(k,l)$ groups for any value of $n > 12$. For the six combinations *FFTF, FTFT, TFTF, TTFT, FFFT, TFFT*, we denote the number of groups by $g^{(abcd)}(n)$, which we define as follow

Definition 5.3.8. *Let*

$$h(n) = f^{(TTFT)}(n) + f^{(FTFT)}(n) + f^{(FFTF)}(n) + f^{(TFTF)}(n) + f^{(FFFT)}(n) + f^{(TFFT)}(n) \tag{5.10}$$

Lemmas 5.3.2, 5.3.3, 5.3.4, 5.3.5 give the values of $f^{(TTFT)}(n), f^{(FTFT)}(n), f^{(FFTF)}(n)$ and $f^{(TFTF)}(n)$, so the problem of calculating $f^{(abcd)}(n)$ (in Definition 5.0.8) is reduced to calculating $f^{(FFFF)}(n)$ and $f^{(TFFF)}(n)$, i.e., $f(n) = h(n) + f^{(FFFF)}(n) + f^{(TFFF)}(n)$.

5.4 $f^{(abcd)}(n)$ for the cases $(FFFF),(TFFF)$

It has shown in [EW10] that in the case $(FFFF)$, the group $\Gamma_n(k,l)$ is infinite, and has a finite abelianization of order greater than one, and the presentation $P_n(k,l)$ is aspherical. In the case $(TFFF)$, the group $\Gamma_n(k,l)$ is large, and has an infinite abelianization, and the presentation $P_n(k,l)$ is aspherical. We are unable to obtain the exact value of $f(n)^{(FFFF)}, f(n)^{(TFFF)}$, but we do obtain lower bounds for certain values of n. In Chapter 6 we obtain upper bounds for $f(n)$ when n has at most two distinct prime factors, and upper bounds for $f(n)$ when n has at most three distinct prime factors.

Lemma 5.4.1.

$$f^{(TFFF)}(n)\begin{cases} \geq 1 & \text{If} \quad \text{n} \equiv 0 \bmod 3, \qquad \text{n} \geq 21 \\ = 0 & \text{otherwise} \end{cases}$$

Proof. If $n \equiv 0 \bmod 3, n \geq 21$. Then $\Gamma_n(1,5)$ implies that $(A),(B),(C),(D)$ are $TFFF$, so $f^{(TFFF)}(n) \geq 1$. If $n \equiv 1$ or $2 \bmod 3$, this contradicts A is T therefore $f^{(TFFF)}(n) = 0$. □

Lemma 5.4.2. *If* $n \geq 10$ *then* $f^{(FFFF)}(n) \geq 1$.

Proof.
1. Let n be even if $k = 1, l = \frac{n}{2}$, then $(A)(B)(C)(D)$ are $FFFF$, then $f^{(FFFF)}(n) \geq 1$.
2. Let n be odd if $k = 1, l = 3$, then $(A)(B)(C)(D)$ are $FFFF$, then $f^{(FFFF)}(n) \geq 1$.

□

In the following lemma d denotes the minimum number of generators of $\Gamma_n(k,l)^{\text{ab}}$ (see Definition 4.2.1)

Lemma 5.4.3. *Let* $n = 8q, q \geq 1$, *then* $d(\Gamma_n(1,3))^{\text{ab}}) \geq 3$.

Proof. We will prove that $\Gamma_n(1,3)^{\text{ab}}$ maps to $\Gamma_8(1,3)^{\text{ab}}$, first lets suppose that $n=8q, q\geq 1$

$$\begin{aligned}\Gamma_{8q}(1,3)^{ab} &\cong \langle x_0,x_1,...,x_{8q-1} \mid x_ix_{i+1}x_{i+3},\ i=0,1,...,8q-1\rangle^{\text{ab}}\\ &\twoheadrightarrow \langle x_0,x_1,...,x_{8q-1} \mid x_ix_{i+1}x_{i+3}, x_i=x_{i+8},\ i=0,1,...,8q-1\rangle^{\text{ab}}\\ &\twoheadrightarrow \langle x_0,x_1,...,x_7 \mid x_ix_{i+1}x_{i+3},\ i=0,1,...,7\rangle^{\text{ab}}=\Gamma_8(1,3)^{\text{ab}}\end{aligned}$$

Now since $\Gamma_8(1,3)^{ab}\cong \mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$ (see Table 6.1) which generated by 3 elements, then $d(\Gamma_n(1,3)^{\text{ab}})\geq 3$. □

Lemma 5.4.4. *If $n=8q, q>2$ Then $\Gamma_n(1,n/2)\ncong\Gamma_n(1,3)$, so $f^{(F,F,F,F)}(n)\geq 2$.*

Proof. Let $k=1, l=n/2$ and $k=1, l=3$ then $\Gamma_n(1,n/2),\Gamma_n(1,3)$ imply that $(A),(B),(C),(D)$ are (F,F,F,F). By Lemmas (5.0.9, 5.4.3) we know that $\Gamma_n(1,n/2)^{\text{ab}}\cong\mathbb{Z}_m, m=2^{n/2}-1$ therefore $d(\Gamma_n(1,n/2)^{\text{ab}})=1$, and $d(\Gamma_n(1,3)^{\text{ab}})\geq 3$, therefore $\Gamma_n(1,n/2)\ncong\Gamma_n(1,3)$, therefore $f^{(F,F,F,F)}(n)\geq 2$. □

Lemma 5.4.5. *Let $4|n, n\not\equiv 0$ mod 16, then $\Gamma_n(1,\frac{n}{2})\ncong\Gamma_n(1,\frac{n}{4})$ so, $f^{(F,F,F,F)}(n)\geq 2$.*

Proof. $\Gamma_n(1,\frac{n}{2}),\Gamma_n(1,\frac{n}{4})$ imply that $(A),(B),(C),(D)$ are *FFFF*, and since $G_{pk}(x_0x_kx_l)^{\text{ab}}\cong G_{pk}(x_0x_lx_k)^{\text{ab}}$. Then by Corollary 3.3.3 (1), (3) we have

a. $|\Gamma_{2k}(x_0x_kx_l)^{\text{ab}}|=|\Gamma_{2k}(x_0x_lx_k)^{\text{ab}}|=|(-1)^{k+l+1}(2^k-(-1)^k)|$, and then $|\Gamma_n(1,\frac{n}{2})^{\text{ab}}|=|\Gamma_n(x_0x_1x_{\frac{n}{2}})^{\text{ab}}|=|(-1)^{\frac{n}{2}+1+1}(2^{\frac{n}{2}}-(-1)^{\frac{n}{2}})|$, since $\frac{n}{2}$ is even, therefore $|\Gamma_n(1,\frac{n}{2})^{\text{ab}}|=|2^{\frac{n}{2}}-1|=|(2^{\frac{n}{4}}+1)(2^{\frac{n}{4}}-1)|$

b. $|\Gamma_{4k}(x_0x_kx_l)^{\text{ab}}|=|\Gamma_{4k}(x_0x_lx_k)^{\text{ab}}|=|(-1)^{k+l+1}(2^k-(-1)^k)((2^k+1)-(-1)^k(\sqrt{2})^k.2\cos\frac{(2l-k)\pi}{4})|$, and then $|\Gamma_n(1,\frac{n}{4})^{\text{ab}}|=|\Gamma_n(x_0x_1x_{\frac{n}{4}})^{\text{ab}}|=|(-1)^{\frac{n}{4}+1+1}[2^{\frac{n}{4}}-(-1)^{\frac{n}{4}}][(2^{\frac{n}{4}}+1)-(-1)^{\frac{n}{4}}(\sqrt{2})^{\frac{n}{4}}.2\cos\frac{(n-8)\pi}{16}]|$.

To prove that $|\Gamma_n(1,\frac{n}{2})^{\text{ab}}|\neq|\Gamma_n(1,\frac{n}{4})^{\text{ab}}|$ for given values of n, we will consider the absolute value of the abelianization and we have two cases

1. If $n\equiv 4$ or 12 mod 16 then $\frac{n}{4}$ is odd so $|\Gamma_n(1,\frac{n}{4})^{\text{ab}}|=|[2^{\frac{n}{4}}+1][(2^{\frac{n}{4}}+1)+(\sqrt{2})^{\frac{n}{4}}.2\cos\frac{(n-8)\pi}{16}]|$ and $\cos\frac{(n-8)\pi}{16}=\pm\frac{1}{\sqrt{2}}$ so $|\Gamma_n(1,\frac{n}{4})^{\text{ab}}|=|[2^{\frac{n}{4}}+1][(2^{\frac{n}{4}}+1)\pm 2.(\sqrt{2})^{\frac{n}{4}-1}]|$, and since $\pm 2.(\sqrt{2})^{\frac{n}{4}-1}\neq\mp 2$ and hence $|\Gamma_n(1,\frac{n}{2})^{\text{ab}}|\neq|\Gamma_n(1,\frac{n}{4})^{\text{ab}}|$.

2. If $n \equiv 8 \mod 16$ then $\frac{n}{4}$ is even so $|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |[2^{\frac{n}{4}} - 1][(2^{\frac{n}{4}} + 1) - (\sqrt{2})^{\frac{n}{4}}.2\cos\frac{(n-8)\pi}{16}]|$ and $\cos\frac{(n-8)\pi}{16} = \pm 1$ implies that $|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |[2^{\frac{n}{4}} - 1][(2^{\frac{n}{4}} + 1) \pm 2.(\sqrt{2})^{\frac{n}{4}}]|$, and since $\pm 2.(\sqrt{2})^{\frac{n}{4}} \neq 0$ we have that $|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| \neq |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}|$, therefore $\Gamma_n(1, \frac{n}{2}) \ncong \Gamma_n(1, \frac{n}{4})$. Note that when $n = 0 \mod 16$, therefore $\cos\frac{(n-8)\pi}{16} = 0$ and then $|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| \cong |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |2^{\frac{n}{2}} - 1|$, so we can not conclude that $\Gamma_n(1, \frac{n}{2}) \ncong \Gamma_n(1, \frac{n}{4})$.

□

From Proposition 5.0.9 we know that $\Gamma_n(1, \frac{n}{2})^{ab} = \mathbb{Z}_{2^{\frac{n}{2}} \pm 1}$. This leads us to leave the following question

Question. Is the group $\Gamma_n(1, \frac{n}{4})^{ab} \cong \mathbb{Z}_{2^{\frac{n}{2}}-1}$ or not?

Lemma 5.4.6. *Let* $12|n, \frac{n}{6} \equiv 0$ *or* $4 \mod 6$*, then* $\Gamma_n(1, \frac{n}{2}), \Gamma_n(1, \frac{n}{4}), \Gamma_n(1, \frac{n}{6})$ *are pairwise non-isomorphic, so* $f^{(F,F,F,F)}(n) \geq 3$.

Proof. $\Gamma_n(1, \frac{n}{2}), \Gamma_n(1, \frac{n}{4}), \Gamma_n(1, \frac{n}{6})$ imply that $(A), (B), (C), (D)$ are *FFFF*, and by using Corollary 3.3.3 (similar to proof of Lemma 5.4.5) we have

a. $|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| = |2^{\frac{n}{2}} - 1 = (2^{\frac{n}{4}} + 1)(2^{\frac{n}{4}} - 1)|$

b. $|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |[2^{\frac{n}{4}} + (-1)^{\frac{n}{4}+1}][(2^{\frac{n}{4}} + 1) + (-1)^{\frac{n}{4}+1}(\sqrt{2})^{\frac{n}{4}}.2\cos\frac{(n-8)\pi}{16}]|$

c. $|\Gamma_n(1, \frac{n}{6})^{\mathrm{ab}}| = |[2^{\frac{n}{6}} + (-1)^{\frac{n}{6}+1}][(3^{\frac{n}{6}} + 1) + (-1)^{\frac{n}{6}+1}(\sqrt{3})^{\frac{n}{6}}.2\cos\frac{(n-12)\pi}{36}][2 + (-1)^{\frac{n}{6}+1}.2\cos\frac{(n-12)\pi}{18}]|$

We shall prove that $\Gamma_n(1, \frac{n}{2}), \Gamma_n(1, \frac{n}{4}), \Gamma_n(1, \frac{n}{6})$ are pairwise non-isomorphic for given values of n. Working mod 8 we get,

$$|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| = |2^{\frac{n}{2}} - 1| \equiv 1 \mod 8 \tag{5.11}$$

$$\begin{aligned}|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| &= |[2^{\frac{n}{4}} + (-1)^{\frac{n}{4}+1}][(2^{\frac{n}{4}} + 1) + (-1)^{\frac{n}{4}+1}(\sqrt{2})^{\frac{n}{4}}.2\cos\frac{(n-8)\pi}{16}]| \\ &= |[2^{\frac{n}{4}} + (-1)^{\frac{n}{4}+1}][(2^{\frac{n}{4}} + 1) + (-1)^{\frac{n}{4}+1}(2)^{\frac{(n+8)}{8}}\cos\frac{(n-8)\pi}{16}]| \\ &\equiv [0 + (-1)^{\frac{n}{4}+1}][0 + 1 + 0] \mod 8\end{aligned}$$

$$|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| \equiv (-1)^{\frac{n}{4}+1} \mod 8, \text{this gives} \tag{5.12}$$
$$|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| \equiv 1 \mod 8 \quad \text{If} \quad n \equiv 12 \text{ or } 36 \mod 48, n \geq 24 \tag{5.13}$$
$$|\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| \equiv -1 \mod 8 \quad \text{If} \quad n \equiv 24 \mod 48 \tag{5.14}$$

$$\begin{aligned}
|\Gamma_n(1, \frac{n}{6})^{\mathrm{ab}}| &= |[2^{\frac{n}{6}} + (-1)^{\frac{n}{6}+1}][(3^{\frac{n}{6}} + 1) + (-1)^{\frac{n}{6}+1}(\sqrt{3})^{\frac{n}{6}}.2\cos\frac{(n-12)\pi}{36}][2 + (-1)^{\frac{n}{6}+1}.2\cos\frac{(n-12)\pi}{18}]| \\
&\equiv [0 + (-1)^{\frac{n}{6}+1}][(1+1) + (-1)^{\frac{n}{6}+1}.(1).2\cos\frac{(n-12)\pi}{36}][2 - 2\cos\frac{(n-12)\pi}{18}] \mod 8 \\
&\equiv -1.[2 - 2\cos\frac{(n-12)\pi}{36}][2 - 2\cos\frac{(n-12)\pi}{18}] \mod 8 \\
&\equiv 4.[1 - \cos\frac{(n-12)\pi}{36}][1 - \cos\frac{(n-12)\pi}{18}] \mod 8 \text{ which is even} \\
&\not\equiv \pm 1 \mod 8
\end{aligned} \tag{5.15}$$

Then by (5.11),(5.12), (5.15), we have $|\Gamma_n(1, \frac{n}{6})^{\mathrm{ab}}| \neq |\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}|$, $|\Gamma_n(1, \frac{n}{6})^{\mathrm{ab}}| \neq |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}|$, therefore $\Gamma_n(1, \frac{n}{6}) \not\cong \Gamma_n(1, \frac{n}{2})$, $\Gamma_n(1, \frac{n}{6}) \not\cong \Gamma_n(1, \frac{n}{4})$

Now if $n \equiv 12 \mod 24$ then by (5.11),(5.12), we have $|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| \neq |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}|$, therefore $\Gamma_n(1, \frac{n}{2}) \not\cong \Gamma_n(1, \frac{n}{4})$.

If $n \equiv 24 \mod 48$ then $\frac{n}{4}$ is even, and

$$\begin{aligned}
&\cos\frac{(n-8)\pi}{16} = \mp 1 \Rightarrow |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |[2^{\frac{n}{4}} - 1][(2^{\frac{n}{4}} + 1) \pm 2^{\frac{n+8}{8}}]| \\
&\text{, and since } \mp 2^{\frac{n+8}{8}} \neq 0 \Rightarrow |\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| \neq |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}|
\end{aligned} \tag{5.16}$$

Therefore $\Gamma_n(1, \frac{n}{2}) \not\cong \Gamma_n(1, \frac{n}{4})$. Note that when $n = 0 \mod 48$, therefore $\cos\frac{(n-8)\pi}{16} = 0$ and then $|\Gamma_n(1, \frac{n}{2})^{\mathrm{ab}}| \cong |\Gamma_n(1, \frac{n}{4})^{\mathrm{ab}}| = |2^{\frac{n}{2}} - 1|$, so we can not conclude that $\Gamma_n(1, \frac{n}{2}) \not\cong \Gamma_n(1, \frac{n}{4})$ (here we can also ask same question that we asked in Lemma 5.4.5). □

5.5 The group $\Gamma_n(1, \frac{n}{2} - 1)$

Lemma 5.2.4 shows that in the two cases *TFFT*, *FFFT* the groups $\Gamma_n(k, l) \cong \Gamma_n(1, \frac{n}{2} - 1)$. Here we study this group in each case as part of our investigations into $\Gamma_n(k, l)$ groups.

The case *TFFT* requires that $n \equiv 0 \mod 6$, and [EW10, Lemma 2.2] implies that $\Gamma_n(1, \frac{n}{2} -$

$1)^{ab}$ is infinite. We prove the following theorem, and we will use in the proof the following equation which is the equation (1.12), where $\beta(G_n(\omega)^{ab})$ is a torsion-free rank of $\Gamma_n(1, \frac{n}{2}-1)^{ab}$ (see Definition 1.4.5).

$$\beta(G_n(\omega)^{ab}) = deg(gcd(f(t), g(t))) \tag{5.17}$$

Where $f(t)$ is the polynomial associated with the abelianization of the group (see Definition 1.4.1), $g(t) = t^n - 1$.

Theorem 5.5.1. *If $n \equiv 0$ mod 6 then the torsion-free rank of $\Gamma_n(1, \frac{n}{2} - 1)^{ab}$ is 2.*

Proof. Let $n = 6m$ then $\Gamma_n(1, \frac{n}{2}-1)^{ab} = \Gamma_{6m}(1, 3m-1)^{ab}$, and now the associated polynomial of $\Gamma_{6m}(1, 3m-1)^{ab}$ is $f(t) = 1 + t + t^{3m-1}$. By (5.17) we have $\beta(\Gamma_n(1, \frac{n}{2} - 1)^{ab}) = deg(gcd(f(t), g(t)))$, where $g(t) = t^n - 1 = t^{(6m)} - 1$. Now by simplifying $f(t), g(t)$ we get

$$\begin{aligned}
f(t) &= 1 + t + t^{3m-1} \\
&= (1 + t + t^2)(t^{3m-3} - t^{3m-4} + t^{3m-6} - t^{3m-7} + ... + t^3 - t^2 + 1) \\
f(t) &= h(t).F(t) \quad \text{where} \\
h(t) &= 1 + t + t^2, \\
F(t) &= t^{3m-3} - t^{3m-4} + t^{3m-6} - t^{3m-7} + ... + t^3 - t^2 + 1
\end{aligned}$$

Now $F(t)$ can be simplified as follow

$$\begin{aligned}
F(t) &= t^{3m-3} - t^{3m-4} + t^{3m-6} - t^{3m-7} + ... + t^3 - t^2 + 1 \\
&= (t - 1)(t^{3m-4} + t^{3m-7} + ... + t^5 + t^2) + 1 \\
&= (t - 1)t^2(t^{3m-6} + t^{3m-9} + ... + t^3 + 1) + 1
\end{aligned}$$

Since

$$\begin{aligned}
(t^{3m-6} + t^{3m-9} + ... + t^3 + 1)(t^2 + t + 1) &= t^{3m-4} + t^{3m-7} + ... + t^5 + t^2 \\
&\quad + t^{3m-5} + t^{3m-8} + ... + t^4 + t \\
&\quad + t^{3m-6} + t^{3m-9} + ... + t^3 + 1 \\
&= 1 + t + t^2 + ... + t^{3m-5} + t^{3m-4}
\end{aligned}$$

so

$$F(t) = (t-1)t^2(\frac{1+t+t^2+...+t^{3m-5}+t^{3m-4}}{t^2+t+1}) + 1, \quad t^2+t+1 \neq 1$$
$$= \frac{(t^{3m-3}-1).t^2}{t^2+t+1} + 1, \quad t^2+t+1 \neq 1$$

$$F(t) = \frac{t^{3m-1}-t^2+t^2+t+1}{t^2+t+1}, \quad t^2+t+1 \neq 1$$
$$= \frac{t^{3m-1}+t+1}{t^2+t+1}, \quad t^2+t+1 \neq 1$$
$$g(t) = t^{6m} - 1$$
$$= (1+t+t^2)(t^{6m-2} - t^{6m-3} + t^{6m-5} - t^{6m-6} + ... - t^3 + t - 1)$$
$$= h(t).G(t) \quad \text{where}$$
$$G(t) = t^{6m-2} - t^{6m-3} + t^{6m-5} - t^{6m-6} + ... - t^3 + t - 1$$

Therefore $\big(f(t), g(t)\big) = h(t)\big(F(t), G(t)\big)$.

Now $G(t)$ has the roots $\lambda_q = e^{\frac{2q\pi i}{6m}}, \quad q = 0, 1, ..., 6m-1, \quad q \neq 2m, 4m$ (since these values give the roots of $h(t)$ which are $\frac{-1+\sqrt{3}i}{2} = e^{\frac{2\pi}{3}}, \frac{-1-\sqrt{3}i}{2} = e^{\frac{4\pi}{3}}$). Now $1 = |\lambda_q| = \lambda_q\overline{\lambda_q}$ so $\lambda_q^{-1} = \overline{\lambda_q}$. Now assume for contradiction that λ_q is a root of $F(t)$ therefore

$$F(\lambda_q) = 0 \Rightarrow \quad \frac{\lambda_q^{3m-1}+\lambda_q+1}{\lambda_q^2+\lambda_q+1} = 0$$
$$\Rightarrow \quad \lambda_q^{3m-1}+\lambda_q+1 = 0 \Rightarrow \quad \lambda_q^{3m-1} = -(\lambda_q+1) \tag{5.18}$$

Similarly we have

$$F(\lambda_q^{-1}) = 0 \Rightarrow \quad \lambda_q^{-(3m-1)} = -(\lambda_q^{-1}+1) \tag{5.19}$$

By multiplying (5.18) and (5.19) we get

$$\begin{aligned} 1 &= 1 + \lambda_q + \lambda_q^{-1} + 1 \\ \lambda_q + \lambda_q^{-1} &= -1 \\ \Rightarrow e^{\frac{2q\pi i}{6m}} + e^{\frac{-2q\pi i}{6m}} &= -1 \\ 2\cos\frac{2q\pi i}{6m} = -1 \Rightarrow \cos\frac{2q\pi i}{6m} &= \frac{-1}{2} \end{aligned}$$

therefore $q = 2m$ or $4m$ (contradiction), so λ_q is not a root of $F(t)$ so $F(t), G(t)$ have no common roots and the greatest common divisor of $f(t), g(t)$ is $1 + t + t^2$ therefore the torsion-free rank of $\Gamma_n(1, \frac{n}{2} - 1)^{ab}$ is 2. □

The case *FFFT* requires that $n \equiv 2$ or $4 \bmod 6$, and [EW10, Lemma 2.2] imply that $\Gamma_n(1, \frac{n}{2} - 1)^{ab}$ is finite. We provide in the following theorem a formula that computes the order of abelianization of this group, which is similar to the formula that given in Theorem 1.5.7 for computing the order of abeelianization of $F(2, n)$ groups.

Theorem 5.5.2. *Let $n \equiv 2$ or $4 \bmod 6$, then $|\Gamma_n(1, \frac{n}{2} - 1)^{ab}| = 3\Big(L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}}\Big)$ where L_n is Lucas number for order n.*

In order to obtain the order of $\Gamma_n(1, \frac{n}{2} - 1)^{ab}$, we will use the following formula of Corollary 3.2.4 (c) .

$$|\Gamma_{pk}(x_0 x_k x_l)^{ab}| = \prod_{j=0}^{p-1} P^{1,1}_{j,p} \text{ where } P^{1,1}_{j,n}(k, l) = (1 + \zeta_p^j)^k + (-1)^{k+1}(\zeta_p^j)^l \tag{5.20}$$

where k, l are modulo n, $1 \leq j \leq n - 1$ and $\zeta_p = e^{\frac{2\pi i}{p}}$. For the proof we need

Lemma 5.5.3. *Let $n \equiv 2$ or $4 \bmod 6$, then*

$$\prod_{\substack{j=0 \\ j\, odd}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\alpha=0}^{\frac{n}{2}-1} \Big(1 + 2i \sin\Big(\frac{2\pi(2\alpha + 1)}{n}\Big)\Big)$$

Proof.

$$\prod_{\substack{j=0 \\ j\, odd}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\substack{j=1 \\ j\, odd}}^{n-1} \left(1 + \zeta_n^j + \zeta_n^{j(\frac{n}{2}-1)}\right) \quad \text{by (5.20)}$$

For j odd $\zeta_n^{j(\frac{n}{2})} = -1$ then

$$= \prod_{\substack{j=1 \\ j\, odd}}^{n-1} \left(1 + \zeta_n^j - \zeta_n^{-j}\right)$$

let $j = 2\alpha + 1$

$$\prod_{\substack{j=0 \\ j\, odd}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\alpha=0}^{\frac{n}{2}-1} \left(1 + e^{i(\frac{2\pi}{n})(2\alpha+1)} - e^{-i(\frac{2\pi}{n})(2\alpha+1)}\right) = \prod_{\alpha=0}^{\frac{n}{2}-1} \left(1 + 2i \sin\left(\frac{2\pi(2\alpha+1)}{n}\right)\right)$$

□

And we need

Lemma 5.5.4. *Let $n \equiv 2$ or 4 mod 6, then* $\prod_{\substack{j=1 \\ j\, even}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = -1$

Proof.

$$\prod_{\substack{j=1 \\ j\, even}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\substack{j=1 \\ j\, even}}^{n-1} \left(1 + \zeta_n^j + \zeta_n^{j(\frac{n}{2}-1)}\right)$$

For j even $\zeta_n^{j(\frac{n}{2})} = 1$ then

$$\prod_{\substack{j=1 \\ j\, even}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\substack{j=1 \\ j\, even}}^{n-1} \left(1 + \zeta_n^j + \zeta_n^{-j}\right)$$

$$= \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{-j}\right) . \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{j} + \zeta_n^{2j} + 1\right)$$

$$= \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{-j}\right) . \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\frac{\zeta_n^{3j} - 1}{\zeta_n^{j} - 1}\right)$$

$$= \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{-j}\right) . \frac{\prod\limits_{\substack{j=1 \\ j\,even}}^{n-1} (\zeta_n^{3j} - 1)}{\prod\limits_{\substack{j=1 \\ j\,even}}^{n-1} (\zeta_n^{j} - 1)}$$

n ≡ 2 or 4 mod 6 therefore $(3, n) = 1$ and for j ∈ {1, .., n − 1} then $3j \in \{1, .., n-1\}$

$$\prod_{\substack{j=1 \\ j\,even}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{-j}\right) . \frac{\prod\limits_{\substack{j=1 \\ j\,even}}^{n-1} (\zeta_n^{j} - 1)}{\prod\limits_{\substack{j=1 \\ j\,even}}^{n-1} (\zeta_n^{j} - 1)} \tag{5.21}$$

so numerator and denominator above cancel and we get

$$\prod_{\substack{j=1 \\ j\,even}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1) = \prod_{\substack{j=1 \\ j\,even}}^{n-1} \left(\zeta_n^{-j}\right) = \zeta_n^{-(\sum\limits_{\substack{j=1 \\ j\,even}}^{n-1} j)} = \zeta_n^{\frac{-n(n-1)}{2}}$$

$$= \left(\zeta_n^{\frac{n}{2}}\right)^{-(n-1)} = (-1)^{-(n-1)} = -1$$

□

We also need the following equation which is equation (4.2)

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n \tag{5.22}$$

We can now proof Theorem 5.5.2

Proof of Theorem 5.5.2. Set $k = 1, l = \frac{n}{2} - 1, p = n$ in (5.20), we get

$$|\Gamma_n(1, \frac{n}{2} - 1)^{ab}| = |\prod_{j=0}^{n-1} P_{j,n}(1, \frac{n}{2} - 1)|, \quad \text{but}$$

$$P_{j,n}(1, \frac{n}{2} - 1) = 1 + \zeta_n^j + \zeta_n^{j(\frac{n}{2}-1)} \quad \text{for} \quad j = 0, 1, ...(n-1)$$

$$P_{0,n}(1, \frac{n}{2} - 1) = 1 + 1 + 1 = 3$$

$$\text{therefore } |\Gamma_n(1, \frac{n}{2} - 1)^{ab}| = |3.\prod_{j=1}^{n-1} P_{j,n}(1, \frac{n}{2} - 1)|$$

$$= |3.\prod_{\substack{j=1 \\ j\, odd}}^{n-1} P_{j,n}(1, \frac{n}{2} - 1).\prod_{\substack{j=2 \\ j\, even}}^{n-2} P_{j,n}(1, \frac{n}{2} - 1)|$$

from lemmas (5.5.3), (5.5.4)

$$= |3.\prod_{\alpha=0}^{\frac{n}{2}-1} \left(1 + 2i \sin\left(\frac{2\pi(2\alpha+1)}{n}\right).(-1)\right|$$

$$= |-3.\prod_{\alpha=0}^{\frac{n}{2}-1} \left(1 + 2i \sin\left(\frac{2\pi(2\alpha+1)}{n}\right)\right)|$$

$$= |3.R| \tag{5.23}$$

$$\text{where} \qquad \mathrm{R} = \prod_{\alpha=0}^{\frac{\mathrm{n}}{2}-1} \left(1 + 2\mathrm{i} \sin\left(\frac{2\pi(2\alpha+1)}{\mathrm{n}}\right)\right) \tag{5.24}$$

Now the following relation is Equation (10) of [GR08]

$$\prod_{j=0}^{n-1} \left(1 + 2i \sin\frac{2j\pi}{n}\right) = 1 + F_n - 2F_{n+1} + (-1)^n = 1 - L_n + (-1)^n \tag{5.25}$$

where F_n is Fibonacci number of order n. Therefore

$$\prod_{j=0}^{n-1} \left(1 + 2i \sin\frac{2j\pi}{n}\right) = \prod_{\substack{j=0 \\ j\, odd}}^{n-1} \left(1 + 2i \sin\frac{2j\pi}{n}\right).\prod_{\substack{j=0 \\ j\, even}}^{n-1} \left(1 + 2i \sin\frac{2j\pi}{n}\right)$$

$$= \prod_{\alpha=0}^{\frac{n}{2}-1}\Big(1+2i\sin\Big(\frac{2\pi(2\alpha+1)}{n}\Big)\Big). \prod_{\alpha=0}^{\frac{n}{2}-1}\Big(1+2i\sin\Big(\frac{2\pi\alpha}{\frac{n}{2}}\Big)\Big)$$

$$= R. \prod_{\alpha=0}^{\frac{n}{2}-1}\Big(1+2i\sin\Big(\frac{2\pi\alpha}{\frac{n}{2}}\Big)\Big) \qquad \text{by (5.23)}$$

$$R = \frac{\prod_{j=0}^{n-1}\Big(1+2i\sin\frac{2j\pi}{n}\Big)}{\prod_{\alpha=0}^{\frac{n}{2}-1}\Big(1+2i\sin\Big(\frac{2\pi\alpha}{\frac{n}{2}}\Big)\Big)} = \frac{L_n - 1 - (-1)^n}{L_{\frac{n}{2}} - 1 - (-1)^{\frac{n}{2}}} \quad \text{by (5.25)}$$

and since $n \equiv 2$ or $4 \bmod 6$, therefore

$$R = \frac{L_n - 2}{L_{\frac{n}{2}} - 1 - (-1)^{\frac{n}{2}}}. \tag{5.26}$$

By (5.22) we have

$$L_n = \Big(\frac{1+\sqrt{5}}{2}\Big)^n + \Big(\frac{1-\sqrt{5}}{2}\Big)^n \text{therefore } L_{\frac{n}{2}} = \Big(\frac{1+\sqrt{5}}{2}\Big)^{\frac{n}{2}} + \Big(\frac{1-\sqrt{5}}{2}\Big)^{\frac{n}{2}}$$

$$\text{then } L_{\frac{n}{2}}^2 = \Big(\frac{1+\sqrt{5}}{2}\Big)^n + \Big(\frac{1-\sqrt{5}}{2}\Big)^n + 2(-1)^{\frac{n}{2}} = L_n + 2(-1)^{\frac{n}{2}} \tag{5.27}$$

When $n \equiv 2$ or $10 \bmod 12$ therefore $\frac{n}{2} \equiv 1$ or $5 \bmod 6$, which is odd number. By substituting (5.27) in (5.26) we get

$$R = \frac{L_n - 2}{L_{\frac{n}{2}} - 1 - (-1)^{\frac{n}{2}}} = \frac{L_n - 2}{L_{\frac{n}{2}}} = \frac{L_{\frac{n}{2}}^2}{L_{\frac{n}{2}}} = L_{\frac{n}{2}} \tag{5.28}$$

When $n \equiv 4$ or $8 \bmod 12$ therefore $\frac{n}{2} \equiv 2$ or $4 \bmod 6$, which is even number. By substituting (5.27) in (5.26) we get

$$\begin{aligned} R = \frac{L_n - 2}{L_{\frac{n}{2}} - 2} &= \frac{L_n - 2 + 4 - 4}{L_{\frac{n}{2}} - 2} \\ &= \frac{(L_n + 2) - 4}{L_{\frac{n}{2}} - 2} \\ &= \frac{L_{\frac{n}{2}}^2 - 4}{L_{\frac{n}{2}} - 2} = L_{\frac{n}{2}} + 2 \end{aligned} \tag{5.29}$$

Now following on from (5.28), (5.29), when $n \equiv 2$ or 4 mod 6, therefore

$$R = L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}} \tag{5.30}$$

□

5.6 When does $\Gamma_n(k,l) \cong \Gamma'_n(k',l')$ imply $n = n'$?

This is similar to Question 1.5.16, but here we consider $\Gamma_n(k,l)$ instead of $G_n(m,k)$.

From Table 5.2 $\Gamma_n(k,l) \cong \Gamma'_n(k',l')$ implies $n = n'$ in the following cases

(a) For the case (*FFFT*), since if $n \equiv 2$ or 4 mod 6, $|\Gamma_n(k,l)^{ab}| = 3\Big(L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}}\Big)$.

It is clear since Lucas numbers are increasing that $3\Big(L_{\frac{n}{2}} + 1 + (-1)^{\frac{n}{2}}\Big) = 3\Big(L_{\frac{n'}{2}} + 1 + (-1)^{\frac{n'}{2}}\Big)$ implies $n = n'$.

(b) The case (*FFTF*), since if $n \equiv 0$ mod 3, $\Gamma_n(k,l)^{ab} \cong \mathbb{Z}_\alpha, \alpha = 3(2^{n/3} - (-1)^{n/3})$. It is clear that $(2^{n/3} - (-1)^{n/3})/3 = (2^{n'/3} - (-1)^{n'/3})/3$ implies $n = n'$.

(c) The case (*TFTF*), since if $n \equiv 3$ or 6 mod 9, $\Gamma_n(k,l)^{ab} \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_\gamma, \gamma = (2^{n/3} - (-1)^{n/3})/3$. It is clear that $(2^{n/3} - (-1)^{n/3})/3 = (2^{n'/3} - (-1)^{n'/3})/3$ implies $n = n'$.

In the following cases $\Gamma_n(k,l) \cong \Gamma'_n(k',l')$ does not imply that $n = n'$

(a) The case (*FTFT*), for example $\Gamma_7(1,2) \cong \Gamma_{14}(1,2) \cong \mathbb{Z}_3$.

(b) The case (*TTFT*), for example $\Gamma_9(1,2) \cong \Gamma_{12}(1,2) \cong \mathbb{Z} * \mathbb{Z}$.

In the other cases (*TFFT*), (*FFFF*), (*TFFF*), we do not know if $\Gamma_n(k,l) \cong \Gamma'_n(k',l')$ implies $n = n'$ or not.

Chapter 6

$G_n(m,k), \Gamma_n(k,l)$ groups when n has few prime factors

The number of isomorphism types $f(n)$ of the groups $G_n(m,k), n = p^l, p$ is prime, was conjectured in [COS08, Conjecture 8] (restated as Conjecture 1.5.14). In Section 6.1, we show that the values given in the conjecture are an upper bound for $f(n)$. Results about the upper bound of $f(n)$ of $\Gamma_n(k,l)$ groups, where $n = p^\alpha q^\beta$, and p, q are distinct primes, will be seen in Section 6.2. In Section 6.3, we give in Table 6.1 the structure of the $\Gamma_n(k,l)$ groups for $n \leq 29$, and the precise value of $f(n)$. Finally in Section 6.4 (this section requires (A) true), we give an upper bound for $f(n)$ of $\Gamma_n(k,l)$ groups, where n has at most three distinct prime factors.

6.1 f(n) for $G_n(m,k), n = p^l, p$ is prime , $l \geq 1$

The following proposition was stated in [COS08]

Proposition 6.1.1. *[COS08, Proposition 6]*

1. $G_n(m,k) \cong G_n(m, n+m-k) \cong G_n(n-m, n-m+k)$.

2. *If* $(n,t) = 1$*, then* $G_n(m,k) \cong G_n(mt, kt)$.

3. $G_{2h}(2h-1, 2h-2) \cong G_{2h}(2h-1, 1) \cong G_{2h}(1, 2h-1) \cong G_{2h}(1,2) \cong F(2,2h)$.

The following lemma is stated in [BV03], [Wil09]

Lemma 6.1.2. *[BV03, Lemma 1.3], [Wil09, Lemma 7]*

- *If* $(n,k) = 1$ *then* $G_n(m,k) \cong H(n,t) = G_n(t,1)$ *where* $tk \equiv m \bmod n$.
- *If* $(n,m) = 1$ *then* $G_n(m,k) \cong G_n(1,k')$ *where* $k' = kt$ *where* $tm \equiv 1 \bmod n$.
- *If* $(n,k-m) = 1$ *then* $G_n(m,k) \cong H(n,t)$ *where* $t(k-m) \equiv n-m \bmod n$.

We recall the following conjecture from introduction, which is Conjecture 1.5.14.

Conjecture 6.1.3. *If* $n = p^l$ *for an odd prime* p *and positive integer* l*, then* $f(n) = p^l - \frac{(p-1)}{2}p^{(l-1)} - 1$. *If* $l > 2$ *then* $f(2^l) = 3(2^{l-2})$.

Theorems 6.1.4 and 6.1.5 show that the values given in Conjecture 6.1.3 are upper bounds for $f(n)$

Theorem 6.1.4. *If* $n = p^l$ *where* p *is an odd prime, and* l *is a positive integer, then* $G_n(m,k)$ *is isomorphic to* $G_n(1,k')$ *for some* $k' \in \{2, ..., \frac{p^l+1}{2}\}$ *or to* $G_n(t'p,1)$ *for some* $t' \in \{2, ..., p^{l-1}-1\}$*, and hence* $f(n) \leq p^l - \frac{(p-1)}{2}p^{(l-1)} - 1$.

Proof. **Case 1** If $(p^l,m) = 1$ then Lemma 6.1.2 implies that $G_{p^l}(m,k) \cong G_{p^l}(1,k')$ for some k', $2 \leq k' \leq p^l - 1$. But if $\frac{p^l+1}{2} \leq k' \leq p^l - 1$ then $k'' = p^l + 1 - k'$ satisfies $2 \leq k'' \leq \frac{p^l+1}{2}$. Proposition 6.1.1 implies that $G_{p^l}(1,k') \cong G_{p^l}(1,p^l+1-k') = G_{p^l}(1,k'')$. So the isomorphism types arise by choosing $k' \in \{2, ..., \frac{p^l+1}{2}\}$, This gives $\frac{p^l-1}{2}$ isomorphism types.

Case 2 When $(p^l,m) > 1$ then $(p^l,k) = 1$ since $(p^l,m,k) = 1$, therefore Lemma 6.1.2 implies that $G_{p^l}(m,k) \cong G_{p^l}(t_i,1)$ where $t_ik \equiv m \bmod p^l$. Since $p|m$ we have that $p|t_ik$ but p does not divides k therefore $p|t_i$, therefore $t_i \in A = \{ip, 1 \leq i \leq (p^{l-1}-1)\}$ (this contains an even number of elements), this gives $p^{l-1} - 1$ isomorphism types. Now Proposition 6.1.1 implies $G_{p^l}(t_i,1) \cong G_{p^l}(t_i,t_i-1)$.

Let $\alpha_i = -(t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)$ then

$$\begin{aligned}
\alpha_i(t_i - 1) &= -(t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)(t_i - 1) \\
&= (t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)(1 - t_i) \\
&= 1 - t_i^l \\
&\equiv 1 \mod p^l
\end{aligned}$$

By Proposition 6.1.1, and since $(p^l, \alpha_i) = 1$ we have $G_{p^l}(t_i, t_i - 1) \cong G_{p^l}(\alpha_i t_i, \alpha(t_i - 1)) = G_{p^l}(\alpha_i t_i, 1)$, therefore

$$G_{p^l}(t_i, 1) \cong G_{p^l}(\alpha_i t_i, 1) \tag{6.1}$$

Let $t_j = \alpha_i t_i \bmod p^l$ then $t_j \in A$, and let $\alpha_j = -(t_j^{l-1} + t_j^{l-2} + ... + t_j^2 + t_j + 1)$, therefore similarly

$$\alpha_j(t_j - 1) \equiv 1 \bmod p^l \tag{6.2}$$

Now define a function $f : A \longrightarrow A$ by $f(t_i) = \alpha_i t_i \bmod p^l$ then $f(t_i) = t_j$ (by hypotheses). We will show that $f(f(t_i)) \equiv t_i \bmod p^l$.

We have

$$\begin{aligned}
t_j &\equiv \alpha_i t_i \mod p^l \quad \text{by hypotheses} \\
\Rightarrow t_j &\equiv \alpha_i t_i - \alpha_i + \alpha_i \mod p^l \\
\Rightarrow t_j &\equiv \alpha_i(t_i - 1) + \alpha_i \mod p^l \\
\Rightarrow t_j &\equiv 1 + \alpha_i \mod p^l \\
\Rightarrow t_j - 1 &\equiv \alpha_i \mod p^l
\end{aligned} \tag{6.3}$$

therefore

$$\begin{aligned}
f(f(t_i)) &\equiv f(\alpha_i t_i) \mod p^l \\
&\equiv f(t_j) \mod p^l \\
&\equiv \alpha_j t_j \mod p^l \quad \text{by definition} \\
&\equiv \alpha_j \alpha_i t_i \mod p^l \\
&\equiv \alpha_j (t_j - 1) t_i \mod p^l \quad \text{by (6.3)} \\
&\equiv t_i \mod p^l \quad \text{by (6.2)}
\end{aligned}$$

Then $f(t_j) \equiv f(f(t_i)) \equiv t_i \bmod p^l$, and $f(t_i) = t_j \bmod p^l$. Now let Γ be a directed graph with a set of vertices $V = A$ and a set of edges $E = \{(t_i, f(t_i)), 1 \leq i \leq p^{(l-1)} - 1\}$. Then for each vertex $t_i \in V$, we have that the outdegree outdeg$(t_i) = 1$, and we will show that the indegree indeg$(t_i) = 1$. Let $t_k \in A$ and $\alpha_k = -(t_k^{l-1} + t_k^{l-2} + ... + t_k^2 + t_k + 1)$, then $\alpha_k(t_k - 1) \equiv 1 \bmod p^l$.

Suppose for contradiction that indeg$(t_i) \geq 2$. Then there are edges $(t_j, t_i)(t_k, t_i)$, with $t_j \not\equiv t_k$ mod p^l. Then

$$
\begin{aligned}
f(t_k) &\equiv t_i \bmod p^l \\
\Rightarrow \alpha_k t_k &\equiv f(t_j) \bmod p^l \\
\Rightarrow \alpha_k t_k &\equiv \alpha_j t_j \bmod p^l \\
\Rightarrow \alpha_k t_k - \alpha_k + \alpha_k &\equiv \alpha_j t_j - \alpha_j + \alpha_j \bmod p^l \\
\Rightarrow \alpha_k(t_k - 1) + \alpha_k &\equiv \alpha_j(t_j - 1) + \alpha_j \bmod p^l \\
\Rightarrow 1 + \alpha_k &\equiv 1 + \alpha_j \bmod p^l \\
\Rightarrow \alpha_k &\equiv \alpha_j \bmod p^l \\
\Rightarrow \alpha_k(t_k - 1) &\equiv \alpha_j(t_k - 1) \bmod p^l \\
\Rightarrow 1 &\equiv \alpha_j(t_k - 1) \bmod p^l \\
\Rightarrow \alpha_j(t_j - 1) &\equiv \alpha_j(t_k - 1) \bmod p^l \\
\Rightarrow t_j - 1 &\equiv t_k - 1 \bmod p^l \quad \text{since } (\alpha_j, p^l) = 1 \\
\Rightarrow t_j &\equiv t_k \bmod p^l \quad \text{a contradiction}
\end{aligned}
$$

Therefore indeg$(t_i) = 1$, and Γ consists of $|A|/2$ connected components, each has two vertices $\{t_i, f(t_i)\}$. Further by (6.1) we have $G_{p^l}(t_i, 1) \cong G_{p^l}(\alpha t_i, 1) = G_{p^l}(f(t_i), 1)$. Let $S \subset A$ be a set formed by taking one vertex from each components of Γ. Then $|S| = |A|/2 = \frac{p^{l-1}-1}{2}$, and for any $t_i \in A$, the group $G_{p^l}(t_i, 1) \cong G_{p^l}(s, 1)$ for some $s \in S$. Hence there are at most $\frac{p^{l-1}-1}{2}$ isomorphisms types amongst the groups $G_p^l(t_i, 1), 1 \leq i \leq p^{l-1} - 1$. From the first part of the proof we have at most $\frac{p^l-1}{2}$ isomorphism types when $(p^l, m) = 1$, and from second part we have at most $\frac{p^{l-1}-1}{2}$ ones when $(p^l, m) > 1$. Therefore there is at most $\frac{p^l-1}{2} + \frac{p^{l-1}-1}{2} = \frac{p^l+p^{l-1}}{2} - 1 = \frac{2p^l-p^l+p^{l-1}}{2} - 1 = p^l - \frac{(p-1)}{2}p^{(l-1)} - 1$ isomorphism types among the irreducible groups $G_{p^l}(m, k)$. □

Theorem 6.1.5. *If $n = 2^l$ where l is a positive integer, $l > 2$. Then $f(2^l) \leq 3(2^{l-2})$.*

Proof. **Case 1** If m is odd, then Lemma 6.1.2 implies that $G_{2^l}(m, k) \cong G_{2^l}(1, k')$ for some k', $2 \leq k' \leq 2^l - 1$. But if $2^{l-1} + 1 \leq k' \leq 2^l - 1$ then $k'' = 2^l + 1 - k'$ satisfies $2 \leq k'' \leq 2^l - 2^{l-1} = 2^{l-1}$. Proposition 6.1.1 implies that $G_{2^l}(1, k') \cong G_{2^l}(1, 2^l + 1 - k') = G_{2^l}(1, k'')$. So the isomorphisms types arise by choosing $k' \in \{2, ..., 2^{l-1}\}$. This gives $(2^{l-1} - 1)$ isomorphism types.

Case 2 When m is even then, $(2^l, k) = 1$ since $(2^l, m, k) = 1$, therefore Lemma 6.1.2 implies that $G_{2^l}(m,k) \cong G_{2^l}(t_i, 1)$ where $t_i k \equiv m \mod 2^l$. Since $2|m$ we have that $2|t_i k$ but 2 does not divides k therefore $2|t_i$, therefore $t_i \in A = \{2i, 1 \leq i \leq (2^{l-1} - 1)\} = \{2, 4, ..., 2^l - 2\}$ (this contains an odd number of elements), this gives $2^{l-1} - 1$ isomorphism types.

Now Proposition 6.1.1 implies $G_{2^l}(t_i, 1) \cong G_{2^l}(t_i, t_i - 1)$. Let $\alpha_i = -(t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)$ then

$$\begin{aligned}\alpha_i(t_i - 1) &\equiv -(t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)(t_i - 1) \mod 2^l \\ &\equiv (t_i^{l-1} + t_i^{l-2} + ... + t_i^2 + t_i + 1)(1 - t_i) \mod 2^l \\ &\equiv 1 - t_i^l \equiv 1 \mod 2^l \qquad \text{since } 2|t_i\,. \end{aligned} \tag{6.4}$$

That is, α_i is the (unique) multiplication inverse of $(t_i - 1) \mod n$. By Proposition 6.1.1, and since $(2^l, \alpha_i) = 1$ we have $G_{2^l}(t_i, t_i - 1) \cong G_{2^l}(\alpha_i t_i, \alpha(t_i - 1)) = G_{2^l}(\alpha_i t_i, 1)$, therefore

$$G_{2^l}(t_i, 1) \cong G_{2^l}(\alpha_i t_i, 1) \tag{6.5}$$

When $t_i = 2$, then $t_i - 1 = 1$, and by(6.4) then $\alpha_i \equiv 1 \mod 2^l$, therefore $\alpha_i t_i = 1.2 = 2 = t_i$. When $t_i = 2^{l-1}$, then $t_i - 1 = 2^{l-1} - 1 \mod 2^l$ and since

$$(2^{l-1} - 1)(2^{l-1} - 1) \equiv 2^{2(l-1)} - 2^l + 1 \mod 2^l \equiv 1 \mod 2^l$$

then $\alpha_i = 2^{l-1} - 1$ and

$$\begin{aligned}\alpha_i t_i &\equiv (2^{l-1} - 1).2^{l-1} \mod 2^l \\ &\equiv 2^{2(l-1)} - 2^{l-1} \mod 2^l \\ &\equiv 2^{l-1} \mod 2^l \\ &\equiv t_i \mod 2^l\,. \end{aligned}$$

When $t_i = 2^{l-1} + 2$, then $t_i - 1 = 2^{l-1} + 1$ and since

$$(2^{l-1} + 1)(2^{l-1} + 1) \equiv 2^{2(l-1)} + 2^l + 1 \mod 2^l \equiv 1 \mod 2^l$$

$$\begin{aligned} &\text{then } \alpha_i = 2^{l-1} + 1 \text{ and} \\ &\alpha_i t_i \equiv (2^{l-1}+1)(2^{l-1}+2) \mod 2^l \\ &\quad \equiv 2^{2(l-1)} + 2^l + 2^{l-1} + 2 \mod 2^l \\ &\quad \equiv 2^{l-1} + 2 \mod 2^l \equiv t_i \mod 2^l . \end{aligned}$$

Therefore the permutation α_i takes the values of $\{2, 2^{l-1}, 2^{l-1}+2\}$ to them self. Now we chose all $t_i \in B = A - \{2, 2^{l-1}, 2^{l-1}+2\}$ (this group contains $2^{l-1} - 4$ elements, which is even number), we can consider the permutation α_i as permuting B. Let $t_j = \alpha_i t_i \mod 2^l$ then $t_j \in B$, and let $\alpha_j = -(t_j^{l-1} + t_j^{l-2} + \dots + t_j^2 + t_j + 1)$, therefore as before

$$\alpha_j(t_j - 1) \equiv 1 \mod 2^l . \tag{6.6}$$

Now define a function $f : B \longrightarrow B$ by $f(t_i) = \alpha_i t_i \mod p^l$ then $f(t_i) = t_j$. We will show that $f(f(t_i)) \equiv t_i \mod 2^l$.

We have

$$\begin{aligned} & t_j \equiv \alpha_i t_i \mod 2^l \quad \text{by definition of } t_j \\ \Rightarrow\ & t_j \equiv \alpha_i t_i - \alpha_i + \alpha_i \mod 2^l \\ \Rightarrow\ & t_j \equiv \alpha_i(t_i - 1) + \alpha_i \mod 2^l \\ \Rightarrow\ & t_j \equiv 1 + \alpha_i \mod 2^l \qquad \text{by (6.4)} \end{aligned}$$

$$\Rightarrow t_j - 1 \equiv \alpha_i \mod 2^l \tag{6.7}$$

therefore

$$\begin{aligned}
f(f(t_i)) &\equiv f(\alpha_i t_i) \mod 2^l \\
&\equiv f(t_j) \mod 2^l \\
&\equiv \alpha_j t_j \mod 2^l \quad \text{by definition} \\
&\equiv \alpha_j f(t_i) \mod 2^l \\
&\equiv \alpha_j \alpha_i t_i \mod 2^l \\
&\equiv \alpha_j (t_j - 1) t_i \mod 2^l \quad \text{by (6.7)} \\
&\equiv t_i \mod 2^l \quad \text{by (6.6)}
\end{aligned}$$

Then $f(t_j) \equiv f(f(t_i)) \equiv t_i \bmod 2^l$, and $f(t_i) \equiv t_j \bmod 2^l$. Now let Γ be a directed graph with a set of vertices $V = B$ and a set of edges $E = \{(t_i, f(t_i)), 1 \leq i \leq 2^{(l-1)} - 1\}$. Then for each vertex $t_i \in V$ outdeg$(t_i) = 1$, and we will show that indeg$(t_i) = 1$. Suppose for contradiction that indeg$(t_i) \geq 2$. Then there are edges $(t_j, t_i), (t_k, t_i)$, with $t_j \not\equiv t_k \bmod 2^l$, $t_j \in B, t_k \in B$. Let $\alpha_k = -(t_k^{l-1} + t_k^{l-2} + ... + t_k^2 + t_k + 1)$, then

$$\alpha_k(t_k - 1) \equiv 1 \bmod 2^l . \tag{6.8}$$

Now

$$
\begin{aligned}
f(t_k) &\equiv t_i \bmod 2^l \\
\Rightarrow \alpha_k t_k &\equiv f(t_j) \bmod 2^l \\
\Rightarrow \alpha_k t_k &\equiv \alpha_j t_j \bmod 2^l \\
\Rightarrow \alpha_k t_k - \alpha_k + \alpha_k &\equiv \alpha_j t_j - \alpha_j + \alpha_j \bmod 2^l \\
\Rightarrow \alpha_k (t_k - 1) + \alpha_k &\equiv \alpha_j (t_j - 1) + \alpha_j \bmod 2^l \\
\Rightarrow 1 + \alpha_k &\equiv 1 + \alpha_j \bmod 2^l \quad \text{by (6.6), (6.8)} \\
\Rightarrow \alpha_k &\equiv \alpha_j \bmod 2^l \\
\Rightarrow \alpha_k (t_k - 1) &\equiv \alpha_j (t_k - 1) \bmod 2^l \\
\Rightarrow 1 &\equiv \alpha_j (t_k - 1) \bmod 2^l \quad \text{by (6.8)} \\
\Rightarrow \alpha_j (t_j - 1) &\equiv \alpha_j (t_k - 1) \bmod 2^l \quad \text{by (6.6)} \\
\Rightarrow t_j - 1 &\equiv t_k - 1 \bmod 2^l \quad \text{since } (\alpha_j, 2^l) = 1 \\
\Rightarrow t_j &\equiv t_k \bmod 2^l \quad \text{a contradiction.}
\end{aligned}
$$

Therefore indeg$(t_i) = 1$, and Γ consists of $|B|/2$ connected components, each has two vertices $\{t_i, f(t_i)\}$. Further by (6.5) we have $G_{2^l}(t_i, 1) \cong G_{2^l}(\alpha_i t_i, 1) = G_{2^l}(f(t_i), 1)$. Let $S \subset B$ be a set formed by taking one vertex from each components of Γ. Then $|S| = |B|/2 = \frac{2^{l-1}-4}{2} = 2^{l-2} - 2$, and for any $t_i \in B$, the group $G_{2^l}(t_i, 1) \cong G_{2^l}(s, 1)$ for some $s \in S$. Therefore for each $t_i \in A$ the group $G_{2^l}(t_i, 1)$ is isomorphic to $G_{2^l}(2, 1)$ or $G_{2^l}(2^{l-1}, 1)$ or $G_{2^l}(2^{l-1} + 2, 1)$ or to $G_{2^l}(s, 1)$ for some $s \in S$. Hence there are at most $2^{l-2} - 2 + 3 = 2^{l-2} + 1$ isomorphisms types amongst the groups $G^l_2(t_i, 1), 1 \leq i \leq 2^{l-1} - 1$. From the first part of the proof, we have that if m is odd then $G_{2^l}(m, k)$ is isomorphic to (at least) one of $(2^{l-1} - 1)$ groups. From second part, we have that if m even then $G_{2^l}(m, k)$ is isomorphic to (at least) one of $(2^{l-2} + 1)$ groups. Therefore there is at most $(2^{l-1} - 1) + (2^{l-2} + 1) = 2^{l-2}(2 + 1) = 3.2^{l-2}$ isomorphism types among the irreducible groups $G_{2^l}(m, k)$. □

6.2 f(n) of $\Gamma_n(k,l)$ where n has at most two distinct prime factors

We consider $\Gamma_n(k,l)$ for $n = p^\alpha q^\beta$, where p, q are distinct primes and $\alpha \geq 0, \beta \geq 0$. We apply the isomorphisms identified in Lemma 5.0.10, to obtain an upper bound of $f(n)$.

Lemma 6.2.1. *If $(n,k) = 1$ or $(n,l) = 1$ or $(n,k-l) = 1$ then $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some l'.*

Proof. When $(n,k) = 1$, part (5) of Lemma 5.0.10 implies that $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some l'. When $(n,l) = 1$, then parts (2), (5) of Lemma 5.0.10 imply that $\Gamma_n(k,l) \cong \Gamma_n(l,k) \cong \Gamma_n(1,l')$ for some l'. When $(n,k-l) = 1$, then parts (4), (2), (5) of Lemma 5.0.10 imply that $\Gamma_n(k,l) \cong \Gamma_n(k,k-l) \cong \Gamma_n(k-l,k) \cong \Gamma_n(1,l')$ for some l'. □

Lemma 6.2.2. If $n = p^\alpha q^\beta$, where p, q are distinct prime and $\alpha \geq 0, \beta \geq 0$, and $(n,k,l) = 1$ then $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some $2 \leq l' \leq (n-1)$.

Proof. We shall show that at least one of the following holds: $(n,k) = 1, (n,l) = 1, (n,k-l) = 1$. For then the result follows Lemma 6.2.1. Suppose for a contradiction $(n,k) > 1, (n,l) > 1, (n,k-l) > 1$. Now $(n,k) > 1$ implies that $p|k$ or $q|k$, and $(n,l) > 1$ implies that $p|l$ or $q|l$, and $(n,k-l) > 1$ implies that $p|(k-l)$ or $q|(k-l)$. Since $(n,k) > 1$ we have $p|k$ or $q|k$. Without loss of generality suppose $p|k$ then $p \nmid l$ since $(n,k,l) = 1$, and then $p \nmid (k-l)$. Therefore $q|l$ and $q|(k-l)$, so $q|k$ so $q|(n,k,l) = 1$, a contradiction. Therefore $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some $2 \leq l' \leq (n-1)$. □

Lemma 6.2.3 is proved by using Lemma 5.0.10

Lemma 6.2.3. a. *Suppose $(n,l) = 1$, then $\Gamma_n(1,l) \cong \Gamma_n(1,l')$ where $l'l \equiv 1$ mod n.*

b. *Suppose $(n,l) = 1$, then $\Gamma_n(1,l) \cong \Gamma_n(1,n+1-\bar{l})$ where $l\bar{l} \equiv 1$ mod n.*

c. *Suppose $(n,l-1) = 1$, then $\Gamma_n(1,l) \cong \Gamma_n(1,1+\bar{l})$ where $(l-1).\bar{l} \equiv 1$ mod n.*

d. *Suppose $(n,l) = 1$, then $\Gamma_n(1,l) \cong \Gamma_n(1,\bar{l})$ where $l\bar{l} \equiv 1$ mod n.*

Proof. **a.** By using parts 2, 5 of Lemma 5.0.10 we get $\Gamma_n(1,l) \cong \Gamma_n(l,1) \cong \Gamma_n(l.l', 1.l') \cong \Gamma_n(1,l')$.

b. By using parts 2, 5, 4 of Lemma 5.0.10 we get $\Gamma_n(1,l) \cong \Gamma_n(l,1) \cong \Gamma_n(l.\bar{l},1.\bar{l}) \cong \Gamma_n(1,\bar{l}) \cong \Gamma_n(1,1-\bar{l}) \cong \Gamma_n(1,n+1-\bar{l})$.

c. By using 1, 5, 4 of Lemma 5.0.10 we get $\Gamma_n(1,l) \cong \Gamma_n(l-1,-1) \cong \Gamma_n((l-1)\bar{l},-1.\bar{l}) \cong \Gamma_n(1,-\bar{l}) \cong \Gamma_n(1,1+\bar{l})$.

d. By using 2, 5 of Lemma 5.0.10 we get $\Gamma_n(1,l) \cong \Gamma_n(l,1) \cong \Gamma_n(l.\bar{l},1.\bar{l}) \cong \Gamma_n(1,\bar{l})$.

□

We give here an upper bound of $f(n)$, for $\Gamma_n(1,l)$

Lemma 6.2.4. Let $n = p^\alpha q^\beta$, where p, q are distinct prime and $\alpha \geq 0, \beta \geq 0$ then,

a. If n is even, we have two cases

1. If $n \equiv 0 \bmod 4$, then $\Gamma_n(1,l) \cong \Gamma_n(1,l'')$ for some $1 < l'' \leq \frac{n}{2}$, and there are at most $\frac{n}{2} - 1$ isomorphisms types amongst the groups $\Gamma_n(1,l)$.
2. If $n \equiv 2 \bmod 4$, $n \geq 10$ then $\Gamma_n(k,l) \cong \Gamma_n(1,l'')$ for some $1 < l'' \leq \frac{n}{2}$, and there are at most $\frac{n}{2} - 2$ isomorphisms types amongst the groups $\Gamma_n(1,l)$.

b. If n is odd we have two cases

1. If $n \equiv 3 \bmod 6$ and $n \geq 9$, then $\Gamma_n(1,l) \cong \Gamma_n(1,l'')$ for some $1 < l'' \leq \frac{n-3}{2}$, and there are at most $\frac{n-5}{2}$ isomorphisms types amongst the groups $\Gamma_n(k,l)$.
2. If $n \equiv 1$ or $5 \bmod 6$, $n \geq 11$ then $\Gamma_n(1,l) \cong \Gamma_n(1,l'')$ for some $1 < l'' \leq \frac{n-3}{2}$, and there are at most $\frac{n-7}{2}$ isomorphisms types amongst the groups $\Gamma_n(1,l)$.

Proof. **a.** n even

If $\frac{n}{2} < l' \leq n-1$ then $-\frac{n}{2} > -l' \geq 1-n$, so $(n+1) - \frac{n}{2} > (n+1) - l' \geq (n+1) + 1 - n$. Now let $l'' = (n+1) - l'$, then $\frac{n}{2} + 1 > l'' \geq 2$, and then by Lemma 6.2.2 and (4) in Lemma 5.0.10, we have $\Gamma_n(k,l) \cong \Gamma_n(1,l') \cong \Gamma_n(1,1-l') = \Gamma_n(1,n+1-l') = \Gamma_n(1,l'')$, therefore $\Gamma_n(k,l) \cong \Gamma_n(1,l'')$ for some $1 < l'' \leq \frac{n}{2}$, this gives at most $\frac{n}{2} - 1$ isomorphism types.

If $n \equiv 2 \bmod 4$ and $n \geq 10$, then we can find one more isomorphism as follow. Since $1 < \frac{n+2}{4} < \frac{n}{2} - 1 < \frac{n}{2}$ we have two cases

1. When $n \equiv 2 \mod 8$ ($\frac{n+2}{4}$ is odd), we shall show that $\Gamma_n(1, \frac{n+2}{4}) \cong \Gamma_n(1, \frac{n}{2} - 1)$, therefore there are at most $\frac{n}{2} - 2$ isomorphisms types amongst the groups $\Gamma_n(k, l)$. Observe first $(\frac{n+2}{4})(\frac{n+4}{2}) = \frac{n^2+6n+8}{8} = \frac{n.(n+6)}{8} + 1 \equiv 1 \mod n$, therefore

$$\begin{aligned}\Gamma_n(1, \frac{n+2}{4}) &\cong \Gamma_n(1, n+1-\frac{n+4}{2}) \qquad \text{by part } (b) \text{ of Lemma 6.2.3}\\ &\cong \Gamma_n(1, \frac{2n+2-n-4}{2})\\ &\cong \Gamma_n(1, \frac{n}{2} - 1)\end{aligned}$$

2. When $n \equiv 6 \mod 8$ ($\frac{n+2}{4}$ is even), similarly we shall show that $\Gamma_n(1, \frac{n+2}{4}) \cong \Gamma_n(1, \frac{n}{2} - 1)$, therefore there are at most $\frac{n}{2} - 2$ isomorphisms types amongst the groups $\Gamma_n(k, l)$.
   Observe that $(\frac{n-2}{4})(\frac{n-4}{2}) = \frac{n^2-6n+8}{8} = \frac{n.(n-6)}{8} + 1 \equiv 1 \mod n$, therefore

$$\begin{aligned}\Gamma_n(1, \frac{n+2}{4}) &\cong \Gamma_n(1, 1+\frac{n-4}{2}) \qquad \text{by part } (c) \text{ of Lemma 6.2.3}\\ &\cong \Gamma_n(1, \frac{n}{2} - 1)\end{aligned}$$

b. n odd

If $\frac{n+1}{2} \le l' \le n-1$, therefore $-(\frac{n+1}{2}) \ge -l' \ge 1-n$, and then $(n+1) - (\frac{n+1}{2}) \ge (n+1) - l' \ge (n+1) + 1 - n$. Now let $l'' = (n+1) - l'$, then $\frac{n+1}{2} \ge l'' \ge 2$, by Lemma 6.2.2 and (4) in Lemma 5.0.10, we have $\Gamma_n(k, l) \cong \Gamma_n(1, l') \cong \Gamma_n(1, 1-l') = \Gamma_n(1, n+1-l') = \Gamma_n(1, l'')$, therefore $\Gamma_n(k, l) \cong \Gamma_n(1, l'')$ for some $1 < l'' \le \frac{n+1}{2}$.

When $l'' = \frac{n+1}{2}$, and by Lemma 6.2.3 we have that $\Gamma_n(1, \frac{n+1}{2}) \cong \Gamma_n(1, 2)$, also when $l'' = \frac{n-1}{2}$, and by (2), (5), (4) in Lemma 5.0.10 we have that $\Gamma_n(1, \frac{n-1}{2}) \cong \Gamma_n(\frac{n-1}{2}, 1) \cong \Gamma_n(\frac{n-1}{2}.(n-2), 1(n-2)) \cong \Gamma_n(1, n-2) \cong \Gamma_n(1, 1-(n-2)) \cong \Gamma_n(1, 3)$, therefore $\Gamma_n(k, l) \cong \Gamma_n(1, l'')$ for some $1 < l'' \le \frac{n-3}{2}$, this gives at most $\frac{n-3}{2} - 1 = \frac{n-5}{2}$ isomorphism types.

If $n \equiv 1$ or $5 \mod 6$, $n \ge 11$, then we can find one more isomorphism as follow. There are two cases

1. When $n \equiv 1 \mod 6$, we shall show that $\Gamma_n(1, \frac{n+2}{3}) \cong \Gamma_n(1, 3)$, therefore there are at

most $\frac{n-5}{2} - 1 = \frac{n-7}{2}$ isomorphisms types amongst the groups $\Gamma_n(k,l)$.

$$\begin{aligned}\Gamma_n(1,\frac{n+2}{3}) &\cong \Gamma_n(1,1-\frac{n+2}{3}) \qquad \text{by (4) in Lemma 5.0.10}\\ &\cong \Gamma_n(1,\frac{-n+1}{3})\\ &\cong \Gamma_n(1,\frac{2n+1}{3})\\ &\cong \Gamma_n(1,3) \qquad \text{by part } (d) \text{ of Lemma 6.2.3}\end{aligned}$$

2. When $n \equiv 5 \bmod 6$, we shall show that $\Gamma_n(1,\frac{n+1}{3}) \cong \Gamma_n(1,3)$, therefore there are at most $\frac{n-5}{2} - 1 = \frac{n-7}{2}$ isomorphisms types amongst the groups $\Gamma_n(k,l)$.

$$\Gamma_n(1,3) \cong \Gamma_n(1,\frac{n+1}{3}) \qquad \text{by part } (d) \text{ of Lemma 6.2.3}$$

□

Lemma 6.2.4 will allow us to obtain the upper bound of $f^{FFFF}(n)$, when $n = p^\alpha q^\beta$, where $p \geq 5, q \geq 5$ are distinct primes

Lemma 6.2.5. *Let $n = p^\alpha q^\beta, n \geq 11, p, q$ are distinct primes and $p \geq 5, q \geq 5, \alpha \geq 0, \beta \geq 0$, then $f^{(F,F,F,F)}(n) \leq \frac{n-7}{2}$.*

Proof. Lemmas(6.2.2 and 6.2.4) showed that $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some l', and when n is odd, then $f(n) \leq \frac{n-5}{2}$, so we consider $\Gamma_n(1,l)$. Since $p, q \geq 5$ therefore $n \not\equiv 0 \bmod 2$ and $n \not\equiv 0 \bmod 3$, therefore A, B, C, D are always $FFFF$ for $0 \leq l \leq n-1$, except when $l = 2$, $l = n-1$ or $l = \frac{n+1}{2}$ then B holds, but $\Gamma_n(1,2) \cong \Gamma_n(1,n-1) \cong \Gamma_n(1,\frac{n+1}{2}) \cong \mathbb{Z}_3$ by number $2,4,5$ Lemma 5.0.10, so by excluding this isomorphism from the number of isomorphisms types we get $f^{(F,F,F,F)}(n) = f(n) - f^{(F,T,F,F)}(n) \leq \frac{n-5}{2} - 1 = \frac{n-7}{2}$. □

6.3 Investigating $\Gamma_n(k,l)$ for $n \leq 29$

The smallest number that has more than 2 distinct prime factors is 30, so we investigate $\Gamma_n(k,l)$ for $n \leq 29$. This means $n = p^\alpha q^\beta$, for distinct primes p, q and $\alpha \geq 0, \beta \geq 0$. We present our results in Table 6.1 in terms of $(A),(B),(C),(D)$ conditions, we give in that table the structure of the $\Gamma_n(k,l)$ groups, and the precise value of $f(n)$. The problem of counting $\Gamma_n(k,l)$ is reduced by Lemma 6.2.2 to counting $\Gamma_n(1,l)$. In order to obtain this we use Lemma 6.2.4 which gives an upper bound of $f(n)$ of $\Gamma_n(1,l)$, and it can be seen from the table that the bounds in Lemma 6.2.4 are attained for $n = 4,6,8,9,10,11,12,13,15,21,24$, therefore no cases of Lemma 6.2.4 can be directly improved. For other values of n we use Lemma 5.0.10 (which is Lemma 2.1. of [EW10]), and Lemma 6.2.3 to obtain isomorphisms, and we distinguish isomorphisms by calculating abelianization of group by using [GAP]. Sometimes the abelianization is not enough to distinguish two groups, so we use [GAP] to calculate $\frac{\Gamma'_n(k,l)}{\Gamma''_n(k,l)} = (\Gamma'_n(k,l))^{ab}$ as well. For example $\Gamma_{10}(1,3)^{ab} = \mathbb{Z}_{33}, \Gamma_{10}(1,5)^{ab} = \mathbb{Z}_{33}$, but $(\Gamma'_{10}(1,3))^{ab} = \mathbb{Z}_{31}, (\Gamma'_{10}(1,5))^{ab} = \mathbb{Z}_9^5 \oplus \mathbb{Z}_{61}$, therefore $\Gamma_{10}(1,3) \not\cong \Gamma_{10}(1,5)$. By Lemma 6.2.4 we have that $\Gamma_n(k,l) \cong \Gamma_n(1,l')$ for some $l' \in S$, where

$$S = \begin{cases} \{2,3,...,\frac{n}{2}\} \text{ when } n \text{ when } n \equiv 0 \bmod 4 \\ \{2,3,...,\frac{n}{2}\} \setminus \{\frac{n}{2}-1\} \text{ when } n \equiv 2 \bmod 4 \\ \{2,3,...,\frac{n-3}{2}\} \text{ when } n \text{ when } n \equiv 3 \bmod 6 \\ \{2,3,...,\frac{n-3}{2}\} \setminus \{\frac{n+2}{3}\} \text{ when } n \equiv 1 \bmod 6 \\ \{2,3,...,\frac{n-3}{2}\} \setminus \{\frac{n+1}{3}\} \text{ when } n \equiv 5 \bmod 6 \end{cases}$$

For giving more information about the group, we use Corollary 1.6.2 to show when the group contains free subgroup, which we denote in the table by free sbgp.

All isomorphisms indicated in Table 6.1 are a direct applications of Lemma 6.2.3, and we give below full details.

- When $n = 14$, part (a) of Lemma 6.2.3 implies that $\Gamma_{14}(1,3) \cong \Gamma_{14}(1,5)$.
- When $n = 16$, part (b) of Lemma 6.2.3 implies that $\Gamma_{16}(1,3) \cong \Gamma_{16}(1,6)$ where $l = 3, \bar{l} = 11$, and implies that $\Gamma_{16}(1,4) \cong \Gamma_{16}(1,5)$ where $l = 5, \bar{l} = 13$.
- When $n = 17$, we have $\Gamma_{17}(1,4) \cong \Gamma_{17}(1,5) \cong \Gamma_{17}(1,7)$. This holds because part (b) of Lemma 6.2.3 implies that $\Gamma_{17}(1,4) \cong \Gamma_{17}(1,5)$, where $l = 4, \bar{l} = 13$, and part (c) of Lemma 6.2.3 implies that $\Gamma_{17}(1,7) \cong \Gamma_{17}(1,4)$, where $l = 7, \bar{l} = 3$.
- When $n = 18$, part (b) of Lemma 6.2.3 implies that $\Gamma_{18}(1,6) \cong \Gamma_{18}(1,7)$, where $l = 7, \bar{l} = 13$.

- When $n = 19$, We have $\Gamma_{19}(1,4) \cong \Gamma_{19}(1,5) \cong \Gamma_{19}(1,6)$. This holds because part (a) of Lemma 6.2.3 implies that $\Gamma_{19}(1,4) \cong \Gamma_{19}(1,5)$, and part ($b$) of Lemma 6.2.3 implies that $\Gamma_{19}(1,6) \cong \Gamma_{19}(1,4)$, where $l = 6, \bar{l} = 16$.

- When $n = 20$, part (a) of Lemma 6.2.3 implies that $\Gamma_{20}(1,3) \cong \Gamma_{20}(1,7)$. Part ($c$) of Lemma 6.2.3 implies that $\Gamma_{20}(1,4) \cong \Gamma_{20}(1,8)$, where $l = 4, \bar{l} = 7$.

- When $n = 22$, part (b) of Lemma 6.2.3 implies that $\Gamma_{22}(1,3) \cong \Gamma_{22}(1,8)$ where $l = 3, \bar{l} = 15$, and implies that $\Gamma_{22}(1,4) \cong \Gamma_{22}(1,7)$ where $l = 7, \bar{l} = 19$. Part (d) of Lemma 6.2.3 implies that $\Gamma_{22}(1,5) \cong \Gamma_{22}(1,9)$ where $l = 5, \bar{l} = 9$.

- When $n = 23$, we have $\Gamma_{23}(1,4) \cong \Gamma_{23}(1,6) \cong \Gamma_{23}(1,9)$. This holds because part (a) of Lemma 6.2.3 implies that $\Gamma_{23}(1,4) \cong \Gamma_{23}(1,6)$, and part ($a$) of Lemma 6.2.3 implies that $\Gamma_{23}(1,9) \cong \Gamma_{23}(1,6)$, where $l = 9, \bar{l} = 18$. We also have $\Gamma_{23}(1,5) \cong \Gamma_{23}(1,7) \cong \Gamma_{23}(1,10)$. This holds because part (b) of Lemma 6.2.3 implies that $\Gamma_{23}(1,5) \cong \Gamma_{23}(1,10)$, where $l = 5, \bar{l} = 14$, and part (d) of Lemma 6.2.3 implies that $\Gamma_{23}(1,7) \cong \Gamma_{23}(1,10)$ where $l = 7, \bar{l} = 10$.

- When $n = 25$, we have $\Gamma_{25}(1,4) \cong \Gamma_{25}(1,7) \cong \Gamma_{25}(1,8)$. This holds because part (b) of Lemma 6.2.3 implies that $\Gamma_{25}(1,4) \cong \Gamma_{25}(1,7)$ where $l = 4, \bar{l} = 19$, and implies that $\Gamma_{25}(1,8) \cong \Gamma_{25}(1,4)$ where $l = 8, \bar{l} = 22$, and also implies that $\Gamma_{25}(1,5) \cong \Gamma_{16}(1,6)$ where $l = 6, \bar{l} = 21$.

- When $n = 26$, part (a) of Lemma 6.2.3 implies that $\Gamma_{26}(1,3) \cong \Gamma_{26}(1,9)$, and part ($c$) of Lemma 6.2.3 implies that $\Gamma_{26}(1,4) \cong \Gamma_{26}(1,10)$ where $l = 4, \bar{l} = 9$. part (b) of Lemma 6.2.3 implies that $\Gamma_{26}(1,5) \cong \Gamma_{26}(1,6)$ where $l = 5, \bar{l} = 21$, and also implies that $\Gamma_{26}(1,8) \cong \Gamma_{26}(1,11)$ where $l = 11, \bar{l} = 19$.

- When $n = 27$, part (a) of Lemma 6.2.3 implies that $\Gamma_{27}(1,4) \cong \Gamma_{26}(1,7)$. We also have $\Gamma_{27}(1,5) \cong \Gamma_{27}(1,8) \cong \Gamma_{27}(1,11)$. This holds because part (c) of Lemma 6.2.3 implies that $\Gamma_{27}(1,5) \cong \Gamma_{27}(1,8)$, where $l = 5, \bar{l} = 7$, and part (b) of Lemma 6.2.3 implies that $\Gamma_{27}(1,8) \cong \Gamma_{27}(1,11)$ where $l = 8, \bar{l} = 17$. Part (c) of Lemma 6.2.3 implies that $\Gamma_{27}(1,6) \cong \Gamma_{27}(1,12)$ where $l = 6, \bar{l} = 11$. Part (d) of Lemma 6.2.3 implies that $\Gamma_{27}(1,9) \cong \Gamma_{27}(1,10)$ where $l = 9, \bar{l} = 10$.

- When $n = 28$, Lemma 6.2.3 implies that $\Gamma_{28}(1,3) \cong \Gamma_{28}(1,10)$ where $l = 3, \bar{l} = 19$, and implies $\Gamma_{28}(1,4) \cong \Gamma_{28}(1,9)$ where $l = 9, \bar{l} = 25$, and implies that $\Gamma_{28}(1,5) \cong \Gamma_{28}(1,12)$ where $l = 5, \bar{l} = 17$, it is also implies that $\Gamma_{28}(1,6) \cong \Gamma_{28}(1,11)$ where $l = 11, \bar{l} = 23$.

- When $n = 29$, we have $\Gamma_{29}(1,4) \cong \Gamma_{29}(1,8) \cong \Gamma_{29}(1,11)$. This holds because part (*b*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,4) \cong \Gamma_{29}(1,8)$, where $l = 4, \bar{l} = 22$, and part (*d*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,8) \cong \Gamma_{29}(1,11)$, where $l = 8, \bar{l} = 11$. We also have $\Gamma_{29}(1,5) \cong \Gamma_{29}(1,6) \cong \Gamma_{29}(1,7)$. This holds because part (*a*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,5) \cong \Gamma_{29}(1,6)$ and part (*a*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,5) \cong \Gamma_{29}(1,7)$ where $l = 7, \bar{l} = 25$. We also have $\Gamma_{29}(1,9) \cong \Gamma_{29}(1,12) \cong \Gamma_{29}(1,13)$. This holds because part (*c*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,9) \cong \Gamma_{29}(1,12)$, where $l = 9, \bar{l} = 11$, and part (*d*) of Lemma 6.2.3 implies that $\Gamma_{29}(1,9) \cong \Gamma_{27}(1,13)$, where $l = 9, \bar{l} = 13$.

Table 6.1: $\Gamma_n(1,l)$, $l \in S$

n	A	B	C	D	$f(n)$	l	$\Gamma_n(1,l)^{ab}$	$\Gamma_n(1,l)$	$\Gamma'/\Gamma'' = (\Gamma')^{ab}$
3	T	T	T	T	1	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
4	F	T	F	T	1	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
5	F	T	F	T	1	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
6	T	T	T	T	2	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	T	F		3	$\mathbb{Z}_9$	Metacyclic	
7	F	T	F	T	2	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$	Free sbgp	
8	F	T	F	T	3	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	T		3	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	∞	
	F	F	F	F		4	$\mathbb{Z}_{15}$	Free sbgp	
9	T	T	F	T	2	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	T	F		3	$\mathbb{Z}_{27}$	Metacyclic	
10	F	T	F	T	3	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	T		3	$\mathbb{Z}_{33}$	∞	$\mathbb{Z}_{31}$
	F	F	F	F		5	$\mathbb{Z}_{33}$	Free sbgp	$\mathbb{Z}_9^5 \oplus \mathbb{Z}_{61}$
11	F	T	F	T	2	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{69}$	Free sbgp	
12	T	T	F	T	5	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_{117}$	Free sbgp	
	F	F	T	F		4	$\mathbb{Z}_{45}$	Metacyclic	
	T	F	T	T		5	$\mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_5 * \mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		6	$\mathbb{Z}_{63}$	Free sbgp	
13	F	T	F	T	3	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{159}$	Free sbgp	
	F	F	F	F		4	$\mathbb{Z}_3^4$	Free sbgp	
14	F	T	F	T	4	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 5	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	Free sbgp	
	F	F	F	T		4	$\mathbb{Z}_{87}$	∞	
	F	F	F	F		7	$\mathbb{Z}_{129}$	Free sbgp	
15	T	T	F	T	5	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_{279}$	Free sbgp	
	F	F	F	F		4	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{18}$	Free sbgp	
	T	F	T	F		5	$\mathbb{Z}_{11} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_{11} * \mathbb{Z} * \mathbb{Z}$	
	F	F	T	F		6	$\mathbb{Z}_{99}$	Metacyclic	
16	F	T	F	T	5	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 6	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{51}$	Free sbgp	
	F	F	F	F		4, 5	$\mathbb{Z}_{255}$	Free sbgp	$\mathbb{Z}_2^{16} \oplus \mathbb{Z}_3^{16} \oplus \mathbb{Z}_7^2 \oplus \mathbb{Z}_{17}^8 \oplus \mathbb{Z}_{19}^{16} \oplus \mathbb{Z}^8$
	F	F	F	T		7	$\mathbb{Z}_7 \oplus \mathbb{Z}_{21}$	∞	
	F	F	F	F		8	$\mathbb{Z}_{255}$	Free sbgp	$\mathbb{Z}_2^{64} \oplus \mathbb{Z}_4^{26} \oplus \mathbb{Z}_8^{16} \oplus \mathbb{Z}_{13}^{12} \oplus \mathbb{Z}_{41}^2$
17	F	T	F	T	3	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{717}$	Free sbgp	
	F	F	F	F		4, 5, 7	$\mathbb{Z}_{309}$	Free sbgp	

n	A	B	C	D	$f(n)$	l	$\Gamma_n(1,l)^{ab}$	$\Gamma_n(1,l)$	$\Gamma'/\Gamma'' = (\Gamma')^{ab}$
18	T	T	F	T	6	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_{999}$	Free sbgp	
	F	F	F	F		4	$\mathbb{Z}_{513}$	Free sbgp	$\mathbb{Z}_7^4 \oplus \mathbb{Z}_{19}^4 \oplus \mathbb{Z}_{37}$
	T	F	F	T		5	$\mathbb{Z}_{19} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_{19} * \mathbb{Z} * \mathbb{Z}$	
	F	F	T	F		6, 7	$\mathbb{Z}_{189}$	Metacyclic	
	F	F	F	F		9	$\mathbb{Z}_{513}$	Free sbgp	$\mathbb{Z}_5^9 \oplus \mathbb{Z}_7^4 \oplus \mathbb{Z}_{19}^8 \oplus \mathbb{Z}_{37} \oplus \mathbb{Z}_{361} \oplus$ $\mathbb{Z}_{130987}^7 \oplus \mathbb{Z}_{100673737384483}^7$
19	F	T	F	T	4	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{1371}$	Free sbgp	
	F	F	F	F		4, 5, 6	$\mathbb{Z}_{573}$	Free sbgp	
	F	F	F	F		8	$\mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{21}$	Free sbgp	
20	F	T	F	T	7	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 7	$\mathbb{Z}_{2013}$	Free sbgp	
	F	F	F	F		4, 8	$\mathbb{Z}_{825}$	Free sbgp	$\mathbb{Z}_4^{20} \oplus \mathbb{Z}_7^{20} \oplus \mathbb{Z}_{11}^{12} \oplus \mathbb{Z}_{31} \oplus \mathbb{Z}_{601}$
	F	F	F	F		5	$\mathbb{Z}_{825}$	Free sbgp	$\mathbb{Z}_5^{25} \oplus \mathbb{Z}_9^5 \oplus \mathbb{Z}_{61} \oplus \mathbb{Z}_{181}$
	F	F	F	F		6	$\mathbb{Z}_{1353}$	Free sbgp	
	F	F	F	T		9	$\mathbb{Z}_5 \oplus \mathbb{Z}_{75}$	∞	
	F	F	F	F		10	$\mathbb{Z}_{1023}$	Free sbgp	
21	T	T	F	T	8	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{774}$	Free sbgp	
	F	F	F	F		4	$\mathbb{Z}_{1143}$	Free sbgp	
	T	F	F	F		5	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z} \oplus \mathbb{Z}$	Free sbgp	
	F	F	T	F		7	$\mathbb{Z}_{387}$	Metacyclic	
	T	F	T	F		8	$\mathbb{Z}_{43} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_{43} * \mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		9	$\mathbb{Z}_{13} \oplus \mathbb{Z}_{117}$	Free sbgp	
22	F	T	F	T	6	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 8	$\mathbb{Z}_{4623}$	Free sbgp	
	F	F	F	F		4, 7	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{69}$	Free sbgp	$\mathbb{Z}_4^{11} \oplus \mathbb{Z}_{23}^{23} \oplus \mathbb{Z}_{67} \oplus \mathbb{Z}_{463} \oplus \mathbb{Z}^{22}$
	F	F	F	F		5, 9	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{69}$	Free sbgp	$\mathbb{Z}_2^{11} \oplus \mathbb{Z}_5^{22} \oplus \mathbb{Z}_{23}^{24} \oplus \mathbb{Z}_{67} \oplus \mathbb{Z}_{859} \oplus \mathbb{Z}^{22}$
	F	F	F	T		6	$\mathbb{Z}_{597}$	∞	
	F	F	F	F		11	$\mathbb{Z}_{2049}$	Free sbgp	
23	F	T	F	T	4	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{47} \oplus \mathbb{Z}_{141}$	Free sbgp	
	F	F	F	F		4, 6, 9	$\mathbb{Z}_{2487}$	Free sbgp	
	F	F	F	F		5, 7, 10	$\mathbb{Z}_{2073}$	Free sbgp	

n	A	B	C	D	$f(n)$	l	$\Gamma_n(1,l)^{ab}$	$\Gamma_n(1,l)$	$\Gamma'/\Gamma'' = (\Gamma')^{ab}$
24	T	T	F	T	11	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{117}$	Free sbgp	
	F	F	F	F		4	$\mathbb{Z}_{3285}$	Free sbgp	
	T	F	F	F		5	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$	Free sbgp	
	F	F	F	F		6	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{63}$	Free sbgp	
	F	F	F	F		7	$\mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{63}$	Free sbgp	
	T	F	T	F		8	$\mathbb{Z}_{85} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_{85} * \mathbb{Z} * \mathbb{Z}$	
	F	F	T	F		9	$\mathbb{Z}_{765}$	Metacyclic	
	F	F	F	F		10	$\mathbb{Z}_5 \oplus \mathbb{Z}_{585}$	Free sbgp	
	T	F	F	T		11	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z} \oplus \mathbb{Z}$	∞	
	F	F	F	F		12	$\mathbb{Z}_{4095}$	Free sbgp	
25	F	T	F	T	5	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{13953}$	Free sbgp	
	F	F	F	F		4, 7, 8	$\mathbb{Z}_{4803}$	Free sbgp	
	F	F	F	F		5, 6	$\mathbb{Z}_{3333}$	Free sbgp	
	F	F	F	F		10, 11	$\mathbb{Z}_{4983}$	Free sbgp	
26	F	T	F	T	7	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 9	$\mathbb{Z}_{20829}$	Free sbgp	
	F	F	F	F		4, 10	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{237}$	Free sbgp	
	F	F	F	F		5, 6	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{159}$	Free sbgp	
	F	F	F	T		7	$\mathbb{Z}_{1563}$	∞	
	F	F	F	F		8, 11	$\mathbb{Z}_{53} \oplus \mathbb{Z}_{159}$	Free sbgp	
	F	F	F	F		13	$\mathbb{Z}_{8193}$	Free sbgp	
27	T	T	F	T	6	2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$	
	F	F	F	F		3	$\mathbb{Z}_{30699}$	Free sbgp	
	F	F	F	F		4, 7	$\mathbb{Z}_{8829}$	Free sbgp	
	T	F	F	F		5, 8, 11	$\mathbb{Z}_{271} \oplus \mathbb{Z} \oplus \mathbb{Z}$	Free sbgp	
	F	F	F	F		6, 12	$\mathbb{Z}_{13203}$	Free sbgp	
	F	F	T	F		9, 10	$\mathbb{Z}_{1539}$	Metacyclic	
28	F	T	F	T	9	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3, 10	$\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{696}$	Free sbgp	
	F	F	F	F		4, 9	$\mathbb{Z}_{29} \oplus \mathbb{Z}_{435}$	Free sbgp	
	F	F	F	F		5, 12	$\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{120}$	Free sbgp	
	F	F	F	F		6, 11	$\mathbb{Z}_{17139}$	Free sbgp	
	F	F	F	F		7	$\mathbb{Z}_{14577}$	Free sbgp	
	F	F	F	F		8	$\mathbb{Z}_{18705}$	Free sbgp	
	F	F	F	T		13	$\mathbb{Z}_{13} \oplus \mathbb{Z}_{195}$	∞	
	F	F	F	F		14	$\mathbb{Z}_{16383}$	Free sbgp	
29	F	T	F	T	5	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$	
	F	F	F	F		3	$\mathbb{Z}_{64731}$	Free sbgp	
	F	F	F	F		4, 8, 11	$\mathbb{Z}_{17403}$	Free sbgp	
	F	F	F	F		5, 6, 7	$\mathbb{Z}_{59} \oplus \mathbb{Z}_{177}$	Free sbgp	
	F	F	F	F		9, 12, 13	$\mathbb{Z}_{21579}$	Free sbgp	

6.4 $f(n)$ of $\Gamma_n(k,l)$ groups when n has three distinct prime factors

Here only when A hold we give an upper bound for $f(n)$ of $\Gamma_n(k,l)$ where $n = p^\alpha q^\beta r^\gamma$, where p, q and r are distinct primes

Lemma 6.4.1. *Let $n = p^\alpha q^\beta r^\gamma$, where p, q and r are distinct primes and $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$, and $(n,k,l) = 1, k \neq l, k \neq 0, l \neq 0$. If (A) holds then $\Gamma_n(k,l) \cong \Gamma_n(1,l'')$ for some l'', where*

1. *If n is even then $l'' \in \{2,5,8,...,\frac{n}{2}-1\}$, and there are at most $\frac{n}{6}$ isomorphism types.*
2. *If n is odd then $l'' \in \{2,5,8,...,\frac{n-1}{2}\}$, and there are at most $\frac{n-3}{6}$ isomorphisms types amongst the groups $\Gamma_n(k,l)$.*

Proof. We shall show that at least one of the following hold: $(n,k) = 1, (n,l) = 1, (n,k-l) = 1$. For then the result follows Lemma 6.2.1. Suppose for contradiction $(n,k) > 1, (n,l) > 1, (n,k-l) > 1$. Now (A) holds so $3|n$, therefore $(p = 3, \alpha \geq 1)$ or $(q = 3, \beta \geq 1)$ or $(r = 3, \gamma \geq 1)$, without loss of generality suppose $r = 3, \gamma \geq 1$. Now $(n,k) > 1$ implies that $p|k$ or $q|k$ or $r|k$, and $(n,l) > 1$ implies that $p|l$ or $q|k$ or $r|l$, and $(n,k-l) > 1$ implies that $p|(k-l)$ or $q|(k-l)$ or $r|(k-l)$. If $3|k$ then since (A) holds we have $3|l$, so $3|(n,k,l) = 1$, a contradiction. Therefore $3 \nmid k$. Similarly $3 \nmid l, 3 \nmid (k-l)$.

Since $(n,k) > 1$ we have $p|k$ or $q|k$. Without loss of generality suppose $p|k$ then $p \nmid l$ since$(n,k,l) = 1$, and then $p \nmid (k-l)$. Therefore $q|l$ and $q|(k-l)$, so $q|k$ so $q|(n,k,l) = 1$, a contradiction. Therefore

$$\Gamma_n(k,l) \cong \Gamma_n(1,l') \tag{6.9}$$

for some $2 \leq l' \leq (n-1)$.

Therefore by Theorem 5.1.1, the parameters n, k', l' satisfy (A) so $1 + l' \equiv 0 \bmod 3$, so $l' \equiv 2 \bmod 3$, therefore $l' \in \{2,5,8,...,(n-1)\}$ which has $\frac{n}{3}$ isomorphism types.

Now suppose n is even, then $l' \in \{2,5,8,...,\frac{n}{2}-1,\frac{n}{2}+2,...,n-1\}$ (which has even numbers of elements $= \frac{n}{3}$). If $\frac{n}{2}+2 \leq l' \leq n-1$ then $-\frac{n}{2}-2 \geq -l' \geq 1-n$, so $(n+1)-\frac{n}{2}-2 \geq (n+1)-l' \geq (n+1)+1-n$ gives $\frac{n}{2}-1 \geq (n+1)-l' \geq 2$. Now let $l'' = (n+1)-l'$, then $\frac{n}{2}-1 \geq l'' \geq 2$, and then by part (4) in Lemma 5.0.10, we have $\Gamma_n(k,\ell) \cong \Gamma_n(1,l') \cong \Gamma_n(1,1-l') = \Gamma_n(1,n+1-l') = \Gamma_n(1,l'')$, therefore $\Gamma_n(k,\ell) \cong \Gamma_n(1,l'')$ for some $2 \le l'' \le \frac{n}{2}-1$, which means $l'' \in \{2,5,8,...,\frac{n}{2}-1\}$ this gives at most $\frac{n}{6}$ isomorphism types.

When n odd, $l' \in \{2,5,8,...,\frac{n-5}{2},\frac{n+1}{2},\frac{n+7}{2},...,n-1\}$ (which has odd numbers of elements $= \frac{n}{3}$). When $l' = \frac{n+1}{2}$, we have $\Gamma_n(1,\frac{n+1}{2}) \cong \Gamma_n(1,2)$ by number (2,5) in Lemma 5.0.10. Now if $\frac{n+7}{2} \le l' \le n-1$, therefore $-(\frac{n+7}{2}) \ge -l' \ge 1-n$, and then $(n+1)-(\frac{n+7}{2}) \ge (n+1)-l' \ge (n+1)+1-n$. Now let $l'' = (n+1)-l'$, then $\frac{n-5}{2} \ge l'' \ge 2$, and by (6.9) and (4) in Lemma 5.0.10, we have $\Gamma_n(k,l) \cong \Gamma_n(1,l') \cong \Gamma_n(1,1-l') = \Gamma_n(1,n+1-l') = \Gamma_n(1,l'')$, therefore $\Gamma_n(k,l) \cong \Gamma_n(1,l'')$ for some $2 \le l'' \le \frac{n-5}{2}$, which means $l'' \in \{2,5,8,...,\frac{n-5}{2}\}$ this gives at most $\frac{n-3}{6}$ isomorphism types. □

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Appendix A

Table of isomorphisms classes of $G_n(m,k)$ groups for $n \leq 27$

Table A.1: Isomorphisms classes of $G_n(m,k)$ groups for $n \leq 27$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
3	1	1	$H(3,2)$	-	Q_8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
4	2	2	$H(4,2)$	$S(2,4)$	$SL(2,3)$	$\mathbb{Z}_3$
			$H(4,3)$	$F(2,4)$	$\mathbb{Z}_5$	$\mathbb{Z}_5$
5	2	2	$H(5,2)$	$S(2,5)$	$SL(2,5)$	1
			$H(5,3)$	$F(2,5)$	$\mathbb{Z}_{11}$	$\mathbb{Z}_{11}$
6	5	4	$H(6,2)$	$S(2,6)$	infinite	$\mathbb{Z} \oplus \mathbb{Z}$
			$H(6,3)$	-	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$	$\mathbb{Z}_7$
			$H(6,4)$	-	$\mathbb{Z}_9$	$\mathbb{Z}_9$
			$H(6,5)$	$F(2,6)$	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$
			$G_6(1,3)$	-	$\mathbb{Z}_7$	$\mathbb{Z}_7$
7	3	3	$H(7,2)$	$S(2,7)$	infinite	1
			$H(7,3)$	-	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(7,4)$	$F(2,7)$	$\mathbb{Z}_{29}$	$\mathbb{Z}_{29}$
8	6	6	$H(8,2)$	$S(2,8)$	infinite	$\mathbb{Z}_3$
			$H(8,3)$	-	group of order $3^{10}.5$	$\mathbb{Z}_5$
			$H(8,4)$	-	infinite	$\mathbb{Z}_{15}$
			$H(8,5)$	-	$\mathbb{Z}_{17}$	$\mathbb{Z}_{17}$
			$H(8,6)$	-	infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
			$H(8,7)$	$F(2,8)$	infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_{15}$
9	5	5	$H(9,2)$	$S(2,9)$	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(9,3)$	-	infinite	$\mathbb{Z}_7$
			$H(9,4)$	-	Unknown	$\mathbb{Z}_{19}$
			$H(9,5)$	$F(2,9)$	infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{38}$
			$H(9,7)$	-	Unknown	$\mathbb{Z}_{37}$
10	8	5	$H(10,2)$	$S(2,10)$	Infinite	$\mathbb{Z}_3$
			$H(10,3)$	-	Infinite	$\mathbb{Z}_{11}$
			$H(10,4)$	-	Infinite	$\mathbb{Z}_{33}$
			$H(10,5)$	-	Infinite	$\mathbb{Z}_{31}$
			$H(10,6)$	-	$\mathbb{Z}_{33}$	$\mathbb{Z}_{33}$
			$H(10,7)$	-	Infinite	$\mathbb{Z}_{11}$
			$H(10,9)$	$F(2,10)$	Infinite	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$
			$G_{10}(1,5)$	-	$\mathbb{Z}_{31}$	$\mathbb{Z}_{31}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
11	5	4	$H(11,2)$	$S(2,11)$	Infinite	1
			$H(11,3)$	-	Infinite	$\mathbb{Z}_{23}$
			$H(11,4)$	-	Infinite	$\mathbb{Z}_{23}$
			$H(11,6)$	$F(2,11)$	Infinite	$\mathbb{Z}_{199}$
			$H(11,8)$	-	Infinite	$\mathbb{Z}_{67}$
12	12	10	$H(12,2)$	$S(2,12)$	Infinite	$\mathbb{Z} \oplus \mathbb{Z}$
			$H(12,3)$	-	Infinite	$\mathbb{Z}_{35}$
			$H(12,4)$	-	Infinite	$\mathbb{Z}_{45}$
			$H(12,5)$	-	Infinite	$\mathbb{Z}_8 \oplus \mathbb{Z}_8$
			$H(12,6)$	-	Infinite	$\mathbb{Z}_{63}$
			$H(12,7)$	-	$\mathbb{Z}_{65}$	$\mathbb{Z}_{65}$
			$H(12,8)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$
			$H(12,9)$	-	Infinite	$\mathbb{Z}_{91}$
			$H(12,10)$	-	Infinite	$\mathbb{Z}_{117}$
			$H(12,11)$	$F(2,12)$	Infinite	$\mathbb{Z}_8 \oplus \mathbb{Z}_{40}$
			$G_{12}(1,3)$	-	Infinite	$\mathbb{Z}_{35}$
			$G_{12}(1,4)$	-	Infinite	$\mathbb{Z}_{91}$
13	6	6	$H(13,2)$	$S(2,13)$	Infinite	1
			$H(13,3)$	-	Infinite	$\mathbb{Z}_{53}$
			$H(13,4)$	-	Infinite	$\mathbb{Z}_{79}$
			$H(13,5)$	-	Infinite	$\mathbb{Z}_{131}$
			$H(13,6)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
			$H(13,7)$	$F(2,13)$	Infinite	$\mathbb{Z}_{521}$
14	11	8	$H(14,2)$	$S(2,14)$	Infinite	$\mathbb{Z}_3$
			$H(14,3)$	-	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$
			$H(14,4)$	-	Infinite	$\mathbb{Z}_{87}$
			$H(14,5)$	-	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$
			$H(14,7)$	-	Infinite	$\mathbb{Z}_{127}$
			$H(14,8)$	-	$\mathbb{Z}_{129}$	$\mathbb{Z}_{129}$
			$H(14,9)$	-	Infinite	$\mathbb{Z}_{29}$
			$H(14,10)$	-	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$
			$H(14,11)$	-	Infinite	$\mathbb{Z}_{29}$
			$H(14,13)$	$F(2,14)$	Infinite	$\mathbb{Z}_{29} \oplus \mathbb{Z}_{29}$
			$G_{14}(1,7)$	-	$\mathbb{Z}_{127}$	$\mathbb{Z}_{127}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
15	12	12	$H(15,2)$	$S(2,15)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(15,3)$	-	Infinite	$\mathbb{Z}_{77}$
			$H(15,4)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{22}$
			$H(15,5)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{62}$
			$H(15,6)$	-	Infinite	$\mathbb{Z}_{217}$
			$H(15,7)$	-	Infinite	$\mathbb{Z}_{61}$
			$H(15,8)$	$F(2,15)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{682}$
			$H(15,10)$	-	Infinite	$\mathbb{Z}_{31}$
			$H(15,11)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{122}$
			$H(15,12)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$
			$H(15,13)$	-	Infinite	$\mathbb{Z}_{341}$
			$G_{15}(1,6)$	-	Infinite	$\mathbb{Z}_{271}$
16	12	9	$H(16,2)$	$S(2,16)$	Infinite	$\mathbb{Z}_3$
			$H(16,3)$	-	Infinite	$\mathbb{Z}_{85}$
			$H(16,4)$	-	Infinite	$\mathbb{Z}_{255}$
			$H(16,5)$	-	Infinite	$\mathbb{Z}_{17} \oplus \mathbb{Z}_{17}$
			$H(16,6)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{51}$
			$H(16,7)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_{15}$
			$H(16,8)$	-	Infinite	$\mathbb{Z}_{255}$
			$H(16,9)$	-	$\mathbb{Z}_{257}$	$\mathbb{Z}_{257}$
			$H(16,10)$	-	Infinite	$\mathbb{Z}_7 \oplus \mathbb{Z}_{21}$
			$H(16,11)$	-	Infinite	$\mathbb{Z}_{85}$
			$H(16,12)$	-	Infinite	$\mathbb{Z}_{255}$
			$H(16,15)$	$F(2,16)$	Infinite	$\mathbb{Z}_{21} \oplus \mathbb{Z}_{105}$
17	7 or 8	7	$H(17,2)$	$S(2,17)$	Infinite	1
			$H(17,3)$	-	Infinite	$\mathbb{Z}_{103}$
			$H(17,4)$	-	Infinite	$\mathbb{Z}_{307}$
			$H(17,5)$	-	Infinite	$\mathbb{Z}_{409}$
			$H(17,9)$	$F(2,17)$	Infinite	$\mathbb{Z}_{3571}$
			$H(17,12)$	-	Infinite	$\mathbb{Z}_{613}$
			$H(17,6) \cong$ $H(17,11)$?	- -	Infinite Infinite	$\mathbb{Z}_{137}$ $\mathbb{Z}_{137}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	Abelianization
18	17	10	$H(18,2)$	$S(2,18)$	Infinite	$\mathbb{Z} \oplus \mathbb{Z}$
			$H(18,3)$	-	Infinite	$\mathbb{Z}_{133}$
			$H(18,4)$	-	Infinite	$\mathbb{Z}_{513}$
			$H(18,5)$	-	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_{76}$
			$H(18,6)$	-	Infinite	$\mathbb{Z}_{189}$
			$H(18,7)$	-	Infinite	$\mathbb{Z}_{703}$
			$H(18,8)$	-	Infinite	$\mathbb{Z}_{19} \oplus \mathbb{Z} \oplus \mathbb{Z}$
			$H(18,9)$	-	Infinite	$\mathbb{Z}_{511}$
			$H(18,10)$	-	$\mathbb{Z}_{513}$	$\mathbb{Z}_{513}$
			$H(18,11)$	-	Infinite	$\mathbb{Z}_4 \oplus \mathbb{Z}_{76}$
			$H(18,13)$	-	Infinite	$\mathbb{Z}_{703}$
			$H(18,15)$	-	Infinite	$\mathbb{Z}_{259}$
			$H(18,16)$	-	Infinite	$\mathbb{Z}_{999}$
			$H(18,17)$	$F(2,18)$	Infinite	$\mathbb{Z}_{76} \oplus Z_{76}$
			$G_{18}(1,3)$	-	Infinite	$\mathbb{Z}_{133}$
			$G_{18}(1,4)$	-	Infinite	$\mathbb{Z}_{259}$
			$G_{18}(1,9)$	-	$\mathbb{Z}_{511}$	$\mathbb{Z}_{511}$
19	8 or 9	8	$H(19,2)$	$S(2,19)$	Infinite	1
			$H(19,3)$	-	Infinite	$\mathbb{Z}_{191}$
			$H(19,4)$	-	Infinite	$\mathbb{Z}_{647}$
			$H(19,5)$	-	Infinite	$\mathbb{Z}_{761}$
			$H(19,7)$	-	Infinite	$\mathbb{Z}_{1483}$
			$H(19,8)$	-	Infinite	$\mathbb{Z}_{419}$
			$H(19,10)$	$F(2,18)$	Infinite	$\mathbb{Z}_{9349}$
			$H(19,9) \cong$ $H(19,15)?$	- -	Infinite Infinite	$\mathbb{Z}_{229}$ $\mathbb{Z}_{229}$
20	18	13	$H(20,2)$	$S(2,20)$	Infinite	$\mathbb{Z}_3$
			$H(20,3)$	-	Infinite	$\mathbb{Z}_{275}$
			$H(20,4)$	-	Infinite	$\mathbb{Z}_{825}$
			$H(20,5)$	-	Infinite	$\mathbb{Z}_{1271}$
			$H(20,6)$	-	Infinite	$\mathbb{Z}_{1353}$
			$H(20,7)$	-	Infinite	$\mathbb{Z}_{275}$
			$H(20,9)$	-	Infinite	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$
			$H(20,10)$	-	Infinite	$\mathbb{Z}_{1023}$
			$H(20,11)$	-	$\mathbb{Z}_{1025}$	$\mathbb{Z}_{1025}$
			$H(20,12)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z}_{75}$
			$H(20,13)$	-	Infinite	$\mathbb{Z}_{671}$
			$H(20,14)$	-	Infinite	$\mathbb{Z}_{2013}$
			$H(20,15)$	-	Infinite	$\mathbb{Z}_{775}$
			$H(20,16)$	-	Infinite	$\mathbb{Z}_{825}$
			$H(20,17)$	-	Infinite	$\mathbb{Z}_{671}$
			$H(20,19)$	$F(2,20)$	Infinite	$\mathbb{Z}_{55} \oplus \mathbb{Z}_{275}$
			$G_{20}(1,5)$	-	Infinite	$\mathbb{Z}_{1271}$
			$G_{20}(1,6)$	-	Infinite	$\mathbb{Z}_{775}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	Abelianization
21	15 or 16	15	$H(21,2)$	$S(2,21)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(21,3)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{98}$
			$H(21,5)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{86}$
			$H(21,6)$	-	Infinite	$\mathbb{Z}_{1421}$
			$H(21,7)$	-	Infinite	$\mathbb{Z}_{127}$
			$H(21,8)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{1094}$
			$H(21,9)$	-	Infinite	$\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$
			$H(21,10)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{86}$
			$H(21,11)$	$F(2,21)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12238}$
			$H(21,14)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{254}$
			$H(21,15)$	-	Infinite	$\mathbb{Z}_{2107}$
			$H(21,16)$	-	Infinite	$\mathbb{Z}_{463}$
			$H(21,19)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{758}$
			$G_{21}(1,7)$	-	Infinite	$\mathbb{Z}_{2269}$
			$H(21,4) \cong$ $H(21,13)$?	- -	Infinite Infinite	$\mathbb{Z}_{1247}$ $\mathbb{Z}_{1247}$
22	17	10	$H(22,2)$	$S(2,22)$	Infinite	$\mathbb{Z}_3$
			$H(22,3)$	-	Infinite	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{23}$
			$H(22,4)$	-	Infinite	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{69}$
			$H(22,5)$	-	Infinite	$\mathbb{Z}_{1541}$
			$H(22,6)$	-	Infinite	$\mathbb{Z}_{597}$
			$H(22,7)$	-	Infinite	$\mathbb{Z}_{1541}$
			$H(22,8)$	-	Infinite	$\mathbb{Z}_{4623}$
			$H(22,9)$	-	Infinite	$\mathbb{Z}_{1541}$
			$H(22,11)$	-	Infinite	$\mathbb{Z}_{2047}$
			$H(22,12)$	-	$\mathbb{Z}_{2049}$	$\mathbb{Z}_{2049}$
			$H(22,13)$	-	Infinite	$\mathbb{Z}_{199}$
			$H(22,14)$	-	Infinite	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{69}$
			$H(22,15)$	-	Infinite	$\mathbb{Z}_{23} \oplus \mathbb{Z}_{23}$
			$H(22,17)$	-	Infinite	$\mathbb{Z}_{199}$
			$H(22,19)$	-	Infinite	$\mathbb{Z}_{1541}$
			$H(22,21)$	$F(2,22)$	Infinite	$\mathbb{Z}_{199} \oplus \mathbb{Z}_{199}$
			$G_{22}(1,11)$	-	$\mathbb{Z}_{2047}$	$\mathbb{Z}_{2047}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
23	10 or 11	9	$H(23,2)$	$S(2,23)$	Infinite	1
			$H(23,3)$	-	Infinite	$\mathbb{Z}_{691}$
			$H(23,5)$	-	Infinite	$\mathbb{Z}_{3313}$
			$H(23,6)$	-	Infinite	$\mathbb{Z}_{2347}$
			$H(23,12)$	$F(2,23)$	Infinite	$\mathbb{Z}_{64079}$
			$H(23,16)$	-	Infinite	$\mathbb{Z}_{6533}$
			$H(23,18)$	-	Infinite	$\mathbb{Z}_{1151}$
			$H(23,4)$	-	Infinite	$\mathbb{Z}_{47} \oplus \mathbb{Z}_{47}$
			$H(23,14)$	-	Infinite	$\mathbb{Z}_{47} \oplus \mathbb{Z}_{47}$
			$H(23,8) \cong$ $H(23,10)$?	- -	Infinite Infinite	$\mathbb{Z}_{599}$ $\mathbb{Z}_{599}$
24	26	22	$H(24,2)$	$S(2,24)$	Infinite	$\mathbb{Z} \oplus \mathbb{Z}$
			$H(24,3)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{35}$
			$H(24,4)$	-	Infinite	$\mathbb{Z}_{3285}$
			$H(24,5)$	-	Infinite	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{272}$
			$H(24,6)$	-	Infinite	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{63}$
			$H(24,7)$	-	Infinite	$\mathbb{Z}_9 \oplus \mathbb{Z}_{585}$
			$H(24,8)$	-	Infinite	$\mathbb{Z}_{85} \oplus \mathbb{Z} \oplus \mathbb{Z}$
			$H(24,9)$	-	Infinite	$\mathbb{Z}_{6643}$
			$H(24,10)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z}_{585}$
			$H(24,11)$	-	Infinite	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{80}$
			$H(24,12)$	-	Infinite	$\mathbb{Z}_{4095}$
			$H(24,13)$	-	$\mathbb{Z}_{4097}$	$\mathbb{Z}_{4097}$
			$H(24,14)$	-	Infinite	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z} \oplus \mathbb{Z}$
			$H(24,15)$	-	Infinite	$Z_9 \oplus \mathbb{Z}_{315}$
			$H(24,16)$	-	Infinite	$\mathbb{Z}_{765}$
			$H(24,17)$	-	Infinite	$\mathbb{Z}_{80} \oplus \mathbb{Z}_{80}$
			$H(24,18)$	-	Infinite	$\mathbb{Z}_7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{63}$
			$H(24,19)$	-	Infinite	$\mathbb{Z}_7 \oplus \mathbb{Z}_{455}$
			$H(24,20)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$
			$H(24,21)$	-	Infinite	$\mathbb{Z}_{1547}$
			$H(24,22)$	-	Infinite	$\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{117}$
			$H(24,23)$	$F(2,24)$	Infinite	$\mathbb{Z}_{144} \oplus \mathbb{Z}_{720}$
			$G_{24}(1,3)$	-	Infinite	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{35}$
			$G_{24}(1,4)$	-	Infinite	$\mathbb{Z}_{1547}$
			$G_{24}(1,9)$	-	Infinite	$\mathbb{Z}_{6643}$
			$G_{24}(1,10)$	-	Infinite	$\mathbb{Z}_9 \oplus \mathbb{Z}_{315}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
25	14	13	$H(25,2)$	$S(2,25)$	Infinite	1
			$H(25,3)$	-	Infinite	$\mathbb{Z}_{1111}$
			$H(25,4)$	-	Infinite	$\mathbb{Z}_{4411}$
			$H(25,5)$	-	Infinite	$\mathbb{Z}_{4681}$
			$H(25,6)$	-	Infinite	$\mathbb{Z}_{6101}$
			$H(25,7)$	-	Infinite	$\mathbb{Z}_{3851}$
			$H(25,8)$	-	Infinite	$\mathbb{Z}_{2761}$
			$H(25,9)$	-	Infinite	$\mathbb{Z}_{14311}$
			$H(25,10)$	-	Infinite	$\mathbb{Z}_{3131}$
			$H(25,11)$	-	Infinite	$\mathbb{Z}_{1951}$
			$H(25,12)$	-	Infinite	$\mathbb{Z}_{1151}$
			$H(25,13)$	$F(2,25)$	Infinite	$\mathbb{Z}_{167761}$
			$H(25,16)$	-	Infinite	$\mathbb{Z}_{5801}$
			$H(25,21)$	-	Infinite	$\mathbb{Z}_{1151}$
26	20	14	$H(26,2)$	$S(2,26)$	Infinite	$\mathbb{Z}_3$
			$H(26,3)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{159}$
			$H(26,4)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{237}$
			$H(26,5)$	-	Infinite	$\mathbb{Z}_{6943}$
			$H(26,6)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{159}$
			$H(26,7)$	-	Infinite	$\mathbb{Z}_{521}$
			$H(26,8)$	-	Infinite	$\mathbb{Z}_{53} \oplus \mathbb{Z}_{159}$
			$H(26,9)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{159}$
			$H(26,11)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{393}$
			$H(26,12)$	-	Infinite	$\mathbb{Z}_{1563}$
			$H(26,13)$	-	Infinite	$\mathbb{Z}_{8191}$
			$H(26,14)$	-	$\mathbb{Z}_{8193}$	$\mathbb{Z}_{8193}$
			$H(26,15)$	-	Infinite	$\mathbb{Z}_{521}$
			$H(26,17)$	-	Infinite	$\mathbb{Z}_{79} \oplus \mathbb{Z}_{79}$
			$H(26,18)$	-	Infinite	$\mathbb{Z}_{20829}$
			$H(26,19)$	-	Infinite	$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{393}$
			$H(26,21)$	-	Infinite	$\mathbb{Z}_{6943}$
			$H(26,23)$	-	Infinite	$\mathbb{Z}_{79} \oplus \mathbb{Z}_{79}$
			$H(26,25)$	$F(2,26)$	Infinite	$\mathbb{Z}_{521} \oplus \mathbb{Z}_{521}$
			$G_{26}(1,13)$	-	$\mathbb{Z}_{8191}$	$\mathbb{Z}_{8191}$

n	$f(n)$	$g(n)$	Groups	type of group	Finite or infinite	G/G'
27	17	17	$H(27,2)$	$S(2,27)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
			$H(27,3)$	-	Infinite	$\mathbb{Z}_{1897}$
			$H(27,4)$	-	Infinite	$\mathbb{Z}_{9253}$
			$H(27,5)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{6194}$
			$H(27,6)$	-	Infinite	$\mathbb{Z}_{5299}$
			$H(27,7)$	-	Infinite	$\mathbb{Z}_{6031}$
			$H(27,9)$	-	Infinite	$\mathbb{Z}_{511}$
			$H(27,10)$	-	Infinite	$\mathbb{Z}_{19927}$
			$H(27,11)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{4538}$
			$H(27,13)$	-	Infinite	$\mathbb{Z}_{2071}$
			$H(27,14)$	$F(2,27)$	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{219602}$
			$H(27,16)$	-	Infinite	$\mathbb{Z}_{4033}$
			$H(27,17)$	-	Infinite	$\mathbb{Z}_2 \oplus \mathbb{Z}_{4142}$
			$H(27,19)$	-	Infinite	$\mathbb{Z}_{19441}$
			$H(27,21)$	-	Infinite	$\mathbb{Z}_{6433}$
			$H(27,22)$	-	Infinite	$\mathbb{Z}_{8227}$
			$H(27,25)$	-	Infinite	$\mathbb{Z}_{30007}$