# On the discrete nonlinear Schrödinger equation with $\mathcal{P J}$-Symmetry 



Amal Jasim Mohammed
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Department of Mathematical Sciences
University of Essex

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#### Abstract

The purpose of this thesis is to develop the inverse scattering method for the nonlocal semi-discrete nonlinear Schrödinger equation (known as Ablowitz-Ladik equation) with parity-time symmetry proposed in Ablowitz and Musslimani's paper.

This includes the eigenfunctions (Jost solutions) of the associated Lax pair, the scattering data and the fundamental analytic solutions. In addition, we study the spectral properties of the associated discrete Lax operator. Based on the formulated Riemann-Hilbert problem, we derive the one- and two-soliton solutions to the nonlocal Ablowitz-Ladik equation. Finally, we prove the completeness relation for the associated Jost solutions. Based on this, we derive the expansion formula over the Jost solutions is evaluated. This allows interpreting the inverse scattering method as a generalised Fourier transform.

We derive the dressing method based on the seed solution to the discrete nonlinear Schrödinger equation. Explicit relations are obtained amongst the spectrum problem associated with the expansion over the negative and positive power of the eigenvalues. We show a general formula for the Riemann-Hilbert problem based dressing method in terms of the Lax representation associated with a given nonlinear equation.

Next, we study square barrier potentials for the Ablowitz-Ladik like of the discrete nonlinear Schrödinger equation and a certain class of integrable systems of multi-component generalisation of the Manakov model. We are interested in conditions distinguishing blow up and not blow up solutions. From considering single and double excitations as initial conditions, we conjecture the following: 1) if the Lax operator has no spectrum outside nor inside the unit circle, there is no blow up; 2) when it does, mirror symmetric initial conditions are sufficient, but not necessary, for bounded solutions; 3 ) to obtain bounded solutions, each spectrum outside the unit circle needs (but is not sufficient) to have a reciprocal counterpart on the inside. Numerical method used to evaluate the eigenvalues of the Ablowitz-Ladik problem. Numerical simulations are also presented, illustrating our analytical results.


## Publications

The results in chapters 3 and 4 have been published in:

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## Chapter 1

## Introduction

In this chapter, we will begin by setting the scene with a history of the soliton and integrable equations. We will also introduce some related concepts for the inverse scattering method (ISM): Lax representation, the direct and the inverse scattering transform (IST). Also, we will show the connection between the Riemann-Hilbert problem (RHP) and the ISM. In addition, we will outline important concepts: the $\mathcal{P T}$ and the $\mathcal{C P J}$-symmetry. Finally, we will give an overview of the thesis itself, highlighting the main results and describing the aspects which have effectively motivated the research.

### 1.1 Soliton and the nonlinear evolution equations

Soliton theory has been the centre of the developments of nonlinear wave propagation since the nineteenth century, in almost all applications. It began with the pioneering work of J Scott Russell [38] who observed solitons. He built a wave tank in his laboratory in order to study this phenomenon more closely; he made further important observations of the properties of the solitary wave. At first, the scientific community of his time did not seem impressed by Russel's discovery; later scientists showed how Russell's solitary wave arose and can also be explained mathematically.

A soliton associates with a certain solution of a nonlinear equation with some specific properties. Here are some definitions: solitons are caused by a cancellation of nonlinear and dispersive effects in the medium [87]. A soliton is a solitary wave with finite energy and the necessary conditions for its existence include nonlinearity and dispersion [71]. A soliton is a solution to a nonlinear equation or system which represents a wave of permanent form, is localized and decaying at infinity and interacts with other solitons so that after the interaction it retains its form [110]. Later, the argument over solitary water waves was finally resolved by Korteweg et al. [38, 78] who published their theoretical treatment when they introduced their famous equation. The Korteweg de Vries equation
$(\mathrm{KdV})$ is,

$$
\begin{equation*}
u_{t}+u_{x x x}-6 u_{x} u(x, t)=0 \tag{1.1.1}
\end{equation*}
$$

and the solitary wave solution of equation (1.1.1) is

$$
\begin{equation*}
u(x, t)=a \operatorname{sech}^{2}(\gamma(x-V t)), \quad V=2 a=4 \gamma^{2} \tag{1.1.2}
\end{equation*}
$$

where $u(x, t)$ is a real valued function, $\gamma$ is the wave number, $V$ is the speed, $a$ is the wave amplitude and $x, t \in \mathbb{R}$ where $t$ is the time. This was the most important event of the development of soliton theory. After that, this phenomenon attracted the attention of scientists who tried to find the solution of the KdV equation numerically. They found that the soliton of this equation has manifested waves with sharp peaks. These waves move almost independently with constant speeds and pass through each other after collisions [20, 54]. Furthermore, the solitary waves behave like stable particles; thus this helped to discover the soliton. The behaviour of the soliton-like solution (solitary wave) is very stable in the following sense: it is localised, the solution decays as $x \rightarrow \pm \infty$. It is of a travelling wave type. The wave profile $u$ at any time $t \geq 0$ and $x \in \mathbb{R}$ is calculated by $u(x, t)=f(x-c t)$, where $f$ is a function on $\mathbb{R}$ and $c$ denotes the speed of prevalence (spread) [38]. Soliton solutions satisfy some kind of dispersion relation (taller waves travel faster than smaller one). Furthermore, they obtain a sort of overlap principle. This can be explained as follows: in linear waves when two solitary waves with different speed travel together, the taller one will catch up after some finite time; an interaction of both waves takes place and moments later they separate again; both keep their speeds and shape but now the taller wave is in front of the smaller one. The nonlinear partial differential equation (PDE) (1.1.1) governs reasonably shallow water waves which is applicable to the situation that Scott Russell saw [77]. This equation also acts as a model in many physical phenomena. The KdV equation is a well-known equation for providing single-soliton solutions (a right moving soliton $u(x, t)=f(x-c t)$ ). The discovery of the soliton phenomenon led scientists and researchers to develop an interest in systems that are integrable. Meanwhile, they have not yet reached a comprehensive definition of integrability. Notable examples are the KdV, nonlinear Schrödinger equation (NLS), sineGordon (SG) equations and many others. Some of integrable nonlinear partial differential equations (PDEs):

- the cubic NLE describing waves on the surface of shallow water,

$$
\begin{equation*}
\mathrm{i} q_{t}+q_{x x}+\epsilon 2|q|^{2} q=0, \quad \epsilon= \pm 1 \tag{1.1.3a}
\end{equation*}
$$

where $q(x, t)$ is a complex valued function. When $\epsilon=1$, (1.1.3a) is called the focusing NLS equation which admits soliton (kink) solutions and for $\epsilon=-1$, (1.1.3a) is called the defocusing NLS equation which admits anti-kink solitons. This equation is a key model describing optic propagation in Kerr media [42] and models quasi-monochromatic wave packets propagating in nonlinear media, and also the NLS equation describes Langmuir waves in plasma, while the NLS equation is used in different applications, for instance in fluid mechanics [112]. The NLS equation is presented in different formula; each one explains what the researchers are trying to express; for instance (1.1.3b) [112] with constant boundary condition $q(x, t)=\rho e^{\mathrm{i} \theta_{ \pm}}, \theta_{ \pm} \in[0,2 \pi]$

$$
\begin{equation*}
\mathrm{i} q_{t}+q_{x x}-2\left(|q|^{2}-\rho^{2}\right) q=0, \quad \rho \in \mathbb{R}_{+} \tag{1.1.3b}
\end{equation*}
$$

- the SG equation is a nonlinear hyperbolic PDE of the form:

$$
\begin{equation*}
u_{x t}=\sin (u(x, t)), \tag{1.1.4}
\end{equation*}
$$

where equation (1.1.4) models solitons and many other problems in quantum mechanics, including applications; for instance, in relativistic field theory, it models Josephson junctions [74].

A single-soliton solution to the NLS equation (1.1.3a) $(\epsilon=1)$ [118] is

$$
\begin{equation*}
u(x, t)=2 \mathrm{i}\left(\zeta_{1}^{*}-\zeta_{1}\right) \frac{c_{1} e^{\left(\theta_{1}-\theta_{1}^{*}\right)}}{e^{-\left(\theta_{1}+\theta_{1}^{*}\right)}+\left|c_{1}\right|^{2} e^{\left(\theta_{1}+\theta_{1}^{*}\right)}}, \tag{1.1.5a}
\end{equation*}
$$

by taking $\zeta_{1}=\xi+\mathrm{i} \eta$ and $c_{1}=e^{-2 \eta x_{0}+\mathrm{i} \sigma_{0}}$, then (1.1.5a) becomes:

$$
\begin{equation*}
u(x, t)=2 \eta \operatorname{sech}\left[2 \eta\left(x+4 \xi t-x_{0}\right)\right] \exp \left\{-2 \mathrm{i} \xi x-4 \mathrm{i}\left(\xi^{2}-\eta^{2}\right) t+\mathrm{i} \sigma_{0}\right\} \tag{1.1.5b}
\end{equation*}
$$

where $\xi, \eta$ are the real and imaginary parts of $\zeta_{1}$, and $x_{0}, \sigma_{0}$ are real parameters. Equation (1.1.5b) is a solitary wave solution with $(2 \eta)$ amplitude and $(-4 \xi)$ velocity.

### 1.2 The inverse scattering method

Often, linear partial differential equations can be solved by a Fourier transform; however, it is difficult to solve nonlinear PDEs. Therefore, the IST is nowadays a subject of intense research when it comes to solve nonlinear PDEs. In an explicit sense of this method, the rapidly and decreasing boundary conditions make the dynamics quite simple because of the recovery of the time evolution of the transition coefficients; both continuous and
discrete spectra become linear. The KdV equation (1.1.1) undoubtedly the first nonlinear PDE which is solved by the inverse scattering method. This method is the best key in solving the Cauchy problem for the KdV equation. Gardner, Greene, Kruskal and Miura created a method to derive the exact solution of the Cauchy problem for the KdV equation, for rapidly decaying initial values [10, 45, 93]. Historically, equation (1.1.1) was derived by Diederik Korteweg and Gustav de Vries as a mathematical model of water-waves in shallow channels.

### 1.2.1 Lax representations

Peter D. Lax [80] considered a nonlinear PDE in evolutionary form

$$
\begin{equation*}
u_{t}=K(u), \quad u=u(x, t), \tag{1.2.1}
\end{equation*}
$$

where the nonlinear differential operator $K$ is a time independent operator (not considered derivative with respect to $t$ ). He considered a pair of linear differential operators, $L$ and $A$. To find the eigenvalues and the eigenfunctions, $L$ operator associated to (1.2.2a) spectral problem

$$
\begin{equation*}
L \psi=\lambda \psi, \tag{1.2.2a}
\end{equation*}
$$

meanwhile, the operator $A$ is related to the time evolution of the eigenfunctions

$$
\begin{equation*}
\psi_{t}=A \psi \tag{1.2.2b}
\end{equation*}
$$

Proposition 1 If the spectral parameter is a fixed point, i.e $\lambda_{t}=0$, then the relations (1.2.2) lead to

$$
\begin{equation*}
L_{t}+[L, A]=0, \quad[L, A]=L A-A L \tag{1.2.3}
\end{equation*}
$$

Proof: First, differentiate (1.2.2a) with respect to $t$

$$
\begin{equation*}
L_{t} \psi+L \psi_{t}=\lambda \psi_{t} \tag{1.2.4a}
\end{equation*}
$$

substitute (1.2.2b) in (1.2.4a)

$$
\begin{equation*}
L_{t} \psi+L A \psi=A \lambda \psi, \tag{1.2.4b}
\end{equation*}
$$

then,

$$
\begin{equation*}
\left(L_{t}+L A-A L\right) \psi=0 . \tag{1.2.4c}
\end{equation*}
$$

To sum up, equation (1.2.3) is called the Lax equation and (1.2.2) are called a Lax pair (or Lax operators) for (1.2.1). Writing nonlinear evolution equations (NLEEs) as a compati-
bility condition of the linear equations (1.2.2), plays an important role for the solvability of the equation under the IST.

Example 1 Lax operators for the nonlinear KdV and NLS equations.

- the KdV equation (1.1.1) can be written as a compatibility condition of the form (1.2.3), of a system of linear equations (1.2.2), where $L$ and $M \equiv A$ are given by

$$
\begin{align*}
L & =-\frac{\partial^{2}}{\partial x^{2}}+u(x, t)  \tag{1.2.5a}\\
M & =4 \partial_{x}^{3}-3 u \partial_{x}-3 \partial_{x} u \tag{1.2.5b}
\end{align*}
$$

where $u(x, t)$ is a real valued function.

- for the NLS equation (1.1.3a) the form of the two operators $L$ and $M \equiv A$ are given

$$
\begin{align*}
L & =\mathrm{i} \frac{\partial}{\partial x}+\left(q(x, t)-\lambda \sigma_{3}\right),  \tag{1.2.5c}\\
M & =\mathrm{i} \frac{\partial}{\partial x}+\sum_{k=0}^{2} V_{k} \lambda^{k} \tag{1.2.5d}
\end{align*}
$$

where $q(x, t)$ is a complex valued function.

### 1.2.2 The direct and inverse scattering transform

Researchers have shown that the IST is not yet formulated to be uniformly applicable to all NLEEs. However, there is a class of nonlinear PDEs that can be solved/integrated by considering the scattering problem for a given linear differential operator. For PDEs having a Lax representation, one can start with the corresponding scattering problem for the operator $L(\lambda)$. The Zakharov-Shabat (ZS) system used the following $L(\lambda)$ operator for the continuous NLS equation [105].

$$
\begin{equation*}
L(\lambda) \chi \equiv\left(\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} x}+U(x, t, \lambda)\right) \chi(x, t, \lambda)=0 \tag{1.2.6a}
\end{equation*}
$$

where

$$
U(x, t, \lambda)=q(x, t)-\lambda \sigma_{3}, \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0  \tag{1.2.6b}\\
0 & -1
\end{array}\right), \quad q(x, t)=\left(\begin{array}{cc}
0 & q^{+} \\
q^{-} & 0
\end{array}\right)
$$

The scattering problem is determined by the asymptotics of the eigenfunctions of the operator $L$, called here, Jost solutions ${ }^{1}$, and their ratio, known as the scattering matrix

[^0]$T(\lambda, t)$, and its elements are called scattering data. The transformation to the scattering data linearises the PDE. The whole IST, in fact, can be considered as a nonlinear analogue of the standard Fourier transform. The function $u(x, t)$ in Fig. 1.1 is a real valued function


Figure 1.1: The Fourier transform scheme.
equivalent to $q(x, t)$ and $\hat{u}(\lambda, t)$ equivalent to $T(\lambda, t)$. Consider the general formula of the Cauchy problem for the NLS equation

$$
\begin{equation*}
q_{t}=K(q), \quad q(x, 0)=f(x), \quad q:=q(x, t), \tag{1.2.7}
\end{equation*}
$$

with boundary conditions $q(x, 0) \rightarrow \pm \infty$. This method is demonstrated as three basic steps. The following steps will explain briefly how it works. Furthermore, in Fig. 1.2 we present these steps for the NLS; a similar scheme for the discrete NLS (DNLS) equation.


Figure 1.2: The scattering problem scheme

1. Step I: The direct problem: The direct problem consists of finding the scattering data $\left(a^{ \pm}(\lambda), b^{ \pm}(\lambda)\right.$; see (1.4.24) and $\left.a^{ \pm}(z), b^{ \pm}(z)\right)$ of operator $L$ at a fixed value of temporal parameter, say $t$ from $\left.\left.T(\lambda, t)\right|_{t=0}\right)$, by using the initial condition $q(x, 0)=$ $f(x)$.
2. Step II: Time evolution of the scattering data: For the second step, we need to determine the scattering data at an arbitrary time $t \in \mathbb{R}$. From (1.2.2b), and from the first step we have $\left.T(\lambda, t)\right|_{t=0}$, we will determine $\left.T(\lambda, t)\right|_{t \in \mathbb{R}}$. We are now dealing with a linear problem (1.2.2b) rather than a nonlinear one.
3. Step III: The inverse problem: This step follows from the Fourier transform method; here, we need to recover $q=q(x, t)$ from $\left.T(\lambda, t)\right|_{t \in \mathbb{R}}$.

Remark 1 In this chapter and the next chapters, t-dependence is not always shown.

### 1.2.3 The AKNS scheme

Ablowitz, Kaup, Newell and Segur (AKNS) at Clarkson College (in Potsdam, New York State, 1974) were developed some techniques for applying the IST to obtain the solution of many equations such as the KdV, NLS and mKdV equations; this development is called the AKNS scheme [6, 12, 34, 38, 54].

This section shows how the AKNS method scheme one to derive the family of integrable NLEEs. Alternatively to the original Lax formulation (1.2.2), one can rewrite both Lax operators as differential operators (1.2.5). Thus, the isospectrality condition (1.2.3) is equivalent to the compatibility condition $[L(\lambda), M(\lambda)]=0, \lambda_{t}=0$, where $L, M$ are two linear operators. The KdV, mKdV, NLS, SG, and sinh-Gordon equations [54, 80, 118] are all NLEEs because each of these equations can be written as the compatibility condition of a Lax pair. We will show the compatibility condition for DNLS equation in chapter 3. Lax equation hold for particular types of eigenfunctions and it works better when the operator $L(\lambda)$ is the ZS system. There are two important conditions that the potential should satisfy in (1.2.6a):

Condition 1. $q(x)$ belongs to the space $\mathcal{M}$ of off-diagonal $2 \times 2$ matrix valued functions, whose matrix elements are complex Schwartz-type ${ }^{1}$ functions. This condition is made for simplification and will help to outline the main ideas of the IST [49] .

Condition 2. The potential $q(x)$ is such that the corresponding transition coefficients $a^{+}(\lambda)$ and $a^{-}(\lambda)$ have a finite number of simple zeros in their regions of analyticity located at $\lambda_{k}^{ \pm}$:

$$
\begin{equation*}
\left\{\lambda_{k}^{ \pm}: a_{k}^{ \pm}=0, \operatorname{Im} \lambda_{k}^{ \pm} \gtrless 0, \quad k=1,2, \ldots, N\right\} . \tag{1.2.8}
\end{equation*}
$$

[^1]The general formula of the second operator $M$ of the ZS system is;

$$
\begin{equation*}
M \chi \equiv\left(\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} t}+V(x, t, \lambda)\right) \chi(x, t, \lambda), \tag{1.2.9a}
\end{equation*}
$$

where $M(\lambda)$ is a polynomial in $\lambda$,

$$
\begin{equation*}
V(x, t, \lambda)=\sum_{k=0}^{N} \lambda^{N-k} V_{k}(x, t) \tag{1.2.9b}
\end{equation*}
$$

then, apply the two operators $L, M$ in Lax representation

$$
\begin{equation*}
[L(\lambda), M(\lambda)]=0, \tag{1.2.10}
\end{equation*}
$$

and equate the coefficients of the position powers of $\lambda$ to zero; this gives:

$$
\begin{align*}
{\left[V_{0}(x, t), \sigma_{3}\right] } & =0,  \tag{1.2.11a}\\
\mathrm{i} \frac{\mathrm{~d} V_{k}}{\mathrm{~d} x}+\left[q, V_{k}(x, t)\right]-\left[\sigma_{3}, V_{k+1}(x, t)\right] & =0, \tag{1.2.11b}
\end{align*}
$$

for $k=0,1, \ldots, N-1$ and the $\lambda$-independent term gives:

$$
\begin{equation*}
-\mathrm{i} \frac{\partial q}{\partial t}+\mathrm{i} \frac{\partial V_{N}}{\partial x}+\left[q(x, t), V_{N}(x, t)\right]=0 \tag{1.2.12}
\end{equation*}
$$

this is called the zero curvature equation.
Example 2 The compatibility condition with $V(x, t, \lambda)=\lambda^{2} V_{0}+\lambda V_{2}+V_{3}$ can lead to the NLS equation formula.

The special case of $V(x, t, \lambda)$ is based on the constants that were obtained from (1.2.11) and (1.2.12)

$$
\begin{equation*}
V(x, t, \lambda)=-\mathrm{i} \sigma_{3} q_{x}-q^{+} q^{-} \sigma_{3}-2 \lambda q(x, t)+2 \lambda^{2} \sigma_{3} \tag{1.2.13}
\end{equation*}
$$

by matrix form

$$
V(x, t, \lambda)=\left(\begin{array}{cc}
-q^{+}(x, t) q^{-}(x, t)+2 \lambda^{2} & -\mathrm{i} q_{x}^{+}(x, t)-2 \lambda q^{+}(x, t)  \tag{1.2.14}\\
\mathrm{i} q_{x}^{-}(x, t)-2 \lambda q^{-}(x, t) & q^{+}(x, t) q^{-}(x, t)-2 \lambda^{2}
\end{array}\right)
$$

then, from (1.2.12) one can easily derive the NLS equation

$$
\begin{equation*}
-\mathrm{i} q_{t}+\sigma_{3} q_{x x}+2 q^{+} q^{-} \sigma_{3} q(x, t)=0 \tag{1.2.15}
\end{equation*}
$$

In this case, the dispersion law $f(\lambda)=2 \lambda^{2} \sigma_{3}$, is calculated from (1.2.13) when $q \rightarrow \pm \infty$.

Example 3 This example confirms how the compatibility condition can also derive the KdV equation. In this case, we need to define a cubic polynomial of $\lambda: V(x, t, \lambda)=$ $\lambda^{3} V_{0}+\lambda^{2} V_{1}+\lambda V_{2}+V_{3}$.

As in the first example, we need a special case of $V(x, t, \lambda)$ based on the constants:

$$
\begin{align*}
V_{0}(x, t, \lambda) & =-4 \sigma_{3}, \quad V_{1}=4 q(x, t), \quad V_{2}=2 q^{+} q^{-} \sigma_{3}+2 \mathrm{i} \sigma_{3} q_{x} \\
V_{3} & =-\mathrm{i}\left(q^{+} q_{x}^{-}-q^{-} q_{x}^{+}\right) \sigma_{3}-q_{x x}-2 q^{+} q^{-} q(x, t) . \tag{1.2.16}
\end{align*}
$$

In this case, the dispersion law for the $\operatorname{KdV}$ equation is $f(\lambda)=-4 \lambda^{3} \sigma_{3}$, which is calculated from (1.2.16) when $q \rightarrow \pm \infty$. From the compatibility condition (1.2.10), one can find the general form of the higher NLS equation

$$
\begin{align*}
\sigma_{3} q_{t}+\sigma_{3} q_{x x x}+2 \sigma_{3}\left(q_{x}^{+} q^{-} q(x, t)\right. & \left.+q^{+} q_{x}^{-} q(x, t)+q^{+} q^{-} q_{x}(x, t)\right) \\
& -2\left(q^{+} q_{x}^{-} q(x, t)-q^{-} q_{x}^{+} q(x, t)\right)=0 \tag{1.2.17}
\end{align*}
$$

and for each $q^{ \pm}(x, t)$ we have the higher-NLS system

$$
\begin{align*}
& q_{t}^{+}+q_{x x x}^{+}+6\left(q^{+} q^{-}\right) q_{x}^{+}(x, t)=0,  \tag{1.2.18a}\\
& q_{t}^{-}+q_{x x x}^{-}+6\left(q^{+} q^{-}\right) q_{x}^{-}(x, t)=0 . \tag{1.2.18b}
\end{align*}
$$

We can find also the KdV equation by reducing the system (1.2.18) and this done by imposing a relation $q^{+}=v(x, t)$ and $q^{-}=1$ [93]

$$
\begin{equation*}
v_{t}+v_{x x x}+6 v_{x} v(x, t)=0 \tag{1.2.19}
\end{equation*}
$$

Analogously, we can set the involution $q^{+}=q^{-}=p(x, t)$, then this leads to the modified $\mathrm{KdV}(\mathrm{mKdV})$ equation, where $p(x, t)$ is a real valued function,

$$
\begin{equation*}
p_{t}+p_{x x x}+6 p_{x} p^{2}(x, t)=0 . \tag{1.2.20}
\end{equation*}
$$

### 1.3 Riemann-Hilbert problem and the inverse scattering method

The inverse scattering problem (ISP) is the problem of determining the potential that corresponds to a given set of scattering data in a differential equation. ISP uses different strategies to find the solutions of differential equation; we are interested in the way that ISP uses the RHP. The classical approach to the inverse problem based on the Gel'fandLevitan Marchenko (GLM) equation, is also called Volterra integral equation [41]. With-


Figure 1.3: The inverse scattering transform
out going into the details of the derivation of the GLM equation (1.3.1) (for more details see $\left.[38]^{1},[54]^{2}\right)$, it is based on the analytic properties of the Jost solutions of the operator $L$. The GLM equation has the following form

$$
\begin{equation*}
K(x, z)=F(x, z)+\int_{x}^{\infty} \mathrm{d} y K(x, y) F(y+z) . \tag{1.3.1}
\end{equation*}
$$

From the scattering problem, there is a relation between the Jost solutions and the scattering data. Given scattering data, one determines the kernel $F(x, z)$ of the GLM equation. The last step of the solution of the ISP is the determination of the potential $(u(x, t)$ in [38] and $q(x)$ in [54]), that correspond to the scattering data.

### 1.3.1 Sokhotski-Plemelj formula

For the basic tools for solving the RHP, we need to use the Sokhotski-Plemelj formula. We will start with a brief introduction to the Cauchy integral theorem [94]. Let $F(z)$ be defined by

$$
\begin{equation*}
F(z)=\frac{1}{2 \pi \mathrm{i}} \int_{C} \frac{f(t) \mathrm{d} t}{t-z} \tag{1.3.2}
\end{equation*}
$$

where $C$ is a curve in the complex $t$-plane, $f(t)$ is a complex-valued function prescribed on $C$, and $z$ is a point not on $C$. The curve $C$ may be an arc or a closed contour. For proper curves $C$ and functions $f, F(z)$ will be an analytic function of $z$. If $C$ is a closed contour, the positive orientation will be counterclockwise and $t_{0} \in C$, then the limits of $F(z)$ as $z \rightarrow t_{0}$ from the left and from the right (or from the inside and outside of $C$ ) if limits exist, will be denoted by $F^{+}\left(t_{0}\right)$ and $\left.F^{-}\left(t_{0}\right)\right)$, respectively. The principal value of

[^2]

Figure 1.4: The contours in Cauchy theorem.
the integral (2.2.21a) denoted by $F_{p}\left(t_{0}\right)$ has the following form

$$
\begin{align*}
F_{p}\left(t_{0}\right) & \equiv \frac{1}{2 \pi \mathrm{i}}(P) \int_{C} \frac{f(t) \mathrm{d} t}{t-t_{0}}  \tag{1.3.3}\\
& =\frac{1}{2 \pi \mathrm{i}} \lim _{\epsilon \rightarrow 0} \int_{C-C_{\epsilon}} \frac{f(t) \mathrm{d} t}{t-t_{0}}, \tag{1.3.4}
\end{align*}
$$

where $C_{\epsilon}$ is part of the curve $C$ contained within a small circle of radius $\epsilon$, centered on $t_{0}$. The next step is to find a relation between $F_{P}\left(t_{0}\right), F^{+}\left(t_{0}\right)$, and $F^{-}\left(t_{0}\right)$. To derive such a relation, if $f(t)$ is analytic at point $t_{0}$, then there exists a circle $\xi$ with center $t_{0}$ and radius $r>0$ such that $f(t)$ is analytic in the semicircle $\left\{t:\left|t-t_{0}\right|<r\right\}$. Let $z$ be a point in the left side of the curve $C$ and inside the circle $\xi$, with the Cauchy theorem stating that

$$
\begin{equation*}
\int_{C_{\epsilon}} \frac{f(t)}{t-z} \mathrm{~d} t=\int_{C_{1}} \frac{f(t)}{t-z} \mathrm{~d} t \tag{1.3.5}
\end{equation*}
$$

where $C_{1}$ is the arc of a circle with center $t_{0}$ and radius $\epsilon<r$, then

$$
\begin{equation*}
F(z)=\int_{C-C_{\epsilon}} \frac{f(t)}{t-z} \mathrm{~d} t+\frac{1}{2 \pi \mathrm{i}} \int_{C_{1}} \frac{f(t)}{t-z} \mathrm{~d} t \tag{1.3.6}
\end{equation*}
$$

therefore letting $z \rightarrow t_{0}$ from left(+) in (1.3.6), then $F^{+}\left(t_{0}\right)$ has the following form

$$
\begin{equation*}
F^{+}\left(t_{0}\right)=\int_{C-C_{\epsilon}} \frac{f(t)}{t-t_{0}} \mathrm{~d} t+\frac{1}{2 \pi \mathrm{i}} \int_{C_{1}} \frac{f(t)}{t-t_{0}} \mathrm{~d} t, \tag{1.3.7}
\end{equation*}
$$

by letting $\epsilon \rightarrow 0$, and equation (1.3.7) becomes:

$$
\begin{equation*}
F^{+}\left(t_{0}\right)=F_{p}\left(t_{0}\right)+\frac{1}{2} f\left(t_{0}\right) . \tag{1.3.8}
\end{equation*}
$$

We can follow the same steps to obtain $F^{-}\left(t_{0}\right)$ after changing the sides of $z$ and $C_{1}$,

$$
\begin{equation*}
F^{-}\left(t_{0}\right)=F_{p}\left(t_{0}\right)-\frac{1}{2} f\left(t_{0}\right) . \tag{1.3.9}
\end{equation*}
$$

The relations (1.3.8) and (1.3.9) are known as Plemelj formulas.

### 1.3.2 Multiplicative Riemann-Hilbert problem

Our work deals with the RHP with boundary conditions. The beginning of its use had an original formula which deals with an $N \times N$ linear system of partial differential equations (called Fuchsian system)

$$
\begin{equation*}
\frac{\mathrm{d} \Psi(\lambda)}{\mathrm{d} \lambda}=A(\lambda) \Psi(\lambda) \tag{1.3.10}
\end{equation*}
$$

where the rational function of $\lambda, A(\lambda)$ is an $N \times N$ matrix whose singularities are simple poles. The Fuchsian system (1.3.10) generates a representation of the fundamental group of the Riemann sphere (punctured at the poles of $A(\lambda)$ ) in the group, via its fundamental solution $\Psi(\lambda)$ along closed curves. The first application of the Riemann Hilbert method to integrable partial differential equations is found in the works of Manakov, Shabat, and Zakharov [120] and since then it has been widely used in soliton theory [94]. In general, the RHP can be stated as follows: let $\Gamma$ be an oriented contour, which divides the complex $\lambda$-plane into two regions the positive direction will be counterclockwise $\Gamma^{+} \backslash \Gamma$ and $\Gamma^{-} \backslash \Gamma$. The contour $\Gamma$ might have points of self-intersection, and a priori might have more than one connected component. Also, let $G(\lambda)$ be a matrix function defined on the contour $\Gamma$, i.e. a map from $\Gamma$ into the set of $N \times N$ invertible matrices. The RHP determined by the pair $(\Gamma ; G)$ consists of finding an $N \times N$ matrix-valued function $\tilde{F}(\lambda) \in \mathbb{C}^{N \times N}$ with the following properties. In particular, for the NLS equation, the classical RHP can be presented as a matrix form (for example $2 \times 2$ ); we take $\Gamma=\mathbb{R}$.

1. The function $\tilde{F}(\lambda)$ is analytic for $\lambda \in \mathbb{C} / \mathbb{R}$.
2. The following jump condition is satisfied:

$$
\begin{equation*}
\tilde{F}^{+}(\lambda)=\tilde{F}^{-}(\lambda) \tilde{G}(\lambda), \quad \lambda \in \mathbb{R} \tag{1.3.11}
\end{equation*}
$$

where $\tilde{F}^{+}(\lambda)$ and $\tilde{F}^{-}(\lambda)$ are supposed to be analytic functions into the upper $\Gamma^{+}$ and lower $\Gamma^{-}$half-planes, respectively, see Fig. 1.5. A matrix function $\tilde{G}(\lambda)$, which
is defined on the real line $-\infty<x<\infty$, has the following form

$$
\tilde{G}(\lambda)=\left(\begin{array}{cc}
\mathbb{1}, & -\rho^{-}(\lambda)  \tag{1.3.12}\\
\rho^{+}(\lambda), & \mathbb{1}
\end{array}\right) .
$$

3. As $\lambda \rightarrow \infty$ along any direction outside $\mathbb{R}$, then $\tilde{F}^{+}(\lambda) \rightarrow \mathbb{1}$.


Figure 1.5: The close contours $\Gamma^{ \pm}$are denoted for the upper and lower half planes. The contours $\Gamma^{ \pm}=\mathbb{R} \cup \Gamma_{\epsilon, \pm} \cup \Gamma_{ \pm, \infty}$ of integrations in the case where $\lambda$ is on the real axis.

### 1.3.3 The additive Riemann-Hilbert problem

The original multiplicative jump condition (1.3.11) can be written in the additive form

$$
\begin{equation*}
\log \tilde{F}^{+}(\lambda)-\log \tilde{F}^{-}(\lambda)=\log \tilde{G}(\lambda) \tag{1.3.13}
\end{equation*}
$$

An additive jump relation of the form $F^{+}(\lambda)=F^{-}(\lambda)+f(\lambda)$ can always be resolved by means of the contour integral (2.2.21a).

$$
\begin{equation*}
F(z)=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma} \frac{f(t) \mathrm{d} t}{t-z} . \tag{1.3.14}
\end{equation*}
$$



Figure 1.6: The contour $\Gamma$ and the regions $\Gamma^{ \pm} / \Gamma$ for a generic RHP.

If we assume $\log \tilde{F}^{ \pm}(\lambda)=F^{ \pm}(\lambda)$ and $\log \tilde{G}(\lambda)=G(\lambda)$, re-write (1.3.13) as $F^{+}(\lambda)-$ $F^{-}(\lambda)=G(\lambda)$, then the RHP has the following solutions:

$$
\begin{array}{ll}
F^{+}(\lambda)=H(\lambda)+\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \frac{F(\mu) \mathrm{d} \mu}{\mu-\lambda}, & \lambda \in \Gamma^{+} / \Gamma, \\
F^{-}(\lambda)=H(\lambda)+\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \frac{F(\mu) \mathrm{d} \mu}{\mu-\lambda}, & \lambda \in \Gamma^{-} / \Gamma, \tag{1.3.15b}
\end{array}
$$

where $H(\lambda)$ is an arbitrary entire (analytic) function of $\lambda$. A normalisation needs to have a unique solution for the RHP; this is done by fixing the solutions at $\lambda=\lambda_{0} \in \Gamma^{+}, F^{+}(\lambda)=$ $F_{0}^{+}:$

$$
\begin{equation*}
H\left(\lambda_{0}\right)=F_{0}^{+}-\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \frac{F(\mu) \mathrm{d} \mu}{\mu-\lambda_{0}} . \tag{1.3.16}
\end{equation*}
$$

From (1.3.7), (1.3.8) and Loiuville's theorem, the solution of normalised RHP is provided by:

$$
\begin{array}{ll}
F^{+}(\lambda)=F_{0}^{+}(\lambda)+\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} F(\mu) \mathrm{d} \mu\left(\frac{1}{\mu-\lambda}-\frac{1}{\mu-\lambda_{0}}\right), & \lambda \in \Gamma^{+} / \Gamma, \\
F^{-}(\lambda)=F_{0}^{+}(\lambda)+\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} F(\mu) \mathrm{d} \mu\left(\frac{1}{\mu-\lambda}-\frac{1}{\mu-\lambda_{0}}\right), & \lambda \in \Gamma^{-} / \Gamma . \tag{1.3.17b}
\end{array}
$$

The additive RHP can be solved also for function $F^{ \pm}(x, \lambda)$, which may depend on another parameter $x$. In chapter 3, we use the RHP in an additive form. In this, the spectral representation for the Jost solutions obtained in Sec.1.3 are solutions of an additive RHP for functions that are two-component vector functions depending analytically on the spectral parameter $z$. In the DNLS equation, $\Omega=|z|=1$ is the continuous spectrum, $\Gamma^{ \pm} \equiv \Omega^{ \pm}, z_{0}=\infty$. If we compare (3.6.3a) with the RHP, then $F^{+}(\lambda), F^{-}(\lambda), F(\lambda), F_{0}^{+}$
are:

$$
\begin{align*}
F^{+}(\lambda) & \equiv \tilde{\chi}_{n}^{+}(z)=\tilde{\varphi}_{n}^{+}=\frac{\varphi_{n}^{+}}{a^{+}}(z),  \tag{1.3.18a}\\
F^{-}(\lambda) & \equiv \xi_{n}^{-}(z)  \tag{1.3.18b}\\
F(\lambda) & \equiv z^{-2 n} \rho^{+}(z) \xi_{n}^{+}(z)  \tag{1.3.18c}\\
F_{0}^{+} & \equiv\binom{1}{0} \tag{1.3.18d}
\end{align*}
$$

Then equation (3.6.10) can be considered as an additive RHP (1.3.13) and the spectral representation (3.6.15c) provides the solution to $\xi_{n}^{-}(z, t)$ of this RHP. Analogously, taking:
$F^{+}(\lambda), F^{-}(\lambda), F(\lambda), F_{0}^{+}$to be

$$
\begin{align*}
F^{+}(\lambda) & \equiv \xi_{n}^{+}(z)  \tag{1.3.19a}\\
F^{-}(\lambda) & \equiv \tilde{\chi}_{n}^{-}(z)=\tilde{\varphi}_{n}^{-}=\frac{\varphi_{n}^{-}}{a^{-}}(z),  \tag{1.3.19b}\\
F(\lambda) & \equiv-z^{2 n} \rho^{-}(z) \xi_{n}^{-}(z)  \tag{1.3.19c}\\
F_{0}^{+} & \equiv\binom{0}{1} \tag{1.3.19d}
\end{align*}
$$

### 1.4 Generating operators and integrable hierarchies

In this section, we will solve the set of recursion relations (1.2.11b), which come from the AKNS to find the coefficients of the polynomial and the NLEE equation. This method starts with the following initial condition:

$$
\begin{equation*}
V_{0}=c_{0} \sigma_{3}, \tag{1.4.1}
\end{equation*}
$$

where $c_{0}$ is a constant. We need to define a form, which can help to find the off-diagonal part of a traceless matrix ${ }^{1}$, called the projectors:

$$
\begin{equation*}
\pi_{0} \equiv \frac{1}{4}\left[\sigma_{3},\left[\sigma_{3}, .\right]\right] . \tag{1.4.2}
\end{equation*}
$$

[^3]So, each matrices $V_{k}(x, t)$ can be split into a diagonal and an off-diagonal matrix:

$$
\begin{align*}
V_{k}(x, t) & =\omega_{k}(x, t) \sigma_{3}+V_{k}^{f}(x, t),  \tag{1.4.3a}\\
V_{k}^{f}(x, t) & =\pi_{0} V_{k}(x, t),  \tag{1.4.3b}\\
\omega_{k}(x, t) & =\frac{1}{2} \operatorname{tr}\left(V_{k}(x, t) \sigma_{3}\right) . \tag{1.4.3c}
\end{align*}
$$

When the recursion relation (1.2.11b) starts with $k=0$ :

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} c_{0}}{\mathrm{~d} x} \sigma_{3}+\left[q(x, t), \sigma_{3}\right]-\left[\sigma_{3}, V_{1}(x, t)\right]=0 \tag{1.4.4}
\end{equation*}
$$

the first term $\frac{\mathrm{d} c_{0}}{\mathrm{~d} x}=0, c_{0}$ is a constant, then $V_{1}^{f}$ will be

$$
\begin{equation*}
V_{1}^{f}(x, t)=-c_{0} q(x, t) . \tag{1.4.5}
\end{equation*}
$$

For general $k$,

$$
\begin{align*}
\mathrm{i} \frac{\mathrm{~d}\left(\omega_{k}(x, t) \sigma_{3}+V_{k}^{f}(x, t)\right)}{\mathrm{d} x} & +\left[q,\left(\omega_{k}(x, t) \sigma_{3}+V_{k}^{f}(x, t)\right)\right] \\
& -\left[\sigma_{3},\left(\omega_{k+1}(x, t) \sigma_{3}+V_{k+1}^{f}(x, t)\right)\right]=0,  \tag{1.4.6a}\\
\mathrm{i} \frac{\mathrm{~d} \omega_{k}(x, t) \sigma_{3}}{\mathrm{~d} x}+\mathrm{i} \frac{\mathrm{~d} V_{k}^{f}(x, t)}{\mathrm{d} x} & +\left[q, \omega_{k}(x, t) \sigma_{3}\right]+\left[q, V_{k}^{f}(x, t)\right] \\
& -\left[\sigma_{3}, \omega_{k+1}(x, t) \sigma_{3}\right]-\left[\sigma_{3}, V_{k+1}^{f}(x, t)\right]=0, \tag{1.4.6b}
\end{align*}
$$

the diagonal and off-diagonal parts are:

$$
\begin{array}{r}
\mathrm{i} \frac{\mathrm{~d} \omega_{k}(x, t) \sigma_{3}}{\mathrm{~d} x}+\left[q, V_{k}^{f}(x, t)\right]=0 \\
\mathrm{i} \frac{\mathrm{~d} V_{k}^{f}(x, t)}{\mathrm{d} x}+\left[q, \omega_{k}(x, t) \sigma_{3}\right]-\left[\sigma_{3}, V_{k+1}^{f}(x, t)\right]=0 \tag{1.4.7b}
\end{array}
$$

respectively. Multiply both sides of (1.4.7a) by $\sigma_{3}$ and taking the trace, then equation (1.4.7a) becomes:

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} \omega_{k}}{\mathrm{~d} x}+\frac{1}{2} \operatorname{tr}\left(\sigma_{3}\left[q(x, t), V_{k}^{f}(x, t)\right]\right)=0 \tag{1.4.8}
\end{equation*}
$$

and by integrating both sides with respect to $x$

$$
\begin{equation*}
\omega_{k}(x, t)=c_{k}+\frac{\mathrm{i}}{2} \int_{ \pm \infty}^{x} \mathrm{~d} \lambda \operatorname{tr}\left(\sigma_{3}\left[q(y, t), V_{k}^{f}(y, t)\right]\right), \tag{1.4.9}
\end{equation*}
$$

where $c_{k}$ is an integration real constant. In the next step, we need to apply ( $\frac{1}{4}\left[\sigma_{3},.\right]$ ) to both sides of equation (1.4.7b), we obtain:

$$
\begin{equation*}
\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\mathrm{~d} V_{k}^{f}}{\mathrm{~d} x}\right]+\frac{1}{4}\left[\sigma_{3},\left[q(x, t), \sigma_{3}\right] \omega_{k}(x, t)\right]=\frac{1}{4}\left[\sigma_{3},\left[\sigma_{3}, V_{k+1}^{f}(x, t)\right]\right], \tag{1.4.10}
\end{equation*}
$$

we can see the RHS of (1.4.10) gives the off-diagonal part. We only need to change the commutator of the second term on the LHS and use equation (1.4.9) that yields:

$$
\begin{align*}
V_{k+1}^{f}(x, t)= & \frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\mathrm{~d} V_{k}^{f}}{\mathrm{~d} x}\right]-\frac{1}{4}\left[\sigma_{3},\left[\sigma_{3}, q(x, t)\right] \omega_{k}(x, t)\right] \\
= & \frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\mathrm{~d} V_{k}^{f}}{\mathrm{~d} x}\right] \\
& -\frac{\mathrm{i}}{2} q(x, t) \int_{ \pm \infty}^{x} \mathrm{~d} \lambda \operatorname{tr}\left(\sigma_{3}\left[q(y, t), V_{k}^{f}(y, t)\right]\right)-c_{k} q(x, t), \tag{1.4.11}
\end{align*}
$$

where $k=1, \ldots, N$. We will denote a new operator $\Lambda_{ \pm}$which represents an integrodifferential (recursion) operator

$$
\begin{equation*}
\Lambda_{ \pm} X=\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\mathrm{~d} X}{\mathrm{~d} x}\right]-\frac{\mathrm{i}}{2} q(x, t) \int_{ \pm \infty}^{x} \mathrm{~d} \lambda \operatorname{tr}\left(\sigma_{3}[q(y, t), X(y, t)]\right) \tag{1.4.12}
\end{equation*}
$$

Therefore, the recurrent relation (1.2.11b) can be written in the following form:

$$
\begin{align*}
V_{k+1}^{f}(x, t) & =\Lambda_{ \pm} V_{k}^{f}(x, t)-c_{k} q(x, t)  \tag{1.4.13a}\\
V_{1}(x, t) & =-c_{0} q(x, t) \tag{1.4.13b}
\end{align*}
$$

and because the NLS equation comes from the off-diagonal part of $V(x, t)$, we need to use (1.4.3a), then the lambda-independence equation (1.2.12) becomes:

$$
\begin{equation*}
-\mathrm{i} \frac{\partial q}{\partial t}+\mathrm{i} \frac{\partial V_{N}}{\partial x}+\left[q(x, t), \sigma_{3}\right] \omega_{N}(x, t)=0 \tag{1.4.14}
\end{equation*}
$$

and by applying $-\frac{1}{4}\left[\sigma_{3},.\right]$ to both sides of equation (1.4.14) and using (1.4.9), one can obtain the NLS equation with the special case:

$$
\begin{equation*}
\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\partial q}{\partial t}\right]-\Lambda_{ \pm} V_{N}^{f}(x, t)+c_{N} q(x, t)=0 \tag{1.4.15}
\end{equation*}
$$

or using $\lambda \equiv \Lambda_{ \pm}$, then (1.4.15) becomes:

$$
\begin{equation*}
\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\partial q}{\partial t}\right]+f\left(\Lambda_{ \pm}\right) q(x, t)=0 \tag{1.4.16a}
\end{equation*}
$$

where

$$
\begin{equation*}
f(\lambda)=\sum_{p=0}^{N} c_{p} \lambda^{N-p} . \tag{1.4.16b}
\end{equation*}
$$

Example 4 The NLS equation: We present an example of using the recursion operators (1.4.12) and (1.4.13) to obtain the NLS equation, we start to find $V_{N}^{f}(x, t), N=2$ :

$$
\begin{equation*}
V_{2}^{f}(x, t)=-\frac{\mathrm{i}}{2} c_{0} \sigma_{3} q_{x}-c_{0} q(x, t) . \tag{1.4.17}
\end{equation*}
$$

Substituting $V_{2}^{f}(x, t)$ into (1.4.15) gives:

$$
\begin{equation*}
\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\partial q}{\partial t}\right]-\Lambda_{ \pm}\left(-\frac{\mathrm{i} c_{0}}{2} \sigma_{3} q_{x}-c_{1} q(x, t)\right)+c_{2} q(x, t)=0 \tag{1.4.18}
\end{equation*}
$$

Finally, using the recursion operators (1.4.12) again, we have

$$
\begin{equation*}
-\mathrm{i} q_{t}-\frac{c_{0}}{2} \sigma_{3} q_{x x}+\mathrm{i} c_{1} q_{x}+c_{0} q^{+} q^{-} \sigma_{3} q-2 c_{2} \sigma_{3} q(x, t)=0, \tag{1.4.19}
\end{equation*}
$$

when $c_{0}=1$ and $c_{1}=c_{2}=0$, then equation (1.4.19) becomes the NLS equation:

$$
\begin{equation*}
-\mathrm{i} q_{t}+\sigma_{3} q_{x x}+2 q^{+} q^{-} \sigma_{3} q(x, t)=0 \tag{1.4.20}
\end{equation*}
$$

since $q(x, t)=\left(\begin{array}{cc}0 & q^{+} \\ q^{-} & 0\end{array}\right)$, then equation (1.4.20) becomes:

$$
\begin{align*}
-\mathrm{i} q_{t}^{+}+q_{x x}^{+}+2 q^{+} q^{-} q^{+}(x, t) & =0,  \tag{1.4.21}\\
\mathrm{i} q_{t}^{-}+q_{x x}^{-}+2 q^{+} q^{-} q^{-}(x, t) & =0, \tag{1.4.22}
\end{align*}
$$

and when we apply $q^{+}(x, t)=\epsilon\left(q^{-}\right)^{*}(x, t)$ on (1.4.22), we will obtain local NLS equation (1.1.3a).

### 1.4.1 Hierarchies and integrals of motion of soliton equations

As we showed in the previous subsection, for each NLEE, we have a different dispersion law $f(\lambda)$. Applying the IST to the ZS system (1.2.6a), one can solve each of the systems of the form [54]:

$$
\begin{equation*}
\frac{\mathrm{i}}{4}\left[\sigma_{3}, \frac{\mathrm{~d} q}{\mathrm{~d} t}\right]+f(\Lambda) q(x, t)=0 \tag{1.4.23}
\end{equation*}
$$

where $\Lambda$ is one of the so-called recursion operators $\Lambda_{+}, \Lambda_{-}$which is defined in (1.4.12) On the other hand the scattering matrix $T(\lambda, t)$ is

$$
T(\lambda, t)=\left(\begin{array}{cc}
a^{+}(\lambda) & -b^{-}(\lambda)  \tag{1.4.24}\\
b^{+}(\lambda) & a^{-}(\lambda)
\end{array}\right)
$$

we can calculate the scattering data for any moment $t>0$ from

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} T}{\mathrm{~d} t}+\left[f(\lambda) \sigma_{3}, T(\lambda, t)\right]=0 \tag{1.4.25}
\end{equation*}
$$

then, we have

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} a^{ \pm}}{\mathrm{d} t}=0, \quad \mathrm{i} \frac{\mathrm{~d} b^{ \pm}}{\mathrm{d} t} \mp 2 f(\lambda) b^{ \pm}(t, \lambda)=0 \tag{1.4.26}
\end{equation*}
$$

$a^{ \pm}(\lambda)$ are $t$-independent and also we can prove that they are analytic functions of $\lambda$ for $\lambda \in \mathbb{C}_{ \pm}$. Then, $a^{ \pm}(\lambda)$ can be expanded as a Taylor series:

$$
\begin{equation*}
\pm \ln a^{ \pm}(\lambda)=\sum_{k=1}^{\infty} C_{k} \lambda^{-k} \tag{1.4.27}
\end{equation*}
$$

where $C_{k}$ is an infinite number of integrals of motion. Then, $\pm \ln a^{ \pm}(\lambda)$ can be expressed as generating functionals of the integrals of motion for the hierarchy of NLEE.

### 1.5 Spectral theory of Lax operator

Here, we present briefly the basic concepts of the spectral theory of linear operators acting on Hilbert space (for more details see [16, 68, 99]). For a linear operator acting on the Hilbert space $\mathcal{H}$ (here, a linear transformation $\mathfrak{L}(\mathcal{H})$, is a vector space in its own right (from, and to a Hilbert space)), $T \in \mathfrak{L}(\mathcal{H})$, we say $\lambda \in \mathbb{C}$ is a regular point of $T$ if ( $T-\lambda \mathbb{1}$ ) is bounded ${ }^{1}$ and invertible. We denote the set of all regular points $\rho(T)$ the resolvent set of $T$ and $\sigma(T)$ is the complement of $\rho(T)$, called the spectrum of $T$.

### 1.5.1 Resolvent

Let $T \in \mathfrak{L}(\mathcal{H})$ be a linear operator and $\lambda \in \rho(T)$, then $R_{\lambda}(T)=(T-\lambda \mathbb{1})^{-1}$ is a resolvent of $T$ at $\lambda$. We need to look into the spectral decomposition from an analytical viewpoint. For instance, for finite-dimensional space, let $A$ be an arbitrary $N \times N$ matrix and $\lambda \in \mathbb{C}$ that is a larger eigenvalue $\lambda>\lambda_{1}>\lambda_{2}>\lambda_{3} \ldots$ of $A$. Then, the compact operator can

[^4]expand $R_{\lambda}(T)$ into a convergent power series as ${ }^{1}$ [68]:
\[

$$
\begin{align*}
R_{\lambda}(T)=(A-\lambda \mathbb{1})^{-1}=\frac{1}{-\lambda\left(\mathbb{1}-\frac{A}{\lambda}\right)} & =-\frac{1}{\lambda} \sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^{n} \\
& =-\frac{1}{\lambda}+\left(-\frac{1}{\lambda}\right) \frac{A}{\lambda}+\left(-\frac{1}{\lambda}\right)\left(\frac{A}{\lambda}\right)^{2}+\ldots \tag{1.5.1a}
\end{align*}
$$
\]

and then the residue of $R_{\lambda}(A)$ is the coefficient of $\frac{1}{\lambda}$

$$
\begin{equation*}
\operatorname{Res}\left[R_{\lambda}(A)\right]=-1, \quad \frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda R_{\lambda}(A)=-1 \tag{1.5.1b}
\end{equation*}
$$

If we change the position of $\lambda$ in $(1.5 .1 \text { a })^{2}$

$$
\begin{aligned}
-\lambda R_{\lambda}(A) & =1+\frac{A}{\lambda}+\left(\frac{A}{\lambda}\right)^{2}+\ldots \\
-\operatorname{Res}\left(\lambda R_{\lambda}(A)\right) & =A
\end{aligned}
$$

then,

$$
\begin{equation*}
-\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda \lambda R_{\lambda}(A)=A \tag{1.5.1c}
\end{equation*}
$$

and in general

$$
\begin{equation*}
-\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda(\lambda)^{n} R_{\lambda}(A)=A^{n}, \quad \text { for } \quad n=0,1, \ldots \tag{1.5.2}
\end{equation*}
$$

If we assume $f(A)$ and $f(\lambda)$ are power series ${ }^{3}$, then equation (1.5.2) can be written as:

$$
\begin{align*}
f(A) & =-\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda f(\lambda) R_{\lambda}(A) \\
& =\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda \frac{-f(\lambda)}{A-\lambda \mathbb{1}} \\
& =\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda \frac{f(\lambda)}{\lambda \mathbb{1}-A} . \tag{1.5.3}
\end{align*}
$$

Formula (1.5.3) is looks as the generalisation of the Cauchy integral formula to operator value functions. Next, we need to know the analytic behaviour of $R_{\lambda}(A)$. Using the theorem of the inverse of a matrix [84], we can write the resolvent of $A$ as:

$$
\begin{equation*}
\left[R_{\lambda}(A)\right]_{j k}=\left[(A-\lambda \mathbb{1})^{-1}\right]_{j k}=\frac{C_{j k}(\lambda)}{\operatorname{det}(A-\lambda \mathbb{1})} \tag{1.5.4}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\left[R_{\lambda}(A)\right]_{j k}=\frac{C_{j k}(\lambda)}{p(\lambda)} \tag{1.5.5}
\end{equation*}
$$

[^5]where $p(\lambda)$ is the characteristic polynomial of $A$ and $C_{j k}(\lambda)$ is the cofactor of the elements of the matrix $(A-\lambda \mathbb{1})$ and also a polynomial. Then, the rational function $\left[R_{\lambda}(A)\right]_{j k}$ of $\lambda$ has only poles as singularities and these poles are the eigenvalues of $A$. We defined a contour $\Gamma$, which consists of small circles $\gamma_{j}$ which circulate the isolated eigenvalues $\lambda_{j}$, then with $f(A)=1$, (see Fig. $1.7^{1}$ )
\[

$$
\begin{equation*}
1=-\frac{1}{2 \pi \mathrm{i}} \oint_{\Gamma} \mathrm{d} \lambda R_{\lambda}(A)=-\frac{1}{2 \pi \mathrm{i}} \sum_{j=1}^{r} \oint_{\Gamma} \mathrm{d} \lambda R_{\lambda}(A), \tag{1.5.6}
\end{equation*}
$$

\]

then,

$$
\begin{equation*}
1=\sum_{j=1}^{r} P_{j}, \quad P_{j}=-\frac{1}{2 \pi \mathrm{i}} \sum_{j=1}^{r} \oint_{\gamma_{j}} \mathrm{~d} \lambda R_{\lambda}(A) \tag{1.5.7}
\end{equation*}
$$

where $P_{j}$ are orthogonal projection operators. $\left\{P_{j}\right\}$ is a set of orthogonal projection operators. Then, equation (1.5.7) is a resolution of the identity (partition as spectral decomposition theorem) ${ }^{2}$.

(a) The contour $\Gamma$ containing all eigenvalues.

(b) the small contours $\gamma_{j}$ circulate the isolated eigenvalues $\lambda_{j}$.

Figure 1.7: The arrows direction represent the integration in the positive sense.

### 1.6 Integrable discretisation: The Ablowitz-Ladik equation

Completely integrable infinite-dimensional systems are a subject of constant interest and many investigations in different areas of Mathematics and Physics over the last five decades

[^6][40,54, 93] and they appear in a wide range of applications - from differential geometry to classical and quantum field theory, fluid mechanics and optics.

Nonlinear partial differential equations are notoriously difficult to solve and so far there is no general method to find solutions. However, there are certain types of nonlinear PDEs whose initial value problems can be solved using the IST method [38, 40, 93, 120]. A special a class of completely integrable infinite-dimensional systems is the PDEs which possess class of special solutions (soliton). The ISM is used to integrate the solutions to this kind of PDEs [40, 93]. The NLS equation

$$
\begin{equation*}
\mathrm{i} q_{t}+q_{x x}+2\left|q^{2}\right| q=0, \quad q=q(x, t) \tag{1.6.1}
\end{equation*}
$$

appeared at the very early stage of the development of the ISM [40, 93, 106] as one of the classical examples of integrable equations by the ISM and has attracted significant attention from the scientific community [30, 46, 120, 121]. It appears as a universal model for weakly nonlinear dispersive waves, nonlinear optics and plasma physics [12]. The NLS model has been generalised in several directions. The first one is to consider multicomponent generalisations. The first multi-component/vector generalisation of (1.6.1) was proposed by S. V. Manakov in 1974 (see [93])

$$
\begin{equation*}
i \mathbf{v}_{t}+\mathbf{v}_{x x}+2\left(\mathbf{v}^{\dagger}, \mathbf{v}\right) \mathbf{v}=0, \quad \mathbf{v}=\mathbf{v}(x, t) \tag{1.6.2}
\end{equation*}
$$

Here, $\mathbf{v}$ is an $n$-component complex-valued vector and $(\cdot, \cdot)$ is the standard scalar product. It is again integrable by the ISM [12, 40, 54, 93]. The multi-component NLS equation (called the Manakov model) appears in studies of electromagnetic waves in optical media. Another direction, motivated by the applications of the differential geometric and Lie algebraic methods to soliton type equations [18, 43, 52, 55, 57, 58, 63, 64, 85, 111, 119] (for a detailed review see e.g., [54]), has led to the discovery of a close relationship between the multicomponent (matrix) NLS equations and the homogeneous and symmetric spaces [43]. The first integrable discretisation of the NLS equation (1.6.1) was proposed by Ablowitz and A. Ladik (AL) and has the form [2, 3, 4]:

$$
\begin{equation*}
\mathrm{i} Q_{n, t}=\frac{1}{h^{2}}\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right) \pm\left|Q_{n}\right|^{2}\left(Q_{n+1}+Q_{n-1}\right) \tag{1.6.3}
\end{equation*}
$$

It is a differential-difference or semi-discrete equation (discrete in space and continuous in time), and is in fact a $O\left(h^{2}\right)$ finite difference approximation of (1.6.1). The corresponding scattering problem is usually called the AL scattering problem [5, 12, 13, 27, 47, 50, 51, 61]. Equation (1.6.3) also has several physical applications: it describes the dynamics
of anharmonic lattices [107], self-trapping on a dimer [75], and various types of Heisenberg spin chains [72, 95] amongst other. Various discretisations of the NLS models were studied [11, 29, 30, 109, 113, 114, 115, 116] including perturbation effects [35, 36]. The nonlocal reductions of the NLS equation (1.6.1) and the AL equation (1.6.3) are of particular interest in regards to applications in $\mathcal{P J}$-symmetric optics, especially in developing a theory of electromagnetic waves in artificial heterogenic media[1, 19]. For an up-to-date review, see for example [44, 122]. The initial interest in such systems was motivated by quantum mechanics [23, 24, 88]. In [23, 24] it was shown that quantum systems with a non-hermitian Hamiltonian admit states with real eigenvalues, i.e. the hermiticity of the Hamiltonian is not a necessary condition to have a real spectrum. Using such Hamiltonians, one can build up new quantum mechanics [22, 23, 24, 88, 89, 90]. The point is that in the case of a non-hermitian Hamiltonian with a real spectrum, the modulus of the wave function for the eigenstates is time-independent even in the case of complex potentials. There are a number of ideas that have a deep meaning which classify themselves with symmetries. Symmetry can be described as a transformation that leaves the system fundamentally unchanged after the transformation has been performed. Furthermore, symmetries have been an essential ingredient in the understanding of the physical laws of Nature. In this section, I will briefly concentrate on the Discrete Symmetries $\mathcal{C P}, \mathcal{T}$ and CPI. These three essential symmetries are defined as $[28,86,108]$

- Charge conjugation $(\mathcal{C})$ : reversing the electric charge and all the internal quantum numbers. We can also say that the symmetry between positive and negative charges causes the transformation of a particle into the corresponding anti-particle.
- Parity $(\mathcal{P})$ : space inversion; reversal of the space coordinates, but not the time.
- Time reversal $(\mathcal{T})$ : replacing $t$ by $-t$. This reverses time-derivatives such as momentum.

Space inversion (parity) symmetry was discovered to be broken. Then, there was the hope that the combination of $\mathcal{P}$ with charge conjugation ( $(\mathcal{P})$ was a good symmetry. Examination of the case of the neutrino is instructive at this point. The parity operation on a neutrino would leave its spin in the same direction while reversing space coordinates. Neither of these things is observed in nature; neutrinos are always left-handed, anti-neutrinos always right-handed. But if you add the charge conjugation operation, the result of the combined operation gives you back the original particle [26, 66, 96]. In quantum mechanics, time-reversal transformation, $\mathfrak{T}(\psi(t)) \rightarrow \psi^{*}(-t)$ keeps the Schrodinger


Figure 1.8: Mirror symmetry (inversion).
equation, $\mathrm{i} \hbar \partial \psi(t) / \partial t=\mathcal{H} \psi(t)$, invariant under a $\mathcal{T}$ transformation if the Hamiltonian $\mathcal{H}$ is real [26].

The $\mathcal{C P S}$ theorem appeared for the first time in the work of Julian Schwinger [104], and indicated that there is a connection between spin and statistics. For a short time, it was thought that the $\mathcal{C P}$-symmetry would always leave a system invariant, when the notable example of the neutral Kaons has shown a slightly violation of the $\mathcal{C P}$-symmetry, which implied violation of $\mathcal{T}$-symmetry as well. Then, the combination of all three symmetries ( CPJ ) is invariant if each of the independents is invariant [25,33]. CPJ -symmetry is recognised to be a fundamental property of physical laws, which has the implication that a "mirror-image" of our universe with all objects having their positions is reflected by an arbitrary plane (corresponding to a parity inversion), all momenta reversed (corresponding to a time inversion) and with all matter replaced by antimatter (corresponding to a charge inversion) would evolve under exactly our physical laws [15, 117].

### 1.7 Parity-time symmetry and integrable equations

Integrable systems with parity-time ( $\mathcal{P J}$ )-symmetry were studied extensively over the last two decades [21, 23, 102]. Recently, in [9, 49] the nonlocal integrable equation of NLS type with $\mathcal{P J}$-symmetry was proposed, due to the invariance of the so-called self-induced potential $V(x, t)=\psi(x, t) \psi^{*}(-x,-t)$ under the combined action of parity and time reversal symmetry. The one-soliton solution for this model is derived and it was shown that it develops singularities in finite time [7]. Soon after this, nonlocal $\mathcal{P T}$-symmetric generalisations were found for the AL model in [8]. The nonlocal reductions of the NLS (1.6.1) and the AL equations (1.6.3) are of particular interest in regards to applications in $\mathcal{P T}$-symmetric optics, especially in developing the theory of electromagnetic waves in artificial heterogenic media[1, 19]. For an up-to-date review, see for example [44, 122]. The initial interest in such systems was motivated by quantum mechanics [23, 24, 88]. In [23, 24], it was shown that quantum systems with a non-hermitian Hamiltonian admit
states with real eigenvalues, i.e. the hermiticity of the Hamiltonian is not a necessary condition to have a real spectrum. Using such Hamiltonians, one can build up new quantum mechanics [22, 23, 24, 88, 89, 90]. The starting point is the fact that in the case of a non-hermitian Hamiltonian with real spectrum, the modulus of the wave function for the eigenstates is time-independent even in the case of complex potentials.

The first pseudo-hermitian Hamiltonian with real spectrum historically was the $\mathcal{P J}$ symmetric Hamiltonian in [23, 24, 59, 100]. Pseudo-hermiticity here, means that the Hamiltonian $\mathcal{H}$ commutes with the operators of spatial reflection $\mathcal{P}$ and time reversal $\mathfrak{T}$ : $\mathcal{P J T}=\mathcal{H P T}$. The action of these operators is defined as $\mathcal{P}: x \rightarrow-x$ and $\mathcal{T}: t \rightarrow$ $-t$. If we assume that the wave function is a scalar, then this leads to the action of the operator of spatial reflection on the space of states: $\mathcal{P} \psi(x, t)=\psi(-x, t)$ and $\mathcal{T} \psi(x, t)=$ $\psi^{*}(x,-t)$. As a result, the Hamiltonian and the wave function are $\mathcal{P J}$-symmetric, if $\mathcal{H}(x, t)=\mathcal{H}^{*}(-x,-t)$ and $\psi(x, t)=\psi^{*}(-x,-t)$. Here, we also used the fact that the parity operator $\mathcal{P}$ is linear and unitary while the time reversal operator $\mathcal{T}$ is antilinear and antiunitary ${ }^{1}$. The action of the $\mathcal{P}$ and $\mathcal{T}$ operators on the Hamiltonian induces

$$
\begin{equation*}
\mathcal{P} Q_{n}(t)=Q_{-n}(t), \quad \mathcal{T} Q_{n}(t)=Q_{n}^{*}(-t), \tag{1.7.1}
\end{equation*}
$$

an action on the associated scattering problem (1.7.2) and to its potential (1.7.3)

$$
\begin{align*}
& \Psi_{n+1}(z, t)=\mathcal{L}_{n}(z, t) \Psi_{n}(z, t),  \tag{1.7.2}\\
& \Psi_{n+1}(z, t)=\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}(t)\right) \Psi_{n}(z, t),
\end{align*}
$$

where

$$
\mathbf{Z}=\left(\begin{array}{cc}
z & 0  \tag{1.7.3}\\
0 & z^{-1}
\end{array}\right), \quad \tilde{\mathbf{Q}}_{n}=\left(\begin{array}{cc}
0 & Q_{n}^{+} \\
Q_{n}^{-} & 0
\end{array}\right)
$$

This leads to the reduction (symmetry) condition [65]

$$
\begin{equation*}
Q_{n}^{-}(t)= \pm\left(Q^{+}\right)_{-n}^{*}(t) \tag{1.7.4}
\end{equation*}
$$

The generic differential-difference system (1.7.2) satisfies the symmetry of the form [65]:

$$
\begin{equation*}
\mathbf{C}\left[\mathcal{L}_{n}(z)\right]:=B \mathcal{L}_{-n}\left(z^{*}\right)^{\dagger} B^{-1}=\mathcal{L}_{n}(z), \tag{1.7.5}
\end{equation*}
$$

where $\mathbf{C}$ is an automorphism of the Lie group $S L(2, \mathbb{C})$. The particular choice $B=$

[^7]$\operatorname{diag}(1,-1)$ of the realisation of $\mathbf{C}$ will give $Q_{n}^{-}(t)=\epsilon\left(Q_{-n}^{+}(t)\right)^{*}$.
As a result, we obtain the nonlocal AL equation with $\mathcal{P T}$-symmetry, proposed in [8]:
\[

$$
\begin{equation*}
\mathrm{i} Q_{n, \tau}^{+}=\left(Q_{n+1}^{+}-2 Q_{n}^{+}+Q_{n-1}^{+}\right)-\epsilon Q_{n}^{+}\left(Q^{+}\right)_{-n}^{*}\left(Q_{n+1}^{+}+Q_{n-1}^{+}\right), \quad \epsilon= \pm 1 \tag{1.7.6}
\end{equation*}
$$

\]

### 1.8 Aims of thesis

This thesis aims:

1. to study the spectral properties of the semi-discrete Lax operators with nonlocal symmetries (resolvent, the completeness of Jost solutions);
2. to develop the IST for differential-difference Lax-operators of the AL type. This includes: the associated Jost solutions, scattering matrix, and the fundamental analytic solutions (FASs);
3. to obtain one-and two-soliton solutions to the RHP;
4. to study not just single box initial data to calculate the soliton solution to the nonlocal discrete and continuous NLS equations. The blow up and bounded solutions are discussed;
5. to study not just single box initial data to find the type of the solution to the nonlocal discrete Manakov NLS equations.

### 1.9 Outline of the thesis

We started our work in chapter 1 by discussing the history of the discovery of solitons and the NLEEs. Undoubtedly, the Riemann problem is the key tool to solve the ISP; for that, a short introduction was presented. Furthermore, we introduced a short introduction about the spectral theory, as basic information for what we will introduce in chapter 4. In addition, integrable discritisation of the AL equation, the $\mathcal{P T}$ and $\mathcal{C P} \mathcal{T}$-symmetry are introduced.

The core of this thesis, is to study the spectral properties of the semi-discrete with nonlocal symmetries; therefore we start in chapter 2 with the ZS and all the properties that can be related to build the presentation of the DNLS equation. This chapter includes: an introduction to the direct scattering problem of the ZS system. Next, the resolvent of the spectral problem is introduced which includes the Wronskian relations, completeness
of the "Squared" solutions and expansions over the "Squared" solutions. Finally, the involution of the local and nonlocal ZS system are presented.

The AL system with $\mathcal{P J}$-symmetry is introduced in chapter 3 . We start with the IST for the AL problem. This includes: the scattering problem, the Jost solutions and the FASs. Taking into consideration the time evolution equation using the zero curvature equation is derived. Such analysis is based on Lax operators for the DNLS equation. Evidently, the calculation of the time evolution of the scattering data is introduced. We study and outline the local and nonlocal DNLS equations. Additionally, the Riemann problem is used to analyse the one-and two- soliton solutions to the nonlocal DNLS equation that takes into account the nonlocal involution. We find the relation between the matrix RHP and the ISP for the AL system. This is the main idea of the dressing method, which is used to find the solution of the nonlocal DNLS equation. The results of this chapter appear in [65].

The completeness of the Jost solutions is intended for the expansion over the Jost solution in chapter 4 . We start with the spectral theory of the discrete Lax operator; this includes the resolvent of $\mathcal{L}_{n}$ and the new formula of the FASs. In this chapter we prove the formula of the completeness of the Jost solutions for the nonlocal DNLS equation. The results of this chapter appear in [65].

In chapter 5, the square barrier potential for the nonlocal NLS equation for two models continuous and discrete types is studied. We illustrate and analysis the general mathematical steps for two models to outline the conditions of having blow up or not blow up solutions. Next, we have presented numerical simulations for each model which are supported by providing different examples.

The discrete Manakov nonlocal NLS equation is presented in chapter 6. To continue what we have studied in chapter 5 the condition of having blow up or not blow up solutions for the nonlocal DNLS equation, in this chapter we have used one more condition to confirm whether the solution of the Manakov equation is blow up or not. The analytic approach and numerical examples are presented.

Finally, chapter 7 summarises the main results in the thesis and concludes with suggestions for further research.

## Chapter 2

## The Zakharov-Shabat system

### 2.1 Introduction

In this chapter, we will present some properties for the ZS system (see [54]). Here, we start to outline the direct scattering problem for the system as well as the properties of the FASs $\chi^{ \pm}(x, \lambda)$ which are used to construct the kernel of the resolvent $R^{ \pm}(x, y, \lambda)$ of the operator $L(\lambda)$. We also show the completeness of the square solutions and the expansions of the square solutions.

### 2.2 Direct scattering transform

### 2.2.1 Jost solutions and scattering matrix

Here, we present a scattering problem for the ZS system (1.2.6a). From the theory of linear differential equations, a solution $\chi(x, \lambda)$ of (1.2.6a) is called fundamental when $\operatorname{det} \chi(x, \lambda) \neq 0$ and does not depend on $x$. Another advantage of the theory of linear differential equations, is that any two fundamental solutions $\psi(x, \lambda)$ and $\phi(x, \lambda)$ must be linearly dependent.

$$
\begin{equation*}
\phi(x, \lambda)=\psi(x, \lambda) T(\lambda), \quad \lambda \in \mathbb{R} \tag{2.2.1a}
\end{equation*}
$$

where $T(\lambda, t)$ is the scattering matrix

$$
T(\lambda)=\left(\begin{array}{cc}
a^{+}(\lambda) & -b^{-}(\lambda)  \tag{2.2.1b}\\
b^{+}(\lambda) & a^{-}(\lambda)
\end{array}\right)
$$

and

$$
\begin{equation*}
\operatorname{det} \psi(x, \lambda)=\operatorname{det} \phi(x, \lambda)=1 \quad \text { and } \quad \operatorname{det} T(\lambda)=1 \tag{2.2.1c}
\end{equation*}
$$

Both solutions $\psi(x, t, \lambda)$ and $\phi(x, t, \lambda)$ are called Jost solutions (functions), and, in particular, they are introduced by their asymptotics for $x \rightarrow \infty$ or $x \rightarrow-\infty$ to be plane waves:

$$
\begin{align*}
& \lim _{x \rightarrow \infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \psi(x, \lambda)=\mathbb{1}  \tag{2.2.2a}\\
& \lim _{x \rightarrow-\infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \phi(x, \lambda)=\mathbb{1} \tag{2.2.2b}
\end{align*}
$$

where $\mathbb{1}$ is a $2 \times 2$ identity matrix.

### 2.2.2 Analytic properties of the Jost solution

In this section, we will define $\xi(x, \lambda)$ and $\varphi(x, \lambda)$ as an eigenfunctions which are successfully satisfied as associated systems, whose solutions are related to $\psi(x, \lambda)$ and $\phi(x, \lambda)$

$$
\begin{align*}
& \xi(x, \lambda)=\psi(x, \lambda) e^{\mathrm{i} \lambda \sigma_{3} x}  \tag{2.2.3a}\\
& \varphi(x, \lambda)=\phi(x, \lambda) e^{\mathrm{i} \lambda \sigma_{3} x} \tag{2.2.3b}
\end{align*}
$$

Furthermore, $\xi(x, \lambda)$ and $\varphi(x, \lambda)$ are solutions of the following linear differential equations:

$$
\begin{array}{r}
\mathrm{i} \frac{\mathrm{~d} \xi}{d x}+q(x) \xi(x, \lambda)-\lambda\left[\sigma_{3}, \xi(x, \lambda)\right]=0 \\
\mathrm{i} \frac{\mathrm{~d} \varphi}{d x}+q(x) \varphi(x, \lambda)-\lambda\left[\sigma_{3}, \varphi(x, \lambda)\right]=0 \tag{2.2.4b}
\end{array}
$$

with the boundary conditions:

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \xi(x, \lambda)=\mathbb{1}, \quad \lim _{x \rightarrow-\infty} \varphi(x, \lambda)=\mathbb{1} \tag{2.2.5}
\end{equation*}
$$

The system (2.2.4) with (2.2.5) can be represented as a system of integral equations

$$
\begin{align*}
& \xi(x, \lambda)=\mathbb{1}+\mathrm{i} \int_{\infty}^{x} \mathrm{~d} y e^{-\mathrm{i} \lambda \sigma_{3}(x-y)} q(y) \xi(y, \lambda) e^{\mathrm{i} \lambda \sigma_{3}(x-y)},  \tag{2.2.6a}\\
& \varphi(x, \lambda)=\mathbb{1}+\mathrm{i} \int_{-\infty}^{x} \mathrm{~d} y e^{-\mathrm{i} \lambda \sigma_{3}(x-y)} q(y) \varphi(y, \lambda) e^{\mathrm{i} \lambda \sigma_{3}(x-y)} . \tag{2.2.6b}
\end{align*}
$$

In addition, equation (2.2.6a) can be written in detail as

$$
\xi(x, \lambda)=\mathbb{1}+\mathrm{i} \int_{\infty}^{x} \mathrm{~d} y\left(\begin{array}{cc}
q^{+}(y) \xi_{21}(y, \lambda) & q^{+}(y) \xi_{22}(y, \lambda) e^{-2 \mathrm{i} \lambda(x-y)}  \tag{2.2.7}\\
q^{-}(y) \xi_{11}(y, \lambda) e^{2 \mathrm{i} \lambda(x-y)} & q^{-}(y) \xi_{12}(y, \lambda)
\end{array}\right)
$$

Similarly, the explicit representation for $\varphi(x, \lambda)$ can be written as $\xi(x, \lambda)$ in equation (2.2.7) but the lower limit will be $-\infty$.

### 2.2.3 The fundamental analytic solutions and its inverse

The properties of the fundamental solutions ( $\phi$ and $\psi$ ) are supposed to construct the FASs of the ZS system which are obtained by combining the pairs of columns of the Jost functions with the same analyticity properties:

$$
\begin{align*}
& \chi^{+}(x, \lambda)=\left(\varphi^{+}, \xi^{+}\right) e^{-\mathrm{i} \lambda \sigma_{3} x}=\left(\phi^{+}, \psi^{+}\right),  \tag{2.2.8a}\\
& \chi^{-}(x, \lambda)=\left(\xi^{-}, \varphi^{-}\right) e^{-\mathrm{i} \lambda \sigma_{3} x}=\left(\psi^{-}, \phi^{-}\right), \tag{2.2.8b}
\end{align*}
$$

since $\xi^{+}, \xi^{-}$are fundamental solutions to (1.2.6a), they are therefore linearly dependent and can be represented as:

$$
\begin{align*}
& \chi^{+}(x, \lambda)=\psi(x, \lambda)\left(\begin{array}{ll}
a^{+} & 0 \\
b^{+} & 1
\end{array}\right)=\phi(x, \lambda)\left(\begin{array}{ll}
1 & b^{-} \\
0 & a^{+}
\end{array}\right),  \tag{2.2.9a}\\
& \chi^{-}(x, \lambda)=\psi(x, \lambda)\left(\begin{array}{cc}
1 & -b^{-} \\
0 & a^{-}
\end{array}\right)=\phi(x, \lambda)\left(\begin{array}{cc}
a^{-} & 0 \\
-b^{+} & 1
\end{array}\right) . \tag{2.2.9b}
\end{align*}
$$

Now, we can find a relation between the scattering data and the determinant of the fundamental analytic solution. Since $\operatorname{det} \psi=\operatorname{det} \phi=1$, we can find easily from equation (2.2.9b) that

$$
\operatorname{det}\left(\chi^{+}\right)(x, \lambda)=\operatorname{det}(\psi) \operatorname{det}\left(\begin{array}{cc}
a^{+} & 0  \tag{2.2.10}\\
b^{+} & 1
\end{array}\right)=a^{+}(\lambda), \quad \operatorname{det}\left(\chi^{-}\right)(x, \lambda)=a^{-}(\lambda)
$$

In order to show that the inverse of the FASs, ( $\chi^{+}$and $\chi^{-}$), are also FASs, we will use the cooperation of the inverse of the fundamental solution $\psi(x, \lambda)$. First we need to calculate the inverse of matrix $\psi$ as $\psi^{-1}=\hat{\psi}$ which satisfies the ZS system (1.2.6a)

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} \hat{\psi}}{\mathrm{~d} x}-\hat{\psi}(x, \lambda)\left(q(x)-\lambda \sigma_{3}\right)=0 \tag{2.2.11}
\end{equation*}
$$

since

$$
\psi^{-1}(x, \lambda)=\hat{\psi}(x, \lambda)=\left(\begin{array}{cc}
\psi_{2}^{+} & -\psi_{1}^{+}  \tag{2.2.12}\\
-\psi_{2}^{-} & \psi_{1}^{-}
\end{array}\right)(x, \lambda) .
$$

Then, substituting equation (2.2.12) in equation (2.2.1a) yields

$$
\begin{align*}
T(\lambda, t)=\left(\begin{array}{cc}
a^{+}(\lambda) & -b^{-}(\lambda) \\
b^{+}(\lambda) & a^{-}(\lambda)
\end{array}\right) & =\left(\begin{array}{cc}
\psi_{2}^{+} & -\psi_{1}^{+} \\
-\psi_{2}^{-} & \psi_{1}^{-}
\end{array}\right)\left(\begin{array}{cc}
\phi_{1}^{+} \phi_{1}^{-} \\
\phi_{2}^{+} & \phi_{2}^{-}
\end{array}\right)  \tag{2.2.13a}\\
& =\binom{\tilde{\psi}^{+}(x, \lambda)}{-\tilde{\psi}^{-}(x, \lambda)}\left(\phi^{+}, \phi^{-}\right)(x, \lambda) \tag{2.2.13b}
\end{align*}
$$

where vector $\tilde{H}$ is defined as $\tilde{H}=\left(H_{2},-H_{1}\right)$. One of the scattering data $a^{+}(\lambda)$ can be found from equation (2.2.13b)

$$
\begin{equation*}
a^{+}(\lambda)=\tilde{\psi}^{+}(x, \lambda) \phi^{+}(x, \lambda) \tag{2.2.14}
\end{equation*}
$$

or $a^{+}(\lambda)$ can be obtain from equation (2.2.13a),

$$
\begin{equation*}
a^{+}(\lambda)=\left(\psi_{2}^{+} \phi_{1}^{+}-\psi_{1}^{+} \phi_{2}^{+}\right)=-W\left(\psi^{+}, \phi^{+}\right) . \tag{2.2.15}
\end{equation*}
$$

Therefore, the inverse of $\chi^{+}$and $\chi^{-}$can be obtained by equating both sides of equation (2.2.9a) and (2.2.9b), respectively

$$
\hat{\chi}^{+}=\frac{1}{a^{+}}\left(\begin{array}{cc}
\psi_{2}^{+} & -\psi_{1}^{+}  \tag{2.2.16}\\
-\phi_{2}^{+} & \phi_{1}^{+}
\end{array}\right)=\frac{1}{a^{+}}\binom{\tilde{\psi}^{+}(x, \lambda)}{-\tilde{\phi}^{+}(x, \lambda)},
$$

where $\tilde{\psi}^{+}=\left(\psi_{2},-\psi_{1}\right)$ and

$$
\hat{\chi}^{-}=\frac{1}{a^{-}}\left(\begin{array}{cc}
\phi_{2}^{-} & -\phi_{1}^{-}  \tag{2.2.17}\\
-\psi_{2}^{-} & \psi_{1}^{-}
\end{array}\right)=\frac{1}{a^{-}}\binom{\tilde{\phi}^{-}(x, \lambda)}{-\tilde{\psi}^{-}(x, \lambda)} .
$$

### 2.2.4 Asymptotic behaviour of FASs for $\lambda \rightarrow \infty$

Contour integration methods played an important role in this section. This section will point out that the analytic property of $\chi^{ \pm}(x, \lambda)$ is not enough. So, defining new functions $\eta^{ \pm}(x, \lambda)$

$$
\begin{equation*}
\eta^{ \pm}(x, \lambda)=\chi^{ \pm}(x, \lambda) e^{\mathrm{i} \lambda \sigma_{3} x} \tag{2.2.18}
\end{equation*}
$$

will help us to find additional properties to check what will happen when $\lambda \rightarrow \infty$. When we differentiate (2.2.18) with respect to $x$, we get:

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} \eta^{+}}{\mathrm{d} x}=\mathrm{i} \frac{\mathrm{~d} \chi^{+}}{\mathrm{d} x} e^{\mathrm{i} \lambda \sigma_{3} x}-\chi^{+} e^{\mathrm{i} \lambda \sigma_{3} x}\left(\lambda \sigma_{3}\right), \tag{2.2.19}
\end{equation*}
$$

and as $\chi^{ \pm}$satisfies the ZS system, $\eta^{ \pm}(x, \lambda)$ satisfies

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} \eta^{ \pm}}{\mathrm{d} x}+q(x) \eta^{ \pm}-\lambda\left[\sigma_{3}, \eta^{ \pm}\right]=0 . \tag{2.2.20}
\end{equation*}
$$

From the FASs $\chi^{+}$and $\chi^{-}$(see (2.2.9b)), the following equations (2.2.21) are hold only for all $\lambda \in \mathbb{C}$

$$
\begin{gather*}
\chi^{+}(x, \lambda) \underset{x \rightarrow \infty}{\longrightarrow} e^{-\mathrm{i} \lambda \sigma_{3} x}\left(\begin{array}{cc}
a^{+} & 0 \\
b^{+} & 1
\end{array}\right) \\
\chi^{+}(x, \lambda) \underset{x \rightarrow-\infty}{\longrightarrow} e^{-\mathrm{i} \lambda \sigma_{3} x}\left(\begin{array}{ll}
1 & b^{-} \\
0 & a^{+}
\end{array}\right)  \tag{2.2.21a}\\
\chi^{-}(x, \lambda) \underset{x \rightarrow \infty}{\longrightarrow} e^{-\mathrm{i} \lambda \sigma_{3} x}\left(\begin{array}{cc}
1 & -b^{-} \\
0 & a^{-}
\end{array}\right), \\
\chi^{-}(x, \lambda) \underset{x \rightarrow-\infty}{\longrightarrow} e^{-\mathrm{i} \lambda \sigma_{3} x}\left(\begin{array}{cc}
a^{-} & 0 \\
-b^{+} & 1
\end{array}\right) \tag{2.2.21b}
\end{gather*}
$$

Now, we will find the asymptotic behaviour for the scattering data from $\lambda=\infty$. For that, the asymptotic expansions of $\eta^{ \pm}(x, \lambda)$ over the inverse powers of $\lambda$ have the form:

$$
\begin{equation*}
\eta^{ \pm}(x, \lambda)=\mathbb{1}+\sum_{k=1}^{\infty} \eta_{k}^{ \pm}(x) \lambda^{-k}, \quad \lambda \in \mathbb{C}_{ \pm} \tag{2.2.22}
\end{equation*}
$$

If we insert (2.2.22) into (2.2.20), we get the first relation from the term $\lambda^{0}$ is

$$
\begin{equation*}
q(x)-\left[\sigma_{3}, \eta_{1}^{ \pm}\right]=0, \tag{2.2.23}
\end{equation*}
$$

which can be considered as the initial condition. The second relation from the terms $\lambda^{-k}$ is

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} x} \eta_{k}^{ \pm}(x)+q(x) \eta_{k}^{ \pm}(x)-\left[\sigma_{3}, \eta_{(k+1)}^{ \pm}(x)\right]=0 . \tag{2.2.24}
\end{equation*}
$$

Equation (2.2.24) is close to the one for $V_{k}(x, t)$ in the zero curvature equation (1.2.12). So, they are solved in an analogous way by splitting $\eta_{k}^{ \pm}(x)$,

$$
\begin{equation*}
\eta_{k}^{ \pm}(x)=\left(\eta_{k}^{ \pm}(x)\right)^{d}+\left(\eta_{k}^{ \pm}(x)\right)^{f} . \tag{2.2.25}
\end{equation*}
$$

So, at $k=1$,

$$
\begin{equation*}
\eta_{1}^{ \pm}(x)=\left(\eta_{1}^{ \pm}(x)\right)^{d}+\left(\eta_{1}^{ \pm}(x)\right)^{f}, \tag{2.2.26}
\end{equation*}
$$

substituting the above equation into (2.2.23), then we get:

$$
\begin{equation*}
\left(\eta^{ \pm}(x)\right)^{f}=\frac{1}{2} \sigma_{3} q(x)=\frac{1}{4}\left[\sigma_{3}, q(x)\right], \tag{2.2.27}
\end{equation*}
$$

and the diagonal part of $\left(\eta^{ \pm}\right)^{d}(x)$ is calculated using (2.2.24) with $k=1$, using the behaviour of $\eta^{ \pm}(x, \lambda)$ for $x \rightarrow \pm \infty$ and since $\eta^{ \pm}=\chi^{ \pm} e^{\mathrm{i} \lambda \sigma_{3} x}$

$$
\begin{equation*}
\left(\eta_{1}^{+}(x)\right)^{d}=-\frac{\mathrm{i}}{2} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y) \sigma_{3}, \tag{2.2.28}
\end{equation*}
$$

therefore,

$$
\begin{align*}
& \left(\eta_{1}^{+}\right)(x)=\frac{1}{2}\left(\begin{array}{cc}
-\mathrm{i} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y) & q^{+}(x) \\
-q^{-}(x) & \mathrm{i} \int_{\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y)
\end{array}\right),  \tag{2.2.29a}\\
& \left(\eta_{1}^{-}\right)(x)=\frac{1}{2}\left(\begin{array}{cc}
-\mathrm{i} \int_{\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y) & q^{+}(x) \\
-q^{-}(x) & \mathrm{i} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y)
\end{array}\right) . \tag{2.2.29b}
\end{align*}
$$

Continuing this procedure, we can subsequently find $\eta_{2}^{ \pm}(x), \eta_{3}^{ \pm}(x), \ldots$. When we take $k=1$ in (2.2.22) then,

$$
\begin{equation*}
\eta^{+}(x, \lambda)=\mathbb{1}+\eta_{1}^{+}(x) \lambda^{-1}, \quad \lambda \in \mathbb{C}, \tag{2.2.30}
\end{equation*}
$$

and from (2.2.21) $\eta^{+}(x)$ we will obtain:

$$
\left(\begin{array}{cc}
a^{+}(\lambda) & 0  \tag{2.2.31}\\
b^{+}(\lambda) e^{2 \mathrm{i} \lambda x} & 1
\end{array}\right)=\left(\begin{array}{cc}
1-\frac{\mathrm{i}}{2 \lambda} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y) & \frac{1}{2 \lambda} q^{+}(x) \\
-\frac{1}{2 \lambda} q^{-}(x) & 1+\frac{\mathrm{i}}{2 \lambda} \int_{\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y)
\end{array}\right),
$$

and

$$
\begin{align*}
& a^{+}(\lambda)=\lim _{x \rightarrow \infty}\left(\eta^{+}(x, \lambda)\right)_{11}=1-\frac{\mathrm{i}}{2 \lambda} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y)+O\left(\lambda^{-2}\right),  \tag{2.2.32a}\\
& b^{+}(\lambda)=\lim _{x \rightarrow \infty}\left(\eta^{+}(x, \lambda)\right)_{21}=\frac{-1}{2 \lambda} q^{-}(x) e^{-2 \mathrm{i} \lambda x}=O\left(\lambda^{-1}\right) . \tag{2.2.32b}
\end{align*}
$$

By analogy, we can find $a^{-}(\lambda), b^{-}(\lambda)$ from equation (2.2.29b)

$$
\begin{align*}
& a^{-}(\lambda)=\lim _{x \rightarrow \infty}\left(\eta^{-}(x, \lambda)\right)_{22}=1+\frac{\mathrm{i}}{2 \lambda} \int_{-\infty}^{x} \mathrm{~d} y q^{-}(y) q^{+}(y)+O\left(\lambda^{-2}\right),  \tag{2.2.33a}\\
& b^{-}(\lambda)=-\lim _{x \rightarrow \infty}\left(\eta^{-}(x, \lambda)\right)_{12}=\frac{-1}{2 \lambda} q^{-}(x) e^{-2 \mathrm{i} \lambda x}=O\left(\lambda^{-1}\right) \tag{2.2.33b}
\end{align*}
$$

### 2.3 The spectrum of $L(\lambda)$

In this section, the FASs are used to construct new functions called the resolvent of $L$. Our aim is to find the discrete eigenvalues, i.e all $\lambda$ that do not belong to the spectrum of $L$. The spectrum of $L$ is the complement set of the set of the points that render the resolvent of $L$ bounded. The resolvent functions are introduced as:

$$
\begin{align*}
& R^{+}(x, y, \lambda)=\frac{1}{\mathrm{i}} \chi^{+}(x, \lambda)\left(\begin{array}{cc}
-\theta(y-x) & 0 \\
0 & \theta(x-y)
\end{array}\right) \hat{\chi}^{+}(y, \lambda),  \tag{2.3.1a}\\
& R^{-}(x, y, \lambda)=\frac{1}{\mathrm{i}} \chi^{-}(x, \lambda)\left(\begin{array}{cc}
\theta(x-y) & 0 \\
0 & -\theta(y-x)
\end{array}\right) \hat{\chi}^{-}(y, \lambda), \tag{2.3.1b}
\end{align*}
$$

where $\theta(x)$ is the step function and $\frac{\mathrm{d} \theta}{\mathrm{d} x}=\delta(x)$

$$
\theta(x)=\left\{\begin{array}{cl}
1, & \text { for } x>0  \tag{2.3.2}\\
1 / 2, & \text { for } x=0 \\
0, & \text { for } x<0
\end{array}\right.
$$

Then,

- the differentiation of equation (2.3.1) with respect to $x$ makes $R^{+}$satisfy

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} R^{+}}{\mathrm{d} x}+\left(q(x)-\lambda \sigma_{3}\right) R^{+}(x, y, \lambda)=\delta(x-y) \mathbb{1} \tag{2.3.3}
\end{equation*}
$$

- $R^{ \pm}$are analytic functions of $\lambda$ for $\operatorname{Im} \lambda \lessgtr 0$, respectively, at all points $\lambda \neq \lambda_{k}^{ \pm}$and at $\lambda_{k}^{ \pm}$, the function $R^{ \pm}(x, y, \lambda)$ has poles.

The kernel of the resolvent of the operator of $L, R(x, y, \lambda)$, must satisfy the above conditions where:

$$
\begin{align*}
R_{\lambda} f & \equiv \int_{-\infty}^{\infty} \mathrm{d} y R(x, y, \lambda) f(y),  \tag{2.3.4}\\
R(x, y, \lambda) & = \begin{cases}R^{+}(x, y, \lambda), & \operatorname{Im} \lambda>0 \\
R^{-}(x, y, \lambda), & \operatorname{Im} \lambda<0\end{cases} \tag{2.3.5}
\end{align*}
$$

In order to ensure that the integral operator $R_{\lambda}$ is well defined. Then, $R^{ \pm}(x, y, \lambda)$ should fall fast enough for $x, y \rightarrow \pm \infty$. So, when $x, y \rightarrow \infty$, then

$$
R^{+}(x, y, \lambda) \Rightarrow \lim _{x, y \rightarrow \infty} \frac{1}{\mathrm{i}}\left(\begin{array}{cc}
-\theta(y-x) e^{-\mathrm{i} \lambda(x-y)} & 0  \tag{2.3.6}\\
-\frac{b^{+}}{a^{+}} e^{\mathrm{i} \lambda(x+y)} & \theta(x-y) e^{\mathrm{i} \lambda(x-y)}
\end{array}\right) .
$$

For $\operatorname{Im} \lambda \neq 0, \operatorname{Im} \lambda>0$ all matrix elements of $R^{+}$fall off exponentially when $x$ and $y$ tend independently to $\infty$. In other words, $\lambda_{k}^{ \pm}$are simple discrete eigenvalue of the operator $L$. In addition, the behaviour of the solutions $\chi^{ \pm}$is described in the neighborhood of the points $\lambda_{k}^{ \pm}$from (2.2.9a) $\operatorname{det} \chi^{ \pm}=a^{ \pm}(\lambda)$. According to the condition that we mentioned in Condition 2 (in chapter 1), both $a^{ \pm}\left(\lambda_{k}^{ \pm}\right)$have a simple pole, so $a^{ \pm}\left(\lambda_{k}^{ \pm}\right)=0$ and by equating both side of (2.2.9a)

$$
\begin{equation*}
\phi_{k}^{ \pm}(x)= \pm b_{k}^{ \pm} \psi_{k}^{ \pm}(x), \tag{2.3.7a}
\end{equation*}
$$

where

$$
\begin{equation*}
\phi_{k}^{ \pm}(x)=\phi^{ \pm}\left(x, \lambda_{k}^{ \pm}\right) \quad \text { and } \quad \psi_{k}^{ \pm}(x)=\psi^{ \pm}\left(x, \lambda_{k}^{ \pm}\right) . \tag{2.3.7b}
\end{equation*}
$$

### 2.3.1 The Wronskian relations

The Wronskian relation plays an essential role in proving that the IST is a generalised Fourier transform. The Wronskian relations are the map from the potentials $\mathcal{M}$ and the scattering data of $L$, ( $L$-operator). This can help formulate the idea that the IST is a generalised Fourier transform.

The first Wronskian relation starts with the identity representation

$$
\begin{equation*}
\left.\left(\hat{\chi} \sigma_{3} \chi(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} x \frac{\mathrm{~d}}{\mathrm{~d} x}\left(\mathrm{i} \hat{\chi} \sigma_{3} \chi\right)(x, \lambda), \tag{2.3.8}
\end{equation*}
$$

the ZS system associated with (2.2.11), then $\hat{\chi}$ satisfies:

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d} \hat{\chi}}{\mathrm{~d} x}-\hat{\chi}(x, \lambda)\left(q(x)-\lambda \sigma_{3}\right)=0 \tag{2.3.9}
\end{equation*}
$$

and (2.3.8) becomes:

$$
\begin{equation*}
\left.\left(\hat{\chi} \sigma_{3} \chi(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} x \hat{\chi}\left[U(x, \lambda), \sigma_{3}\right] \chi(x, \lambda), \tag{2.3.10a}
\end{equation*}
$$

or

$$
\begin{equation*}
\left.\left(\hat{\chi} \sigma_{3} \chi(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} x \hat{\chi}\left[q(x), \sigma_{3}\right] \chi(x, \lambda) . \tag{2.3.10b}
\end{equation*}
$$

In addition, the LHS of equation (2.3.10b) can be represented by the coefficients of the

Jost functions by using the asymptotic behaviours of $\chi^{ \pm}(x, \lambda)(2.2 .21)$

$$
\begin{align*}
& \left.\left(\hat{\chi}^{+} \sigma_{3} \chi^{+}(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-2\left(\begin{array}{cc}
0 & b^{-}(\lambda) \\
b^{+}(\lambda) & 0
\end{array}\right),  \tag{2.3.11a}\\
& \left.\left(\hat{\chi}^{-} \sigma_{3} \chi^{-}(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-2\left(\begin{array}{cc}
0 & b^{-}(\lambda) \\
b^{+}(\lambda) & 0
\end{array}\right) . \tag{2.3.11b}
\end{align*}
$$

Meanwhile, equation (2.3.11) will help show that the reflection coefficients can be expressed by the potential functions. The Wronskian relations (2.3.10b) (with $\chi \equiv \chi^{+}$) can define $\rho^{ \pm}$and $\tau^{ \pm}$as integrals of the potential $q(x)$ multiplied by some bilinear combination of the eigenfunctions of $L$. In order to express this reflection

$$
\begin{equation*}
\rho^{ \pm}(\lambda)=\frac{b^{ \pm}(\lambda)}{a^{ \pm}(\lambda)}, \quad \tau^{ \pm}(\lambda)=\frac{b^{\mp}(\lambda)}{a^{ \pm}(\lambda)}, \quad \lambda \in \mathbb{R} \tag{2.3.12}
\end{equation*}
$$

The following relation is to represent the Wronskian relation by the reflection coefficient function $\rho^{+}(\lambda)$. We need to replace $\chi \equiv \chi^{+}(x, \lambda)$ and multiply both sides of equation (2.3.10b) by $\sigma_{+}$. Dividing by $a^{+}(\lambda)$, then the trace yields

$$
\begin{align*}
&\left.\left(\hat{\chi}^{+} \sigma_{3} \chi^{+}(x, \lambda)-\sigma_{3}\right)\right|_{-\infty} ^{\infty}=-\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} x \hat{\chi^{+}}\left[q(x), \sigma_{3}\right] \chi^{+}(x, \lambda),  \tag{2.3.13a}\\
& \frac{2 b^{+}(\lambda)}{a^{+}(\lambda)}=\frac{\mathrm{i}}{a^{+}(\lambda)} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(\left[q(x), \sigma_{3}\right]\right. \\
&\left.\chi^{+}(x, \lambda) \sigma_{+} \hat{\chi^{+}}(x, \lambda)\right), \tag{2.3.13b}
\end{align*}
$$

where $\sigma_{+}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$. To simplify (2.3.13b), we need to define a new function $\mathcal{E}_{+}^{+}(x, y)$

$$
\begin{equation*}
\mathcal{E}_{+}^{+}(x, \lambda)=\chi^{+}(x, \lambda) \sigma_{+} \hat{\chi^{+}}(x, \lambda), \tag{2.3.14}
\end{equation*}
$$

where $\operatorname{tr}\left(\left[q(x), \sigma_{3}\right] \mathcal{E}_{+}^{+}\right)=\operatorname{tr}\left(q(x)\left[\sigma_{3}, \mathcal{E}_{+}^{+}\right]\right)$and the skew-symmetric scalar product $\llbracket .$, . $\rrbracket$ for any two matrices is defined as:

$$
\begin{equation*}
\llbracket X, Y \rrbracket \equiv \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(X(x),\left[\sigma_{3}, Y\right]\right)=-\llbracket Y, X \rrbracket . \tag{2.3.15}
\end{equation*}
$$

By applying the properties of the matrices, we will obtain a new formula for the reflection
coefficient function $\rho^{+}(\lambda)$. Then, the RHS of equation (2.3.13b) becomes:

$$
\begin{align*}
& \frac{2 b^{+}(\lambda)}{a^{+}(\lambda)}=\frac{\mathrm{i}}{a^{+}(\lambda)} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(\left[q(x), \sigma_{3}\right] \mathcal{E}_{+}^{+}\right), \\
& 2 \rho^{+}(\lambda)=\frac{\mathrm{i}}{a^{+}(\lambda)} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(q(x)\left[\sigma_{3}, \mathcal{E}_{+}^{+}\right]\right), \tag{2.3.16a}
\end{align*}
$$

then,

$$
\begin{equation*}
\rho^{+}(\lambda)=\frac{\mathrm{i}}{a^{+}(\lambda)} \llbracket q(x), \varepsilon_{+}^{+}(x, \lambda) \rrbracket . \tag{2.3.16b}
\end{equation*}
$$

Another Wronskian relation can be defined from equation (2.3.16a)

$$
\begin{equation*}
\rho^{+}(\lambda)=\frac{\mathrm{i}}{a^{+}(\lambda)} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(q(x)\left[\sigma_{3}, \mathcal{E}_{+}^{+}(x, \lambda)\right]\right) \tag{2.3.17}
\end{equation*}
$$

By defining $\Phi^{+}(x, \lambda)$ to be the off-diagonal of $\mathcal{E}_{+}^{+}$, the so-called "squared" solutions of $L$

$$
\boldsymbol{\Phi}^{+}(x, \lambda)=a^{+}(\lambda) \varepsilon_{+}^{+f}(x, \lambda)=\left(\begin{array}{cc}
0 & \left(\phi_{1}^{+}(x, \lambda)\right)^{2}  \tag{2.3.18}\\
-\left(\phi_{2}^{+}(x, \lambda)\right)^{2} & 0
\end{array}\right) .
$$

Then, applying equation (2.3.18) in equation (2.3.17), and a new formula will appear for the reflection coefficient $\rho^{+}(\lambda)$

$$
\begin{align*}
\rho^{+}(\lambda) & =\frac{\mathrm{i}}{a^{+}(\lambda)} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(\frac{1}{a^{+}} q(x)\left[\sigma_{3}, \boldsymbol{\Phi}^{+}\right](x, \lambda)\right) \\
& =\frac{\mathrm{i}}{\left(a^{+}(\lambda)\right)^{2}} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(q(x)\left[\sigma_{3}, \boldsymbol{\Phi}^{+}\right](x, \lambda)\right), \tag{2.3.19}
\end{align*}
$$

by using the properties of $\sigma_{3}$ in equation (2.3.19)

$$
\begin{equation*}
\rho^{+}(\lambda)=\frac{\mathrm{i}}{\left(a^{+}(\lambda)\right)^{2}} \int_{-\infty}^{\infty} \mathrm{d} x \operatorname{tr}\left(2 q(x) \Phi^{+}(x, \lambda)\right) . \tag{2.3.20}
\end{equation*}
$$

In case $q(x) \simeq 0$ as $x \rightarrow \pm \infty$ the limits of the Jost function $\phi(x, \lambda)$ and $\psi(x, \lambda)$ in (2.2.2) confirm that $\phi_{2}^{+}(x, \lambda)=0, \phi_{1}^{+}(x, \lambda)=e^{-\mathrm{i} \lambda x}, \psi_{1}^{+}(x, \lambda)=0$ and $\psi_{2}^{+}(x, \lambda)=e^{\mathrm{i} \lambda x}$. Then, from equation (2.2.15), $a^{+}(\lambda) \rightarrow 1$. The off-diagonal matrix $\boldsymbol{\Phi}^{+}(x, \lambda)=\left(\phi_{1}^{+}(x, \lambda)\right)^{2} \sigma^{+}$. Therefore the final representation for equation (2.3.20) will be

$$
\begin{equation*}
\rho^{+}(\lambda)=2 \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d} x q^{-}(x) e^{-2 \mathrm{i} \lambda x} \tag{2.3.21}
\end{equation*}
$$

The main goal is to show that this interpretation holds true for any potential $q(x) \in \mathcal{M}$. For our purpose, we need to find a new formula for any potential from (2.3.10b). Let us recall (2.3.16b) acting with the off-diagonal $\Phi^{ \pm}$and the skew-symmetric scalar product

$$
\begin{equation*}
\rho^{ \pm}(\lambda)=\frac{\mathrm{i}}{\left(a^{ \pm}(\lambda)\right)^{2}} \llbracket q(x), \boldsymbol{\Phi}^{ \pm}(x, \lambda) \rrbracket, \tag{2.3.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau^{ \pm}(\lambda)=\frac{\mathrm{i}}{\left(a^{ \pm}(\lambda)\right)^{2}} \llbracket q(x), \boldsymbol{\Psi}^{ \pm}(x, \lambda) \rrbracket . \tag{2.3.23}
\end{equation*}
$$

Then, the general "squared" solutions $\boldsymbol{\Phi}^{ \pm}(x, \lambda)$ and $\boldsymbol{\Psi}^{ \pm}(x, \lambda)$ are defined as

$$
\begin{gather*}
\boldsymbol{\Phi}^{ \pm}(x, \lambda)=a^{ \pm}(\lambda) \mathcal{E}_{ \pm}^{ \pm}(x, \lambda)^{f}=\left(\begin{array}{cc}
0 & \pm\left(\phi_{1}^{ \pm}(x, \lambda)\right)^{2} \\
\mp\left(\phi_{2}^{ \pm}(x, \lambda)\right)^{2} & 0
\end{array}\right),  \tag{2.3.24a}\\
\boldsymbol{\Psi}^{ \pm}(x, \lambda)=a^{ \pm}(\lambda) \mathcal{E}_{\mp}^{ \pm}(x, \lambda)^{f}=\left(\begin{array}{cc}
0 & \mp\left(\psi_{1}^{ \pm}(x, \lambda)\right)^{2} \\
\pm\left(\psi_{2}^{ \pm}(x, \lambda)\right)^{2} & 0
\end{array}\right) . \tag{2.3.24b}
\end{gather*}
$$

### 2.3.2 Completeness of the "Squared" solutions

The Wronskian relation is the transition from the potential $q(x)$ to the scattering data $\mathcal{T}_{k}, k=1,2$, which is related to the expansion of the "squared" solutions

$$
\begin{array}{lll}
\mathcal{T}_{1}=\left\{\rho^{+}(\lambda), \rho^{-}(\lambda),\right. & \lambda \in \mathbb{R}, & \left.\lambda_{k}^{ \pm}, C_{k}^{ \pm}, \quad k=1, \ldots, N\right\} \\
\mathcal{T}_{2}=\left\{\tau^{+}(\lambda), \tau^{-}(\lambda),\right. & \left.\lambda \in \mathbb{R}, \quad \lambda_{k}^{ \pm}, M_{k}^{ \pm}, \quad k=1, \ldots, N\right\}, \tag{2.3.26}
\end{array}
$$

where

$$
\begin{equation*}
C_{k}^{ \pm}=\frac{b_{k}^{ \pm}}{\dot{a}_{k}^{ \pm}}, \quad M_{k}^{ \pm}=\frac{1}{b_{k}^{ \pm} \dot{a}_{k}^{ \pm}}, \quad \dot{a}_{k}^{ \pm}=\left.\frac{\mathrm{d} a^{ \pm}}{\mathrm{d} \lambda}\right|_{\lambda=\lambda_{k}^{ \pm}} . \tag{2.3.27}
\end{equation*}
$$

The completeness and uniqueness of the solution of the ISP depends on the invertibility of the map $q(x) \rightarrow \mathcal{T}$. Therefore, the contour integration method is used with the following Green function

$$
G(x, y, \lambda)=\left\{\begin{array}{cl}
G^{+}(x, y, \lambda), & \text { for } \lambda \in \mathbb{C}_{+}  \tag{2.3.28}\\
1 / 2\left(G^{+}(x, y, \lambda)+G^{-}(x, y, \lambda)\right), & \text { for } \lambda \in \mathbb{R} \\
G^{-}(x, y, \lambda), & \text { for } \lambda \in \mathbb{C}_{-}
\end{array}\right.
$$

where

$$
\begin{align*}
& G^{ \pm}(x, y, \lambda)=G_{1}^{ \pm}(x, y, \lambda) \theta(x-y)-G_{2}^{ \pm}(x, y, \lambda) \theta(y-x),  \tag{2.3.29}\\
& G_{1}^{ \pm}(x, y, \lambda)=\frac{1}{\left(a^{ \pm}(\lambda)\right)^{2}} \boldsymbol{\Psi}^{ \pm}(x, \lambda) \otimes \boldsymbol{\Phi}^{ \pm}(y, \lambda),  \tag{2.3.30}\\
& G_{2}^{ \pm}(x, y, \lambda)=\frac{1}{\left(a^{ \pm}(\lambda)\right)^{2}}\left(\boldsymbol{\Phi}^{ \pm}(x, \lambda) \otimes \boldsymbol{\Psi}^{ \pm}(y, \lambda)+\frac{1}{2} \boldsymbol{\Theta}^{ \pm}(x, \lambda) \otimes \boldsymbol{\Theta}^{ \pm}(y, \lambda)\right), \tag{2.3.31}
\end{align*}
$$

where $\otimes$ denotes the tensor product ${ }^{1}$ [84] and

$$
\begin{equation*}
\mathbf{\Theta}^{ \pm}(x, \lambda)=a^{+}(\lambda)\left(\chi^{ \pm}(x, \lambda) \sigma_{3} \hat{\chi}^{ \pm}(x, \lambda)\right) . \tag{2.3.32}
\end{equation*}
$$

The poles of the Green functions $G^{ \pm}$are concurrent with $\lambda_{k}^{ \pm}$. So, when $a^{ \pm}(\lambda)$ have firstorder zeros at $\lambda_{k}^{ \pm}$, then $G^{ \pm}$would have second-order poles at these points

$$
\begin{align*}
\mathcal{J}_{G}(x, y) & =\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\mathbb{C}_{+}} \mathrm{d} \lambda G^{+}(x, y, \lambda)-\oint_{\mathbb{C}_{-}} \mathrm{d} \lambda G^{-}(x, y, \lambda)\right), \\
& =\sum_{k=1}^{N}\left(\operatorname{Res}_{\lambda=\lambda_{k}^{+}} G^{+}(x, y, \lambda)+\underset{\lambda=\lambda_{k}^{-}}{\operatorname{Res}} G^{-}(x, y, \lambda)\right), \tag{2.3.33}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{J}_{G}(x, y)= & -\frac{\mathrm{i}}{2} \delta(x-y) \Pi_{0} \\
& -\frac{\mathrm{i}}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} \lambda\left(\frac{\boldsymbol{\Psi}^{+}(x, y) \otimes \boldsymbol{\Phi}^{+}(y, x)}{\left(a^{+}(\lambda)\right)^{2}}-\frac{\boldsymbol{\Psi}^{-}(x, y) \otimes \boldsymbol{\Phi}^{-}(y, x)}{\left(a^{-}(\lambda)\right)^{2}}\right), \\
= & \sum_{k=1}^{N}\left(X_{k}^{+}(x, y)+X_{k}^{-}(x, y)\right) \tag{2.3.34}
\end{align*}
$$

where

$$
\begin{align*}
& \operatorname{Res}_{\lambda=\lambda_{k}^{ \pm}} G^{ \pm}(x, y, \lambda)=X_{k}^{ \pm}(x, y) \\
& \quad=\frac{1}{\left(\dot{a}_{k}^{ \pm}\right)^{2}}\left(\boldsymbol{\Psi}_{k}^{ \pm}(x) \otimes \dot{\boldsymbol{\Phi}}_{k}^{ \pm}(y)+\dot{\boldsymbol{\Psi}}_{k}^{ \pm}(x) \otimes \boldsymbol{\Phi}_{k}^{ \pm}(y)-\frac{2 \ddot{a}_{k}^{ \pm}}{\dot{a}_{k}^{ \pm}} \boldsymbol{\Psi}_{k}^{ \pm}(x) \otimes \boldsymbol{\Phi}_{k}^{ \pm}(y)\right) . \tag{2.3.35}
\end{align*}
$$

Then, the completeness relation for the "squared" solutions are

$$
\begin{align*}
\delta(x-y) \Pi_{0}= & -\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \lambda\left(\frac{\boldsymbol{\Psi}^{+}(x, y) \otimes \boldsymbol{\Phi}^{+}(y, x)}{\left(a^{+}(\lambda)\right)^{2}}-\frac{\boldsymbol{\Psi}^{-}(x, y) \otimes \boldsymbol{\Phi}^{-}(y, x)}{\left(a^{-}(\lambda)\right)^{2}}\right) \\
& +2 \mathrm{i} \sum_{k=1}^{N}\left(X_{k}^{+}(x, y)+X_{k}^{-}(x, y)\right)  \tag{2.3.36a}\\
\Pi_{0} & =\sigma_{+} \otimes \sigma_{-}-\sigma_{-} \otimes \sigma_{+} \tag{2.3.36b}
\end{align*}
$$

where $\sigma_{+}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$ and $\sigma_{-}=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right)$.

[^8]
### 2.3.3 Expansions over the "Squared" solutions

The completeness relation allows one to expand any element $X(x)$ of the phase space $\mathcal{M}$. If $X(x)$ is an off-diagonal matrix valued function, which falls off for $|x| \rightarrow \infty$. So, we can rewrite the $X(x)$ matrix as

$$
\begin{equation*}
X(x)=X_{+}(x) \sigma_{+}+X_{-}(x) \sigma_{-} \tag{2.3.37}
\end{equation*}
$$

Using (2.3.36b) we can obtain $X(x)$ from the following relation:

$$
\begin{equation*}
\frac{1}{2} \operatorname{tr}_{1}\left(\left[\sigma_{3}, X(x)\right] \otimes \mathbb{1}\right) \Pi_{0}=-X(x) \tag{2.3.38}
\end{equation*}
$$

where $\operatorname{tr}_{1}$ is the first position of the tensor product. Therefore, when we multiply equation (2.3.36a) on the right by $\frac{1}{2}\left[\sigma_{3}, X(x)\right] \otimes \mathbb{1}$, take $\operatorname{tr}_{1}$, and integrate over $\mathrm{d} x$, the following relation is the expansion of $X(x)$ over the system $\boldsymbol{\Phi}^{ \pm}$

$$
\begin{align*}
X(x)= & \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \lambda\left(\psi_{X}^{+}(\lambda) \boldsymbol{\Phi}^{+}(x, \lambda)-\psi_{X}^{-}(\lambda) \boldsymbol{\Phi}^{-}(x, \lambda)\right) \\
& -2 \mathrm{i} \sum_{k=1}^{N}\left(\psi_{X, k}^{ \pm} \dot{\boldsymbol{\Phi}}_{k}^{ \pm}+\dot{\psi}_{X, k}^{ \pm} \boldsymbol{\Phi}_{k}^{ \pm}\right) \tag{2.3.39}
\end{align*}
$$

where

$$
\begin{gather*}
\psi_{X}^{ \pm}(\lambda)=\frac{\llbracket \Psi^{ \pm}(x, \lambda), X(x) \rrbracket}{\left(a^{ \pm}(\lambda)\right)^{2}}, \quad \psi_{X, k}^{ \pm}(\lambda)=\frac{\llbracket \boldsymbol{\Psi}_{k}^{ \pm}(x, \lambda), X(x) \rrbracket}{\left(\dot{a}_{k}^{ \pm}\right)^{2}},  \tag{2.3.40a}\\
\dot{\psi}_{X, k}^{ \pm}=\frac{1}{\left(\dot{a}_{k}^{ \pm}\right)^{2}} \llbracket \dot{\Psi}_{k}^{ \pm}(x)-\frac{\ddot{a}_{k}^{ \pm}}{\dot{a}_{k}^{ \pm}} \boldsymbol{\Psi}_{k}^{ \pm}(x), X(x) \rrbracket . \tag{2.3.40b}
\end{gather*}
$$

Analogously, the expansions of $X(x)$ over the system $\boldsymbol{\Psi}^{ \pm}$can be obtained by multiplying (2.3.36a) from the right by $\frac{1}{2} \mathbb{1} \otimes\left[\sigma_{3}, X(x)\right]$, taking $\operatorname{tr}_{2}$, and integrating over $\mathrm{d} x$

$$
\begin{align*}
X(x)= & -\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \lambda\left(\phi_{X}^{+}(\lambda) \Psi^{+}(x, \lambda)-\phi_{X}^{-}(\lambda) \Psi^{-}(x, \lambda)\right) \\
& -2 \mathrm{i} \sum_{k=1}^{N}\left(\phi_{X, k}^{ \pm} \dot{\Psi}_{k}^{ \pm}+\dot{\phi}_{X, k}^{ \pm} \Psi_{k}^{ \pm}\right) \tag{2.3.41}
\end{align*}
$$

where

$$
\begin{equation*}
\phi_{X}^{ \pm}(\lambda)=\frac{\llbracket \boldsymbol{\Phi}^{ \pm}(x, \lambda), X(x) \rrbracket}{\left(a^{ \pm}(\lambda)\right)^{2}}, \quad \psi_{X, k}^{ \pm}(\lambda)=\frac{\llbracket \boldsymbol{\Phi}_{k}^{ \pm}(x, \lambda), X(x) \rrbracket}{\left(\dot{a}_{k}^{ \pm}\right)^{2}}, \tag{2.3.42a}
\end{equation*}
$$

$$
\begin{equation*}
\dot{\phi}_{X, k}^{ \pm}=\frac{1}{\left(\dot{a}_{k}^{ \pm}\right)^{2}} \llbracket \dot{\boldsymbol{\Phi}}_{k}^{ \pm}(x)-\frac{\ddot{a}_{k}^{ \pm}}{\dot{a}_{k}^{ \pm}} \boldsymbol{\Phi}_{k}^{ \pm}(x), X(x) \rrbracket . \tag{2.3.42b}
\end{equation*}
$$

For example, when $X(x)=q(x)$ then, equations (2.3.41) and (2.3.42) become

$$
\begin{align*}
q(x)= & -\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d} \lambda\left(\phi_{q}^{+}(\lambda) \Psi^{+}(x, \lambda)-\phi_{q}^{-}(\lambda) \Psi^{-}(x, \lambda)\right) \\
& -2 \mathrm{i} \sum_{k=1}^{N}\left(\phi_{q, k}^{ \pm} \dot{\Psi}_{k}^{ \pm}+\dot{\phi}_{q, k}^{ \pm} \Psi_{k}^{ \pm}\right) \tag{2.3.43}
\end{align*}
$$

where

$$
\begin{gather*}
\phi_{q}^{ \pm}(\lambda)=\frac{\llbracket \boldsymbol{\Phi}^{ \pm}(x, \lambda), q(x) \rrbracket}{\left(a^{ \pm}(\lambda)\right)^{2}}, \quad \psi_{q, k}^{ \pm}(\lambda)=\frac{\llbracket \boldsymbol{\Phi}_{k}^{ \pm}(x, \lambda), q(x) \rrbracket}{\left(\dot{a}_{k}^{ \pm}\right)^{2}},  \tag{2.3.44a}\\
\dot{\phi}_{q, k}^{ \pm}=\frac{1}{\left(\dot{a}_{k}^{ \pm}\right)^{2}} \llbracket \dot{\Phi}_{k}^{ \pm}(x)-\frac{\ddot{a}_{k}^{ \pm}}{\dot{a}_{k}^{ \pm}} \boldsymbol{\Phi}_{k}^{ \pm}(x), q(x) \rrbracket . \tag{2.3.44b}
\end{gather*}
$$

Therefore, any element $q(x)$ in the corresponding function space $\mathcal{M}$ can be written as a combination of the eigenfunctions $\phi(x, \lambda)$ and $\psi(x, \lambda)$.

### 2.4 Involution of the Zaharov-Shabat system

In this section, we will introduce the local and nonlocal involution of the ZS system.

### 2.4.1 Canonical/Local involution

The independent-complex-value functions $\left(q^{+}, q^{-}\right)$are potentials of the operator $L$. This section outlines some possibilities for the potentials, called involutions. These reductions lead to symmetries of the second order of the ZS system.

The first involution for (1.2.15) has the form:

$$
\begin{equation*}
q^{-}(x, t)=\varepsilon_{0}\left(q^{+}(x, t)\right)^{*}, \quad \varepsilon_{0}= \pm 1 \tag{2.4.1}
\end{equation*}
$$

and the second form

$$
\begin{equation*}
q^{-}(x, t)=\eta_{0} q^{+}(x, t), \quad \eta_{0}= \pm 1 . \tag{2.4.2}
\end{equation*}
$$

This section will describe the first option only (2.4.1) (for the second option see [47]); this provides the symmetry matrix for $U(x, t, \lambda)$ in (1.2.6a)

$$
U^{*}\left(x, t, \lambda^{*}\right)=-\epsilon^{-1} U(x, t, \lambda) \epsilon, \quad \epsilon=\left(\begin{array}{cc}
0 & 1  \tag{2.4.3}\\
-\varepsilon_{0} & 0
\end{array}\right)
$$

Then, the scattering matrix and the FASs must satisfy

$$
\begin{align*}
& \left(\chi^{+}\left(x, t, \lambda^{*}\right)\right)^{*}=\epsilon^{-1} \chi^{-}(x, t, \lambda) \epsilon,  \tag{2.4.4a}\\
& \left(\chi^{-}\left(x, t, \lambda^{*}\right)\right)^{*}=\epsilon^{-1} \chi^{+}(x, t, \lambda) \epsilon, \tag{2.4.4b}
\end{align*}
$$

and

$$
\begin{equation*}
T^{*}\left(\lambda^{*}, t\right)=\epsilon^{-1} T(\lambda, t) \epsilon, \tag{2.4.5a}
\end{equation*}
$$

in components

$$
\begin{align*}
& a^{-}(\lambda)=\left(a^{+}\left(\lambda^{*}\right)\right)^{*}  \tag{2.4.5b}\\
& b^{-}(\lambda)=\varepsilon_{0}\left(b^{+}\left(\lambda^{*}\right)\right)^{*}, \tag{2.4.5c}
\end{align*}
$$

and therefore the reflection coefficients become

$$
\begin{align*}
& \rho^{-}(t, \lambda)=\varepsilon_{0}\left(\rho^{+}\left(t, \lambda^{*}\right)\right)^{*},  \tag{2.4.6a}\\
& \tau^{-}(t, \lambda)=\varepsilon_{0}\left(\tau^{+}\left(t, \lambda^{*}\right)\right)^{*}, \tag{2.4.6b}
\end{align*}
$$

where $\lambda$ is complex in (2.4.5b) and real in (2.4.5c) and (2.4.6). These formulas show the effect of the involutions on the scattering data, related to the continuous spectrum of $L$. Therefore when $\varepsilon_{0}=-1$, this can allow one to reformulate the ZS system into:

$$
\begin{equation*}
\tilde{L} \psi(x, t, \lambda)=\left(\mathrm{i} \sigma_{3} \frac{\mathrm{~d} \psi}{\mathrm{~d} x}+U_{0}(x, t) \psi(x, t, \lambda)\right)=\lambda \psi(x, t, \lambda), \tag{2.4.7}
\end{equation*}
$$

where $U_{0}(x, t)=\sigma_{3} q(x, t)=U_{0}^{\dagger}(x, t)$ is a hermitian matrix. Then, the system (2.4.7) is an eigenvalue problem for the self-adjoint operator $\tilde{L}$. So, for real $\lambda$, the unitary condition for the scattering matrix is:

$$
\begin{equation*}
a^{+}(\lambda) a^{-}(\lambda)+b^{+}(t, \lambda) b^{-}(t, \lambda)=1, \quad \lambda \in \mathbb{R} \tag{2.4.8a}
\end{equation*}
$$

Furthermore, subtitling (2.4.5) and $\varepsilon_{0}=-1$ into (2.4.8a), we will have

$$
\begin{equation*}
a^{+}(\lambda)\left(a^{+}\left(\lambda^{*}\right)\right)^{*}+b^{+}(t, \lambda) \varepsilon_{0}\left(b^{+}\left(t, \lambda^{*}\right)\right)=1 \tag{2.4.8b}
\end{equation*}
$$

and then,

$$
\begin{equation*}
|a(\lambda)|^{2}=1+|b(\lambda)|^{2} \tag{2.4.8c}
\end{equation*}
$$

which means that for all $\lambda \in \mathbb{R},|a(\lambda)|^{2} \geq 1$, and the absence of the discrete eigenvalue means, the operator $\tilde{L}$ does not have discrete eigenvalues on the real $\lambda$-axis. Therefore,
the NLS equation do not have soliton solutions. Then, the minimal set of scattering data consists of

$$
\begin{align*}
& \mathcal{T}_{1} \equiv\left\{\rho^{ \pm}(t, \lambda), \lambda \in \mathbb{R}\right\},  \tag{2.4.9a}\\
& \mathcal{T}_{2} \equiv\left\{\tau^{ \pm}(t, \lambda), \lambda \in \mathbb{R}\right\} \tag{2.4.9b}
\end{align*}
$$

However, the involution (2.4.1) with $\varepsilon_{0}=1$ allows the existence of a discrete spectrum of $L$ with some restrictions. Since $a^{ \pm}(\lambda)$ have simple zeros and the Taylor series of $a^{ \pm}(\lambda)$

$$
\begin{align*}
& a^{+}(\lambda)=\left(\lambda-\lambda_{k}^{+}\right)\left(\dot{a}_{k}^{+}+\frac{1}{2}\left(\lambda-\lambda_{k}^{+}\right) \ddot{a}_{k}^{+}+\ldots\right),  \tag{2.4.10a}\\
& a^{-}(\lambda)=\left(\lambda-\lambda_{k}^{-}\right)\left(\dot{a}_{k}^{-}+\frac{1}{2}\left(\lambda-\lambda_{k}^{-}\right) \ddot{a}_{k}^{-}+\ldots\right), \tag{2.4.10b}
\end{align*}
$$

taking complex conjugate for equation (2.4.10a) and using (2.4.5b)

$$
\begin{align*}
\left(a^{+}\left(\lambda^{*}\right)\right)^{*} & =\left(\lambda^{*}-\lambda_{k}^{+}\right)^{*}\left(\dot{a}_{k}^{+}+\frac{1}{2}\left(\lambda^{*}-\lambda_{k}^{+}\right) \ddot{a}_{k}^{+}+\ldots\right)^{*},  \tag{2.4.11a}\\
\left(a^{+}\left(\lambda^{*}\right)\right)^{*}=a^{-}(\lambda) & =\left(\lambda-\left(\lambda_{k}^{+}\right)^{*}\right)\left(\left(\dot{a}_{k}^{+}\right)^{*}+\frac{1}{2}\left(\lambda-\left(\lambda_{k}^{+}\right)^{*}\right)\left(\ddot{a}_{k}^{+}\right)^{*}+\ldots\right), \tag{2.4.11b}
\end{align*}
$$

then, compare with (2.4.10b) we have

$$
\begin{equation*}
\lambda_{k}^{-}=\left(\lambda_{k}^{+}\right)^{*}, \quad \dot{a}_{k}^{-}=\left(\dot{a}_{k}^{+}\right)^{*}, \quad \ddot{a}_{k}^{-}=\left(\ddot{a}_{k}^{+}\right)^{*} . \tag{2.4.12}
\end{equation*}
$$

Thus, the data on the discrete spectrum should satisfy

$$
\begin{equation*}
b_{k}^{-}=\left(b_{k}^{+}\right)^{*}, \quad C_{k}^{-}=\left(C_{k}^{+}\right)^{*}, \quad M_{k}^{-}=\left(M_{k}^{+}\right)^{*} \tag{2.4.13}
\end{equation*}
$$

We deduce that the minimal sets of scattering data consist of

$$
\begin{equation*}
\mathfrak{T}_{1} \equiv\left\{\rho^{+}(t, \lambda), \lambda \in \mathbb{R}, \lambda_{k}^{+}, C_{k}^{+}(t), \quad k=1, \ldots, N\right\}, \tag{2.4.14a}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{T}_{2} \equiv\left\{\tau^{+}(t, \lambda), \lambda \in \mathbb{R}, \lambda_{k}^{+}, M_{k}^{+}(t), \quad k=1, \ldots, N\right\} . \tag{2.4.14b}
\end{equation*}
$$

Furthermore, the involution also has an effect on the dispersion laws. Both (1.4.25) and (2.4.5a) are consistent when $f(\lambda)$ satisfies:

$$
\begin{equation*}
\sum_{p} f_{p} \lambda^{p}=f(\lambda)=\left(f\left(\lambda^{*}\right)\right)^{*}=\sum_{p} f_{p}^{*} \lambda^{p} . \tag{2.4.15}
\end{equation*}
$$

This can happen when the coefficients $f_{p}$ are real.

### 2.4.2 The nonlocal involution of the Zakharov-Shabat system

A nonlocal NLS equation was recently introduced [7] and shown to be an integrable infinite dimensional evolution equation. The symmetry reduction was first noted by Ablowitz and Ladik [7] in 2013. Later on, in 2015 they introduced the key symmetries of the eigenfunctions and the scattering data. Hence they obtained conserved quantities [9].

The following nonlocal NLS equation is

$$
\begin{equation*}
\mathrm{i} q_{t}(x, t)=q_{x x}(x, t) \pm q(x, t) q^{*}(-x, t) q(x, t) \tag{2.4.16}
\end{equation*}
$$

where $*$ denotes complex conjugation and $q(x, t)$ is a complex value function of the real variables $x$ and $t$. When one compares (2.4.16) with (1.1.3a), the nonlinear term $q^{*}(x, y) q(x, t)$ will change in to $q^{*}(-x, y) q(x, t)$. In the local and nonlocal NLS equation the potential $q(x, t)$ tends to zero when $|x| \rightarrow \infty$. The involution of the nonlocal ZS system is simplified by assuming $q^{-}(x, t)=\left(q^{+}(-x, t)\right)^{*}$. To establish symmetry properties of the Jost functions of (1.2.6a), let us assume that if

$$
\begin{equation*}
\binom{\psi_{1}(x, \lambda)}{\psi_{2}(x, \lambda)} \tag{2.4.17a}
\end{equation*}
$$

is an eigenfunction of (1.2.6a). Then,

$$
\sigma_{1}\binom{\psi_{1}^{*}\left(-x,-\lambda^{*}\right)}{\psi_{2}^{*}\left(-x,-\lambda^{*}\right)}=\binom{a_{1} \psi_{2}^{*}\left(-x,-\lambda^{*}\right)}{a_{2} \psi_{1}^{*}\left(-x,-\lambda^{*}\right)} \quad \text { with } \quad \sigma_{1}=\left(\begin{array}{cc}
0 & a_{1}  \tag{2.4.17b}\\
a_{2} & 0
\end{array}\right)
$$

is also an eigenfunction of (1.2.6a), where $\sigma_{1}^{2}=\mathbb{1}$; in this case $a_{1}=a_{2}=1$. Therefore, $U(x, t, \lambda)$ in (1.2.6a) must satisfy

$$
\begin{equation*}
\sigma_{1} U^{*}\left(-x, t,-\lambda^{*}\right) \sigma_{1}^{-1}=U(x, t, \lambda), \tag{2.4.17c}
\end{equation*}
$$

and the Jost solutions $\psi(x, t, \lambda)$ and $\phi(x, t, \lambda)$ for the operator $L$ must satisfy

$$
\begin{align*}
\psi^{-}(x, t, \lambda) & =\sigma_{1}^{-1}\left(\phi^{-}\right)^{*}\left(-x, t,-\lambda^{*}\right)  \tag{2.4.18}\\
\psi^{+}(x, t, \lambda) & =\sigma_{1}\left(\phi^{+}\right)^{*}\left(-x, t,-\lambda^{*}\right)
\end{align*}
$$

The general form is

$$
\begin{equation*}
\psi(x, t, \lambda)=\sigma_{1} \phi^{*}\left(-x, t,-\lambda^{*}\right) \sigma_{1}^{-1} \tag{2.4.19a}
\end{equation*}
$$

with their asymptotics:

$$
\begin{align*}
\sigma_{1}\left(\lim _{x \rightarrow-\infty} \exp \left(-\mathrm{i} \lambda \sigma_{3} x\right) \psi^{*}\left(-x, t,-\lambda^{*}\right)\right) \sigma_{1}^{-1}= & \lim _{x \rightarrow-\infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \phi(x, t, \lambda) \\
& =\mathbb{1}  \tag{2.4.19b}\\
\sigma_{1}\left(\lim _{x \rightarrow \infty} \exp \left(-\mathrm{i} \lambda \sigma_{3} x\right) \phi^{*}\left(-x, t,-\lambda^{*}\right)\right) \sigma_{1}^{-1}= & \lim _{x \rightarrow \infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \psi(x, t, \lambda) \\
& =\mathbb{1} \tag{2.4.19c}
\end{align*}
$$

The symmetry in the Jost solutions in turn imposes symmetry in the scattering data $\left(a^{+}(\lambda), b^{+}(\lambda), b^{-}(\lambda), a^{-}(\lambda)\right)$. The scattering matrix also needes to be constructed

$$
\begin{equation*}
\psi^{*}\left(-x, t,-\lambda^{*}\right)=\phi^{*}\left(-x, t,-\lambda^{*}\right) \widehat{T}^{*}\left(-\lambda^{*}\right) \tag{2.4.20a}
\end{equation*}
$$

where

$$
\begin{equation*}
T(\lambda, t)=\sigma_{1} \widehat{T}^{*}\left(-\lambda^{*}, t\right) \sigma_{1}^{-1} \tag{2.4.20b}
\end{equation*}
$$

and when equating both sides of (2.4.20b) one can see that

$$
\begin{equation*}
a^{ \pm}(\lambda)=\left(a^{ \pm}\left(-\lambda^{*}\right)\right)^{*}, \quad b^{ \pm}(\lambda)=\left(b^{\mp}\left(-\lambda^{*}\right)\right)^{*} \tag{2.4.20c}
\end{equation*}
$$

The determinant of $\hat{T}\left(-\lambda^{*}, t\right)$ is

$$
\begin{equation*}
\operatorname{det} \hat{T}(\lambda, t) \equiv\left(a^{+}\left(-\lambda^{*}\right)^{*}\left(a^{-}\left(-\lambda^{*}\right)\right)^{*}+\left(b^{+}\left(-\lambda^{*}\right)\right)^{*}\left(b^{-}\left(-\lambda^{*}\right)\right)^{*}=1, \quad \lambda \in \mathbb{C} .\right. \tag{2.4.20d}
\end{equation*}
$$

Similarly, for the FASs, we get:

$$
\begin{equation*}
\chi^{ \pm}(x, t, \lambda)=\sigma_{1}\left(\chi^{ \pm}\left(-x, t,-\lambda^{*}\right)\right)^{*} \sigma_{1}^{-1} \tag{2.4.20e}
\end{equation*}
$$

The dispersion law of the NLEE satisfies the following relation

$$
\begin{equation*}
\mathrm{f}(\lambda)=\left(\mathrm{f}\left(-\lambda^{*}\right)\right)^{*} \tag{2.4.20f}
\end{equation*}
$$

### 2.5 Summary

The following table is a summary of the illustration that we have demonstrated in the previous sections to show the difference between the two types of the NLS equation.

| Local NLS, $\quad \epsilon=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ | Nonlocal NLS, $\quad \sigma_{1}=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ |
| :---: | :---: |
| $q^{-}(x, t)=\left(q^{+}(x, t)\right)^{*}, \quad U(x, t, \lambda)=\left(\begin{array}{c}-\lambda \\ \left(q^{+}(x, t)\right)^{*}\end{array} \begin{array}{c}q^{+}(x, t) \\ \lambda\end{array}\right)$ | $q^{-}(x, t)=\left(q^{+}(-x, t)\right)^{*}, \quad U(x, t, \lambda)=\left(\begin{array}{cc}-\lambda \\ \left(q^{+}(-x, t)\right)^{*} & q^{+}(x, t) \\ \lambda\end{array}\right)$ |
| $U^{*}\left(x, t, \lambda^{*}\right)=\left(\begin{array}{cc} -\lambda & \left(q^{+}(x, t)\right)^{*} \\ q^{+}(x, t) & \lambda \end{array}\right)$ | $U^{*}\left(-x, t,-\lambda^{*}\right)=\left(\begin{array}{cc}-\lambda \\ q^{+}(x, t) & \left(q^{+}(-x, t)\right)^{*} \\ \lambda\end{array}\right)$ |
| $U^{*}\left(x, t, \lambda^{*}\right)=-\epsilon^{-1} U(x, t, \lambda) \epsilon$ | $U(x, t, \lambda)=\sigma_{1} U^{*}\left(-x, t,-\lambda^{*}\right) \sigma_{1}^{-1}$ |
| $\left(\psi\left(x, t, \lambda^{*}\right)\right)^{*}=\epsilon^{-1} \phi(x, t, \lambda) \epsilon$ | $\psi(x, t, \lambda)=\sigma_{1} \phi^{*}\left(-x, t,-\lambda^{*}\right) \sigma_{1}^{-1}$ |
| $T^{*}\left(\lambda^{*}, t\right)=\epsilon^{-1} T(\lambda, t) \epsilon$ | $T(\lambda, t)=\sigma_{1} \widehat{T}^{*}\left(-\lambda^{*}, t\right) \sigma_{1}^{-1}$ |
| $\begin{aligned} & a^{\mp}(\lambda)=\left(a^{ \pm}\left(\lambda^{*}\right)\right)^{*} \\ & b^{\mp}(\lambda)=\left(b^{ \pm}\left(\lambda^{*}\right)\right)^{*} \end{aligned}$ | $\begin{aligned} & a^{ \pm}(\lambda)=\left(a^{ \pm}\left(-\lambda^{*}\right)\right)^{*} \\ & b^{ \pm}(\lambda)=\left(b^{\mp}\left(-\lambda^{*}\right)\right)^{*} \end{aligned}$ |
| $\begin{aligned} \left(\chi^{+}\left(x, t, \lambda^{*}\right)\right)^{*} & =\epsilon^{-1} \chi^{-}(x, t, \lambda) \epsilon \\ \left(\chi^{-}\left(x, t, \lambda^{*}\right)\right)^{*} & =\epsilon^{-1} \chi^{+}(x, t, \lambda) \epsilon \end{aligned}$ | $\begin{aligned} & \chi^{-}(x, \lambda)=\sigma_{1}\left(\chi^{-}\left(-x,-\lambda^{*}\right)\right)^{*} \sigma_{1}^{-1} \\ & \chi^{+}(x, \lambda)=\sigma_{1}\left(\chi^{+}\left(-x,-\lambda^{*}\right)\right)^{*} \sigma_{1}^{-1} \end{aligned}$ |
| $\begin{aligned} & V(x, t, \lambda)= \\ & \left(\begin{array}{cc} -q^{+}(x, t)\left(q^{+}(x, t)\right)^{*}+2 \lambda^{2} & -\mathrm{i} q_{x}^{+}(x, t)-2 \lambda q^{+}(x, t) \\ \mathrm{i}\left(q_{x}^{+}(x, t)\right)^{*}-2 \lambda\left(q^{+}(x, t)\right)^{*} & q^{+}(x, t)\left(q^{+}(x, t)\right)^{*}-2 \lambda^{2} \end{array}\right) \end{aligned}$ | $\begin{gathered} V(x, t, \lambda)= \\ \left(\begin{array}{cc} -q^{+}(x, t)\left(q^{+}(-x, t)\right)^{*}+2 \lambda^{2} & -\mathrm{i} q_{x}^{+}(x, t)-2 \lambda q^{+}(x, t) \\ \mathrm{i}\left(q_{x}^{+}(-x, t)\right)^{*}-2 \lambda\left(q^{+}(-x, t)\right)^{*} & q^{+}(x, t)\left(q^{+}(-x, t)\right)^{*}-2 \lambda^{2} \end{array}\right) \end{gathered}$ |
| $-\mathrm{i} q_{t}(x, t)+q_{x x}(x, t)+2 q\|q\|^{2}(x, t)=0$ | $-\mathrm{i} q_{t}(x, t)+q_{x x}(x, t)+2 q(x, t) q^{*}(-x, t) q(x, t)=0$ |

Table 2.1: Local and nonlocal involutios of Zakharov-Shabat system (A summary).

## Chapter 3

## The Ablowitz-Ladik system with $\mathcal{P T}$-symmetry: the self-induced potential

### 3.1 Introduction

Discrete nonlinear Schrödinger (DNLS) equation is an essential equation in many branches of physics and applied mathematics, particularly in water waves. The first example of integrable discretisation of the NLS equation was introduced by M. Ablowitz and F. Ladik $[2,3]$

$$
\begin{equation*}
\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} t} Q_{n}=\frac{1}{h^{2}}\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right) \pm\left|Q_{n}\right|^{2}\left(Q_{n+1}+Q_{n-1}\right) \tag{3.1.1}
\end{equation*}
$$

In general, there is no guarantee that any given discretisation of integrable PDE yields an integrable equation. However, there are integrable discretisations of integrable PDEs. For instance, equation (3.1.2) also describes a particular case for a lattice of coupled harmonic oscillators in one spatial dimension ${ }^{1}[32,69]$

$$
\begin{equation*}
\mathrm{i} Q_{n, t}+\gamma\left|Q_{n}\right|^{2} Q_{n}+\varepsilon\left(Q_{n+1}+Q_{n-1}\right)=0 \tag{3.1.2}
\end{equation*}
$$

where $\gamma$ is a harmonic parameter. In this chapter we will discuss the IST for DNLS of the AL type. We will pay special attention to reductions in the potential of local and nonlocal types. The latter will give rise to integrable systems with $\mathcal{P J}$-symmetries [8, 21] recall equation (1.6.3),

$$
\begin{equation*}
\mathrm{i} \frac{d}{d t} Q_{n}=\frac{1}{h^{2}}\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right) \pm Q_{n} Q_{-n}^{*}\left(Q_{n+1}+Q_{n-1}\right) \tag{3.1.3}
\end{equation*}
$$

[^9]A $\mathcal{P J}$-symmetry NLS equation was found in 2013 [7]. The following year, an integrable discrete $\mathcal{P T}$-symmetric (DNLS) equation was obtained from a new nonlocal $\mathcal{P T}$ symmetric reduction of the Ablowitz-Ladik scattering problem [8].

The core of this chapter aims to find the 2 -soliton solutions of nonlocal DNLS equation, using the RHP developed for solving a vast variety of problems in pure and applied mathematics. This development comes from the theory of integrable systems. The notion of integrable systems was found in the original work of Lax, Faddev and Zakharov, that is now known as the IST in soliton theory. The modern problems related to the theory of soliton, play crucial role in the new formulation of the ISM. The main target of RHP is to reduce a particular problem to the reconstruction of an analytic function from jump conditions or to the analytic valued factorisation of a given matrix-scalar value function defined on a curve [40, 73, 93]. The results of this chapter appear in [65].

### 3.2 Inverse scattering transform for the Ablowitz-Ladik system

Here, we will introduce the scattering problem subject to boundary conditions. The scattering data are reformulated in terms of eigenfunctions having constant boundary conditions, the so-called Jost functions.

### 3.2.1 Scattering problem

Consider the spectral problem, following [12]:

$$
\begin{align*}
\Psi_{x}(x, t, \lambda) & =U(x, t, \lambda) \Psi(x, t, \lambda), \\
\text { where } \quad U(x, t, \lambda) & =\left(\begin{array}{cc}
-\mathrm{i} \lambda & q^{+}(x, t) \\
q^{-}(x, t) & \mathrm{i} \lambda
\end{array}\right) . \tag{3.2.1}
\end{align*}
$$

Then, the natural discretisation of the scattering problem (3.2.1) is

$$
\begin{align*}
\left(\frac{\Psi_{n+1}-\Psi_{n}}{h}\right) & =\left(\begin{array}{cc}
-\mathrm{i} \lambda & q_{n}^{+}(t) \\
q_{n}^{-}(t) & \mathrm{i} \lambda
\end{array}\right) \Psi_{n}+O\left(h^{2}\right)  \tag{3.2.2a}\\
\Psi_{n+1} & =\left(\begin{array}{cc}
1-\mathrm{i} h \lambda & h q_{n}^{+}(t) \\
h q_{n}^{-}(t) & 1+\mathrm{i} h \lambda
\end{array}\right) \Psi_{n}+O\left(h^{2}\right) \tag{3.2.2b}
\end{align*}
$$

where $\Psi_{n}=\Psi(n h)=\left(\Psi_{n}^{(1)}, \Psi_{n}^{(2)}\right)^{T}, q_{n}^{+}=q^{+}(n h)$ and $q_{n}^{-}=q^{-}(n h)$. Then, rewrite the finite difference ${ }^{1}$ (3.2.2a) as:

$$
\Psi_{n+1}=\left(\begin{array}{cc}
z & Q_{n}^{+}  \tag{3.2.3a}\\
Q_{n}^{-} & z^{-1}
\end{array}\right) \Psi_{n}
$$

where

$$
\begin{align*}
Q_{n}^{+} & =h q_{n}^{+}, \quad Q_{n}^{-}=h q_{n}^{-}, \\
z & =e^{-\mathrm{i} \lambda h}=1-\mathrm{i} \lambda h+O\left(h^{2}\right), \quad z^{-1}=e^{\mathrm{i} \lambda h}=1+\mathrm{i} \lambda h+\boldsymbol{O}\left(h^{2}\right) . \tag{3.2.3b}
\end{align*}
$$

Equation (3.2.3a) refers to the AL scattering problem. In our work we will denote the matrix in (3.2.3a) as:

$$
\begin{equation*}
\mathcal{L}_{n}(z, t)=\mathbf{Z}+\tilde{\mathbf{Q}}_{n}, \tag{3.2.4}
\end{equation*}
$$

where

$$
\mathbf{Z}=\left(\begin{array}{cc}
z & 0  \tag{3.2.5}\\
0 & z^{-1}
\end{array}\right), \quad \tilde{\mathbf{Q}}_{n}=\left(\begin{array}{cc}
0 & Q_{n}^{+} \\
Q_{n}^{-} & 0
\end{array}\right)
$$

### 3.2.2 Jost function and the scattering matrix

The infinite number of conserved quantities in equation (3.1.1) can be derived by assuming that $\left|Q_{n}^{+}\right|,\left|Q_{n}^{-}\right| \rightarrow 0$ as $n \rightarrow \pm \infty$. Then, the eigenfunctions are asymptotic to the solution of the AL scattering problem

$$
\Psi_{n+1}=\left(\begin{array}{cc}
z & 0  \tag{3.2.6}\\
0 & z^{-1}
\end{array}\right) \Psi_{n}
$$

The scattering problem for DNLS equation is also determined by the asymptotics of the eigenfunctions of $\mathcal{L}_{n}(z, t)$, and their ratio, known as the scattering matrix $T(z, t)$, and its elements are called scattering data. The transformation of the scattering data linearises the PDE. The whole IST, in fact, can be considered as a nonlinear analogue of the standard Fourier transform. Function $u(x, t)$ in Fig. 1.1 is a real value function equivalent to $q_{n}(t)$ and $\hat{u}(\lambda, t)$ is equivalent to $T(z, t)$. In this case, $\phi_{n}=\left(\phi_{n}^{+}, \phi_{n}^{-}\right)$, and $\psi_{n}=\left(\psi_{n}^{-}, \psi_{n}^{+}\right)$are

[^10]eigenfunctions which satisfy the following boundary conditions
\[

$$
\begin{align*}
& \psi_{n}(z)=\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right), \quad \text { as } \quad n \rightarrow+\infty \\
& \phi_{n}(z)=\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right), \quad \text { as } \quad n \rightarrow-\infty \tag{3.2.7}
\end{align*}
$$
\]

The pairs $\phi^{+}, \phi^{-}$and $\psi^{-}, \psi^{+}$are linearly dependent and one can write $\phi^{+}, \phi^{-}$as linear combinations of $\psi^{-}, \psi^{+}$where the coefficients of these linear combinations depend on $z$.

$$
\phi_{n}(z)=\psi_{n}(z) T(z), \quad T(z)=\left(\begin{array}{cc}
a^{+}(z) & -b^{-}(z)  \tag{3.2.8}\\
b^{+}(z) & a^{-}(z)
\end{array}\right), \quad \text { when } \quad|z|=1
$$

We can rewrite the scattering problem (3.2.3a) as:

$$
\begin{equation*}
\Psi_{n+1}-\mathbf{Z} \Psi_{n}=\tilde{\mathbf{Q}}_{n} \Psi_{n} \tag{3.2.9}
\end{equation*}
$$

then, the following functions are defined as the Jost functions

$$
\begin{equation*}
\xi_{n}(z)=\psi_{n}(z) \mathbf{Z}^{-n}, \quad \varphi_{n}(z)=\phi_{n}(z) \mathbf{Z}^{-n} \tag{3.2.10}
\end{equation*}
$$

The Jost functions are solutions of the respective difference equations

$$
\begin{align*}
\xi_{n+1} & =\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right) \xi_{n} \mathbf{Z}^{-1}  \tag{3.2.11a}\\
\varphi_{n+1} & =\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right) \varphi_{n} \mathbf{Z}^{-1} \tag{3.2.11b}
\end{align*}
$$

with the constant boundary conditions

$$
\xi_{n}(z) \rightarrow\left(\begin{array}{ll}
1 & 0  \tag{3.2.12}\\
0 & 1
\end{array}\right), \text { as } n \rightarrow \infty, \quad \varphi_{n}(z) \rightarrow\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), \text { as } n \rightarrow-\infty
$$

Green's functions are used to construct a set of summation equations whose solutions satisfy, is respectively needed, the difference equations (3.2.11) with the constant boundary conditions (3.2.12). Green's function corresponding to (3.2.11a) is a solution of the summation equation

$$
\begin{equation*}
\mathbf{G}_{n+1}-\mathbf{Z G}_{n} \mathbf{Z}^{-1}=\mathbb{1} \delta_{0, n} \mathbf{Z}^{-1} \tag{3.2.13a}
\end{equation*}
$$

where

$$
\delta_{0, n}= \begin{cases}0 & n \neq 0  \tag{3.2.13b}\\ 1 & n=0\end{cases}
$$

Now, if $v_{n}(z)$ satisfies the summation equation

$$
\begin{equation*}
v_{n}(z)=\omega+\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{k} \tilde{\mathbf{Q}}_{k} v_{k} \mathbf{Z}^{-1} \tag{3.2.13c}
\end{equation*}
$$

where $\tilde{\mathbf{G}}_{k}=\mathbf{G}_{n-k} \tilde{\mathbf{Q}}_{k} \mathbf{G}_{n-k} \hat{\mathbf{Q}}_{k}$ is a solution of (3.2.13a) and $\omega$ satisfies the following equation

$$
\begin{equation*}
\omega-\mathbf{Z} \omega \mathbf{Z}^{-1}=\mathbf{0} \tag{3.2.13d}
\end{equation*}
$$

then, $v_{n}$ is a solution of the difference equation (3.2.11a), and we can see in the following paragraph, that the choice of the Green's function and the choice of the inhomogeneous term $\omega$ together determine the eigenfunctions and its analytical properties. To find the Green's function explicitly, multiply both sides of (3.2.13a), from the right, by the matrix $\mathbf{Z}$, then the summation equation becomes:

$$
\begin{equation*}
\mathbf{G}_{n+1} \mathbf{Z}-\mathbf{Z} \mathbf{G}_{n}=\mathbb{1} \delta_{0, n} \tag{3.2.14}
\end{equation*}
$$

Let us define the diagonal matrix

$$
\mathbf{G}_{n}=\left(\begin{array}{cc}
g_{n}^{1} & 0  \tag{3.2.15}\\
0 & g_{n}^{2}
\end{array}\right)
$$

substitute it in equation (3.2.14)

$$
\left(\begin{array}{cc}
g_{n+1}^{1} & 0  \tag{3.2.16}\\
0 & g_{n+1}^{2}
\end{array}\right)\left(\begin{array}{cc}
z & 0 \\
0 & z^{-1}
\end{array}\right)-\left(\begin{array}{cc}
z & 0 \\
0 & z^{-1}
\end{array}\right)\left(\begin{array}{cc}
g_{n}^{1} & 0 \\
0 & g_{n}^{2}
\end{array}\right)=\mathbb{1} \delta_{0, n},
$$

equation (3.2.16) becomes the system

$$
\begin{gather*}
z g_{n+1}^{1}-z g_{n}^{1}=1, \quad z^{-1} g_{n+1}^{2}-z^{-1} g_{n}^{2}=1,  \tag{3.2.17}\\
b_{j}\left(g_{n+1}^{(j)}-g_{n}^{(j)}\right)=\delta_{0, n} \tag{3.2.18}
\end{gather*}
$$

where $b_{1}=z, b_{2}=z^{-1}$. Next, let us represent $g_{n}^{(j)}$ and $\delta_{0, n}$ as Fourier integrals

$$
\begin{align*}
& g_{n}^{(j)}=\frac{1}{2 \pi \mathrm{i}} \oint_{|p|=1} p^{n-1} \hat{g}^{(j)}(p) \mathrm{d} p,  \tag{3.2.19}\\
& \delta_{0, n}=\frac{1}{2 \pi \mathrm{i}} \oint_{|p|=1} p^{n-1} \mathrm{~d} p . \tag{3.2.20}
\end{align*}
$$

What we need now is to apply equations (3.2.19) and (3.2.20) in equation (3.2.18)

$$
\begin{array}{r}
\frac{1}{2 \pi \mathrm{i}} \oint_{|p|=1} p^{(n+1)-1} \hat{g}^{(j)}(p) \mathrm{d} p-\frac{1}{2 \pi \mathrm{i}} \oint_{||p|=1} p^{n-1} \hat{g}^{(j)}(p) \mathrm{d} p \\
=\frac{1}{b_{j}} \frac{1}{2 \pi \mathrm{i}} \oint_{|p|=1} p^{n-1} \mathrm{~d} p \tag{3.2.21a}
\end{array}
$$

then,

$$
\begin{equation*}
\hat{g}^{(j)}(p)=\frac{1}{b_{j}(p-1)} . \tag{3.2.21b}
\end{equation*}
$$

Next, apply equation (3.2.21b) in equation (3.2.19), and we will obtain:

$$
\begin{equation*}
g_{n}^{(j)}=\frac{1}{b_{j}} \frac{1}{2 \pi \mathrm{i}} \oint_{|p|=1} \frac{p^{(n-1)}}{p-1} \mathrm{~d} p . \tag{3.2.22}
\end{equation*}
$$

It is clear that the integral in equation (3.2.22) has a simple pole at $p=1$ which is on the contour $|p|=1$. To avoid the singularity, let $C^{\text {out }}$ be a contour enclosing $p=1$ and $C^{\text {in }}$ be a contour excluding $p=1$

$$
g_{n}^{(j), \text { out }}=\frac{1}{b_{j}} \frac{1}{2 \pi \mathrm{i}} \oint_{\text {Cout }} \frac{p^{(n-1)}}{p-1} \mathrm{~d} p=\frac{1}{b_{j}} \begin{cases}1, & n \geq 1  \tag{3.2.23}\\ 0, & n \leq 0\end{cases}
$$

and

$$
g_{n}^{(j), \text { in }}=\frac{1}{b_{j}} \frac{1}{2 \pi \mathrm{i}} \oint_{C^{\text {in }}} \frac{p^{(n-1)}}{p-1} \mathrm{~d} p=\frac{1}{b_{j}}\left\{\begin{array}{rr}
0, & n \geq 1  \tag{3.2.24}\\
-1, & n \leq 0
\end{array}\right.
$$

Substituting equation (3.2.23) or equation (3.2.24) into equation (3.2.15) and we will obtain two Green's functions satisfying (3.2.14)

$$
\mathbf{G}_{n}^{\text {out }}=\theta(n-1)\left(\begin{array}{cc}
z^{-1} & 0  \tag{3.2.25a}\\
0 & z
\end{array}\right), \quad \mathbf{G}_{n}^{\text {in }}=-\theta(-n)\left(\begin{array}{cc}
z^{-1} & 0 \\
0 & z
\end{array}\right)
$$

where

$$
\theta(n)=\sum_{k=-\infty}^{n} \delta_{0, n}= \begin{cases}1, & n \geq 0  \tag{3.2.25b}\\ 0, & n<0\end{cases}
$$

Then, taking into account the boundary conditions (3.2.12) and relation (3.2.13d) for the inhomogeneous term in (3.2.13c), if both $Q_{n}^{+}, Q_{n}^{-} \rightarrow 0$ as $n \rightarrow \pm \infty$, we obtain the
following summation equations for $\xi_{n}(z)$ and $\varphi_{n}(z)$ :

$$
\begin{align*}
& \xi_{n}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{k}^{1} \tilde{\mathbf{Q}}_{k} \xi_{k} \mathbf{Z}^{-1}  \tag{3.2.26a}\\
& \varphi_{n}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{k}^{2} \tilde{\mathbf{Q}}_{k} \varphi_{k} \mathbf{Z}^{-1} \tag{3.2.26b}
\end{align*}
$$

where

$$
\begin{align*}
& \tilde{\mathbf{G}}_{k}^{1}=\mathbf{G}_{n-k}^{(\mathrm{in})} \tilde{\mathbf{Q}}_{n-k} \mathbf{G}_{n-k}^{(\mathrm{in})} \hat{\tilde{\mathbf{Q}}}_{k},  \tag{3.2.26c}\\
& \tilde{\mathbf{G}}_{k}^{2}=\mathbf{G}_{n-k}^{(\text {out) })} \tilde{\mathbf{Q}}_{n-k} \mathbf{G}_{n-k}^{(\text {out })}  \tag{3.2.26d}\\
& \hat{\mathbf{Q}}_{k}
\end{align*}
$$

### 3.2.3 Analytic properties of the eigenfunctions

Using a lemma [12] which states: if $\left\|Q^{+}\right\|_{1}=\sum_{-\infty}^{\infty}\left|Q_{n}^{+}\right|<\infty$ and $\left\|Q^{-}\right\|_{1}=\sum_{-\infty}^{\infty}\left|Q_{n}^{-}\right|<$ $\infty$, then $\xi_{n}(z)$ and $\varphi_{n}(z)$ defined by (3.2.26) are analytic functions of $z$ for $|z| \lessgtr 1$ and continuous for $|z| \leqslant 1$ and $|z| \geqslant 1$, respectively. In addition, the solution of the summation equations (3.2.26) is unique in the space of bounded functions. The Neumann series of the eigenfunction $\varphi_{n}(z)$ is represented as:

$$
\begin{align*}
\varphi_{n}(z) & =\sum_{j=0}^{\infty} C_{n}^{j}(z),  \tag{3.2.27a}\\
C_{n}^{j+1}(z) & =\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{n-k}^{(2)}(z) \tilde{\mathbf{Q}}_{k} C_{k}^{j}(z) \mathbf{Z}^{-1}, \quad j \geq 0, \tag{3.2.27b}
\end{align*}
$$

where

$$
C_{n}^{0}(z)=\left(\begin{array}{ll}
1 & 0  \tag{3.2.27c}\\
0 & 1
\end{array}\right)
$$

Using $\tilde{\mathbf{G}}_{n}^{(2)}$ function in (3.2.26d), equation (3.2.27b) in component form will be

$$
\begin{align*}
C_{n}^{j+1}= & \sum_{k=-\infty}^{n-1} \theta(n-k-1)\left(\begin{array}{cc}
z^{-1} & 0 \\
0 & z
\end{array}\right)\left(\begin{array}{cc}
0 & Q_{n-k}^{+} \\
Q_{n-k}^{-} & 0
\end{array}\right)\left(\begin{array}{cc}
z^{-1} & 0 \\
0 & z
\end{array}\right) \\
& C_{k}^{j}\left(\begin{array}{cc}
z^{-1} & 0 \\
0 & z
\end{array}\right) \tag{3.2.28a}
\end{align*}
$$

Then, for $k=1, n>2$, the eigenfunction $\varphi_{n}(z)$ is of the form:

$$
\begin{align*}
\varphi_{n}(z)= & \left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
0 & z Q_{n-1}^{+} \\
z^{-1} Q_{n-1}^{-} & 0
\end{array}\right) \\
& +\left(\begin{array}{cc}
z^{-2} Q_{n-2}^{-} Q_{n-1}^{+} & 0 \\
0 & z^{2} Q_{n-1}^{-} Q_{n-2}^{+}
\end{array}\right)+\ldots  \tag{3.2.28b}\\
\varphi_{n}(z)= & \left(\begin{array}{cc}
1+O\left(z^{-2}, \text { even }\right) & z Q_{n-1}^{+}+O\left(z^{3}, \text { odd }\right) \\
z^{-1} Q_{n-1}^{-}+O\left(z^{-3}, \text { odd }\right) & 1+O\left(z^{2}, \text { even }\right)
\end{array}\right) \simeq \mathbb{1}+\mathbf{Z} \tilde{\mathbf{Q}}_{n-1} \tag{3.2.28c}
\end{align*}
$$

We can see from equation (3.2.28c) that the first column of $\varphi_{n}(z)$ called $\varphi_{n}^{+}(z)$ is analytic when $|z| \rightarrow \infty$ and the second column called $\varphi_{n}^{-}(z)$ is analytic when $|z| \rightarrow 0$; "even" indicates that the higher order terms are even powers of $z^{-1}$ while "odd" that the higher order terms are odd powers. Analogously, one can obtain the Neumann series expansion of $\xi_{n}(z)$ but first, we need to rewrite the difference equation (3.2.11a)

$$
\begin{align*}
\xi_{n+1} & =\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right) \xi_{n} \mathbf{Z}^{-1},  \tag{3.2.29a}\\
\xi_{n+1} \mathbf{Z} & =\mathbf{Z} \xi_{n}+\tilde{\mathbf{Q}}_{n} \xi_{n}, \tag{3.2.29b}
\end{align*}
$$

as:

$$
\begin{equation*}
\left(1-Q_{n}^{-} Q_{n}^{+}\right) \xi_{n}-\mathbf{Z}^{-1} \xi_{n+1} \mathbf{Z}=-\tilde{\mathbf{Q}}_{n} \xi_{n+1} \mathbf{Z} \tag{3.2.30}
\end{equation*}
$$

To define the modified Jost function

$$
\begin{align*}
\hat{\xi}_{n}(z) & =c_{n} \xi_{n}(z), \\
\text { where } \quad c_{n} & =\prod_{k=n}^{+\infty}\left(1-Q_{k}^{-} Q_{k}^{+}\right) . \tag{3.2.31}
\end{align*}
$$

Then, equation (3.2.30) becomes:

$$
\begin{equation*}
\hat{\xi}_{n}-\mathbf{Z}^{-1} \hat{\xi}_{n+1} \mathbf{Z}=-\tilde{\mathbf{Q}}_{n} \hat{\xi}_{n+1} \mathbf{Z} \tag{3.2.32}
\end{equation*}
$$

and the modified Jost solution $\hat{\xi}_{n}(z)$ must satisfy the following difference equation

$$
\begin{equation*}
\hat{\xi}_{n} \mathbf{Z}^{-1}-\mathbf{Z}^{-1} \hat{\xi}_{n+1}=-\tilde{\mathbf{Q}}_{n} \hat{\xi}_{n+1} \tag{3.2.33a}
\end{equation*}
$$

with the boundary condition

$$
\hat{\xi}_{n}(z) \rightarrow\left(\begin{array}{ll}
1 & 0  \tag{3.2.33b}\\
0 & 1
\end{array}\right) .
$$

Next, we need to find the Green's function. To do so, we modify the summation equation (3.2.13c)

$$
\begin{array}{r}
\hat{\mathbf{G}}_{n} \mathbf{Z}^{-1}-\mathbf{Z}^{-1} \hat{\mathbf{G}}_{n+1}=-\mathbb{1} \delta_{0, n}, \\
v_{n}(z)=\omega+\sum_{k=-\infty}^{+\infty} \hat{\mathbf{G}}_{n-k} \tilde{\mathbf{Q}}_{k} v_{k} \mathbf{Z}, \tag{3.2.35a}
\end{array}
$$

where $\hat{\mathbf{G}}_{n}$ is a solution of (3.2.34), $\omega$ satisfies

$$
\begin{equation*}
\omega-\mathbf{Z}^{-1} \omega \mathbf{Z}=\mathbf{0} \tag{3.2.35b}
\end{equation*}
$$

and

$$
\hat{\mathbf{G}}_{n}=\theta(-n)\left(\begin{array}{cc}
z & 0  \tag{3.2.36}\\
0 & z^{-1}
\end{array}\right) .
$$

Then, the summation equation for the $\hat{\xi}_{n}$ function becomes:

$$
\begin{equation*}
\hat{\xi}_{n}(z)=\mathbb{1}+\sum_{-\infty}^{\infty} \tilde{\mathbf{G}}_{n-k} \tilde{\mathbf{Q}}_{k} \hat{\xi}_{k}(z) \mathbf{Z} \tag{3.2.37}
\end{equation*}
$$

where $\tilde{\mathbf{G}}_{k}=\hat{\mathbf{G}}_{k} \tilde{\mathbf{Q}}_{k} \hat{\mathbf{G}}_{k} \hat{\tilde{\mathbf{Q}}}_{k}$. The Neumann series for the modified function $\hat{\xi}_{n}$ (3.2.33) is

$$
\begin{align*}
\hat{\xi}_{n}(z) & =\sum_{j=0}^{\infty} C_{n}^{j}(z),  \tag{3.2.38a}\\
C_{n}^{j+1}(z) & =\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{n-k} \tilde{\mathbf{Q}}_{k} C_{k}^{j}(z) \mathbf{Z}, \quad j \geq 0, \tag{3.2.38b}
\end{align*}
$$

where

$$
C_{n}^{0}(z)=\left(\begin{array}{ll}
1 & 0  \tag{3.2.38c}\\
0 & 1
\end{array}\right)
$$

and equation (3.2.38b) in component form is

$$
\begin{align*}
& C_{n}^{j+1}=\sum_{k=n}^{\infty} \theta(k-n)\left(\begin{array}{cc}
z & 0 \\
0 & z^{-1}
\end{array}\right)\left(\begin{array}{cc}
0 & Q_{k}^{+} \\
Q_{k}^{-} & 0
\end{array}\right)\left(\begin{array}{cc}
z & 0 \\
0 & z^{-1}
\end{array}\right) C_{k}^{j}\left(\begin{array}{cc}
z & 0 \\
0 & z^{-1}
\end{array}\right),  \tag{3.2.39a}\\
& \hat{\xi}_{n}(z)=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
0 & z^{-1} Q_{n}^{+} \\
z Q_{n}^{-} & 0
\end{array}\right)+\left(\begin{array}{cc}
z^{2} Q_{n}^{+} Q_{n}^{-} & 0 \\
0 & z^{-2} Q_{n}^{-} Q_{n}^{+}
\end{array}\right)+\ldots,  \tag{3.2.39b}\\
& \hat{\xi}_{n}(z)=\left(\begin{array}{cc}
1+O\left(z^{2}, \text { even }\right) & z^{-1} Q_{n}^{+}+O\left(z^{-3}, \text { odd }\right) \\
z Q_{n}^{-}+O\left(z^{3}, \text { odd }\right) & 1+O\left(z^{-2}, \text { even }\right)
\end{array}\right) \simeq \mathbb{1}+\mathbf{Z}^{-1} \tilde{\mathbf{Q}}_{n} \tag{3.2.39c}
\end{align*}
$$

Taking into account that $\hat{\xi}_{n}=c_{n} \xi_{n}$. Then, $\xi(z)_{n}$ is of the form:

$$
\begin{align*}
\xi_{n}(z) & =\left(\begin{array}{cc}
c_{n}^{-1}+O\left(z^{2}, \text { even }\right) & c_{n}^{-1} z^{-1} Q_{n}^{+}+O\left(z^{-3}, \text { odd }\right) \\
c_{n}^{-1} z Q_{n}^{-}+O\left(z^{3}, \text { odd }\right) & c_{n}^{-1}+O\left(z^{-2}, \text { even }\right)
\end{array}\right) \\
& \simeq c_{n}^{-1}\left(\mathbb{1}+\mathbf{Z}^{-1} \tilde{\mathbf{Q}}_{n}\right) . \tag{3.2.40}
\end{align*}
$$

In equation (3.2.40), the first column of $\xi_{n}(z)$ called $\xi_{n}^{-}(z)$ which is analytic when $|z| \rightarrow 0$ and the second column $\xi_{n}^{+}(z)$ is analytic when $|z| \rightarrow \infty$.

### 3.2.4 Fundamental analytic solutions

The properties of the fundamental solutions $\phi_{n}$ and $\psi_{n}$ are supposed to construct the fundamental analytic solutions (FASs) $\left(\chi_{n}^{+}(z)\right.$ and $\left.\chi_{n}^{-}(z)\right)$ of the finite difference system (3.2.3a) which are obtained by combining the pairs of columns of the Jost functions with the same analyticity properties

$$
\begin{align*}
& \chi_{n}^{+}(z)=\left(\varphi_{n}^{+}, \xi_{n}^{+}\right),  \tag{3.2.41a}\\
& \chi_{n}^{-}(z)=\left(\xi_{n}^{-}, \varphi_{n}^{-}\right) . \tag{3.2.41b}
\end{align*}
$$

Then, the FASs $\chi_{n}^{ \pm}(z)$ have the following form:

$$
\begin{align*}
& \lim _{|z| \rightarrow \infty} \chi_{n}^{+}(z)=\left(\begin{array}{cc}
1+O\left(z^{-2}, \text { even }\right) & c_{n}^{-1} z^{-1} Q_{n}^{+}+O\left(z^{-3}, \text { odd }\right) \\
z^{-1} Q_{n-1}^{-}+O\left(z^{-3}, \text { odd }\right) & c_{n}^{-1}+O\left(z^{-2}, \text { even }\right)
\end{array}\right),  \tag{3.2.42a}\\
& \lim _{|z| \rightarrow 0} \chi_{n}^{-}(z)=\left(\begin{array}{cc}
c_{n}^{-1}+O\left(z^{2}, \text { even }\right) & z Q_{n-1}^{+}+O\left(z^{3}, \text { odd }\right) \\
c_{n}^{-1} z Q_{n}^{-}+O\left(z^{3}, \text { odd }\right) & 1+O\left(z^{2}, \text { even }\right)
\end{array}\right) \tag{3.2.42b}
\end{align*}
$$

Since $\phi_{n}(z)$ and $\psi_{n}(z)$ are solutions for the scattering problem (3.2.3a), then for any integer $s \geq 1$, their Wronskian relation satisfies the recursive relation,

$$
\begin{align*}
W\left(\phi_{n}^{+}(z), \phi_{n}^{-}(z)\right) & =\left\{\prod_{k=n-s}^{n-1}\left(1-Q_{k}^{-} Q_{k}^{+}\right)\right\} W\left(\phi_{n-s}^{+}(z), \phi_{n-s}^{-}(z)\right),  \tag{3.2.43a}\\
& =\left\{\prod_{k=n-s}^{n-1}\left(1-Q_{k}^{-} Q_{k}^{+}\right)\right\} W\left(\varphi_{n-s}^{+}(z), \varphi_{n-s}^{-}(z)\right), \tag{3.2.43b}
\end{align*}
$$

and when $s \rightarrow+\infty$

$$
\begin{equation*}
W\left(\phi_{n}^{+}(z), \phi_{n}^{-}(z)\right)=\prod_{k=-\infty}^{n-1}\left(1-Q_{k}^{-} Q_{k}^{+}\right) . \tag{3.2.43c}
\end{equation*}
$$

Similarly, for any integer $s \geq 1$,

$$
\begin{align*}
W\left(\psi_{n}^{-}(z), \psi_{n}^{+}(z)\right) & =\left\{\prod_{k=n}^{n+s-1}\left(1-Q_{k}^{-} Q_{k}^{+}\right)^{-1}\right\} W\left(\psi_{n+s}^{-}(z), \psi_{n+s}^{+}(z)\right),  \tag{3.2.44a}\\
& =\left\{\prod_{k=n}^{n+s-1}\left(1-Q_{k}^{-} Q_{k}^{+}\right)^{-1}\right\} W\left(\xi_{n+s}^{-}(z), \xi_{n+s}^{+}(z)\right), \tag{3.2.44b}
\end{align*}
$$

and as $s \rightarrow+\infty$,

$$
\begin{equation*}
W\left(\psi_{n}^{-}(z), \psi_{n}^{+}(z)\right)=\prod_{k=n}^{+\infty}\left(1-Q_{k}^{-} Q_{k}^{+}\right)^{-1} . \tag{3.2.44c}
\end{equation*}
$$

Then, $\chi_{n}^{ \pm}(z)$ is analytic when $|z| \gtrless 1$. The idea comes from the linear combination

$$
\left(\varphi_{n}^{+}, \varphi_{n}^{-}\right)(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)(z)\left(\begin{array}{cc}
a^{+}(z) & -z^{2 n} b^{-}(z)  \tag{3.2.45}\\
z^{-2 n} b^{+}(z) & a^{-}(z)
\end{array}\right)
$$

Then, we can write $\chi_{n}^{+}(z)$ as:

$$
\chi_{n}^{+}(z)=\left(\varphi_{n}^{+}, \xi_{n}^{+}\right)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)\left(\begin{array}{cc}
a^{+}(z) & 0  \tag{3.2.46}\\
z^{-2 n} b^{+}(z) & 1
\end{array}\right)
$$

and since $\operatorname{det} \xi_{n}=c_{n}^{-1}$. Then, from equation (3.2.46) $\operatorname{det}\left(\chi^{+}(z)\right)=c_{n}^{-1} a^{+}(z)$. Recalling the analytic properties of the Jost functions and the expressions (3.2.28c) and (3.2.40), we can find the analytic properties of $\chi_{n}^{+}(z)$. Taking the determinant for both sides, we can see that $a^{+}(z)$ has an analytic extension in the region $|z| \rightarrow \infty$

$$
\operatorname{det}\left(\varphi_{n}^{+}, \xi_{n}^{+}\right)=\operatorname{det}\left(\begin{array}{cc}
1+O\left(z^{-2}, \text { even }\right) & c_{n}^{-1} z^{-1} Q_{n}^{+}+O\left(z^{3}, \text { odd }\right)  \tag{3.2.47a}\\
z^{-1} Q_{n-1}^{-}+O\left(z^{-3}, \text { odd }\right) & c_{n}^{-1}+O\left(z^{-2}, \text { even }\right)
\end{array}\right)
$$

since $\operatorname{det}\left(\chi_{n}^{+}(z)\right)=c_{n}^{-1} a^{+}(z)$ and $\chi_{n}^{+}(z)$ is analytic when $|z| \rightarrow \infty$. Then,

$$
\begin{equation*}
a^{+}(z)=1-O\left(z^{-2}, \text { even }\right) \quad \text { as }|z| \rightarrow \infty . \tag{3.2.47b}
\end{equation*}
$$

Using a similar algorithm for $\chi_{n}^{-}(z)$,

$$
\chi_{n}^{-}(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)\left(\begin{array}{cc}
1 & -z^{2 n} b^{-}(z)  \tag{3.2.47c}\\
0 & a^{-}(z)
\end{array}\right)
$$

we obtain $\operatorname{det}\left(\chi_{n}^{-}(z)\right)=c_{n}^{-1} a^{-}(z)$. Similar idea, we can find that $a^{-}(z)$ is analytic in $|z| \rightarrow 0$

$$
\begin{align*}
\operatorname{det}\left(\chi_{n}^{-}(z)\right) & =c_{n}^{-1} a^{-}(z)  \tag{3.2.48a}\\
\operatorname{det}\left(\xi_{n}^{-}, \varphi_{n}^{-}\right) & =\operatorname{det}\left(\begin{array}{cc}
c_{n}^{-1}+O\left(z^{2}, \text { even }\right) & z Q_{n-1}^{+}+O\left(z^{3}, \text { odd }\right) \\
c_{n}^{-1} z Q_{n}^{-}+O\left(z^{3}, \text { odd }\right) & 1+O\left(z^{2}, \text { even }\right)
\end{array}\right) . \tag{3.2.48b}
\end{align*}
$$

Since $\chi_{n}^{-}(z)$ is an analytic function as $|z| \rightarrow 0$. Then,

$$
\begin{equation*}
a^{-}(z)=1-O\left(z^{2}, \text { even }\right) \quad \text { as }|z| \rightarrow 0 \tag{3.2.48c}
\end{equation*}
$$

we can conclude that $a^{+}(z) \rightarrow 1$ as $|z| \rightarrow \infty$ and $a^{-}(z) \rightarrow 1$ as $|z| \rightarrow 0$. Furthermore, the Jost functions are continuous up to $|z|=1$, and the functions $a^{+}(z)$ and $a^{-}(z)$ are also continuous up to $|z|=1$. The scattering coefficients can be written as explicit sums of the Jost functions. They are derived as follows: from equation (3.2.26c) and equation (3.2.26d), we can find the identity relation for $n>0$ :

$$
\begin{equation*}
\tilde{\mathbf{G}}_{k}^{2}=\tilde{\mathbf{G}}_{k}^{1}+\mathbb{1} \tag{3.2.49a}
\end{equation*}
$$

Substituting both equations (3.2.26) in the linear combination between $\varphi_{n}$ and $\xi_{n}$, (3.2.45)

$$
\begin{align*}
\mathbb{1} & +\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{k}^{2} \tilde{\mathbf{Q}}_{k} \varphi_{k} \mathbf{Z}^{-1} \\
& =\left[\mathbb{1}+\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{k}^{1} \tilde{\mathbf{Q}}_{k} \xi_{k} \mathbf{Z}^{-1}\right]\left(\begin{array}{cc}
a^{+}(z) & -z^{2 n} b^{-}(z) \\
z^{-2 n} b^{+}(z) & a^{-}(z)
\end{array}\right), \tag{3.2.49b}
\end{align*}
$$

and using equation (3.2.49a) then, we have

$$
\begin{align*}
\left(\begin{array}{cr}
1-a^{+}(z) & z^{2 n} b^{-}(z) \\
-z^{-2 n} b^{+}(z) & 1-a^{-}(z)
\end{array}\right) & =-\sum_{k=-\infty}^{+\infty} \tilde{\mathbf{Q}}_{k} \varphi_{k} \mathbf{Z}^{-1} \\
& =-\sum_{k=-\infty}^{+\infty}\binom{z^{-1} Q_{k}^{+} \varphi_{k}^{(2),+} z Q_{k}^{+} \varphi_{k}^{(2),-}}{z^{-1} Q_{k}^{-} \varphi_{k}^{(1),+} z Q_{k}^{-} \varphi_{k}^{(1),-}} \tag{3.2.49c}
\end{align*}
$$

and if we compare each column in equation (3.2.49c), the corresponding expressions for the scattering coefficients are

$$
\begin{array}{ll}
a^{+}(z)=1+\sum_{k=-\infty}^{+\infty} z^{-1} Q_{k}^{+} \varphi_{k}^{(2),+}, & b^{+}(z)=\sum_{k=-\infty}^{+\infty} z^{2 k-1} Q_{k}^{-} \varphi_{k}^{(1),+}, \\
a^{-}(z)=1+\sum_{k=-\infty}^{+\infty} z Q_{k}^{-} \varphi_{k}^{(1),-}, & b^{-}(z)=-\sum_{k=-\infty}^{+\infty} z^{-2 k+1} Q_{k}^{+} \varphi_{k}^{(2),-} . \tag{3.2.51}
\end{array}
$$

### 3.3 Deriving evolution equations of DNLS

In this section, we will derive the operator of the time-dependence equation in an analogous way as in [17]. We start with the scattering problem

$$
\begin{equation*}
\Psi_{n+1}(z, t)=\mathcal{L}_{n} \Psi_{n}, \quad n \in \mathbb{N}, \tag{3.3.1}
\end{equation*}
$$

and time-evolution is given by the linear problem for a family of Lax operators related to the $s l(2, \mathbb{C})$ algebra

$$
\begin{equation*}
\Psi_{n, t}(z, t)=M_{n} \Psi_{n} . \tag{3.3.2}
\end{equation*}
$$

The compatibility condition for the integrable DNLS equation with two linear operators (the two Lax operators) $\mathcal{L}_{n}, M_{n}$, the so-called differential-difference zero curvature representation, takes the form:

$$
\begin{equation*}
M_{n+1} \mathcal{L}_{n}=\mathcal{L}_{n, t}+\mathcal{L}_{n} M_{n} \tag{3.3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{n}(z, t)=\frac{1}{z^{2}} M_{-2, n}+\frac{1}{z} M_{-1, n}+M_{0, n}+z M_{1, n}+z^{2} M_{2, n} \tag{3.3.4a}
\end{equation*}
$$

where $M_{\mathrm{i}, n}, \mathrm{i}=-2,-1,0,1,2$ have the forms:

$$
\begin{gather*}
M_{-2, n}=\left(\begin{array}{cc}
0 & 0 \\
0 & M_{-2, n}^{d}
\end{array}\right), \quad M_{-1, n}=\left(\begin{array}{cc}
0 & M_{-1, n}^{+} \\
M_{-1, n}^{-} & 0
\end{array}\right), \\
M_{0, n}=\left(\begin{array}{cc}
M_{0, n}^{d 1} & 0 \\
0 & M_{0, n}^{d 2}
\end{array}\right)  \tag{3.3.4b}\\
M_{1, n}=\left(\begin{array}{cc}
0 & M_{1, n}^{+} \\
M_{1, n}^{-} & 0
\end{array}\right), \quad M_{2, n}=\left(\begin{array}{cc}
M_{2, n}^{d} & 0 \\
0 & 0
\end{array}\right) .
\end{gather*}
$$

So, $M_{n}(z, t)$ will be

$$
M_{n}(z, t)=\left(\begin{array}{cc}
M_{0, n}^{d 1}+z^{2} M_{2, n}^{d} & z M_{1, n}^{+}+\frac{1}{z} M_{-1, n}^{+}  \tag{3.3.4c}\\
z M_{1, n}^{-}+\frac{1}{z} M_{-1, n}^{-} & M_{0, n}^{d 2}+z^{-2} M_{-2, n}^{d}
\end{array}\right)
$$

Substituting each $\mathcal{L}_{n}, M_{n}$ in equation (3.3.3) and equating powers of $z$ to zero, yields a system of twelve equations. The following points show how we fixed the coefficients of the powers of $z$.

- The coefficients of the highest $\left(z^{3}\right)$ and the lowest $\left(z^{-3}\right)$ powers of $z$ are started and estimated as constants in complex plane $\mathbb{C}$

$$
\begin{gather*}
z^{3}: \triangle M_{2, n}^{d}=M_{2, n+1}^{d}-M_{2, n}^{d}=C_{1},  \tag{3.3.5a}\\
z^{-3}: \triangle M_{-2, n}^{d}=M_{-2, n+1}^{d}-M_{-2, n}^{d}=C_{2}, \tag{3.3.5b}
\end{gather*}
$$

where $C_{1}$ and $C_{2}$ are constants.

- Moving on to the coefficients of $z^{2}$, and $z^{-2}$, we can find two relations for the positive powers

$$
\begin{align*}
& z^{2}: M_{1, n}^{+}=Q_{n}^{+} C_{1}  \tag{3.3.6a}\\
& z^{2}: M_{1, n+1}^{-}=Q_{n}^{-} C_{1} . \tag{3.3.6b}
\end{align*}
$$

Analogously, for the negative powers

$$
\begin{align*}
& z^{-2}: M_{-1, n+1}^{+}=Q_{n}^{+} C_{2},  \tag{3.3.6c}\\
& z^{-2}: M_{-1, n}^{-}=Q_{n}^{-} C_{2} . \tag{3.3.6d}
\end{align*}
$$

- For the positive powers $z$, we find two relations; one of them is identically satisfied and the other one is

$$
\begin{align*}
z: \triangle M_{0, n}^{d 1} & =\triangle\left(-Q_{n}^{+} Q_{n-1}^{-} C_{1}\right),  \tag{3.3.7a}\\
M_{0, n+1}^{d 1}-M_{0, n}^{d 1} & =\left(-Q_{n+1}^{+} Q_{n}^{-}+Q_{n}^{+} Q_{n-1}^{-}\right) C_{1} . \tag{3.3.7b}
\end{align*}
$$

For simplification, this term $M_{0, n+1}^{d 1}+Q_{n+1}^{+} Q_{n}^{-} C_{1}$ is assumed to be equal to a constant $C_{3}$ in the complex plane $\mathbb{C}$,

$$
\begin{equation*}
M_{0, n}^{d 1}=C_{3}-Q_{n}^{+} Q_{n-1}^{-} C_{1} . \tag{3.3.7c}
\end{equation*}
$$

- In a similar way, one can find the coefficients of $z^{-1}$ as

$$
\begin{equation*}
M_{2, n}^{d 2}=C_{4}-Q_{n}^{-} Q_{n-1}^{+} C_{2} . \tag{3.3.8}
\end{equation*}
$$

- Finally, the two remaining equations which are the coefficients of $z^{0}$, are the evolution equations

$$
\begin{align*}
Q_{n, t}^{+} & =Q_{n}^{+} M_{0, n+1}^{d 1}+M_{1, n+1}^{+}-Q_{n}^{+} M_{0, n}^{d 2}-M_{-1, n}^{+} \\
& =Q_{n}^{+}\left(C_{3}-C_{4}\right)-\left(1-Q_{n}^{+} Q_{n}^{-}\right)\left(Q_{n-1}^{+} C_{2}-Q_{n+1}^{+} C_{1}\right), \tag{3.3.9a}
\end{align*}
$$

and

$$
\begin{align*}
Q_{n, t}^{-} & =Q_{n}^{-} M_{0, n+1}^{d 2}+M_{-1, n+1}^{-}-Q_{n}^{-} M_{0, n}^{d 1}-M_{1, n}^{-} \\
& =-Q_{n}^{-}\left(C_{3}-C_{4}\right)-\left(1-Q_{n}^{+} Q_{n}^{-}\right)\left(Q_{n-1}^{-} C_{1}-Q_{n+1}^{-} C_{2}\right) \tag{3.3.9b}
\end{align*}
$$

Next, substituting $M_{i, n}$, the time dependent matrix $M_{n}$ becomes:

$$
M_{n}=\left(\begin{array}{cc}
C_{3}-Q_{n}^{+} Q_{n-1}^{-} C_{1}+z^{2} C_{1} & z Q_{n}^{+} C_{1}+\frac{1}{z} Q_{n-1}^{+} C_{2}  \tag{3.3.10a}\\
z Q_{n-1}^{-} C_{1}+\frac{1}{z} Q_{n}^{-} C_{2} & C_{4}-Q_{n}^{-} Q_{n-1}^{+} C_{2}+z^{-2} C_{2}
\end{array}\right)
$$

After finding the NLEEs (3.3.9), the spatial type of constants $\left(C_{3}-C_{4}\right)=\frac{2 \mathrm{i}}{h^{2}}$ and $C_{4}=\frac{-\mathrm{i}}{h^{2}}$ $\left(C_{3}=\frac{\mathrm{i}}{h^{2}}\right.$ ), are used to find the the NLEE. Furthermore, $C_{2}=\frac{\mathrm{i}}{h^{2}}$ and $C_{1}=\frac{-\mathrm{i}}{h^{2}}$. Then, the time dependence matrix becomes:

$$
M_{n}=\frac{1}{h^{2}}\left(\begin{array}{rr}
\mathrm{i} Q_{n}^{+} Q_{n-1}^{-}+\mathrm{i}\left(1-z^{2}\right) & -\mathrm{i}\left(z Q_{n}^{+}-z^{-1} Q_{n-1}^{+}\right)  \tag{3.3.10b}\\
\mathrm{i}\left(z^{-1} Q_{n}^{-}-z Q_{n-1}^{-}\right) & -\mathrm{i} Q_{n}^{-} Q_{n-1}^{+}-\mathrm{i}\left(1-z^{-2}\right)
\end{array}\right)
$$

with some transformations being built on the matrix $M_{n}$, (see the Appendix), the time dependence equation (3.3.2) becomes:

$$
\frac{\mathrm{d} \Psi_{n}}{\mathrm{~d} \tau}=\left(\begin{array}{cc}
\mathrm{i} Q_{n}^{+} Q_{(n-1)}^{-}-\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} & -\mathrm{i}\left(z Q_{n}^{+}-z^{-1} Q_{n-1}^{+}\right)  \tag{3.3.11}\\
\mathrm{i}\left(z^{-1} Q_{n}^{-}-z Q_{n-1}^{-}\right) & -\mathrm{i} Q_{n}^{-} Q_{n-1}^{+}+\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2}
\end{array}\right) \Psi_{n} .
$$

### 3.4 Time evolution of the scattering data

In this section, we will find the explicit form for the scattering data for any time [12]. Let's start from equation (3.3.11). Since $Q_{n}^{ \pm} \rightarrow 0$ when $n \rightarrow \pm \infty$, the time-dependence (3.3.11) is asymptotically of the form:

$$
\partial_{\tau} \Psi_{n}=\left(\begin{array}{cc}
-\mathrm{i} \omega & 0  \tag{3.4.1}\\
0 & \mathrm{i} \omega
\end{array}\right) \Psi_{n}, \quad \text { as } \quad n \rightarrow \pm \infty
$$

where $\omega=\frac{1}{2}\left(z-z^{-1}\right)^{2}$. As the solutions of (3.4.1) are not compatible with the boundary conditions of the Jost functions (3.2.12), we will define new time-dependence functions

$$
\mathcal{M}_{n}(z, \tau)=\varphi_{n}(z, \tau)\left(\begin{array}{cc}
e^{-\mathrm{i} \omega \tau} & 0  \tag{3.4.2}\\
0 & e^{\mathrm{i} \omega \tau}
\end{array}\right), \quad \mathcal{N}_{n}(z, \tau)=\xi_{n}(z, \tau)\left(\begin{array}{cc}
e^{-\mathrm{i} \omega \tau} & 0 \\
0 & e^{\mathrm{i} \omega \tau}
\end{array}\right)
$$

Then, from the linear combination

$$
\varphi_{n}(z, \tau)=\xi_{n}(z, \tau)\left(\begin{array}{cc}
a^{+}(z) & -z^{2 n} b^{-}(z)  \tag{3.4.3}\\
z^{-2 n} b^{+}(z) & a^{-}(z)
\end{array}\right)
$$

the new form of equation (3.4.2) becomes:

$$
\mathcal{M}_{n}(z, \tau)=\mathcal{N}_{n}(z, \tau)\left(\begin{array}{cc}
a^{+}(z, \tau) & -z^{2 n} e^{2 \mathrm{i} \omega \tau} b^{-}(z, \tau)  \tag{3.4.4}\\
z^{-2 n} e^{-2 \mathrm{i} \omega \tau} b^{+}(z, \tau) & a^{-}(z, \tau)
\end{array}\right) .
$$

The aim is now to find the expression of the scattering coefficients $a^{ \pm}(z, t)$ and $b^{ \pm}(z, t)$. The following steps will explain all the details:

- we first differentiate equations (3.4.4) with respect to $\tau$

$$
\begin{align*}
& \quad \mathcal{M}_{\{n, \tau\}}(z, \tau)=\mathcal{N}_{\{n, \tau\}}(z, \tau)\left(\begin{array}{cc}
a^{+}(z, \tau) & -z^{2 n} e^{2 \mathrm{i} \omega \tau} b^{-}(z, \tau) \\
z^{-2 n} e^{-2 \mathrm{i} \omega \tau} b^{+}(z, \tau) & a^{-}(z, \tau)
\end{array}\right)(z, \tau) \\
& +\mathcal{N}_{n}\left(\begin{array}{cc}
a_{\tau}^{+}(z, \tau) & -z^{2 n} e^{2 \mathrm{i} \omega \tau}\left[2 \mathrm{i} \omega \mathrm{~b}^{-}(\mathrm{z}, \tau)+\mathrm{b}_{\tau}^{-}(\mathrm{z}, \tau)\right] \\
z^{-2 n} e^{-2 \mathrm{i} \omega \tau}\left[-2 \mathrm{i} \omega \mathrm{~b}^{+}(\mathrm{z}, \tau)+\mathrm{b}_{\tau}^{+}(\mathrm{z}, \tau)\right] & a_{\tau}^{-}(z, \tau)
\end{array}\right), \tag{3.4.5a}
\end{align*}
$$

- since $\mathcal{M}_{n}(z, \tau)$ and $\mathcal{N}_{n}(z, \tau)$ satisfy equation (3.4.1), that means we can obtain
another relation for $\mathcal{M}_{n, \tau}(z, \tau)$ and $\mathcal{N}_{n, \tau}(z, \tau)$

$$
\begin{align*}
& \mathcal{M}_{\{n, \tau\}}(z, \tau)=\left(\begin{array}{cc}
-\mathrm{i} \omega & 0 \\
0 & \mathrm{i} \omega
\end{array}\right) \mathcal{M}_{n}(z, \tau),  \tag{3.4.5b}\\
& \mathcal{N}_{\{n, \tau\}}(z, \tau)=\left(\begin{array}{cc}
-\mathrm{i} \omega & 0 \\
0 & \mathrm{i} \omega
\end{array}\right) \mathcal{N}_{n}(z, \tau),
\end{align*}
$$

using both equations (3.4.5b) in equation (3.4.4)

$$
\begin{align*}
\mathcal{M}_{\{n, \tau\}}(z, \tau) & =\left(\begin{array}{cc}
-i \omega & 0 \\
0 & i \omega
\end{array}\right) \mathcal{N}_{n}(z, \tau)\left(\begin{array}{cc}
a^{+}(z, \tau) & -z^{2 n} e^{2 i \omega \tau} b^{-}(z, \tau) \\
z^{-2 n} e^{-2 i \omega \tau} b^{+}(z, \tau) & a^{-}(z, \tau)
\end{array}\right) \\
& =\mathcal{N}_{\{n, \tau\}}(z, \tau)\left(\begin{array}{cc}
a^{+}(z, \tau) & -z^{2 n} e^{2 i \omega \tau} b^{-}(z, \tau) \\
z^{-2 n} e^{-2 i \omega \tau} b^{+}(z, \tau) & a^{-}(z, \tau)
\end{array}\right), \tag{3.4.5c}
\end{align*}
$$

- by comparing relations (3.4.5a) and (3.4.5c), one can obtain:

$$
\begin{align*}
a_{\tau}^{ \pm}(z, \tau) & =0, \quad a^{ \pm}(z, \tau)=a^{ \pm}(z, 0)  \tag{3.4.6a}\\
\mp 2 \mathrm{i} \omega b^{ \pm}(\mathrm{z}, \tau)+b_{\tau}^{ \pm}(\mathrm{z}, \tau) & =0 \quad \text { and } \quad b^{ \pm}(z, \tau)=e^{ \pm 2 i \omega \tau} b^{ \pm}(z, 0) . \tag{3.4.6b}
\end{align*}
$$

### 3.5 Symmetries of the Ablowitz-Ladik system

## 1. Local involution

An important and harmonic tool to construct new integrable NLEE is using the reduction condition: $Q_{n}^{-}= \pm\left(Q_{n}^{+}\right)^{*}$. Then, $\mathcal{L}_{n}$ must satisfy

$$
\begin{equation*}
\mathbf{C} \mathcal{L}_{n}(z, t)=\mathcal{L}_{n}^{*}\left((1 / z)^{*}, t\right) \tag{3.5.1}
\end{equation*}
$$

and the $T(z, t)$ satisfies

$$
\begin{equation*}
\mathbf{C} T(z, t)=T^{*}\left((1 / z)^{*}, t\right), \tag{3.5.2}
\end{equation*}
$$

and the eigenfunctions become

$$
\begin{equation*}
\mathbf{C} \phi(z, t)=\psi_{n}^{*}\left((1 / z)^{*}, t\right) \tag{3.5.3}
\end{equation*}
$$

where the transformation is $\mathbf{C} X=\sigma X \sigma^{-1}, \sigma=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. Thus, the local DNLS
equation has the following form

$$
\begin{equation*}
\mathrm{i} \frac{d}{d t} q_{n}=\frac{1}{h^{2}}\left(q_{n+1}-2 q_{n}+q_{n-1}\right) \pm\left|q_{n}\right|^{2}\left(q_{n+1}+q_{n-1}\right) \tag{3.5.4}
\end{equation*}
$$

2. Nonlocal involution

The spectral problem (3.2.3a) satisfies the symmetry (the nonlocal involution) of the form:

$$
\begin{equation*}
Q_{n}^{-}(t)=\epsilon\left(Q_{-n}^{+}(t)\right)^{*}, \quad \epsilon= \pm 1 \tag{3.5.5}
\end{equation*}
$$

Thus, the potentials $\tilde{\mathbf{Q}}_{n}$ (3.2.5) and matrix $M_{n}$ in equation (3.3.2) reduce to

$$
\begin{align*}
& \tilde{\mathbf{Q}}_{n}=\left(\begin{array}{cc}
0 & Q_{n}^{+} \\
\epsilon\left(Q_{-n}^{+}\right)^{*}(t) & 0
\end{array}\right),  \tag{3.5.6}\\
& M_{n}(z, t)= \\
& \mathrm{i}\left(\begin{array}{c}
\epsilon Q_{n}^{+}\left(Q_{1-n}^{+}\right)^{*}-\frac{1}{2}\left(z-z^{-1}\right)^{2} \\
\epsilon\left(z^{-1}\left(Q_{-n}^{+}\right)^{*}-z\left(Q_{1-n}^{+}\right)^{*}\right) \\
z^{-1} Q_{n-1}^{+}-\epsilon\left(Q_{-n}^{+}\right)^{*} Q_{n-1}^{+}+\frac{1}{2}\left(z-Q_{n}^{-1}\right)^{2}
\end{array}\right) . \tag{3.5.7}
\end{align*}
$$

The nonlocal involution imposes a symmetry condition on the eigenfunctions, FASs and scattering matrix

$$
\begin{align*}
\mathbf{C}\left(\psi_{-n}^{\dagger}\left((z)^{*}, t\right)\right) & :=B \psi_{-n}^{\dagger}\left((z)^{*}, t\right) B^{-1}=\phi_{n}(z, t),  \tag{3.5.8a}\\
\mathbf{C}\left(\chi_{-n}^{\{-, \dagger\}}\left((z)^{*}, t\right)\right) & :=B \chi_{-n}^{\{-, \dagger\}}\left((z)^{*}, t\right) B^{-1}=\chi_{n}^{+}(z, t), \tag{3.5.8b}
\end{align*}
$$

and

$$
\begin{equation*}
\mathbf{C}\left(T^{\dagger}\left((z)^{*}, t\right)\right):=B T^{\dagger}\left((z)^{*}, t\right) B^{-1} . \tag{3.5.8c}
\end{equation*}
$$

As a result, we obtain:

$$
\begin{equation*}
a^{ \pm}(z, t)=\left(a^{ \pm}\left(z^{*}, t\right)\right)^{*}, \quad b^{ \pm}(z, t)=\left(b^{\mp}\left(z^{*}, t\right)\right)^{*}, \tag{3.5.8d}
\end{equation*}
$$

where the particular choice of $B$ is a zero diagonal matrix and the off-diagonal entries are $(1,-1)$, and $\mathbf{C}$ is an automorphism of the Lie group $S L(2, \mathbb{C})$. Then, equation (3.5.9) is a nonlocal differential-difference equation that possesses the same integrable structure as the local integrable DNLS equation

$$
\begin{equation*}
\text { i } Q_{n, \tau}^{+}=\left(Q_{n+1}^{+}-2 Q_{n}^{+}+Q_{n-1}^{+}\right)-\epsilon Q_{n}^{+}\left(Q_{-n}^{+}\right)^{*}\left(Q_{n+1}^{+}+Q_{n-1}^{+}\right) \tag{3.5.9}
\end{equation*}
$$

### 3.6 Soliton solutions with $\mathcal{P}$ - - symmetry

In this section, we will use the RHP (3.2.45) to find the soliton solution to the nonlocal DNLS equation (3.1.3).

### 3.6.1 Case of no poles

In this part, we can rewrite equation (3.2.45) as:

$$
\begin{align*}
& \left(\frac{\varphi_{n}^{+}}{a^{+}}, \frac{\varphi_{n}^{-}}{a^{-}}\right)(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)(z)\left(\begin{array}{cc}
1 & -z^{2 n} \rho^{-} \\
z^{-2 n} \rho^{+} & 1
\end{array}\right),  \tag{3.6.1a}\\
& \left(\tilde{\varphi}_{n}^{+}, \tilde{\varphi}_{n}^{-}\right)(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)(z)\left(\begin{array}{cc}
1 & -z^{2 n} \rho^{-} \\
z^{-2 n} \rho^{+} & 1
\end{array}\right), \tag{3.6.1b}
\end{align*}
$$

where $\rho^{ \pm}(z)=\frac{b^{ \pm}(z)}{a^{ \pm}(z)}$, are the reflection coefficients. Furthermore, $\tilde{\varphi}_{n}^{+}(z)$ and $\tilde{\varphi}_{n}^{-}(z)$, have the same canonical normalisation as $\varphi_{n}^{+}(z)$ and $\varphi_{n}^{-}(z)$ in equations (3.2.28c) and (3.2.40) and are meromorphic ${ }^{1}$ in the region $|z| \rightarrow \infty,|z| \rightarrow 0$, respectively. Then, we can define new functions $\tilde{\chi}^{ \pm}$, of the following form

$$
\begin{align*}
& \tilde{\chi}_{n}^{+}(z)=\left(\tilde{\varphi}_{n}^{+}, \xi_{n}^{+}\right)(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)(z)\left(\begin{array}{cc}
1 & 0 \\
z^{-2 n} \rho^{+} & 1
\end{array}\right),  \tag{3.6.2a}\\
& \tilde{\chi}_{n}^{-}(z)=\left(\xi_{n}^{-}, \tilde{\varphi}_{n}^{-}\right)(z)=\left(\xi_{n}^{-}, \xi_{n}^{+}\right)(z)\left(\begin{array}{cc}
1-z^{2 n} \rho^{-} \\
0 & 1
\end{array}\right) . \tag{3.6.2b}
\end{align*}
$$

Let us now first consider the case when there are no discrete eigenvalues, that is when $\tilde{\chi}_{n}^{ \pm}(z)$ have no poles, which means $\tilde{\chi}_{n}^{+}(z)$ is analytic outside the unit circle $|z|=1$ and $\tilde{\chi}_{n}^{-}(z)$ is analytic inside the unit circle. We can write the jump conditions (3.6.1b) as:

$$
\begin{equation*}
\tilde{\chi}_{n}^{+}(z)-\tilde{\chi}_{n}^{-}(z)=\tilde{\chi}_{n}^{-}(z) \mathbf{G}_{n}(z), \quad|z|=1 \tag{3.6.3a}
\end{equation*}
$$

where

$$
\mathbf{G}_{n}(z)=\left(\begin{array}{cc}
\rho^{+} \rho^{-} & z^{2 n} \rho^{-}  \tag{3.6.3b}\\
z^{-2 n} \rho^{+} & 0
\end{array}\right), \quad \tilde{\chi}_{n}^{+} \rightarrow \mathbb{1} \quad \text { as } \quad|z| \rightarrow \infty .
$$

Note that (3.6.3a)hold, in general, only for $|z|=1$. Therefore, equation (3.6.3a) can be regarded as a generalised RHP on $|z|=1$ with the boundary conditions given by (3.6.3b).

[^11]We start with the projection integral operators [12][91]

$$
\begin{equation*}
\bar{P}(f)(z)=\lim _{\substack{\zeta \rightarrow z \\|\zeta|<1}} \frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \frac{\mathrm{~d}(\omega) f(\omega)}{\omega-\zeta}, \tag{3.6.4}
\end{equation*}
$$

applying $\bar{P}$ on the left hand side of equation (3.6.3a),

$$
\begin{equation*}
\bar{P}\left(\tilde{\chi}_{n}^{+}-\tilde{\chi}_{n}^{-}\right)(z)=\lim _{\substack{\zeta \rightarrow z \\|\zeta|<1}} \frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \frac{\left(\tilde{\chi}_{n}^{+}-\tilde{\chi}_{n}^{-}\right)(\omega)}{\omega-\zeta} \mathrm{d}(\omega), \tag{3.6.5}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
\tilde{\chi}_{n}^{-}(z)=\mathbb{1}+\lim _{\substack{\zeta \rightarrow z \\|\zeta|<1}} \frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \frac{\tilde{\chi}_{n}^{-} \mathbf{G}_{n}(\omega)}{\omega-\zeta} \mathrm{d}(\omega), \tag{3.6.6}
\end{equation*}
$$

which is a linear integral equation on $|z|=1$, for $\xi_{n}^{-}(z)$ and $\tilde{\varphi}_{n}^{-}(z)$. We can rewrite (3.6.6) equivalently as:

$$
\begin{equation*}
\tilde{\chi}_{n}^{-}(z)=\mathbb{1}+\lim _{\substack{\zeta \rightarrow z \\|\zeta|<1}} \frac{z}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \omega^{-2} \tilde{\chi}_{n}^{-} \mathbf{G}_{n}(\omega) \mathrm{d}(\omega)+O\left(z^{2}\right), \tag{3.6.7}
\end{equation*}
$$

where $O\left(z^{2}\right)$ is defined in Sec. 3.2.3. Functions $\tilde{\chi}_{n}^{+}(z)$ and $\tilde{\chi}_{n}^{-}(z)$ are meromorphic for $|z| \gtrless 1$ and have the limits as in equations (3.2.42). We can see that both functions contain the term $c_{n}=\prod_{k=n}^{\infty}\left(1-Q_{k}^{+} Q_{k}^{-}\right)$. The boundary condition for $\tilde{\chi}_{n}^{ \pm}$depends on $Q_{k}^{+}$and $Q_{k}^{-}$for all $k \geq n$. However, $Q_{n}^{+}$and $Q_{n}^{-}$are unknowns in the inverse problem. To remove this dependence, we need to multiply both sides of (3.2.42) by $\left(\begin{array}{cc}1 & 0 \\ 0 & c_{n}\end{array}\right)$,

$$
\tilde{\chi}_{n}^{-}(z)=\left(\begin{array}{cc}
c_{n}^{-1}+O\left(z^{2}, \text { even }\right) & z Q_{n-1}^{+}+O\left(z^{3}, \text { odd }\right)  \tag{3.6.8a}\\
z Q_{n}^{-}+O\left(z^{3}, \text { odd }\right) & c_{n}+O\left(z^{2}, \text { even }\right)
\end{array}\right), \quad z \rightarrow 0
$$

then,

$$
\tilde{\chi}_{n}^{-}(z) \mathbf{G}_{n}(z)=\left(\begin{array}{cc}
\rho^{+} \rho^{-} \xi_{n}^{-, 1}+z^{-2 n} \rho^{+} \tilde{\varphi}_{n}^{-, 1} & z^{2 n} \rho^{-} \xi_{n}^{-, 1}  \tag{3.6.8b}\\
\rho^{+} \rho^{-} \xi_{n}^{-, 2}+z^{-2 n} \rho^{+} \tilde{\varphi}_{n}^{-, 2} & z^{2 n} \rho^{-} \xi_{n}^{-, 2}
\end{array}\right) .
$$

To recover $Q_{n}^{+}$and $Q_{n}^{-}$by comparing the expansion in equation (3.6.8a) about $z=0$ with
the RHS of equation (3.6.7), we obtain:

$$
\begin{align*}
Q_{n-1}^{+} & =\frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \omega^{2(n-1)} \rho^{-}(\omega) \xi_{n}^{-,, 1}(\omega) \mathrm{d} \omega  \tag{3.6.9a}\\
Q_{n}^{+} & =\frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \omega^{2 n} \rho^{-}(\omega) \xi_{n+1}^{-, 1}(\omega) \mathrm{d} \omega  \tag{3.6.9b}\\
Q_{n}^{-} & =\frac{1}{2 \pi \mathrm{i}} \oint_{|\omega|=1} \omega^{-2} \rho^{+}(\omega) \rho^{-}(\omega) \xi_{n}^{-, 2}(\omega)+\omega^{-2(n+1)} \rho^{+}(\omega) \tilde{\varphi}_{n}^{-, 2}(\omega) \mathrm{d} \omega \tag{3.6.9c}
\end{align*}
$$

The formulation of the inverse problem is now complete. We recovered the potentials from the eigenfunctions that are defined for $|z| \leq 1$, namely $\tilde{\chi}_{n}^{-}(z)$.

### 3.6.2 Case of poles

The method of solution requires an extra step if $\tilde{\varphi}_{n}^{ \pm}(z)$ have poles. Here, we need to apply the contour integration method. This method starts from the relations between the Jost functions (3.6.1a),

$$
\begin{align*}
& \tilde{\varphi}_{n}^{+}=\frac{\varphi_{n}^{+}}{a^{+}}(z)=\xi_{n}^{-}(z)+z^{-2 n} \rho^{+}(z) \xi_{n}^{+}(z),  \tag{3.6.10a}\\
& \tilde{\varphi}_{n}^{-}=\frac{\varphi_{n}^{-}}{a^{-}}(z)=\xi_{n}^{+}(z)-z^{2 n} \rho^{-}(z) \xi_{n}^{-}(z) \tag{3.6.10b}
\end{align*}
$$

We will apply the contour integration method on equations (3.6.10) to find the following integral representations

$$
\begin{align*}
& \mathcal{J}_{1, n}(z)=\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\gamma^{+}} \frac{\mathrm{d} \omega \varphi_{n}^{+}(\omega)}{(\omega-z) a^{+}(\omega)}-\oint_{\gamma^{-}} \frac{\mathrm{d} \omega \xi_{n}^{-}(\omega)}{(\omega-z)}\right),  \tag{3.6.11a}\\
& \mathcal{J}_{2, n}(z)=\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\gamma^{+}} \frac{\mathrm{d} \omega \xi_{n}^{+}(\omega)}{(\omega-z)}-\oint_{\gamma^{-}} \frac{\mathrm{d} \omega \varphi_{n}^{-}(\omega)}{(\omega-z) a^{-}(\omega)}\right) . \tag{3.6.11b}
\end{align*}
$$

We will outline one of the cases ( $\mathcal{J}_{2, n}$ ), and analogously, one can derive the other. As a reminder (i) $\frac{1}{a^{+}(z)}$ has simple pole at $z=z_{j}^{+}$; (ii) $\frac{1}{a^{-(z)}}$ has simple pole at $z=z_{j}^{-}$and (iii) $\xi_{n}^{-}, \xi_{n}^{+}$have no poles, and therefore the integrand of the first integral in $\mathcal{J}_{2, n}(z)$ has only a pole at $z=\omega$ and (iv) outside the contour is negatively oriented, while inside the contour is positively oriented. Thus, when $z \in \Omega_{+}$, we find

$$
\begin{equation*}
\mathcal{J}_{2, n}(z)=\xi_{n}^{+}(z)-\sum_{j=1}^{S}\left[\frac{\varphi_{n}^{-}\left(z_{j}^{-}\right)}{\left(z-z_{j}^{-}\right) \dot{a}_{j}^{-}}+\frac{\varphi_{n}^{-}\left(-z_{j}^{-}\right)}{\left(z+z_{j}^{-}\right) \dot{a}_{j}^{-}}\right], \tag{3.6.12}
\end{equation*}
$$

and we need to use the asymptotic behaviour of $\xi_{n}^{+}$and $\varphi_{n}^{-}$to obtain:

$$
\begin{equation*}
\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\gamma^{+} \rightarrow \Omega(|\omega|=1)} \frac{\mathrm{d} \omega}{(\omega-z)}\binom{0}{1}-\oint_{\gamma^{-} \rightarrow \Omega} \frac{\mathrm{d} \omega}{(\omega-z)}\binom{0}{1}\right)=\binom{0}{1} \tag{3.6.13}
\end{equation*}
$$



Figure 3.1: The thick line represents the continuous spectrum $(\Omega=|z|=1)$ of $\mathcal{L}_{n}(z)$ in the complex $z$ plane. $\Omega_{+}$represents the region $|z|>1$ and $\Omega_{-}$represents the region $|z|<1$. $\gamma_{ \pm}$are contours for both regions $|z| \gtrless$, respectively and have the same orientation around the continuous spectrum of $\mathcal{L}_{n}(z)$.

Finally, the integral along the unit circle $(\Omega)$ is evaluated from equation (3.6.10b) and is equal to

$$
\begin{align*}
& \frac{1}{2 \pi \mathrm{i}}\left(\oint_{\gamma^{+}} \frac{\mathrm{d} \omega \xi_{n}^{+}(\omega)}{(\omega-z)}-\oint_{\gamma^{-}} \frac{\mathrm{d} \omega \varphi_{n}^{-}(\omega)}{(\omega-z) a^{-}(\omega)}\right) \\
& =\frac{1}{2 \pi \mathrm{i}} \oint_{\Omega} \frac{\mathrm{d} \omega}{(\omega-z)} \omega^{2 n} \rho^{-}(\omega) \xi_{n}^{-}(\omega) . \tag{3.6.14}
\end{align*}
$$

This will lead to the following representation for $\xi_{n}^{+}(z)$ : When $a^{ \pm}(z)$ have zeros at $z=$ $z_{j}^{ \pm}$, then (3.2.45) remains

$$
\begin{equation*}
\varphi_{n, j}^{ \pm}= \pm\left(z_{j}^{ \pm}\right)^{\mp 2 n} b_{j}^{ \pm} \xi_{n, j}^{ \pm}, \tag{3.6.15a}
\end{equation*}
$$

where $\varphi_{n, j}^{ \pm}=\varphi_{n}^{ \pm}\left(z_{j}^{ \pm}\right)$and $\xi_{n, j}^{ \pm}=\xi_{n}^{ \pm}\left(z_{j}^{ \pm}\right)$, thus

$$
\begin{align*}
\xi_{n}^{+}(z) & =\binom{0}{1}+\frac{1}{2 \pi \mathrm{i}} \oint_{\Omega} \frac{\mathrm{d} \omega}{(\omega-z)} \omega^{2 n} \rho^{-}(\omega) \xi_{n}^{-}(\omega) \\
& -\sum_{j=1}^{S} C_{j}^{-}\left(z_{j}^{-}\right)^{2 n}\left[\frac{\xi_{n}^{-}\left(z_{j}^{-}\right)}{\left(z-z_{j}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{j}^{-}\right)}{\left(z+z_{j}^{-}\right)}\right] . \tag{3.6.15b}
\end{align*}
$$

The same structures are used to find the integral $\mathcal{J}_{1, n}$ when $z \in \Omega_{-}$, we can find $\xi_{n}^{-}(z)$

$$
\begin{align*}
\xi_{n}^{-}(z) & =\binom{1}{0}+\frac{1}{2 \pi \mathrm{i}} \oint_{\Omega} \frac{\mathrm{d} \omega}{(\omega-z)} \omega^{-2 n} \rho^{+}(\omega) \xi_{n}^{+}(\omega) \\
& +\sum_{j=1}^{S} C_{j}^{+}\left(z_{j}^{+}\right)^{-2 n}\left[\frac{\xi_{n}^{+}\left(z_{j}^{+}\right)}{\left(z-z_{j}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{j}^{+}\right)}{\left(z+z_{j}^{+}\right)}\right] \tag{3.6.15c}
\end{align*}
$$

where $C_{j}^{+}=C^{+}\left(z_{j}^{+}\right)$and $C_{j}^{-}=C^{-}\left(z_{j}^{-}\right)$are the norming constants defined as:

$$
\begin{equation*}
C_{j}^{+}=\frac{b_{j}^{+}}{\dot{a}_{j}^{+}}, \quad \text { and } \quad C_{j}^{-}=\frac{b_{j}^{-}}{\dot{a}_{j}^{-}} . \tag{3.6.16}
\end{equation*}
$$

Then, the minimal set $\left(\mathcal{T}_{1}=\left\{\rho^{+}(z), \rho^{-}(z), z \in|z|=1\right\}\right.$, is considered as the minimal set of scattering data. Thus, the system of singular integral equations (3.6.15) admits a unique solution, so $\mathcal{T}_{1}$ determines uniquely the Jost solutions $\xi_{n}^{ \pm}$.

### 3.6.3 Reflectionless potentials

The case where the scattering data observe proper eigenvalues, $\rho^{+}(z)=\rho^{-}(z)$ must be equal 0 on $\Omega(|z|=1)$, then the system (algebraic-integral) (3.6.15) reduces to the linear algebraic system

$$
\begin{align*}
& \xi_{n}^{+}(z)=\binom{0}{1}-\sum_{j=1}^{S} C_{j}^{-}\left(z_{j}^{-}\right)^{2 n}\left[\frac{\xi_{n}^{-}\left(z_{j}^{-}\right)}{\left(z-z_{j}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{j}^{-}\right)}{\left(z+z_{j}^{-}\right)}\right]  \tag{3.6.17a}\\
& \xi_{n}^{-}(z)=\binom{1}{0}+\sum_{j=1}^{S} C_{j}^{+}\left(z_{j}^{+}\right)^{-2 n}\left[\frac{\xi_{n}^{+}\left(z_{j}^{+}\right)}{\left(z-z_{j}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{j}^{+}\right)}{\left(z+z_{j}^{+}\right)}\right], \tag{3.6.17b}
\end{align*}
$$

where $\xi_{n}^{ \pm}(z)$ are defined on $z_{j}^{ \pm}$and $-z_{j}^{ \pm}$as:

$$
\begin{align*}
\xi_{n}^{+}\left(z_{j}^{+}\right) & =\binom{0}{1}-\sum_{k=1}^{S} C_{k}^{-}\left(z_{k}^{-}\right)^{2 n}\left[\frac{\xi_{n}^{-}\left(z_{k}^{-}\right)}{\left(z_{j}^{+}-z_{k}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{k}^{-}\right)}{\left(z_{j}^{+}+z_{k}^{-}\right)}\right],  \tag{3.6.18a}\\
\xi_{n}^{+}\left(-z_{j}^{+}\right) & =\binom{0}{1}+\sum_{k=1}^{S} C_{k}^{-}\left(z_{k}^{-}\right)^{2 n}\left[\frac{\xi_{n}^{-}\left(z_{k}^{-}\right)}{\left(z_{j}^{+}+z_{k}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{k}^{-}\right)}{\left(z_{j}^{+}-z_{k}^{-}\right)}\right],  \tag{3.6.18b}\\
\xi_{n}^{-}\left(z_{j}^{-}\right) & =\binom{1}{0}+\sum_{k=1}^{S} C_{k}^{+}\left(z_{k}^{+}\right)^{-2 n}\left[\frac{\xi_{n}^{+}\left(z_{k}^{+}\right)}{\left(z_{j}^{-}-z_{k}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{k}^{+}\right)}{\left(z_{j}^{-}+z_{k}^{+}\right)}\right],  \tag{3.6.18c}\\
\xi_{n}^{-}\left(-z_{j}^{-}\right) & =\binom{1}{0}-\sum_{k=1}^{S} C_{k}^{+}\left(z_{k}^{+}\right)^{-2 n}\left[\frac{\xi_{n}^{+}\left(z_{k}^{+}\right)}{\left(z_{j}^{-}+z_{k}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{k}^{+}\right)}{\left(z_{j}^{-}-z_{k}^{+}\right)}\right] . \tag{3.6.18d}
\end{align*}
$$

The above relations show that

$$
\begin{array}{lll}
\xi_{n}^{+, 1}\left(-z_{j}^{+}\right)=-\xi_{n}^{+, 1}\left(z_{j}^{+}\right), & \text {if and only if } & \xi_{n}^{-,, 1}\left(-z_{j}^{-}\right)=\xi_{n}^{-, 1}\left(z_{j}^{-}\right), \\
\xi_{n}^{+, 2}\left(-z_{j}^{+}\right)=\xi_{n}^{+, 2}\left(z_{j}^{+}\right), & \text {if and only if } & \xi_{n}^{-, 2}\left(-z_{j}^{-}\right)=-\xi_{n}^{-, 2}\left(z_{j}^{-}\right) . \tag{3.6.19b}
\end{array}
$$

### 3.6.4 one - Soliton solution

The one - soliton solution for (3.1.3), there are 4 eigenvalues $\left(\left\{ \pm z_{1}^{+}\right\},\left\{ \pm z_{1}^{-}\right\}\right)$. Starting from (3.6.8a) and (3.6.17b), we will obtain the formulas of $Q_{n}^{-}(z)$

$$
\begin{align*}
z Q_{n}^{-} & =\sum_{k=1}^{S} C_{k}^{+}\left(z_{k}^{+}\right)^{-2 n}\left[\frac{\xi_{n}^{+, 2}\left(z_{k}^{+}\right)}{\left(z-z_{k}^{+}\right)}+\frac{\xi_{n}^{+, 2}\left(-z_{k}^{+}\right)}{\left(z+z_{k}^{+}\right)}\right] \\
Q_{n}^{-} & =2 \sum_{j=1}^{S} \frac{C_{k}^{+}\left(z_{k}^{+}\right)^{-2 n}}{z^{2}-\left(z_{k}^{+}\right)^{2}} \xi_{n}^{+, 2}\left(z_{k}^{+}\right) \tag{3.6.20}
\end{align*}
$$

and when $S=1$,

$$
\begin{equation*}
Q_{1 n}^{-}=2 C_{1}^{+}\left(z_{1}^{+}\right)^{-2 n-2} \xi_{n}^{+, 2}\left(z_{1}^{+}\right) \tag{3.6.21}
\end{equation*}
$$

Solving the linear algebraic system (3.6.17a) and (3.6.17b) we find

$$
\begin{align*}
& \xi_{n}^{-, 1}\left(z_{1}^{-}\right)=\left[1-4 C_{1}^{+} C_{1}^{-} \frac{\left(z_{1}^{+}\right)^{-2(n-1)}\left(z_{1}^{-}\right)^{2 n}}{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)^{2}}\right]^{-1}  \tag{3.6.22}\\
& \xi_{n}^{+, 2}\left(z_{1}^{+}\right)=\left[1+4 C_{1}^{+} C_{1}^{-} \frac{\left(z_{1}^{-}\right)^{2(n+1)}\left(z_{1}^{+}\right)^{-2 n}}{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)^{2}}\right]^{-1} \tag{3.6.23}
\end{align*}
$$

Substituting equation (3.6.23) in equation (3.6.21), with the involution condition $Q_{1 n}^{-}(t)=$ $-\left(Q^{+}\right)_{-1 n}^{*}$ we have

$$
\begin{gather*}
-\left(Q^{+}\right)_{-1 n}^{*}=2 C_{1}^{+}\left(z_{1}^{+}\right)^{-2 n-2} \xi_{n}^{+, 2}\left(z_{1}^{+}\right),  \tag{3.6.24}\\
Q_{1 n}^{+}=-2\left(C_{1}^{+}\right)^{*}\left(\left(z_{1}^{+}\right)^{*}\right)^{2 n-2}\left((\xi)_{-n}^{+, 2}\left(z_{1}^{+}\right)\right)^{*},  \tag{3.6.25}\\
Q_{1 n}^{+}(\tau)=\frac{-2\left(C_{1}^{+}\right)^{*}\left(\left(z_{1}^{+}\right)^{*}\right)^{2(n-1)}}{1+4\left(C_{1}^{+}\right)^{*}\left(C_{1}^{-}\right)^{*}\left(\left(z_{1}^{+}\right)^{*}\right)^{2 n}\left(\left(\left(z_{1}^{+}\right)^{*}\right)^{2}-\left(\left(z_{1}^{-}\right)^{*}\right)^{2}\right)^{-2}\left(\left(z_{1}^{-}\right)^{*}\right)^{2(-n+1)}}, \tag{3.6.26}
\end{gather*}
$$

where the norminig constants $C_{j}^{ \pm}(z, \tau)(3.6 .16)$ are defined as

$$
\begin{align*}
C_{1}^{+}(z, \tau)=C_{1}^{+}(0) e^{2 \mathrm{i} \omega_{1}^{+} \tau}, & C_{1}^{-}(z, \tau)=C_{1}^{-}(0) e^{-2 \mathrm{i} \omega_{1}^{-} \tau} \\
C_{1}^{+}(0)=\frac{z_{1}^{+}\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right) e^{\mathrm{i} \alpha_{1}^{+}}}{2 z_{1}^{-}}, & C_{1}^{-}(0)=\frac{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right) e^{\mathrm{i} \alpha_{1}^{-}}}{2 z_{1}^{-} z_{1}^{+}} . \tag{3.6.27}
\end{align*}
$$

where $\alpha_{1}^{ \pm}$are positive numbers and $\omega_{1}^{ \pm}=\frac{1}{2}\left(z_{1}^{ \pm}-\left(z_{1}^{ \pm}\right)^{-1}\right)^{2}$. Equation (3.6.26) is the solution of the nonlocal DNLS equation (3.1.3) where $Q_{1 n}^{+}(\tau)=h q_{n}(t)$ and $\tau \in h^{-2} t$.

### 3.6.5 Two - Soliton solutions

In the same way as before (case of one pole) we will apply RHP for two - soliton solutions. We can formulate the corresponding two poles problem and find the following linear integral equations $\xi_{n}^{ \pm}(z)$

$$
\begin{align*}
\xi_{n}^{+}(z)=\binom{0}{1} & +\frac{1}{2 \pi \mathrm{i}} \oint_{\Omega}\left(\frac{\mathrm{d} \omega}{(\omega-z)}(\omega)^{2 n} \rho^{-}(\omega) \xi_{n}^{-}(\omega)\right)-\left[C _ { 1 } ^ { - } ( z _ { 1 } ^ { - } ) ^ { 2 n } \left(\frac{\xi_{n}^{-}\left(z_{1}^{-}\right)}{\left(z-z_{1}^{-}\right)}\right.\right. \\
& \left.\left.+\frac{\xi_{n}^{-}\left(-z_{1}^{-}\right)}{\left(z+z_{1}^{-}\right)}\right)\right]+\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{2}^{-}\right)}{\left(z-z_{2}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{2}^{-}\right)}{\left(z+z_{2}^{-}\right)}\right)\right] \tag{3.6.28a}
\end{align*}
$$

In a similar fashion

$$
\begin{align*}
\xi_{n}^{-}(z)=\binom{1}{0} & -\frac{1}{2 \pi \mathrm{i}} \oint_{\Omega}\left(\frac{\mathrm{d} \omega}{(\omega-z)}(\omega)^{-2 n} \rho^{+}(\omega) \xi_{n}^{+}(\omega)\right)+\left[C _ { 1 } ^ { + } ( z _ { 1 } ^ { + } ) ^ { - 2 n } \left(\frac{\xi_{n}^{+}\left(z_{1}^{+}\right)}{\left(z-z_{1}^{+}\right)}\right.\right. \\
& \left.\left.+\frac{\xi_{n}^{+}\left(-z_{1}^{+}\right)}{\left(z+z_{1}^{+}\right)}\right)\right]+\left[C_{2}^{+}\left(z_{2}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{2}^{+}\right)}{\left(z-z_{2}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{2}^{+}\right)}{\left(z+z_{2}^{+}\right)}\right)\right] . \tag{3.6.28b}
\end{align*}
$$

The soliton solutions correspond to zero reflection coefficients $b_{j}^{ \pm}=0$; this can be found when $\left(\rho^{+}(z)=\rho^{-}(z)=0\right.$ on $\Omega(|z|=1)$ ). Then, the system (algebraic-integral) (3.6.28) is reduced to the linear system

$$
\begin{align*}
\xi_{n}^{+}(z)=\binom{0}{1} & -\left[C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{1}^{-}\right)}{\left(z-z_{1}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{1}^{-}\right)}{\left(z+z_{1}^{-}\right)}\right)\right] \\
& -\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{2}^{-}\right)}{\left(z-z_{2}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{2}^{-}\right)}{\left(z+z_{2}^{-}\right)}\right)\right] . \tag{3.6.29a}
\end{align*}
$$

Similarly

$$
\begin{align*}
\xi_{n}^{-}(z)=\binom{1}{0} & +\left[C_{1}^{+}\left(z_{1}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{1}^{+}\right)}{\left(z-z_{1}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{1}^{+}\right)}{\left(z+z_{1}^{+}\right)}\right)\right] \\
& +\left[C_{2}^{+}\left(z_{2}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{2}^{+}\right)}{\left(z-z_{2}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{2}^{+}\right)}{\left(z+z_{2}^{+}\right)}\right)\right], \tag{3.6.29b}
\end{align*}
$$

where $\xi_{n}^{+}\left( \pm z_{j}^{+}\right)$are evaluated at the eigenvalue $\pm z_{j}^{+}$, and similarly for $\xi_{n}^{-}\left( \pm z_{j}^{-}\right)$are evaluated at the eigenvalue $\pm z_{j}^{-}$. We can find the expressions for these vectors by evaluating equation (3.6.29a) at the points $\pm z_{\{1,2\}}^{+}$and equation (3.6.29b) at the points $\pm z_{\{1,2\}}^{-}$. This
results in a linear algebraic system composed of equations (3.6.29a) and (3.6.29b)

$$
\begin{align*}
& \xi_{n}^{+}\left( \pm z_{1}^{+}\right)=\binom{0}{1}- {\left[C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{1}^{-}\right)}{\left( \pm z_{1}^{+}-z_{1}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{1}^{-}\right)}{\left( \pm z_{1}^{+}+z_{1}^{-}\right)}\right)\right] } \\
&-\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{2}^{-}\right)}{\left( \pm z_{1}^{+}-z_{2}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{2}^{-}\right)}{\left( \pm z_{1}^{+}+z_{2}^{-}\right)}\right)\right],  \tag{3.6.30a}\\
& \xi_{n}^{+}\left( \pm z_{2}^{+}\right)=\binom{0}{1}- {\left[C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{1}^{-}\right)}{\left( \pm z_{2}^{+}-z_{1}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{1}^{-}\right)}{\left( \pm z_{2}^{+}+z_{1}^{-}\right)}\right)\right] } \\
&-\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{2}^{-}\right)}{\left( \pm z_{2}^{+}-z_{2}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{2}^{-}\right)}{\left( \pm z_{2}^{+}+z_{2}^{-}\right)}\right)\right],  \tag{3.6.30b}\\
& \xi_{n}^{-}\left( \pm z_{1}^{-}\right)=\binom{1}{0}+ {\left[C_{1}^{+}\left(z_{1}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{1}^{+}\right)}{\left( \pm z_{1}^{-}-z_{1}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{1}^{+}\right)}{\left( \pm z_{1}^{-}+z_{1}^{+}\right)}\right)\right] } \\
&+\left[C_{2}^{+}\left(z_{2}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{2}^{+}\right)}{\left( \pm z_{1}^{-}-z_{2}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{2}^{+}\right)}{\left( \pm z_{1}^{-}+z_{2}^{+}\right)}\right)\right],  \tag{3.6.31a}\\
& \xi_{n}^{-}\left( \pm z_{2}^{-}\right)=\binom{1}{0}+\left[C_{1}^{+}\left(z_{1}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{1}^{+}\right)}{\left( \pm z_{2}^{-}-z_{1}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{1}^{+}\right)}{\left( \pm z_{2}^{-}+z_{1}^{+}\right)}\right)\right] \\
&+\left[C_{2}^{+}\left(z_{2}^{+}\right)^{-2 n}\left(\frac{\xi_{n}^{+}\left(z_{2}^{+}\right)}{\left( \pm z_{2}^{-}-z_{2}^{+}\right)}+\frac{\xi_{n}^{+}\left(-z_{2}^{+}\right)}{\left( \pm z_{2}^{-}+z_{2}^{+}\right)}\right)\right] . \tag{3.6.31b}
\end{align*}
$$

From equations (3.6.30) and (3.6.31), the following relations hold and this helps us to find the two - soliton solutions,

$$
\begin{array}{lll}
\xi_{n}^{+, 1}\left(-z_{j}^{+}\right)=-\xi_{n}^{+, 1}\left(z_{j}^{+}\right) & \text {if, and only if, } & \xi_{n}^{-,, 1}\left(-z_{j}^{-}\right)=\xi_{n}^{-, 1}\left(z_{j}^{-}\right) \\
\xi_{n}^{+, 2}\left(-z_{j}^{+}\right)=\xi_{n}^{+, 2}\left(z_{j}^{+}\right) & \text {if, and only if, } & \xi_{n}^{-, 2}\left(-z_{j}^{-}\right)=-\xi_{n}^{-, 2}\left(z_{j}^{-}\right) \tag{3.6.32b}
\end{array}
$$

We can recover $Q_{n}^{+}$from the power series expansion of the RHS of $\varphi_{n}^{-}(z)$ in equation (3.6.29a). However, it is difficult to find the potential. For that reason, we will multiply both sides of $\chi_{n}^{-}(z)$ by $\left(\begin{array}{cc}1 & 0 \\ 0 & c_{n}\end{array}\right)$

$$
\begin{align*}
\left(\tilde{\xi}_{n}^{-}, \tilde{\varphi}_{n}^{-}\right) \simeq \tilde{\chi}_{n}^{-}(z) & =\left(\begin{array}{cc}
1 & 0 \\
0 & c_{n}
\end{array}\right)\left(\begin{array}{cc}
c_{n}^{-1} & z Q_{n-1}^{+} \\
c_{n}^{-1} z Q_{n}^{-} & 1
\end{array}\right) \\
& =\left(\begin{array}{cc}
c_{n}^{-1} & z Q_{n-1}^{+} \\
z Q_{n}^{-} & c_{n}
\end{array}\right) \tag{3.6.33}
\end{align*}
$$

where $\tilde{\chi}_{n}^{-}(z)$ has the same power series expansions as $\chi_{n}^{-}(z)$. Then, we can find the potential $Q_{n-1}^{+}$from $\tilde{\varphi}_{n}^{-, 1}$ in equation (3.6.33).

Since $b^{ \pm}(z)=0$, then from equation (3.6.10b), we have $\tilde{\varphi}_{n}^{-}(z)=\xi_{n}^{+}(z)$. As a result

$$
\begin{align*}
\tilde{\varphi}_{n}^{-}(z)=\binom{0}{1} & -\left[C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{1}^{-}\right)}{\left(z-z_{1}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{1}^{-}\right)}{\left(z+z_{1}^{-}\right)}\right)\right] \\
& -\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-}\left(z_{2}^{-}\right)}{\left(z-z_{2}^{-}\right)}+\frac{\xi_{n}^{-}\left(-z_{2}^{-}\right)}{\left(z+z_{2}^{-}\right)}\right)\right] . \tag{3.6.34}
\end{align*}
$$

Now, by comparing the power series expansion of the RHS of equation (3.6.34) to the expansion (3.6.33), we obtain:

$$
\begin{align*}
z Q_{n-1}^{+}= & -\left[C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-,, 1}\left(z_{1}^{-}\right)}{\left(z-z_{1}^{-}\right)}+\frac{\xi_{n}^{-,, 1}\left(-z_{1}^{-}\right)}{\left(z+z_{1}^{-}\right)}\right)\right] \\
& -\left[C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}\left(\frac{\xi_{n}^{-, 1}\left(z_{2}^{-}\right)}{\left(z-z_{2}^{-}\right)}+\frac{\xi_{n}^{-, 1}\left(-z_{2}^{-}\right)}{\left(z+z_{2}^{-}\right)}\right)\right], \\
= & -2 z\left[\frac{C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}}{\left(z^{2}-\left(z_{1}^{-}\right)^{2}\right)} \xi_{n}^{-, 1}\left(z_{1}^{-}\right)+\frac{C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}}{\left(z^{2}-\left(z_{2}^{-}\right)^{2}\right)} \xi_{n}^{-, 1}\left(z_{2}^{-}\right)\right] . \tag{3.6.35}
\end{align*}
$$

We have $\pm z_{j}^{+}$and $\pm z_{j}^{-}$eigenvalues with $\left|z_{j}^{+}\right|>1$ and $\left|z_{j}^{-}\right|<1$, so we can solve the linear system of equations (3.6.30a) and (3.6.31b) for $\xi_{n}^{-}\left(z_{1}^{-}\right), \xi_{n}^{-}\left(z_{2}^{-}\right), \xi_{n}^{+}\left(z_{1}^{+}\right)$and $\xi_{n}^{+}\left(z_{2}^{+}\right)$. In particular, we will find $Q_{n}^{+}$; to do so, we need $\xi_{n}^{-}\left(z_{1}^{-}\right)$and $\xi_{n}^{-}\left(z_{2}^{-}\right)$

$$
\begin{gather*}
\xi_{n}^{+, 1}\left(z_{1}^{+}\right)=-2 z_{1}^{+}\left[\frac{C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}}{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)} \xi_{n}^{-,, 1}\left(z_{1}^{-}\right)+\frac{C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}}{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{2}^{-}\right)^{2}\right)} \xi_{n}^{-, 1}\left(z_{2}^{-}\right)\right],  \tag{3.6.36a}\\
\xi_{n}^{+, 1}\left(z_{2}^{+}\right)=-2 z_{2}^{+}\left[\frac{C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}}{\left(\left(z_{2}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)} \xi_{n}^{-,, 1}\left(z_{1}^{-}\right)+\frac{C_{2}^{-}\left(z_{2}^{-}\right)^{2 n}}{\left(\left(z_{2}^{+}\right)^{2}-\left(z_{2}^{-}\right)^{2}\right)} \xi_{n}^{-, 1}\left(z_{2}^{-}\right)\right],  \tag{3.6.36b}\\
\xi_{n}^{-, 1}\left(z_{1}^{-}\right)=1+\frac{2 C_{1}^{+}\left(z_{1}^{+}\right)^{-2(n-1)}}{\left(\left(z_{1}^{-}\right)^{2}-\left(z_{1}^{+}\right)^{2}\right)} \xi_{n}^{+, 1}\left(z_{1}^{+}\right)+\frac{2 C_{2}^{+}\left(z_{2}^{+}\right)^{-2(n-1)}}{\left(\left(z_{1}^{-}\right)^{2}-\left(z_{2}^{+}\right)^{2}\right)} \xi_{n}^{+, 1}\left(z_{2}^{+}\right)  \tag{3.6.36c}\\
\xi_{n}^{-, 1}\left(z_{2}^{-}\right)=1+\frac{2 C_{1}^{+}\left(z_{1}^{+}\right)^{-2(n-1)}}{\left(\left(z_{2}^{-}\right)^{2}-\left(z_{1}^{+}\right)^{2}\right)} \xi_{n}^{+, 1}\left(z_{1}^{+}\right)+\frac{2 C_{2}^{+}\left(z_{2}^{+}\right)^{-2(n-1)}}{\left(\left(z_{2}^{-}\right)^{2}-\left(z_{2}^{+}\right)^{2}\right)} \xi_{n}^{+, 1}\left(z_{2}^{+}\right) \tag{3.6.36d}
\end{gather*}
$$

Then, we can obtain the potential $Q_{n}^{+}$from (3.6.35) by taking the residue when $z \rightarrow z_{1}^{-}$ or $z \rightarrow z_{2}^{-}$, respectively [65]:

$$
\begin{align*}
Q_{n}^{+} & =\frac{1}{2} C_{1}^{-}\left(z_{1}^{-}\right)^{2 n} \xi_{n+1}^{-, 1}\left(z_{1}^{-}\right)-\frac{2 C_{2}^{-}\left(z_{2}^{-}\right)^{2(n+1)}}{\left(\left(z_{1}^{-}\right)^{2}-\left(z_{2}^{-}\right)^{2}\right)} \xi_{n+1}^{-, 1}\left(z_{2}^{-}\right)  \tag{3.6.37a}\\
Q_{n}^{+} & =\frac{-2 C_{1}^{-}\left(z_{1}^{-}\right)^{2(n+1)}}{\left(\left(z_{2}^{-}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)} \xi_{n+1}^{-, 1}\left(z_{1}^{-}\right)+\frac{1}{2} C_{2}^{-}\left(z_{2}^{-}\right)^{2 n} \xi_{n+1}^{-, 1}\left(z_{2}^{-}\right) \tag{3.6.37b}
\end{align*}
$$

where $C_{1}^{ \pm}(0)$ are defined in equation (3.6.27) and $C_{2}^{ \pm}(0)$ are defined as:

$$
\begin{align*}
C_{2}^{+}(z, \tau)=C_{2}^{+}(0) e^{2 \mathrm{i} \omega_{2}^{+} \tau}, & C_{2}^{-}(z, \tau)=C_{2}^{-}(0) e^{-2 \mathrm{i} \omega_{2}^{-} \tau} \\
C_{2}^{+}(0)=\frac{z_{2}^{+}\left(\left(z_{2}^{+}\right)^{2}-\left(z_{2}^{-}\right)^{2}\right) e^{\mathrm{i} \alpha_{2}^{+}}}{2 z_{2}^{-}}, & C_{2}^{-}(0)=\frac{\left(\left(z_{2}^{+}\right)^{2}-\left(z_{2}^{-}\right)^{2}\right) e^{\mathrm{i} \alpha_{2}^{-}}}{2 z_{2}^{-} z_{2}^{+}} . \tag{3.6.38}
\end{align*}
$$

Finally, (3.6.37a) (or (3.6.37b)) is the solution to the nonlocal DNLS equation (3.1.3).

### 3.7 Dressing method for the discrete nonlocal nonlinear Schrödinger equation

In this section, we will approach to solve the integrable equation (3.1.3) by using the notion of the dressing method [37]. This method covers soliton solutions for local DNLS [35], but it is not proved yet for nonlocal DNLS equation. Here, we derive the dressing method from the spectral problem (3.2.3a) and study the class of solutions to equation (3.1.3). The solutions of the scattering problem (3.2.3a) are uniquely determined by their respective boundary conditions (3.2.7). The symmetry relation $Q_{n}^{-}(t)=-\left(Q_{-n}^{+}(t)\right)^{*}$ leads to

$$
\begin{align*}
\left(\xi_{n-1}^{-}, \xi_{n-1}^{+}\right)(z) & =\left(1 /(\tilde{v})_{-n}^{*}\right) B\left(\left(\varphi_{1-n}^{+}\right)^{*},\left(\varphi_{1-n}^{-}\right)^{*}\right)\left(z^{*}\right) B^{-1},  \tag{3.7.1}\\
\tilde{v}_{n} & =\prod_{k=-\infty}^{\infty} \tilde{g}_{k}, \quad \tilde{g}_{n}=1+Q_{n}^{+}\left(Q_{-n}^{+}\right)^{*} . \tag{3.7.2}
\end{align*}
$$

Now, we will show that the functions $\tilde{g}_{n}$ are nothing more than the determinants of $\varphi_{n}(z)$ and $\xi_{n}(z)$. From equation (3.2.8), $\operatorname{det}\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right)$ is $\tilde{g}_{n}$. Then,

$$
\begin{align*}
\operatorname{det} \xi_{n+1}(z) & =\tilde{g}_{n} \operatorname{det} \xi_{n}(z), \\
\operatorname{det} \xi_{n}(z) & =\left(\tilde{g}_{n}\right)^{-1} \operatorname{det} \xi_{n+1}(z), \\
& =\left(\tilde{g}_{n}\right)^{-1}\left(\tilde{g}_{n+1}\right)^{-1} \operatorname{det} \xi_{n+2}(z) \\
& \vdots  \tag{3.7.3}\\
& =\prod_{k=n}^{\infty} \tilde{g}_{k}^{-1}=\tilde{v}_{n}^{+} .
\end{align*}
$$

In the same way, we can obtain that

$$
\begin{equation*}
\operatorname{det} \varphi_{n}(z)=\prod_{k=-\infty}^{n-1} \tilde{g}_{k}=\tilde{v}_{n}^{-} \tag{3.7.4}
\end{equation*}
$$

Next, the determinants of the scattering matrix $T(z)$ can be found from the linear combination of the eigenfunctions (3.2.11)

$$
\begin{align*}
\phi_{n}(z) \mathbf{Z}^{-n} & =\psi_{n}(z) \mathbf{Z}^{-n} \mathbf{Z}^{n} T \mathbf{Z}^{-n},  \tag{3.7.5a}\\
\varphi_{n}(z) & =\xi_{n}(z) \mathbf{Z}^{n} T \mathbf{Z}^{-n} . \tag{3.7.5b}
\end{align*}
$$

As a result,

$$
\begin{align*}
\operatorname{det} \varphi_{n}(z) & =\operatorname{det} \xi_{n}(z) \operatorname{det} \mathbf{Z}^{n} \operatorname{det} T \operatorname{det} \mathbf{Z}^{-n}  \tag{3.7.6a}\\
\tilde{v}_{n}^{-} & =\tilde{v}_{n}^{+} \operatorname{det} T  \tag{3.7.6b}\\
\operatorname{det} T & =\left(\tilde{v}_{n}^{+}\right)^{-1} \tilde{v}_{n}^{-}=\tilde{v}_{n} . \tag{3.7.6c}
\end{align*}
$$

The next step is to study the eigenfunctions $\xi_{n}(z)$ and $\varphi_{n}(z)$ and their asymptotic behaviour. Let us write the spectral equation (3.2.3a) in the explicit form

$$
\left(\begin{array}{cc}
\varphi_{11} & \varphi_{12}  \tag{3.7.7}\\
\varphi_{21} & \varphi_{22}
\end{array}\right)_{n+1}=\left(\begin{array}{cc}
\varphi_{11}+z^{-1} Q_{n}^{+} \varphi_{21} & z^{2} \varphi_{12}+z Q_{n}^{+} \varphi_{22} \\
z^{-2} \varphi_{21}-z^{-1}\left(Q_{-n}^{+}\right)^{*} \varphi_{11} & \varphi_{22}-z\left(Q_{-n}^{+}\right)^{*} \varphi_{12}
\end{array}\right)_{n}
$$

Let us assume we have (at least asymptotically) the expansion over the negative power of $z$

$$
\begin{equation*}
\varphi_{n}(z)=\varphi_{n}^{(0)}(z)+z^{-1} \varphi_{n}^{(1)}(z)+z^{-2} \varphi_{n}^{(2)}(z)+\ldots \tag{3.7.8}
\end{equation*}
$$

by substituting equation (3.7.8) in equation (3.7.7), we obtain:

$$
\begin{align*}
& \left(\begin{array}{cc}
\varphi_{11}^{(0)}(z)+z^{-1} \varphi_{11}^{(1)}(z)+z^{-2} \varphi_{11}^{(2)}(z) & \varphi_{12}^{(0)}(z)+z^{-1} \varphi_{12}^{(1)}(z)+z^{-2} \varphi_{12}^{(2)}(z) \\
\varphi_{21}^{(0)}(z)+z^{-1} \varphi_{21}^{(1)}(z)+z^{-2} \varphi_{21}^{(2)}(z) & \varphi_{22}^{(0)}(z)+z^{-1} \varphi_{22}^{(1)}(z)+z^{-2} \varphi_{22}^{(2)}(z)
\end{array}\right)_{n+1}= \\
& \left(\begin{array}{cc}
\left(\varphi_{11}^{(0)}+\ldots\right)+z^{-1} Q_{n}^{+}\left(\varphi_{21}^{(0)}+\ldots\right) & z^{2}\left(\varphi_{12}^{(0)}+\ldots\right)+z Q_{n}^{+}\left(\varphi_{22}^{(0)}+\cdots+\right) \\
z^{-2}\left(\varphi_{21}^{(0)}+\ldots\right)-z^{-1}\left(Q_{-n}^{+}\right)^{*}\left(\varphi_{11}^{(0)}+\ldots\right)\left(\varphi_{22}^{(0)}+\ldots\right)-z\left(Q_{-n}^{+}\right)^{*}\left(\varphi_{12}^{(0)}+\ldots\right)
\end{array}\right)_{n} . \tag{3.7.9}
\end{align*}
$$

After simplifying the last equation, and taking the $\lim z \rightarrow \infty$, equation (3.7.9) becomes:

$$
\begin{align*}
& \binom{\varphi_{11}^{(0)}(z) \varphi_{12}^{(0)}(z)}{\varphi_{21}^{(0)}(z) \varphi_{22}^{(0)}(z)}_{n+1} \\
& =\left(\begin{array}{cc}
\varphi_{11}^{(0)} z^{2} \varphi_{12}^{(0)}+z \varphi_{12}^{(1)}+\varphi_{12}^{(2)}+z Q_{n}^{+} \varphi_{22}^{(0)}+Q_{n}^{+} \varphi_{22}^{(1)} \\
0 & \varphi_{22}^{(0)}-z\left(Q_{-n}^{+}\right)^{*} \varphi_{12}^{(0)}-\left(Q_{-n}^{+}\right)^{*} \varphi_{12}^{(1)}
\end{array}\right)_{n} . \tag{3.7.10}
\end{align*}
$$

If we assume the potential

$$
\begin{equation*}
Q_{n}^{+}=-\frac{\varphi_{12}^{(1)}}{\varphi_{22}^{(0)}}, \quad Q_{n}^{+}=-\frac{\varphi_{12}^{(2)}}{\varphi_{22}^{(1)}}, \tag{3.7.11}
\end{equation*}
$$

then, equation (3.7.10) becomes

$$
\begin{align*}
\varphi_{n+1}^{(0)}(z) & =\left(\begin{array}{cc}
1 & 0 \\
0 & 1+Q_{n}^{+}\left(Q_{-n}^{+}\right)^{*}
\end{array}\right)\binom{\varphi_{11}^{(0)}(z) \varphi_{12}^{(0)}(z)}{\varphi_{21}^{(0)}(z) \varphi_{22}^{(0)}(z)}_{n} \\
& =\left(\begin{array}{cc}
1 & 0 \\
& \\
0 & \tilde{g}_{n}
\end{array}\right) \varphi_{n}^{(0)}(z) . \tag{3.7.12}
\end{align*}
$$

In the $\lim z \rightarrow 0$, for which $\xi_{n}(z)=\xi_{n}^{(0)}(z)+z \xi_{n}^{(1)}(z)+z^{2} \xi_{n}^{(2)}(z)+\ldots$, using the same structures, we have

$$
\xi_{n+1}^{(0)}(z)=\left(\begin{array}{cc}
\tilde{g}_{n} & 0  \tag{3.7.13}\\
& \\
0 & 1
\end{array}\right) \xi_{n}^{(0)}(z)
$$

### 3.7.1 The dressing method and the fundamental analytic solutions

Important tools for reducing the ISP to a RHP are the $\mathrm{FASs}^{1} \mathfrak{X}_{n}^{+}(z)$ and $\mathfrak{X}_{n}^{-}(z)$. Their construction is based on the Gauss decomposition of $T(z)$

$$
\begin{gather*}
\mathfrak{X}_{n}^{+}(z)=\left(\varphi_{n}^{+}, \xi_{n}^{+}\right)=\psi_{n}(z) T^{-}(z)=\phi_{n}(z) S^{+}(z)  \tag{3.7.14a}\\
\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)=\binom{\tilde{\varphi}_{n}}{\tilde{\xi}_{n}}(z)=T^{+}(z)\left(\psi_{n}\right)^{-1}(z)=S^{-}(z)\left(\phi_{n}\right)^{-1}(z), \tag{3.7.14b}
\end{gather*}
$$

where $\tilde{X}=\left(X_{2},-X_{1}\right)$ is a column vector, and

$$
\begin{align*}
& T^{-}(z)=\left(\begin{array}{cc}
a^{+}(z) & 0 \\
b^{+}(z) & 1
\end{array}\right), \quad T^{+}(z)=\left(\begin{array}{cc}
a^{-} / \tilde{v}_{n}(z) & b^{-}(z) / \tilde{v}_{n} \\
0 & 1
\end{array}\right),  \tag{3.7.14c}\\
& S^{+}(z)=\left(\begin{array}{cc}
1 & b^{-}(z) / \tilde{v}_{n} \\
0 & a^{+}(z) / \tilde{v}_{n}
\end{array}\right), \quad S^{-}(z)=\left(\begin{array}{cc}
1 & 0 \\
b^{+}(z) & a^{-}(z)
\end{array}\right), \tag{3.7.14d}
\end{align*}
$$

[^12]are the factors in the Gauss decomposition of the associated scattering matrix $T(z)$
\[

$$
\begin{equation*}
T(z)=T^{-}(z) \hat{S}^{+}(z)=T^{+}(z) \hat{S}^{-}(z), \tag{3.7.15}
\end{equation*}
$$

\]

and are expressed in terms of the matrix elements of the scattering matrix $T(z)$. This construction ensures that $\xi^{ \pm}(z)$ are analytic functions of $z$ for $z \in \Omega_{ \pm}$, where $\Omega_{ \pm}$represent the regions of outside and inside the unit circle $|z|=1$, respectively. From relations (3.7.14), we can conclude that

$$
\begin{align*}
\operatorname{det} \mathfrak{X}_{n}^{+}(z) & =\tilde{v}_{n}^{+} a^{+}(z)=\tilde{v}^{-} a^{+} / \tilde{v}_{n}=\tilde{v}_{n}^{+} a^{+}(z),  \tag{3.7.16a}\\
\operatorname{det}\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z) & =\left(\tilde{v}_{n}^{+}\right)^{-1} a^{-}(z) / \tilde{v}_{n}=a^{-}\left(\tilde{v}_{n}^{-}\right)^{-1}=\left(\tilde{v}_{n}^{-}\right)^{-1} a^{-}(z) . \tag{3.7.16b}
\end{align*}
$$

### 3.7.2 The Riemann-Hilbert problem and the dressing method

Studying the RHP, we will outline the important symmetry between the eigenfunctions $\varphi_{n}(z)$ and $\xi_{n}(z)$. Both $\varphi_{n}(z)$ and $\xi_{n}(z)$ are solutions to the spectral problem (3.2.3a) which are determined from their boundary conditions (3.2.7). Suppose $\tilde{\mathbf{Q}}_{0}$ is one of the seed solutions of the nonlinear equation (3.3.3). In special cases, trivial solutions, such as zero can be called a seed solution. We, therefore, know explicitly the matrices $\mathcal{L}_{0}$ and $M_{0}$ which correspond to this solution. As a result, we can solve the system of linear equations

$$
\begin{equation*}
\Psi_{n, x}(z)=\mathcal{L}_{0} \Psi_{n}(z), \quad \Psi_{n, t}(z)=M_{0} \Psi_{n}(z) \tag{3.7.17}
\end{equation*}
$$

for the matrix function $\Psi_{n}(z)$. Next, we will demonstrate that there exists a possibility to build a class of new solutions of the nonlinear equation (3.3.3), this class being parameterised by a closed oriented contour $\Omega_{ \pm}$on the plane $\mathbb{C}$ and by a nondegenerate bounded matrix function $G(z)$ defined on the contour $|z|=1$ (see Fig. 3.1). For this purpose, the normalisation of the RHP for the matrix $G_{n, 0}(z)$ on the contour $|z|=1$ has the following form:

$$
\begin{equation*}
\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z) \mathfrak{X}_{n}^{+}(z)=G(z), \quad z \in|z|=1, \tag{3.7.18}
\end{equation*}
$$

where

$$
G(z)=T^{+} T^{-}=S^{-} S^{+}=\left(\begin{array}{cc}
1 & b^{-} / \tilde{v}_{n}  \tag{3.7.19}\\
b^{+} & 1
\end{array}\right) .
$$

In other words, $\left.\mathfrak{X}_{n}^{+}(z)\left(\left(\mathfrak{X}_{n}^{-}(z)\right)^{-1}\right)\right)$ is analytic outside (inside) the unit circle $|z|=1$ and both of them satisfy equation (3.7.18). Asymptotic formulas for analytic solutions
are derived directly from those for the Jost functions. In particular, for $|z| \rightarrow \infty$

$$
\mathfrak{X}_{n}^{+}(z)=\binom{\varphi_{11}^{(0),+}(z) \xi_{12}^{(0),+}(z)}{\varphi_{21}^{(0),+}(z) \xi_{22}^{(0),+}(z)} \rightarrow\left(\begin{array}{cc}
\lim _{z \rightarrow \infty} \varphi_{11}^{(0),+}(z) & 0  \tag{3.7.20a}\\
0 & \lim _{z \rightarrow \infty} \xi_{22}^{(0),+}(z)
\end{array}\right)
$$

From equation (3.7.12),

$$
\begin{align*}
& \lim _{z \rightarrow \infty} \varphi_{11}^{(0),+}(n+1, z)=\varphi_{11}^{(0),+}(n, z)=\varphi_{11}^{(0),+}(n-1, z)=\cdots=1,  \tag{3.7.20b}\\
& \text { and } \begin{aligned}
\lim _{z \rightarrow \infty} \xi_{22}^{(0),+}(n, z) & =\tilde{g}_{n}^{-1} \xi_{22}^{(0),+}(n+1, z)=\tilde{g}_{n}^{-1} \tilde{g}_{n+1}^{-1} \xi_{22}^{(0),+}(n+2, z)=\ldots \\
& =\prod_{k=n}^{\infty} \tilde{g}_{k}^{-1}=v_{n}^{+} .
\end{aligned} . \tag{3.7.20c}
\end{align*}
$$

Then, for $z \rightarrow \infty$

$$
\lim _{z \rightarrow \infty} \mathfrak{X}_{n}^{+}(z) \rightarrow\left(\mathfrak{X}_{n}^{+, 0}\right)(z)=\left(\begin{array}{cc}
1 & 0  \tag{3.7.21a}\\
& \\
0 & \tilde{v}_{n}^{+}
\end{array}\right)
$$

and analogously, the asymptotic behaviour for $\left(\mathfrak{X}_{n}^{-}\right)^{-1}$ for $z \rightarrow 0$ is

$$
\lim _{z \rightarrow 0}\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z) \rightarrow\left(\mathfrak{X}_{n}^{-, 0}\right)^{-1}(z)=\left(\begin{array}{cc}
\left(\tilde{v}_{n}^{-}\right)^{-1} & 0  \tag{3.7.21b}\\
0 & 1
\end{array}\right) .
$$

At this point we arrived at the RHP. In [37], the authors have noted that there exists a possibility to reformulate the local AL spectral problem to arrive at the RHP with the canonical normalisation but this is obtained at the cost of nonlinear dependence of the spectral problem on the potential $\tilde{\mathbf{Q}}_{n}(t)$.

The next step is to build the normalisation condition. This means we should define one of the matrices $\mathfrak{X}_{n}^{+}(z)\left(\right.$ or $\left.\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)\right)$ at the infinite point $z \rightarrow \infty$ of the $z$-plane. If $\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)=\mathbb{1}$, then this normalisation is called canonical.

### 3.7.3 Derivation of the dressing method

The dressing Method is the most effective method for deriving the soliton solutions of the corresponding NLEE. The method starts with the regular solutions $\mathfrak{X}_{n}^{+, 0}(z)$ and $\left(\mathfrak{X}_{n}^{-, 0}\right)^{-1}(z)$, i.e. when they have no singularity or zeros in their regions of analyticity. As it is known, the DNLS equation has four eigenvalues $\left\{z_{1}^{ \pm},-z_{1}^{ \pm}\right\}$for the one soliton solution. We are going to construct the corresponding singular solutions $\mathfrak{X}_{n}^{+}(z)$ and $\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)$ of the RHP using the dressing factors. Let us represent the FAS $\mathfrak{X}_{n}^{+}(z)$
$\left(\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)\right)$ in a factorised form

$$
\begin{array}{r}
\mathfrak{X}_{n}^{+}(z)=u_{n, j}(z) \mathfrak{X}_{n}^{+, 0}, \\
\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)=\frac{1}{c_{j}(z)} u_{n, j}(z)\left(\mathfrak{X}_{n}^{-, 0}\right)^{-1}, \tag{3.7.22b}
\end{array}
$$

or

$$
\begin{equation*}
\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)=\mathbb{1}+\left(\frac{1}{c_{j}}-1\right)\left(\mathbb{1}-P_{j}\right)\left(\mathfrak{X}_{n}^{-, 0}\right)^{-1} \tag{3.7.22c}
\end{equation*}
$$

where $u_{n, j}$ is the dressing factor

$$
\begin{equation*}
u_{n, j}(z)=\mathbb{1}+\left(c_{j}(z)-1\right) P_{j, n}, \quad c_{j}(z)=\frac{z-z_{j}^{+}}{z-z_{j}^{-}} \tag{3.7.22d}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{j, n}=\frac{\left|s_{j, n}\right\rangle\left\langle m_{j, n}\right|}{\left\langle m_{j, n} \mid s_{j, n}\right\rangle}, \quad j=1, \ldots, S . \tag{3.7.22e}
\end{equation*}
$$

Here, $P_{j, n}$ is the rank 1 projector, $P_{j, n}^{2}=P_{j, n}$ and $\left|s_{j, n}(z)\right\rangle$ is a two column vector and $\left\langle m_{j, n}(z)\right|$ is a two row vector. Our aim is to find out how these eigenvectors depend on $z$ and how they are determined by the scattering data of $\mathcal{L}_{n}$. Here, we will take $j=1$, for the following calculations. From equation (3.7.16), the determinant of both sides of equations (3.7.22a) and (3.7.22b) is

$$
\begin{align*}
v_{n}^{+} a^{+}(z) & =c_{1}(z) v_{n}^{+} a_{0}^{+}(z),  \tag{3.7.23a}\\
\left(\tilde{v}_{k}^{-}\right)^{-1} a^{-}(z) & =\frac{1}{c_{1}(z)}\left(v_{n}^{-}\right)^{-1} a_{0}^{-}(z), \tag{3.7.23b}
\end{align*}
$$

where $a_{0}^{ \pm}(z)$ has no zeros in $z \in \Omega_{ \pm}$(regular). Thus we find that $a^{ \pm}(z)$ have zeros at $z_{1}^{ \pm}$ respectively.

$$
\begin{equation*}
a^{+}(z)=c_{1}(z) a_{0}^{+}(z) \quad \text { and } \quad a^{-}(z)=\frac{a_{0}^{-}(z)}{c_{1}(z)} \tag{3.7.23c}
\end{equation*}
$$

The next step is to find the general form of the two vectors $\left|s_{j}(x)\right\rangle$ and $\left\langle m_{j}(x)\right|$. First, we will find a relation between $\mathcal{L}_{0, n}(z)$ and $\mathcal{L}_{1, n}(z)$. Since $\mathfrak{X}_{n}^{+, 0}(z)$ and $\mathfrak{X}_{n}^{+, 1}(z)$ are both satisfy (3.2.3a), then

$$
\begin{align*}
& \mathfrak{X}_{n+1}^{\{+, 0\}}(z)=\mathcal{L}_{0, n}(z) \mathfrak{X}_{n}^{\{+, 0\}}(z),  \tag{3.7.24a}\\
& \mathfrak{X}_{n+1}^{\{+, 1\}}(z)=\mathcal{L}_{1, n}(z) \mathfrak{X}_{n}^{\{+, 1\}}(z), \tag{3.7.24b}
\end{align*}
$$

and if $\mathfrak{X}_{n}^{\{+, 1\}}(z)$ satisfies the RHP equation (3.7.24a), then $\mathfrak{X}_{n+1}^{\{+, 1\}}(z)$ is also

$$
\begin{equation*}
\mathfrak{X}_{n+1}^{\{+, 1\}}(z)=u_{n, 1}(z) \mathfrak{X}_{n+1}^{\{+, 0\}}(z) . \tag{3.7.24c}
\end{equation*}
$$

We start with substituting (3.7.24c) to equation (3.7.24b):

$$
\begin{equation*}
u_{n+1,1} \mathfrak{X}_{n+1}^{\{+, 0\}}(z)=\mathcal{L}_{1, n}(z) \mathfrak{X}_{n}^{\{+, 1\}}(z), \tag{3.7.25a}
\end{equation*}
$$

equation (3.7.24c) to equation (3.7.25a)

$$
\begin{equation*}
u_{n+1,1} \mathfrak{X}_{n+1}^{\{+, 0\}}(z)=\mathcal{L}_{1, n}(z) u_{n, 1}(z) \mathfrak{X}_{n}^{\{+, 0\}}(z), \tag{3.7.25b}
\end{equation*}
$$

and equation (3.7.24a) to equation (3.7.25b)

$$
\begin{equation*}
u_{n+1,1} \mathcal{L}_{0, n}(z) \mathfrak{X}_{n}^{\{+, 0\}}(z)=\mathcal{L}_{1, n}(z) u_{n, 1}(z) \mathfrak{X}_{n}^{\{+, 0\}}(z) \tag{3.7.25c}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\mathcal{L}_{1, n}(z) u_{n, 1}(z)=u_{n+1,1} \mathcal{L}_{0, n}(z) . \tag{3.7.25d}
\end{equation*}
$$

We will solve the RHP (3.7.18) with zeros by means of its regularisation. This procedure consists of extracting rational factors from $\mathfrak{X}_{n}^{+}(z)$ which are responsible for the existence of zeros. Indeed, if $\operatorname{det} \mathfrak{X}_{n}^{+}\left(z_{j}^{+}\right)=0$, then at the point $z_{j}^{+}$there exists an eigenvector $|\iota\rangle$ with zero eigenvalue, $\mathfrak{X}_{n}^{+}\left(z_{j}^{+}\right)|\iota\rangle=0$. Taking the limit of $z \rightarrow z_{1}^{-}$, to equation (3.7.25d)

$$
\begin{align*}
& \lim _{z \rightarrow z_{1}^{-}}\left(z-z_{1}^{-}\right)\left(\mathcal{L}_{1, n}(z)\left(\mathbb{1}+\left(\frac{z_{1}^{-}-z_{1}^{+}}{z-z_{1}^{-}}\right) P_{1, n}\right)\right)  \tag{3.7.26}\\
& =\lim _{z \rightarrow z_{1}^{-}}\left(z-z_{1}^{-}\right)\left(\mathbb{1}+\left(\frac{z_{1}^{-}-z_{1}^{+}}{z-z_{1}^{-}}\right) P_{n+1,1}\right) \mathcal{L}_{0, n}(z)
\end{align*}
$$

take limit for each part

$$
\begin{align*}
0+\left(z_{1}^{-}-z_{1}^{+}\right) \mathcal{L}_{1, n}\left(z_{1}^{-}\right) P_{n, 1} & =\left(z_{1}^{-}-z_{1}^{+}\right) P_{n+1,1} \mathcal{L}_{0, n}\left(z_{1}^{-}\right),  \tag{3.7.27}\\
\mathcal{L}_{1, n}\left(z_{1}^{-}\right) P_{n, 1} & =P_{n+1,1} \mathcal{L}_{0, n}\left(z_{1}^{-}\right) .
\end{align*}
$$

To find $\left|s_{j}\right\rangle$ and $\left\langle m_{j}\right|$, we need to substitute equation (3.7.22e) in equation (3.7.27)

$$
\begin{equation*}
\mathcal{L}_{1, n}\left(z_{1}^{-}\right) \frac{\left|s_{1, n}\right\rangle\left\langle m_{1, n}\right|}{\left\langle m_{1, n} \mid s_{1, n}\right\rangle}=\frac{\left|s_{1, n+1}\right\rangle\left\langle m_{1, n+1}\right|}{\left\langle m_{1, n+1} \mid s_{1, n+1}\right\rangle} \mathcal{L}_{0, n}\left(z_{1}^{-}\right), \tag{3.7.28a}
\end{equation*}
$$

so,

$$
\begin{align*}
\left\langle m_{1, n}\right| & =\left\langle m_{n+1}\right| \mathcal{L}_{0, n}\left(z_{1}^{-}\right) \\
& =\left\langle m_{n+1}\right|\left(\begin{array}{cc}
z_{1}^{-} & 0 \\
0 & \left(z_{1}^{-}\right)^{-1}
\end{array}\right) . \tag{3.7.28b}
\end{align*}
$$

Then, the general form for the vector $\left\langle m_{1, n}\right|$ is

$$
\begin{equation*}
\left\langle m_{1, n}\right|=\langle\mu| \mathbf{Z}^{-n}\left(z_{1}^{-}\right), \tag{3.7.28c}
\end{equation*}
$$

where $\langle\mu|$ is a $t$-dependent vector. On the other hand, we can write equation (3.7.25d) as:

$$
\begin{equation*}
\hat{u}_{n+1,1} \mathcal{L}_{1, n}(z)=\mathcal{L}_{0, n}(z) \hat{u}_{n, 1}(z), \tag{3.7.29a}
\end{equation*}
$$

this time we will take the limit of $z \rightarrow z_{1}^{+}$, and with similar calculations, we will have

$$
\begin{equation*}
P_{n+1,1} \mathcal{L}_{1, n}\left(z_{1}^{+}\right)=\mathcal{L}_{0, n}\left(z_{1}^{+}\right) P_{n, 1} \tag{3.7.29b}
\end{equation*}
$$

$$
\begin{align*}
\frac{\left|s_{1, n+1}\right\rangle\left\langle m_{1, n+1}\right|}{\left\langle m_{1, n+1} \mid s_{1, n+1}\right\rangle} \mathcal{L}_{1, n}\left(z_{1}^{+}\right) & =\mathcal{L}_{0, n}\left(z_{1}^{+}\right) \frac{\left|s_{1, n}\right\rangle\left\langle m_{1, n}\right|}{\left\langle m_{1, n} \mid s_{1, n}\right\rangle} \\
\left|s_{1, n+1}\right\rangle & =\mathcal{L}_{0, n}\left(z_{1}^{+}\right)\left|s_{1, n}\right\rangle \\
& =\left(\begin{array}{cc}
z_{1}^{+} & 0 \\
0 & \left(z_{1}^{+}\right)^{-1}
\end{array}\right)\left|s_{1, n}\right\rangle . \tag{3.7.29c}
\end{align*}
$$

Then, the general form for the vector $\left|s_{1, n}\right\rangle$ is

$$
\begin{equation*}
\left|s_{1, n}\right\rangle=\mathbf{Z}^{n}\left(z_{1}^{+}\right)|\vartheta\rangle \tag{3.7.29d}
\end{equation*}
$$

where $|\vartheta\rangle$ is a $t$-dependent vector. The following paragraph is to determine $\left|s_{1, n}\right\rangle$ and $\left\langle m_{1, n}\right|$.

Let us differentiate the equation $\mathfrak{X}_{n}^{+}(z)\left|s_{1, n}\right\rangle=0$ with respect to $t$. Since $\mathfrak{X}_{n}^{+}(z)$ is a solution of the spectral equation (3.3.11), and at the boundary it satisfies (3.4.1), we obtain:

$$
\begin{align*}
\left(\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{n}\right\rangle\right)_{t} & =\mathfrak{X}_{t, n}^{+}\left(z_{1}^{+}\right)\left|s_{n}\right\rangle+\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{t, n}\right\rangle, \\
0 & =\omega\left(z_{1}^{+}\right) \mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{n}\right\rangle+\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{t, n}\right\rangle, \\
\omega\left(z_{1}^{+}\right) \mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{n}\right\rangle & =\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{t, n}\right\rangle,  \tag{3.7.30}\\
\left|s_{t, n}\right\rangle & =\omega\left(z_{1}^{+}\right)\left|s_{n}\right\rangle, \\
\left|s_{n}\right\rangle & =\omega\left(z_{1}^{+}\right)|\tilde{s}\rangle,
\end{align*}
$$

where $\omega(z)$ is defined in Sec. 3.4 and $|\tilde{s}\rangle$ is a $t$-dependent (but $n$-independent) vector. In the same manner, we find the evolutionary equation for $\left\langle m_{1, n}\right|$. We explicitly find the
coordinate dependence of the vectors $\left|s_{1, n}\right\rangle$ and $\left\langle m_{1, n}\right|$ by

$$
\begin{array}{ll}
\left|s_{1, n}\right\rangle=\mathbf{Z}^{n} e^{\omega\left(z_{1}\right)}\left|s_{0}\right\rangle, & \left|s_{0}\right\rangle \text { const }, \\
\left\langle m_{1, n}\right|=\left\langle\mu_{0}\right| \mathbf{Z}^{-n} e^{\omega\left(z_{1}\right)}, & \left\langle\mu_{0}\right| \text { const. } \tag{3.7.31b}
\end{array}
$$

The linear combination between the Jost solutions at $z=z_{1}^{ \pm}$, is defined in (3.6.15a), then from the definition of the FASs (3.7.14) and (3.6.15a), the two vectors have the following form

$$
\begin{align*}
\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right) & =\psi_{1}^{+}(z)\left(b_{1}^{+}, 1\right),  \tag{3.7.32a}\\
\left(\mathfrak{X}^{-}\right)^{-1}\left(z_{1}^{-}\right) & =-\binom{b_{1}^{-}}{1} \tilde{\psi}_{j}^{-}\left(z_{1}^{-}\right) . \tag{3.7.32b}
\end{align*}
$$

Thus, we conclude that

$$
\begin{array}{rr}
\mathfrak{X}_{n}^{+}\left(z_{1}^{+}\right)\left|s_{0,1}\right\rangle=0, & \left|s_{0,1}\right\rangle=\binom{1}{-b_{1}^{+}}, \\
\left\langle m_{0,1}\right|\left(\mathfrak{X}_{n}^{-}\right)^{-1}\left(z_{1}^{-}\right)=0, & \left\langle m_{0,1}\right|=\left(1,-b_{1}^{-}\right) . \tag{3.7.33b}
\end{array}
$$

### 3.7.4 A solution formula of $Q_{n}^{+}(z)$ for the nonlocal DNLS equation

As we know, solitons correspond to the discrete eigenvalues of the RHP with zeros of the scattering coefficients $a^{ \pm}(z)$. From equation (3.7.16), the determinants of $\mathfrak{X}_{n}^{+}(z)$ and $\left(\mathfrak{X}_{n}^{-}\right)^{-1}(z)$ are $a^{ \pm}(z)$, then possible zeros of $a^{ \pm}(z)$ at some points lead to $\operatorname{det} \mathfrak{X}_{n}^{+}\left(z_{j}^{+}\right)=$ $0, z_{j}^{+} \in \Omega_{+}, \Omega_{+}=|z|>1, j=1, \ldots, S$, where $S$ is the number of solitons (the discrete eigenvalues) and $\operatorname{det}\left(\mathfrak{X}_{n}^{-}\right)^{-1}\left(z_{\ell}^{-}\right)=0, z_{\ell}^{-} \in \Omega_{-}, \Omega_{-}=|z|<1, \ell=1, \ldots, S$. In accordance with equations (3.7.11), (3.7.21) and $\mathfrak{X}_{n}^{+}(z)=\mathfrak{X}_{n}^{+, 0} u_{n, 1}(z)$, a solution (at $z_{1}^{ \pm}$) of the AL equation can be obtained from the solution of the RHP by

$$
\begin{equation*}
Q_{n}^{+}(t)=-\lim _{z \rightarrow \infty} \frac{z \mathfrak{X}_{n, 12}^{+}(z)}{\mathfrak{X}_{n, 22}^{+}(z)}=-\frac{\mathfrak{X}_{n, 12}^{+,, 1}(z)}{\mathfrak{X}_{n, 22}^{+, 0}(z)}=-\frac{u_{n, 12}}{\tilde{v}_{n}^{+}} . \tag{3.7.34}
\end{equation*}
$$

To find the solution to equation (1.7.6), we need to find the dressing factor $u_{n}(z)$. Our restriction is to obtain the soliton solutions of equation (1.7.6), i.e., $G(z)=\mathbb{1}$. In another sense

$$
\begin{equation*}
\left(\mathfrak{X}_{n}^{-}\right)^{-1} \mathfrak{X}_{n}^{+}(z)=\mathbb{1}, \tag{3.7.35a}
\end{equation*}
$$

then, when $z \rightarrow 0$

$$
\begin{gather*}
\mathfrak{X}_{n}^{+}(z)=\mathfrak{X}_{n}^{-}(z)=\left(\begin{array}{rr}
\tilde{v}_{k}^{-} & 0 \\
0 & 1
\end{array}\right),  \tag{3.7.35b}\\
\mathfrak{X}_{n}^{+}(z)=\left(\begin{array}{rr}
\tilde{v}_{k}^{-} & 0 \\
0 & 1
\end{array}\right), \tag{3.7.35c}
\end{gather*}
$$

from (3.7.22a), and (3.7.21)

$$
\begin{align*}
u_{n}(z) & =\mathfrak{X}_{n}^{+}(z)\left(\mathfrak{X}_{n}^{+, 0}\right)^{-1}(z),  \tag{3.7.35d}\\
\lim _{z \rightarrow 0} u_{n}(z) & =\left(\begin{array}{cc}
\tilde{v}_{k}^{-} & 0 \\
0 & \left(\tilde{v}_{k}^{+}\right)^{-1}
\end{array}\right), \tag{3.7.35e}
\end{align*}
$$

then, $u_{n, 22}(z)=\left(\tilde{v}_{k}^{+}\right)^{-1}$; in this case the solution of the DNLS equation depends on the components of the dressing factors. Therefore, equation (3.7.34) becomes

$$
\begin{equation*}
Q_{n}^{+}(t)=-\lim _{z \rightarrow 0} u_{n, 12} u_{n, 22}(z) \tag{3.7.36}
\end{equation*}
$$

Remark 2 We need to compute the dressing factor $u_{n}(z)$ (in exact form) to write the final form of the one soliton solution of (3.1.3) which will be addressed in a future work.

### 3.8 Summary

The following table is a summary of the illustration that we have demonstrated in the previous sections to show the difference between the two types of the DNLS equation.

| Local DNLS , $\quad \sigma=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ | Nonlocal DNLS, $\quad \sigma=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ |
| :---: | :---: |
| $Q_{n}^{-}(t)=\left(Q_{n}^{+}(t)\right)^{*}$, | $Q_{n}^{-}(t)=-\left(Q^{+}(t)\right)_{-n}^{*}$ |
| $\tilde{\mathbf{Q}}_{n}^{*}=\sigma \tilde{\mathbf{Q}}_{n} \sigma^{-1}$ | $\tilde{\mathbf{Q}}_{-n}^{\dagger}=\sigma \tilde{\mathbf{Q}}_{n} \sigma^{-1}$ |
| $L_{n}(z, t)=\left(\begin{array}{cc}z & Q_{n}^{+}(t) \\ -\left(Q_{n}^{+}(t)\right)^{*} & z^{-1}\end{array}\right)$ | $L_{n}(z, t)=\left(\begin{array}{cc}z & Q_{n}^{+}(t) \\ -\left(Q_{-n}^{+}(t)\right)^{*} & z^{-1}\end{array}\right)$ |
| $\left(\psi_{n}\left((1 / z)^{*}, t\right)\right)^{*}=\sigma \phi_{n}(z, t) \sigma^{-1}$ | $\psi_{n}(z, t)=\sigma \psi_{-n}^{\dagger}\left(z^{*}, t\right) \sigma^{-1}$ |
| $T^{*}\left((1 / z)^{*}, t\right)=\sigma T(z, t) \sigma^{-1}$ | $T(z, t)=\sigma T^{*}\left(z^{*}, t\right) \sigma^{-1}$ |
| $\begin{aligned} & a^{\mp}(z)=\left(a^{ \pm}(1 / z)^{*}\right)^{*} \\ & b^{\mp}(z)= \pm\left(b^{ \pm}(1 / z)^{*}\right)^{*} \end{aligned}$ | $\begin{aligned} & a^{ \pm}(z)=\left(a^{ \pm}\left(z^{*}\right)\right)^{*} \\ & b^{ \pm}(z)=\left(b^{\mp}\left(z^{*}\right)\right)^{*} \end{aligned}$ |
| $\begin{aligned} & \frac{\mathrm{d}}{\tau} \Psi_{n}(z, \tau)= \\ & \left(\begin{array}{cc} -\mathrm{i} Q_{n}^{+}\left(Q^{+}\right)_{n-1}^{*}-\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} & -\mathrm{i}\left(z Q_{n}^{+}-z^{-1} Q_{n-1}^{+}\right) \\ \left.-\mathrm{i}\left(z^{-1}\left(Q_{n}^{+}\right)^{*}-z\left(Q^{+}\right)_{n-1}^{*}\right)\right) & \mathrm{i}\left(Q^{+}\right)_{n}^{*} Q_{n-1}^{+}+\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} \end{array}\right) \Psi_{n} . \end{aligned}$ | $\begin{gathered} \frac{\mathrm{d}}{\tau} \Psi_{n}(z, \tau)= \\ \left(\begin{array}{cc} -\mathrm{i} Q_{n}^{+}\left(Q_{(-\{n-1\}}^{+}\right)^{*}-\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} & -\mathrm{i}\left(z Q_{n}^{+}-z^{-1} Q_{n-1}^{+}\right) \\ -\mathrm{i}\left(z^{-1}\left(Q_{-n}^{+}\right)^{*}-z\left(Q_{(-\{n-1\}}^{+}\right)^{*}\right) & \mathrm{i}\left(Q_{-n}^{+}\right)^{*} Q_{n-1}^{+}+\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} \end{array}\right) \Psi_{n} . \end{gathered}$ |
| $\mathrm{i} Q_{n, \tau}=\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right)+Q_{n} Q_{n}^{*}\left(Q_{n+1}+Q_{n-1}\right)$ | $\mathrm{i} Q_{n, \tau}=\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right)+Q_{n} Q_{-n}^{*}\left(Q_{n+1}+Q_{n-1}\right)$ |

Table 3.1: Involution/reduction conditions for Ablowitz-Ladik system (A summary).

## Chapter 4

## Complete integrability of the discrete nonlinear Schrödinger equation

### 4.1 Introduction

Gerdjikov et al. [50, 61], used an equivalent eigenvalue problem to build the completeness relation for the discrete block ZS system. Here, we will present the spectrum of the $L_{n}(z)$ operator for DNLS equation in a different form. The spectrum of the $L(\lambda)$ operator for continuous NLS equation is introduced [49]. The spectral theory condition is to find a family of self-adjoint commuting projectors acting on the Hilbert space $\mathcal{H}$. That means for any self-adjoint operator $\mathcal{A}$, one can introduce a spectral measure and then prove the spectral theory. However, our operator $\mathcal{L}_{n}$ is not self-adjoint, so we will prove the completeness relation of the Jost solutions which is equivalent to the spectral decomposition of the operator. The derivation of the completeness relation of the Jost solutions starts by introducing the new functions $R_{n}^{ \pm}(z)$, which have singularities at $z^{ \pm}$, respectively [49, 54]. The results of this chapter appear in [65].

### 4.2 The spectral theory of the discrete Lax operator

Sections 1.5.1 and 2.3 have introduced the advantage of the resolvent and its properties in the continuous NLS. Here, we will show how the spectrum of the discrete operator $\mathcal{L}_{n}$ will be derived. To do so, we need to describe the FASs $\chi_{n}^{ \pm}(z)$ with respect to the Jost solutions $\phi_{n}(z)$ and $\psi_{n}(z)$.

### 4.2.1 The resolvent of case $\mathcal{L}_{n}$

The FASs $\chi_{n}^{ \pm}(z)$ are used to find the spectrum of the operator $\mathcal{L}_{n}$. In this section, we will present the new functions $R_{\{n, m\}}^{ \pm}(z) . R_{\{n, m\}}^{ \pm}(z)$ are analytic in $\gamma^{ \pm}$see Fig. 4.1. The kernel of $R_{\{n, m\}}^{ \pm}(z)$ is chosen in such a way that it is automatically compatible with the class of admissible potentials for $\mathcal{L}_{n}(z)$ (3.2.4).

$$
\begin{align*}
& R_{\{n, m\}}^{+}(z)=\chi_{n+1}^{+}(z)\left(\begin{array}{cc}
\theta(m-n) & 0 \\
0 & \theta(n-m)
\end{array}\right) \hat{\chi}_{m}^{+}(z),  \tag{4.2.1a}\\
& R_{\{n, m\}}^{-}(z)=\chi_{n+1}^{-}(z)\left(\begin{array}{cc}
\theta(n-m) & 0 \\
0 & \theta(m-n)
\end{array}\right) \hat{\chi}_{m}^{-}(z) . \tag{4.2.1b}
\end{align*}
$$

Then, $R_{\{n+1, m+1\}}^{+}(z)$ satisfies the equation (4.2.2):

$$
\begin{align*}
R_{\{n+1, m+1\}}^{+}(z) & =\chi_{n+2}^{+}(z)\left(\begin{array}{cc}
\theta(m+1-n-1) & 0 \\
0 & \theta(n+1-m-1)
\end{array}\right) \hat{\chi}_{m+1}^{+}(z) \\
& =\mathbf{Z} \chi_{n+1}^{+}(z)\left(\begin{array}{cc}
\theta(m-n) & 0 \\
0 & \theta(n-m)
\end{array}\right) \hat{\chi}_{m}^{+}(z) \hat{\mathbf{Z}} \\
& =\mathbf{Z} R_{\{n, m\}}^{+}(z) \hat{\mathbf{Z}} \tag{4.2.2}
\end{align*}
$$

Theorem 4.2.1 The Resolvent function $R_{\{n, m\}}^{+}(z)$ should tend to zero when $|z| \rightarrow \infty$ and $R_{\{n, m\}}^{-}(z)$ should tend to zero when $|z| \rightarrow 0$.

Proof: We will use the asymptotic behaviour of $R_{\{n, m\}}^{+}(z)$

$$
\left.\begin{array}{rl}
R_{\{n, m\}}^{+}(z) & \longrightarrow \\
& =\left(\begin{array}{cc}
z^{(n+1)} & 0 \\
0 & z^{-(n+1)}
\end{array}\right)\left(\begin{array}{cc}
\theta(m-n) & 0 \\
0 & \theta(n-m)
\end{array}\right)\left(\begin{array}{cc}
z^{-m} & 0 \\
0 & z^{m}
\end{array}\right)  \tag{4.2.3a}\\
0 & z^{-(n-m+1)} \theta(m-n) \\
0
\end{array}\right),
$$

and from the definition of the Heaviside function,

$$
\begin{align*}
\theta(n)= & \sum_{k=-\infty}^{n} \delta_{0, n}
\end{align*}=\left\{\begin{array}{ll}
1, & n \geq 0, \\
0, & n<0,
\end{array}, \quad \theta(n-m)=\sum_{k=-\infty}^{n} \delta_{0, n}=\left\{\begin{array}{ll}
1, & n \geq m  \tag{4.2.3b}\\
0, & n<m,
\end{array}, ~ \begin{cases}1, & m \geq n, \\
0, & m<n,\end{cases}\right.\right.
$$

Here, we must verify that $R_{\{n, m\}}^{+}(z)$ tends to zero when $|z| \rightarrow \infty$. When $n, m \rightarrow+\infty$ and $n-m>1, n>m$, the matrix elements of $R_{\{n, m\}}^{+}(z)$ tend to zero when $|z| \rightarrow \infty$

$$
R_{\{n, m\}}^{+}(z) \longrightarrow\left(\begin{array}{cc}
0 & 0  \tag{4.2.4}\\
0 & z^{-(n-m+1)}
\end{array}\right)
$$

Then, $R_{\{n+1, m+1\}}^{+}(z)$ is analytic when $|z| \rightarrow \infty$. Analogously, one can verify that $R_{\{n, m\}}^{-}(z)$ falls to zero and analytic when $|z| \rightarrow 0$.

### 4.2.2 The Jost solutions, scattering matrix and FASs

As a basic tool to find the completeness relation in the next section, we need to use the FASs of $\mathcal{L}_{n}(z)$, obtained by combining the pairs of columns of the Jost solutions with the same analyticity properties. From equation (3.2.7), using the eigenfunctions

$$
\begin{align*}
\left(\varphi_{n}^{+}, \varphi_{n}^{-}\right) & =\left(\phi_{n}^{+}, \phi_{n}^{-}\right)\left(\begin{array}{cc}
z^{-n} & 0 \\
0 & z^{n}
\end{array}\right), & \left(\xi_{n}^{-}, \xi_{n}^{+}\right) & =\left(\psi_{n}^{-}, \psi_{n}^{+}\right)\left(\begin{array}{cc}
z^{-n} & 0 \\
0 & z^{n}
\end{array}\right), \\
\varphi_{n}^{+}(z) & =\phi_{n}^{+}(z) z^{-n}, & \xi_{n}^{+}(z) & =\psi_{n}^{+}(z) z^{n}, \tag{4.2.5}
\end{align*}
$$

then, we will combine the columns

$$
\chi_{n}^{+}(z)=\left(\varphi_{n}^{+}, \xi_{n}^{+}\right)\left(\begin{array}{cc}
z^{n} & 0  \tag{4.2.6}\\
0 & z^{-n}
\end{array}\right)=\left(\phi_{n}^{+}, \psi_{n}^{+}\right)(z)
$$

and from the linear combination $\left(\phi_{n}(z)=\psi_{n}(z) T(z)\right)$, between the Jost solutions and the scattering matrix we can obtain the following relations:

$$
\begin{align*}
& \chi_{n}^{+}(z)=\psi_{n}(z)\left(\begin{array}{cc}
a^{+} & 0 \\
b^{+} & 1
\end{array}\right)=\phi_{n}(z)\left(\begin{array}{ll}
1 & \beta^{-} \\
0 & \alpha^{+}
\end{array}\right)  \tag{4.2.7a}\\
& \chi_{n}^{-}(z)=\psi_{n}(z)\left(\begin{array}{cc}
1 & -b^{-} \\
0 & a^{-}
\end{array}\right)=\phi_{n}(z)\left(\begin{array}{cc}
\alpha^{-} & 0 \\
-\beta^{+} & 1
\end{array}\right), \tag{4.2.7b}
\end{align*}
$$

where

$$
\hat{T}(z)=\left(\begin{array}{cc}
\alpha^{-}(z) & \beta^{-}(z)  \tag{4.2.7c}\\
-\beta^{+}(z) & \alpha^{+}(z)
\end{array}\right)
$$

The diagonal matrix of both $T(z)$ and $\hat{T}(z)$ allows for the analytic continuation of the unit circle, namely $a^{+}(z), \alpha^{+}(z)$ are analytic functions of $z$ for $z \rightarrow \infty$, while $a^{-}(z), \alpha^{-}(z)$ are analytic functions of $z$ for $z \rightarrow 0$.

$$
\begin{equation*}
\lim _{z \rightarrow \infty} a^{+}(z)=\lim _{z \rightarrow \infty} \alpha^{+}(z)=\mathbb{1}, \quad \lim _{z \rightarrow 0} a^{-}(z)=\lim _{z \rightarrow 0} \alpha^{-}(z)=\mathbb{1} \tag{4.2.8}
\end{equation*}
$$

Therefore, in order to find the inverse of $\chi_{n}^{+}(z)$ and $\chi_{n}^{-}(z)$, we need to equate both sides of equations (4.2.7a), (4.2.7b). Thus, we have

$$
\begin{equation*}
\hat{\chi}_{n}^{+}(z)=\frac{1}{\alpha^{+}(z)}\binom{\tilde{\Psi}_{n}^{+}(z)}{-\tilde{\Phi}_{n}^{+}(z)} \tag{4.2.9a}
\end{equation*}
$$

where $\tilde{\Psi}_{n}^{+}(z)=\left(\Psi_{\{2, n\}}(z),-\Psi_{\{1, n\}}(z)\right)$, and analogously we can find the inverse of $\chi_{n}^{-}(z)$

$$
\begin{equation*}
\hat{\chi}_{n}^{-}(z)=\frac{1}{\alpha^{-}(z)}\binom{\tilde{\Phi}_{n}^{-}(z)}{-\tilde{\Psi}_{n}^{-}(z)} . \tag{4.2.9b}
\end{equation*}
$$

We will use equations (4.2.7), (4.2.9) in the next section.

### 4.3 Completeness of the Jost solutions

We will derive the completeness relation for the Jost solutions of $\mathcal{L}_{n}$. Our derivation uses again the contour integration method (see Fig. 4.1). We also note that the contours $\Gamma^{ \pm}$ have the same orientation, contrary to the continuous case in Sec. 1.3 (see Fig. 1.5) where the analogous contours $\gamma_{ \pm}$should have also the same orientations. So, we start with the integral

$$
\begin{align*}
\mathcal{J}_{R,\{n, m\}}(z) & =\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\gamma_{+}} \mathrm{d} z R_{n}^{+}(z)-\oint_{\gamma_{-}} \mathrm{d} z R_{n}^{-}(z)\right) \\
& =\sum_{j=1}^{S}\left(\underset{z= \pm z_{j}^{+}}{\operatorname{Res}} R_{n}^{+}(z)+\underset{z= \pm z_{j}^{-}}{\operatorname{Res}} R_{n}^{-}(z)\right) \tag{4.3.1}
\end{align*}
$$

where $R_{n}^{ \pm}(z)$ provided by (4.2.1a), determines the kernel of the resolvent of $\mathcal{L}_{n}$. At the point of the discrete spectrum, the neighbourhood of $\pm z_{j}^{ \pm}$, and the orientations of the contours $\gamma_{\{-, \text {in }\}}$ and $\gamma_{\{-, \text {out }\}}$ are opposite (as well as those of $\gamma_{\{+, \text {in }\}}$ and $\gamma_{\{+, \text {out }\}}$ ). We

(a) The contours $\gamma_{ \pm}$before avoiding the singularities.

(b) The contours $\gamma_{ \pm}$after avoiding the singularities.

Figure 4.1: (a) The contours $\gamma_{ \pm}$for the integral (4.3.1). (b) The contours $\gamma_{ \pm}$for the integral (4.3.1) that avoid singularities when $|z| \rightarrow 0$ and $|z| \rightarrow \infty$. In (a) and (b) the solid (red) line $|z|=1$ is the continuous spectrum (no eigenvalues). The contour $\gamma_{+}$ contains all eigenvalues outside $|z|=1$. The contour $\gamma_{-}$contains all eigenvalues inside $|z|=1$. The outer contour $\gamma_{-, \text {out }}$ and the inner contour $\gamma_{+, \text {in }}$ have the same orientation around $|z|=1$.
derived the following relations from equation (4.2.7):

$$
\begin{align*}
& a^{ \pm}(z)=\left(z-\left( \pm z_{j}^{ \pm}\right)\right) \dot{a}_{j}^{ \pm}+\frac{1}{2}\left(z-\left( \pm z_{j}^{ \pm}\right)\right)^{2} \ddot{a}_{j}^{ \pm}+\ldots,  \tag{4.3.2a}\\
& \alpha^{ \pm}(z)=\left(z-\left( \pm z_{j}^{ \pm}\right)\right) \dot{\alpha}_{j}^{ \pm}+\frac{1}{2}\left(z-\left( \pm z_{j}^{ \pm}\right)\right)^{2} \ddot{\alpha}_{j}^{ \pm}+\ldots,  \tag{4.3.2b}\\
& \chi_{n}^{+}\left(z_{j}^{+}\right)=\psi_{n, j}^{+}(z)\left(b_{j}^{+}, 1\right)=\phi_{n, j}^{+}(z)\left(1,1 / b_{j}^{+}\right),  \tag{4.3.2c}\\
& \hat{\chi}_{n}^{+}\left(z_{j}^{+}\right)=\binom{1}{-\beta_{j}^{+}} \frac{\tilde{\Psi}_{n, j}^{+}(z)}{\left(z- \pm z_{j}^{+}\right) \dot{\alpha}_{j}^{+}}=\binom{1 / \beta_{j}^{+}}{-1} \frac{\tilde{\Phi}_{n, j}^{+}(z)}{\left(z- \pm z_{j}^{+}\right) \dot{\alpha}_{j}^{+}},  \tag{4.3.2d}\\
& \chi_{n}^{-}\left(z_{j}^{-}\right)=\psi_{n, j}^{-}(z)\left(1,-b_{j}^{-}\right)=\phi_{n, j}^{-}(z)\left(-1 / b_{j}^{-}, 1\right),  \tag{4.3.2e}\\
& \hat{\chi}_{n}^{-}\left(z_{j}^{-}\right)=-\binom{\beta_{j}^{-}}{1} \frac{\tilde{\Psi}_{n, j}^{-}(z)}{\left(z- \pm z_{j}^{-}\right) \dot{\alpha}_{j}^{-}}=\binom{1}{1 / \beta_{j}^{-}} \frac{\tilde{\Phi}_{n, j}^{-}(z)}{\left(z- \pm z_{j}^{-}\right) \dot{\alpha}_{j}^{-}}, \tag{4.3.2f}
\end{align*}
$$

where $\tilde{\Psi}_{n}(z)$ and $\tilde{\Phi}_{n}(z)$ are related to $\psi_{n}(z)$ and $\phi_{n}(z)$, respectively. Then, the residues of the resolvent kernel (when $n>m, n-m>1$ ) are:

$$
\begin{align*}
\operatorname{Res}_{z= \pm z_{j}^{-}} R_{n}^{-}(z)= & \lim _{z \rightarrow \pm z_{j}^{-}}\left(z-\left( \pm z_{j}^{-}\right)\right) R_{n}^{-}(z) \\
= & \lim _{z \rightarrow \pm z_{j}^{-}}\left(z-\left( \pm z_{j}^{-}\right)\right) \chi_{n+1}^{-}(z)\left(\begin{array}{cc}
\theta(n-m) & 0 \\
0 & \theta(m-n)
\end{array}\right) \hat{\chi}_{m}^{-}(z) \\
= & \lim _{z \rightarrow \pm z_{j}^{-}}\left(z-\left( \pm z_{j}^{-}\right)\right) \phi_{n+1, j}^{-}(z)\left(-1 / b_{j}^{-}, 1\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \\
& \binom{-\beta_{j}^{-}}{-1} \frac{\tilde{\Psi}_{m, j}^{-}(z)}{\left(z- \pm z_{j}^{-}\right) \dot{\alpha}_{j}^{-}} \tag{4.3.3a}
\end{align*}
$$

then,

$$
\begin{equation*}
\operatorname{Res}_{z= \pm z_{j}^{-}} R_{n}^{-}(z)=\frac{\beta_{j}^{-} \phi_{n+1, j}^{-}(z) \tilde{\Psi}_{m, j}^{-}(z)}{b_{j}^{-} \dot{\alpha}_{j}^{-}(z)}, \tag{4.3.3b}
\end{equation*}
$$

and, if $\beta^{-}(z)=b^{-}(z)$,

$$
\begin{equation*}
\underset{z= \pm z_{j}^{-}}{\operatorname{Res}_{n}} R_{n}^{-}(z)=\frac{\phi_{n+1, j}^{-}(z) \tilde{\Psi}_{m, j}^{-}(z)}{\dot{\alpha}_{j}^{-}(z)} . \tag{4.3.3c}
\end{equation*}
$$

A similar calculation for $\operatorname{Res}_{z= \pm z_{j}^{+}} R_{n}^{+}(z)$, if $\beta^{+}(z)=b^{+}(z)$ shows that

$$
\begin{equation*}
\underset{z= \pm z_{j}^{+}}{\operatorname{Res}} R_{n}^{+}(z)=-\frac{\phi_{n+1, j}^{+}(z) \tilde{\Psi}_{m, j}^{+}(z)}{\dot{\alpha}_{j}^{+}(z)} . \tag{4.3.3d}
\end{equation*}
$$

Next, we can calculate the gap between the outer contour $\gamma_{-, \text {out }}$ and the inner contour $\gamma_{+, \text {in }}$ which is called the jump of $R_{n}(z)$ on the unit circle $(|z|=1)$. From equation (4.2.1a) and (4.2.4), we have

$$
\begin{align*}
R_{\{n, m\}}^{ \pm}= & \frac{\epsilon(n-m)}{2} \chi_{n+1}^{ \pm}(z) \hat{\chi}_{m}^{ \pm}(z) \mp \frac{1}{2} \chi_{n+1}^{ \pm}(z) \sigma_{3} \hat{\chi}_{m}^{ \pm}(z), \\
R_{\{n, m\}}^{+}-R_{\{n, m\}}^{-}= & \frac{\epsilon(n-m)}{2}\left(\chi_{n+1}^{+}(z) \hat{\chi}_{m}^{+}(z)-\chi_{n+1}^{-}(z) \hat{\chi}_{m}^{-}(z)\right) \\
& -\frac{1}{2}\left(\chi_{n+1}^{+}(z) \sigma_{3} \hat{\chi}_{m}^{+}(z)+\chi_{n+1}^{-}(z) \sigma_{3} \hat{\chi}_{m}^{-}(z)\right), \tag{4.3.4a}
\end{align*}
$$

since $\chi_{n+1}^{+}(z) \hat{\chi}_{m}^{+}(z)=\chi_{n+1}^{-}(z) \hat{\chi}_{m}^{-}(z)$. Thus,

$$
\begin{align*}
R_{\{n, m\}}^{+}-R_{\{n, m\}}^{-} & =-\frac{1}{2}\left(\chi_{n+1}^{+}(z) \sigma_{3} \hat{\chi}_{m}^{+}(z)+\chi_{n+1}^{-}(z) \sigma_{3} \hat{\chi}_{m}^{-}(z)\right) \\
& =-\frac{1}{2}\left(\chi_{n+1}^{+}(z)\left(\mathbb{1}+\sigma_{3}\right) \hat{\chi}_{m}^{+}(z)-\chi_{n+1}^{-}(z)\left(\mathbb{1}-\sigma_{3}\right) \hat{\chi}_{m}^{-}(z)\right) \\
& =-\left(\frac{\phi_{n+1}^{+}(z) \tilde{\Psi}_{m}^{+}(z)}{\alpha^{+}(z)}+\frac{\phi_{n+1}^{-}(z) \tilde{\Psi}_{m}^{-}(z)}{\alpha^{-}(z)}\right) . \tag{4.3.4b}
\end{align*}
$$

### 4.3.1 Asymptotic behaviour for $z \rightarrow \infty$ and $z \rightarrow 0$

We will show in this section, the relation between $R_{n}^{ \pm}(z)$ and their behaviour for $z \rightarrow \infty$ and $z \rightarrow 0$.

Theorem 4.3.1 To study and calculate the analytic properties of the solutions $R_{n}^{ \pm}(z)$ and also their behaviour for $z \rightarrow \infty$ and $z \rightarrow 0$, the contour integration method is used to calculate the regular and singular points of $R_{n}^{ \pm}(z)$.

Proof: The idea is shown in Fig. 4.1, where $R_{n}^{+}(z)$ is an analytic function in $\gamma_{+}$and has simple poles at $z_{j}^{+}$when $z \rightarrow \infty$ and $R_{n}^{-}(z)$ is an analytic function in $\gamma_{-}$and has simple poles at $z_{j}^{-}$when $z \rightarrow 0$ (we need to make sure that we have the same numbers of discrete eigenvalue in each region $\left(\gamma_{-}\right.$and $\left.\gamma_{+}\right)$). It follows directly from Cauchy's theorem for analytic functions that the integrals along the cuts cancel (see Fig. 4.1). Now, we need to find the second term of equation (4.3.1). This needs to use the asymptotic ${ }^{1}$ behaviour of $\chi_{n}^{ \pm}(z)$ when $z \rightarrow \infty$ and $z \rightarrow 0$, respectively:

$$
\begin{align*}
& \chi_{\{\text {asy }, n\}}^{+}(z)=\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right)+\mathcal{O}(1 / z) \quad z \rightarrow \infty  \tag{4.3.5a}\\
& \chi_{\{\text {asy }, n\}}^{-}(z)=\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right)+\mathcal{O}(z), \quad z \rightarrow 0, \tag{4.3.5b}
\end{align*}
$$

where $\mathcal{O}(1 / z)$ are the terms of $1 / z$ order and higher do not contribute to the integrals and $\mathcal{O}(z)$ are the terms of $z$ order and higher do not contribute to the integrals. That means

$$
\begin{align*}
& R_{\{\text {asy }, n, m\}}^{+}=\left(\begin{array}{cc}
z^{(n+1)} & 0 \\
0 & z^{-(n+1)}
\end{array}\right)\left(\begin{array}{cc}
\theta(m-n) & 0 \\
0 & \theta(n-m)
\end{array}\right)\left(\begin{array}{cc}
z^{-m} & 0 \\
0 & z^{m}
\end{array}\right),  \tag{4.3.6a}\\
& R_{\{\text {asy }, n, m\}}^{-}=\left(\begin{array}{cc}
z^{(n+1)} & 0 \\
0 & z^{-(n+1)}
\end{array}\right)\left(\begin{array}{cc}
\theta(n-m) & 0 \\
0 & \theta(m-n)
\end{array}\right)\left(\begin{array}{cc}
z^{-m} & 0 \\
0 & z^{m}
\end{array}\right) . \tag{4.3.6b}
\end{align*}
$$

[^13]Then, by the Cauchy integral formula, one obtains:

$$
\begin{align*}
\oint_{\gamma_{+}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+} & =\left(\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}-\oint_{\left\{\gamma_{+}, \text {in }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}\right) \\
& =2 \pi \mathrm{i} \sum_{j=1}^{S} \operatorname{Res} R_{\{\text {asy }, n, m\}}^{+},  \tag{4.3.7a}\\
\oint_{\gamma_{-}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-} & =\left(\oint_{\left\{\gamma_{-}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}-\oint_{\left\{\gamma_{-}, \text {in }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}\right) \\
& =2 \pi \mathrm{i} \sum_{j=1}^{S} \operatorname{Res} R_{\{\text {asy }, n, m\}}^{-} . \tag{4.3.7b}
\end{align*}
$$

We need to subtract equation (4.3.7b) from equation (4.3.7a) as follows:

$$
\begin{align*}
\oint_{\gamma_{+}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+} & -\oint_{\gamma_{-}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}=\left(\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}+\oint_{\left\{\gamma_{-}, \text {in }\right\}} \mathrm{d} z R_{\{\text {ass }, n, m\}}^{-}\right) \\
& -\left(\oint_{\left\{\gamma_{+}, \text {in }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}+\oint_{\left\{\gamma_{-}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}\right) \\
& =2 \pi \mathrm{i} \sum_{j=1}^{S}\left(\operatorname{Res} R_{\{\text {asy }, n, m\}}^{+}+\operatorname{Res} R_{\{\text {asy }, n, m\}}^{-}\right) . \tag{4.3.7c}
\end{align*}
$$

Finally, we have three terms

$$
\begin{align*}
& \frac{1}{2 \pi \mathrm{i}}\left(\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}+\oint_{\{\gamma-, \mathrm{in}\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}\right)+\frac{1}{2 \pi \mathrm{i}} \oint_{|z|=1} \mathrm{~d} z\left(R^{+}+R^{-}\right)_{\{\text {ass }, n, m\}} \\
& =\sum_{j=1}^{S} \operatorname{Res}\left(R_{\{\text {asy }, n, m\}}^{+}+R_{\{\text {asy }, n, m\}}^{-}\right) . \tag{4.3.7d}
\end{align*}
$$

Next, the first term needs more calculations (we have simplified the second and the third terms). From Fig. 4.1 $z_{j}^{+}$is outside the contour $\gamma_{\{+ \text {,in }}$ and $z_{j}^{-}$is inside the contour $\gamma_{\{-, \text {in }\}}$, so the first term of equation (4.3.7d) for $n>m$, becomes:

$$
\begin{aligned}
& \frac{1}{2 \pi \mathrm{i}}\left(\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}+\oint_{\left\{\gamma_{-}, \text {in }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}\right) \\
= & \frac{1}{2 \pi \mathrm{i}} \oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z\left[\left(\begin{array}{cc}
z^{(n+1)} & 0 \\
0 & z^{-(n+1)}
\end{array}\right)\left(\begin{array}{cc}
\theta(m-n) & 0 \\
0 & \theta(n-m)
\end{array}\right)\left(\begin{array}{cc}
z^{-m} & 0 \\
0 & z^{m}
\end{array}\right)\right] \\
& +\frac{1}{2 \pi \mathrm{i}} \oint_{\left\{\gamma_{-}, \text {in }\right\}} \mathrm{d} z\left[\left(\begin{array}{cc}
z^{(n+1)} & 0 \\
0 & z^{-(n+1)}
\end{array}\right)\left(\begin{array}{cc}
\theta(n-m) & 0 \\
0 & \theta(m-n)
\end{array}\right)\left(\begin{array}{cc}
z^{-m} & 0 \\
0 & z^{m}
\end{array}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
& =\frac{1}{2 \pi \mathrm{i}}\left[\left(\begin{array}{ll}
0 & 0 \\
0 \oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z z^{-(n-m+1)}
\end{array}\right)+\left(\begin{array}{cc}
\oint_{\{\gamma-, \text { in }\}} \mathrm{d} z z^{(n-m+1)} & 0 \\
0 & 0
\end{array}\right)\right] \\
& =\frac{1}{2 \pi \mathrm{i}}\binom{\oint_{\left\{\gamma_{-}, \text {in }\right\}} \mathrm{d} z z^{(n-m+1)}}{0} \tag{4.3.8}
\end{align*}
$$

Here, we are trying to show that (4.3.8) is equivalent to the Dirac delta function

$$
\delta(x)= \begin{cases}0 & x \neq 0  \tag{4.3.9a}\\ \infty & x=0\end{cases}
$$

If we set $z=r e^{\mathrm{i} \theta}$, where $r$ is the radius of contour $\gamma_{-}$, then

$$
\begin{align*}
\oint_{\{\gamma-, \mathrm{in}\}} \mathrm{d} \theta\left(r e^{\mathrm{i} \theta}\right)^{(n-m+1)} \mathrm{i} r e^{\mathrm{i} \theta} & =\mathrm{i} r^{(n-m+2)} \int_{-\infty}^{\infty} \mathrm{d} \theta e^{\mathrm{i} \theta(n-m+2)} \\
& =0, \quad \text { when } \quad r \rightarrow 0 . \tag{4.3.9b}
\end{align*}
$$

However, if we set $z=G e^{\mathrm{i} \theta}$ where $G$ is a radius of contour $\gamma_{+}$, then

$$
\begin{align*}
\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} \theta\left(G e^{\mathrm{i} \theta}\right)^{-(n-m+1)} \mathrm{i} G e^{\mathrm{i} \theta} & =\mathrm{i} G^{(n-m)} \int_{-\infty}^{\infty} \mathrm{d} \theta e^{-\mathrm{i} \theta(n-m)}, \\
& \approx 2 \pi \mathrm{i} G^{(n-m)} \delta(n-m) . \tag{4.3.9c}
\end{align*}
$$

In this case, we can estimate that

$$
\begin{equation*}
\frac{1}{2 \pi \mathrm{i}}\left(\oint_{\left\{\gamma_{+}, \text {out }\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{+}+\oint_{\left\{\gamma_{-}, \mathrm{in}\right\}} \mathrm{d} z R_{\{\text {asy }, n, m\}}^{-}\right) \approx \delta(n-m) . \tag{4.3.9d}
\end{equation*}
$$

Thus, equation (4.3.7d) becomes:

$$
\begin{align*}
& \delta(n-m)+\frac{1}{2 \pi \mathrm{i}} \oint_{|z|=1} \mathrm{~d} z\left(R^{+}+R^{-}\right)_{\{\text {asy }, n, m\}} \\
& =\sum_{j=1}^{S} \operatorname{Res}\left(R_{\{\text {asy }, n, m\}}^{+}+R_{\{\text {asy }, n, m\}}^{-}\right) . \tag{4.3.10}
\end{align*}
$$

In the next section will use equations (4.3.3c), (4.3.3d), (4.3.4b) and (4.3.10) to present the expansion over the Jost solutions.

### 4.3.2 Expansion over the Jost solutions

Combining the relations (4.3.3c), (4.3.3d), (4.3.4b) and (4.3.10) leads to the following expressions for $\mathcal{J}_{R,\{n, m\}}(z)$ in (4.3.1)

$$
\begin{align*}
& \mathcal{J}_{R,\{n, m\}}(z)= \\
& \delta(n-m)-\frac{1}{2 \pi \mathrm{i}} \int_{|z|=1} \mathrm{~d} z\left(\frac{\phi_{n+1}^{+}(z) \tilde{\Psi}_{m}^{+}(z)}{\alpha^{+}(z)}+\frac{\phi_{n+1}^{-}(z) \tilde{\Psi}_{m}^{-}(z)}{\alpha^{-}(z)}\right) \\
& =\sum_{j=1}^{S}\left(\frac{\phi_{n+1, j}^{-}(z) \tilde{\Psi}_{m, j}^{-}(z)}{\dot{\alpha}_{j}^{-}(z)}-\frac{\phi_{n+1, j}^{+}(z) \tilde{\Psi}_{m, j}^{+}(z)}{\dot{\alpha}_{j}^{+}(z)}\right) . \tag{4.3.11}
\end{align*}
$$

Then, we obtain the completeness relation

$$
\begin{align*}
\delta(n-m) \mathbb{1}= & \frac{1}{2 \pi \mathrm{i}} \int_{|z|=1} \mathrm{~d} z\left(\frac{\phi_{n+1}^{+}(z) \tilde{\Psi}_{m}^{+}(z)}{\alpha^{+}(z)}+\frac{\phi_{n+1}^{-}(z) \tilde{\Psi}_{m}^{-}(z)}{\alpha^{-}(z)}\right) \\
& +\sum_{j=1}^{S}\left(\frac{\phi_{\{n+1, j\}}^{+}(z) \tilde{\Psi}_{m, j}^{+}(z)}{\dot{\alpha}_{j}^{+}(z)}-\frac{\phi_{\{n+1, j\}}^{-}(z) \tilde{\Psi}_{m, j}^{-}(z)}{\dot{\alpha}_{j}^{-}(z)}-\right) . \tag{4.3.12}
\end{align*}
$$

Therefore, the Jost solutions $\phi_{n}^{ \pm}(z)$ form a complete set of functions over the space of fundamental solutions of $\mathcal{L}_{n}(z)$. Based on the completeness relation (4.3.12), one can expand every function $Y(z)=\binom{Y_{1}}{Y_{2}}$ from the space of solutions of $\mathcal{L}_{n}(z)$ over the complete set Jost solutions by the following expansion formulas:

$$
\begin{align*}
Y_{n}(z)= & \frac{1}{2 \pi \mathrm{i}} \int_{|z|=1} \mathrm{~d} z\left(\phi_{n+1}^{+}(z) y_{m}^{+}(z)+\phi_{n+1}^{-}(z) y_{m}^{-}(z)\right) \\
& +\sum_{j=1}^{S}\left(\phi_{\{n+1, j\}}^{+}(z) y_{\{m, j\}}^{+}(z)-\phi_{\{n+1, j\}}^{-}(z) y_{\{m, j\}}^{-}(z)\right), \tag{4.3.13}
\end{align*}
$$

where

$$
\begin{align*}
& y_{m}^{ \pm}(z)=\frac{1}{\alpha^{ \pm}(z)} \int_{|z|=1} \mathrm{~d} z \tilde{\Psi}_{m}^{ \pm}(z) Y_{m}(z),  \tag{4.3.14a}\\
& y_{\{m, j\}}^{ \pm}=\frac{1}{\dot{\alpha}_{j}^{ \pm}} \int_{|z|=1} \mathrm{~d} z \tilde{\Psi}_{\{m, j\}}^{ \pm}(z) Y_{m}(z) . \tag{4.3.14b}
\end{align*}
$$

Thus, the spectral decomposition of the operator $\mathcal{L}_{n}$ is

$$
\begin{align*}
\mathcal{L}_{n} Y_{n}(z)= & \frac{1}{2 \pi \mathrm{i}} \int_{|z|=1} \mathrm{~d} z z\left(\phi_{n+1}^{+}(z) y_{m}^{+}(z)+\phi_{n+1}^{-}(z) y_{m}^{-}(z)\right) \\
& +\sum_{j=1}^{S}\left(z_{j}^{+} \phi_{\{n+1, j\}}^{+}(z) y_{\{m, j\}}^{+}(z)-z_{j}^{-} \phi_{\{n+1, j\}}^{-}(z) y_{\{m, j\}}^{-}(z)\right) . \tag{4.3.15}
\end{align*}
$$

## Chapter 5

## Square barrier potential for continuous and discrete nonlocal NLS equations

In this chapter, two equations (5.0.1) are used to study the NLS equation

$$
\begin{align*}
\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} t} q & =q_{x x}+2 q^{2}(x, t) q^{*}(-x, t)  \tag{5.0.1a}\\
\mathrm{i} \frac{\mathrm{~d}}{\mathrm{~d} \tau} Q_{n} & =\left(Q_{n+1}-2 Q_{n}+Q_{n-1}\right)+Q_{n} Q_{-n}^{*}\left(Q_{n+1}+Q_{n-1}\right) . \tag{5.0.1b}
\end{align*}
$$

The main purpose of this chapter is to present a direct and systematic way of finding solutions to the nonlocal continuous and discrete NLS equations (5.0.1). We analysed the problem of the existence of a dynamical barrier that needs to be excited on a lattice site to lead to the formulation of the nonlocal continuous and discrete NLS equation. We determined the boundary conditions for creating solitons with such initial conditions for the continuous and discrete nonlocal NLS equations. In [39, 76], the authors introduced the square barrier initial potentials for discrete local NLS equation. They also introduced the Manakov model with a 2-components vector. They also present the boundary conditions for the non-integrable discrete NLS model. The presentation of this work is based on the use of the linear combination of the boundary conditions to evaluate functions (the scattering data $a^{ \pm}(\lambda)$ and $\left.a^{ \pm}(z)\right)$ which can lead to the eigenvalues of the Ablowitz-Ladik system. Our calculations are in agreement with chapters 2 and 3. For the continuous NLS equation, the eigenvalues of the spectral problem must lie in the upper (lower) half plane and for the DNLS equation, the eigenvalues must lie outside (inside) the unit circle $(|z|=1)$.

This chapter is organised as follows: Sec. 5.1 illustrates the general mathematical steps of the continuous spectral problem and presents two examples of this model. Section 5.2 considers the general mathematical steps of the problem of the discrete AblowitzLadik problem and provides different examples. We present our analytical and numerical
results for each example.

### 5.1 Blow up or not to blow up solutions to the continuous NLS equation

In this section, we will introduce the spectral problem with its Jost solutions. Following [9], the model (5.1.1) is slightly different from the ZS system (1.2.6a)

$$
\begin{align*}
& \Psi_{x}(x, \lambda)=L(x, \lambda) \Psi(x, \lambda)  \tag{5.1.1a}\\
& \Psi_{t}(x, \lambda)=M(x, \lambda) \Psi(x, \lambda) . \tag{5.1.1b}
\end{align*}
$$

Recall the Jost solutions (2.2.2)

$$
\begin{align*}
& \lim _{x \rightarrow \infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \psi(x, t, \lambda)=\mathbb{1},  \tag{5.1.2}\\
& \lim _{x \rightarrow-\infty} \exp \left(\mathrm{i} \lambda \sigma_{3} x\right) \phi(x, t, \lambda)=\mathbb{1} \tag{5.1.3}
\end{align*}
$$

and the scattering matrix (2.2.1b)

$$
T(\lambda, t)=\psi^{-1}(x, \lambda) \phi(x, \lambda)=\left(\begin{array}{cc}
a^{+}(\lambda) & -b^{-}(\lambda)  \tag{5.1.4}\\
b^{+}(\lambda) & a^{-}(\lambda)
\end{array}\right)
$$

where $\operatorname{det} T(\lambda) \equiv a^{+}(\lambda) a^{-}(\lambda)+b^{+}(\lambda) b^{-}(\lambda)=1$, for $\lambda \in \mathbb{R}$. The nonlocal involution for the ZS system (5.1.1a)

$$
\Psi_{x}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & q^{+}(x, t)  \tag{5.1.5}\\
-\left(q^{+}\right)^{*}(-x, t) & \mathrm{i} \lambda
\end{array}\right) \Psi(x, \lambda),
$$

and for the scattering data is

$$
\begin{equation*}
a^{ \pm}(\lambda)=\left(a^{ \pm}\left(-\lambda^{*}\right)\right)^{*} \quad \text { and } \quad b^{ \pm}(\lambda)=\left(b^{\mp}\left(-\lambda^{*}\right)\right)^{*} \tag{5.1.6}
\end{equation*}
$$

The time evolution of the solution to the NLS equation is done by calculating the evolution of the scattering data which is explained in [9, 12]. The authors calculated the scattering data as:

$$
\begin{equation*}
a^{ \pm}(\lambda)=a^{ \pm}(\lambda, 0), \quad b^{ \pm}=e^{\mp 4 i\left(\lambda^{ \pm}\right)^{2} t} b^{ \pm}\left(\lambda^{ \pm}, 0\right) \tag{5.1.7}
\end{equation*}
$$

where the phases $b^{+}\left(\lambda^{+}, 0\right)=e^{\mathrm{i} \alpha}$ and $b^{-}\left(\lambda^{-}, 0\right)=e^{\mathrm{i} \bar{\alpha}}$, are arbitrary with $\alpha, \bar{\alpha}$ both being real and positive. The derivation of the one soliton solution for the nonlocal NLS is given
in [7,9] which is the most general one soliton solution of breathing type

$$
\begin{equation*}
q^{+}(x)=-\frac{2\left(\lambda_{1}^{+}+\lambda_{1}^{-}\right) e^{-2 \lambda_{1}^{-} x} e^{\mathrm{i} \bar{\alpha}} e^{-4 \mathrm{i}\left(\lambda_{1}^{-}\right)^{2} t}}{1+e^{-2\left(\lambda_{1}^{+}+\lambda_{1}^{-}\right) x} e^{\mathrm{i}(\alpha+\bar{\alpha})} e^{4 \mathrm{i}\left(\left(\lambda_{1}^{+}\right)^{2}-\left(\lambda_{1}^{-}\right)^{2}\right) t}} \tag{5.1.8}
\end{equation*}
$$

In the next section, we use two examples to show the case where we have a blow up or not solution to equation (5.1.8) for the square barrier potential type.

### 5.1.1 Eigenvalues of square barrier potentials with numerical results

Here, we will use numerical method to find the eigenvalues of the spectral problem correspond to the zeros of the scattering data $a^{ \pm}(\lambda)$. We will use iteration method on the spectral problem (5.1.5) for two types of potentials; to do so, we will use the Jost solution $\phi(x, \lambda)$ in different positions (for example see Fig. 5.1). Since we defined that $\phi(x, \lambda)=\left(\phi^{+}, \phi^{-}\right)(x, \lambda)$ is a solution to the spectral problem when $x \rightarrow-\infty$, we need to find the solution in each region; this will implies finding the $a^{ \pm}(\lambda)$ functions.

Example 5 The first type of the regions is defined as:

$$
\begin{gather*}
q^{+}(x)=\left\{\begin{array}{cc}
k & \ell_{1}<x<\ell_{2} \\
0 & \text { otherwise }
\end{array}\right.  \tag{5.1.9a}\\
\left(q^{+}\right)^{*}(-x)=\left\{\begin{array}{cc}
k & -\ell_{2}<x<-\ell_{1} \\
0 & \text { otherwise }
\end{array}\right. \tag{5.1.9b}
\end{gather*}
$$

where $k$ is a positive constant, and $\ell_{1}$ and $\ell_{2}$ are positive numbers on the x-axis. Figure 5.1 indicates five regions $(I-V)$ of potentials $q^{+}(x)$ and $\left(q^{+}\right)^{*}(-x)$. In region $I, q(x) \rightarrow 0$


Figure 5.1: Example 5: The square barrier potentials for (5.1.9).
when $x \rightarrow-\infty$, then the spectral problem becomes:

$$
\phi_{x}^{(+, I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.10}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I)}(x, \lambda),
$$

where $\phi(x, \lambda)=\binom{\phi_{1}}{\phi_{2}}(x, \lambda)$ is a column vector. We find the solutions of the system
(5.1.10) are

$$
\begin{align*}
& \phi_{1}^{(+, I)}(x, \lambda)=f_{1} e^{-\mathrm{i} \lambda x}, \\
& \phi_{2}^{(+, I)}(x, \lambda)=f_{2} e^{\mathrm{i} \lambda x} . \tag{5.1.11}
\end{align*}
$$

Using the boundary conditions (5.1.2) of the Jost solution $\phi(x, \lambda)$ as $x \rightarrow-\infty$, we can find the constants $f_{1}=1, f_{2}=0$. Therefore, in this case the solutions in region $I$ will be

$$
\begin{align*}
& \phi_{1}^{(+, I)}(x, \lambda)=e^{-\mathrm{i} \lambda x}, \\
& \phi_{2}^{(+, I)}(x, \lambda)=0 . \tag{5.1.12}
\end{align*}
$$

In region $I I$, the spectral problem becomes:

$$
\phi_{x}^{(+, I I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.13}\\
-k & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I I)}(x, \lambda) .
$$

We can rewrite the system (5.1.13) as:

$$
\begin{align*}
& \dot{\phi}_{1}^{(+, I I)}(x)=-\mathrm{i} \lambda \phi_{1}^{(+, I I)}(x),  \tag{5.1.14a}\\
& \dot{\phi}_{2}^{(+, I I)}(x)=-k \phi_{1}^{(+, I I)}(x)+\mathrm{i} \lambda \phi_{2}^{(+, I I)}(x) \tag{5.1.14b}
\end{align*}
$$

The solutions of (5.1.14) are:

$$
\begin{align*}
\phi_{1}^{(+, I I)}(x, \lambda) & =f_{3} e^{-\mathrm{i} \lambda x},  \tag{5.1.15a}\\
\phi_{2}^{(+, I I)}(x, \lambda) & =e^{\mathrm{i} \lambda x}\left[\frac{k}{2 \mathrm{i} \lambda} e^{-2 \mathrm{i} \lambda x}+f_{4}\right] . \tag{5.1.15b}
\end{align*}
$$

Since the solutions $\phi_{1}^{(+, I)}(x, \lambda)$ and $\phi_{1}^{(+, I I)}(x, \lambda)$ are equals at $x=-\ell_{2},\left(\phi_{1}\right.$ is continuous at $x=-\ell_{2}$ ), then $f_{3}=1$. Therefore, the first solution (5.1.15a) will be

$$
\begin{equation*}
\phi_{1}^{(+, I I)}(x, \lambda)=e^{-\mathrm{i} \lambda x} \tag{5.1.16}
\end{equation*}
$$

Next, we can find $f_{4}$ at $x=-\ell_{2}$, using the relation between the two regions $(I)$ and (II), ( $\phi_{2}$ is continuous at $x=-\ell_{2}$ ), then $f_{4}=\frac{-k}{2 i \lambda} e^{2 i \lambda \ell_{2}}$. Therefore, the second solution (5.1.15b) will be

$$
\begin{equation*}
\phi_{2}^{(+, I I)}(x, \lambda)=e^{\mathrm{i} \lambda x}\left[\frac{k}{2 \mathrm{i} \lambda} e^{-2 \mathrm{i} \lambda x}-\frac{k}{2 \mathrm{i} \lambda} e^{2 \mathrm{i} \lambda \ell_{2}}\right] . \tag{5.1.17}
\end{equation*}
$$

In region $I I I$, the spectral problem read as in $(I)$

$$
\phi_{x}^{(+, I I I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.18}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I I I)}(x, \lambda) .
$$

The solutions of (5.1.18) are:

$$
\begin{align*}
& \phi_{1}^{(+, I I I)}(x, \lambda)=e^{-\mathrm{i} \lambda x}  \tag{5.1.19a}\\
& \phi_{2}^{(+, I I I)}(x, \lambda)=\frac{k}{2 \mathrm{i} \lambda}\left(e^{2 \mathrm{i} \lambda \ell_{1}}-e^{2 \mathrm{i} \lambda \ell_{2}}\right) e^{\mathrm{i} \lambda x} . \tag{5.1.19b}
\end{align*}
$$

In region $I V$, the spectral problem becomes:

$$
\phi_{x}^{(+, I V)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & k  \tag{5.1.20}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I V)}(x, \lambda)
$$

and its solutions are

$$
\begin{align*}
\phi_{1}^{(+, I V)}(x, \lambda)= & e^{-\mathrm{i} \lambda x}\left[1-\frac{k^{2}}{4 \lambda^{2}}\left(e^{2 \mathrm{i} \lambda \ell_{1}}-e^{2 \mathrm{i} \lambda \ell_{2}}\right) e^{2 \mathrm{i} \lambda x}\right. \\
& \left.+\frac{k^{2}}{4 \lambda^{2}}\left(e^{2 \mathrm{i} \lambda \ell_{1}}-e^{2 \mathrm{i} \lambda \ell_{2}}\right) e^{2 \mathrm{i} \lambda \ell_{1}}\right] \tag{5.1.21a}
\end{align*}
$$

and

$$
\begin{equation*}
\phi_{2}^{(+, I V)}(x, \lambda)=\frac{k}{2 \mathrm{i} \lambda}\left(e^{2 \mathrm{i} \lambda \ell_{1}}-e^{2 \mathrm{i} \lambda \ell_{2}}\right) e^{\mathrm{i} \lambda x} . \tag{5.1.21b}
\end{equation*}
$$

The spectral problem in region $V$ is

$$
\phi_{x}^{(+, V)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.22}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, V)}(x, \lambda) .
$$

The solution of the first eigenfunction of (5.1.22) is

$$
\begin{align*}
\phi_{1}^{(+, V)}(x, \lambda)= & \left(\left(1 / 2 \mathrm{i}\left(\frac{-1 / 2 \mathrm{i} k\left(-e^{\mathrm{i} \lambda(l 1+2 l 2)}+e^{3 \mathrm{i} \lambda l 1}\right)}{\lambda}\right) k\left(e^{\mathrm{i} \lambda(l 1-l 2)}-e^{-\mathrm{i} \lambda(l 1-l 2)}\right)\right.\right. \\
& \left.\left.+\lambda e^{-\mathrm{i} \lambda l 2}\right) \lambda^{-1}\right)\left(e^{\mathrm{i} \lambda(l 2-x)}\right) \tag{5.1.23a}
\end{align*}
$$

and the second eigenfunction

$$
\begin{equation*}
\phi_{2}^{(-, V)}(x, \lambda)=e^{\mathrm{i} \lambda x} \tag{5.1.23b}
\end{equation*}
$$

From the solution in region $V$, we can find $a^{ \pm}(\lambda)$. We remind that the linear combination of the Jost solutions $\phi(x, \lambda)$ and $\psi(x, \lambda)$ are connected with the scattering matrix $T(\lambda, t)$,
we obtain:

$$
\begin{align*}
a^{+}(\lambda) & =\left(\left(1 / 2 \mathrm{i}\left(\frac{-1 / 2 \mathrm{i} k\left(-e^{\mathrm{i} \lambda(l 1+2 l 2)}+e^{3 i \lambda l 1}\right)}{\lambda}\right) k\left(e^{\mathrm{i} \lambda(l 1-l 2)}-e^{-\mathrm{i} \lambda(l 1-l 2)}\right)\right.\right. \\
& \left.\left.+\lambda e^{-\mathrm{i} \lambda l 2}\right) \lambda^{-1}\right)\left(e^{\mathrm{i} \lambda l 2}\right)  \tag{5.1.24a}\\
a^{-}(\lambda) & =1 . \tag{5.1.24b}
\end{align*}
$$

Then, the zeros of $a^{+}(\lambda)\left(a^{+}(\lambda)=0\right)$ will be the discrete eigenvalues $\lambda_{j}^{+}$. Since $a^{+}(\lambda)$ is analytic in the upper half complex $\lambda$-plain $\left(\lambda \in \mathbb{C}_{+}\right)$, we have discrete eigenvalues when $k>0.5$ (see Fig. 5.2). However, $a^{-}(\lambda)=1 \neq 0$, means, this function can not be zero for all $\lambda \in \mathbb{C}_{-}$. Then, we have no discrete eigenvalues in the lower half complex $\lambda$-plain. In this case we have blow up solution.


Figure 5.2: (a) and (b) are contour plots of the real and imaginary parts of the solution of $a^{+}(\lambda)=0$ when $\ell_{1}=1, \ell_{2}=3$ at $k=0.4$ and $k=0.8$, respectively. The blue dashed lines represent the imaginary part of equations $\operatorname{Im}\left(\mathrm{a}^{+}(\lambda)\right)$, while the red solid lines are for the real part of the equation $\operatorname{Re}\left(\mathrm{a}^{+}(\lambda)\right)$. The dotted line is the real line $(-\infty<x<\infty)$. Intersections between the blue and red lines represent uniform solutions of $a^{+}(\lambda)=0, \lambda=\lambda_{j}, a^{+}\left(\lambda_{j}\right)$.

Example 6 The second type of the potentials is defined as:

$$
\begin{align*}
q^{+}(x) & =\left\{\begin{array}{cc}
k & -\ell_{1}<x<\ell_{2} \\
0 & \text { otherwise }
\end{array}\right.  \tag{5.1.25a}\\
\left(q^{+}\right)^{*}(-x) & =\left\{\begin{array}{cc}
k & -\ell_{2}<x<-\ell_{1} \\
0 & \text { otherwise }
\end{array}\right. \tag{5.1.25b}
\end{align*}
$$

where $k$ is a positive constant, $\ell_{1}$ and $\ell_{2}$ are positive numbers on the x-axis. Figure 5.3 indicates five regions $(I-V)$ of potentials $q^{+}(x)$ and $\left(q^{+}\right)^{*}(-x)$. In region $I, q(x) \rightarrow 0$


Figure 5.3: Example 6: The square barrier potentials (5.1.25)
when $x \rightarrow-\infty$. Then, the spectral problem becomes an eigenvalue problem

$$
\phi_{x}^{(+, I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.26}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I)}(x, \lambda) .
$$

According to the Jost solution $\phi(x, \lambda)$, when $x \rightarrow-\infty$, the solutions of the system (5.1.26) will be

$$
\begin{align*}
& \phi_{1}^{(+, I)}(x, \lambda)=e^{-\mathrm{i} \lambda x},  \tag{5.1.27}\\
& \phi_{2}^{(+, I)}(x, \lambda)=0 .
\end{align*}
$$

In region $I I$, the spectral problem is

$$
\phi_{x}^{(+, I I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.28}\\
-k & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I I)}(x, \lambda),
$$

in this case we have the system of equations

$$
\begin{align*}
& \dot{\phi}_{1}^{(+, I I)}(x)=-\mathrm{i} \lambda \phi_{1}^{(+, I I)}(x),  \tag{5.1.29a}\\
& \dot{\phi}_{2}^{(+, I I)}(x)=-k \phi_{1}^{(+, I I)}(x)+\mathrm{i} \lambda \phi_{2}^{(+, I I)}(x) \tag{5.1.29b}
\end{align*}
$$

the solution of (5.1.29) are

$$
\begin{align*}
\phi_{1}^{(+, I I)}(x, \lambda) & =e^{-\mathrm{i} \lambda x},  \tag{5.1.30a}\\
\phi_{2}^{(+, I I)}(x, \lambda) & =e^{\mathrm{i} \lambda x}\left[\frac{k}{2 \mathrm{i} \lambda} e^{-2 \mathrm{i} \lambda x}-\frac{k}{2 \mathrm{i} \lambda} e^{2 \mathrm{i} \lambda \ell_{2}}\right] . \tag{5.1.30b}
\end{align*}
$$

In region $I I I$, the spectral problem is

$$
\phi_{x}^{(+, I I I)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & k  \tag{5.1.31}\\
-k & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I I I)}(x, \lambda) .
$$

We have the system of equations,

$$
\begin{align*}
\dot{\phi}_{1}^{(+, I I I)} & =-\mathrm{i} \lambda \phi_{1}^{(+, I I I)}+k \phi_{2}^{(+, I I I)},  \tag{5.1.32a}\\
\dot{\phi}_{2}^{(+, I I I)} & =-k \phi_{1}^{(+, I I I)}+\mathrm{i} \lambda \phi_{2}^{(+, I I I)} . \tag{5.1.32b}
\end{align*}
$$

Simplifying this system, we obtain:

$$
\begin{gather*}
\phi_{1}^{(+, I I I)}=\frac{\mathrm{i} \lambda}{k} \phi_{2}^{(+, I I I)}-\frac{1}{k} \dot{\phi}_{2}^{(+, I I I)},  \tag{5.1.33a}\\
\ddot{\phi}_{2}^{(+, I I I)}+\left(\lambda^{2}+k^{2}\right) \phi_{2}^{(+, I I I)}=0 . \tag{5.1.33b}
\end{gather*}
$$

The solution to (5.1.33) is

$$
\begin{align*}
& \phi_{1}^{(+, I I I)}(x)=e^{-\mathrm{i} \lambda x},  \tag{5.1.34}\\
& \phi_{2}^{(+, I I I)}(x)=\frac{1}{2} \frac{-\mathrm{i} k e^{-\mathrm{i} \lambda x}+\mathrm{i} e^{\mathrm{i} \lambda x} k e^{2 \mathrm{i} \lambda l 2}}{\lambda} . \tag{5.1.35}
\end{align*}
$$

In region $I V$,

$$
\phi_{x}^{(+, I V)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & k  \tag{5.1.36}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, I V)}(x, \lambda),
$$

and the solutions are

$$
\begin{align*}
\phi_{1}^{(+, I V)}(x, \lambda)= & -\frac{1}{2 \lambda}\left(\frac { 1 } { e ^ { - \mathrm { i } \lambda \ell _ { 1 } } } \left(\mathrm { i } \left(\frac { 1 } { 2 } \frac { 1 } { \sqrt { M } \lambda } \left(k \left(-2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \mathbf{P}\right.\right.\right.\right.\right. \\
& +2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M} \mathbf{P}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}-2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}-4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N} \\
& \left.\left.\left.\left.+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}}\right)\right)\right) k e^{\mathrm{i} \lambda \ell_{1}}\right)+\frac{1}{e^{\mathrm{i} \lambda \ell_{1}}}\left(\mathrm { i } e ^ { - \mathrm { i } \lambda x } \left(2 \mathrm { i } \left(\frac { 1 } { \sqrt { M } \lambda } \left(-\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}\right.\right.\right.\right. \\
& \left.\left.+\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}-2 \mathrm{i} \lambda^{2} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}+2 \lambda \sqrt{M} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{P}-\sqrt{M} e^{\mathrm{i} \lambda \ell_{1}} \lambda\right)\right) \lambda \\
& -\left(\frac { 1 } { 2 \sqrt { M } \lambda } \left(k \left(-2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \mathbf{P}+2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M} \mathbf{P}\right.\right.\right. \\
& +2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}-2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}-4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \\
& \left.\left.\left.\left.\left.\left.-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right)\right) k\right)\right)\right), \tag{5.1.37a}
\end{align*}
$$

and

$$
\begin{align*}
\phi_{2}^{(+, I V)}(x, \lambda)= & \frac{1}{e^{-\mathrm{i} \lambda \ell_{1}}}\left(\left(\frac { 1 } { 2 \sqrt { M } \lambda } \left(k \left(-2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \mathbf{P}+2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M} \mathbf{P}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}\right.\right.\right.\right. \\
& \left.\left.\left.-2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}-4 e^{\mathrm{i} \lambda \ell_{1}} \lambda \mathbf{N}+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right)\right) e^{\mathrm{i} \lambda x}, \tag{5.1.37b}
\end{align*}
$$

where $M=k^{2}+\lambda^{2}, \mathbf{N}=\sin \left(\sqrt{M} \ell_{1}\right) \cos \left(\sqrt{M} \ell_{1}\right)$ and $\mathbf{P}=\cos ^{2}\left(\sqrt{M} \ell_{1}\right)$.
The spectral problem in region $V$ is

$$
\phi_{x}^{(+, V)}(x, \lambda)=\left(\begin{array}{cc}
-\mathrm{i} \lambda & 0  \tag{5.1.38}\\
0 & \mathrm{i} \lambda
\end{array}\right) \phi^{(+, V)}(x, \lambda) .
$$

Then, the solution of $\phi_{1}^{(+, V)}(x, \lambda)$ in (5.1.38) is

$$
\begin{align*}
\phi_{1}^{(+, V)}(x, \lambda)= & \frac{1}{e^{-\mathrm{i} \lambda \ell_{2}}}\left(\left(\frac { 1 } { \lambda } \left(\frac { \mathrm { i } } { 2 } \left(-2 \mathrm{i} e^{-\mathrm{i} \lambda \ell}\left(\frac { - 1 } { \sqrt { M } } \left(\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+2 \mathrm{i} \lambda^{2} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}\right.\right.\right.\right.\right.\right. \\
& \left.\left.-2 \lambda \sqrt{M} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{P}+\lambda \sqrt{M} e^{\mathrm{i} \lambda \ell_{1}}\right)\right)+\left(\frac { k ^ { 2 } } { 2 \sqrt { M } \lambda } \left(2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \mathbf{P}\right.\right. \\
& -2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M} \mathbf{P}-2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \\
& \left.\left.+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right) e^{\mathrm{i} \lambda \ell}+e^{-\mathrm{i} \lambda \ell}\left(-\frac{k^{2}}{2 \sqrt{M} \lambda}\left(2 \mathrm{i} \sqrt{M} e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{P}-2 \mathrm{i} \sqrt{M} e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{P}\right.\right. \\
& -2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \\
& \left.\left.\left.\left.\left.\left.+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right)\right)\right)\right) e^{-\mathrm{i} \lambda x}\right), \tag{5.1.39a}
\end{align*}
$$

where $\boldsymbol{\ell}=\ell_{1}+\ell_{2}$. Analogously, we calculate $a^{-}(\lambda)$ from $\phi^{-}(x, \lambda)$. Then, the eigenfunc-
tion in region $V$, has the form:

$$
\begin{equation*}
\phi_{2}^{(-, V)}(x, \lambda)=\frac{e^{-3 \mathrm{i} \lambda \ell 1} e^{\mathrm{i} \lambda \ell 2}}{\sqrt{M}}(2 \mathrm{i} \lambda \mathbf{N}+2 \sqrt{M} \mathbf{P}-\sqrt{M}) e^{\mathrm{i} \lambda x} \tag{5.1.39b}
\end{equation*}
$$

Then, $a^{ \pm}(\lambda)$ are given by

$$
\begin{align*}
a^{+}(\lambda)= & \frac{1}{e^{-\mathrm{i} \lambda \ell_{2}}}\left(\frac { 1 } { \lambda } \left(\frac { \mathrm { i } } { 2 } \left(-2 \mathrm{i} e^{-\mathrm{i} \lambda \ell}\left(\frac { - 1 } { \sqrt { M } } \left(\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} k^{2} e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+2 \mathrm{i} \lambda^{2} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}\right.\right.\right.\right.\right. \\
& \left.\left.-2 \lambda \sqrt{M} e^{\mathrm{i} \lambda \ell_{1}} \mathbf{P}+\lambda \sqrt{M} e^{\mathrm{i} \lambda \ell_{1}}\right)\right)+\left(\frac { k ^ { 2 } } { 2 \sqrt { M } \lambda } \left(2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \mathbf{P}-2 \mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M} \mathbf{P}\right.\right. \\
& \left.\left.-2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M}+\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right) e^{\mathrm{i} \lambda \ell} \\
& +e^{-\mathrm{i} \lambda \ell}\left(-\frac{k^{2}}{2 \sqrt{M} \lambda}\left(2 \mathrm{i} \sqrt{M} e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{P}-2 \mathrm{i} \sqrt{M} e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{P}\right.\right. \\
& -2 \lambda e^{2 \mathrm{i} \lambda \ell_{1}} \mathbf{N}+2 \lambda e^{2 \mathrm{i} \lambda \ell_{2}} \mathbf{N}+4 \lambda e^{\mathrm{i} \lambda \ell_{1}} \mathbf{N}-\mathrm{i} e^{2 \mathrm{i} \lambda \ell_{1}} \sqrt{M} \\
& \left.\left.\left.\left.\left.+\mathrm{i}^{2 \mathrm{i} \lambda \ell_{2}} \sqrt{M}\right)\right)\right)\right)\right), \tag{5.1.40a}
\end{align*}
$$

and

$$
\begin{equation*}
a^{-}(\lambda)=\frac{e^{-3 \mathrm{i} \lambda \ell 1} e^{\mathrm{i} \lambda \ell 2}}{\sqrt{M}}(2 \mathrm{i} \lambda \mathbf{N}+2 \sqrt{M} \mathbf{P}-\sqrt{M}) \tag{5.1.40b}
\end{equation*}
$$

respectively. Computing the eigenvalues in this case, we expect our solution to blow up


Figure 5.4: (a) and (b) are contour plots of the real and imaginary parts of the solution of $a^{ \pm}(\lambda)=0$ when $k=0.6, \ell_{1}=1, \ell_{2}=3$, respectively. The blue dashed lines represent the imaginary part of equations $\operatorname{Im}\left(\mathrm{a}^{ \pm}(\lambda)\right)$, while the red solid lines are for the real part of the equation $\operatorname{Re}\left(\mathrm{a}^{ \pm}(\lambda)\right)$. The dotted line is the real line $(-\infty<x<\infty)$. Intersections between the blue and red lines represent uniform solutions of $a^{ \pm}(\lambda)=0, \lambda=\lambda_{j}$.
when $x=0$ and

$$
\begin{equation*}
t=\frac{(2 n+1) \pi-(\alpha+\bar{\alpha})}{4\left(\left(\lambda_{1}^{+}\right)^{2}+\left(\lambda_{1}^{-}\right)^{2}\right)}, \quad n \in \mathbb{N} . \tag{5.1.41}
\end{equation*}
$$

### 5.2 Blow up or not to blow up solutions to the nonlocal DNLS equation

The discrete system of equations

$$
\begin{align*}
\mathrm{i} Q_{n, \tau}^{+} & =\left(Q_{n+1}^{+}-2 Q_{n}^{+}+Q_{n-1}^{+}\right)-Q_{n}^{+} Q_{n}^{-}\left(Q_{n+1}^{+}+Q_{n-1}^{+}\right),  \tag{5.2.1}\\
-\mathrm{i} Q_{n, \tau}^{-} & =\left(Q_{n+1}^{-}-2 Q_{n}^{-}+Q_{n-1}^{-}\right)-Q_{n}^{-} Q_{n}^{-}\left(Q_{n+1}^{-}+Q_{n-1}^{-}\right),
\end{align*}
$$

are the compatibility condition (or differential-difference zero curvature) of two operators $L_{n}$ and $M_{n}$ for a $2 \times 2$ function $\Psi_{n}(z, \tau)[2,3]:$

$$
\begin{align*}
\Psi_{n+1}(z, \tau) & =\mathcal{L}_{n}(z, \tau) \Psi_{n}(z, \tau), \quad n \in \mathbb{N}  \tag{5.2.2a}\\
\dot{\Psi}_{n}(z, \tau) & =M_{n}(z, \tau) \Psi_{n}(z, \tau) \tag{5.2.2b}
\end{align*}
$$

where $\Psi_{n+1}(z, \tau)$ is a $2 \times 2$ matrix eigenfunction and $\mathcal{L}_{n}(z, \tau)$ in (5.2.2a) is

$$
\begin{equation*}
\mathcal{L}_{n}(z, \tau)=\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right), \tag{5.2.3a}
\end{equation*}
$$

where

$$
\mathbf{Z}=\left(\begin{array}{cc}
z & 0  \tag{5.2.3b}\\
0 & z^{-1}
\end{array}\right) \quad \text { and } \quad \tilde{\mathbf{Q}}_{n}(\tau)=\left(\begin{array}{cc}
0 & Q_{n}^{+}(\tau) \\
-\left(Q^{+}\right)_{-n}^{*}(\tau) & 0
\end{array}\right)
$$

To make the system simpler, we denoted $Q_{n}^{+}(\tau)=Q_{n}(\tau)$ then,

$$
\tilde{\mathbf{Q}}_{n}(\tau)=\left(\begin{array}{cc}
0 & Q_{n}(\tau)  \tag{5.2.4}\\
-Q_{-n}^{*}(\tau) & 0
\end{array}\right)
$$

The matrix $M_{n}(z, \tau)$ in (5.2.2a) is

$$
M_{n}(z, \tau)=\left(\begin{array}{cc}
-\mathrm{i} Q_{n} Q_{-n+1}^{*}-\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2} & -\mathrm{i}\left(z Q_{n}-z^{-1} Q_{n-1}\right)  \tag{5.2.5}\\
\mathrm{i}\left(-z^{-1} Q_{-n}^{*}+z Q_{-n+1}^{*}\right) & \mathrm{i} Q_{-n}^{*} Q_{n-1}+\frac{\mathrm{i}}{2}\left(z-z^{-1}\right)^{2}
\end{array}\right)
$$

and the zero curvature of (5.2.2) is

$$
\begin{equation*}
M_{n+1} \mathcal{L}_{n}=\dot{\mathcal{L}}_{n}+\mathcal{L}_{n} M_{n} \tag{5.2.6}
\end{equation*}
$$

where dot ${ }^{\circ}$ in (5.2.2b) and (5.2.6) correspond to the derivative with respect to $t$. We shortly introduce the key concepts from chapter 3 that we need in this section. From
(5.2.2a), we have the two solutions when the potential $\left|Q_{n}\right| \rightarrow 0$ as $n \rightarrow \pm \infty$

$$
\begin{align*}
& \psi_{n}(z) \rightarrow\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right), \quad \text { as } \quad n \rightarrow+\infty  \tag{5.2.7a}\\
& \phi_{n}(z) \rightarrow\left(\begin{array}{cc}
z^{n} & 0 \\
0 & z^{-n}
\end{array}\right), \quad \text { as } \quad n \rightarrow-\infty \tag{5.2.7b}
\end{align*}
$$

These solutions are the Jost functions (or solutions) to (5.2.2a). The pairs $\phi_{n}=\left(\phi_{n}^{+}, \phi_{n}^{-}\right)$ and $\psi_{n}=\left(\psi_{n}^{-}, \psi_{n}^{+}\right)$are related to the scattering matrix $T(z)$, i.e,

$$
\begin{equation*}
\phi_{n}(z)=\psi_{n}(z) T(z), \quad \text { when } \quad|z|=1 \tag{5.2.8a}
\end{equation*}
$$

where

$$
T(z)=\left(\begin{array}{cc}
a^{+}(z) & -b^{-}(z)  \tag{5.2.8b}\\
b^{+}(z) & a^{-}(z)
\end{array}\right)
$$

The operator (5.2.2b) determines the evolution of the Jost solutions, which leads us to the time evolution of the component of the scattering matrix (6.1.14). For more information see [12]. Since $\tilde{\mathbf{Q}}_{n}(\tau) \rightarrow 0$ as $n \rightarrow \pm \infty$, one gets

$$
\begin{align*}
& a^{ \pm}(z, \tau)=a^{ \pm}(z, 0),  \tag{5.2.9a}\\
& b^{ \pm}(z, \tau)=e^{ \pm 2 i \omega \tau} b^{ \pm}(z, 0), \tag{5.2.9b}
\end{align*}
$$

where $\omega^{ \pm}=\frac{1}{2}\left(z^{ \pm}-\left(z^{ \pm}\right)^{-1}\right)^{2}$. Ablowitz et al. [9, 12] used a generalised Riemann-Hilbert boundary value problem to find the soliton solution to the nonlocal NLS equation (5.0.1b). The eigenvalues in the nonlocal NLS equation appear in pairs $\left\{z_{j}^{+}, z_{j}^{+, *}\right\}$ and $\left\{z_{j}^{-}, z_{j}^{-, *}\right\}$, where $j \in \mathbb{N}$; $\mathbb{N}$ being the number of eigenvalues. For a one - soliton solution, they present real eigenvalues $z_{1}^{+}>1$ and $0<z_{1}^{-}<1$, and the solution has the following form,

$$
\begin{equation*}
Q_{n, 1}(\tau)=\frac{-2 C_{1}^{-}\left(z_{1}^{-}\right)^{2 n}}{1+4 C_{1}^{+} C_{1}^{-}\left(z_{1}^{+}\right)^{-2 n}\left(z_{1}^{-}\right)^{2(n+1)}\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right)^{-2}}, \tag{5.2.10}
\end{equation*}
$$

where $C_{1}^{ \pm}(z, \tau)$ are called norming constant, $C_{1}^{+}=\frac{b_{1}^{+}(z, \tau)}{\dot{a}_{1}^{+}(z, 0)}=\frac{z_{1}^{+}\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right) e^{2 i \omega_{1}^{+} \tau} e^{i \beta}}{2 z_{1}^{-}}$and $C_{1}^{-}=\frac{b_{1}^{-}(z, \tau)}{\bar{a}_{1}^{-}(z, 0)}=\frac{\left(\left(z_{1}^{+}\right)^{2}-\left(z_{1}^{-}\right)^{2}\right) e^{-2 i \omega_{1}^{-} \tau} e^{i \bar{\beta}}}{2 z_{1}^{+} z_{1}^{-}}, \omega_{1}^{ \pm}=\frac{1}{2}\left(z_{1}^{ \pm}-\left(z_{1}^{ \pm}\right)^{-1}\right)^{2}$, with $\beta, \bar{\beta}$ both being real and positive.

Upon inspection, the one soliton will not blow up when the eigenvalues satisfy the relation $z_{j}^{+}=1 / z_{j}^{-}$. Here, we conjecture that such a condition will be met by initial conditions that are mirror symmetric with respect to the vertical axis or those that are $\mathcal{P J}$-symmetric at $t=0$. In the following section, we will illustrate the conjecture through
analysing specific cases.

### 5.3 Eigenvalues of square barrier potentials with numerical results

The eigenvalues of the spectral problem correspond to the zeros of $a^{ \pm}(z)$ with $|z| \neq 1$. We will use the iteration method on equation (5.2.2a), to calculate the zeros of the scattering data $a^{ \pm}(z)$. Since we defined that $\phi_{n}(z)$ is a solution of (5.2.2a), then

$$
\begin{equation*}
\phi_{n+1}(z, t)=\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{n}\right) \phi_{n}(z) \tag{5.3.1}
\end{equation*}
$$

First, consider a one-site excitation

$$
Q_{n}(0)= \begin{cases}k, & n=n_{0}, \quad k \in \mathbb{C}  \tag{5.3.2}\\ 0, & \text { otherwise }\end{cases}
$$

Example 7 When $n_{0}=0$, we have

$$
\tilde{\mathbf{Q}}_{0}(0)=\left(\begin{array}{cc}
0 & Q_{0}  \tag{5.3.3}\\
-Q_{0}^{*} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & k \\
-k & 0
\end{array}\right), \quad Q_{n \neq 0}(0)=\mathbf{0} .
$$

Since, as $n \rightarrow-\infty, \phi_{n}=\left(\begin{array}{cc}z^{n} & 0 \\ 0 & z^{-n}\end{array}\right)$, we will start with $n=0$ in equation (5.3.1). Then,

$$
\begin{gather*}
\phi_{1}(z)=\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{0}\right) \phi_{0}(z)=\left(\begin{array}{cc}
z & k \\
-k & z^{-1}
\end{array}\right)  \tag{5.3.4a}\\
\phi_{2}(z)=\left(\mathbf{Z}+\tilde{\mathbf{Q}}_{1}\right) \phi_{1}(z)=\left(\begin{array}{cc}
z^{2} & z k \\
-z^{-1} k & z^{-2}
\end{array}\right),  \tag{5.3.4b}\\
\vdots  \tag{5.3.5}\\
\phi_{n}(z)=\left(\begin{array}{cc}
z^{n} & z^{n-1} k \\
z^{-(n-1)} k & z^{-n}
\end{array}\right)=\mathbf{Z}^{n}\left(\mathbb{1}+\mathbf{Z}^{-1} \tilde{\mathbf{Q}}_{0}\right),
\end{gather*}
$$

where $\mathbb{1}$ is an identity $2 \times 2$ matrix. Comparing (5.3.5) with (6.1.14), we can say that $T(z)=\left(\mathbb{1}+\mathbf{Z}^{-1} \tilde{\mathbf{Q}}_{0}\right)$. Since we are looking for the zeros of the function $a^{ \pm}(z)$, which is on the diagonal part of the matrix $T(z)=\mathbb{1}$, we obtain that $a^{ \pm}(z)=1$. Hence, the initial
condition (6.2.1) with $n_{0}=0$ cannot generate solitons. This is in agrement with the local case [39], as considering solutions of (5.0.1b) with the symmetry $Q_{n}(t)=Q_{-n}(t)$ will make the nonlocal system local.

Example 8 When $n_{0}=1$, we have

$$
\begin{align*}
Q_{-1}(0) & =\left(\begin{array}{cc}
0 & 0 \\
-k & 0
\end{array}\right),  \tag{5.3.6a}\\
Q_{1}(0) & =\left(\begin{array}{ll}
0 & k \\
0 & 0
\end{array}\right), \quad Q_{n \neq\{-1,1\}}(0)=\mathbf{0} \tag{5.3.6b}
\end{align*}
$$

Using the same argument, we obtain:

$$
\begin{equation*}
T(z)=\left(\mathbb{1}+\mathbf{Z}^{-2} \tilde{\mathbf{Q}}_{1} \mathbf{Z} \tilde{\mathbf{Q}}_{-1} \mathbf{Z}^{-1}\right) \tag{5.3.7}
\end{equation*}
$$

and hence

$$
\begin{equation*}
a^{+}(z)=z^{4}-k^{2}, \quad a^{-}(z)=\frac{1}{z^{4}} \tag{5.3.8}
\end{equation*}
$$

Under the condition $|z|>1$, we obtain the zeros of $a^{+}(z)$ when $z^{+}= \pm \sqrt{ \pm k}$ at $|k|>1$. On the other hand, $a^{-}(z)$ has no solution under the condition $|z|<1$. In this case, we expect our solution to blow up because at $n=0$, the denominator of (5.2.10) will be zero at a finite time (see Fig. 5.5a and 5.5b).

Example 9 We consider here, two-site excitations $n_{0}=0,1$ that is

$$
Q_{n}= \begin{cases}k, & n=0,1  \tag{5.3.9}\\ 0, & \text { otherwise }\end{cases}
$$

The same calculations as before yield

$$
\begin{equation*}
a^{+}(z)=z^{4}-2 k^{2} z^{2}-k^{2}, \quad a^{-}(z)=\frac{1}{z^{6}}, \tag{5.3.10}
\end{equation*}
$$

which give us $z^{+}= \pm \sqrt{ \pm k \sqrt{k^{2}+1}+k^{2}}$ and no $z^{-}$. Hence, we should obtain a blow up solution to equation (5.2.10).

Example 10 Considering another two excitations of the form:

$$
Q_{n}=k \times \begin{cases}1, & n=-1  \tag{5.3.11}\\ e^{\mathrm{i} \theta}, & n=1 \\ 0, & \text { otherwise }\end{cases}
$$

we obtain $a^{+}(z)=z^{4}-k^{2}$ and $a^{-}(z)=-z^{4} k^{2}+1$, from which we obtain:

$$
\begin{equation*}
z^{+}= \pm \sqrt{ \pm k}, \quad z^{-}= \pm \frac{1}{\sqrt{ \pm k}} \tag{5.3.12}
\end{equation*}
$$

i.e., the eigenvalues are independent of the phase $\theta$. Note that for each $z^{+}$, there is a reciprocal eigenvalue $z^{-}$, i.e., $z^{+}=1 / z^{-}$. Using numerical simulation, we will see below that depending on the phase, the corresponding solution can be bounded or blow up.

Example 11 Finally, we consider three excitations ( $n_{0}=-1,0,1$ ) of the form:

$$
Q_{n}= \begin{cases}k+\mathrm{i} \alpha_{n}, & n=-1,0,1  \tag{5.3.13}\\ 0, & \text { otherwise }\end{cases}
$$

where without loss of generality $\alpha_{0}=0$. We obtain $a^{+}(z)=z^{4}-2 k^{2} z^{2}-k^{2}$ and $a^{-}(z)=-k^{2} z^{4}-2 k^{2} z^{2}+1$, such that

$$
\begin{equation*}
z^{+}= \pm \sqrt{k \sqrt{k^{2}+1}+k^{2}}, \quad, z^{-}= \pm \frac{\sqrt{k\left(-k+\sqrt{k^{2}+1}\right)}}{k} . \tag{5.3.14}
\end{equation*}
$$

This implies that no blow up will be obtained if and only if $\alpha_{-1}= \pm \alpha_{1}$.

### 5.4 Numerical simulations

To compare our analytical results with the numerics, first we solve (5.2.2a) for the spectrum of the Lax operator for a given initial condition. To do so, we rewrite the equation into the generalised eigenvalue problem

$$
z\left(\begin{array}{cc}
\mathbb{1} & 0  \tag{5.4.1}\\
\boldsymbol{Q}_{-n}^{*} & \mathbb{1}_{+}
\end{array}\right) \boldsymbol{\Psi}=\left(\begin{array}{cc}
\mathbb{1}_{+} & -\boldsymbol{Q}_{n} \\
0 & \mathbb{1}
\end{array}\right) \boldsymbol{\Psi}
$$

where $\mathbb{1}=\delta_{i, j}$ is the identity matrix and $\mathbb{1}_{+}=\delta_{i, j-1}$ is an off-diagonal "identity" one. A standard eigenvalue solver in Matlab is then used to determine the spectrum $z$. Time integrations of (5.0.1b) for the initial condition are then obtained numerically using the fourth order Runge-Kutta method.

First, we consider the initial condition (5.3.6) with $n_{0}=1$. Note that $n_{0}=0$ yields trivial dispersion. In Fig. 5.5a, we plot the spectrum, which in agreement with the analytical calculations shows the presence of eigenvalues outside the unit circle and none inside. The location of the eigenvalues coincides with the analytical results. We show in Fig. 5.5b the dynamics of the initial condition, where, as conjectured, the soliton collapses.


Figure 5.5: (a) The spectrum in the complex plane for the initial condition (6.2.1) with $n_{0}=1$ and $k=1.4$. The dots outside the unit circle are the solutions of $a^{+}(z)=0$. (b) Time evolution for the initial condition where the blow up solution of (5.2.10) can be seen at a later time.

(a) The solutions of $a^{+}(z)=0$ in the analytic region $|z|>1$ and the solutions of $a^{-}(z)=0$ in the analytic region $|z|<1$.

(b) Time evolution of the initial condition (5.3.11) .

Figure 5.6: (a) The spectrum in the complex plane for the initial condition (5.3.11) with $k=1.4$ and $\theta=1$. The dots outside and inside the unit circle are the solutions of $a^{ \pm}(z)=0$, respectively. (b) Time dynamics of the initial condition where a stable soliton breather solution can be clearly seen to (5.2.10).


Figure 5.7: Time evolution of the initial condition (5.3.11) with $k=1.4$ and $\theta=3.1$.

Next, we study the initial conditions (5.3.9) and (5.3.11). We do not present the former because it yields a rapid blow up at finite time. The latter initial condition with $\theta=0$ will yield the same dynamics as the local NLS equation. In Fig. 5.6 we present the spectrum and time dynamics of the latter initial condition for $\theta=1$. It is clear that the solution is still bounded, despite the fact that it is no longer mirror symmetric.

An interesting result is presented in Fig. 5.7, for $\theta=3.1$, (note that the spectrum of the Lax operator is still the same as in Fig. 5.6a) for which the oscillation has a higher amplitude than with $\theta=1$. When $\theta=\pi$, we obtain a blow up solution. This is an example where the reciprocality of eigenvalues outside and inside the unit circle is necessary, but not sufficient.

### 5.5 Conclusion

Our study has examined the possibility for the creation of solitons in the discrete integrable lattices of the nonlocal NLS equation type with or without a blow up. Our conjecture has based on the box type initial conditions, which were successfully tested in numerical simulations. To sum up, we conjectured that: 1) if the Lax operator has no spectrum outside nor inside the unit circle, there is no blow up; 2) when it does have a spectrum outside or inside the unit circle, mirror symmetric initial conditions are sufficient, but not necessary, for bounded solutions; 3) to obtain bounded solutions, each spectrum outside the unit circle needs (but is not sufficient) to have a reciprocal counterpart on the inside. When the initial condition for the potential is asymmetric, then initial
excitations with power above some threshold will lead to blow up solutions. Numerical simulations supporting the conjecture have been presented.

The next question will be to provide a rigorous proof of the conjecture, which will be addressed in future work.

## Chapter 6

## Discrete Manakov nonlocal nonlinear Schrödinger equations

### 6.1 Introduction

The most commonly used mathematical model and completely integrable nonlinear PDE is the NLS equation (1.1.3a) [40,54, 93, 120]. Here, $q(x, t)$ is a complex valued function tending fast enough to zero as $|x| \rightarrow \infty$ [12, 54]. Furthermore, the NLS arises in more than one area in Physics, such as in the evolution of small amplitude slowly varying wave packets in deep water, nonlinear optics and Plasma Physics [2, 10, 11]. The AblowitzLadik equation [3], which is an integrable form of the discretised NLS equation (recall equation (1.6.3)), has solutions on a zero background level(soliton)

$$
\begin{equation*}
\mathrm{i} Q_{n, \tau}=Q_{n+1}-2 Q_{n}+Q_{n-1}+\left|Q_{n}\right|^{2}\left(Q_{n+1}+Q_{n-1}\right) \tag{6.1.1}
\end{equation*}
$$

Here, $\tau$ is the continuous evolution variable and $n=0, \pm 1, \pm 2, \ldots$ are integers. The inverse scattering technique with zero boundary condition for equations (1.1.3a) and (6.1.1), have been developed in $[3,9,10,70,92,93]$. At the same time, the inverse scattering technique for non-vanishing boundary conditions has been developed as well in [31, 98]. In recent years there has been wide interest in the study of soliton solutions of certain discrete systems associated with the vector extensions of the NLS. Manakov in [83] showed that the vector NLS, is

$$
\begin{equation*}
\mathrm{i} \mathbf{q}_{t}=\mathbf{q}_{x x}+2\|\mathbf{q}\|^{2} \mathbf{q}(x, t) \tag{6.1.2}
\end{equation*}
$$

where $\mathbf{q}$ is a $P$-component vector and $\|$.$\| denotes the vector norm. Equation (6.1.2)$ also possessed solitons and could be integrated via the IST. When $P=2$, $\mathbf{q}$ corresponds to components of the electric field and this is relevant in the study of electromagnetic waves in optical media. In this case, when $P=2$ equation (6.1.2) is called the Manakov
equation and is presented as the first integrable multi-component generalisation of the scalar NLS equation (1.1.3a)

$$
\begin{align*}
& \mathrm{i} q_{1, t}+q_{1, x x}+2\left(\left|q_{1}\right|^{2}+\left|q_{2}\right|^{2}\right) q_{1}(x, t)=0,  \tag{6.1.3a}\\
& \mathrm{i} q_{2, t}+q_{1, x x}+2\left(\left|q_{1}\right|^{2}+\left|q_{2}\right|^{2}\right) q_{2}(x, t)=0 . \tag{6.1.3b}
\end{align*}
$$

Equation (6.1.3) are associated with a scattering problem of the ZS type in [83, 93]. It was proposed by S. V. Manakov as an asymptotic model for the propagation of the electric field in a waveguide. In this chapter, we are dealing with the discrete vector NLS. The choice of discretisation for vector NLS (6.1.2) is either symmetric or asymmetric. These discretisations are as follows:

Symmetric discretisation

$$
\begin{equation*}
\mathrm{i} \mathbf{Q}_{n, \tau}=\mathbf{Q}_{n-1}-2 \mathbf{Q}_{n}+\mathbf{Q}_{n+1}+\mathbf{Q}_{n} \mathbf{Q}_{-n}^{*}\left(\mathbf{Q}_{n+1}+\mathbf{Q}_{n-1}\right), \tag{6.1.4}
\end{equation*}
$$

where $\mathbf{Q}_{n}$, is a P-component vector. The general form of $\mathbf{Q}_{n}(\tau)$ is $\mathbf{Q}_{n}=\left(\begin{array}{cc}0 & \mathbf{Q}_{n}^{+} \\ \mathbf{Q}_{n}^{-} & \mathbf{0}\end{array}\right)$.
Asymmetric discretisation

$$
\begin{align*}
& \mathrm{i} \mathbf{Q}_{n, \tau}^{+}=\mathbf{Q}_{n-1}^{+}-2 \mathbf{Q}_{n}^{+}+\mathbf{Q}_{n+1}^{+}-\mathbf{Q}_{n}^{+} \mathbf{Q}_{n}^{-}\left(\mathbf{Q}_{n+1}^{+}+\mathbf{Q}_{n-1}^{+}\right),  \tag{6.1.5a}\\
& \mathrm{i} \mathbf{Q}_{n, \tau}^{-}=\mathbf{Q}_{n-1}^{-}-2 \mathbf{Q}_{n}^{-}+\mathbf{Q}_{n+1}^{-}-\mathbf{Q}_{n}^{+} \mathbf{Q}_{n}^{-}\left(\mathbf{Q}_{n+1}^{-}+\mathbf{Q}_{n-1}^{-}\right), \tag{6.1.5b}
\end{align*}
$$

where $\mathbf{Q}_{n}^{+}$and $\mathbf{Q}_{n}^{-}$are $N \times M$ and $M \times N$ matrices, respectively. Using the symmetry condition (6.1.6), equations (6.1.5) become local multi-component NLS (MNLS) equation discrete type (6.1.7)

$$
\begin{gather*}
\mathbf{Q}_{n}^{-}(\tau)=-\mathbf{B}_{-}\left(\mathbf{Q}_{n}^{+}\right)^{\dagger}\left(\mathbf{B}_{+}\right)^{-1}, \quad \mathbf{B}=\left(\begin{array}{cc}
\mathbf{B}_{+} & 0 \\
0 & \mathbf{B}_{-}
\end{array}\right),  \tag{6.1.6}\\
\mathrm{i} \mathbf{Q}_{n, \tau}=\mathbf{Q}_{n-1}-2 \mathbf{Q}_{n}+\mathbf{Q}_{n+1}+\left\|\mathbf{Q}_{n}\right\|^{2}\left(\mathbf{Q}_{n+1}+\mathbf{Q}_{n-1}\right) . \tag{6.1.7}
\end{gather*}
$$

Here, we assumed that blocks $\mathbf{B}_{+}$and $\mathbf{B}_{-}$are nonsingular matrices and the superscript $\dagger$ denotes the transpose complex conjugate. We also concern with the nonlocal integrable discrete MNLS equation for such initial conditions. For this we need the nonlocal condition on the potentials $Q_{n}^{ \pm}(\tau)$,

$$
\begin{equation*}
\mathbf{Q}_{n}^{-}(\tau)=-\mathbf{B}_{-}\left(\mathbf{Q}_{-n}^{+}\right)^{\dagger}\left(\mathbf{B}_{+}\right)^{-1} \tag{6.1.8}
\end{equation*}
$$

where $N=1, M=2$, and a matrix $\mathbf{B}$ will be

$$
B=\left(\begin{array}{c|cc}
1 & 0 & 0  \tag{6.1.9}\\
\hline 0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \mathbf{B}_{+}=1, \quad \mathbf{B}_{-}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

The particular choice in equation (6.1.4) is when $P=2$, so $\mathbf{Q}_{n}^{+}=\left(U_{n}, V_{n}\right)$. By applying the condition (6.1.8), the nonlocal discrete MNLS model and focusing type has the following form

$$
\begin{align*}
\mathrm{i} U_{n, \tau} & =\left(U_{n+1}-2 U_{n}+U_{n-1}\right)+\left(U_{n} U_{-n}^{*}+V_{n} V_{-n}^{*}\right)\left(U_{n+1}+U_{n-1}\right),  \tag{6.1.10}\\
-\mathrm{i} V_{n, \tau} & =\left(V_{n+1}-2 V_{n}+V_{n-1}\right)+\left(U_{n} U_{-n}^{*}+V_{n} V_{-n}^{*}\right)\left(V_{n+1}+V_{n-1}\right) .
\end{align*}
$$

The symmetric system (6.1.10) is associated with a linear operator pair and can be written as

$$
\begin{equation*}
\Psi_{n+1}=\left(\mathbf{Z}+\mathbf{Q}_{n}\right) \Psi_{n}, \tag{6.1.11}
\end{equation*}
$$

with

$$
\frac{\mathrm{d} \Psi_{n}}{\mathrm{~d} \tau}=
$$

$$
\left(\begin{array}{ccc}
-\mathrm{i}\left(U_{n} U_{-(n-1)}^{*}+V_{n} V_{-(n-1)}^{*}\right)-\frac{2}{3} \mathrm{i}\left(z-z^{-1}\right)^{2} & -\mathrm{i}\left(z U_{n}-z^{-1} U_{n-1}\right) & -\mathrm{i}\left(z V_{n}-z^{-1} V_{n-1}\right)  \tag{6.1.12a}\\
\mathrm{i}\left(-z^{-1} U_{-n}^{*}+z U_{-(n-1)}^{*}\right) & \mathrm{i} U_{n-1} U_{-n}^{*}+\frac{1}{3} \mathrm{i}\left(z-z^{-1}\right)^{2} & \mathrm{i} U_{-n}^{*} V_{n-1} \\
\mathrm{i}\left(-z^{-1} V_{-n}^{*}+z V_{-(n-1)}^{*}\right) & \mathrm{i} U_{n-1} V_{-n}^{*} & \mathrm{i} V_{n-1} V_{-n}^{*}+\frac{1}{3} \mathrm{i}\left(z-z^{-1}\right)^{2}
\end{array}\right),
$$

where

$$
\mathbf{Z}=\left(\begin{array}{ccc}
z & 0 & 0  \tag{6.1.12b}\\
0 & z^{-1} & 0 \\
0 & 0 & z^{-1}
\end{array}\right), \quad \mathbf{Q}_{n}=\left(\begin{array}{ccc}
0 & U_{n} & V_{n} \\
-U_{-n}^{*} & 0 & 0 \\
-V_{-n}^{*} & 0 & 0
\end{array}\right)
$$

We refer to solutions of the discrete scattering problem (6.1.11) as Jost functions with respect to the parameter $z$. When the potentials $\left|U_{n}\right|,\left|V_{n}\right| \rightarrow 0$ as $n \rightarrow \mp \infty$, the Jost functions are asymptotic to the solutions of

$$
\Psi_{n+1}=\left(\begin{array}{ccc}
z^{n} & 0 & 0  \tag{6.1.13a}\\
0 & z^{-n} & 0 \\
0 & 0 & z^{-n}
\end{array}\right) \Psi_{n}
$$

Therefore, the Jost functions defined by the following boundary conditions

$$
\begin{align*}
& \psi_{n}(z)=\left(\begin{array}{ccc}
z^{n} & 0 & 0 \\
0 & z^{-n} & 0 \\
0 & 0 & z^{-n}
\end{array}\right), \quad \text { as } \quad n \rightarrow+\infty  \tag{6.1.13b}\\
& \phi_{n}(z)=\left(\begin{array}{ccc}
z^{n} & 0 & 0 \\
0 & z^{-n} & 0 \\
0 & 0 & z^{-n}
\end{array}\right), \quad \text { as } n \rightarrow-\infty
\end{align*}
$$

the pairs $\phi_{n}(z)=\left(\phi_{n}^{+}, \phi_{n}^{-}\right)$and $\psi_{n}(z)=\left(\psi_{n}^{-}, \psi_{n}^{+}\right)$are linearly dependent and one can write them as linear combinations. The coefficients of these linear combinations depend on $z$. Then, the following relation holds on $|z|=1$ and defines the scattering coefficients $a^{+}(z), \mathbf{a}^{-}(z), \mathbf{b}^{+}(z)$, and $\mathbf{b}^{-}(z)$

$$
\phi_{n}(z)=\psi_{n}(z) T(z), \quad T(z)=\left(\begin{array}{cc}
a^{+}(z) & -\mathbf{b}^{-}(z)  \tag{6.1.14}\\
\mathbf{b}^{+}(z) & \mathbf{a}^{-}(z)
\end{array}\right), \quad \text { when } \quad|z|=1
$$

and when $z=z_{j}$ then, $a^{+}\left(z_{j}\right)=0=\mathbf{a}^{-}\left(z_{j}\right)$ and

$$
\begin{equation*}
\phi_{n}^{ \pm}\left(z_{j}\right)= \pm \mathbf{b}^{ \pm}\left(z_{j}\right) \psi_{n}^{ \pm}\left(z_{j}\right), \tag{6.1.15}
\end{equation*}
$$

$z_{j}$ is an eigenvalue of (6.1.11). Here, $a^{+}(z)$ is a single element but $\mathbf{a}^{-}(z)$ is a block $2 \times 2$ matrix, $\mathbf{b}^{-}(z)$ is a block $1 \times 2$ matrix and $\mathbf{b}^{+}(z)$ is a block $2 \times 1$ matrix. The time evolution of the Jost solution for the nonlocal discrete MNLS equation can be determined as in [12]

$$
\begin{align*}
& a_{\tau}^{+}(z, \tau)=0, \quad \mathbf{a}_{\tau}^{-}(z, \tau)=0, \\
& a^{+}(z, \tau)=a^{+}(z, 0), \quad \mathbf{a}^{-}(z, \tau)=\mathbf{a}^{-}(z, 0),  \tag{6.1.16a}\\
& \mp 2 \mathrm{i} 7 \mathbf{b}^{ \pm}(\mathrm{z}, \tau)+\mathbf{b}_{\tau}^{ \pm}(\mathrm{z}, \tau)=0, \quad \quad \mathbf{b}^{ \pm}(z, \tau)=e^{ \pm 2 \mathrm{i} \omega \tau} \mathbf{b}^{ \pm}(z, 0) . \tag{6.1.16b}
\end{align*}
$$

### 6.2 Square barrier potentials with nonlocal reduction

In this section, we use specific initial conditions for the potential for which the theory of $\mathcal{P T}$-symmetry is applicable. In [9, 101], the authors used the spectral problem, for the continuous nonlocal NLS equation, with the boundary conditions for the case of the so-called single box initial data to calculate the coefficients of the Jost solutions $a^{ \pm}(\lambda)$. In [39, 76], the authors introduced the square barrier initial potentials for the local DNLS
system in single component as well as in the discrete MNLS equation .
In this section we will find the the critical point of three examples of square barrier potentials to determine the discrete eigenvalues $z_{k}^{ \pm}$of the discrete Manakov vector NLS equation which corresponds to the system (6.1.11). The strategy is to use the iteration method, which uses the asymptotic behaviour for the Jost solutions. This will lead to the coefficients of the Jost solutions of $a^{+}(z)$ and $\mathbf{a}^{-}(z)$. We related the potentials in each example to the type of the solution that we are expecting. We have defined two sufficient conditions that have no blow up solutions to equation (6.1.10). The first condition is when the potentials satisfy the parity condition ( $\mathcal{P}$ for parity), which means $V_{n}=U_{-n}$. The second condition is a relation between the solutions of the scattering data $a^{+}(z)$ and $\mathbf{a}^{-}(z)$. The solutions of the determinant of $\mathbf{a}^{-}(z)$ must be the inverse of the solutions of $a^{+}(z)$. In order to explain these conditions the following examples will show the type of solution between $a^{+}(z)$ and $\mathbf{a}^{-}(z)$. Here, we will start using the spectral problem to calculate the zeros of the scattering data $a^{+}(z)$ and $\mathbf{a}^{-}(z)$. Note that $\mathbf{a}^{-}(z)$ is a $2 \times 2$ matrix and that we will find the zeros of $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$.

### 6.2.1 Eigenvalues of square barrier potentials with numerical results

We will use numerical method to find the eigenvalues of the spectral problem correspond to the zeros of $a^{+}(z)$ and $\mathbf{a}^{-}(z)$ with $|z| \neq 1$. We will use the iteration method on equation (6.1.11), to calculate the zeros of the scattering data $a^{+}(z)$ and $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$.

Example 12 We consider $n_{0}=-1, v_{0}=1$. We have

$$
U_{n}(0)=\left\{\begin{array}{cc}
k, & n=n_{0}, \quad k \in \mathbb{C},  \tag{6.2.1}\\
0, & \text { otherwise }
\end{array} \quad V_{n}(0)=\left\{\begin{array}{cc}
k, & n=v_{0}, \quad k \in \mathbb{C} \\
0, & \text { otherwise }
\end{array}\right.\right.
$$

In this case, according to the position $n$, the potentials have the following forms

$$
\mathbf{Q}_{-1}=\left(\begin{array}{ccc}
0 & U_{-1} & 0  \tag{6.2.2}\\
0 & 0 & 0 \\
-V_{1}^{*} & 0 & 0
\end{array}\right), \quad \mathbf{Q}_{1}=\left(\begin{array}{ccc}
0 & 0 & V_{1} \\
-U_{-1}^{*} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

We will start with $n=-1$ in equation (6.1.11) and as $n \rightarrow-\infty$, the eigenfunction $\phi_{n}(z)$ read

$$
\phi_{n}(z)=\left(\begin{array}{ccc}
z^{n} & 0 & 0  \tag{6.2.3a}\\
0 & z^{n} & 0 \\
0 & 0 & z^{n}
\end{array}\right)
$$

then, the algorithm to find the eigenvalues will start as follows:

$$
\begin{align*}
n=-1: \quad \phi_{0}(z) & =\left(\mathbf{Z}+\mathbf{Q}_{-1}\right) \phi_{-1}(z)  \tag{6.2.3b}\\
& \phi_{0}(z)=\left(\begin{array}{ccc}
1 & z k & 0 \\
0 & 1 & 0 \\
-z^{-1} k & 0 & 1
\end{array}\right),  \tag{6.2.3c}\\
& \vdots \\
& \phi_{2}(z)=\left(\begin{array}{ccc}
z^{2}-\frac{k^{2}}{z^{2}} & k z^{3} & \frac{k}{z} \\
-k z & -k^{2} z^{2}+z^{-2} & 0 \\
-\frac{k}{z^{3}} & 0 & z^{-2}
\end{array}\right), \tag{6.2.3d}
\end{align*}
$$

$\phi_{2}(z)$ can be written also as a matrix form

$$
\begin{equation*}
\phi_{2}(z)=\left(\mathbf{Z}+\mathbf{Q}_{1}\right) \phi_{1}(z), \tag{6.2.3e}
\end{equation*}
$$

where $\phi_{1}(z)=\mathbf{Z}+\mathbf{Z} \mathbf{Q}_{1} \mathbf{Z}^{-1}$ and $\mathbf{Q}_{1}$ is defined in (6.2.2), so

$$
\begin{equation*}
\phi_{2}(z)=\mathbf{Z}^{2}\left(\mathbb{1}+\mathbf{Q}_{-1} \mathbf{Z}^{-1}+\mathbf{Z}^{-2} \mathbf{Q}_{1} \mathbf{Z}+\mathbf{Z}^{-2} \mathbf{Q}_{1} \mathbf{Z} \mathbf{Q}_{-1} \mathbf{Z}^{-1}\right), \tag{6.2.4a}
\end{equation*}
$$

where $\mathbb{1}$ is a $3 \times 3$ matrix. Now, if we compare equation (6.2.4) with equation (6.1.14), the $T(z)$ matrix is the terms between the brackets. Since the $a^{+}(z)$ and $\mathbf{a}^{-}(z)$ are allocated in the diagonal parts of $T(z)$, so, we need the matrices which have diagonal terms only. Then, the diagonal of $T(z)$ :

$$
\begin{equation*}
\operatorname{diag} \cdot T(z)=\left(\mathbb{1}+\mathbf{Z}^{-2} \mathbf{Q}_{1} \mathbf{Z} \mathbf{Q}_{-1} \mathbf{Z}^{-1}\right) \tag{6.2.4b}
\end{equation*}
$$

it is easy to find out the $a^{+}(z)$ is

$$
\begin{equation*}
a^{+}(z)=z^{4}-k^{2}, \tag{6.2.5a}
\end{equation*}
$$

taking into mind the $\mathbf{a}^{-}(z)$ function is a $2 \times 2$ matrix. To find the solution of $\mathbf{a}^{-}(z)$, we need to take the determinant of $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$. Then, the $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ function is

$$
\begin{equation*}
\operatorname{det}\left(\mathbf{a}^{-}(z)\right)=1-z^{4} k^{2} \tag{6.2.5b}
\end{equation*}
$$

Next, we will find the solutions for both $a^{+}(z)$ and $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ functions. The scattering data $a^{+}(z)$ function has four solutions. Since $a^{+}(z)$ is analytic when $|z|>1$, so the solutions of $a^{+}(z)=0$ when $z= \pm \sqrt{ \pm k}$. Then, the eigenvalues of the system (6.1.11)
will be when $|k|>1$ and $n_{0}=-1, v_{0}=1$ (see Fig. 6.1). Since $\mathbf{a}^{-}(z)$ is analytic when $|z|<1$, we find that when $|k|<1$, the solutions of $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ are $z= \pm \frac{1}{\sqrt{ \pm k}}$. As we can see from the solutions of both functions $a^{+}(z)$ and $\mathbf{a}^{-}(z)$, the number of the eigenvalues outside the unit circle is the same as inside the unit circle and the solutions of $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ are the inverse of the solutions of $a^{+}(z)$.

Example 13 Next, we consider the example where $n_{0}=2, v_{0}=-2$ :

$$
\begin{array}{cc}
U_{n}(0)=\left\{\begin{array}{cc}
k, & n=2, \\
0, & \text { otherwise },
\end{array}\right. & V_{n}(0)=\left\{\begin{array}{cc}
k, & n=-2, \\
0, & \text { otherwise },
\end{array}\right. \\
\mathbf{Q}_{-2}=\left(\begin{array}{ccc}
0 & 0 & V_{-2} \\
-U_{2}^{*} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \mathbf{Q}_{2}=\left(\begin{array}{ccc}
0 & U_{2} & 0 \\
0 & 0 & 0 \\
-V_{2}^{*} & 0 & 0
\end{array}\right) . \tag{6.2.6b}
\end{array}
$$

In this case we will start with $n=-2$ in equation (6.1.11). Then, the $a^{+}(z)$ and $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ functions read

$$
\begin{align*}
a^{+}(z) & =z^{8}-k^{2},  \tag{6.2.7a}\\
\operatorname{det}\left(\mathbf{a}^{-}(z)\right) & =1-z^{8} k^{2} . \tag{6.2.7b}
\end{align*}
$$

The $a^{+}(z)$ function has eight solutions. Here, the $a^{+}(z)$ function is zeros when $z=$ $\pm k^{1 / 4}, z= \pm(-k)^{1 / 4}, z= \pm \mathrm{i}(k)^{1 / 4}$ and $z= \pm \mathrm{i}(-k)^{1 / 4}$. Under the condition $|z|>1$, the soliton solution will be when $|k|>1$ (see Fig. 6.2). Furthermore, the zeros of $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ are $z= \pm \frac{1}{k^{1 / 4}}, z= \pm \frac{1}{(-k)^{1 / 4}}, z= \pm \frac{1}{k^{1 / 4}}$, and $z= \pm \mathrm{i} \frac{1}{(-k)^{1 / 4}}$, then when $|k|<1$, the eigenvalues are inside the unit circle. Then, from the above solutions, $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)=$ $\left(1 /\left(\right.\right.$ the solutions of $\left.\left.a^{+}(z)\right)\right)$.

Example 14 We consider $n_{0}=1, v_{0}=2$. We have:

$$
\begin{array}{ll}
\mathbf{Q}_{-2}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
-V_{2}^{*} & 0 & 0
\end{array}\right), & \mathbf{Q}_{-1}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
-U_{1}^{*} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
\mathbf{Q}_{-1}=\left(\begin{array}{ccc}
0 & U_{1} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), & \mathbf{Q}_{2}=\left(\begin{array}{ccc}
0 & 0 & V_{2} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \tag{6.2.8b}
\end{array}
$$

Then, the $a^{+}(z)$ and $\mathbf{a}^{-}(z)$ has the following form:

$$
\begin{align*}
a^{+}(z) & =z^{8}-k^{2} z^{4}-k^{2},  \tag{6.2.9a}\\
\operatorname{det}\left(\mathbf{a}^{-}(z)\right) & =\frac{1}{z^{3}} . \tag{6.2.9b}
\end{align*}
$$

The scattering data $a^{+}(z)$ function has eight solutions. Since $a^{+}(z)$ is analytic when $|z|>1$, so the solutions of $a^{+}(z)=0$ must be outside the unit circle $|z|=1$. In this case, we have only four eigenvalues from eight (see Fig. 6.4). Here, the $a^{+}(z)$ function is zeros when $z= \pm \frac{1}{2} \mathcal{G}_{1}, z= \pm \mathrm{i} \frac{1}{2} \mathcal{G}_{1}, z= \pm \frac{1}{2} \mathcal{G}_{2}$ and $z= \pm \mathrm{i} \frac{1}{2} \mathcal{G}_{2}$, where $\mathcal{G}_{1}=\sqrt{8 k \sqrt{k^{2}+4}+8 k^{2}}$ and $\mathcal{G}_{1}=\sqrt{-8 k \sqrt{k^{2}+4}+8 k^{2}}$. However, $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ has no solution.

Example 15 Here, we consider $n_{0}=1,-1, v_{0}=1,-1$ :

$$
U_{n}(0)=k \times\left\{\begin{array}{cc}
1, & n_{0}=1,  \tag{6.2.10}\\
e^{\mathrm{i} \theta}, & n_{0}=-1, \\
0, & \text { otherwise },
\end{array} \quad V_{n}(0)=k \times\left\{\begin{array}{cc}
e^{\mathrm{i} \theta}, & v_{0}=1 \\
1, & v_{0}=-1 \\
0, & \text { otherwise }
\end{array}\right.\right.
$$

in this example the $a^{+}(z)$ and $\mathbf{a}^{-}(z)$ read as

$$
\begin{align*}
a^{+}(z) & =z^{4}-2 k^{2},  \tag{6.2.11a}\\
\operatorname{det}\left(\mathbf{a}^{-}(z)\right) & =1-2 z^{4} k^{2} . \tag{6.2.11b}
\end{align*}
$$

The $a^{+}(z)$ function which has four solutions is zeros when $z= \pm(2)^{1 / 4} \sqrt{k}$ and $z=$ $\mathrm{i} \pm(2)^{1 / 4} \sqrt{k}$. Under the condition $|z|>1$, the eigenvalues of the system (6.1.11) are existed when $k>\frac{1}{2} \sqrt{2}$ shown in Fig. 6.3. In addition, the zeros of the $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)$ are $z= \pm \frac{1}{(2)^{1 / 4} \sqrt{k}}$, and $z= \pm \mathrm{i} \frac{1}{(2)^{1 / 4} \sqrt{k}}$. In this example, we can also see that for each $z^{+}$, there is a reciprocal eigenvalue $z^{-}$, i.e., $z^{+}=1 / z^{-}$.

### 6.3 Conclusion

In this chapter, we have considered the impact of square barrier initial conditions on the type of the solution of the integrable discretisation of the Manakov lattice. We found that one-site excitations can give a blow up solution (see Fig. 6.4) because one of the $a^{+}(z)$ and $\mathbf{a}^{-}(z)$ functions has no solution. However, when we have symmetric potentials, the spectral operator has balanced eigenvalues (see Fig. 6.1-6.3), i.e., if one of the eigenvalues is inside the unit circle then, there is one outside. In this case, we have a bounded solution (but not sufficient). We are interested in the case where the initial condition is similar to the initial condition for the potential in (6.2.10), where there is a relation between the
solutions of $a^{+}(z)$, and $\mathbf{a}^{-}(z)$ despite the fact that it is no longer mirror symmetric. The idea used in this chapter may also be extended to find other new rigorous proofs of the conjecture about the solution whether it can be a blow up or not solution.


Figure 6.1: (a) and (b) are contour plots of the real and imaginary parts of equations $a^{+}(z)=0$ and $\operatorname{det}\left(\mathbf{a}^{-}(z)\right)=0$ when $n_{0}=-1$ and $v_{0}=1$ at $k=0.8$, respectively. The blue dashed lines represent the imaginary part of equations $\operatorname{Im}\left(\mathrm{a}^{+}(\mathrm{z})\right)$ and $\operatorname{Im}\left(\operatorname{det}\left(\mathbf{a}^{-}(\mathrm{z})\right)\right)$, while the red solid lines are for the real part of the equation $\operatorname{Re}\left(\mathrm{a}^{+}(\mathrm{z})\right)$ and $\operatorname{Re}\left(\operatorname{det}\left(\mathbf{a}^{-}(\mathrm{z})\right)\right)$. The dotted curve is the unit circle $|z|=1$. Intersections between the blue and red lines represent uniform solutions of $a^{+}(z)$ and $\mathbf{a}^{-}(z), z=z_{j}$. As we can see that there is no discrete eigenvalues for both $a^{+}(z)=0$ and $\mathbf{a}^{-}(z)=0$. Panels (c) and (d) show contour plots of the real and imaginary parts of the equation of $a^{+}(z)=0$ and $\mathbf{a}^{-}(z)=0$ when $n_{0}=-1$ and $v_{0}=1$ at $k=1.2$, respectively. The red, blue and black (solid, dashed and dots) curves are defined as in (a) and (b). In panels (c) and (d), we have 4-discrete eigenvalues.


Figure 6.2: The same as Fig. 6.1 but for $n_{0}=-2$ and $v_{0}=2$ at $k=1.6$. There are 8 -discrete eigenvalues.


Figure 6.3: The same as Fig. 6.1, but for $n_{0}=v_{0}=-1,1$ at $k=0.9$. There are 4-discrete eigenvalues.


Figure 6.4: The same as Fig. 6.1. Here, the solution of $a^{+}(z)=0$ for $n_{0}=1$ and $v_{0}=2$ at $k=1.5$. We have only four eigenvalues from eight because $a^{+}(z)$ is analytic in $|z|>1$.

## Chapter 7

## Conclusions and future work

In this chapter, we are going to summarize the work and the main results obtained in chapters 3-6 and also point out several problems that would be interesting as future works.

### 7.1 Summery of results

In this thesis, we have studied a nonlocal version [8] of the semi-discrete NLS equation in the Ablowitz-Ladik form. This equation appears to be $\mathcal{P J}$-symmetric. We formulated the direct scattering problem for the nonlocal Ablowitz-Ladik equation. This incudes the construction of the Jost solutions, the minimal set of scattering data and the construction of the FASs. Based on the formulation of the IST for (3.5.5) in the form of the additive Riemann-Hilbert boundary value problem, the one- and two-soliton solutions were derived.

It was shown in [8] that the one-soliton solution develops a singularity in finite time. This was due to the imbalance of the associated the RHP: the number of zeros of the FASs on the boundary of the contour is not equal to the number of zeros inside the contour. The nonlocal involution requires that if $z_{j}$ is a discrete eigenvalue, then $z_{j}^{*}$ must also be an eigenvalue, i.e., both $z_{j}$ and $z_{j}^{*}$ must be either inside or outside the unit circle. Depending on the positions of the discrete eigenvalues $z_{j}^{ \pm}$in the spectral plane, there are two regimes for the two-soliton solution: if one of the discrete eigenvalues is inside the unit circle and the other outside, then the nonlocal involution preserves their number balanced inside and outside the contour, and the corresponding two-soliton solutions are consequently regular for all $t$. Otherwise we found that the two-soliton solution will develop a singularity in finite time.

We briefly outlined the spectral properties of the Lax operator $\mathcal{L}_{n}(z)$. We have derived the completeness relations for the Jost solutions and obtained expansions over the complete set of Jost solutions for a generic function from the space of solutions of $\mathcal{L}_{n}(z)$.

Our study has examined the possibility for the creation of solitons in the discrete integrable lattices of the nonlocal NLS equation type with or without blow up solutions. Our conjecture was based on the box type initial conditions, and was successfully tested in numerical simulations. To sum up, we conjectured that: 1) if the Lax operator has no spectrum outside or inside the unit circle, there is no blow up solutions; 2) when it does have a spectrum outside or inside the unit circle, mirror symmetric initial conditions are sufficient, but not necessary, for bounded solutions; 3 ) to obtain bounded solutions, each spectrum outside the unit circle needs (but is not sufficient) to have a reciprocal counterpart inside the unite circle. When the initial condition for the potential is asymmetric, then initial excitations with power above some threshold will lead to blow up solutions. Numerical simulations that support the conjecture have been presented in chapter 5.

Finally, the two-component vector, discrete, nonlocal NLS equation (Manakov model) was also studied for box type initial conditions. We found that when one of the functions $a^{+}(z), \mathbf{a}^{-}(z)$ has no solutions, we have a blow up solution for the model. Meanwhile, when both functions $a^{+}(z), \mathbf{a}^{-}(z)$ have solutions with a particular relation, the model has bounded solutions. Numerical simulations that support the conjecture have been presented in chapter 6.

### 7.2 Future work

The results of this thesis can be extended in several directions:

- To construct gauge covariant formulations of the ISM for the nonlocal AblowitzLadik equation (3.6.26), including the generating (recursion) operator [48] and its spectral decomposition [56], the description of the class of the differentialdifference equations solvable by the spectral problem (5.2.2a) (i.e. the corresponding integrable hierarchy) and the description of the infinite set of integrals of motion and the hierarchy of Hamiltonian structures.
- To study the gauge-equivalent systems [55, 57, 58].
- To study the associated Darboux transformations and their generalisations for both local and nonlocal Ablowitz-Ladik equations. This will provide an algebraic method for constructing and classifying possible soliton solutions, including also rational solutions [37].
- To extend the results of this project for the case of non-vanishing boundary conditions (a non-trivial background) [14, 81, 82, 97]. In the local case, such solutions are
of interest in nonlinear optics; they arise in the theory of ultrashort femto-second nonlinear pulses in optical fibers. The nonlocal reduction of the Ablowitz-Ladik equation can be of particular interest in the theory of electromagnetic waves in artificial heterogenic media [122]. The considerations required in this case are more complicated and will be discussed elsewhere.
- To study multi-component generalisations [43, 52, 53, 60, 63, 64, 67] for both local and nonlocal semi-discrete NLS equations. This includes the block Ablowitz-Ladik system $[47,61]$ and generalisations to homogeneous and symmetric spaces. Such multi-component generalisations are much more complicated than in the continuous case and, to the best of our knowledge, they have not yet been studied.
- To find an elegant form for the two soliton solutions of the nonlocal DNLS equation and plot the time evolution of the solution.
- To complete the final form of the solution of the nonlocal DNLS equation by using the dressing method. Additionally, to compare the solution obtained by the dressing method with the solution obtained in [8].
- To provide a rigorous proof of the conjecture that is used to find the type of solution to the nonlocal DNLS and discrete MNLS equations. In addition, to find analytically the solution of the discrete Manakov system using the Cauchy integral and the dressing methods. Numerical simulations will be used to justify and support the solutions obtained for both methods.


## Appendix

## Lie groups and Lie algebras

## Basic definitions:

The elements $A, B, C, \ldots$ form a group $\mathcal{G}$ if they satisfy $[62,103]$ the following:

Identity There exists a unique element $\mathbb{1} \in \mathcal{G}$ such that for every other element $A \in \mathcal{G}$, it satisfies

$$
\mathbb{1} \cdot A=A \cdot \mathbb{1}=A ;
$$

Closure There exists a group multiplication (a binary operation defined on $\mathcal{G}$ ) which is closed in $\mathcal{G}$, i.e. to each pair $A, B \in \mathcal{G}$ it puts into a correspondence their product $A \cdot B \in \mathcal{G}$ which is again an element in $\mathcal{G}$;

Inverse To each element $A \in \mathcal{G}$ there corresponds a unique element $A^{-1} \in \mathcal{G}$ such that:

$$
A \cdot A^{-1}=A^{-1} \cdot A=\mathbb{1} ;
$$

Associativity The group multiplication is associative, i.e.:

$$
A \cdot(B \cdot C)=(A \cdot B) \cdot C=A \cdot B \cdot C
$$

Definition 1 A vector space over a number field $F$ with an operation $\mathrm{g} \times \mathrm{g} \rightarrow \mathrm{g}$, denoted [ $X, Y]$ and called the commutator of $X$ and $Y$, is called a Lie algebra over $F$ if the following points are satisfied:

1. $X, Y \in \mathrm{~g}$ implies $[X, Y] \in \mathrm{g}$.
2. $[X, \alpha Y+\beta Z]=\alpha[X, Y]+\beta[X, Z]$ for $\alpha, \beta \in F$ and $X, Y, Z \in g$.
3. Skew symmetry $[X, Y]=-[Y, X]$.
4. Jacobi identity $[X,[Y, Z]]+[Y,[Z, X]]+[Z,[X, Y]]=0$.

Definition 2 When the field $F$ is of real numbers $\mathbb{R}$, we say $g$ is a real Lie algebra, and we say g is complex if $F=\mathbb{C}$.

Definition 3 A matrix or linear Lie algebra is an algebra of matrices with commutator

$$
X Y-Y X
$$

taken as the Lie bracket $[X, Y]$. In addition, the commutator satisfies properties 2 through 4 of definition 1 .

Definition 4 The set of all $n \times n$ matrices with entries in $F$ is a Lie algebra, known as $g l(n, F)$.

Definition 5 The set of all matrices of trace zero with entries in $F$ forms a Lie algebra, known as $s l(n, F)$. The Pauli Matrices $\sigma_{j}, j=1,2,3$ belong to the Lie algebra,

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

Definition 6 The set of skew-Hermitian matrices $\left(A^{*}=-A\right.$ ) forms a Lie algebra over $\mathbb{R}$ known as $s u(n)$. Then, $\sigma_{j}, j=1,2,3$ are the basis in $s u(2, \mathbb{C})$.

Definition 7 The set of $2 \times 2$ complex matrices with trace zero form the complex Lie algebra $s l(2, \mathbb{C})$.

$$
\sigma_{+}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right), \quad \sigma_{-}=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)
$$

where $\sigma_{+}, \sigma_{-}$, are the basis for $s l(2, \mathbb{C})$ and the commutative relations of these matrices are

$$
\left[\sigma_{3}, \sigma_{ \pm}\right]= \pm 2 \sigma_{ \pm}, \quad\left[\sigma_{+}, \sigma_{-}\right]=\sigma_{3} \text { and } \sigma_{1} \pm \mathrm{i} \sigma_{2}=\sigma_{ \pm}
$$

Example 16 Let's assume the matrix has the form:

$$
X=\left[\begin{array}{cc}
x_{3} & x_{+} \\
x_{-} & -x_{3}
\end{array}\right],
$$

where trace of $X$ is $\operatorname{tr}(X)=0$. Recalling the matrices $\sigma 1, \sigma 2, \sigma 3$

$$
\begin{array}{cc}
\sigma_{3}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right], \quad \sigma_{2}=\left[\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right], \quad \sigma_{1}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \\
\sigma_{-}=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right], \quad \sigma_{+}=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right],
\end{array}
$$

we can rewrite $X$ as

$$
X=x_{+} \sigma_{+}+x_{-} \sigma_{-}+x_{3} \sigma_{3} .
$$

The following relations are founded using the above matrices

$$
x_{ \pm}=\operatorname{tr}\left(X \sigma_{\mp}\right), \text { and } x_{3}=\frac{1}{2} \operatorname{tr}\left(X \sigma_{3}\right) .
$$

Example 17 If A is a $2 \times 2$ matrix, then the commutator relation between $\sigma_{3}$ and $A$ is

$$
\left[\sigma_{3}, A\right]=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)-\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)=\left(\begin{array}{cc}
0 & 2 a_{12} \\
-2 a_{21} & 0
\end{array}\right)=2 \sigma_{3} A^{f}
$$

where $A^{f}$ is the off-diagonal matrix of $A$. Taking once again a commutator with $\sigma_{3}$ from both sides, one obtains:

$$
\left[\sigma_{3},\left[\sigma_{3}, A\right]\right]=4 A^{f}
$$

Therefore, one can write the projector $\pi$, extracting the off-diagonal part of the matrix, as a double commutator:

$$
A^{f}=\pi(A)=\frac{1}{4}\left[\sigma_{3},\left[\sigma_{3}, A\right]\right] .
$$

Example 18 This is another interesting example which makes our work easier. If we rewrite the Jost solutions in [54], $\psi(x, \lambda) \sim e^{-\mathrm{i} \sigma_{3} \lambda x}, \phi(x, \lambda) \sim e^{-\mathrm{i} \sigma_{3} \lambda x}$ as a matrix, then the relation between $\sigma_{3}, \phi(x, \lambda)$ and its inverse is

$$
\begin{gathered}
\left(\phi(x, \lambda) \sigma_{3} \phi^{-1}(x, \lambda)\right)= \\
\left(e^{-\mathrm{i} \sigma_{3} \lambda x} \sigma_{3} e^{\mathrm{i} \sigma_{3} \lambda x}\right)=\left(\begin{array}{cc}
e^{-\mathrm{i} k x} & 0 \\
0 & e^{\mathrm{i} k x}
\end{array}\right)\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
e^{\mathrm{i} k x} & 0 \\
0 & e^{-\mathrm{i} k x}
\end{array}\right)=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)=\sigma_{3} .
\end{gathered}
$$

Similarly for $\psi(x, \lambda)$

$$
\left(\psi(x, \lambda) \sigma_{3} \psi^{-1}(x, \lambda)\right)=\left(e^{-\mathrm{i} \sigma_{3} \lambda x} \sigma_{3} e^{\mathrm{i} \sigma_{3} \lambda x}\right)=\sigma_{3}
$$

## Gauss decomposition for Lie groups

We start by listing the special types of matrices. We will also list the corresponding properties of their eigenvalues. By $A$ we denote a generic $n \times n$ matrix and by $\alpha_{1}, \ldots, \alpha_{n}$ we denote its eigenvalues. An excellent exposition of this matter is given in [79].

Symmetric $A=A^{t}$; no restrictions on $\alpha_{j}$ 's;

Skew-symmetric $A=-A^{t}$; if $\alpha_{j}$ is an eigenvalue, then $-\alpha_{j}$ is also an eigenvalue;
Orthogonal $A^{t} A=\mathbb{1}$;
Real $A=A^{*}$; if $\alpha_{j}$ is an eigenvalue, then $\alpha_{j}^{*}$ is also an eigenvalue;
Imaginary $A=-A^{*}$; if $\alpha_{j}$ is an eigenvalue, then $-\alpha_{j}^{*}$ is also an eigenvalue;
Hermitian $A=A^{\dagger}$; if $\alpha_{j}$ is an eigenvalue, then $\alpha_{j}^{*}$ is also an eigenvalue;
Skew-Hermitian $A=-A^{\dagger}$; if $\alpha_{j}$ is an eigenvalue, then $-\alpha_{j}^{*}$ is also an eigenvalue;
Unitary $A A^{\dagger}=1$. if $\alpha_{j}$ is an eigenvalue, then $\left|\alpha_{j}\right|=1, \alpha_{j}=\exp \left(i \phi_{j}\right)$.
Each of the following statements is equivalent to saying that a non-singular matrix $\mathbf{A}$ possesses an $\mathbf{L U}$ factorization [84].

- For each non-singular matrix A, there exists a permutation matrix P such that PA possesses an $\mathbf{L U}$ factorisation $\mathbf{P A}=\mathbf{L} \mathbf{U}$.
- We can factor the diagonal entries out of the upper factor as shown below:

$$
\left(\begin{array}{cccc}
u_{11} & u_{12} & \ldots & u_{1 n} \\
0 & u_{11} & \ldots & u_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & u_{n n}
\end{array}\right)=\left(\begin{array}{cccc}
u_{11} & 0 & \ldots & 0 \\
0 & u_{11} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & u_{n n}
\end{array}\right)\left(\begin{array}{cccc}
1 & u_{12} / u_{11} & \ldots & u_{1 n} / u_{11} \\
0 & 1 & \ldots & u_{2 n} / u_{22} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{array}\right)
$$

- The LDU factorisation. There is some symmetry in an $\mathbf{L U}$ factorisation because the lower factor has 1 's on its diagonal while the upper factor has a nonunit diagonal. Setting $\mathbf{D}=\operatorname{diag}\left(u_{11}, u_{22}, \ldots, u_{n n}\right)$ and redefining $\mathbf{U}$ to be the upper-triangular matrix allows any $\mathbf{L U}$ factorisation to be written as $\mathbf{A}=\mathbf{L D U}$, where $\mathbf{L}$ and $\mathbf{U}$ are lower-and upper-triangular matrices with 1 s on both of their diagonals.

Example 19 Determine the LDU factorisation of $T(\lambda)$, where $T(\lambda)$ is a $2 \times 2$ matrix.

$$
\begin{aligned}
\left(\begin{array}{cc}
a^{+}(\lambda) & -b^{-}(\lambda) \\
b^{+}(\lambda) & a^{-}(\lambda)
\end{array}\right) & =M_{+} M^{d} M_{-} \\
& =\left(\begin{array}{cc}
1 & m_{+} \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
d_{1} & 0 \\
0 & d_{2}
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
m_{-} & 1
\end{array}\right) \\
& =\left(\begin{array}{cc}
d_{1}+m_{+} m_{-}-d_{2} & m_{+} d_{2} \\
m_{-} d_{2} & d_{2}
\end{array}\right)
\end{aligned}
$$

Using backward substitution, we can easily find each of $M_{+}, M^{d}, M_{-}$. Then,

$$
M_{+}=\left(\begin{array}{cc}
1 & -b^{-} / a^{-} \\
0 & 1
\end{array}\right), \quad M^{d}=\left(\begin{array}{cc}
1 / a^{-} & 0 \\
0 & a^{-}
\end{array}\right) \quad \text { and } M_{-}=\left(\begin{array}{rr}
1 & 0 \\
b^{+} / a^{-} & 1
\end{array}\right) .
$$

Example 20 The following example is for a $3 \times 3$ matrix

$$
\left.\begin{array}{c}
M_{0}=\left[\begin{array}{ccc}
d_{1,1} & 0 & 0 \\
0 & d_{2,1} & 0 \\
0 & 0 & d_{3,1}
\end{array}\right], \quad M_{+}=\left[\begin{array}{ccc}
1 & m_{1,2} & m_{1,3} \\
0 & 1 & m_{2,3} \\
0 & 0 & 1
\end{array}\right], \\
M_{-}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
m_{2,1} & 1 & 0 \\
m_{3,1} & m_{3,2} & 1
\end{array}\right] \quad \text { and } T=\left[\begin{array}{l}
T_{1,1} T_{1,2} T_{1,3} \\
T_{2,1} T_{2,2} \\
T_{2,3} \\
T_{3,1}
\end{array} T_{3,2} T_{3,3}\right.
\end{array}\right] .
$$

Then, we can find the inference of $T$ matrix as
$T=M_{+} M_{0} M_{-}=\left[\begin{array}{ccc}m_{1,2} d_{2,1} m_{2,1}+m_{1,3} d_{3,1} m_{3,1}+d_{1,1} & m_{1,3} d_{3,1} m_{3,2}+m_{1,2} d_{2,1} & m_{1,3} d_{3,1} \\ m_{2,3} d_{3,1} m_{3,1}+d_{2,1} m_{2,1} & m_{2,3} d_{3,1} m_{3,2}+d_{2,1} & m_{2,3} d_{3,1} \\ d_{3,1} m_{3,1} & d_{3,1} m_{3,2} & d_{3,1}\end{array}\right]$.

Using the same method in order to find the unknown elements of the $M$.

$$
\begin{aligned}
& T_{3,3}=d_{3,1} \quad m_{3,2}=\frac{T_{3,2}}{T_{3,3}} \quad m_{3,1}=\frac{T_{3,1}}{T_{3,3}} \quad m_{2,3}=\frac{T_{2,3}}{T_{3,3}} \quad m_{1,3}=\frac{T_{1,3}}{T_{3,3}} ; \\
& T=\left[\begin{array}{ccc}
\frac{T_{1,3} T_{3,1}}{T_{3,3}}+m_{1,2} d_{2,1} m_{2,1}+d_{1,1} & \frac{T_{1,3} T_{3,2}}{T_{3,3}}+m_{1,2} d_{2,1} & T_{1,3} \\
\frac{T_{2,3} T_{3,1}}{T_{3,3}}+d_{2,1} m_{2,1} & \frac{T_{2,3} T_{3,2}}{T_{3,3}}+d_{2,1} & T_{2,3} \\
T_{3,1} & T_{3,2} & T_{3,3}
\end{array}\right], \\
& d_{2,1}=\frac{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}}{T_{3,3}} \quad m_{2,1}=\frac{T_{2,1} T_{3,3}-T_{2,3} T_{3,1}}{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}}, \\
& T=\left[\begin{array}{ccc}
\frac{T_{1,3} T_{3,1}}{T_{3,3}}+\frac{m_{1,2}\left(T_{2,1} T_{3,3}-T_{2,3} T_{3,1}\right)}{T_{3,3}}+d_{1,1} & \frac{T_{1,3} T_{3,2}}{T_{3,3}}+\frac{\left(T_{2,2} T_{3,3}-T_{2,3} T_{3,2}\right) m_{1,2}}{T_{3,3}} & T_{1,3} \\
\frac{T_{2,3} T_{3,1}}{T_{3,3}}+\frac{T_{2,1} T_{3,3}-T_{2,3} T_{3,1}}{T_{3,3}} & \frac{T_{2,3} T_{3,2}}{T_{3,3}}+\frac{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}}{T_{3,3}} & T_{2,3} \\
T_{3,1} & T_{3,2} & T_{3,3}
\end{array}\right],
\end{aligned}
$$

$$
\begin{gathered}
m_{1,2}=\frac{T_{1,2} T_{3,3}-T_{1,3} T_{3,2}}{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}} \\
d_{1,1}=\frac{T_{1,1} T_{2,2} T_{3,3}-T_{1,1} T_{2,3} T_{3,2}-T_{1,2} T_{2,1} T_{3,3}+T_{1,2} T_{2,3} T_{3,1}+T_{1,3} T_{2,1} T_{3,2}-T_{1,3} T_{2,2} T_{3,1}}{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}},
\end{gathered}
$$

$$
\begin{aligned}
& M_{+}=\left[\begin{array}{ccc}
1 \overbrace{\frac{T_{1,2} T_{3,3}-T_{1,3} T_{3,2}}{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}}}^{\text {cof }_{T_{2,1}}} & \frac{T_{1,3}}{T_{3,3}} \\
0 & 1 & \frac{T_{2,3}}{T_{3,3}} \\
0 & 0 & 1
\end{array}\right], \\
& M_{-}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
\overbrace{\frac{T_{2,1} T_{3,3}-T_{2,3} T_{3,1}}{T_{2,2} T_{3,3}-T_{2,3} T_{3,2}}}^{\mathrm{CoF}_{T_{1,2}}} & 1 & 0 \\
\frac{T_{3,1}}{T_{3,3}} & \frac{T_{3,2}}{T_{3,3}}
\end{array}\right],
\end{aligned}
$$

where COF means cofactor of entry $T_{i, j}$.

## Time dependent condition

We are looking for special solutions of linearised problems in the form $Q_{n}=z^{2 n} \mathrm{e}^{-\mathrm{i} \omega\left(z^{2}\right) t}$. As an example is the differential-difference NLS for which $\omega\left(z^{2}\right)=2-z^{2}-z^{-2}$. Since the difference of the diagonal of the matrix $M_{n}$ in (3.3.11) at $n \rightarrow \pm \infty$ is

$$
\lim _{n \rightarrow \pm \infty} \frac{1}{2 h^{2}}\left(-\left(z-z^{-1}\right)^{2}-\left(z-z^{-1}\right)^{2}\right)=\frac{1}{h^{2}}\left(2-z^{2}-z^{-2}\right)
$$

then, we satisfied the condition for the special solutions. This idea originally comes from writing the general liner differential-difference equation,

$$
\begin{equation*}
\frac{\mathrm{d} \Psi_{n}}{\mathrm{~d} t}=-\mathrm{i} \omega(z) \Psi_{n} \tag{1}
\end{equation*}
$$

where $z$ is the shift operator $z \Psi_{n}=\Psi_{n+1}$. Then, if we take $\omega(z)=\left(z+z^{-1}-2\right)$ and use the shift operator equation, Equation (1) will be equivalent to

$$
\begin{aligned}
\frac{\mathrm{d} \Psi_{n}}{\mathrm{~d} t} & =-\mathrm{i}\left(z+z^{-1}-2\right) \Psi_{n} \\
& =\mathrm{i}\left(2-z-z^{-1}\right) \Psi_{n} \\
& =\mathrm{i}\left(2 \Psi_{n}-\Psi_{n+1}-\Psi_{n-1}\right)
\end{aligned}
$$

## The story behind the Riemann-Hilbert Problem

Riemann's idea that any function is completely determined by allocating its singularities and behaviours around the singularities [73]. It got the RHP because the contour C in here, is an arc, a closed contour or collection of arcs which is more general than in Hilbert's problem (close curve or circle C). The following points show the different between Riemann and Hilbart problems [94].

- The Riemann problem (RP) is a problem of determining an analytic function $W_{+}(z)$ inside a closed contour $\mathbf{C}$ (the contour is a unit circle) such that the bound values of its real and imaginary parts on the contour $\mathbf{C}$ satisfy the linear relation (2)

$$
\begin{align*}
& W_{+}(z)=u(x, y)+\mathrm{i} v(x, y),  \tag{2}\\
& \quad \alpha(t) u(t)+\beta(t) v(t)=\gamma(t)
\end{align*}
$$

where $t$ on $\mathbf{C}$ and $\alpha(t), \beta(t)$ and $\gamma(t)$ are real functions. This problem introduced another function $W_{-}(z)$ which is analytic outside $\mathbf{C}$ by

$$
W_{-}(z)=\bar{W}_{+}\left(\frac{1}{\bar{z}}\right) .
$$

For $z$ on $\mathbf{C}$ and $z \bar{z}=|z|^{2}=1$, then $z=\frac{1}{\bar{z}}$. This lead to the relations $z \rightarrow t$ and $\frac{1}{\bar{z}} \rightarrow t$. Furthermore, we can obtain both $u(t)$ and $v(t)$. Then, the RP relation (2) becomes:

$$
\frac{\alpha(t)-\mathrm{i} \beta(t)}{2} W_{+}(t)+\frac{\alpha(t)+\mathrm{i} \beta(t)}{2} W_{-}(t)=\gamma(t),
$$

which is reduced to finding the functions $W_{-}(z)$ and $W_{+}(z)$ analytic inside and outside the unit circle C, respectively. Simultaneously, their boundary values on
the circle satisfy Eq. (2), and for $z \rightarrow \infty, W_{-}(z) \rightarrow \bar{W}_{+}(0)$.

- The Hilbert problem: Hilbert generalised the RP. This problem determined a function $W(z)$ analytic for all values of $z$ except on the curve $\mathbf{C}$, such that for $t \rightarrow \mathbf{C}$,

$$
\begin{equation*}
W_{+}(t)=g(t) W_{-}(t)+f(t), \tag{3}
\end{equation*}
$$

where $W_{+}(z), W_{-}(z)$ are limits of $W(z)$ as $z \rightarrow t$ from inside and outside the $\mathbf{C}$, respectively, and $g(t), f(t)$ are complex-valued functions

- Carleman's approach to solving the RHP. Carleman presented the RHP on singular integral equations. The main idea is to find a nonzero holomorphic function $L(z)$ that is analytic everywhere on the $z$ plane except possibly on the curve $\mathbf{C}$, satisfying

$$
\begin{equation*}
L_{+}(t)=g(t) L_{-}(t), \tag{4}
\end{equation*}
$$

since $L_{-}(t), L_{+}(t)$ are known and nonzero. Then, Eq. (3) becomes:

$$
\frac{W_{+}}{L_{+}}(t)-\frac{W_{-}}{L_{-}}(t)=\frac{f}{L_{+}}(t)
$$

Since $L(z) \neq 0$, the function $\frac{W(z)}{L(z)}$ is analytic for $z$ not on C, such that Eq. (4) is satisfied. Since $L(z)$ is known, $W(z)$ is known. If we take

$$
M(z)=\frac{1}{2 \pi \mathrm{i}} \oint_{\mathbf{C}} \frac{f(t) / L_{+}(t)}{t-z} \mathrm{~d} t
$$

then, using the discontinuity theorem we have

$$
\begin{aligned}
\frac{f(t)}{L_{+}(t)} & =M_{+}(t)-M_{-}(t), \\
\frac{W_{+}}{L_{+}}(t)-M_{+}(t) & =\frac{W_{-}}{L_{-}}(t)-M_{-}(t),
\end{aligned}
$$

then, from the above calculation the function $\frac{W(z)}{L(z)}-M(z)$ is continuous on C . To find the function $L(z)$ from Eq. (4) we have

$$
\begin{equation*}
\log \left(\frac{L_{+}}{L_{-}}\right)(t)=\log g(t) \tag{5}
\end{equation*}
$$

The solution to Eq. (5) when $\mathbf{C}$ is an arc, since $g(z)$ is continuous at end points $z_{1}$ and $z_{2}$ of the arc having the values $q\left(z_{1}\right)$ and $g\left(z_{2}\right)$ at their end points. Then,

$$
\log (L(z))=\frac{1}{2 \pi \mathrm{i}} \oint_{\mathbf{C}} \frac{g(t)}{t-z} \mathrm{~d} t=Q(z),
$$

and

$$
L(z)=e^{(Q(z))} \neq 0
$$

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[^0]:    ${ }^{1}$ In quantum theory of scattering, they are also known as $\mid$ in $>$ and $\mid$ out $>$ states.

[^1]:    ${ }^{1}$ The function $f \in \mathbb{C}^{\infty}$ is called a Schwartz function if it goes to zero as $|x| \rightarrow \infty$.

[^2]:    ${ }^{1}$ Here, they have solved the ISP for the Sturm-Liouville equation.
    ${ }^{2}$ Here, they have solved the ISP for the ZS system.

[^3]:    ${ }^{1}$ The trace of an $n \times n$ square matrix $A$ is defined to be the sum of the elements on the main diagonal. We call a matrix $A$ is traceless when the trace of the matrix is equal to zero.

[^4]:    ${ }^{1}$ If $T$ is bounded then automatically $(T-\lambda \mathbb{1})$ is bounded too.

[^5]:    ${ }^{1}$ The idea comes from the geometric series $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^{n}=1+x^{2}+x^{3}+\ldots$.
    ${ }^{2}$ Keeping in mind we are looking for the coefficient of $\frac{1}{\lambda}$.
    ${ }^{3} f(A)=1+A+A^{2}+\ldots$ and $f(\lambda)=1+\lambda+(\lambda)^{2}+\ldots$.

[^6]:    ${ }^{1}$ Integration in the opposite direction acquires a minus sign.
    ${ }^{2}$ Let A be a linear operator on a finite dimension complex inner product space $\mathcal{\nu}$. Then for distinct eigenvalues there exist nonzero (hermitian) projectors $P_{1}, P_{2}, \ldots, P_{r}$, which satisfies $P_{j} P_{j}=$ $0, \sum_{j=1}^{r} P_{j}=1$ and $\sum_{j=1}^{r} \lambda_{j} P_{j}=A$.

[^7]:    ${ }^{1}$ Antilinear operator: Let A be a linear operator in certain Hilbert space $\mathcal{H}$. Let us suppose that $|\Psi\rangle,|\varphi\rangle \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. An antilinear operator A satisfies the condition: $A(\alpha|\Psi\rangle+\beta|\varphi\rangle)=\dot{\alpha} A(|\Psi\rangle)+$ $\beta^{*} A(|\varphi\rangle)$. Antiunitary operator: A is said to be antiunitary if it is antilinear and $A A^{\dagger}=A^{\dagger} A=\mathbb{1} \leftrightarrow$ $A^{-1}=A^{\dagger}$.

[^8]:    ${ }^{1}$ A tensor product of two matrices $A_{m \times n}$ and $B_{p \times q}$ is the $m p \times n q$ block matrix.

[^9]:    ${ }^{1}$ Equation (3.1.2) is derived from equation (3.1.1) by assuming $\varepsilon=\frac{1}{h^{2}}$ with a simple transformation $Q_{n} \rightarrow Q_{n} e^{-2 i t \varepsilon}$.

[^10]:    ${ }^{1}$ The derivative of a function $f$ at a point $x$ is defined by the limit.

[^11]:    ${ }^{1}$ Meromorphic functions are complex functions that have only simple poles.

[^12]:    ${ }^{1}$ In this section the FASs are different than in Sec. 3.2.4.

[^13]:    ${ }^{1}$ asy: asymptotic

