Appendices for Fiscal Policy with Limited-Time Commitment

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A Appendix to Sections 1 and 2

In this appendix, we first provide additional formal definitions and assumptions required for the main proof, **Proposition 1**. In doing so, we provide a generic recursivity proof for models where the realised state next period can depend on the realisation of the shock next period. Finally, we provide a discussion of the necessity of **Assumption 3**, alternative power structures, and a recursive algorithm for FC with policies as state variables, which can be used whenever the equivalence result holds.

A.1 Definition of competitive equilibrium

We define competitive equilibrium in the standard way:

Definition 3. *Given an initial condition* $(b_0, z_0) \in B \times Z$ *and a policy sequence* $\{\tau_t(z^t)\}_{t=0}^{\infty}$, *a* **competitive equilibrium** *is a sequence of functions* $\{c_t(z^t), p_t(z^t), b_t(z^t)\}_{t=0}^{\infty}$ *which satisfy (2), (3) and (4) for all* $t = 0, 1, ...$ *and histories* $z^t \in Z^t$.

We summarise the variables of the economy in a vector $y_t(z^t) \equiv (c_t(z^t), p_t(z^t), b_t(z^t), \tau_t(z^t))$, and use the notation $\mathbf{y} \equiv \{y_t(z^t)\}_{t=0}^{\infty}$ to denote plans. Let $\Pi^*(b_0, z_0)$ denote the set of plans which are competitive equilibria. That is, $\mathbf{y} \in \Pi^*(b_0, z_0)$ if and only if it is a feasible competitive equilibrium from initial state (b_0, z_0).

Define the truncation of any plan from time t as $\mathbf{y}_t \equiv \{y_s(z^s)\}_{s=t}^{\infty}$. Once time t is reached, a plan is feasible if and only if it satisfies the competitive equilibrium constraints at times $t, t+1, ...$ Importantly, constraints dated $t-1$ and before can be ignored, because these are now in the past. Due to the recursive nature of the definition of competitive equilibrium, it must be that any truncated plan $\mathbf{y}_t$ is a valid competitive equilibrium from t onwards if and only if $\mathbf{y}_t \in \Pi^*(b_t, z_t)$.

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A.2 Further details: Full-Commitment Ramsey equilibrium

We maintain the following regularity assumption throughout the text:

Assumption 1. *The environment is such that r is bounded over all competitive equilibrium paths. The discount rate is strictly less than one:* $0 \leq \beta < 1$

This assumption serves two purposes. Firstly, it ensures that discounted utility is well defined in the limit of an infinite horizon, in the sense that the limit is sure to converge. More importantly, it rules out potential extraneous equilibria of the LTC game achieving negative infinite utility.[1] The assumption does not rule out unbounded utility functions, such as the commonly used Constant Relative Risk Aversion (CRRA) form, as long as the environment is such that the possible values of utility in equilibrium are themselves bounded.

In a FC equilibrium, a single benevolent infinitely-lived government endowed with the ability to credibly commit into the infinite future announces a contingent plan at $t = 0$ and then implements it. Denote a policy plan by $\boldsymbol{\tau} \equiv \{\tau_t(z^t)\}_{t=0}^{\infty}$.

In order for the government to be able to pin down a unique competitive equilibrium, the following assumption on the mapping from policy plans to allocations is required.

Assumption 2. *Given an initial condition $(b_0, z_0) \in B^*$ and a policy plan $\boldsymbol{\tau}$, there exists a unique competitive equilibrium, $\mathbf{y}$. That is, there is a unique sequence of functions $\{c_t(z^t), p_t(z^t), b_t(z^t)\}_{t=0}^{\infty}$ which satisfy (2), (3) and (4) for all $t = 0, 1, ...$ and histories $z^t \in G^t$.*

This assumption allows us to map an infinite sequence of policies to a single competitive equilibrium, ensuring that the government knows exactly which equilibrium is selected for a given policy choice.[2] Under this assumption we can equivalently define the government's problem as choosing a conditional path for policies, $\boldsymbol{\tau}$, or simply choosing the associated competitive equilibrium, $\mathbf{y}$. This allows us to state the FC government's problem, for any $(b_0, z_0) \in B^*$, as (5).

Definition 4. *Fix an initial condition $(b_0, z_0) \in B^*$. Let $\boldsymbol{\tau}^{FC}(b_0, z_0)$ be the policy sequence that solves (5). A* **Full-Commitment (FC) Ramsey equilibrium** *is the competitive equilibrium, $\mathbf{y}^{FC}(b_0, z_0)$, associated with $\boldsymbol{\tau}^{FC}(b_0, z_0)$.*

A.3 Further details: LTC game

In the definition of the LTC game in Section 1.3 we restricted the government to choose policies consistent with competitive equilibrium. In particular, for the contingent case we define $C(s_t; \underset{\sim}{\tau})$ as the set of values for $\underset{\sim}{\tau}_{t+L}$ which are consistent with some competitive

[1] If the government today thinks that future governments will play a policy leading to negative infinite utility, whatever policy the current government follows its value is negative infinity. Hence it is weakly optimal for the government to pursue the same policy, leading to this policy being a Markov-Perfect equilibrium.

[2] Bassetto (2005) argues that a full specification of out-of-equilibrium government strategies is necessary to guarantee that the desired equilibrium is obtained. In this paper we implicitly assume that there is a more general formulation of the government strategy, involving binding promises about out-of-equilibrium paths, that guarantees uniqueness of the desired Ramsey equilibrium.

equilibrium. That is, if and only if $\underset{\sim}{\tau}_{t+L} \in C(s_t; \underset{\sim}{\tau})$ then a competitive equilibrium exists and implies the values for (b_{t+1}, c_t, p_t) given in $\phi\left(s_t, \underset{\sim}{\tau}_{t+L}; \underset{\sim}{\tau}\right)$. Then, it is understood that governments are restricted to choosing $\underset{\sim}{\tau}_{t+L} \in C(s_t; \underset{\sim}{\tau})$ in (10).

Given an equilibrium policy function $\underset{\sim}{\tau}$ of either the contingent or non-contingent LTC game and an initial condition s_0, we can iterate the policy function forwards to solve for an implied contingent policy plan $\boldsymbol{\tau}^*(s_0) = \{\tau_t(z^t)\}_{t=0}^{\infty}$. Importantly, when $L > 0$ we must specify as part of our initial conditions the pre-committed policies from the point of time 0: τ_0^L. Thus, we cannot solve for the paths generated by the LTC game without specifying some rule for these initial conditions.

Definition 5. *Given initial conditions $s_0 = \left(b_0, z_0, \tau_0^L\right)$, a* **symmetric Markov-Perfect Limited-Time-Commitment (LTC) equilibrium** *is the competitive equilibrium, $\mathbf{y}^*(s_0)$, associated with the equilibrium policy plan $\boldsymbol{\tau}^*(s_0)$*

Consider either non-contingent LTC in a deterministic environment, or contingent LTC in a stochastic environment. Take as initial conditions the first L periods of the optimal Full Commitment policy. That is, take $\tau_0^L = (\tau_0^{FC}, \tau_1^{FC}(z^1), ..., \tau_{L-1}^{FC}(z^{L-1}))$, where $\tau_t^{FC}(z^t)$ is the time-t element of $\boldsymbol{\tau}^{FC}(b_0, z_0)$. Then, as $L \to \infty$ the equilibrium trivially coincides with the FC equilibrium, as in the limit the entire path of taxes coincides with the initial conditions. In this case, LTC nests both NC ($L = 0$) and FC ($L \to \infty$).

A.4 Equivalence proofs

In this section we provide proofs for our propositions regarding when LTC is able to sustain FC. In the text we provided **Proposition 1** for the case of non-contingent LTC in deterministic economies. Below we restate the key assumption and proposition for contingent LTC in stochastic economies before providing the proposition for this case.

A.4.1 Restatement for contingent LTC in stochastic economies

The equivalents to **Definition 2**, **Assumption 3**, and **Proposition 1** for the case of contingent LTC in general stochastic economies are:

Definition 2*. *For any t and $t' > t$, we say that the natural states (b_t, z_t) and a partial policy sequence $\{\tau_s(z_t^s)\}_{s=t}^{t'}$* **uniquely determine** *a variable $x_{t''}$ from time t if all possible competitive equilibria from time t onwards feature the same value of $x_{t''}(z_t^{t''})$ regardless of the future policy choices $\{\tau_s(z_t^s)\}_{s=t'+1}^{\infty}$.*

Assumption 3*. *There exists an L, with $N \leq L < \infty$, such that the following holds for all $t = 0, 1, ...$ and histories z^t.*

1. *From time t, the state variables $s_t = (b_t, z_t, \tau_t^L)$ uniquely determine all problematic variables dated time t to $t + N - 1$.*

2. *From time t, the state variables $s_t = (b_t, z_t, \tau_t^L)$ and the time-t government's choice $\underset{\sim}{\tau}_{t+L}$ uniquely determine c_t, p_t and b_{t+1}.*

Proposition 1*. *Consider an L such that* **Assumption 3*** *holds, and fix initial conditions $(b_0, z_0) \in B^*$. If, in the contingent LTC game, either*

1. $\tau_0^L = (\tau_0^{FC}, \tau_1^{FC}(z_1), \tau_2^{FC}(z_1^2), ..., \tau_{L-1}^{FC}(z_1^{L-1}))$, *such that the initial L periods of policies are restricted to be the optimal values from the FC solution, or*

2. *the time-0 government, in addition to choosing* $\underset{\sim}{\tau}_L$, *is also allowed to choose* τ_0^L

then the unique equilibrium of the LTC game induces the value $V^{FC}(b_0, z_0)$ *and generates the Full Commitment policy sequence* $\boldsymbol{\tau}^{FC}(b_0, z_0)$, *and the Full Commitment Ramsey equilibrium path,* $\boldsymbol{y}^{FC}(b_0, z_0)$.

A.4.2 Proofs of Propositions 1 and 1*

We begin with the proof of **Proposition 1*** for the contingent LTC case, since **Proposition 1** follows as an immediate corollary. The proof is in two steps. Firstly, we apply a standard recursivity proof (given in Appendix A.5) to show that the LTC equilibrium is equivalent to the solution to a sequence problem we call the "Modified Problem". Secondly, we show that, with the right initial conditions, the Modified Problem is equivalent to the Full Commitment solution. For expositional simplicity, we proceed under the assumption that all maxima are attained by unique optimal polices, so that policy correspondences are instead policy functions. All results go through if optimal policies are not unique. Note that **Assumption 2**, which is required for the FC problem to be well defined, is implied by **Assumption 3***.

To build the recursivity proof, we must first define a transition correspondence in terms of the state variable s_t. In the contingent LTC game, the government at t takes the state $s_t \equiv (b_t, z_t, \tau_t^L)$ as given and chooses the contingent values $\underset{\sim}{\tau}_{t+L}$. This is equivalent to choosing contingent values for the state tomorrow, $\underset{\sim}{s}_{t+1} \equiv ((b_{t+1}, \bar{z}^1, \tau_{t+1}^L(z^t, \bar{z}^1)), (b_{t+1}, \bar{z}^2, \tau_{t+1}^L(z^t, \bar{z}^2), ...)$, as long as the choice of $\underset{\sim}{\tau}_{t+L}$ uniquely pins down b_{t+1}, which is guaranteed by **Assumption 3***. Let T_L denote the set of possible values for the state τ_t^L. Define the set

$$S = \{s \in B \times Z \times T_L : s \text{ lies on at least one path } \mathbf{y} \in \Pi^*(b_0, z_0) \text{ for some } (b_0, z_0) \in B^*\} \tag{A.1}$$

This is the set of values for the state s_t which are compatible with competitive equilibrium. Let $S_Z \equiv S^{N_z}$ denote the set of possible choices for $\underset{\sim}{s}_{t+1}$, restricted such that each choice lies on some competitive equilibrium. We use this to define the transition correspondence for s_t:

Lemma 1. *There exists a time-invariant transition correspondence* $\Gamma : S \mapsto S_Z$ *such that* $\underset{\sim}{s}_{t+1}$ *and* s_t *are consistent with competitive equilibrium iff* $\underset{\sim}{s}_{t+1} \in \Gamma(s_t)$. *For all* $s_t \in S$, $\Gamma(s_t)$ *is non-empty.*

Proof. To show that such a definition is possible we first need to show that we can check if $\underset{\sim}{s}_{t+1}$ is on at least one competitive equilibrium plan given s_t without knowledge of $\underset{\sim}{s}_{t+2}, \underset{\sim}{s}_{t+3}, ...$ This is ensured by the restriction that $\underset{\sim}{s}_{t+1} \in S_Z$. $(s_t, \underset{\sim}{s}_{t+1})$ then lies on a competitive equilibrium plan from t onwards iff it is consistent with equations (2), (3), and (4) taken at time t. **Assumption 3*** implies that if $(s_t, \underset{\sim}{s}_{t+1})$ does lie on any competitive equilibrium plans, they must all have the same values of (c_t, p_t, b_{t+1}) and all problematic variables dated t to $t + N$.[3] These are all the variables needed to check

[3] Knowing the time-t choice of $\underset{\sim}{\tau}_{t+L}$ allows us to take the first half of **Assumption 3*** forward one

whether equations (2), (3), and (4) hold, and thus we can establish whether $\underset{\sim}{s}_{t+1}$ lies on at least one competitive equilibrium plan given s_t with knowledge of only $(s_t, \underset{\sim}{s}_{t+1})$. Since equations (2), (3), and (4) are time invariant, defining Γ without time dependence is possible. $\Gamma(s_t)$ is non-empty for all $s_t \in S$ because being in S implies that s_t lies on at least one competitive equilibrium plan. $\square$

Let $A = \{(s, \underset{\sim}{s}') \in S \times S_Z : \underset{\sim}{s}' \in \Gamma(s)\}$ be the graph of $\Gamma(s)$, and redefine the utility function as $F : A \mapsto \mathbb{R}$ such that $F(s_t, \underset{\sim}{s}_{t+1}) = r(c_t, b_t, z_t, \tau_t)$. Again, note that **Assumption 3*** allows us to back out unique values of (c_t, p_t, b_{t+1}) given $(s_t, \underset{\sim}{s}_{t+1})$, which is what allows this reformulation.

We can use **Lemma 1** to restate the equations defining the Markov-Perfect equilibrium, (10) and (11), as a standard Bellman equation:

$$v(s_t) = \max_{\underset{\sim}{s}_{t+1} \in \Gamma(s_t)} F(s_t, \underset{\sim}{s}_{t+1}) + \beta \mathbb{E}_t v(s_{t+1}) \tag{A.2}$$

The key to this restatement is that we have shown that under **Assumption 3*** the transition correspondence Γ and return function F can be written as functions only of the state and current government's choice. Without this assumption, these functions also potentially depended on the actions of future governments (recall the dependence on $\underset{\sim}{\tau}$ in the function $(b_{t+1}, c_t, p_t) = \phi\left(s_t, \underset{\sim}{\tau}_{t+L}; \underset{\sim}{\tau}\right)$. With this assumption, this dependence disappears). Let $\underset{\sim}{s}_{t+1} = G(s_t)$ denote the optimal policy function associated with the solution to (A.2). This defines a policy function for government policies, $\underset{\sim}{\tau}_{t+L} = \tau(s_t)$, and for the associated competitive equilibrium, $y_t = y(s_t)$.

We next define the Modified Problem (MP). Intuitively, this is the problem of a government with Full Commitment choosing a sequence of policies, but who must take the first L periods' policies as given. Since **Lemma 1** holds, we can write the problem in terms of sequences of the state variable, s_t. The initial state variable, $s_0 = (b_0, z_0, \tau_0^L)$ is taken as given, and includes the given initial policies in the variable τ_0^L.

The Modified Problem chooses paths for s_t, summarised in a plan $\boldsymbol{\pi} \equiv \{\pi_t(z^t)\}_{t=0}^{\infty}$ such that $s_t = \pi_t(z^t)$. Feasible plans are defined as in Definition 7 and discounted expected utility in (A.4). This allows us to define the maximised value function as

$$V^{MP}(s_0) = \max_{\boldsymbol{\pi} \in \Pi(s_0)} u(\boldsymbol{\pi}, s_0) \tag{A.3}$$

This is solved by a policy sequence $\boldsymbol{\tau}^{MP}(s_0)$ with implied competitive equilibrium $\boldsymbol{y}^{MP}(s_0)$. We can now state the first half of the proof:

Lemma 2. (LTC = MP) *The function V^{MP} is the unique solution to the LTC game in (A.2). For any $s_0 \in S$, the policy and competitive equilibrium sequences generated by iterating forward on the LTC policy functions τ and y are equal to $\boldsymbol{\tau}^{MP}(s_0)$ and $\boldsymbol{y}^{MP}(s_0)$.*

Proof. The proof follows immediately from the generic recursivity proof, **Proposition 2**, after noting that the LTC game in (A.2) is simply the recursive form of the Modified Problem in (A.3). **Assumption A1** and **Assumption A2** are implied by **Lemma 1**

period. Thus we uniquely determine all problematic variables from both t to $t + N - 1$ and $t + 1$ to $t + N$.

and **Assumption 1**. This implies that the unique value function which solves the LTC game is V^{MP}, and that the policy function describing the transition for the state, G, generates paths for the state equal to the MP paths. Since the paths for the state, s_t, are the same, it follows immediately that the paths for the policies and competitive equilibria are also the same, since 1) the paths for the policies are directly defined by the path for the state, and 2) these imply a unique path for the competitive equilibrium according to **Assumption 3***. □

Importantly, while in general there may be multiple symmetric Markov-Perfect equilibria of the LTC game, this result implies that whenever **Assumption 3*** holds there is a unique equilibrium. This establishes equivalence between LTC and a Modified Problem which is a sequence problem constrained so that the initial policies must take some arbitrary values. The second half of the proof, given below, establishes that as long as these initial policies are chosen correctly, the same paths solve the Modified Problem and Full Commitment problem, and lead to the same maximised value.

Lemma 3. (MP = FC) *For any* $(b_0, z_0) \in B^*$,

1. *If* $\tau_0^L = (\tau_0^{FC}, \tau_1^{FC}(z_1), \tau_2^{FC}(z_1^2), ..., \tau_{L-1}^{FC}(z_1^{L-1}))$, *then* $V^{MP}((b_0, z_0, \tau_0^L)) = V^{FC}(b_0, z_0)$.
2. $\sup_{\tau_0^L} V^{MP}((b_0, z_0, \tau_0^L)) = V^{FC}(b_0, z_0)$

In both cases, the implied optimal policies and competitive equilibria are equal: $\boldsymbol{\tau}^{MP}(s_0) = \boldsymbol{\tau}^{FC}(b_0, z_0)$ *and* $\boldsymbol{y}^{MP}(s_0) = \boldsymbol{y}^{FC}(b_0, z_0)$.

Proof. The FC problem is a less restricted maximisation than the MP problem, being identical in all respects except that the FC problem is free to choose τ_0^L while the MP problem is not. In both cases, if equality did not hold this would imply that one of either the MP or FC problems had not achieved their maximum, which is a contradiction. Equality of $\boldsymbol{y}^{MP}(s_0) = \boldsymbol{y}^{FC}(b_0, z_0)$ follows from the equality of policies given **Assumption 3***. □

Combining the two equivalences between FC and MP, and MP and LTC establishes the equivalence between LTC and FC, and delivers the proof of **Proposition 1**.

Proof of Proposition 1*. Equivalence between FC and MP is established in **Lemma 2**, and between MP and contingent LTC in **Lemma 3**, establishing the equivalence between FC and contingent LTC. □

Finally, the proof of **Proposition 1** follows as an immediate corollary of **Proposition 1*** in the case without uncertainty.

Proof of Proposition 1. This follows as a corollary of **Proposition 1*** in the case without uncertainty, since in deterministic economies their is no distinction between contingent and non-contingent LTC. Additionally, note that **Assumption 3** is simply the restatement of **Assumption 3*** in the case without uncertainty. □

A.5 A generic recursivity proof

This is a proof of recursivity in a stochastic environment. It draws heavily on Lucas and Stokey's (1989, Chapter 9) proofs, but with a few tweaks to make it applicable to our setup. Most importantly, the proofs are written for the case where the state tomorrow can depend on the shock tomorrow, in contrast to their core proof, which is written for the case where the state tomorrow does not depend on the realisation of uncertainty.

For simplicity, we also make the technical assumptions that: 1) Uncertainty is restricted to the discrete probability model. Thus, shocks are finite or countably infinite, but not uncountably infinite. 2) We assume a bounded state space and discount factor less than one. This removes the need for additional proofs in the case where value is infinite, and ensures that the only solution to the recursive formulation is the sequence solution without having to also impose a transversality condition on the value function. 3) We assume the supremum is always attained, and write the problems instead in terms of the "max" operator.

A.5.1 Environment

The analysis is restricted to a discrete probability model, where the shocks each period are drawn from either a finite or countably infinite set. Each period, the value of the shock, z_t, is drawn from the sample space $Z = \{\bar{z}^1, \bar{z}^2, ...\}$, with N_z denoting the (integer, possibly infinite) number of elements of Z. The σ-field $\mathscr{Z}$ is the set of all subsets of Z.[4] Denote the product spaces $(Z^t, \mathscr{Z}^t)$, $t = 1, 2....$ We work with the discrete probability spaces $(Z^t, \mathscr{Z}^t, \mu^t)$ where $\mu^t : \mathscr{Z}^t \mapsto [0, 1]$ are probability measures. Let $z^t = (z_1, ..., z_t) \in Z^t$ denote a partial history of shocks from periods 1 to t. Let P_Z denote the Markov probabilities of transitioning from state $\bar{z}^i$ to $\bar{z}^j$ between periods t and $t + 1$. For any state i at time t, μ_i^t computes the unconditional (time-0) probability that the system will be at that state at that time, which can be computed by iterating on the Markov chain once the initial state, z_0, is known.

Let X denote the set of possible values for the endogenous state, x_t. Let $s_t = (x_t, z_t)$ denote the combined state variable. This is restricted to take values from some set $S \subset X \times Z$.

At time t the agent chooses a value for the endogenous state tomorrow, x_{t+1}, conditional on each possible shock realisation. These are denoted by x^i_{t+1}, which is defined as the choice of x_{t+1} conditional on the realisation $z_{t+1} = \bar{z}^i$ tomorrow. These choices are summarised in a vector $\underset{\sim}{x}_{t+1} = (x^1_{t+1}, x^2_{t+1}, ...)$. Analogously, the agent chooses a vector for the overall state $\underset{\sim}{s}_{t+1} = (s^1_t, s^2_t, ...)$, where $s^i_t \equiv (x^i_{t+1}, \bar{z}^i)$. We denote by $S_Z \equiv S^{N_z}$ the set of possible values for $\underset{\sim}{s}_t$.

This is done subject to restrictions, summarised by the transition correspondence $\Gamma : S \mapsto S_Z$. That is, a vector of next-period states $\underset{\sim}{s}_{t+1}$ is permissible if and only if $\underset{\sim}{s}_{t+1} \in \Gamma(s_t)$. Note that this correspondence does not determine the value of z_{t+1} given the value of z_t, since z is stochastic: the probability of transitioning to different states is given by the Markov matrix P_Z. The correspondence Γ captures any restrictions on

[4] This is the natural choice of all possible combinations of events that could happen. With this choice of σ-field, all functions defined on Z are $\mathscr{Z}$ measurable, so we can ignore measurability discussions (See Lucas and Stokey (1989) Exercise 7.10).

what can and cannot be state contingent. For example, if one of the states is capital, k_t, produced with time-to-build, then Γ would specify that k_{t+1} must be the same in all states tomorrow: $k^1_{t+1} = k^2_{t+1} = ... = k_{t+1}$.

Let A be the graph of Γ. That is, $A = \{(s, \underset{\sim}{s}') \in S \times S_Z : \underset{\sim}{s}' \in \Gamma(s)\}$. Let $F : A \mapsto \mathbb{R}$ be the one-period return function. $F(s, \underset{\sim}{s}')$ is the return if the state is s and $\underset{\sim}{s}'$ is chosen. The discount factor is $\beta \geq 0$. The primatives for this problem are X, Z, P_Z, Γ, F, and β.

A.5.2 Sequence form problem

The agent chooses a contingent plan for the evolution of the state variables:

Definition 6. *Denote a policy plan by* $\boldsymbol{\pi} \equiv \{\pi_t(z^t)\}_{t=0}^{\infty}$. *The functions* $\pi_t : Z^t \mapsto S$ *map shocks into contingent values for the state:* $s_t = \pi_t(z^t)$.

Define the vector $\underset{\sim}{\pi}_{t+1}(z^t) = (\pi_{t+1}(z^t, \bar{z}^1), \pi_{t+1}(z^t, \bar{z}^2), ...)$ as the collection of planned values for the state next period (from the perspective of time t) depending on the realised shock z_{t+1}. Any chosen plan must be feasible:

Definition 7. *A plan* $\boldsymbol{\pi}$ *is feasible from* $s_0 \in S$ *if:* 1) $\underset{\sim}{\pi}_1 \in \Gamma(s_0)$, *and* 2) $\underset{\sim}{\pi}_{t+1} \in \Gamma(\pi_t)$, *for all* $t = 1, 2, ...$ *and* $z_t \in Z^t$. *Denote by* $\Pi(s_0)$ *the set of plans that are feasible from* s_0.

Trivially, any feasible plan must feature $\pi_0 = s_0$. The first assumption is to ensure that there are always feasible plans starting from any state:

Assumption A1. $\Gamma(s)$ *is nonempty for all* $s \in S$.

This means that there is at least one feasible path starting from any state $s_0 \in S$, and hence also $\Pi(s_0)$ is non-empty on S. The second and final assumption is a boundedness restriction:

Assumption A2. *The environment is such that the return function* F *is bounded over all feasible paths. The discount rate is such that* $0 \leq \beta < 1$.

This assumption does two things. Firstly, it rules out cases of infinite utility for which additional proofs must be supplied, and ensures that the limit of expected utilities is well defined in the limit of an infinite horizon. Secondly, it ensures that the unique solution to the recursive problem is the sequence problem, without the need for additional transversality conditions on the solution.

Expected discounted utility is defined as

$$u(\boldsymbol{\pi}, s_0) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t F(\pi_t(z^t), \underset{\sim}{\pi}_{t+1}(z^t)). \tag{A.4}$$

The expectations operator is defined as the sum over discrete probabilities: $\mathbb{E}_0(x_t) = \sum_{i \in Z} x_t(z_t = z^i)\mu^t_i$, and conditional expectations are defined in the usual way. Under the above two assumptions utility is well defined: the limit in the infinite sum exists, and there is at least one feasible path on which utility can be calculated for any initial state $s_0 \in S$.

The sequence problem maximises the discounted sum of expected utility by choosing the whole path of policies $\boldsymbol{\pi}$ at time 0. We restrict ourselves, by assumption, to situations where there is a well defined optimal path (i.e. the supremum is assumed to be achieved for all $s_0 \in S$) and hence express the optimisation in terms of a maximisation. We can define the maximum function $v^* : S \mapsto R$ by

$$v^*(s_0) = \max_{\boldsymbol{\pi} \in \Pi(s_0)} u(\boldsymbol{\pi}, s_0). \tag{A.5}$$

A.5.3 Recursive problem

The recursive problem is defined by the functional equation

$$v(s_t) = \max_{\utilde{s}_{t+1} \in \Gamma(s_t)} F(s_t, \utilde{s}_{t+1}) + \beta \mathbb{E}_t v(s_{t+1}). \tag{A.6}$$

The optimal policy correspondence is given by the set

$$G(s_t) = \underset{\utilde{s}_{t+1} \in \Gamma(s_t)}{\arg\max}\, F(s_t, \utilde{s}_{t+1}) + \beta \mathbb{E}_t v(s_{t+1}). \tag{A.7}$$

Any optimal policy $\utilde{g}(s_t) \in G(s_t)$ specifies a vector of values for the state tomorrow, $\utilde{s}_{t+1} = \utilde{g}(s_t)$, alternatively written as $s_{t+1} = g(s_t, z_{t+1})$. The set of optimal policies is always non-empty, by our maintained assumption that the maximum is achieved. We say that a plan $\boldsymbol{\pi}$ is generated from G if it is formed by recursively applying policies from G.

A.5.4 Proof of recursivity

For our purposes it is only necessary to prove that the solution to the recursive problem solves the sequence problem, and not vice versa.[5] Thus, we summarise the required result in the following proposition.

Proposition 2. *Let* **Assumption A1** *and* **Assumption A2** *hold, and let v^* be defined by the sequence problem,* (A.5)*. Let v be a function satisfying the functional equation* (A.6)*, and let G be the policy correspondence defined by* (A.7)*. Then $v = v^*$ and any plan $\boldsymbol{\pi}^*$ generated by G attains the maximum in* (A.5)*.*

Proof. To show that $v = v^*$ we must show that v solves (A.5). Start by iterating forward on (A.6) one period:

$$\begin{aligned}
v(s_0) &= \max_{\utilde{s}_1 \in \Gamma(s_0)} F(s_0, \utilde{s}_1) + \beta \mathbb{E}_0 v(s_1) \\
&= \max_{\utilde{s}_1 \in \Gamma(s_0)} F(s_0, \utilde{s}_1) + \beta \mathbb{E}_0 \left[\max_{\utilde{s}_2 \in \Gamma(s_1)} F(s_1, \utilde{s}_2) + \beta \mathbb{E}_1 v(s_2) \right] \\
&= \max_{\substack{\utilde{s}_1 \in \Gamma(s_0) \\ \utilde{s}_2 \in \Gamma(s_1^i)}} \left\{ F(s_0, \utilde{s}_1) + \beta \mathbb{E}_0 \left[F(s_1, \utilde{s}_2) + \beta \mathbb{E}_0 v(s_2) \right] \right\},
\end{aligned}$$

[5] The inverse proof is very similar, and closely follows the related proof in Lucas and Stokey (1989) Chapter 9.

where we have combined the two maximisations using the Law of the Iterated Supremum, and combined the expectations using the Law of Iterated Expectations. It is understood that the constraint $\underset{\sim}{s}_2 \in \Gamma(s_1^i)$ actually represents N_z constraints, one for each $i = 1, ..., N_z$. Repeating this step n times, taking the limit as $n \to \infty$, and writing this in plan notation, this is equivalent to

$$v(s_0) = \max_{\boldsymbol{\pi} \in \Pi(s_0)} \left\{ u(\boldsymbol{\pi}, s_0) + \lim_{n \to \infty} \beta^{n-1} \mathbb{E}_0 v(s_n) \right\}. \tag{A.8}$$

According to **Assumption A2**, the return function is bounded along any feasible path, which implies that there is a finite $\bar{F} < \infty$ for which $|F(s, \underset{\sim}{s}')| < \bar{F}$ for any feasible s and $\underset{\sim}{s}'$. This implies that $|v(s)| \leq \bar{F}/(1-\beta)$, and hence that the limit of discounted value must converge to zero since $\beta < 1$ and hence $\lim_{n\to\infty} \beta^{n-1} = 0$. That is, $\lim_{n\to\infty} \beta^{n-1} \mathbb{E}_0 v(s_n) = 0$. Plugging this in gives the final result:

$$v(s_0) = \max_{\boldsymbol{\pi} \in \Pi(s_0)} u(\boldsymbol{\pi}, s_0). \tag{A.9}$$

Which, by comparison to (A.5) proves that $v = v^*$. Additionally, any plan in $G(s_0)$, which by definition achieves value $v(s_0)$, must thus also achieve value $v^*(s_0)$ and hence achieve the maximum in the sequence problem. □

A.6 Necessity of Assumption 3

Our equivalence result, **Proposition 1**, is one of sufficiency: **Assumption 3** is *sufficient* for LTC to support the FC path as its only equilibrium. One may ask whether the assumption is also *necessary* for LTC to support the FC path as an equilibrium. In particular, we focus on the following question: for a FC equilibrium outcome to be supported with a finite commitment horizon, does it have to be the case that policy plans over such a horizon uniquely pin down all problematic variables? A failure of necessity would mean that it may sometimes be possible to support the FC path as the outcome of some Markov-Perfect equilibrium of the LTC game even if **Assumption 3** does not hold.

It is worth pointing out a simple counterexample first, to show why **Assumption 3** is not always necessary. This is that, even in models which have structures which may lead to time inconsistency, some FC paths may in fact be time consistent. As an example, consider the economy considered in Section 3.2 (as in Lucas and Stokey, 1983, but with endogenous government spending). In general, the FC plan is time inconsistent in this model. However, in the case where initial government debt is zero ($\mathrm{b}_0 = 0$) the FC solution features debt which remains constant at zero forever. Without debt there is no incentive to twist interest rates, and a future government would not reoptimise if given the chance. This is true if a government was allowed to reset the sequence at any time $t > 0$ (and retain commitment from then on). Debortoli and Nunes (2013) analyse a smooth Markov-Perfect equilibrium of the NC game in this model and show that when initial government debt is zero ($\mathrm{b}_0 = 0$) the NC equilibrium also features zero debt in all periods. However, the NC and FC solutions differ for initial debt not equal to zero. Here, NC represents a special case of LTC (with $L = 0$) where **Assumption 3** fails, but the equilibrium still supports the FC path for a specific initial condition.

Answering this question more generally is challenging due to the possibility of there being multiple Markov-Perfect equilibria of the LTC game when **Assumption 3** fails.

We start by considering a one-shot deviation argument in a model featuring differentiability and argue that in general it is not possible to support the FC solution with limited commitment if the assumption fails. We then move on to discussing equilibria of the LTC game. We argue that there are special cases where LTC can support FC without **Assumption 3**, but provide proof of what we call a weaker "functional necessity".

A.6.1 Does time inconsistency in FC imply time inconsistency in MP?

Suppose that the government at time 0 announces an FC plan for a sequence of policies, expecting that no future government will reoptimise. We now allow for an ex-post reoptimisation by the government at time 1, but suppose that we have L periods of commitment, so that the government at time 1 cannot change policies dated time 1 to L. Is this finite commitment enough to remove the incentive for the time-1 government to deviate from the original FC plan, even in the absence of **Assumption 3**?

It is relatively straightforward to show that this is not generically the case, at least in problems with continuous choice variables for which Lagrangian methods can be used. For simplicity, we restrict our exposition to a model with no uncertainty and a single policy choice. Suppose we want to convert the return function, $r(c_t, b_t, z_t, \tau_t)$, and competitive equilibrium restrictions (2), (3) and (4) into indirect functions, solving out for endogenous variables and expressed only in terms of policy sequences. If **Assumption 3** fails, then (by the definition of the assumption) this implies that it is not possible to pin down the current equilibrium of the economy as a function of a finite sequence of policies.

We split the economy into equations at and after time 0 in order to facilitate the analysis of the deviation. Let $\tilde{u}^0(b_0, \tau_0, \tau_1, ...) = r(c_0, b_0, z_0, \tau_0)$ denote the indirect period return at time 0, computed as a function of the infinite sequence of policies, recognising that the endogenous variables in c_0 could potentially depend on all future policies. This is constructed by using the competitive equilibrium restrictions (2), (3), and (4) for $t = 0, 1, ...$ to calculate c_0 given a sequence $(\tau_0, \tau_1, ...)$. Similarly, let $\tilde{v}^1(b_1, \tau_1, \tau_2, ...)$ be the discounted sum of utility starting at time 1 computed by again using the competitive equilibrium sequence to convert a sequence of policies in to utilities. $\tilde{v}^1(b_1, \tau_1, \tau_2, ...)$ is calculated using all of the competitive equilibrium constraints from $t = 1$ onwards, and the state b_1 inherited from the last government. Finally, let $b_1 = \tilde{h}(b_0, \tau_0, \tau_1, ...)$ denote the implied value of b_1 given the initial government's plan and all competitive equilibrium constraints from $t = 0$ onwards. Note that $\tilde{v}^1(b_1, \tau_1, \tau_2, ...)$ depends only on the inherited state, and does not need to respect the forward looking constraint (4) from $t = 0$, which is now in the past. The effect of this constraint on the equilibrium from $t = 1$ onwards is summarised only by the constraint $b_1 = \tilde{h}(b_0, \tau_0, \tau_1, ...)$.

The FC problem of maximising discounted utility at time 0 can be expressed using a Lagrangian:

$$\mathcal{L} = \min_{\lambda} \max_{\{\tau_t\}_{t=0}^{\infty}} \tilde{u}^0(b_0, \tau_0, \tau_1, ...) + \beta \tilde{v}^1(b_1, \tau_1, \tau_2, ...) + \lambda(\tilde{h}(b_0, \tau_0, \tau_1, ...) - b_1) \tag{A.10}$$

The first order condition for any policy at time $t \geq 1$ is given by

$$\tilde{u}^0_t + \beta \tilde{v}^1_t + \lambda \tilde{h}_t = 0 \tag{A.11}$$

where $\tilde{u}^0_t$ refers to the derivative of $\tilde{u}^0(\tau_0, \tau_1, ...)$ with respect to the time-t policy, and similarly for $\tilde{v}^1_t$ and $\tilde{h}_t$. Consider now the problem of the government at $t = 1$ who is allowed to reoptimise. We first ignore any limited commitment, and suppose they are allowed to change all policies dated $t = 1$ and onwards. They ignore the initial forward looking constraint, $\tilde{h}$, and simply take the inherited state b_1 as given. They perform the unconstrained maximisation

$$\max_{\{\tau_t\}_{t=1}^{\infty}} \tilde{v}^1(b_1, \tau_1, \tau_2, ...). \tag{A.12}$$

This gives the first order condition

$$\tilde{v}_t^1 = 0. \tag{A.13}$$

Comparing the first order conditions (A.11) and (A.13) reveals the source of the time inconsistency. The initial FC government takes into account the effect of its choices of $\tau_1, \tau_2, ...$ on both initial utility, $u^0(b_0, \tau_0, \tau_1, ...)$, and the initial constraint, $b_1 = \tilde{h}(b_0, \tau_0, \tau_1, ...)$, while the government at $t = 1$ does not. More concretely, the reoptimising government's optimal plan sets the marginal change in discounted utility from adjusting any policy equal to zero. This may not be the case in the initial FC plan, if the sum $\tilde{u}_t^0 + \lambda \tilde{h}_t$ is non-zero. In the following, we assume that the there is meaningful time inconsistency in all future periods, so that the FC and reoptimised solutions differ for every $t = 1, 2, ...$

We can now ask, given that there is time inconsistency in this model, can adding a limited amount of commitment deal with this? Specifically, suppose that we introduce limited commitment in the spirit of LTC, so that the reoptimising government cannot change the L initial policies $(\tau_1, ..., \tau_L)$ from their FC values. In this case, the reoptimising government inherits the constraints $\tau_t = \tau_t^{FC}$ for $t = 1, ..., L$, and the Lagrangian becomes

$$\mathcal{L} = \min_{\{\delta_t\}_{t=1}^{L}} \max_{\{\tau_t\}_{t=1}^{\infty}} \tilde{v}^1(b_1, \tau_1, \tau_2 ...) + \sum_{t=1}^{L} \delta_t(\tau_t - \tau_t^{FC}) \tag{A.14}$$

with first order conditions

$$\tilde{v}_t^1 + \delta_t = 0 \tag{A.15}$$

for $t = 1, ..., L$ and

$$\tilde{v}_t^1 = 0 \tag{A.16}$$

for $t = L+1$ onwards. The first order conditions now contain an additional multiplier, δ_t, from the constraint fixing the value of the initial policies for the first L periods. However, since these multipliers are absent for $t > L$, the optimal solution still does not replicate the initial FC solution for the same reason that for $t > L$ we must have $v_t^1 = 0$ while this may not be the case in the initial FC solution.

Suppose that a reoptimising government without limited commitment would adjust all future policies, $(\tau_1, \tau_2, ...)$. Then adding a limited amount of commitment, so that $(\tau_1, ..., \tau_L)$ cannot be changed, will not remove the incentive to adjust $(\tau_{L+1}, \tau_{L+2}, ...)$ in a model with continuous choice variables. While removing the ability to change $(\tau_1, ..., \tau_L)$ might reduce or alter the incentives to change $(\tau_{L+1}, \tau_{L+2}, ...)$, it can never remove it fully. For example, if c_{t+1} appears in the time-t constraints, a deviating government will still have the incentive to adjust policies hundreds of periods in the future to influence this variable if it can. Hence reoptimisations will still lead to deviations from the FC path.

How does this analysis differ when **Assumption 3** holds? In this case, it is possible to express the initial utility and constraint as functions of only a finite sequence of L taxes: $\tilde{u}^0(b_0, \tau_0, \tau_1, ..., \tau_L)$ and $b_1 = \tilde{h}(b_0, \tau_0, \tau_1, ..., \tau_L)$. This means that the first order conditions in the initial FC problem now become

$$\tilde{u}_t^0 + \beta \tilde{v}_t^1 + \lambda \tilde{h}_t = 0 \tag{A.17}$$

for $0 < t \leq L$ and

$$\tilde{\beta} \tilde{v}_t^1 = 0 \tag{A.18}$$

for $t > L$. Comparing these to the first order conditions of the reoptimising government with L periods of commitment, we can guess and verify that it is optimal to replicate the FC allocation because the multipliers δ_t can soak up the effect of the extra terms by setting $\delta_t = u_t^0 + \lambda \tilde{h}_t$ for $t = 0, ..., L$. Note that these argument also hold for cases where **Assumption 3** holds for some L, but the government only has $L' < L$ periods of commitment. In this case L' periods of commitment will not be able to sustain the FC path.

In terms of the equilibrium concepts introduced in this paper, we have shown that if **Assumption 3** fails, the solution to the Modified Problem will still be time inconsistent for any finite amount of commitment. Of course, this does not answer the ultimate question of whether LTC can support FC when **Assumption 3** fails, because even if the Modified Problem features time inconsistency, it could be that LTC game is able to overcome this, for example by using discontinuous equilibrium policy functions which punish deviations. We turn to this question in the next section.

It is worth noting that the restriction to problems with continuous choice variables is not innocuous. In models with discrete choices, it could be that restricting initial policies to their FC values is enough to reduce the benefit of adjusting future policies below the threshold at which it is beneficial. That is, in the above continuous model we have shown that there is always a marginal incentive to adjust future policies, which can always be realised which choice variables are continuous. If choice variables are discrete, it could be that restricting a few choices is enough to reduce this marginal benefit below the level at which deviating from the FC plan is profitable.

A.6.2 Can LTC support MP when Assumption 3 fails?

The result of the previous subsection seems to suggest that if **Assumption 3** fails there is no way that a finite amount of commitment can support the FC solution. In this section we move on to analysing the LTC game itself, and argue that this is not strictly true. This is because when **Assumption 3** fails there is no guarantee that there is a unique Markov-Perfect equilibrium of the LTC game. Thus, for every possible initial state, it is hard to rule out that there is not some Markov-Perfect equilibrium of the LTC game that could support the FC solution.[6]

We can instead prove a weaker form of *functional* necessity. This form states that, if the Modified Problem remains time-inconsistent when **Assumption 3** does not hold,

[6] Krusell et al. (2004) argue that a discontinuous "step function" equilibrium exists in the NC game in a model without capital (Lucas and Stokey, 1983). This equilibrium replicates the FC path for certain initial conditions even when the path is time inconsistent.

then there is no single LTC policy function which can support the MP paths for all initial states. We provide a discussion below for the case of deterministic economies, but the same result holds in the case of uncertainty.

To see this, consider any initial state $s_0 = (b_0, z_0, \tau_0^L) \in S$ for which the MP path is time-inconsistent, meaning that it would be reoptimised at $t = 1$. Let $\left\{\tau_t^{MP}(s_0)\right\}_{t=0}^{\infty}$ denote the optimal policy sequence in the MP problem, starting from initial state s_0. Construct the time-1 state $s_1^{MP} = (b_1^{MP}, z_1, \tau_1^{L,MP})$, where $\tau_1^{L,MP}$ gives the required policy state variable using the optimal sequence of policies from $t = 1$ to L from the MP policy plan.

Let $\left\{\tau_t^{MPR}(s_0)\right\}_{t=0}^{\infty}$ denote the observed policy sequence starting from state s_0 if the government carries out the original MP plan at time 0, but is allowed to reoptimise everything, except for the pre-committed states in $\tau_1^{L,MP}$, at time 1. By definition, the solution from time 1 onwards must be the same as a government who starts at time 0 with the same state. Hence the whole path is given by $\left\{\tau_t^{MPR}(s_0)\right\}_{t=0}^{\infty} = \left(\tau_0^{MP}(s_0), \left\{\tau_t^{MP}(s_1^{MP}(s_0))\right\}_{t=0}^{\infty}\right)$. Time inconsistency means that

$$\left\{\tau_t^{MP}(s_0)\right\}_{t=1}^{\infty} \neq \left\{\tau_t^{MPR}(s_0)\right\}_{t=1}^{\infty} = \left\{\tau_t^{MP}(s_1^{MP}(s_0))\right\}_{t=0}^{\infty}. \tag{A.19}$$

This states that the continuation of the time-0 plan from time 1 onwards is not the same as the reoptimised problem starting with the state variable $s_1^{MP}(s_0)$ from the MP plan. Suppose that some LTC policy function $\underset{\sim}{\tau}_{t+L} = \underset{\sim}{\tau}(s_t)$ can replicate the original MP plan. This means that we can replicate the sequence $\left\{\tau_t^{MP}(s_0)\right\}_{t=0}^{\infty}$ by iterating forward on $\underset{\sim}{\tau}_{t+L} = \underset{\sim}{\tau}(s_t)$ starting from initial state s_0. In particular, this also means that we can replicate the plan from time 1 onwards, $\left\{\tau_t^{MP}(s_0)\right\}_{t=1}^{\infty}$, starting from state $s_1^{MP}(s_0)$. Suppose as well that this function can also replicate the reoptimised plan starting from time 1, which is another valid MP solution. This means that iterating forward on $\underset{\sim}{\tau}_{t+L} = \underset{\sim}{\tau}(s_t)$ from initial state $s_1^{MP}(s_0)$ must generate the plan $\left\{\tau_t^{MP}(s_1^{MP}(s_0))\right\}_{t=0}^{\infty}$. However, by the definition of time inconsistency these two paths are different, meaning that they cannot both be generated by iterating forward on $\underset{\sim}{\tau}_{t+L} = \underset{\sim}{\tau}(s_t)$ from the same initial state, giving a contradiction.

In conclusion, in this section we have explored the necessity of **Assumption 3** for sustaining **Proposition 1** along several dimensions. While the main focus of our equivalence result is sufficiency, this discussion may further clarify the role of **Assumption 3** in the relationship between LTC and FC.

A.7 Multi-period governments and stochastic changes of government

We proved our equivalence result for a specific power structure, with one-period lived governments: The government at time t chooses only policies dated $t+L$, the government at $t+1$ chooses policies dated $t+L+1$, and so on. Clearly, this assumption may appear restrictive, as there are several alternative plausible power structures one could consider: governments stay in power for multiple years, for example, or one could imagine probabilistic changing of governments, as in Debortoli and Nunes (2010).

We will now argue that, as long as one crucial feature of our setup is preserved, our results go through for a much more general class of power structures. For example, it is straightforward to show that if **Assumption 3** holds for some L, then it also holds for any $L' > L$. That is, if it can be established for some L then this simply represents a lower bound on the amount of commitment required, and the LTC game naturally also supports FC for greater amounts of commitment.

In the next proposition, we focus on the case of governments lasting for multiple periods.

Proposition 3. *Consider a model where* **Assumption 1** *and* **Assumption 3** *hold for some $L > 0$. There is a sequence of governments who remain in power for $M \geq L$ periods each. They have the ability to choose (and commit to) all policy variables while they are in power, except for or the first L periods after coming to power. They also choose the first L policy variables applying during the following term. The unique equilibrium of this game supports the FC solution.*

Informally, the source of time inconsistency here is that the current government disagrees with the past governments about the choice of the policies dated t to $t+L$. As long as these variables are pinned down, giving the government extra commitment, for example inside a fixed term of office, does not change the result that LTC can support FC. A formal proof is available on request from the authors. This proposition clarifies that it is not who is in power, or for how long, that is important, but rather the inability to change initial policies after coming to power. In this sense, our results apply naturally to situations of institutional delay: under sufficient conditions on the environment, a certain amount of institutional delay may improve the commitment ability of the government, and return us to the FC outcome. Equivalent propositions can be established for other power structures, including probabilistic changes of government. The key assumption is that at any time t, the next L periods' policies cannot be changed. It does not matter whether the current government stays in power or whether it is replaced, as long as both governments agree on all the possible policies that both of them could choose.

A.8 Numerical algorithm when LTC supports FC

In this section we briefly discuss how our results can be used to solve for a recursive solution to the time-inconsistent FC problem whenever our theorem holds. In particular, whenever there exists a degree of commitment, L, such that LTC and FC lead to the same outcomes, it is possible to solve for the time-inconsistent solution to the FC problem using standard dynamic programming tools. We first give a general outline of the algorithm, and then apply it to the Lucas and Stokey (1983) model.

A.8.1 Description of algorithm

Whenever our equivalence result holds, the FC plan has a recursive form given by (12). This recursive form takes as state variables $s_t \equiv (b_t, z_t, \tau_t^L)$, which are precisely the state variables of the LTC problem. The policy and value functions of this recursive form of the FC game can be found by applying standard dynamic programming tools to (12). An appealing intuitive feature of this recursive form is that it uses the previously-committed choices (e.g. today's tax rates) as state variables, rather than promised utilities or Lagrange multipliers.

The only complication is that the state space must be appropriately restricted to rule out choices which violate competitive equilibrium. Note that the statement of the LTC game restricts s_t, and consequently choices for s_{t+1}, to lie in the set S, defined as the set of s_t which lie on some competitive equilibrium path. Rather than having to compute all CE paths and checking whether any s_t lies somewhere on one of them, in the next subsection we provide an algorithm which solves for the set S recursively. This can be done “once and for all” before solving for policy and value functions.

Once the set S is solved for, one can apply standard dynamic programming tools to (12) to solve for the “recursified” FC policy functions. Since the LTC policy functions take pre-committed policies as states, one also has to solve for the initial policies, τ_0^L. This is done by by maximising the LTC value function at time zero, $v(s_0) = v((b_0, z_0, \tau_0^L))$, over τ_0^L.

A.8.2 Algorithm to construct feasible set, S

In this section we construct a procedure to calculate the set S, on which the LTC problem is defined. Recall that the set S was defined in (A.1). The procedure is iterative, and based on Phelan and Stacchetti's (2001) procedure for solving for the value correspondence in their setup. First we define a transition correspondence.

Definition 8. $\tilde{\Gamma} : B \times Z \times T_L \mapsto (B \times Z \times T_L)^{N_z}$ *is defined such that* $\underset{\sim}{s}_{t+1} \in \tilde{\Gamma}(s_t)$ *iff there exists some* $\{y_s(z^s)\}_{s=t}^{t+N}$ *consistent with the allocations pinned down by* s_t *and* $\underset{\sim}{s}_{t+1}$*, and which satisfies the constraints (2), (3) and (4) at time* t*.*

Note that this is closely related to the correspondence Γ, but is defined on the more general set $B \times Z \times T_L$, since we do not know the set S at this point. Now, for a generic set $S_n \in B \times Z \times T_L$, define the mapping over sets

$$B(S_n) = \left\{ s \in B \times Z \times T_L : \text{there exists } \underset{\sim}{s}' \in \tilde{\Gamma}(s) \text{ s.t. } \underset{\sim}{s}' \in (S_n)^{N_z} \right\}. \tag{A.20}$$

$B(S_n)$ is the set of $s_t = (b_t, z_t, \tau_t^L)$ such that it is possible to find a $\underset{\sim}{s}_{t+1}$ which satisfies today's CE constraints (given s_t) and leaves next period's states which are in S_n. We first prove two preliminary lemmas.

Lemma 4. $S = B(S)$

Proof. Suppose that $S \supset B(S)$ strictly. Then there exists some $\hat{s} \in B(S)$ and $\hat{s} \notin S$ such that there exists $\underset{\sim}{s}' \in \tilde{\Gamma}(\hat{s})$ with $\underset{\sim}{s}' \in S_Z$. Since $\underset{\sim}{s}'$ is on a CE path and is reachable from $\hat{s}$, $\hat{s}$ must also be on a CE path which features $\hat{s}$ and $\underset{\sim}{s}'$ as its first and second elements. This contradicts $\hat{s} \notin S$. Now suppose that $S \subset B(S)$ strictly. Then there exists some $\hat{s} \in S$ such that there exists no $\underset{\sim}{s}' \in \tilde{\Gamma}(\hat{s})$ with $\underset{\sim}{s}' \in S_Z$. Thus $\hat{s}$ cannot be on a CE path, contradicting $\hat{s} \in S$. Thus it must be that $S = B(S)$. □

Lemma 5. *If* $S_{n+1} \subset S_n$*, then* $B(S_{n+1}) \subset B(S_n)$*.*

Proof. Consider a generic $s \in S_{n+1}$. To be in $B(S_{n+1})$ there must exist an $\underset{\sim}{s}'$ such that $\underset{\sim}{s}' \in (S_{n+1})_z$ and $\underset{\sim}{s}' \in \tilde{\Gamma}(s)$. Since $S_{n+1} \subset S_n$, we must have $\underset{\sim}{s}' \in (S_n)_z$, and hence also $s \in B(S_n)$. Since this is true for any $s \in S_{n+1}$, we have $B(S_{n+1}) \subset B(S_n)$. □

We are now ready to state the main result:

Lemma 6. *Suppose that* $S_0 \supset S$*, and* $B(S_0) \subset S_0$*. Define the recursion* $S_{n+1} = B(S_n)$*. Then* $\lim_{n\to\infty} S_n = S$*.*

Proof. The proof is recursive. 1) $S_1 = B(S_0) \subset S_0 \Rightarrow B(S_1) \subset B(S_0) \Rightarrow S_2 \subset S_1$. Carrying this on, we find that $S_n \subset S_{n-1} \subset ... \subset S_0$ for any n. 2) Also, $S \subset S_0 \Rightarrow B(S) \subset B(S_0) \Rightarrow S \subset S_1$. This inductively implies that $S \subset S_n$ for any n. Combining points 1 and 2 gives $S \subset S_n \subset S_{n-1} \subset ... \subset S_0$. Since this sequence of sets is decreasing it has a limit in the sense of set inclusion: $S_\infty \equiv \lim_{n\to\infty} S_n$. By a simple limit argument, $S_\infty = S$. □

This recursion is thus guaranteed to converge to S for an appropriately chosen S_0. A simple example of an S_0 which satisfies the required conditions is $S_0 = B \times Z \times T_L$.

A.8.3 Illustration: Lucas and Stokey (1983)

This section demonstrates the equivalence of the LTC and FC solutions numerically in the Lucas and Stokey (1983) economy. For simplicity we focus on the case of one period bonds with constant government spending. We consider a government who starts with an initial stock of debt $\mathrm{b}_0 > 0$. As previously discussed, the solution to the FC problem in this case involves a tax cut at time 0, followed by higher but constant taxes from time 1 onwards. We denote the optimal time 0 tax rate by $\tau^l_{0,FC}$ and the time 1 and onwards tax rate by τ^l_{FC}. This policy leads to a constant level of debt from time 1 onwards, which we label b_{FC}.

Utility is parameterised as $u(\mathrm{c}, \mathrm{l}) = \log(\mathrm{c}) - D\mathrm{l}^2/2$. We set $\beta = 0.96$ as standard. We calibrate g, b_0 and D to a fictional steady state were taxes to be constant. We target $\mathrm{l} = 1/3$ by appropriately choosing D, and set g to 20% of output. We target debt to GDP of around 60% and choose b_0 accordingly, leading to required constant taxes of around 22% in the fictional steady state.

The solution to the LTC problem is given by a policy function $\tau^l_{t+1} = g(\mathrm{b}_t, \tau^l_t)$, and an associated transition for debt from the implementability condition. The policy function is solved for on the set S of values of (b_t, τ^l_t) consistent with at least one competitive equilibrium. This set restricts us to values of initial debt such that the government can actually afford to repay without violating any limits on borrowing or taxes.

The recursive form is solved using value function iteration. We discretise the state, (b, τ^l), and maximise over $(\tau^l)'$ using a grid search procedure, with b' solved for explicitly using the implementability constraint. Future values off the grid are interpolated using splines. The set S is solved for using the iterative procedure described above. We solve for explicit maximum and minimum feasible b values on the τ^l grid. During the value function iteration, we restrict the government from making $(\tau^l)'$ choices which would lead $((\tau^l)', \mathrm{b}')$ to be outside of S. To choose grids for the LTC game, we first solve the FC problem starting from b_0. We then take the maximum and minimum values for the grids for b and τ^l to be 5% above and below the largest values observed along the FC path respectively.

Figure D.1 illustrates the solution. The left panel shows the set S, represented as an upper and lower limit for debt for a given tax rate: all values between the dashed and solid lines are consistent with at least one competitive equilibrium. The horizontal portions of the set are simply the exogenously imposed upper and lower bounds, but the top-left corner reveals values which are endogenously inconsistent with competitive equilibrium. Values with high initial debt and low initial taxes are not feasible, since they imply the lowest initial resources for the government, which it is not feasible to finance.

The right panel plots three slices of the policy function $g\left(\mathrm{b}_t, \tau_t^l\right)$ for different values of b_t. The optimal tax rate tomorrow is increasing in government debt, and decreasing in today's pre-committed tax rate. This is because increasing debt or reducing today's tax both worsen the fiscal position of the government, requiring higher taxes tomorrow to balance the intertemporal budget constraint.

Since our theorem holds in this setup, iterating on this policy function starting from initial state $\left(\mathrm{b}_0, \tau_{0,FC}^l\right)$ must replicate the FC path for taxes and debt. That is, we must have that $\tau_{FC}^l = g\left(\mathrm{b}_0, \tau_{0,FC}^l\right)$ for time 0, and $\tau_{FC}^l = g\left(\mathrm{b}_{FC}, \tau_{FC}^l\right)$, meaning that the FC plan appears as points which lie on the LTC policy functions. This is shown in Figure D.2. The left panel plots $g\left(\mathrm{b}_0, \tau_0^l\right)$, giving a slice of the LTC policy function across different τ_0^l values for debt equal to the initial value. The cross plots the (independently calculated) FC optimal taxes for $t = 0$ and $t = 1$. The cross lies on the LTC policy function, confirming that the LTC game replicates the FC choice of τ_1^l if the initial tax is restricted to be $\tau_{0,FC}^l$.

The right panel repeats the exercise for $t = 1$, which is also identical for any $t > 0$. This time the LTC policy function is drawn for debt equal to b_{FC}, which is the value inherited at time 1 for both the LTC and FC policies. Again, the FC cross lies on the LTC line, confirming that the LTC game replicates the FC choice of τ_2^l since the government inherits a pre-committed choice for τ_1^l equal to τ_{FC}^l.

B Appendix to Section 3

In this appendix, we first discuss a regularity condition in the benchmark model. Next, we apply the equivalence results in additional models with respect to the ones discussed in the main text. Finally, we discuss the role of initial policy conditions for equilibrium outcomes under LTC.

B.1 Boundedness restrictions

The boundedness restriction in **Assumption 1** is satisfied in the applications as long as there are no competitive equilibria leading to (positive or negative) infinite value, which is a relatively weak restriction. Competitive equilibrium places an upper bound on period utility, since consumption must ultimately be produced according to the economy's production technology, which is bounded every period if we assume an upper bound for labour supply and capital.

To ensure a lower bound on period utility is less straightforward, since governments in these models may be able to "shut down" the economy by issuing arbitrarily high taxes. This can push consumption towards zero, which leads utility to tend to $-\infty$ without bound with, for example, CRRA utility over consumption. This can be ruled out by an appropriate upper bound on taxation, such as requiring that the government must remain to the left of the peak of the dynamic Laffer curve, or by an arbitrary lower bound on utility. These bounds can always be chosen to be non-binding along an optimal path.

B.2 Equivalence in further special cases of the benchmark model

In this section we provide proofs of equivalence in additional specialisations of the general taxation model of Section 3.

B.2.1 Capital and balanced budgets with Greenwood, Hercowitz and Huffman (1988, GHH) preferences

GHH preferences provide an interesting specialisation of the model with capital and balanced budgets (Case 2) because it is still possible to prove equivalence of LTC and FC in this case, but the required degree of commitment increases to $L = 2$ periods. Assume preferences of the form $u(\mathrm{c}_t - v(\mathrm{l}_t)) + w(\mathrm{g}_t)$. The labour optimality condition and capital Euler equation are now

$$v'(\mathrm{l}_t) = \left(1 - \tau_t^l\right)(1-\alpha)\mathrm{z}_t\mathrm{k}_t^{\alpha}\mathrm{l}_t^{-\alpha} \tag{B.1}$$

$$u'(\mathrm{c}_t - v(\mathrm{l}_t)) = \beta u'(\mathrm{c}_{t+1} - v(\mathrm{l}_{t+1}))\left[1 + \left(\alpha \mathrm{z}_{t+1}\mathrm{k}_{t+1}^{\alpha-1}\mathrm{l}_{t+1}^{1-\alpha} - \delta\right)\left(1 - \tau_{t+1}^k\right)\right] \tag{B.2}$$

and the model is closed with the resource constraint, (18), and balanced budget constraint, (25). Both c_t and l_t are still problematic variables. With separable preferences we had previously shown that $L = 1$ periods of commitment allow LTC to support FC. Importantly, with GHH preferences the labour condition (B.1) no longer contains consumption, c_t. Hence the strategy of determining the problematic variables $(\mathrm{c}_t, \mathrm{l}_t)$ by determining l_t from the budget constraint (25) and consumption from the labour condition (15) no longer works.

However, consider extending to $L = 2$ periods of commitment, so that the government at t chooses policies $(\tau_{t+2}^k, \tau_{t+2}^l, \mathrm{g}_{t+2})$ and takes $(\tau_t^k, \tau_t^l, \mathrm{g}_t, \tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1})$ as fixed. Taking (B.1) and (25) forward one period gives

$$v'(\mathrm{l}_{t+1}) = \left(1 - \tau_{t+1}^l\right)(1-\alpha)\mathrm{z}_{t+1}\mathrm{k}_{t+1}^{\alpha}\mathrm{l}_{t+1}^{-\alpha} \tag{B.3}$$

$$\left(\alpha\tau_{t+1}^k + (1-\alpha)\tau_{t+1}^l\right)\mathrm{z}_{t+1}\mathrm{k}_{t+1}^{\alpha}\mathrm{l}_{t+1}^{1-\alpha} - \tau_{t+1}^k\delta\mathrm{k}_{t+1} = \mathrm{g}_{t+1}. \tag{B.4}$$

Notice that for fixed policies $(\tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1})$ these two equations pin down unique values of $(\mathrm{k}_{t+1}, \mathrm{l}_{t+1})$ consistent with the government's balanced budget and the amount of labour the household will provide. Using the time-t resource constraint, (18), this also pins down current consumption, c_t, since l_t is also pinned down by (B.1). Hence $L =$ 2 periods of commitment to policies pins down both problematic variables, c_t and l_t, allowing LTC to support the FC solution.

B.2.2 Capital and balanced budgets with inelastic labour supply

Here we prove that the LTC supports FC in the model used in the numerical application of Section 4 with $L = 2$ periods of commitment. This model features labour inelastically supplied at $\mathrm{l}_t = 1$ and no labour taxation. The model does not have a labour optimality condition, and the government's balanced budget reduces to

$$\left(\alpha\mathrm{k}_t^{\alpha} - \delta\mathrm{k}_t\right)\tau_t^k = \mathrm{g}_t \tag{B.5}$$

where we also assume for simplicity that $\mathrm{z}_t = 1$. The other equations of the model correspond to (16) and (18) with the appropriate restrictions. Apart from the policy variables, consumption is the only problematic variable. The proof of equivalence with $L = 2$ follows closely the proof for GHH preferences from the last section.

Notice that for fixed policies $(\tau_{t+1}^k, \mathrm{g}_{t+1})$ the balanced budget constraint taken at time $t+1$ pins down a unique value of k_{t+1} consistent with the government's balanced budget. Using the time-t resource constraint, (18), this then pins down current consumption, c_t. Hence, $L = 2$ periods of commitment to policies pins down the problematic variable c_t, allowing LTC to support the FC solution.

B.2.3 Capital and unbalanced budgets with linear utility from consumption

In the main two specialisations, we showed that LTC could support the FC solution if either the resource constraint, (18), or government budget constraint, (19), was converted into a static equation. In this section, we present a special case showing that it is possible to support FC if neither condition holds. This specialisation features production with capital, an arbitrary government budget, but linear utility in consumption.

The model equations are all the same as in the general model of subsection 3.1, with the restriction that $u(\mathrm{c}_t) = \mathrm{c}_t$, implying that $u'(\mathrm{c}_t) = 1$. The equations of the model are now:

$$\mathrm{c}_t + \mathrm{k}_{t+1} - (1-\delta)\,\mathrm{k}_t + \mathrm{g}_t = \mathrm{z}_t \mathrm{k}_t^{\alpha} \mathrm{l}_t^{1-\alpha}. \tag{B.6}$$

$$v'(\mathrm{l}_t) = \left(1 - \tau_t^l\right)(1-\alpha)\mathrm{z}_t \mathrm{k}_t^{\alpha} \mathrm{l}_t^{-\alpha} \tag{B.7}$$

$$\beta \mathrm{b}_{t+1} + \left[\alpha \tau_t^k + (1-\alpha)\,\tau_t^l\right] \mathrm{z}_t \mathrm{k}_t^{\alpha} \mathrm{l}_t^{1-\alpha} = \mathrm{g}_t + \mathrm{b}_t \tag{B.8}$$

$$1 = \beta\left[\alpha \mathrm{z}_t \mathrm{k}_{t+1}^{\alpha-1} \mathrm{l}_{t+1}^{1-\alpha}\left(1 - \tau_{t+1}^k\right) + 1 - \delta\right]. \tag{B.9}$$

Even with linear utility from consumption, there is still time inconsistency and a meaningful distinction between the FC and NC solutions to this model. A government with FC will choose to have the time-0 capital tax at the maximum level, and then set capital taxes from time 1 onwards to zero. A government with NC, on the other hand, will always have the temptation to tax capital once it is installed, and raise capital taxes to the maximum level.

To see that LTC can support FC in this special case, note that the only problematic variables are now labour and the capital tax, coming from $(\mathrm{l}_{t+1}, \tau_{t+1}^k)$ in the Euler equation. In the notation of our general formulation we have $N = 1$, with variables one period ahead appearing in the constraints, and $b_t = (\mathrm{k}_t, \mathrm{b}_t)$, $c_t = (\mathrm{c}_t, \mathrm{l}_t)$, and $\tau_t = (\tau_t^k, \tau_t^l, \mathrm{g}_t)$, and $z_t = \mathrm{z}_t$. We need to show that **Assumption 3**, holds in this model for $L = 1$. To show point (i) of the proof we must show that the state $s_t = (\mathrm{k}_t, \mathrm{b}_t, \mathrm{z}_t, \tau_t^k, \tau_t^l)$ pins down (l_t, τ_t^k). To see that this is the case note that, given k_t, z_t, and τ_t^l, (B.7) pins down a unique l_t, defining a function $\mathrm{l}_t = l(\mathrm{k}_t, \tau_t^l)$. Part (ii) of the proof is left to the reader.

What is the economic reason why one period of commitment can sustain FC in this model, while it cannot when the agent has concave utility from consumption? In the case of linear utility, consumption is no longer a problematic variable. Hence, even though the government cannot pin down c_t with just one period of commitment, this does not matter. The remaining sources of time inconsistency are the capital tax itself, for which commitment is then assumed, and l_{t+1}, which can be pinned down with one period of commitment since the labour supply curve is independent of consumption under linear utility. Intuitively, with linear utility the household does not want to smooth consumption. The Euler equation (B.9) shows that they choose the level of capital only thinking about the return, and hence taxes, at time $t+1$, which is why only one period of commitment is needed.

This example is instructive for two reasons. Firstly, it demonstrates that LTC can sometimes support FC in the absence of static resource or budget constraints. Secondly, it provides an example where the general model was modified to enable LTC to support FC not by adding an extra static equation, but by removing a problematic variable.

B.3 Equivalence in extensions to the benchmark model

In this section we discuss two extensions of the general models where our theorem also holds, and which provide further intuition about the economic forces determining the required length of commitment to sustain the FC solution.

B.3.1 Capital and multi-period balanced budgets

In this section we consider a government who faces the constraint that she must balance her budget every M periods. There are many ways to implement this which lead to LTC supporting FC, and we illustrate one method here. In particular, we suppose that the government can issue one-period bonds, denoted b_t, which are priced according to the agent's Euler equation $\mathrm{q}_t u'(\mathrm{c}_t) = \beta u'(\mathrm{c}_{t+1})$. The government's budget is now

$$\left[\alpha\tau_t^k + (1-\alpha)\,\tau_t^l\right]\mathrm{z}_t\mathrm{k}_t^\alpha \mathrm{l}_t^{1-\alpha} - \tau_t^k\delta\mathrm{k}_t + \beta\frac{u'(\mathrm{c}_{t+1})}{u'(\mathrm{c}_t)}\mathrm{b}_{t+1} = \mathrm{g}_t + \mathrm{b}_t. \tag{B.10}$$

We implement the balanced budget assumption by assuming that the government cannot issue bonds once every M periods. Give each period an index $m_t \in \{1, 2, ..., M\}$ denoting its position in the cycle, with $m_t = M$ denoting the last period of the cycle (where the government can't issue debt) and $m_t = 1$ denoting the first (where there is thus no inherited debt to repay). Whenever $m_t = M$, the government cannot issue debt and hence $\mathrm{b}_{t+1} = 0$. To fix ideas, if one period is a quarter, and the government must balance her yearly budget, this means that $M = 4$, and the government cannot issue any bonds in the fourth quarter of every year.

The rest of the model equations are the same as the second specialisation in Section 3.3: (15), (16), and (18), to which we add the government budget, (B.10), and the restriction that $\mathrm{b}_{t+1} = 0$ if $m_t = M$. We now prove that we can support FC with LTC with $L = M$ periods of commitment. In this case, the policy states are $\tau_t^L = (\tau_t^k, \tau_t^l, \mathrm{g}_t, ..., \tau_{t+M-1}^k, \tau_{t+M-1}^l, \mathrm{g}_{t+M-1})$.

To prove equivalence, we need to show that **Assumption 3**, holds in this model. In other words, we need to show that (i) if we fix $s_t = (\mathrm{k}_t, \mathrm{b}_t, \mathrm{z}_t, \tau_t^L)$ then we pin down the problematic variables $(\mathrm{c}_t, \mathrm{l}_t, \tau_t^k)$, and (ii) s_t and $(\tau_{t+M}^k, \tau_{t+M}^l, \mathrm{g}_{t+M})$ additionally pin down $(\mathrm{k}_{t+1}, \mathrm{b}_{t+1})$.

This has do be done separately for each position in the cycle, but the procedure is similar in all cases. First consider any t where $m_t = 1$, at the beginning of the cycle. Then in period $t + M - 1$ it will be the end of the cycle, meaning that no debt can be issued and $\mathrm{b}_{t+M} = 0$. To check point (i), we can forward (B.10) from t to $t + M - 1$ to yield

$$\sum_{s=0}^{M-1} u'(c_{t+s}) \left(\left[\alpha \tau^k_{t+s} + (1-\alpha) \tau^l_{t+s} \right] z_{t+s} k^{\alpha}_{t+s} l^{1-\alpha}_{t+s} - \tau^k_{t+s} \delta k_{t+s} \right) = \sum_{s=0}^{M-1} u'(c_{t+s}) g_{t+s}, \tag{B.11}$$

where we have used $b_t = 0$ and $b_{t+M} = 0$. Given the committed policy states $(\tau^k_t, \tau^l_t, g_t, ..., \tau^k_{t+M-1}, \tau^l_{t+M-1}, g_{t+M-1})$ and combining this with 1) M resource constraints, (18), from t to $t+M-1$, 2) M labour FOCs, (15), from t to $t+M-1$, and 3) $M-1$ capital Euler equations, (16), from t to $t+M-2$, gives $3 \times M$ equations in $3 \times M$ unknowns, $\{c_{t+s}, l_{t+s}, k_{t+s+1}\}_{s=0}^{M-1}$. While the number of equations does equal the number of unknowns, given the nonlinearity of the system we are unable to generically prove that this system admits a unique solution, and must maintain sufficient functional form restrictions as an additional assumption to be checked case by case. Under this assumption, this system thus pins down (c_t, l_t), as required for point (i). The problematic policy variables τ^k_t are trivially pinned down as part of the policy state τ^L_t. For part (ii), (k_{t+1}, b_{t+1}) were already solved for in the last step.

A similar procedure can be applied for periods at different points in the cycle, by iterating equations forward until a future period where $m_{t+s} = M$. For the first period of the cycle, that involves using equations from t to $t+M-1$, for the second we use equations from t to $t+M-2$ and so on, until in the last period of the cycle we can just use the equations at period t.

B.3.2 Capital and balanced budgets with longer time-to-build

In this section we consider an extension of the model of Section 3.3 with single-period balanced budgets but N period time to build. By N period time to build, we mean that investment today, i_t, creates capital which becomes productive at time $t+N$:

$$k_{t+N} = (1-\delta) k_{t+N-1} + i_t. \tag{B.12}$$

Note that k_{t+N-1} is predetermined at time t, since it is determined by past investment rates. The resource constraint is now

$$c_t + k_{t+N} - (1-\delta) k_{t+N-1} + g_t = z_t k^{\alpha}_t l^{1-\alpha}_t. \tag{B.13}$$

In this case the agent's Euler equation for capital becomes

$$u_{c_t} = \beta u_{c_{t+1}} (1-\delta) + \beta^N u_{c_{t+N}} \alpha z_t k^{\alpha-1}_{t+N} l^{1-\alpha}_{t+N} \left(1 - \tau^k_{t+N}\right). \tag{B.14}$$

The model is now described by (B.13), (B.14), (15), and (25). The problematic variables are now consumption, labour, utilisation, capital taxes, and government spending, due to the appearance of c_{t+N}, l_{t+N}, v_{t+N}, g_{t+N}, and τ^k_{t+N} in the constraints. In the notation of our general formulation we have variables up to N periods ahead appearing in the constraints. The time-to-build structure now means that all of $(k_t, k_{t+1}, ..., k_{t+N-1})$ are natural state variables at time t.

As in the original model with one period time to build, given $\left(k_t, k_{t+1}, ..., k_{t+N-1}, \tau^k_t, \tau^l_t, g_t, z_t\right)$, equations (15) and (25) still form a system of two equations in two unknowns, (c_t, l_t), which admits a unique solution under the original assumptions. However, note that to satisfy part (i) of **Assumption 3** we need problematic variables up to (c_{t+N-1}, l_{t+N-1})

to be uniquely determined by the state $s_t = (\mathrm{k}_t, \mathrm{k}_{t+1}, ..., \mathrm{k}_{t+N-1}, \mathrm{z}_t, \tau_t^L)$. Thus, by the same logic we need commitment to all of $(\tau_t^k, \tau_t^l, \mathrm{g}_t, ..., \tau_{t+N-1}^k, \tau_{t+N-1}^l, \mathrm{g}_{t+N-1})$ to fully satisfy the assumption. Part (ii) of the proof is left to the reader. Thus, N periods of commitment are sufficient to recover FC in this extension.

B.4 The role of initial conditions

The equivalence between LTC and FC outcomes proved in Section 2 relies on initial policy instruments being consistent with the FC plan. We now discuss the consequences of letting governments with LTC inherit arbitrary initial policies. By **Lemma 2**, we know that under our assumptions, the LTC outcome coincides with the outcome of a FC Ramsey plan restricted to start from the same arbitrary initial policy (the Modified Problem). Hence even if the LTC game is initialised with "incorrect" initial policies, the remaining policies will be chosen optimally in the sense that a FC Ramsey planner restricted to the same initial policies would choose the same sequence. In models where the key distinction between the NC and FC solutions is a difference in the average level of a policy rather than the timing (e.g. capital taxes being high on average under NC in a model with capital) this suggests that much of the benefit of LTC in sustaining FC-type policies will be retained even if initial policies are incorrectly set. However, for specific versions of our general model, we are able to provide a more detailed characterisation of the equilibrium.

B.4.1 Initial conditions in Case 1 (Lucas and Stokey (1983) model)

Consider the Lucas and Stokey (1983) model of Section 3.2. We further simplify the exposition assuming exogenous, constant government spending.[7] For this economy, we have established that, starting from initial conditions given by the FC policy sequence, LTC sustains FC outcomes with a single period of commitment. We now ask what happens if a government in the LTC game inherits an arbitrary initial policy, potentially different from the one implied by the FC policy path.

The FC policy for this model is fully characterised by two functions: $\tau_{0,FC}^l(\mathrm{b}_0)$ for $t = 0$ and $\tau_{FC}^l(\mathrm{b}_0)$ with $\tau_t^l = \tau_{FC}^l(\mathrm{b}_0)$ for all $t \geq 1$. The government chooses a perfectly smooth tax from $t = 1$ onwards and uses the tax rate at $t = 0$ to affect the utility value of initial debt by distorting the initial allocation in order to decrease the amount of tax distortions needed to finance expenditure and service the debt. The rest of the allocation is also constant from $t = 1$ onwards.

Moving on to the LTC game, from the intratemporal optimality condition (20), imposing constant g, we can obtain hours as an implicit function of the tax rate only: $\mathrm{l}_t = l(\tau_t^l)$. Using this function, the government budget constraint in period t can then be expressed as

$$u'(l(\tau_t^l) - \mathrm{g}) \left(\mathrm{b}_t + \mathrm{g} - \tau_t^l l(\tau_t^l)\right) = \beta u'(\mathrm{c}_{t+1})\mathrm{b}_{t+1}. \tag{B.15}$$

Let $\mathrm{a}_t \equiv u'\left(l\left(\tau_t^l\right) - \mathrm{g}\right)\left(\mathrm{b}_t + \mathrm{g} - \tau_t^l l(\tau_t^l)\right)$ and note that this variable is a function only of the inherited level of debt and the current tax rate: $\mathrm{a}_t = a\left(\mathrm{b}_t, \tau_t^l\right)$. The economic interpretation of this variable is the (marginal utility) value of the resources that the time-t

[7] This assumption can be easily relaxed without affecting the main insight.

government needs to raise on the bond market. Let $s\left(\tau_t^l\right) \equiv u'\left(l\left(\tau_t^l\right) - \mathrm{g}\right)\left(\mathrm{g} - \tau_t^l l\left(\tau_t^l\right)\right)$. Adding and subtracting $s\left(\tau_{t+1}^l\right)$ on the right-hand side of (B.15) and rearranging yields

$$\mathrm{a}_{t+1} = \beta^{-1}\mathrm{a}_t - s\left(\tau_{t+1}^l\right). \tag{B.16}$$

Note that the problem of the government at $t = 0$ is affected by $\left(\mathrm{b}_0, \tau_0^l\right)$ only through their effect on a_0. This is because this government cannot affect hours worked at $t = 0$. The solution to the LTC game can be formulated in the following recursive form.[8]

$$W(\mathrm{a}_t) = \max_{\mathrm{a}_{t+1}, \tau_{t+1}^l} \beta\left[u\left(l\left(\tau_{t+1}^l\right) - \mathrm{g}\right) - v\left(l\left(\tau_{t+1}^l\right)\right) + W\left(\mathrm{a}_{t+1}\right)\right] \tag{B.17}$$

subject to the transition (B.16). Note that this recursive form ignores contemporaneous utility, which is in any case fixed from the government's point of view. Consider an LTC game starting from arbitrary initial conditions (b_0, τ_0^l), which imply $\mathrm{a}_0 = u'\left(l\left(\tau_0^l\right) - \mathrm{g}\right)\left(\mathrm{b}_0 + \mathrm{g} - \tau_0^l l(\tau_t^l)\right)$. Starting from time 0 and any a_0, the optimal solution to this problem implies that $a_0 = a_1 = ...$ Since optimal (next-period) taxes are a policy function with a_t as the only state, optimal taxes from time 1 onwards must be constant too, and so the solution inherits a key property of the FC solution. We can find the optimal tax rate from time 1 onwards from (B.16): $\tau^*(\mathrm{a}_0) = s^{-1}\left(\frac{1-\beta}{\beta}\mathrm{a}_0\right)$. The value function satisfies

$$W(\mathrm{a}_0) = \frac{\beta}{1-\beta}\left[u\left(l\left(\tau^*\left(\mathrm{a}_0\right)\right) - \mathrm{g}\right) - v\left(l\left(\tau^*\left(\mathrm{a}_0\right)\right)\right)\right]. \tag{B.18}$$

Time-0 welfare starting from arbitrary initial conditions (b_0, τ_0^l) is then given by

$$V\left(\mathrm{b}_0, \tau_0^l\right) = u\left(l\left(\tau_0^l\right) - \mathrm{g}, l\left(\tau_0^l\right)\right) + W(\mathrm{a}_0). \tag{B.19}$$

We now argue that the LTC policy and allocation starting from $\left(\mathrm{b}_0, \tau_0^l\right)$, with $\tau_0^l \neq \tau_{0,FC}^l\left(\mathrm{b}_0\right)$ converges to another FC policy (and allocation), indexed by a different initial debt level. Let $\tilde{\mathrm{b}}_0$ be the solution to the non-linear equation $a\left(\tilde{\mathrm{b}}_0, \tau_{0,FC}^l\left(\tilde{\mathrm{b}}_0\right)\right) = a\left(\mathrm{b}_0, \tau_0^l\right)$.[9] Then, in the equilibrium of the LTC game starting from $\left(\mathrm{b}_0, \tau_0^l\right)$, we will have $\tau_t^l = \tau^*\left(\mathrm{a}_0\right) = \tau_{FC}^l\left(\tilde{\mathrm{b}}_0\right)$ for all $t \geq 1$. That is, the LTC solution replicates the FC solution for the different initial debt level $\tilde{\mathrm{b}}_0$ for all $t \geq 1$. In order to assess the welfare cost of starting from any initial tax, it is sufficient to compare the value attained by the FC policy starting from $t = 0$ with the value defined in (B.19).

B.4.2 Initial conditions with capital, debt, and linear utility from consumption

Consider the special case given in Section B.2.3. For simplicity also assume that $\mathrm{z}_t = 1$ and that government spending is exogenous and equal to a constant g. We can prove that the equilibrium of the LTC game converges to a different FC solution if a generic

[8]This is an alternative formulation relative to the more general recursive formulation used to prove **Proposition 1**. It holds in this model because the welfare-relevant component of the allocation $(\mathrm{c}_t, \mathrm{l}_t)$ is fixed from the point of view of the government dated t.

[9]Under standard parametrisations of the utility function there is a unique solution.

time-t government inherits "incorrect" policies. Recall that in this model the FC solution for $t > 0$ features zero capital taxes and constant labour taxes.

To prove this, we need to characterise the LTC solution. It is convenient to combine the competitive equilibrium constraints into a single implementability constraint:

$$\frac{1}{\beta}\mathrm{k}_t + \mathrm{b}_t = \mathrm{c}_t - v_{l,t}\mathrm{l}_t + \mathrm{k}_{t+1} + \beta\mathrm{b}_{t+1}. \tag{B.20}$$

This constraint combines all the time-t constraints except for the capital Euler equation, and uses the Euler equation from time $t-1$. Along any LTC path with rational expectations we can use $(\mathrm{b}_t, \mathrm{k}_t, \mathrm{l}_t)$ as the only states since we can use the time-t labour condition, (B.7), and $t-1$ Euler, (B.9) to infer what time-t taxes they imply, instead of holding the taxes as additional states. Since the theorem holds, we can write the LTC problem recursively as

$$W(\mathrm{b}_t, \mathrm{k}_t, \mathrm{l}_t) = \max_{\mathrm{b}_{t+1}, \mathrm{k}_{t+1}, \mathrm{l}_{t+1}} \mathrm{k}_t^{\alpha}\mathrm{l}_t^{1-\alpha} + (1-\delta)\,\mathrm{k}_t - \mathrm{g} - \mathrm{k}_{t+1} - v(\mathrm{l}_t) + \beta W(\mathrm{b}_{t+1}, \mathrm{k}_{t+1}, \mathrm{l}_{t+1}) \tag{B.21}$$

where the maximisation is subject to (B.20). This is true for any LTC path from time $t = 1$, since capital and the capital tax will be consistent (through the $t-1$ Euler) allowing us not to use the taxes as states. At time 0 this is not true, since the inherited initial taxes and capital level might not be consistent with the time-0 capital tax, so the following applies for any $t \geq 1$. Denote by λ_t the multiplier on (B.20), then the capital, bond and labour first order conditions give respectively:

$$1 - \lambda_t = \beta\left(\alpha\mathrm{k}_t^{\alpha-1}\mathrm{l}_t^{1-\alpha} + 1 - \delta\right) - \lambda_{t+1} \tag{B.22}$$

$$\lambda_t = \lambda_{t+1} \tag{B.23}$$

$$(1-\alpha)\mathrm{k}_{t+1}^{\alpha}\mathrm{l}_{t+1}^{-\alpha} - v'(\mathrm{l}_{t+1}) - \lambda_{t+1}(v''(\mathrm{l}_{t+1})\mathrm{l}_{t+1} + v'(\mathrm{l}_{t+1})) = 0. \tag{B.24}$$

We can combine the capital and bond first order conditions to give

$$1 = \beta\left(\alpha\mathrm{k}_t^{\alpha-1}\mathrm{l}_t^{1-\alpha} + 1 - \delta\right). \tag{B.25}$$

This is just the household's capital Euler equation with zero capital taxes. Hence we have shown that, for any time $t \geq 1$, regardless of the initial condition, a government with LTC will always immediately set $\tau_{t+1}^k = 0$. Labour taxes will be constant from period $t+1$ onwards because both the multiplier and capital in (B.24) are constant, implying constant hours, and hence constant labour taxes. Taking $t = 1$, the above logic thus implies that by period 2 the solution will have converged to one with zero capital taxes and constant labour taxes. Since the level of labour taxes must satisfy the intertemporal budget constraint, these taxes must be the long-run solution to a FC game for a different value of the initial state variables.

C Appendix to Sections 4 and 5

In this appendix, we first provide the complete set of optimality conditions (including Generalised Euler Equations) under FC, NC and LTC for the model of Section 4. Next, we study the effects of commitment to different fiscal instruments. Finally, we describe our numerical algorithm to compute the LTC equilibrium of the model of Section 4.

C.1 Competitive equilibrium and LTC game

The competitive equilibrium equations of this model can be compactly stated as

$$\mathrm{c}_t + \mathrm{k}_{t+1} + \mathrm{g}_t = f(\mathrm{k}_t) \tag{C.1}$$

$$u'(\mathrm{c}_t) = \beta u'(\mathrm{c}_{t+1}) \left(f'(\mathrm{k}_{t+1}) - \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}} \right) \tag{C.2}$$

where $f(\mathrm{k}_t) = \mathrm{k}_t^\alpha + (1-\delta)\mathrm{k}_t$, and the capital tax rate rate has been solved out for using the balanced budget constraint, $\tau_t^k (f'(\mathrm{k}_t) - 1) \mathrm{k}_t = \mathrm{g}_t$.

In the model of Section 4, and allowing for the shock to the value of government spending, the natural state variables of the LTC equilibrium with L periods of commitment are $s_t = (\mathrm{k}_t, \xi_t, \tau_t^L)$. τ_t^L is the inherited fiscal plan to be enacted from time t to $t+L-1$, consistent of a sequence of (possibly state contingent) pre-committed values for government spending. In a symmetric equilibrium, all governments play the same policy function, $\underset{\sim}{\mathrm{g}}_{t+L} = g(s_t)$, which is either contingent or not depending on the form of LTC chosen. Consumption is given by an associated policy function $\mathrm{c}_t = c(s_t)$ and the value function of the government is $v(s_t)$. If all future government's play this policy function, the time-t government chooses $\underset{\sim}{\mathrm{g}}_{t+L}$ to maximise

$$u(\mathrm{c}_t) + w(\mathrm{g}_t, \xi_t) + \beta \mathbb{E}_t v(s_{t+1}) \tag{C.3}$$

subject to the pre-committed policies, and the value of c_t implied by all competitive equilibrium equations (C.1) and (C.2) from t onwards. The continuation value is given by

$$v(s_t) = u(c(s_t)) + w(\mathrm{g}_t, \xi_t) + \beta \mathbb{E}_t v(s_{t+1}) \tag{C.4}$$

with s_{t+1} computed by updating s_t with the policy function $g(s_t)$.

C.2 Optimality conditions and GEE

We now state three maximisation problems related to the deterministic model of Section 4, namely the optimal policy with FC, with NC and with LTC. We derive the first order conditions for each of these problems.

C.2.1 Full Commitment

Given k_0, the problem of the government with FC is to choose $\{\mathrm{g}_t, \mathrm{k}_{t+1}\}_{t=0}^\infty$ in order to maximise

$$\sum_{t=0}^{\infty} \beta^t \left[u(f(\mathrm{k}_t) - \mathrm{g}_t - \mathrm{k}_{t+1}) + w(\mathrm{g}_t) \right] \tag{C.5}$$

where we have replaced c_t using (C.1), and subject to (C.2) with attached multiplier γ_t^{FC}. The first order conditions for g_0, g_t (with $t>0$), and k_{t+1} are, respectively,

$$w'(\mathrm{g}_0) - u'(\mathrm{c}_0) - \gamma_0^{FC} u''(\mathrm{c}_0) = 0 \tag{C.6}$$

$$w'(\mathrm{g}_t) - u'(\mathrm{c}_t) - \gamma_t^{FC} u''(\mathrm{c}_t) - \gamma_{t-1}^{FC} \left[-u''(\mathrm{c}_t) \left(f'(\mathrm{k}_t) - \frac{\mathrm{g}_t}{\mathrm{k}_t} \right) - \frac{u'(\mathrm{c}_t)}{\mathrm{k}_t} \right] = 0 \tag{C.7}$$

$$u'(\mathrm{c}_t) = \beta\left[u'(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1}) + \gamma_{t+1}^{FC} u''(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1})\right] +$$
$$-\gamma_t^F \left[u''(\mathrm{c}_t) + \beta\left(u''(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1})\left(f'(k_{t+1}) - \frac{g_{t+1}}{k_{t+1}}\right) + u'(c_{t+1})\left(f''(\mathrm{k}_{t+1}) + \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}^2}\right)\right)\right]. \tag{C.8}$$

C.2.2 Limited-Time Commitment

Without uncertainty, the LTC problem with L periods of commitment can be stated as follows. Given states $(\mathrm{k}_t, \mathrm{g}_t, ..., \mathrm{g}_{t+L-1})$, choose $(\mathrm{g}_{t+L}, \mathrm{k}_{t+1})$ to maximise

$$u(f(\mathrm{k}_t) - \mathrm{g}_t - \mathrm{k}_{t+1}) + w(\mathrm{g}_t) + \beta v(\mathrm{k}_{t+1}, \mathrm{g}_{t+1}, ..., \mathrm{g}_{t+L}) \tag{C.9}$$

subject to

$$u'(f(\mathrm{k}_t) - \mathrm{g}_t - \mathrm{k}_{t+1}) = \beta u'(c(\mathrm{k}_{t+1}, \mathrm{g}_{t+1}, ..., \mathrm{g}_{t+L}))\left[f'(\mathrm{k}_{t+1}) - \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}}\right] \tag{C.10}$$

with attached multiplier γ_t, and with continuation value given by

$$v(\mathrm{k}_t, \mathrm{g}_t, ..., \mathrm{g}_{t+L-1}) = u(c(\mathrm{k}_t, \mathrm{g}_t, ..., \mathrm{g}_{t+L-1})) + w(\mathrm{g}_t) + \beta v(\mathrm{k}_{t+1}, \mathrm{g}_{t+1}, ..., \mathrm{g}_{t+L}). \tag{C.11}$$

Taking the first order condition for g_{t+L}, and using the envelope condition to substitute out $v_{\mathrm{g}_{t+L}}(\mathrm{k}_{t+1}, \mathrm{g}_{t+1}, ..., \mathrm{g}_{t+L})$, we can derive a Generalised Euler Equation:

$$w'(\mathrm{g}_{t+L}) - u'(\mathrm{c}_{t+L})\left(1 - \frac{\gamma_{t+L-1}}{\mathrm{k}_{t+L}}\right) - \gamma_{t+L} u''(\mathrm{c}_{t+L}) =$$
$$\sum_{j=0}^{L-1} \beta^{j-L+1} \gamma_{t+j} u''(\mathrm{c}_{t+j+1})\left(f'(\mathrm{k}_{t+j+1}) - \frac{\mathrm{g}_{t+j+1}}{\mathrm{k}_{t+j+1}}\right)\frac{\partial \mathrm{c}_{t+j+1}}{\partial g_{t+L}}. \tag{C.12}$$

Similarly, we can express the first order condition for k_{t+1} as a second Generalised Euler Equation:

$$u'(\mathrm{c}_t) = \beta\left(u'(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1}) + \gamma_{t+1} u''(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1})\right) - ...$$
$$-\gamma_t \left[u''(\mathrm{c}_t) + \beta\left(u''(\mathrm{c}_{t+1})\left(f'(\mathrm{k}_{t+1}) - \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}}\right)\mathrm{c}_{k,t+1} + u'(\mathrm{c}_{t+1})\left(f''(\mathrm{k}_{t+1}) + \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}^2}\right)\right)\right]. \tag{C.13}$$

C.2.3 No Commitment

No commitment is the special case of LTC with $L = 0$ periods of commitment. The GEE for government spending, g_t now reduces to the simpler condition

$$w'(\mathrm{g}_t) = u'(\mathrm{c}_t) + \gamma_t^{NC} u''(\mathrm{c}_t). \tag{C.14}$$

The GEE for capital, k_{t+1} is similar to that under LTC with $L \geq 1$, but now additionally features a derivative of the government spending policy function:

$$u'(\mathrm{c}_t) = \beta\left(u'(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1}) + \gamma_{t+1}^{NC} u''(\mathrm{c}_{t+1}) f'(\mathrm{k}_{t+1})\right) - \gamma_t^{NC}\Bigg[u''(\mathrm{c}_t) + ...$$
$$+\beta\left(u''(\mathrm{c}_{t+1})\left(f'(\mathrm{k}_{t+1}) - \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}}\right) c_{k,t+1}^{NC} + u'(\mathrm{c}_{t+1})\left(f''(\mathrm{k}_{t+1}) + \frac{\mathrm{g}_{t+1}}{\mathrm{k}_{t+1}^2} - \frac{g_{k,t+1}^{NC}}{\mathrm{k}_{t+1}}\right)\right)\Bigg]. \tag{C.15}$$

C.3 Commitment to taxes or spending?

In the model described in Section 4, there is no difference between committing to future taxes or future spending: the government balanced-budget constraint pins down spending given the tax rate and vice versa, because the tax base depends on the capital stock only, which is a pre-determined state variable.

In the real world, we sometimes observe different institutional arrangements for the determination of tax rates and public spending. For instance, there is often a component of government spending that is discretionary and can potentially be changed relative to previous plans without incurring the institutional costs and delays that are typically associated with a reform of the tax code.

In order to introduce a distinction between commitment to taxes and to spending in our model, we now extend the model to allow for endogenous labour supply. This modification of the model implies that the tax base (capital income) is not predetermined, as it depends on hours worked, which respond endogenously to contemporaneous changes in fiscal policy. Hence, we can consider commitment to taxes or spending separately.

We replace the period utility function with $\log(\mathrm{c}_t) + B\log(1 - \mathrm{l}_t) + D\log(\mathrm{g}_t)$, where l_t are hours worked. The government balanced-budget constraint becomes

$$\tau_t^k\left(\alpha \mathrm{k}_t^\alpha \mathrm{l}_t^{1-\alpha} - \delta \mathrm{k}_t\right) = \mathrm{g}_t \tag{C.16}$$

and households set the marginal rate of substitution between leisure and consumption equal to the wage (ie, the marginal product of labour, using firms' profit maximisation):

$$\frac{B\mathrm{c}_t}{1 - \mathrm{l}_t} = (1 - \alpha)\, \mathrm{k}_t^\alpha \mathrm{l}_t^{-\alpha}. \tag{C.17}$$

We solve this model under two alternative specifications of LTC. In the first case, which we denote $LTC_{\mathrm{g}'}(1)$, the government in power at time t takes as given the state vector $(\mathrm{k}_t, \mathrm{g}_t)$ and chooses the tax rate that applies at time t, τ_t^k, and the level of spending that applies at time $t+1$, g_{t+1}. In the second case, which we denote $LTC_{\tau'}(1)$, the government in power at time t takes as given the state vector (k_t, τ_t) and chooses the level of spending that applies at time t, g_t, and the tax rate that applies at time $t+1$, τ_{t+1}^k.

In Table E.2 we compare the steady-state results for these two alternative institutional arrangements with the results obtained under FC.[10] Interestingly, we find that

[10]These are also the results if the government has one period of commitment to both taxes and spending, as our equivalence result applies in this version with endogenous labour supply.

commitment to future taxes lead to lower taxes and spending, inducing a higher level of output and consumption, with a similar amount of hours worked. Overall, the welfare loss of $LTC_{\tau'}(1)$ relative to FC equals 3.9% (in terms of permanent consumption), while $LTC_{\mathrm{g}'}(1)$ induces a larger welfare loss, equal to 5%. To our knowledge, this is the first result in the literature on fiscal policy that could provide support for differential degrees of commitment for taxes and spending. Relatedly, Klein and Ríos-Rull (2003) consider the trade-off between commitment to labour or capital income taxes. Based on these result, we think that LTC is a promising framework to develop a positive theory of the timing of different fiscal instruments.

C.4 Algorithm for LTC

We now describe the key steps of the algorithm we use to compute the LTC equilibrium of the model of Section 4. We refer to the model with shocks and non-contingent LTC with $L = 1$, and briefly explain the differences for the other cases below. We solve the model by approximating the policy functions using projection methods with Chebyshev polynomials of the state variables of the problem, namely capital k, government expenditure g and the shock ξ.

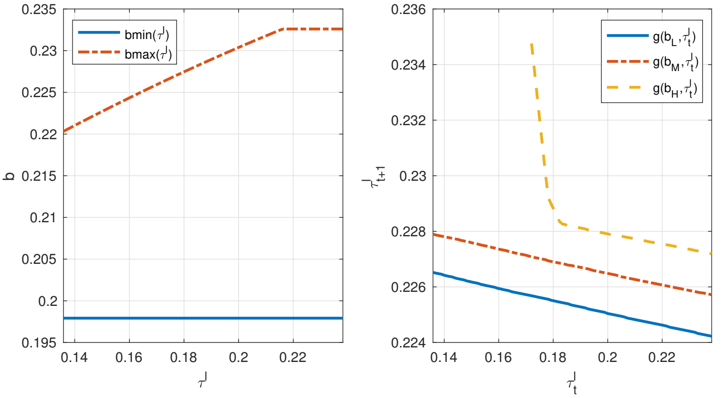
1. We discretise the sets of k, g with Chebyshev nodes. As ξ takes two values, all polynomials mentioned below have two different sets of coefficients on k and g, depending on the realization of ξ.
2. We set a Chebyshev polynomial order S (in our case, $S = 3$) and guess a future policy function $g \approx P(\mathrm{k}, \mathrm{g}, \xi; \phi_g)$, where ϕ_g are the coefficients of the polynomial.
3. We solve the households' consumption-saving problem by iterating on the Euler equation (C.2) and approximating the consumption function $c \approx P(\mathrm{k}, \mathrm{g}, \xi; \phi_c)$, where ϕ_c are the coefficients of the polynomial.
4. We approximate the continuation value function on a grid by iterating on (C.4).
5. We solve the maximisation problem of government t on a grid for $(\mathrm{k}, \mathrm{g}, \xi)$, choosing g' on a fine grid. Next, we approximate the associated decision rule with a polynomial with coefficients ϕ'_g
6. We update the guess for g as follows: $\phi_g = \psi \phi'_g + (1 - \psi)\phi_g$ for some $\psi \in (0, 1]$. We iterate on these steps until convergence of all policy functions.

For the contingent LTC model, we follow the same steps, with the exception that the future policy g' is also a function of the future shock realisation ξ' (the policy function is now formed of two sets of polynomials, one for each ξ'). The deterministic model with $L = 1$ is a special case with constant ξ. For the deterministic model with multiple periods of commitment $L > 1$, we follow the same steps (with a suitably defined state vector that includes all pre-committed policies), with the exception that the consumption policy function is obtained by projection on all the Euler equations between time t and time $t+L$. We jointly minimize the residuals of these private sector optimality conditions and thus obtain current consumption as a function of the state and the announced policy to be implemented at $t + L$.

We verify that higher order polynomials produce quantitatively negligible differences in the solution. We also compute the FC and NC versions of the model using projection with third-order Chebyshev polynomials. For the NC model, we adapt the method described above: the state variables are (k, ξ) and the government maximisation is over g. Klein et al. (2008) solve the NC version of the model numerically exploiting the Generalised Euler Equation (GEE). In the interest of comparison, we also verify that a projection method that uses the GEE for NC produces negligible differences relative to our method. Finally, we check the accuracy of our results for the deterministic LTC model by evaluating the LTC GEE residuals using our solution and verifying that they are sufficiently small.

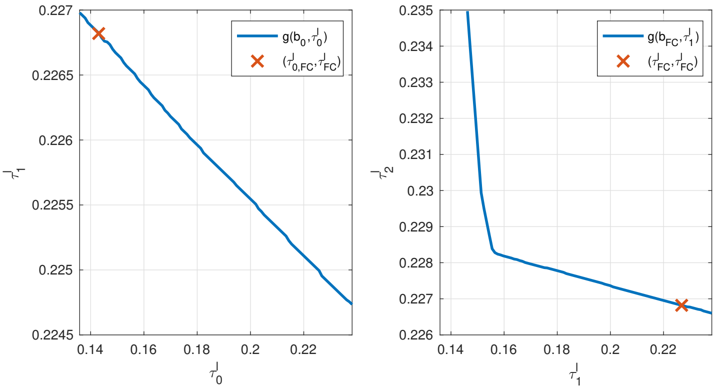
D Additional figures

Figure D.1: *Feasible set and LTC policy function in the LS economy*



Left panel represents the feasible set S via upper and lower limits $bmin(\tau^l)$ and $bmax(\tau^l)$. Right panel plots slices from the policy function $\tau^l_{t+1} = g(\mathrm{b}_t, \tau^l_t)$ at given values of b_t, where τ^l_t and τ^l_{t+1} are represented on the x- and y-axes respectively. b_L, b_M, and b_H represent low, medium, and high debt levels corresponding to 20% above the bottom of the grid for b, 50% above, and 80% above.

Figure D.2: *LTC and FC in the LS economy*



Lines give LTC policy function $\tau^l_{t+1} = g(\mathrm{b}_t, \tau^l_t)$ at a given value of b_t, where τ^l_t and τ^l_{t+1} are represented on the x- and y-axes respectively. Crosses denote the t and $t+1$ values of the optimal FC plan. The left panel plots the $t = 0$ problems, showing the LTC plicy function for state b_0 and the time-0 and 1 FC taxes. The right panel plots the $t > 0$ problems, showing the LTC plicy function for state b_{FC} and the time-1 and 2 FC taxes, which are both equal to τ^l_{FC}.

E Additional tables

Table E.1: *Parameter values*

Parameter	Interpretation	Value
β	discount factor	0.96
ξ	public good utility	0.5
α	capital share	0.36
δ	depreciation	0.08
$p_{HH} = p_{LL}$	transition prob. for ξ_t	0.974
ξ^H	high realization of ξ_t	0.556
ξ^L	low realization of ξ_t	0.444
B	utility from leisure	2.33

Parameter values for the model in Section 4. The first block of the table displays the parameters used in the baseline analysis of the deterministic model. The second block displays the additional parameters used in the stochastic model. The third block displays the additional parameters used in the model with endogenous labour. We follow the yearly calibration in Klein et al. (2008) for the deterministic version of the model, both with exogenous and with endogenous labor. In the stochastic version, we parameterise the process for the shock ξ_t in order to match the volatility and persistence of the ratio between public consumption and private consumption in US data 1960:2017, as described in the text.

Table E.2: *Deterministic model with endogenous labour: steady-state comparison*

Variable	FC	$LTC_{g'}(1)$	$LTC_{\tau'}(1)$
y	1	0.907	0.920
k/y	1.540	1.203	1.244
c/y	0.731	0.705	0.708
g/c	0.200	0.283	0.273
l	0.281	0.289	0.288
τ	0.694	0.840	0.825
welfare loss	–	0.050	0.039

Steady-state results for the model with endogenous labour supply described in Appendix C.3. We consider three versions of the economy: FC, LTC with commitment for one year only to future spending ("$LTC_{g'}(1)$"), and LTC with commitment for one year only to future taxes ("$LTC_{\tau'}(1)$"). We report steady-state output, capital-output ratio, private consumption-output ratio, public consumption-private consumption ratio, hours worked, tax rate and welfare loss, measured as the fraction of permanent consumption that would make the representative household indifferent between the economy considered and the FC economy.