Hyperchaos & labyrinth chaos: revisiting Thomas-Rössler systems

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Abstract

We consider a multidimensional extension of Thomas-Rössler systems, that was inspired by René Thomas' earlier work on biological feedback circuits, and we report on our first results that shows its ability to sustain spatio-temporal behaviour reminiscent of chimera states. The novelty here is that its underlying mechanism is based on "chaotic walks" discovered by René Thomas during the course of his investigations on what he called Labyrinth Chaos. We briefly review the main properties of these systems and their chaotic and hyperchaotic dynamics and discuss the simplest way of coupling, necessary for this spatio-temporal behaviour that allows the emergence of complex dynamical behaviours. We also recall René Thomas' memorable influence and interaction with the authors as we dedicate this work to his memory.

1. Introduction

During the last part of René Thomas' brilliant scientific life, we had the opportunity to collaborate with him working on a class of models that he and, his good friend and equally brilliant scientist, Professor Otto E. Rössler (of the "Rösler attractor" fame) had proposed in the course of their investigations on the fundamentals of chaotic dynamics [1, 2].

Our interaction started by René, "naively" asking questions and seeking for assistance on tricks and tips for his favourite computational platform. He was always presenting the most profound and fundamental questions related

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to his work, as if it was just his joyful occupation, now that he had retired, as he was putting it. You can imagine our enthusiasm when he asked if we could assist him and his friend Otto, with certain issues they were pondering at the time. From the knowledge we acquired from both, we now know, that it is one of those experiences most cherished! Besides René's vast scientific knowledge and contributions, which this volume celebrates, our every-day interaction was a wonderful intellectual journey. Along with his scientific discourse of the highest quality (telling us about his circuits and loops, the necessary conditions of chaos; the utility of graph theory in studying dynamics much before the explosion of the work on network-dynamics; the importance of logic that underlies dynamics), René would interweave the plethora of his intellectual passions. We learned from him about the contributions of amateur astronomers like himself; when things were gloomy he could always encourage us with stories and metaphors from his past climbing expeditions and activism. Of course, he would never cease to mention music even when he, so generously, shared his thoughts and work on some of his favourite toy-models, such as "Labyrinth chaos" and the "Arabesques" [1, 3]. By the way, we also learned that Haydn, his favourite music composer, had a piece on chaos, called "Die Vorstellung des Chaos"! Among all these, we naturally came to consider him as an epitome of the benefits of basic research. Always driven by his intellectual curiosity, René opened avenues that nobody else could see them opened; he had such a great fun doing it! These are fond memories indeed, and will always stay with us to guide us in our future scientific endeavours.

Going now back to René's favourite toy-models: The purpose of this paper is to revisit and propose new directions of research that stem from his seminal investigations on hyperchaos [1, 2, 3].

Within the framework developed by him, M. Kaufman, D. Thieffry and coworkers [4, 6, 7, 5], the dynamical basis of regulatory networks, cell differentiation, multistationarity, homeostasis and memory can be analysed and understood by studying theoretical models based on feedback circuits. The conceptual and analytic tools therein were also extended and can be used in the study of emergence of complex behaviour from simple circuit structures [8], not only in systems pertaining to biological models per se. One of the seminal contributions of René's work is his proposition of general rules in the dynamics of systems. In more details, (i) a positive circuit is necessary to display multiple stable states, and (ii) a negative circuit is necessary to have robust sustained oscillations. Later on, René proposed a necessary condition for chaos and suggested that both a positive and negative circuit are needed to generate deterministic chaotic behaviour.

René and, O. E. Rössler and coworkers [1], have shown further that for dimensions $D \geq 4$, the same simple logical structure of more than one positive and at least one negative circuit can generate "hyperchaos" of arbitrary order m, i.e., chaotic behaviour characterised by more than m positive Lyapunov exponents. As well known, among René's contributions to the theory of dynamical systems (for details see also references within this volume), are "feedback circuits" or simply "circuits", defined as sets of nonzero terms of the Jacobian matrix (linearisation) of the dynamical system such that their row and column indices can form cyclic permutations of each other. In that sense, hyperchaos of order m requires the existence of m positive circuits and at least one negative. What an elegant result!

Moreover, René and O. E. Rössler considered the generation of complex symmetric attractors (which they termed "labyrinth chaos")[1, 2] and its peculiar special case of a chaotic phase-space where a countable-infinite set of unstable fixed points with no attractors is generated by a set of $m \geq 3$ first order differential equations. Based on such systems, René then created a class of conservative systems which he termed "Arabesques" [3].

The discovery and subsequent definition of hyperchaos [10, 9] is due to the seminal work of O. E. Rössler, and describes a more flexible type of chaotic behaviour where the sensitive dependence on initial conditions coexists in more than one directions. In other words, there are more than one positive Lyapunov exponents in the dynamics of such systems. Hyperchaos has been studied initially in experimental works on systems with coupled lasers [11], on Navier-Stokes equations [12] to recent studies on large arrays of coupled oscillators [13]. Surprisingly, the higher dimensional chaotic motion is more amenable to control and synchronisation and quite evidently empowers the design of controllers in circuit theory and applications [14]. It also has good utility in studies in biological modelling in areas such as Bioinformatics [15], biorythms and chronotherapy [20], and neural dynamics [21].

In Sec. 2, we revisit and review the system of ordinary differential equations for hyperchaos introduced initially by René and O. E. Rössler [1, 2]. Since they called this type of dynamics "labyrinth chaos", we shall use the term Thomas-Rössler (TR) systems, as it has been proposed in the literature [18]. In Sec. 3, we extent the original investigations by considering first linearly coupled systems of two and then, of a larger number of TR systems arranged in networks in a circle with nearest-neighbour interactions. In Sec. 4, we conclude and discuss briefly possible future research based on these investigations and provide an outlook of the fruitful continuation of this direction of research that was initiated by René [1, 3].

2. Labyrinth Chaos & Hyperchaos

In [1], René, O. E. Rössler and coworkers proposed a system of coupled ordinary differential equations to elucidate, in terms of feedback circuits, the necessary conditions for chaotic and hyperchaotic motion. Particularly, they considered the equations

$$\frac{dx_i}{dt} = -bx_i + f(x_{i+1}), \qquad i = 1 \dots n \pmod n, \tag{1}$$

where $0 < b < 1$. The circuits of this system can be positive or negative depending on the location in the phase space. This constitutes an "ambiguous" circuit, in the terminology established by René [5, 6, 7]. As it had already been shown, under proper conditions, a single circuit might be sufficient to generate chaotic dynamics. René and O. E. Rössler confirmed and generalised this proposition, and showed that hyperchaos of order $m > 1$ can be generated by a single ambiguous circuit of dimension $2m$. The function f was taken to be nonlinear. In this case, either $f(u) = u^3 - u$ or $f(u) = \sin(u)$ and, used $n = 3$ for chaos and $n = 5$ for hyperchaos. René termed a special class of such systems, with $b = 0$ and f assuming different forms, as "Arabesques" (for a detailed discussion, see [3]).

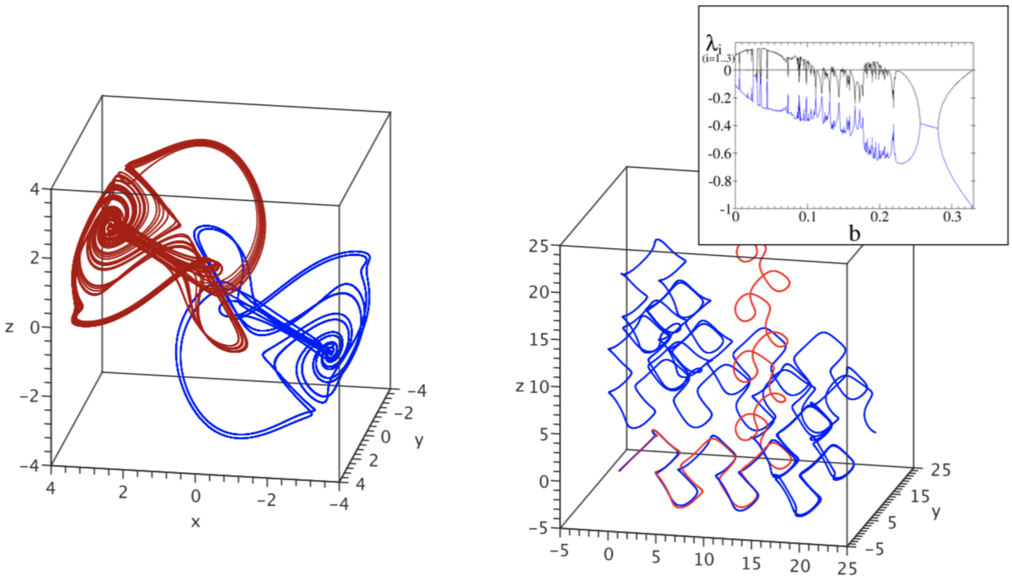
Let us focus now on the case where $f(u) = \sin(u)$ and $n = 3$. The 3-dimensional version of the system then reads

$$\begin{aligned} \frac{dx}{dt} &= -bx + \sin(y), \\ \frac{dy}{dt} &= -by + \sin(z), \\ \frac{dz}{dt} &= -bz + \sin(x), \end{aligned} \tag{2}$$

and its Jacobian is given by

$$\mathbf{J} = \begin{bmatrix} -b & \cos(y_1) & 0 \\ 0 & -b & \cos(z_1) \\ \cos(x_1) & 0 & -b \end{bmatrix}. \tag{3}$$

System (2) exhibits a rich repertoire of dynamical behaviours for different $b > 0$ values [1]. For example, for $b = 0.18$ there is a single chaotic attractor and for $b = 0.19$, a complicated but stable periodic orbit. More complex dynamical regimes can also appear. For example, for $b = 0.2$, the system possesses two coexisting chaotic attractors, which are shown in Fig. 1(a). In the special case where $b = 0$, quite an exotic chaotic behaviour appears, without any attractors. In this case, system (2) is conservative with an infinite lattice of unstable fixed points. The behaviour shown in Fig. 1(b) is what René termed "chaotic walks in labyrinth chaos".



(a)

(b)

Figure 1: Chaotic attractors and labyrinth chaos in the 3-dimensional version of the system of Eq. (1). (a) Two coexisting attractors for $b = 0.2$. (b) "Chaotic walks and labyrinth chaos" for $b = 0$ for which the system is conservative. The red and blue trajectories exhibit sensitive dependence on initial conditions and diverge exponentially in time. In the inlet, the 3 Lyapunov exponents as a function of $0 \leq b < 1/3$ are shown (see [1]).

Similar behaviour appears for $n = 5$ [1] as well. A detailed study of this system was undertaken in [16] and it is presented as a prototype of chaos in [17]. As it is noted in [16], "Despite its mathematical simplicity, this system of ordinary differential equations produces a surprisingly rich dynamic behaviour that can serve as a prototype for chaos studies". It is also noted that in the case of chaotic walks, the approach of an ensemble of initial conditions to equilibrium is by way of fractional Brownian motion with a Hurst exponent approximately equal to 0.61 and a slightly leptokurtic distribution. To the best of our knowledge, this might be the only example of a simple system that links fractional-Brownian motion to nonlinear feedback!

Last but not least, let us note that the simple, underlying structure of the TR system provides for the equally simple and elegant form of the characteristic equation of its Jacobian matrix $\mathbf{J}$. In this case, the $n \times n$ Jacobian of Eq. (1), for a sinusoidal nonlinearity (i.e. for $f(u) = \sin(u)$), reads

$$\det(\mathbf{J} - e_i \mathbf{I}) = (-1)^n \left[(b - e_i)^n - \prod_{i=1}^{n} \cos(x_i) \cos(y_i) \cos(z_i) \right], \qquad (4)$$

where $e_i,\ i = 1, \ldots, n$ are the n eigenvalues of $\mathbf{J}$. Since only those terms in $\mathbf{J}$ that belong to one or more circuits are represented in the characteristic equation and hence, take part in the calculation of the eigenvalues, e_i, it bears significant effect on the calculation of the Lyapunov exponents, λ_i, of the system.

3. Revisiting the TR class of systems

3.1. Two linearly coupled 3-dimensional TR systems

Motivated by the above discussion, we turn here to the study of the effect the simplest *linear* coupling has to two 3-dimensional TR systems, (i.e. $N = 2$, where N denotes the number of coupled systems). As we shall see below, the linearly coupled 6-dimensional system can produce hyperchaotic behaviour. Yet, this is due to a different underlying logic of its feedback circuits.

Obviously, the simplest way to couple two such 3-dimensional systems is by means of a linear coupling involving their x variables. This is considering two copies of Eq. (2)

$$\begin{aligned}
\frac{dx_{1,2}}{dt} &= -b_{1,2} x_{1,2} + \sin(y_{1,2}) + \frac{d}{2}(x_{1,2} - x_{2,1}), \\
\frac{dy_{1,2}}{dt} &= -b_{1,2} y_{1,2} + \sin(z_{1,2}), \\
\frac{dz_{1,2}}{dt} &= -b_{1,2} z_{1,2} + \sin(x_{1,2}),
\end{aligned} \qquad (5)$$

coupled with the last term in the first and fourth equations where $d \geq 0$ and $b_{1,2} \geq 0$. The coupling has a direct effect on the structure of the Jacobian $\mathbf{J}$ of the coupled system. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are the 3×3 Jacobian matrices of the two 3-dimensional copies, then the Jacobian $\mathbf{J}_d$ of the coupled system of Eq. (5) is the 6×6 matrix

$$\mathbf{J}_d = \begin{bmatrix} \mathbf{J}_1 & \mathbf{D} \\ \mathbf{D} & \mathbf{J}_2 \end{bmatrix}, \text{ where } \mathbf{D} = \begin{bmatrix} -\frac{1}{2}d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (6)$$

The characteristic equation of $\mathbf{J}_d$ then reads

$$\det\left(\mathbf{J}_c - e_i\mathbf{I}\right) = \det(\mathbf{J}_2 - e_i\mathbf{I}) \cdot \det((\mathbf{J}_1 - e_i\mathbf{I}) - \mathbf{D}\mathbf{J}_2^{-1}\mathbf{D}) \tag{7}$$

using the Shur complement for the blocks of matrix $\mathbf{J}_c$, where $\mathbf{I}$ is the 6×6 identity matrix. Evidently, Eq. (6) cannot be reduced via Eq. (4) for any value of N. Even setting $b_1 = b_2 = b$, the simplification is insignificant as in Eq. (6) we encounter all terms of $b_{1,2}, e_i, d$, and all their powers combined up to order N, as well as the products of the cosines of all variables in an irreducible manner.

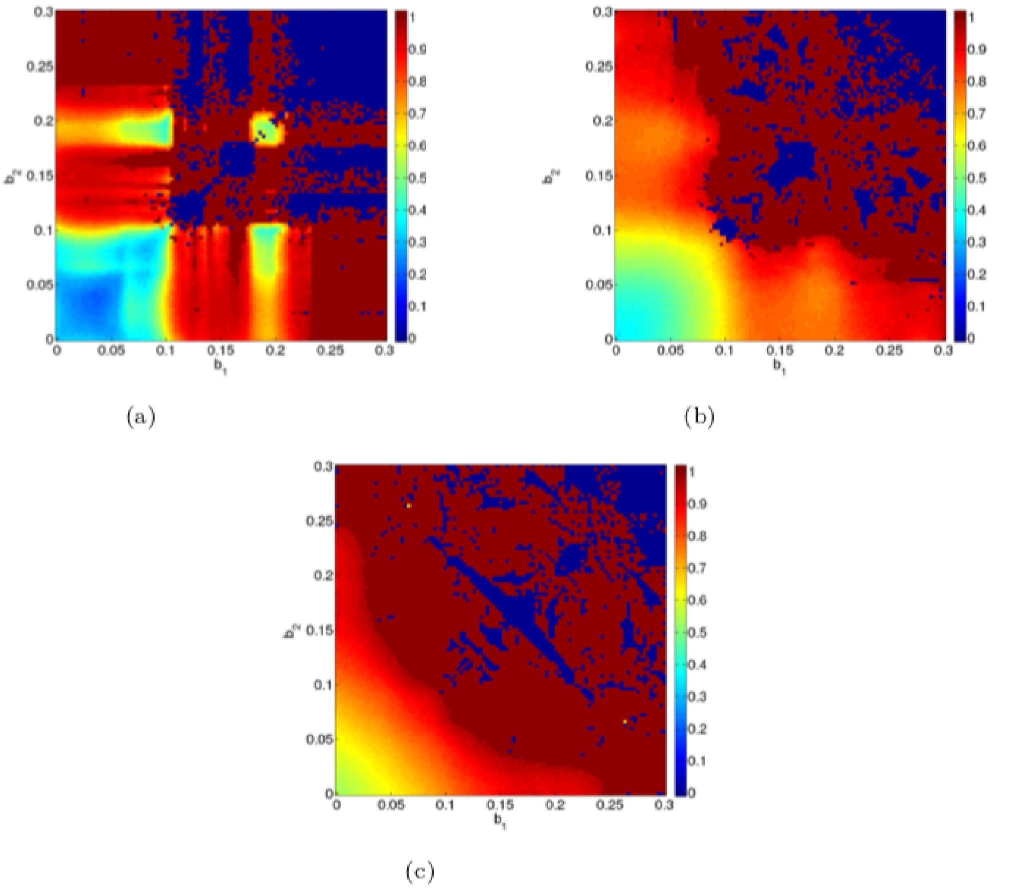


Figure 2: The parameter space (b_1, b_2) for two linearly coupled 3-dimensional TR systems of Eq. (5). The colour-code denotes the relative difference $\Delta\lambda$ (see text) between the first two positive Lyapunov exponents λ_1 and λ_2. Where these exponents are negative, they are assigned the value zero (depicted by dark blue). Hyperchaotic regions are depicted by the colours that correspond to $\Delta\lambda$ between 0 and 1. Light blue: weaker hyperchaos and green to red: stronger hyperchaos. The coupling values were chosen as follows: (a) $d = 0.01$; (b) $d = 0.1$; (c) $d = 0.3$.

As well known, the computation of the Lyapunov exponents, λ_i, of a flow is based on the evaluation of an infinite, or sufficiently large for numerical purposes, product of their Jacobian matrices along their trajectories [18] or equivalently, for a sufficiently long integration time [22]. Figure 2 shows an estimation of the relative difference $\Delta\lambda = \frac{\lambda_1-\lambda_2}{\lambda_1}$ of the first two largest Lyapunov exponents λ_1, λ_2 of Eq. (5). One can see that as d increases (from panel (a) to (c)), the dark blue region where the Lyapunov exponents are non-positive, recedes. This means, the larger the coupling strength, the larger the hyperchaotic region in the parameter space (b_1, b_2) as seen in Fig. (2).

3.2. Can N linearly coupled 3-dimensional TR systems support chimera-like states?

A remarkable, novel discovery in the area of nonlinear dynamics and chaos was made by Kuramoto and Battogtokh in 2002 [23] when they discovered the coexistence of coherent and incoherent behaviour in populations of non-locally coupled phase oscillators. As remarkable as counter intuitive it might sound, since then, their seminal work has triggered a fascinating interest for an ever growing research community witnessed by a rapid growth of publications, spanning the areas of physics, biology and mathematics, among others. As such states were later termed "chimera states" [26], this new synchronisation phenomenon still lacks a complete and rigorous mathematical definition. However, chimera states can be defined as spatio-temporal patterns in networks of coupled oscillators in which synchronous and asynchronous oscillations coexist. This state of broken symmetry, which usually coexists with a stable spatially symmetric state, has intrigued the nonlinear dynamics community since its discovery in the early 2000 [23]. Nevertheless, recent experiments and its relevance to biological networks keeps an unceasing interest in the origin and dynamics of such states (see for example [29, 19, 25, 30] and references therein). For a recent review on chimera states we refer the reader to [27].

Chimera states as phenomena of spatio-temporal patterns in networks of coupled oscillators have apparently the following generic characteristics: (i) robust but varying coherent-incoherent patterns in both space and time (spatio-temporal patterns in which phase-locked oscillators coexist with drifting ones) and (ii) broken symmetry coexisting with a stable spatially symmetric state that depends on initial conditions as well as on the parameters of the system. So far, perfect or imperfect [24] chimera states have not been detected for just local [28] or global coupling but are typical of the intermediate case: a nonlocal coupling comprising of a significant number of nearest neighbours.

In this work, we present preliminary results for N linearly coupled TR systems (see Eq. (2)) which are the generalisation of system (5), that provide evidence they can exhibit spatio-temporal phenomena of coherent and incoherent patterns that alternate dynamically in time, reminiscent of chimera states in networks of non-locally coupled oscillators [23, 26].

In particular, the system of N linearly coupled TR systems is given by

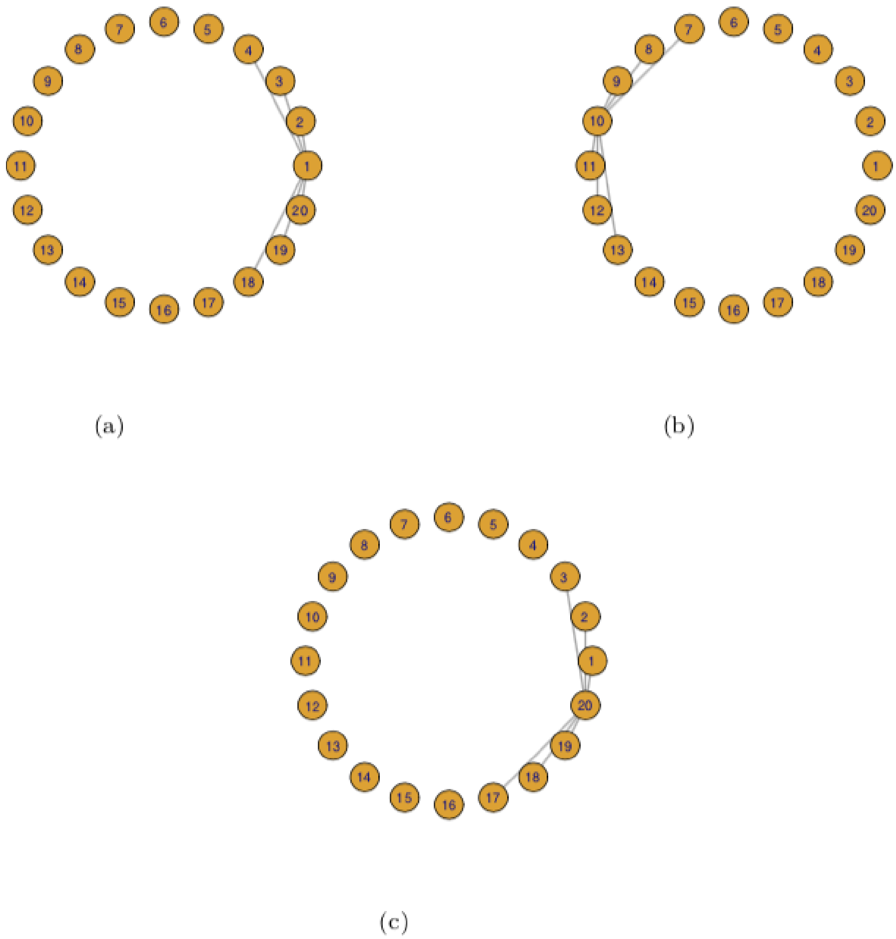


Figure 3: Three examples of a network topology with $N = 20$ nodes arranged in a circle where each node is connected with $P = 3$ nearest-neighbours in either side of the node. For illustration purposes, we only show the 6 nearest neighbours of node 1 in (a), of node 10 in (b) and of node 20 in (c). All other nodes are similarly connected with their 6 nearest neighbours. Notice the periodic boundary conditions for the 1st (panel (a)) and 20th nodes (panel (c)).

$$
\begin{aligned}
\frac{dx_k}{dt} &= -b_k x_k + \sin(y_k) + \frac{d}{2P}\sum_{j=k-P}^{k+P}(x_k - x_j),\\
\frac{dy_k}{dt} &= -b_k y_k + \sin(z_k), \qquad (8)\\
\frac{dz_k}{dt} &= -b_k z_k + \sin(x_k),
\end{aligned}
$$

where $k = 1 \ldots N$, P is the number of nearest neighbours in either side of node k, $d \geq 0$ is the strength of the linear coupling and $b_k \geq 0$ for all k. The term $1/(2P)$ is a normalisation constant for the linear coupling among the N 3-dimensional TR systems and the connectivity can be visualised by a similar network as in Fig. 3.

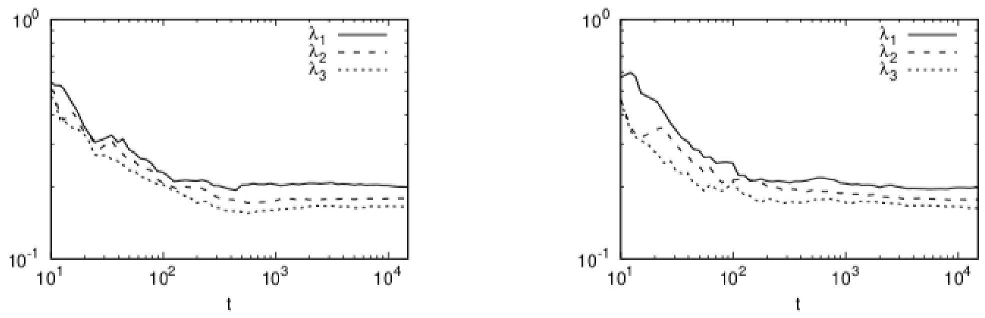
Our studies and numerical simulations have shown that, depending on b_k, d and initial conditions, spatio-temporal phenomena of coherent and incoherent behaviour that alternate dynamically in time, reminiscent of chimera states observed in networks of non-locally coupled oscillators [23, 26, 27], can be seen for P close or equal to $2N$ (i.e. for near-global to global network coupling). For this reason, we have decided to focus in this work on the case where the nodes are globally connected, i.e. for $P = N/2$.

In the next, we study two interesting cases where: (a) the first half of the TR systems are conservative, exhibiting labyrinth chaos (i.e. $b_k = 0$ for $k = 1, \ldots, N/2$) and hyperchaos (i.e. $b_k = 0.18$ for $k = (N/2) + 1, \ldots, N$) when uncoupled and (b) the first half are conservative again and the rest half exhibit complex periodic oscillations (i.e. $b_k = 0.19$ for $k = (N/2)+1, \ldots, N$) when uncoupled. These specific values for b_k where taken from [1].

In both cases, when the systems are coupled together and run for sufficiently long integration times, they exhibit hyperchaotic behaviour as manifested by the convergence of more than one Lyapunov exponents to positive, non-zero, values. We show this in Fig. 4, where we plot the first 3 Lyapunov exponents as a function of time (final integration time is 1.5×10^4) for a system of $N = 40$ TR 3-dimensional systems with $P = 20$ (i.e. global coupling). In both cases, these Lyapunov exponents show a clear tendency to converge to positive, non-zero values.

3.3. Chimera states in a multidimensional TR system with labyrinth chaos and hyperchaos

Here, we focus on the first case where the system in Eq. (8) exhibits labyrinth chaos for half of the 3-dimensional TR systems and hyperchaos for the rest half. In particular, we set $N = 40$, $P = 20$, $d = 0.6$, $b_k = 0$ for



(a)

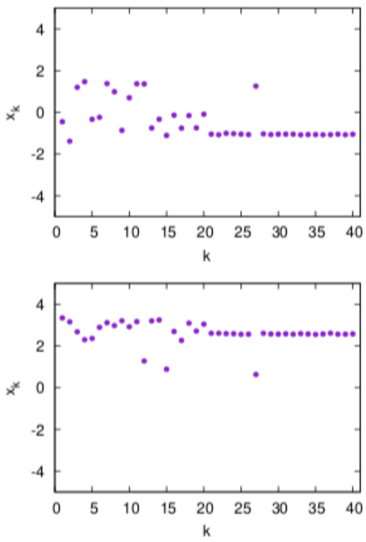
(b)

Figure 4: Hyperchaotic behaviour in a system of 40 linearly coupled 3-dimensional TR systems. The time-evolution of the first 3 largest Lyapunov exponents λ_1, λ_2 and λ_3 for a network of $N = 40$ 3-dimensional linearly coupled TR systems as in Eq. (8), where each node is connected to its $P = 20$ nearest neighbours, in either side of each node. The system resides in a hyperchaotic regime in both cases as more than one Lyapunov exponents are non-zero and positive. In (a) $b_k = 0$ for $k = 1, \ldots, (N/2)$ and $b_k = 0.18$ for $k = (N/2) + 1, \ldots, N$ (labyrinth chaos and hyperchaos) and in (b) $b_k = 0$ for $k = 1, \ldots, (N/2)$ and $b_k = 0.19$ for $k = (N/2)+1, \ldots, N$ (labyrinth chaos and complex periodic oscillations). The final integration time is $t = 1.5 \times 10^4$ at which convergence of the Lyapunov exponents to positive, non-zero, values is observed. Note that in both panels, $d = 0.6$.

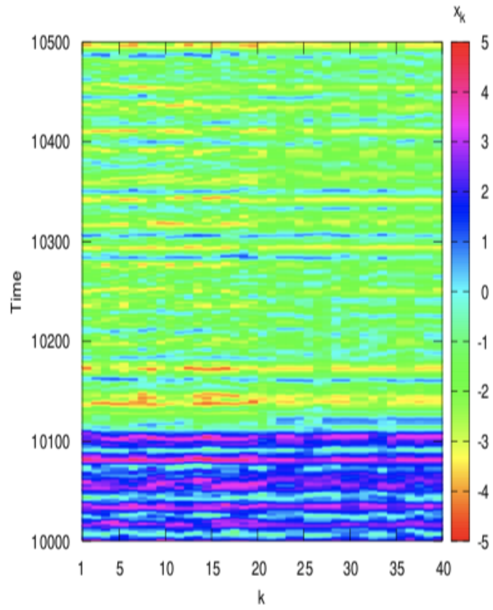
$k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.18$ for $k = 21, \ldots, 40$ (hyperchaos), following René and coworkers [1]. To identify the coherent and incoherent patterns of activity in the system, we compute at each time step of the numerical simulation, which x_k values are locked

$$|x_i - x_j| < h \tag{9}$$

for all $i, j = 1, \ldots, 40,\ i \neq j$, where $h = 10^{-3}$ is a small threshold for locking detection. At each time step of the simulation, when locking is detected, the corresponding $x_k,\ k = 1, \ldots, 40$ values are recorded and the simulation proceeds to the next time step. When the simulation finishes, one obtains a spatio-temporal phenomenon of coherent and incoherent patters such as those depicted in Fig. 5(b). In this plot, we show all x_k values in the time interval $[10^4, 1.05 \times 10^4]$, where blue corresponds to relatively high x_k value and, red, yellow and orange to relatively smaller x_k values. It is evident there are coherent and incoherent groups of x_k variables that alternate in time in a dynamical fashion. To show clearer these patterns, we plot 2 representative examples of spatio-temporal behaviour in panel (a) of the same figure where it is evident the existence of the coherent (locked) and incoherent groups of x_k variables. The plots in panel (a) correspond to times $t = 14462$ (upper plot) and $t = 14515$ (lower plot). We have been able to observe similar patterns of spatio-temporal behaviour at other times as well.



(a)



(b)

Figure 5: Spatio-temporal phenomena of coherent and incoherent patterns, reminiscent of chimera states in 40 3-dimensional TR linearly coupled systems that exhibit labyrinth chaos and hyperchaos with $b_k = 0$ for $k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.18$ for $k = 21, \ldots, 40$ (hyperchaos). The upper plot in panel (a) is for $t = 14462$ and the lower for $t = 14515$. Panel (b) shows the spatio-temporal patterns between $t = 10000$ and $t = 10500$. Note that in these plots, $d = 0.6$.

3.4. Chimera states in a multidimensional TR system with labyrinth chaos and complex periodic oscillations

Finally, we focus on the second case where the system in Eq. (8) exhibits labyrinth chaos for half of the 3-dimensional TR systems and complex periodic oscillations for the rest half. In particular, we set $N = 40$, $P = 20$, $d = 0.6$, $b_k = 0$ for $k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.19$ for $k = 21, \ldots, 40$ (complex periodic oscillations), following again René and coworkers [1]. We follow the same approach as previously to detect locking of the x_k variables and plot in Fig. 6(b) the spatio-temporal patterns of the activity of all x_k values in the time interval $[10^4, 1.05 \times 10^4]$. In this plot again, blue corresponds to relatively high x_k value and, red, yellow and orange to relatively smaller x_k. The coherent and incoherent patterns are again evident and alternate in time in a dynamical fashion. Figure 6(a) shows these patterns clearer where we plot 2 representative examples of spatio-temporal behaviour taken at 2 specific times from panel (b). Again, there exists coherent (locked) and incoherent groups of x_k variables that are reminiscent of chimera states. The plots in panel (a) correspond to times $t = 10184$ (upper plot) and $t = 10371$ (lower plot). As in the previous case, we have been able to observe similar patterns of spatio-temporal behaviour at other times as well.

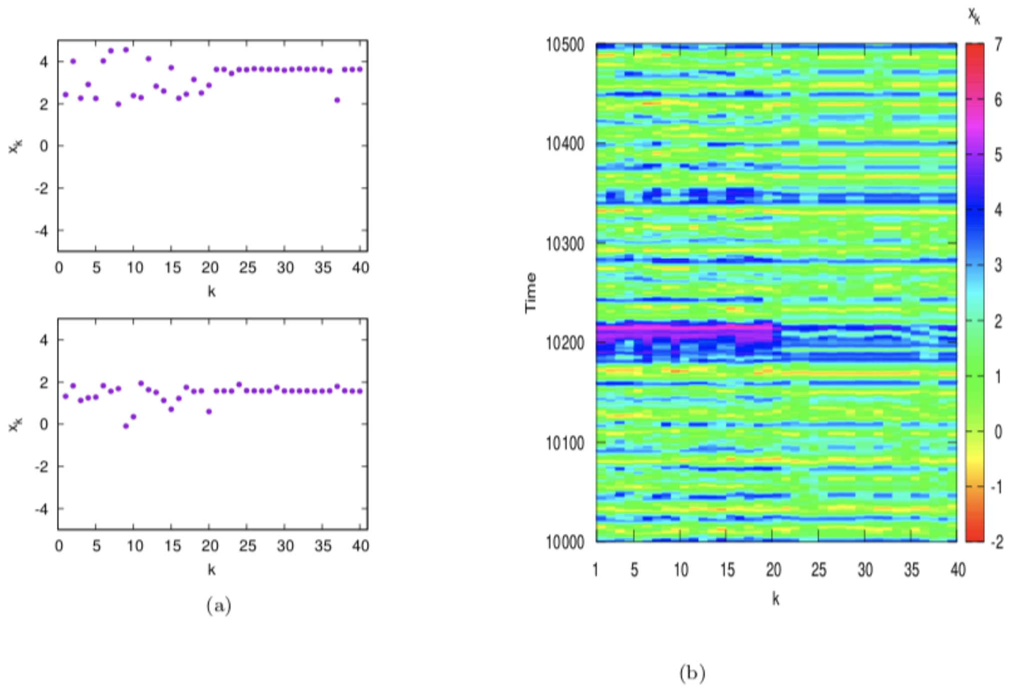


Figure 6: Spatio-temporal phenomena of coherent and incoherent patterns, reminiscent of chimera states in 40 3-dimensional TR linearly coupled systems that exhibit labyrinth chaos and complex periodic oscillations with $b_k = 0$ for $k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.19$ for $k = 21, \ldots, 40$ (complex periodic oscillations). The upper plot in panel (a) is for $t = 10184$ and the lower for $t = 10371$. Panel (b) shows the spatio-temporal patterns between $t = 10000$ and $t = 10500$. Note that in these plots, $d = 0.6$.

4. Conclusions

During the last decade of his life or so, René Thomas was joyfully preoccupied, among other things, with his Arabesque systems, his labyrinth chaos and chaotic walkers. With this contribution to the volume, we had the opportunity to revisit his later work, where we had the honour to contribute to. We extended the Thomas-Rössler systems to a spatio-temporal setting and observed that it can support behaviours reminiscent of chimera states for a wide range of parameter values of the linear coupling term.

What we have here is quite a novel case: the locking and drifting patterns are due to labyrinth chaos and (hyper)chaotic walks, not due to strict periodicity as in other cases in literature. This suggests that the preliminary results in this paper rightfully ask for further investigation as the role of coupling and dissipation are important. Here, we started with the simplest case where the x_k variables of the Thomas-Rössler systems are linearly, globally coupled. This, in effect, sets apart the parameter b, which controls the dissipation for the x_ks, and changes it in time with the $(d/2P)x_k$ part of the coupling. This situation is indeed reminiscent of the case where Arabesques have different dissipation parameters (like the bs) for each variable in this work.

The role of symmetry, both in the linear coupling among the systems and in the structure of each individual system itself, in terms of logical circuits is something worth pursuing in future work. Also, the range of the parameters b_k and d as well as the initial conditions that give rise to various chimera-states is a new, interesting and open question that stems from this initial investigation.

It seems that this is one of the last scientific avenues René opened for further research, and will lead to investigations and spectacular new results that will help shed light on the fundamental aspects of the emergence of chimera states. This research is expected to further provide importance on the mathematical modelling of biological significance. René's work is so hopeful and inspirational, we are sure will lead to new investigations to elucidate the basic logic underlying chimera states, one of the most fascinating aspects of synchronisation in complex systems.

Acknowledgements

We dedicate this paper to the memory of our dear friend, René Thomas, with our fondest memories. We wish to thank his good friends and our teachers, O. E. Rössler and G. Nicolis. Also, Prof. M. Kaufman for her kind patience and for discussing in depth the logic of the "feedback-approach" all these years.

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