Sovereign Debt and Structural Reforms*

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Abstract

We construct a dynamic theory of sovereign debt and structural reforms with limited enforcement and moral hazard. A sovereign country in recession would like to smooth consumption. It can also undertake costly reforms to speed up recovery. The sovereign can renege on contracts by suffering a stochastic cost. The Constrained Optimum Allocation (COA) prescribes non-monotonic dynamics for consumption and effort and imperfect risk sharing. The COA is decentralized by a competitive equilibrium with markets for renegotiable GDP-linked one-period debt. The equilibrium features debt overhang: reform effort decreases in a high debt range. We also consider environments with less complete markets.

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In this paper, we propose a normative and positive dynamic theory of sovereign debt in an environment characterized by informational frictions. The theory rests on two building blocks. First, sovereign debt is subject to limited enforcement, and countries can renege on their obligations subject to real costs as in, e.g., Aguiar and Gopinath (2006), Arellano (2008) and Yue (2010). Second, countries can undertake *structural* policy reforms to speed up recovery from an existing recession.[1] The reform effort is assumed to be unobservable and subject to moral hazard.

The theory is motivated by the recent debt crisis in Europe, where sovereign debt and economic reforms emerged as salient and intertwined policy issues. Greece, for instance, saw its debt-GDP ratio soar from 103% in 2007 to 172% in 2011 (despite a 53% haircut in 2011) and the risk premium on its government debt boom during the same period. While creditors and international organizations pushed the Greek government to introduce structural reforms that would help the economy recover and meet its international financial obligations, such reforms were forcefully resisted internally.[2] Opposers maintained that the reforms would imply major sacrifice for domestic residents while a large share of the benefits would accrue to foreign lenders. Meanwhile, international organizations stepped in to provide financial assistance and access to new loans, asking in exchange fiscal restraint and a commitment to economic reforms. Our theory rationalizes these dynamics.

The model economy is a dynamic endowment economy subject to income shocks following a two-state Markov process. The economy (henceforth, the *sovereign*) starts in a recession with a stochastic duration. Costly structural reforms increase the probability that the recession ends. Consumers' preferences induce a desire for consumption and effort smoothing. We first characterize the solution of two planning problems: the first best and the constrained optimum allocation (COA) subject to limited enforcement and moral hazard. In the first best, the planner provides the sovereign country with full insurance by transferring resources to it during recession and reversing the transfers once the recession ends. The sovereign exerts the efficient level of costly effort as long as the recession lasts.

The first best is not implementable in the presence of informational frictions for two reasons. First, the sovereign has access to a stochastic outside option whose realization is publicly observable. This creates scope for opportunistic deviations involving cashing in transfers for some time, and then unilaterally quit (i.e., *default on*) the contract as soon as the realization of the outside option is sufficiently favorable. Second, the sovereign has an incentive to shirk and rely on the transfers rather than exerting the required reform effort to increase output.

The COA is characterized by means of a promised utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988 and 1990), and Kocherlakota (1996). The optimal contract is subject to an incentive compatibility constraint (IC) that pins down the effort choice and a participation constraint (PC) that captures the limited enforcement. The COA has the following features: throughout recession, within spells of a slack PC (i.e., when the realized cost of default is high), the planner front-loads the sovereign's consumption and decreases it over time in order to provide dynamic incentives for reform effort (as in Hopenhayn and Nicolini 1997). In this case, the solution is dictated by the IC and is history-dependent: consumption and promised utility fall over time, while effort follows non-monotone dynamics. Whenever the PC binds (i.e., the sovereign faces an attractive

[1] Examples of such reforms include labor and product market deregulation, and the establishment of fiscal capacity that allows the government to raise tax revenue efficiently (see, e.g., Ilzkovitz and Dierx 2011). While beneficial in the long run, these reforms entail short-run costs for citizens at large, governments or special-interest groups (see, e.g., Blanchard and Giavazzi 2003).

[2] Italy is another example. The incumbent Italian government – which advocated economic reforms and fiscal stability – was defeated in the elections of March 2018. Since a new populist government which strenuously opposes structural reforms took office, the Italy-Germany 10-year bond spread has increased from 1.3% to 3.2% between March and October 2018.

outside option), the planner increases discretely consumption and promised utility in order to prevent the sovereign from leaving the contract.

Next, we study the decentralization of the COA. We show that the COA can be implemented by a market equilibrium where the sovereign issues two one-period securities paying returns contingent on the aggregate state (*GDP-linked bonds*). The bonds, which are defaultable and renegotiable, are sold to profit-maximizing international creditors who hold well-diversified portfolios. When the sovereign faces a low realization of the default cost, she could, in principle, default, pay a cost, and restart afresh with zero debt. However, costly default can be averted by renegotiation: when a credible default threat is present, a syndicate of creditors offers a take-it-or-leave-it debt haircut. As in Bulow and Rogoff (1989), there is no outright default in equilibrium, but recurrent stochastic debt renegotiations.[3] In order for this market arrangement to attain constrained efficiency, the creditors must, ex-post, have all the bargaining power. We show that a Markov-perfect market equilibrium with these characteristics attains the same allocation as a social planner who can inflict the same punishment in case of a deviation as can the market in case of outright default (i.e., imposing the same stochastic default cost and letting the agent resume in the market equilibrium with zero debt).

In our model, two one-period securities are sufficient to decentralize the COA. This result hinges on two features. First, the process of renegotiation (following a particular protocol) effectively turns the two assets into a continuum of state-contingent securities. Second, the equilibrium debt dynamics and its endogenously evolving price provide efficient dynamic incentives for the sovereign to exert the second-best reform effort. We prove that our environment yields the same allocation as a full set of Arrow-Debreu securities with endogenous borrowing constraints in the spirit of Alvarez and Jermann (2000). Our decentralization is parsimonious and simple, in the sense that it requires only two assets and no need to solve for a set of endogenous borrowing constraints.

In the market equilibrium, debt accumulates and consumption falls over time as long as the recession lingers and debt is not renegotiated. Interestingly, the reform effort is a non-monotone function of debt. This result stems from the interaction between limited enforcement and moral hazard. Under full enforcement, effort would increase monotonically over the recession as debt accumulates. Absent moral hazard, effort would be constant when the PC is slack and decrease every time debt is renegotiated. When both informational constraints are present, effort increases with debt at low levels. However, for sufficiently high debt levels the relationship is flipped: there, issuing more debt deters reforms because, when the probability of renegotiation is high, most of the gains from an economic recovery would accrue to foreign lenders. In this region, the reform effort falls over time as debt further accumulates. This *debt overhang* curtails consumption smoothing: when sovereign debt is high, investors expect low reform effort, are pessimistic about the economic outlook, and request even higher risk premia. Interestingly, in our theory this form of debt overhang is constrained efficient under the postulated informational constraints.

We also study environments with more incomplete markets. In particular, we consider an economy where the sovereign can issue only one asset – a non-contingent bond. This economy fails to attain the COA: the sovereign attains less consumption smoothing and provides an inefficient effort level.[4] This extension is interesting because in reality markets for GDP-linked bonds are often missing. In this (arguably realistic) one-asset environment, there is scope for policy intervention. In particular,

[3] Empirically, unordered defaults are indeed rare events. Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008) documents a substantial heterogeneity in the terms at which debt is renegotiated, which is consistent with our assumption of a stochastic outside option (see also Panizza *et al.* 2009 and Reinhart and Trebesch 2016).

[4] We also study a case where renegotiation is ruled out as in Eaton and Gersovitz (1981). This further curtails consumption smoothing and welfare.

an international institution such as the IMF can improve welfare by means of an assistance program. During the recession, the optimal program entails a persistent budget support through extending loans on favorable terms. When the recession ends, the sovereign is settled with a (large) debt on market terms.

Our analysis is related to a large international and public finance literature. In a seminal contribution, Atkeson (1991) studies the optimal contract in an environment in which an infinitely-lived sovereign borrower faces a sequence of two-period lived lenders subject to repeated moral hazard. The focus of our paper differs from Atkeson's in two regards. First, we emphasize the non-monotone effort dynamics and the role of periodic renegotiations along the equilibrium path. Second, we provide a novel decentralization result through renegotiable bonds.

A number of recent papers deal with the dynamics of sovereign debt under a variety of informational and contractual frictions. Dovis (2019) studies the efficient risk-sharing arrangement between international lenders and a sovereign borrower with limited commitment and private information about domestic productivity. In his model the COA can be implemented as a market equilibrium with non-contingent defaultable bonds of short and long maturity. He does not consider the interaction between structural reforms and limited commitment. Aguiar *et al.* (2017) study a model à la Eaton and Gersovitz (1981) with limited commitment assuming, as we do, that the borrower has a stochastic default cost. Their research is complementary to ours insofar as it focuses on debt maturity in rollover crises, from which we abstract. Jeanne (2009) also studies a rollover crisis in an economy where the government takes a policy action that affects the return to foreign investors (e.g., the enforcement of creditor's right) but this can be reversed within a time horizon that is shorter than that at which investors must commit their resources.

Our work is also related to the literature on debt overhang initiated by Krugman (1988). He constructs a static model with exogenous debt showing that a large debt can deter the borrower from undertaking productive investments. In this regime, it may be optimal for the creditor to forgive debt. This is never optimal in our model. Several papers consider distortions associated with high indebtedness in the presence of informational imperfections. Aguiar and Amador (2014) show that high debt increases the volatility of consumption by reducing risk sharing. Aguiar *et al.* (2009) consider the effect of debt on investment volatility. When an economy is indebted, productivity shocks gives rise to larger dispersion in investment rates. Aguiar and Amador (2011) consider a politico-economic model where capital income can be expropriated ex-post and the government can default on external debt. A country with a large sovereign debt position has a greater temptation to default, and therefore investments are low. Conesa and Kehoe (2017) construct a theory where governments of highly indebted countries may choose to gamble for redemption.

Our research is related also to the literature on endogenous incomplete markets due to limited enforcement or limited commitment. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). Our result that efficiency can be attained by a sequence of one-period debt contract is related to Fudenberg *et al.* (1990) who provide conditions under which commitment problems in repeated moral hazard setups can be resolved by a sequence of one-period contracts. Our theory also builds on the insight of Grossman and Van Huyck (1988) that renegotiation can be used to complete markets. Our focus on a unique Markov equilibrium is related to Auclert and Rognlie (2016) who study the properties of an Eaton-Gersovitz economy. The analysis of constrained efficiency by means of promise-keeping constraints is related to the literature on competitive risk sharing contracts with limited commitment, including, among others, Thomas and Worrall (1988), Marcet and Marimon (1992), Phelan (1995), Kocherlakota (1996), and Krueger and Uhlig (2006).

Finally, our work is related, more generally, to recent quantitative models of sovereign default such as Aguiar and Gopinath (2006), Arellano (2008), and Chatterjee and Eyigungor (2012).[5] Abraham *et al.* (2018) study an Arellano (2008) economy, extended to allow shocks to government expenditures, and compare quantitative outcomes of the market allocation to the optimal design of a Financial Stability Fund. Broner *et al.* (2010) study the incentives to default when parts of the government debt is held by domestic residents. Song *et al.* (2012) and Müller *et al.* (2016a) study the politico-economic determinants of debt in open economies where governments are committed to honor their debt.

The rest of the paper is organized as follows. Section 1 describes the model environment. Section 2 solves for the first best and the COA under limited enforcement and moral hazard. Section 3 provides the main decentralization result. Section 4 considers one-asset economies and discusses policy interventions to restore efficiency. Section 5 concludes. Appendix A contains the proofs of the main lemmas and propositions. Online Appendix B contains additional technical material referred to in the text.

1 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. A benevolent *sovereign* makes decisions on behalf of the representative agent. The stochastic endowment follows a two-state Markov switching process, with realizations $\underline{w}$ and $\bar{w}$, where $0 < \underline{w} < \bar{w}$. We label the two endowment states *recession* and *normal time*, respectively. An economy starting in recession remains in the recession with endogenous probability $1 - p$ and switches to normal time with probability p. Normal time is assumed to be an absorbing state. This assumption aids tractability and enables us to obtain sharp analytical results. During recession, the sovereign can implement a costly reform policy to increase p. In our notation, p denotes both the *reform effort* and the probability that the recession ends. The sovereign can smooth consumption by contracting with a financial intermediary that has access to an international market offering a gross return R.

The sovereign's preferences are given by $\mathbb{E}_0 \sum \beta^t \left[u(c_t) - \phi_t I_{\{\text{default in } t\}} - X(p_t) \right]$, where $\beta = 1/R$. The function u is twice continuously differentiable and satisfies $\lim_{c \to 0} u(c) = -\infty$, $u'(c) > 0$, and $u''(c) < 0$. $I \in \{0, 1\}$ is an indicator switching on when the economy is in a default state and ϕ is an associated utility loss. In the planning allocation, the cost ϕ accrues when the sovereign opts out of the contract offered by the planner. In the market allocation, it accrues when sovereign unilaterally reneges on a debt contract with international lenders.[6] In recession, ϕ follows an *i.i.d.* process drawn from the p.d.f. $f(\phi)$ with an associated c.d.f. $F(\phi)$. We assume that $F(\phi)$ is continuously differentiable everywhere, and denote its support by $\aleph \equiv [\phi_{\min}, \phi_{\max}] \subseteq \mathbb{R}^+$, where $\phi_{\min} < \phi_{\max} < \infty$. The assumption that shocks are independent is for simplicity. In order to focus on debt dynamics in recessions we assume that there is full enforcement in normal time (i.e., in normal time ϕ is arbitrarily large) – see (Müller *et al.* 2016b) for a generalization in which the distribution of ϕ is the same in normal time and in recession.

The function $X(p)$ represents the reform cost, assumed to be increasing and convex in the probability of exiting recession, $p \in [\underline{p}, \bar{p}] \subset [0, 1]$. X is assumed to be twice continuously differentiable, with the following properties: $X(\underline{p}) = 0$, $X'(\underline{p}) = 0$, $X'(p) > 0 \; \forall p > \underline{p}$, $X''(p) > 0$, and $\lim_{p \to \bar{p}} X'(p) = \infty$.

[5] Other papers studying restructuring of sovereign debt include Asonuma and Trebesch (2016), Bolton and Jeanne (2007), Hatchondo *et al.* (2014), Mendoza and Yue (2012), and Yue (2010).

[6] The cost ϕ is exogenous and publicly observed, and captures in a reduced form a variety of shocks including both taste shocks (e.g., the sentiments of the public opinion about defaulting on foreign debt) and institutional shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.). Alternatively, ϕ could be given a politico-economic interpretation, as reflecting special interests of lobbies.

The time line of events is as follows: at the beginning of each period, the endowment state is observed; then, ϕ is realized and publicly observed; finally, effort is exerted.

2 Planning allocation

We first characterize the constrained efficient allocation as the solution to a (*dual*) benevolent social planner problem which maximizes the principal's (i.e., creditors') discounted value of expected future profits subject to a sequence of incentive and limited enforcement constraints. The problem can be written recursively using the agent's (i.e., sovereign's) continuation utility as the state variable and letting the planner maximize profits subject to a promise-keeping constraint.[7]

Let ν denote the promised utility, i.e., the expected utility the sovereign is promised in the beginning of the period, before the realized ϕ is observed. ν is the key state variable of the problem. We denote by ω_ϕ and $\bar{\omega}_\phi$ the promised continuation utilities conditional on the realization ϕ and on the economy staying in recession or switching to normal time, respectively. We denote by $P(\nu)$ the expected present value of profits accruing to the planner conditional on delivering the promised utility ν in the most cost-effective way. The optimal value $P(\nu)$ satisfies the following functional equation:

$$P(\nu) = \max_{\{c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi\}_{\phi \in \aleph}} \int_{\aleph} \left[\underline{w} - c_\phi + \beta \left(p_\phi \bar{P}(\bar{\omega}_\phi) + (1 - p_\phi) P(\omega_\phi)\right)\right] dF(\phi), \tag{1}$$

where the maximization is subject to a promise-keeping constraint (PK)

$$\int_{\aleph} \left[u(c_\phi) - X(p_\phi) + \beta \left(p_\phi \bar{\omega}_\phi + (1 - p_\phi) \omega_\phi\right)\right] dF(\phi) = \nu, \tag{2}$$

the planner's profit function in normal time,

$$\bar{P}(\bar{\omega}_\phi) = \max_{\bar{c} \in [0, \tilde{c}]} \bar{w} - \bar{c} + \beta \bar{P}\left(\beta^{-1}\left[\bar{\omega}_\phi - u(\bar{c})\right]\right), \tag{3}$$

and the boundary conditions $c_\phi \in [0, \tilde{c}], p_\phi \in [\underline{p}, \bar{p}], \nu, \omega_\phi \in [\underline{\omega}, \tilde{\omega}]$, and $\bar{\omega}_\phi \in [\underline{\omega}, \tilde{\bar{\omega}}]$, where $\tilde{c}$, $\tilde{\omega}$, $\tilde{\bar{\omega}}$ and $\underline{\omega}$ are generous bounds that will never bind in equilibrium.

We introduce two informational frictions. The first is limited enforcement captured by a set of PCs: the sovereign can quit the contract when the realization of ϕ makes such action attractive ex-post. The second is moral hazard captured by an IC: the reform effort is chosen by the sovereign and is not observed by the planner.

Under limited enforcement, the planner must provide a utility that exceeds the sovereign's outside option if she quits the contract, $\alpha - \phi$, where ϕ is the shock discussed above and α is the sovereign's value of not being in the contract. Thus, the allocation is subject to the following set of PCs:

$$u(c_\phi) - X(p_\phi) + \beta \left[p_\phi \bar{\omega}_\phi + (1 - p_\phi) \omega_\phi\right] \geq \alpha - \phi, \quad \phi \in \aleph, \tag{4}$$

and to a lower bound on initial and future promised utility, i.e., $\nu \geq \underline{\nu} \equiv \alpha - \mathbb{E}[\phi]$ and

$$\omega_\phi \geq \underline{\nu}, \tag{5}$$

[7] We formulate the planning problem as a one-sided commitment program. The problem would be identical under two-sided lack of commitment under some mild restrictions on the state space. In particular, one should impose an upper bound on the sovereign's initial promised utility to ensure that the principal does not find it optimal to ever exit the contract. We return to this point below.

where $\underline{\nu}$ is the expected utility that the sovereign could attain by quitting the contract in the next period after the realization of ϕ. Thus, it is not feasible for the planner to promise a utility $\omega_\phi < \underline{\nu}$. In addition, we assume that α is sufficiently low to ensure that it is never efficient to terminate the program (even under the realization $\phi = \phi_{\min}$).[8] In the first part of the paper we simply treat α as an exogenous parameter. Later we endogenize α.

When the planning problem is subject to moral hazard, the allocation is also subject to the following IC, reflecting the assumption that the sovereign chooses effort *after* the planner has set the future promised utilities:

$$p_\phi = \arg \max_{p \in [\underline{p}, \bar{p}]} -X(p) + \beta \left[p \bar{\omega}_\phi + (1-p)\, \omega_\phi\right]. \tag{6}$$

2.1 First best

We start by characterizing the first-best allocation given by the program (1)–(3). The proof, which follows standard methods, can be found in Appendix B.

Proposition 1 *Given a promised utility ν, the first best allocation satisfies the following properties. The sequences for consumption and promised utilities are constant at the level $c^{FB}(\nu)$, $\omega^{FB} = \nu$, and $\bar{\omega}^{FB} = \nu + X\left(p^{FB}(\nu)\right) / \left(1 - \beta\left(1 - p^{FB}(\nu)\right)\right)$, implying full consumption insurance, $c^{FB}(\nu) = \bar{c}^{FB}\left(\bar{\omega}^{FB}\right)$. Moreover, effort $p^{FB}(\nu)$ is constant over time throughout recession. $c^{FB}(\nu)$ and $p^{FB}(\nu)$ are strictly increasing and strictly decreasing functions, respectively, satisfying:*

$$\frac{\beta}{1-\beta\left(1-p^{FB}(\nu)\right)} \left(\underbrace{(\bar{w} - \underline{w})}_{\text{output increase if recovery}} \times u'\left(c^{FB}(\nu)\right) + \underbrace{X\left(p^{FB}(\nu)\right)}_{\text{saved effort cost if recovery}} \right) = X'\left(p^{FB}(\nu)\right) \tag{7}$$

$$\frac{u\left(c^{FB}(\nu)\right)}{1-\beta} - \frac{X\left(p^{FB}(\nu)\right)}{1-\beta\left(1-p^{FB}(\nu)\right)} = \nu. \tag{8}$$

The solution to the functional equation (3) in normal time is given by

$$\bar{P}(\bar{\omega}_\phi) = \frac{\bar{w} - \bar{c}^{FB}(\bar{\omega}_\phi)}{1-\beta}, \quad \bar{c}^{FB}(\bar{\omega}_\phi) = u^{-1}((1-\beta)\bar{\omega}_\phi). \tag{9}$$

The first-best allocation yields full insurance: the sovereign enjoys a constant consumption irrespective of the endowment state and exerts a constant reform effort during recession. Moreover, consumption c^{FB} is strictly increasing in ν while effort p^{FB} is strictly decreasing in ν: a higher promised utility is associated with higher consumption and lower effort. $\bar{P}$ is strictly decreasing and strictly concave.

2.2 Constrained Optimum Allocation (COA)

Next, we characterize the COA. The planning problem (1)-(3) is subject to the PC (4), the lower bound on ω_ϕ (5), and the IC (6). Note that the planning problem is evaluated after the uncertainty about the endowment state has been resolved, but before the realization of ϕ. For didactic reasons, we first study a problem with only limited enforcement. Then, we generalize the analysis to the case in which

[8]Note that the constraint (5) is implied by next period's PK (2) and the set of PCs (4). It is convenient to specify it as a separate constraint since this allows us to attach a Lagrange multiplier to (5) instead of (2). The condition on α must guarantee that $\underline{w} - c_{\phi_{\min}} + \beta\left[p_{\phi_{\min}} \bar{P}\left(\bar{\omega}_{\phi_{\min}}\right) + \left(1 - p_{\phi_{\min}}\right) P\left(\omega_{\phi_{\min}}\right)\right] \geq 0$. Otherwise, both parties would gain from terminating the contract. This condition will always hold true when we endogenize α.

there is also moral hazard. We start by establishing a property of the COA that holds true in both environments.[9]

Lemma 1 *Assume that the profit function P is strictly concave. Define the sovereign's discounted utility conditional on the promised utility ν and the realization ϕ as $\mu_\phi(\nu) \equiv u(c_\phi(\nu)) - X(p_\phi(\nu)) + \beta\left[p_\phi(\nu)\bar{\omega}_\phi(\nu) + (1-p_\phi(\nu))\omega_\phi(\nu)\right]$. Then, the COA features a unique threshold function $\tilde{\phi}(\nu)$ such that the PC binds if $\phi < \tilde{\phi}(\nu)$ and is slack if $\phi \geq \tilde{\phi}(\nu)$. Moreover,*

$$\mu_\phi(\nu) = \begin{cases} \alpha - \phi & \text{if} \quad \phi < \tilde{\phi}(\nu), \\ \alpha - \tilde{\phi}(\nu) & \text{if} \quad \phi \geq \tilde{\phi}(\nu). \end{cases}$$

The lemma formalizes the intuitive properties that (i) if the planner provides the agent with higher expected utility than her reservation utility $\alpha - \phi$ in a state ϕ_a, then it is optimal for her to do so for all $\phi > \phi_a$; moreover, promised utility is equalized across all such states; (ii) if the planner provides the agent an expected utility equal to the reservation utility in a state ϕ_b, then it is optimal for her to also do so for all $\phi < \phi_b$. Thus, $\mu_\phi(\nu)$ is linearly decreasing in ϕ for $\phi < \tilde{\phi}(\nu)$, and constant thereafter.

2.2.1 Limited Enforcement without Moral Hazard

In this environment, there is no IC and the planner can control the effort level directly. We prove in Appendix B that the planner's profit function $P(\nu)$ is strictly decreasing, strictly concave and differentiable for all interior ν, i.e., $\nu > \underline{\nu}$. Moreover, the first-order conditions (FOCs) of the program are necessary and sufficient. The proof follows the strategy in Thomas and Worrall (1990).

Combining the FOCs with respect to c_ϕ, ω_ϕ, and $\bar{\omega}_\phi$ with the envelope condition (see the proof of Proposition 2 in Appendix B) yields:[10]

$$\frac{1}{u'(\bar{c}(\bar{\omega}_\phi))} - \frac{1}{u'(c_\phi)} = 0 \qquad (10)$$

$$u'(c_\phi) = -\frac{1}{P'(\omega_\phi)}, \quad \forall \omega_\phi > \underline{\nu}. \qquad (11)$$

Combining these with the FOC with respect to p_ϕ yields

$$X'(p_\phi) = \beta\left[(\bar{\omega}_\phi - \omega_\phi) + u'(c_\phi) \times \left(\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)\right)\right]. \qquad (12)$$

Equation (10) establishes that the planner provides the sovereign with full insurance against the endowment shock, i.e., she sets $\bar{c}(\bar{\omega}_\phi) = c_\phi$ and equates the marginal profit loss associated with promised utilities in the two states. Equation (12) establishes that effort is set at the constrained efficient level: The marginal cost of effort equals the sum of the marginal benefits accruing to the sovereign and to the planner, respectively. Proposition 2 provides a formal characterization of the COA.[11]

[9] Proving that $\bar{P}$ in equation (9) is strictly concave is straightforward. The strict concavity of P is more difficult to establish analytically. We prove below that P is strictly concave in the case without moral hazard, while in the case with moral hazard we will guess and verify it numerically.

[10] Note that since there is full enforcement during normal time, $\bar{c}(\bar{\omega}_\phi)$ and $\bar{P}(\bar{\omega}_\phi)$ are as in the first best (cf. equation 9). Thus, $\bar{P}'(\bar{\omega}_\phi) = -1/u'(\bar{c}(\bar{\omega}_\phi))$. Also, equations (10)-(11) imply that the marginal cost of providing promised utility is equalized across the two endowment states, i.e., $\bar{P}'(\bar{\omega}_\phi) = P'(\omega_\phi)$.

[11] The proof is in Appendix B. The proof of Proposition 3 below encompasses the proof of Proposition 2 as a special case.

Proposition 2 *The COA is characterized as follows. The threshold function* $\tilde{\phi}(\nu)$ *is decreasing and implicitly defined by the condition*

$$\nu = \alpha - \left[\int_{\phi_{\min}}^{\tilde{\phi}(\nu)} \phi dF(\phi) + \tilde{\phi}(\nu) \left[1 - F(\tilde{\phi}(\nu)) \right] \right]. \tag{13}$$

Moreover:

1. *If* $\phi < \tilde{\phi}(\nu)$*, the PC is binding, and the allocation* $(c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi)$ *is determined by (10), (11), (12), and by (4) holding with equality. Moreover,* $\omega_\phi = \nu' > \nu$.

2. *If* $\phi \geq \tilde{\phi}(\nu)$*, the PC is slack, and the allocation* $(c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi)$ *is given by* $\omega_\phi = \nu' = \nu$, $c_\phi = c(\nu)$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, *and* $p_\phi = p(\nu)$*, where the functions* $c(\nu)$, $\bar{\omega}(\nu)$ *and* $p(\nu)$ *are determined by*

$$u(c(\nu)) - X(p(\nu)) + \beta \left[p(\nu)\bar{\omega}(\nu) + (1 - p(\nu))\nu \right] = \alpha - \tilde{\phi}(\nu), \tag{14}$$

(10), and (12), respectively. The solution is history-dependent. The reform effort is strictly decreasing and consumption and future promised utility are strictly increasing in ν.

The COA under limited enforcement has standard back-loading properties. Whenever the PC is slack, consumption, effort and promised utility remain constant over time. Consumption remains constant even when the recession ends. Thus, the COA yields full consumption insurance across all states in which the PC is slack. Whenever the PC binds, the planner increases the sovereign's consumption and promised utilities while reducing her effort in order to meet her PC. In this case, $\nu' > \nu$.[12]

The upper panels of Figure 1 describe the dynamics of consumption and effort (left panel), and promised utilities (right panel) under a particular sequence of realizations of ϕ.[13] In this numerical example, the recessions last for 13 periods. Thereafter, the economy attains full insurance. Consumption is back-loaded and effort is front-loaded. In periods 9 and 11, the PC binds and the planner must increase consumption and promised utility, and reduce effort. Consumption and effort remain constant when the PC is slack. Note also that consumption remains constant when the recession ends.

2.2.2 Limited Enforcement with Moral Hazard

Next, we consider the more interesting case in which the planner cannot observe effort and is subject to both a PC and an IC. The COA features an important qualitative difference from the case without moral hazard: within each spell in which the PC is slack, the planner front-loads consumption and promised utility to incentivize the sovereign to provide effort. Therefore, moral hazard prevents full insurance even across the states of nature in which the PC is slack.

Let us start the analysis from the IC (6). The FOC yields $X'(p_\phi) = \beta(\bar{\omega}_\phi - \omega_\phi)$, or equivalently

$$p_\phi = \Upsilon(\bar{\omega}_\phi - \omega_\phi) \tag{15}$$

where $\Upsilon(x) \equiv (X')^{-1}(\beta x)$. The properties of X imply that p_ϕ is increasing in the promised utility gap $\bar{\omega}_\phi - \omega_\phi$. Equation (15) is the analogue of (12). Effort is distorted because the sovereign does not internalize the benefits accruing to the planner.

[12] Although we have assumed that the planner controls effort directly, the same allocation would obtain if the planner did not control effort ex ante but could observe it ex post and punish deviations. Details are available in the working paper version (Müller *et al.* 2016b).

[13] All figures are generated by a choice of parameters discussed in Appendix B.3.

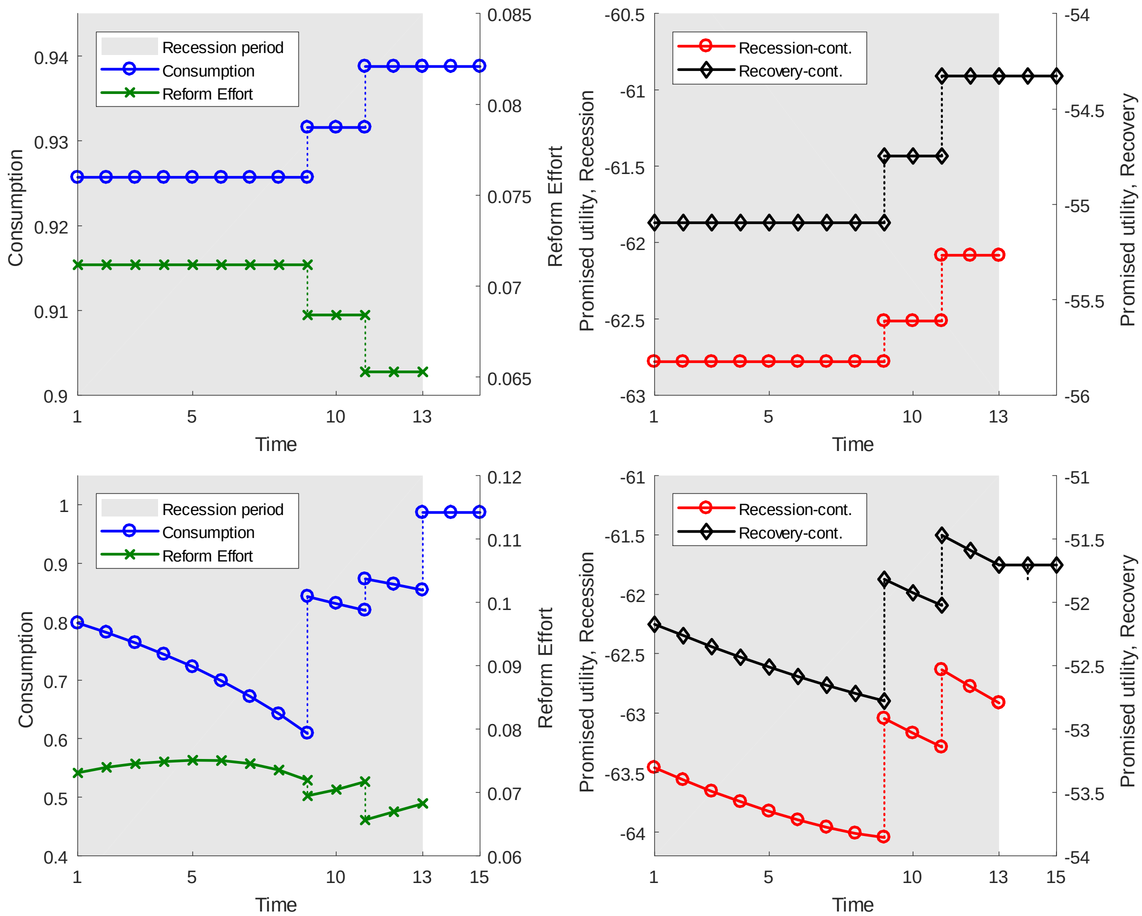

Figure 1: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's. In this particular simulation the recession ends in period 13. The top panels show the planner allocation without moral hazard, the bottom panels with moral hazard.

The FOCs with respect to ω_ϕ and $\bar{\omega}_\phi$ together with the envelope condition yield (see proof of Proposition 3 in Appendix A):

$$\frac{1}{u'(\bar{c}(\bar{\omega}_\phi))} - \frac{1}{u'(c_\phi)} = \frac{\Upsilon'(\bar{\omega}_\phi - \omega_\phi)}{\Upsilon(\bar{\omega}_\phi - \omega_\phi)} \left[\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)\right] \tag{16}$$

$$\frac{1}{u'(c_\phi)} = -P'(\omega_\phi) + \frac{\Upsilon'(\bar{\omega}_\phi - \omega_\phi)}{1 - \Upsilon(\bar{\omega}_\phi - \omega_\phi)} \left[\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)\right], \quad \forall \omega_\phi > \underline{\nu} \tag{17}$$

$$0 = \theta_\phi \times [\omega_\phi - \underline{\nu}], \tag{18}$$

where $\theta_\phi \geq 0$ is the Lagrange multiplier on the constraint $\omega_\phi \geq \underline{\nu}$. This constraint binds in a range of low ν's where the impossibility of promising a utility below $\underline{\nu}$ curbs the planner's ability to provide dynamic incentives.[14] Equation (17) is then replaced by the conditions $\omega_\phi = \underline{\nu}$ and $\theta_\phi > 0$.

The FOC (16) is the analogue of (10). With moral hazard, the planner does no longer provide the sovereign with full insurance against the realization of the endowment shock: by promising a higher

[14] The issue did not arise in the absence of moral hazard because the optimal ω_ϕ was non-decreasing.

consumption if the economy recovers, she provides incentives for effort provision. Note that the effort wedge is proportional to the elasticity Υ'/Υ.

The FOC (17) is the analogue of (11). There, the marginal utility of consumption simply equaled the profit loss associated with an increase in promised utility. Here, the planner finds it optimal to open a wedge that is proportional to the elasticity of effort: she front-loads consumption in order to make the sovereign more eager to leave a recession.

We can now proceed to a full characterization of the COA with moral hazard. As is common for problems with both limited enforcement and moral hazard, it is difficult to prove that the program is globally concave and to establish analytically the curvature of the profit function. We therefore assume that $P(\nu)$ is strictly concave in ν, and verify this property numerically.[15]

Proposition 3 *Assume that P is strictly concave. The COA with unobservable effort is characterized as follows: (i) $p_\phi = \Upsilon(\bar{\omega}_\phi - \omega_\phi)$ as in equation (15); (ii) the threshold function $\tilde{\phi}(\nu)$ is decreasing and implicitly defined by equation (13). Moreover:*

1. *If $\phi < \tilde{\phi}(\nu)$, the PC is binding and the allocation $(c_\phi, \omega_\phi, \bar{\omega}_\phi, \theta_\phi)$ is determined by (16), (17), (18), and by (4) holding with equality.*

2. *If $\phi \geq \tilde{\phi}(\nu)$, the PC is slack and the allocation $(c_\phi, \omega_\phi, \bar{\omega}_\phi, \theta_\phi)$ is determined by (16), (17), (18), and*
$$u(c_\phi) - X(p_\phi) + \beta\left[p_\phi \bar{\omega}_\phi + (1-p_\phi)\omega_\phi\right] = \alpha - \tilde{\phi}(\nu). \quad (19)$$
The solution is history-dependent, i.e., $c_\phi = c(\nu)$, $\omega_\phi = \omega(\nu)$, and $\bar{\omega}_\phi = \bar{\omega}(\nu)$. Promised utility falls over time, i.e., $\omega_\phi = \omega(\nu) = \nu' \leq \nu$ with $\theta_\phi = 0$ when ν is sufficiently large. The function $c(\nu)$ is strictly increasing. The effort function $p(\nu) = \Upsilon(\bar{\omega}(\nu) - \omega(\nu))$ is strictly increasing in a range of low ν. In this range effort declines over time when the PC is slack.

3. *$P(\nu)$ is strictly decreasing, and differentiable for all $\nu > \underline{\nu}$. In particular, $P'(\nu) = -1/u'(c(\nu)) < 0$. Moreover, $\bar{P}(\bar{\omega}_\phi) > P(\omega_\phi)$ for all ϕ and ν.*

Figure 2 illustrates the results of Proposition 3 based on a numerical example. All panels show policy functions for an economy in recession, conditional on a slack PC. Promised utilities $\omega_\phi(\nu)$ and $\bar{\omega}_\phi(\nu)$ and consumption $c(\nu)$ are weakly increasing in ν. The upper left panel shows the law of motion of ν when the recession lingers and the PC remains slack. The fact that $\omega_\phi(\nu)$ is below the 45-degree line implies that promised utility falls over time and converges to the lower bound $\underline{\nu}$.[16] In the range $[\underline{\nu}, \nu^-]$ the planner is constrained by the inability to abase promised utility below $\underline{\nu}$. The two left panels imply that, if the recession lingers and the PC is slack, consumption declines over time. Instead, the dynamics of effort are non-monotone.

We now discuss some additional analytical properties of the COA. We start with consumption dynamics. Combining (17) with the envelope condition $P'(\omega_\phi) = -1/u'(c(\omega_\phi))$ for $\omega_\phi > \underline{\nu}$ and denoting $\nu' = \omega_\phi$ and $\bar{\nu}' = \bar{\omega}_\phi$, yields:
$$\frac{1}{u'(c_\phi)} - \frac{1}{u'(c(\nu'))} = \frac{\Upsilon'(\bar{\nu}' - \nu')}{1 - \Upsilon(\bar{\nu}' - \nu')}\left(\bar{P}(\bar{\nu}') - P(\nu')\right). \quad (20)$$

[15] We prove that under the assumption that P is strictly concave, $P(\nu)$ must be differentiable for all interior ν and that the FOCs are necessary (see Lemma 3.2 in Appendix B). Although we cannot establish in general that they are also sufficient, this turns out to be the case in all parametric examples we considered.

[16] Note that, as ν falls, the probability that the PC binds increases. At $\underline{\nu}$ the PC binds almost surely in the next period, and the sovereign receives the realized reservation utility $(\alpha - \phi')$ if the economy remains in recession.

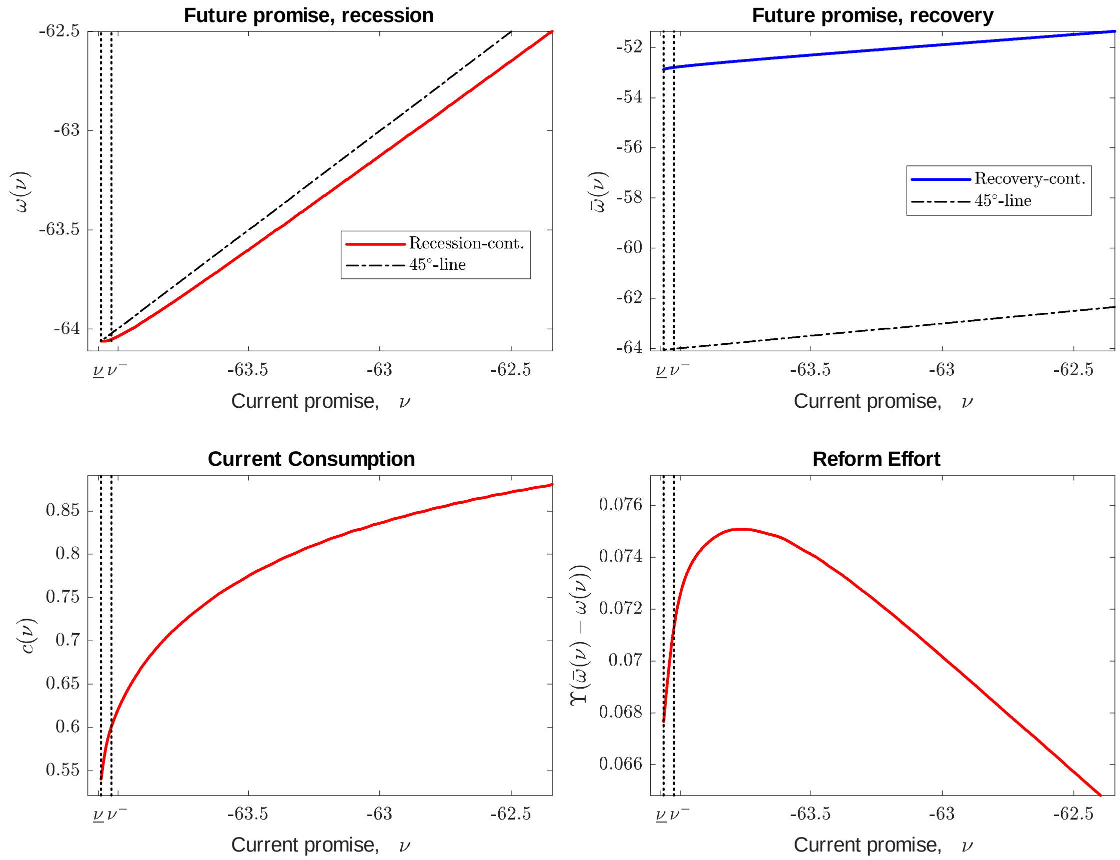

Figure 2: Policy functions for state-contingent promised utility, consumption, and effort conditional on the maximum cost realization $\phi_{\max}$.

We label equation (20) a Conditional Euler Equation (CEE). The CEE describes the optimal consumption dynamics for states where the PC does not bind next period and $\nu > \nu^-$. The right-hand side of (20) is positive. Thus, consumption decreases over time as long as the economy remains in recession and the PC is slack, echoing the optimal consumption dynamics in Hopenhayn and Nicolini (1997).[17]

Combining the CEE (20) with the Euler equation describing the consumption dynamics upon recovery (16), yields a conditional version of the so-called *Inverse Euler Equation* (CIEE):

$$\frac{1}{u'(c_\phi)} = \left(1 - \Upsilon\left(\bar{\nu}' - \nu'\right)\right) \frac{1}{u'(c(\nu'))} + \Upsilon\left(\bar{\nu}' - \nu'\right) \frac{1}{u'(\bar{c}(\bar{\nu}'))}. \tag{21}$$

The CIEE equates the inverse marginal utility in the current period with next period's expected inverse marginal utility conditional on the PC being slack and $\nu > \nu^-$. The key difference relative to the standard inverse Euler equation in the dynamic contract literature (cf. Rogerson 1985) is that our CIEE holds true only in states where the PC is slack. If there were no enforcement problems, then our

[17] The right-hand side of (20) is positive since $\Upsilon' > 0$ and $\bar{P}(\bar{\nu}') > P(\nu')$ (see the proof of Proposition 3 in Appendix A). This implies that the marginal utility of consumption must be rising over time as ν falls. This property differs from some other papers in the repeated moral hazard literature (e.g., Phelan and Townsend (1991)) where the profit function can be non-monotone in promised utility. In our model, the planner would never be better off by delivering more than the promised utility ν (as long as ν is in the feasible set $\nu \geq \underline{\nu}$), since she can reduce current consumption without affecting the effort choice.

CIEE would boil down to a standard inverse Euler equation.

Next, we turn to the effort dynamics. In analogy with consumption, one might expect that, when the PC is slack, the planner would back-load effort to incentivize its provision. However, this conjecture is incorrect. Effort is in fact decreasing in promised utility when ν is high, inducing back-loading of effort.[18] However, when ν is sufficiently low, effort is increasing in promised utility (cf. Proposition 3 and its proof), implying that, as ν falls, effort decreases over time. In this range, the planner front-loads effort even within spells when the PC is slack. The reason is that the planner cannot reduce indefinitely the promised utility. When $\nu \leq \nu^{-}$, the constraint $\omega_\phi \geq \underline{\nu}$ becomes binding, and the planner sets $\omega_\phi = \underline{\nu}$, inducing $p_\phi(\nu) = \Upsilon(\bar{\omega}_\phi(\nu) - \underline{\nu})$. Since both both Υ and $\bar{\omega}_\phi$ are increasing functions (the planner can decrease $\bar{\omega}_\phi$ without bound since there is perfect enforcement in normal time), p_ϕ must be increasing in ν. Hence, effort declines over time if the recession lingers and the PC remains slack for sufficiently long.[19]

The lower panels of Figure 1 illustrate simulated dynamics of consumption, promised utilities, and effort in the case of moral hazard under the same sequence of realizations of ϕ as in the upper panels. Consumption and promised utilities fall over time when the PC is slack, while effort falls or rises over time depending on ν. When ν becomes sufficiently low (i.e., in periods 7, 8, and 9), the reform effort starts falling. Note that consumption increases when the recession ends, implying that the endowment risk is not fully insured.

In conclusion, the combination of limited enforcement and moral hazard delivers effort dynamics qualitatively different from models with only one friction. In our model effort is hump-shaped over time, even when the PC remains slack. In contrast, effort is monotone increasing in many pure moral hazard models and weakly decreasing in pure limited enforcement models. The dynamics of consumption echo the typical properties of models with dynamic moral hazard as long as the PC is slack, namely the planner curtails consumption in order to extract higher effort over time. However, the planner periodically increases consumption and promised utility whenever the PC is binding. This averts the immiseration that would arise in a world of perfect enforcement.

2.3 Primal Formulation

We have characterized the COA by solving a dual planning problem, i.e., maximizing profits subject to a promised-utility constraint. We can alternatively solve a *primal* problem where the planner maximizes discounted utility subject to a promised-expected-profit constraint. We characterize the primal program because it is directly comparable with the market equilibrium studied below.

Let $\mu^{pl}(\pi)$ and $\mu^{pl}_\phi(\pi)$ denote the sovereign's discounted utility before and after the realization of ϕ, respectively. We can write the primal problem as:

$$\begin{aligned} \mu^{pl}(\pi) &= \int_{\aleph} \mu^{pl}_\phi(\pi)\, dF(\phi) \qquad (22) \\ &= \int_{\aleph} \max_{\left\{c_\phi, p_\phi, \pi'_\phi, \bar{\pi}'_\phi\right\}_{\phi\in\aleph}} \left[u(c_\phi) - X(p_\phi) + \beta\left(p_\phi \bar{\mu}^{pl}(\bar{\pi}'_\phi) + (1-p_\phi)\,\mu^{pl}(\pi'_\phi)\right)\right] dF(\phi), \end{aligned}$$

[18] Lemma 3.1 in Appendix B shows that effort is decreasing in promised utility when ν is large. This result is subject to a sufficient condition, namely that $\lim_{p\to\underline{p}} X''(p) > 0$. However, numerical analysis suggests that this is true more generally.

[19] By continuity, this property extends to a contiguous range of ν above ν^{-}.

where $\bar{\mu}^{pl}(\bar{\pi}) = u(\bar{w} - (1-\beta)\bar{\pi}) / (1-\beta)$, subject to the promised-profit constraint

$$\pi = \int_{\aleph} b_\phi dF(\phi), \tag{23}$$

having defined $b_\phi = \underline{w} - c_\phi + \beta \left[p_\phi \bar{\pi}'_\phi + (1-p_\phi)\pi'_\phi\right]$ as the planner's discounted profit after the realization of ϕ. The problem is subject to a set of PCs and ICs

$$u(c_\phi) - X(p_\phi) + \beta\left[p_\phi \bar{\mu}^{pl}(\bar{\pi}'_\phi) + (1-p_\phi)\mu^{pl}(\pi'_\phi)\right] \geq \alpha - \phi, \quad \phi \in \aleph, \tag{24}$$

$$p_\phi = \arg\max_{p\in[\underline{p},\bar{p}]} -X(p) + \beta\left[p\bar{\mu}^{pl}(\bar{\pi}'_\phi) + (1-p)\mu^{pl}(\pi'_\phi)\right], \tag{25}$$

and to the boundary conditions $c_\phi \in [0, \tilde{c}]$, $\bar{\pi}'_\phi \in [\underline{\pi}, \tilde{\pi}]$, and $\pi, \pi'_\phi \in [\underline{\pi}, \pi_{\max}]$, where $\underline{\pi} = (\tilde{c} - \underline{w})/(1-R^{-1})$ and $\tilde{\pi} = \bar{w}/(1-R^{-1})$ are generous bounds that will never bind in equilibrium and $\pi_{\max}$ satisfies $\mu^{pl}(\pi_{\max}) = \alpha - \mathbb{E}[\phi]$.

While the program (22) may in general admit multiple fixed points for μ^{pl}, we can characterize the unique fixed point associated with the COA by exploiting duality properties. First, for the primal and the dual to be equivalent, $\pi = P(\nu)$ and $\nu = \mu^{pl}(\pi)$. Thus, $\mu^{pl} = P^{-1}$. Since P is strictly decreasing and strictly concave, so must be μ^{pl}. Second, the two solutions must feature the same threshold function, i.e., $\Phi^{pl}(\pi) = \tilde{\phi}(P^{-1}(\pi))$.

A complete characterization of the primal COA is deferred to Proposition 4 in Appendix A. Here, we highlight the properties that will be key for the decentralization results. First, the value function of the primal COA satisfies:

$$\mu^{pl}(\pi) = \left[1 - F(\Phi^{pl}(\pi))\right]\left(\alpha - \Phi^{pl}(\pi)\right) + \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} (\alpha - \phi)\, dF(\phi), \tag{26}$$

where $\pi \in [\underline{\pi}, P(\alpha - \mathbb{E}[\phi])]$ and $\Phi^{pl}(\pi) \equiv \tilde{\phi}(P^{-1}(\pi))$. As in the dual, the agent perceives the utility $\alpha - \phi$ if the PC binds and $\alpha - \tilde{\phi}(\nu)$ if the PC is slack, where $\nu = P^{-1}(\pi)$. Second, effort must be the same as in the dual, implying that $p = \Psi^{pl}(\pi', \bar{\pi}') = \Upsilon(\bar{\mu}^{pl}(\bar{\pi}') - \mu^{pl}(\pi'))$. Third, let $b_\phi(\pi)$ denote that optimal choice of b_ϕ in the COA. In Proposition 4, we prove that the COA features[20]

$$b_\phi(\pi) = \begin{cases} b^{pl}(\pi) = \left(W^{pl}\right)^{-1}\left(\alpha - \Phi^{pl}(\pi)\right) & \text{if} \quad \phi \geq \Phi^{pl}(\pi) \\ \hat{b}^{pl}(\phi) = \left(W^{pl}\right)^{-1}(\alpha - \phi) & \text{if} \quad \phi < \Phi^{pl}(\pi) \end{cases}, \tag{27}$$

where

$$\begin{aligned} W^{pl}(x) \equiv & \max_{\bar{\pi}'\in[\underline{\pi},\tilde{\pi}], \pi'\in[\underline{\pi}, P(\alpha-\mathbb{E}[\phi])]} u\left(\underline{w} - x + \beta\left[\Psi^{pl}(\pi', \bar{\pi}')\bar{\pi}' + \left(1 - \Psi^{pl}(\pi', \bar{\pi}')\right)\pi'\right]\right) \\ & -X(\Psi^{pl}(\pi', \bar{\pi}')) + \beta\Psi^{pl}(\pi', \bar{\pi}')\bar{\mu}^{pl}(\bar{\pi}') + \beta(1 - \Psi^{pl}(\pi', \bar{\pi}'))\mu^{pl}(\pi') \end{aligned} \tag{28}$$

is the agent's discounted utility. In particular, $W^{pl}(b_\phi(\pi))$ denotes the continuation utility after the realization of ϕ, conditional on the optimal choice of $b_\phi(\pi)$ given by (27). If the PC is slack, the principal sets $b_\phi = b^{pl}(\pi)$ independently of ϕ. In this case, consumption and continuation utility

[20] Note that the optimal choice of consumption and future promised utility are also identical across the primal and the dual problem. We defer their characterization to the formal statement of the proposition in the appendix.

are also independent of ϕ, and in particular $W^{pl}\left(b^{pl}(\pi)\right) = \alpha - \Phi^{pl}(\pi)$. If, instead, the PC binds, then the principal sets $b_\phi = \hat{b}^{pl}(\phi)$, which ensures that the agent's PC holds with equality. In this case, the agent's continuation utility is $W^{pl}\left(\hat{b}^{pl}(\phi)\right) = \alpha - \phi$. Note that $\pi = \left[1 - F(\Phi^{pl}(\pi))\right] b^{pl}(\pi) + \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} \hat{b}^{pl}(\phi) dF(\phi)$. Namely, the promised expected profit π equals the expected "transfer" she receives from the agent. Note also that inverting this expression allows us to define $b^{pl}(\pi)$ recursively;

$$b^{pl}(\pi) = \frac{\pi - \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} b^{pl}\left(\left(\Phi^{pl}\right)^{-1}(\phi)\right) dF(\phi)}{1 - F\left(\Phi^{pl}(\pi)\right)}. \tag{29}$$

It is easy to show that (29) defines a unique function $b^{pl}(\pi)$ and that this function is increasing.

The primal COA admits the following interpretation. In the initial period, before the state ϕ is realized, the risk-neutral principal is endowed with a claim $b = b^{pl}(\pi)$ on the sovereign whose expected repayment is π.[21] After ϕ is realized, the principal receives the face value of "debt" b in all states ϕ in which the PC is slack. If the PC binds, she takes a haircut, i.e., she receives a lower payment $\hat{b}^{pl}(\phi) < b$. The planner then adjusts optimally the issuance of new claims. The case without moral hazard is especially intuitive. If the PC is slack, the planner keeps the debt constant at its initial level ($b = b'$). If the PC binds, she reduces the future obligation so as to keep the sovereign in the contract. Under moral hazard, b (and, hence, the promised profit) change over time even when the PC is slack in order to provide the optimal dynamic incentives for effort provision. This interpretation is useful to understand the market decentralization to which we now turn.

3 Decentralization

In this section, we show that the COA can be decentralized by a market allocation where the sovereign issues one-period defaultable bonds, held by risk-neutral international creditors. In the market economy, the planner is replaced by a syndicate of international investors (the *creditors*) who can borrow and lend at the gross interest rate R.

We assume that the sovereign can issue two one-period securities. The first is a non-contingent bond, like in Eaton and Gersovitz (1981), which returns one unit of good in all states. However, different from Eaton and Gersovitz (1981), the bond is defaultable and renegotiable according to the protocol discussed below. The second security is a state-contingent bond that yields a return only in the normal state, i.e., a GDP-linked bond. The GDP-linked bond is not subject to any risk of renegotiation, because there is no default in the normal state. In this environment, markets are incomplete for two reasons. First, there exists no market for securities offering a return contingent on the effort level. Second, there is no market to insure explicitly against the realization of ϕ.

The market structure with a non-contingent bond and a GDP-linked bond is equivalent to one in which there exist two state-contingent securities, which we label *recession-contingent debt* and *recovery-contingent debt,* with prices $Q\left(b, \bar{b}\right)$ and $\bar{Q}\left(b, \bar{b}\right)$.[22] We carry out our discussion with this latter notation because it is more standard and convenient. At the beginning of each recession period, the sovereign observes the realization of the default cost ϕ and decides whether to honor the recession-contingent debt that reaches maturity or to announce the intention to default. This announcement triggers a

[21] Note that, since $b^{pl}(\pi)$ is a continuous and monotone increasing function, we can write the inverse function $\pi = \Pi^{pl}\left(b^{pl}\right)$ and define the problem for an initial level of b^{pl} instead of an initial promised profit.

[22] Denote by b_n and b_s the non-contingent and state-contingent security, respectively. Then, $b_n = b$ and $b_s = \bar{b} - b$. Moreover, the prices of these bonds are given by $Q_n(b_n, b_s) = Q\left(b, \bar{b}\right) + \bar{Q}\left(b, \bar{b}\right)$, and $Q_s(b_n, b_s) = \bar{Q}\left(b, \bar{b}\right)$.

renegotiation process. Since debt is honored in normal times, no arbitrage implies that $\bar{Q} = pR^{-1}$. If the country could commit to repay its debt also in recession, the bond price would be $Q = (1-p)R^{-1}$. However, due to the risk of renegotiation, recession-contingent debt sells at a discount, $Q \leq (1-p)R^{-1}$.

We now describe the renegotiation protocol. If the sovereign announces default, the syndicate of creditors can offer a take-it-or-leave-it haircut that we assume to be binding for all creditors.[23] By accepting this offer, the sovereign averts the default cost. In equilibrium, a haircut is offered only if the default threat is credible, i.e., if the realization of ϕ is sufficiently low to make the sovereign prefer default to full repayment.[24] Note that the creditors have, *ex-post*, all the bargaining power, and their offer makes the sovereign indifferent between an outright default and the proposed haircut.

The timing of a debt crisis can be summarized as follows: The sovereign enters the period with the pledged debt b, observes the realization ϕ, and then decides whether to threaten default on all its debt. If the threat is credible, the creditors offer a haircut $\hat{b} \leq b$. Next, the sovereign decides whether to accept or decline this offer. Finally, the sovereign issues new debt subject to the period budget constraint $c = \bar{Q}\left(b', \bar{b}'\right) \times \bar{b}' + Q\left(b', \bar{b}'\right) \times b' + \underline{w} - \hat{b}$.

To facilitate comparison with the COA we start by setting the post-default outside option exogenously equal to α. Thus, in the out-of-equilibrium event that the sovereign declines the offered haircut, the default cost ϕ is triggered, debt is canceled, and realized utility is $\alpha - \phi$. We will later endogenize α by assuming that the sovereign can subsequently resume access to financial markets.

3.1 Market Equilibrium

We focus on market equilibria where agents condition their strategies on pay-off relevant state variables, ruling out reputational mechanisms. We view this assumption as realistic in the context of sovereign debt since it is generally difficult for creditors to commit to punishment strategies, especially when new lenders can enter and make separate deals with the sovereign. In our environment, the pay-off relevant state variables are b, w, and ϕ. For technical reasons, we impose that debt is bounded, $b \in [\underline{b}, \tilde{b}]$ where $\tilde{b} = \bar{w}/\left(1 - R^{-1}\right)$ is the natural borrowing constraint in normal time and $\underline{b} > -\infty$ is such that for $b = \underline{b}$ it is not optimal to default. In equilibrium, these bounds never bind.

The natural equilibrium concept is that of Markov-perfect equilibrium. For didactical reasons, we first define a market equilibrium under the assumption that, in case of outright default, the game ends and the sovereign receives the exogenous utility $\alpha - \phi$. Strictly speaking, this is not a Markov-perfect equilibrium, as different strategies apply on and off the equilibrium path. We will later endogenize α by setting it equal to the value of the reversion to the market equilibrium without debt. In that case, we will refer to the equilibrium as a *Markov-perfect* equilibrium.

Definition 1 *A market equilibrium is a set of value functions $\{V, W\}$, a threshold renegotiation function Φ, a set of equilibrium debt price functions $\{Q, \bar{Q}\}$, and a set of optimal decision rules $\{\mathbb{B}, B, \bar{B}, C, \Psi\}$ such that, given an outside option α and conditional on the state vector $(b, \phi) \in [\underline{b}, \tilde{b}] \times [\phi_{\min}, \phi_{\max}]$, the sovereign maximizes utility, the creditors maximize profits, and markets clear. More formally:*

[23] In our environment there would be no reason for a subset of creditors to deviate and seek a better deal. In Section 4 below, we show that ruling out renegotiation altogether reduces welfare, ex-ante. Therefore, our theory emphasizes the value of making haircut agreements binding for all creditors.

[24] By assumption, the sovereign has always the option to simply honor the debt contract. Thus, the creditors' take-it-or-leave-it offer cannot demand a repayment larger than the face value of outstanding debt.

- *The value function V satisfies*

$$V(b,\phi) = \max\{W(b), \alpha - \phi\}, \tag{30}$$

where $W(b)$ is the value function conditional on the debt level b being honored,

$$W(b) = \max_{(b',\bar{b}')\in[\underline{b},\tilde{b}]^2} u\left(\bar{Q}\left(b',\bar{b}'\right)\times\bar{b}' + Q\left(b',\bar{b}'\right)\times b' + \underline{w} - b\right) + Z\left(b',\bar{b}'\right), \tag{31}$$

with continuation utility Z is defined as

$$Z\left(b',\bar{b}'\right) = \max_{p\in[\underline{p},\bar{p}]}\left\{-X(p) + \beta\left(p\times\bar{\mu}\left(\bar{b}'\right) + (1-p)\times\mu\left(b'\right)\right)\right\}, \tag{32}$$

and the value of starting in recession with debt b and in normal time with debt $\bar{b}$ are $\mu(b) = \int_{\aleph} V(b,\phi)\, dF(\phi)$ and $\bar{\mu}(\bar{b}) = u\left(\bar{w} - \left(1 - R^{-1}\right)\bar{b}\right)/(1-\beta)$, respectively.

- *The threshold renegotiation function Φ satisfies*

$$\Phi(b) = \alpha - W(b). \tag{33}$$

- *The recovery- and recession-contingent debt price functions satisfy the arbitrage conditions:*

$$\bar{Q}\left(b',\bar{b}'\right)\times\bar{b}' = \Psi\left(b',\bar{b}'\right)R^{-1}\times\bar{b}' \tag{34}$$

$$Q\left(b',\bar{b}'\right)\times b' = \left[1 - \Psi\left(b',\bar{b}'\right)\right]R^{-1}\times\Pi\left(b'\right) \tag{35}$$

where $\Pi(b')$ is the expected repayment of the recession-contingent bonds conditional on next period being a recession,

$$\Pi(b) = (1 - F(\Phi(b)))\, b + \int_{\phi_{\min}}^{\Phi(b)} \hat{b}(\phi)\times dF(\phi), \tag{36}$$

and where $\hat{b}(\phi) = \Phi^{-1}(\phi)$ is the new post-renegotiation debt after a realization ϕ.

- *The set of optimal decision rules comprises:*

  1. *A take-it-or-leave-it debt renegotiation offer:*

$$\mathbb{B}(b,\phi) = \begin{cases} \hat{b}(\phi) & \text{if} \quad \phi \le \Phi(b), \\ b & \text{if} \quad \phi > \Phi(b). \end{cases} \tag{37}$$

  2. *An optimal debt accumulation and an associated consumption decision rule:*

$$\left\langle B(\mathbb{B}(b,\phi)), \bar{B}(\mathbb{B}(b,\phi))\right\rangle = \arg\max_{(b',\bar{b}')\in[\underline{b},\tilde{b}]^2}\left\{u\left(\bar{Q}\left(b',\bar{b}'\right)\times\bar{b}' + Q\left(b',\bar{b}'\right)\times b' + \underline{w} - \mathbb{B}(b,\phi)\right) + Z\left(b',\bar{b}'\right)\right\}, \tag{38}$$

$$\begin{aligned} C(\mathbb{B}(b,\phi)) = {} & \bar{Q}\left(B(\mathbb{B}(b,\phi)), \bar{B}(\mathbb{B}(b,\phi))\right)\times\bar{B}(\mathbb{B}(b,\phi)) + \\ & Q\left(B(\mathbb{B}(b,\phi)), \bar{B}(\mathbb{B}(b,\phi))\right)\times B(\mathbb{B}(b,\phi)) + \underline{w} - \mathbb{B}(b,\phi). \end{aligned} \tag{39}$$

3. An optimal effort decision rule:

$$\Psi\left(b',\bar{b}'\right)=\arg\max_{p\in[\underline{p},\bar{p}]}\left\{-X\left(p\right)+\beta\left(p\times\bar{\mu}\left(\bar{b}'\right)+\left(1-p\right)\times\mu\left(b'\right)\right)\right\}. \tag{40}$$

- *The equilibrium law of motion of debt is* $\left(b',\bar{b}'\right)=\left\langle B\left(\mathbb{B}\left(b,\phi\right)\right),\bar{B}\left(\mathbb{B}\left(b,\phi\right)\right)\right\rangle.$
- *The probability that the recession ends is* $p=\Psi\left(b',\bar{b}'\right)$.

Equation (30) implies that there is renegotiation if and only if $\phi<\Phi\left(b\right)$. Since, *ex-post*, creditors have all the bargaining power, the discounted utility accruing to the sovereign equals the value that she would get under outright default. Thus

$$V\left(b,\phi\right)=W\left(\mathbb{B}\left(b,\phi\right)\right)=\begin{cases}W\left(b\right) & \text{if} \quad b\leq\hat{b}\left(\phi\right),\\ \alpha-\phi & \text{if} \quad b>\hat{b}\left(\phi\right).\end{cases}$$

Consider, next, the equilibrium price functions. Since creditors are risk neutral, the expected rate of return on each security must equal the risk-free rate of return. Then, the arbitrage conditions (34) and (35) ensure market clearing in the security markets and pin down the equilibrium price of securities in recession. The function $\Pi\left(b\right)$ defined in equation (36) yields the expected value of the outstanding debt b conditional on the endowment state being a recession but before the realization of ϕ. The obligation b is honored with probability $1-F\left(\Phi\left(b\right)\right)$, where $\Phi\left(b\right)$ denotes the largest realization of ϕ such that the sovereign can credibly threaten to default. With probability $F\left(\Phi\left(b\right)\right)$, debt is renegotiated to a level that depends on ϕ denoted by $\hat{b}\left(\phi\right)$. The haircut $\hat{b}\left(\phi\right)$ keeps the sovereign indifferent between accepting the creditors' offer and defaulting. This implies the following indifference condition,

$$W(\hat{b}\left(\phi\right))=\alpha-\phi. \tag{41}$$

Consider, finally, the set of decision rules. Equations (38)-(39) yield the optimal consumption-saving decisions while equation (40) yields the optimal reform effort. The effort depends on b' and $\bar{b}'$, since it is chosen after the new securities are issued. Note also that since $F\left(\Phi\left(b\right)\right)=0$ for $b'\geq\left(\Phi\right)^{-1}\left(\phi_{\max}\right)$, the bond price function (36) implies that debt exceeding this level will not raise any debt revenue. Thus, it is optimal to choose $b'\leq\left(\Phi\right)^{-1}\left(\phi_{\max}\right)$.

3.2 Decentralization Through Renegotiable One-Period Debt

We now establish a key result of the paper, namely, that there exists a unique market equilibrium with one-period renegotiable bonds satisfying Definition 1 and that this equilibrium decentralizes the COA.

We start by noting the analogy between π in the primal program and the equilibrium function $\Pi\left(b\right)$ in Equation (36). Both define the expected profit accruing to the principal or to the creditors. In particular, one can invert the equilibrium function Π and define $b\left(\pi\right)$ like $b^{pl}\left(\pi\right)$ in Equation (29) – up to replacing Φ^{pl} by the equilibrium function Φ. The decentralization result will be stated under the condition that $\pi=\Pi(b)$ (or, identically, that $b=b\left(\pi\right)$), namely, the sovereign has the same initial obligation in the COA and in the market equilibrium.

For technical reasons, we impose the following additional constraint to (31):

$$\Pi\left(b'\right)\leq\bar{b}'. \tag{LSS}$$

This constraint allows us to prove formally (as a sufficient condition) that the program describing the market equilibrium is a contraction mapping. We label this constraint Limited Short Selling (LSS) because it is equivalent to a limit on the ability of the sovereign to short sell the GDP-linked bond.[25] The LSS constraint has an intuitive interpretation: debt issuance must be such that the expected debt repayment is larger if the economy recovers than if it remains in recession – a natural feature of the allocation given the sovereign's desire to smooth consumption. The LSS constraint is never binding in equilibrium. Under this condition, we can prove that the market equilibrium is unique and decentralizes the COA.

Proposition 5 *(A) The market equilibrium subject to the LSS constraint exists and is unique.*
(B) This equilibrium decentralizes the COA of Proposition 4 conditional on the same outside option α.

We sketch here the strategy of the proof of existence and uniqueness. The complete proof is in Appendix A. We establish that there exists a unique value function W and an associated equilibrium debt threshold function $\hat{b}$ satisfying the maximization in (31) and the market clearing conditions. In particular, $\hat{b}$ and W must satisfy the indifference condition (41) for all $\phi \in \aleph$.

We proceed in two steps. First, we define an *arbitrary* debt threshold function $\delta(\phi)$. We replace the equilibrium debt repayment function $\Pi(b)$ in (36) with the exogenous debt repayment function $\Pi_\delta(b) = \int_\aleph \min\{b, \delta(\phi')\} dF(\phi')$. Then, we show by way of a contraction mapping argument that for any δ (and associated Π_δ) there exists a unique value function W_δ (with associated debt price functions) consistent with the agents' optimizing behavior and market clearing. Note that equation (41) is not necessarily satisfied for an arbitrary δ, and in general, $\delta(\phi) \neq W_\delta^{-1}(\alpha - \phi)$. Second, we prove that the mapping from, say, $\delta_n(\phi)$ to an "updated" debt limit function $\delta_{n+1}(\phi) \equiv W_{\delta_n}^{-1}(\alpha - \phi)$ is a contraction mapping. Therefore, the iteration converges to a unique fixed point $\hat{b}(\phi) = \lim_{n\to\infty} \delta_n(\phi)$ satisfying equation (41), i.e., $W_{\hat{b}}\left(\hat{b}(\phi)\right) = \alpha - \phi$. This is the equilibrium value function, namely, $W = W_{\hat{b}}$. The two contraction mapping arguments together imply that the equilibrium functions $\hat{b}$ and W exist and are unique.[26] This proof strategy may also be of independent interest as a general approach for proving uniqueness of equilibrium in infinite-horizon sovereign-debt models.[27]

Once we have established the existence and uniqueness of the market equilibrium, we show that it decentralizes the COA (part (B) of Proposition 5). To this end, we show that the market equilibrium satisfies the conditions of the primal representation of the COA, in particular $W = W^{pl}$ and $\Phi = \Phi^{pl}$ for $\pi = \Pi(b)$ (see Proposition 4 in Appendix A). The strategy for establishing equivalence between the two programs is similar to Aguiar *et al.* (2017).

The decentralization hinges on the equilibrium price functions Q and $\bar{Q}$. The key features are that (i) there exists two GDP-linked bonds; (ii) the recession-debt is renegotiable; (iii) renegotiation entails no cost; and (iv) the renegotiation protocol involves full ex-post bargaining power for the creditors. When selling one-period bonds at the prices Q and $\bar{Q}$, the sovereign implicitly offers creditors an expected future profit that takes into account the probability of renegotiation. This is equivalent to the profit promised by the social planner in the primal problem. There are two noteworthy features. First, although there is a continuum of states of nature, two securities are sufficient to decentralize the

[25] To see this, recall that in the market structure with a non-state-contingent bond and a GDP-linked bond, $b'_n = b'$ and $b'_s = \bar{b}' - b$. It follows that the LSS constraint $\Pi(b') \leq b'$ is equivalent to a limit on short-selling of the state-contingent claim, namely $b_s \geq \Pi(b_n) - b_n$. Note that the LSS constraint is weaker than a no-short-selling constraint ($b_s \geq 0$) because $\Pi(b_n) - b_n \leq 0$.

[26] We show that the mapping maps arbitrary debt limits into debt limits that are bounded, countinuous, and non-decreasing. The contraction mapping theorem guarantees that there is a unique debt limit and that this debt limit is also bounded, continuous, and non-decreasing.

[27] We thank an anonymous referee for suggesting this proof strategy.

COA. This is due to the state-contingency embedded in the renegotiable bonds. Second, the market equilibrium provides efficient dynamic incentives.

Finally, we briefly return to the discussion about one- vs. two-sided lack of commitment in the planning problem of Section 2.2.1. In footnote 7 we argued that commitment on behalf of the principal is not an issue as long as ν is sufficiently low. In the market equilibrium, this amounts to assuming that $b \geq 0$. In this case, recession debt will always remain positive along the equilibrium path. Since this claim has a non-negative value, the creditors would never unilaterally terminate existing contracts, nor would the principal have any commitment problem if she promised the corresponding utility in the planning program.

3.3 Endogenous Outside Option

Proposition 5 is stated conditional on an exogenous outside option α. In this section, we endogenize α. In a sovereign debt setting, it is natural to focus on an environement in which the sovereign can resume participation in financial markets after defaulting. For simplicity, we assume that access to new borrowing is immediate. This has a natural counterpart in the planning problem, namely, if the sovereign leaves the optimal contract, she reverts to the market equilibrium with zero debt after suffering the punishment ϕ.

With some abuse of notation, let $W(b;\alpha)$ denote the value function conditional on honoring debt b in an economy with outside option α. Similarly, let $W^{pl}(b;\alpha)$ denote the analogue concept in the primal planning allocation. Because W is monotone decreasing in α, there exists a unique α_W satisfying the fixed point $\alpha_W = W(0;\alpha_W)$. Intuitively, $W(b;\alpha_W)$ is the market (Markov-perfect) equilibrium when (out of equilibrium) outright default triggers the payment of the cost ϕ and the reversion to a market equilibrium with zero debt. The next proposition establishes that if the planner and the market can inflict is the same punishment, namely, ϕ and the reversion to a market equilibrium with zero debt, then the planner cannot improve upon the Markov-perfect equilibrium.

Proposition 6 *Consider a planning allocation and a market equilibrium such that, if the sovereign defaults, her continuation utility equals (in both cases) $\alpha_W - \phi$, where α_W is the solution to the fixed point $\alpha_W = W(0;\alpha_W)$. Then, $W(b;\alpha_W) = W^{pl}(b;\alpha_W)$ and $\mu(b;\alpha_W) = \mu^{pl}(\Pi(b);\alpha_W)$. Namely, the Markov-perfect market equilibrium decentralizes the COA conditional on the threat of reversion to a market equilibrium with zero debt.*

Proposition 6 is established under the assumption that neither the planner nor the market can inflict more severe punishments than the reversion to the Markov-perfect equilibrium. One could sustain more efficient subgame perfect equilibria if one could discipline behavior with threats that are subgame perfect but not necessary Markovian (e.g., some forms of temporary market exclusions). Interestingly, such allocations could still be decentralized by a market equilibrium with one-period renegotiable bonds, similar to the equilibrium in Definition 1. Non-Markov equilibria require that the market can coordinate expectations on a sequence of default and effort policies associated with a worse equilibrium. In reality, it may be difficult to achieve such coordination when there exists a multiplicity of anonymous market institutions to which the sovereign can turn after default.

3.3.1 Debt Overhang and Debt Dynamics

The market equilibrium inherits the same properties as the COA. A binding PC in the planning problem corresponds to an episode of sovereign debt renegotiation in the equilibrium. Thus, as long as the recession continues and debt is not renegotiated, consumption falls. When debt is renegotiated down, consumption increases discretely. The debt dynamics mirrors the dynamics of promised utility in the dual representation of the COA. A fall in promised utility corresponds to an increase of sovereign debt. In the COA of Section 2.2.2, ν decreases over time during a recession unless the PC binds and the promised utilities $\omega(\nu)$ and $\bar{\omega}(\nu)$ are increasing functions (where, recall, $\nu' = \omega(\nu)$ if the recession continues and the PC is slack). Correspondingly, as long as debt is honored and the recession continues, both recession- and recovery-contingent debt are increasing over time.[28]

The left panel of Figure 3 shows the equilibrium policy function for recession- and recovery-contingent debt (solid lines) conditional on no renegotiation and on the recession lingering. Recession-contingent debt converges to $b^{\max}$, which corresponds to the lower bound on promised utility $\underline{\nu}$ discussed in Section 2.2.2 and displayed in Figure 2. At this level, debt is renegotiated with certainty if the recession continues, and issuing more debt would not increase the expected repayment in recession. The figure also shows the level of debt b^+ corresponding to ν^- in Figure 2. In the range $b > b^+$ the sovereign would like to issue recession-debt above $b^{\max}$ but is constrained to issue $b' = b^{\max}$. However, she is not subject to any constraints when issuing recovery-contingent debt, so $\bar{b}'$ increases in b in this range.

The right panel of Figure 3 shows the equilibrium effort as a function of b (solid line). This corresponds to the lower right panel in Figure 2: it is increasing at low debt levels and falling at high debt levels. Intuitively, when debt is low, a larger debt strengthens the desire for the sovereign to escape recession (this force is also present in the first best of Proposition 1 where effort is decreasing in promised utility). However, as debt increases, the probability that debt is fully honored in recession falls. At very high debt levels, the lion's share of the returns to the effort investment accrues to creditors making moral hazard rampant. In this region, a *debt overhang* problem arises, reminiscent of Krugman (1988). In our model debt overhang is not a symptom of markets being irrational. To the opposite, it is an equilibrium outcome: a long recession may lead the sovereign to rationally choose to issue debt in the overhang region and creditors to rationally buy it. Creditors are willing to buy recession-contingent debt from a highly indebted countries in the hope of obtaining favorable terms in the renegotiation process.

3.4 Alternative Decentralizations

An alternative decentralization of the COA follows Alvarez and Jermann (2000), henceforth AJ, who show that the COA of a dynamic principal-agent model with enforcement constraints can be attained through a full set of Arrow-Debreu securities subject to appropriate borrowing constraints. In our economy, this requires markets for a continuum of securities paying off in recession – one asset for each $\phi \in \aleph$ – plus one recovery-contingent bond. In this section, we show that the AJ decentralization attains the same allocation as our decentralization through two renegotiable securities.

The proof of Proposition 5 hints at this equivalence. There, the idea is to propose an exogenous repayment schedule conditional on ϕ, denoted by $\hat{b}(\phi)$, and then to show that there exists a unique schedule that satisfies the equilibrium conditions. Such a repayment schedule can be interpreted as the set of borrowing constraints that are necessary for the AJ equilibrium to attain efficiency. While AJ requires explicit trade in a large set of markets, our model achieves the same allocation with two defaultable assets without imposing any explicit borrowing constraint.

To illustrate the result, consider an economy without moral hazard, which is the environment in the original AJ model. Suppose that the sovereign can issue securities $b'_{AJ,\phi}$ that are claims to output in the

[28] In the absence of moral hazard (e.g., if effort were contractible), consumption and both types of debt would remain constant over time unless there is renegotiation.

following period if the recession lingers and state ϕ is realized. These securities are non-renegotiable, and ex-post the sovereign can either deliver the payment $b'_{AJ,\phi}$ or default and pay the penalty ϕ. Under perfect enforcement, the security $b'_{AJ,\phi}$ sells at a price $Q_{AJ,\phi} = (1-p)f(\phi)R^{-1}$. To ensure that the sovereign has an incentive to repay in all states, AJ impose some borrowing constraints. In our model, these constraints are, for all ϕ, $b'_{AJ,\phi} \leq \hat{b}(\phi)$ where $\hat{b}(\phi) = \Phi^{-1}(\phi)$. In Appendix A we prove that the AJ equilibrium yields an allocation that is equivalent to our decentralized equilibrium under the assumption that $b'_{AJ,\phi} = \hat{b}(\phi)$ for all $\phi \leq \Phi(B(b))$ and $b'_{AJ,\phi} = B(b)$ for all $\phi > \Phi(B(b))$, where B is the equilibrium debt function and b is the payment owed in the current period. The crux of the proof is to establish that the revenue obtained from issuing recession-contingent debt in our market equilibrium is identical to that raised by issuing the full set of Arrow-Debreu securities in the AJ economy.

Proposition 7 in Appendix A establishes that an AJ equilibrium with borrowing constraints $b'_{AJ,\phi} \leq \hat{b}(\phi)$ sustains the COA even when the effort choice is subject to an IC. Our decentralization with two renegotiable securities is parsimonious relative to the AJ equilibrium that requires as many securities as there are states (in our environment, this means a continuum of securities). Parsimony is important in realistic extensions in which it is costly for creditors to verify the overall exposure of the sovereign. In our model, creditors must only verify the issued quantity for two assets, while in the AJ framework they must verify that a large number of borrowing constraints are not violated.

We conclude this section by mentioning that, in an earlier draft of this paper, we discussed decentralization with an even richer market structure and stronger informational assumptions. In particular, we showed that if reform effort is observable and verifiable the market equilibrium decentralizes the COA without moral hazard of Proposition 2 as long as there exists markets for *effort-deviation* securities whose return is contingent on the reform effort. In reality, we do not see debt contracts conditional on reform effort, arguably because the extent to which a country passes and, especially, enforces reforms is opaque.

4 Less Complete Markets

In this section, we consider a market equilibrium subject to more severe market frictions, i.e., the sovereign can issue only a one-period non-contingent bond. This environment is interesting because in the real world government bonds typically promise repayments that are independent of the aggregate state. We take this market incompleteness as exogenous and study its effect on the allocation. We first maintain the same renegotiation protocol as in the earlier sections; then, as an extension, we rule out renegotiation as in Eaton and Gersovitz (1981). We use superscripts R and NR for policy functions in the one-asset economy with and without renegotiation, respectively.

The one-asset market equilibrium does not attain the COA conditional on α. In the COA, the planner trades off the gains from risk sharing against the cost of moral hazard in an optimal way. In Section 3, we proved that two securities are sufficient to replicate the COA. However, when only one asset is available, the shortage of instruments forces a particular correlation structure between future consumption in normal time and recession that is generally suboptimal, resulting in less risk sharing in equilibrium.

A formal definition and characterization of equilibrium is deferred to Appendix B, where we also prove existence and uniqueness of equilibrium in the special case of exogenous recovery probability (Proposition 8). Here, we emphasize the salient features of the equilibrium.

Consider, first, the equilibrium policy function for effort $\Psi^R(b)$, where b now denotes a claim to one unit of output next period, irrespective of the endowment state. The FOC for the effort choice yields $X'\left(\Psi^R(b')\right) = \beta\left[\bar{\mu}^R(b') - \mu^R(b')\right]$. The equilibrium features debt overhang, namely, the effort function Ψ^R is decreasing in b in a range of high b.[29] Conversely, $\Psi^R(b)$ is increasing in a range of low b. Thus, effort is non-monotone in debt and shares the qualitative properties of the benchmark allocation with two securities.

Consider, next, consumption dynamics. Even in the one-asset economy the risk of renegotiation introduces some state contingency, since debt is repaid with different probabilities under recession and normal time. This provides a partial substitute for state-contingent contracts, although now this is not sufficient to decentralize the COA. Recall that in the benchmark economy consumption was determined by two CEEs, (16) and (20). Issuing optimally two types of debt allowed the sovereign to mimic the planner's ability to control promised utility in each of the two states. This is not feasible in the one-asset economy: there is only one CEE, which pins down the *expected* marginal rate of substitution in consumption conditional on debt being honored. In Appendix B (Proposition 9), we show that the CEE with non-contingent debt takes the form

$$\mathbb{E}\left\{\frac{u'(c')}{u'(c)}|_{\text{debt is honored at } t+1}\right\} = 1 + \frac{\frac{d}{db'}\Psi^R(b') \times \left[b' - \Pi^R(b')\right]}{\Pr(\text{debt is honored at } t+1)} \tag{42}$$

where $b' - \Pi^R(b')$ is the difference between the expected debt repayment if the economy recovers or remains in recession, and c' denotes future consumption.

Consider, first, the case without moral hazard, i.e., $d\Psi^R/db' = 0$. In the equilibrium with GDP-linked bonds, the sovereign could obtain full insurance against the realization of the endowment shock. In the one-asset economy, this is no longer true: the CEE (42) requires that the expected marginal utility in the CEE be equal to the current marginal utility. For this to be true, consumption growth must be positive if the recession ends and negative if it lingers.

In the general case, the market incompleteness interacts with the moral hazard friction introducing a strategic motive for debt. When the outstanding debt is low, then $d\Psi^R/db' > 0$ and the right-hand side of (42) is larger than unity. In this case, issuing more debt strengthens the *ex-post* incentive to exert reform effort. The CEE implies then higher debt accumulation and lower future consumption growth than in the absence of moral hazard. In contrast, in the region of debt overhang ($d\Psi^R/db' < 0$) more debt weakens the *ex-post* incentive to exert reform effort. To remedy this, the sovereign issues less debt than in the absence of moral hazard. Thus, when debt is large the moral hazard friction magnifies the reduction in consumption insurance.

Figure 3 shows the policy functions for debt and effort in the one-asset economy (dashed lines) and in the two-security economy, with outside options α_R and α_W corresponding, respectively, to the fixed points $\alpha_R = W^R(0;\alpha_R)$ and $\alpha_W = W(0;\alpha_W)$.[30] Debt is always higher (lower) than recession-debt (recovery-debt) in the two-security economy. Therefore, there is less consumption smoothing in the one-asset economy. Effort is hump-shaped in both economies. Conditional on debt, effort is higher

[29] The reason for debt overhang is similar to that in the benchmark economy with GDP-linked bonds. There exists a threshold debt $b_R^{\max} = \Phi^{-1}(\phi_{\max})$ such that debt is renegotiated with probability one if $b' \geq b_R^{\max}$ and the recession continues, while the debt is honored with positive probability if $b' < b_R^{\max}$. Recall that if the recession ends, debt is repaid even for $b' \geq b_R^{\max}$. Thus, for $b' \geq b_R^{\max}$ the difference $\bar{\mu}^R(b') - \mu^R(b')$ is decreasing in b' implying a decreasing effort, i.e., debt overhang. This feature extends to a contiguous range below $b_R^{\max}$.

[30] The results are very similar if we use the same outside option in the two cases. Note that the policy rules for debt accumulation and effort feature discontinuities. Figure 4 in Appendix B illustrates that also consumption has discontinuities. Proposition 9), proves that such discontinuities only arise in correspondence of debt levels that are never chosen in equilibrium (unless there is renegotiation). Moreover, the generalized Envelope Theorem in Clausen and Strub (2016) ensures that the FOCs are necessary for interior optima in spite of the discontinuity of the decision rules.

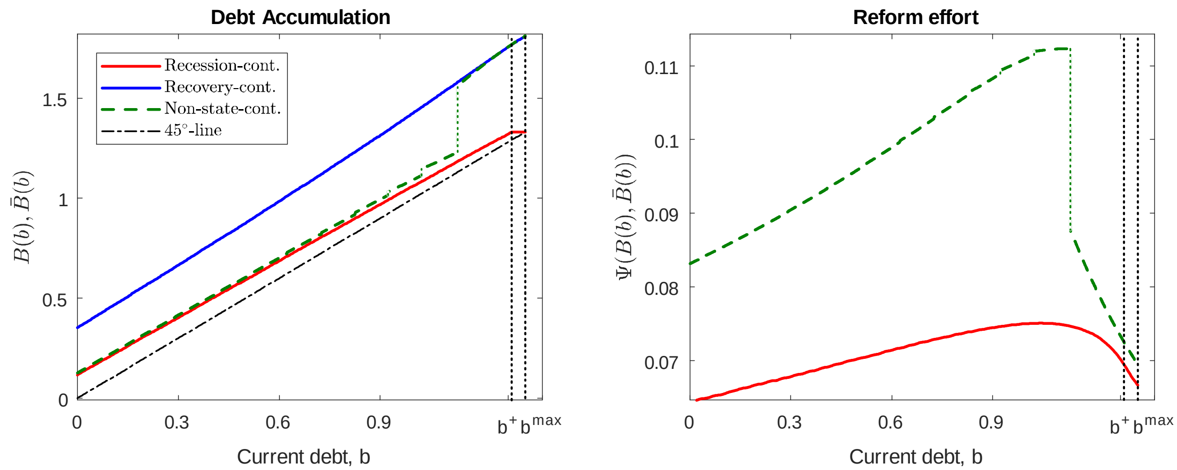

Figure 3: Policy functions conditional on the maximum cost realization ϕ_{max}. The solid lines show the policy functions for the constrained efficient Markov equilibrium. For comparison, the dashed line in each panel shows the policy functions of the one-asset economy.

in the one-asset economy than in the COA reflecting the fact that there is less insurance against the continuation of a recession. This increases the sovereign's incentives to leave the recession.

4.1 No Renegotiation

Next, we consider a version of an Eaton-Gersovitz economy where we shut down renegotiation.[31] Namely, we assume that the sovereign can either default or fully honor the outstanding debt. This alternative environment has three implications. First and foremost, there is costly default in equilibrium. The real costs suffered by the sovereign yields no benefit to creditors, in contrast with the equilibrium with renegotiation, where no real costs accrue and creditors recover a share of the face value of debt. Second, renegotiation affects the price function of debt, and thus the incentive for the sovereign to accumulate debt. In particular, the bond price now becomes $Q^{NR}(b') = R^{-1}\left(\Psi^{NR}(b') + \left[1 - \Psi^{NR}(b')\right]\left(1 - F\left(\Phi^{NR}(b')\right)\right)\right)$. When renegotiation is ruled out the lender expects a lower repayment. Thus, the risk premium associated with each debt level is higher, making it more costly for the sovereign to roll over debt. Therefore, the sovereign will be warier of debt accumulation. This limits consumption smoothing and lowers welfare. Third, conditional on the debt level, the range of ϕ for which debt is honored is different.[32]

Figure 4 in Appendix B compares the policy functions in a one-asset economy without renegotiation (solid lines) with a one-asset economy with renegotiation (dashed lines). We use the outside options α_R (defined above) for the economy with renegotiation and α_{NR} such that $\alpha_{NR} = W^{NR}(0; \alpha_{NR})$ for the no-renegotiation economy. Ruling out renegotiation implies lower consumption for each debt level. The sovereign exerts more effort when it cannot renegotiate (but can default on) its debt, reflecting the fact that the debt price is lower in recession and that there is less insurance. Finally, the probability

[31] Our environment here differs from the standard Eaton-Gersovitz economy insofar as the high endowment state is an absorbing state and we have shocks to both the aggregate state and the outside option.

[32] More formally, in the equilibrium with renegotiation the sovereign renegotiates if $\phi < W^R(0) - W^R(b)$ whereas in the no-renegotiation equilibrium she defaults if $\phi < W^{NR}(0) - W^{NR}(b)$, where W^{NR} is the value function under no renegotiation and recession. As long as W^{NR} is falling more steeply in b than W^R, then, conditional on the debt level, the sovereign is more likely to fully honor the debt in the benchmark equilibrium than in the no-renegotiation equilibrium.

of honoring debt is lower without than with renegotiation.

4.2 Assistance Program

The possibility of market failures provides scope for policy intervention. Consider an assistance program conducted by an international agency like the IMF. The assistance program mimics the COA through a sequence of transfers to the sovereign during recession in exchange of the promise of a repayment once the recession is over. The IMF can commit but has – as the planner – limited enforcement: it can inflict the same stochastic punishment (ϕ) as can markets.

The program has two key features. First, the terms of the program are renegotiable: whenever the country credibly threatens to abandon it, the IMF sweetens the deal by increasing the transfers and reducing the payment the country owes when the recession ends. When there is no credible default threat, the transfer falls over time, implying the constrained optimum sequence of declining consumption and time-varying reform effort prescribed by the COA. Second, when the recession ends, the IMF receives a payment from the sovereign, financed by issuing debt in the market. This payment depends on the length of the recession and on the history of renegotiations.

More formally, let ν^o denote the discounted utility guaranteed to the sovereign when the program is first agreed upon. At the beginning of that period, the IMF buys the debt b_0 with an expenditure $\Pi(b_0)$.[33] Then, the IMF transfers to the country $T(\nu^o) = c(\nu^o) - \underline{w}$ where $c(\nu^o)$ is as in the COA of Proposition 3 unless the realization of ϕ makes the sovereign prefer to terminate the program, in which case the country receives $T_\phi = c_\phi - \underline{w}$, where, again, c_ϕ is as in Proposition 3. In the subsequent periods, consumption and promised utility evolve in accordance with the law of motion of the COA. In other words, the IMF commits to a sequence of state-contingent transfers that mimics the planning allocation, while the sovereign exerts the incentive-compatible reform effort level. As soon as the recession ends, the country owes a debt $\bar{b}_N$ to the international institution, determined by the equation $R^{-1} \times \bar{b}_N = \bar{c}(\bar{\omega}_N) - \bar{w} + \bar{b}_N$, where the discounted utility $\bar{\omega}_N$ depends on the duration of the recession and on the history of realizations of ϕ. Note that $\bar{c}$ is first-best consumption, exactly as in the COA, and that in normal time the country resumes market access to refinance its debt at the constant level $\bar{b}_N$.

The initial promised utility ν^o is determined by how much the IMF is willing to commit to the assistance program. A natural benchmark is the ν^o that makes the IMF break even in expectation. Whether, ex-post, the IMF makes net gains or losses depends on the duration of the recession and on the realized sequence of ϕ's.

Can such a program improve upon the market allocations? The answer hinges on the extent of market and contract incompleteness. If there exists a market for GDP-linked bonds, the renegotiation process is frictionless and efficient, and the sovereign can access markets after leaving the contract, the assistance program cannot improve upon the market allocation. More formally, $\nu^o = P^{-1}(\Pi(b_0))$. This follows immediately from our decentralization result in Section 3. However, in one-asset economies (with or without renegotiation), the assistance program improves efficiency upon the market equilibrium. Interestingly, the assistance program would in this case yield higher utility than the Markov equilibrium with GDP-linked debt. Since market incompleteness makes deviations less attractive, the IMF has access to a more powerful threat to discipline the sovereign. The more incomplete are markets, the lower the sovereign's outside option and the closer to the first best can an assistance program get.

[33] Π depends on the market structure. If there are two securities, then Π is given by equation (36). If there is only one asset, the corresponding definition given in Appendix B applies.

The assistance program would be even more powerful if the IMF could take direct control over the reform process. In this case, the program could implement the COA without moral hazard of Section 2.2.1. Clearly, policies infringing on a country's sovereignty run into severe politico-economic constraints whose discussion goes beyond the scope of our paper.

5 Concluding Remarks

This paper proposes a novel theory of sovereign debt dynamics under limited enforcement and moral hazard. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and endogenous. The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need for structural reforms – are salient features of the European debt crisis during the last decade.

We show that a market equilibrium with renegotiable one-period GDP-linked securities implements the COA. The crux of this result is that, if creditors have full bargaining power *ex-post*, the renegotiable securities stand in for missing markets for state-contingent debt. In addition, these markets provide the optimal trade-off between insurance and incentives that a planner subject to the same informational constraints and the same disciplining instruments as the market can achieve.

We also study the effect of additional exogenous restrictions on market arrangements, such as assuming that the sovereign can only issue non-contingent debt and ruling out renegotiation. In this case, the market equilibrium is not constrained efficient. We discuss an assistance program that can restore efficiency.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. Voters may have an incentive to elect a government that undervalues the cost of default. In our current model, however, the stochastic process governing the creditor's outside option is exogenous, and is outside of the control of the sovereign and creditors.

Second, again for simplicity, we assume that renegotiation is costless and that creditors have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results. However, we are confident that the gist of the results is robust to these extensions.

Third, we make the convenient assumption that reform effort and consumption are separable in the utility function. If reforms are especially costly during recession, our results would be weakened. Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. We leave the exploration of these avenues to future work.

While the current paper has a theoretical focus, in Müller *et al.* (2019), we study a generalized version of the model where we relax some of the assumptions we made here to retain analytical tractability. That paper quantifes the effects of policy intervention under different informational assumptions. We find that a calibrated model can successfully replicate salient empirical moments not targeted in the calibration, including the bond spread, the frequency of renegotiations, the average haircut of the debt's face value, and its variance across renegotiation episodes.

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A Appendix A

This appendix contains the proofs of main lemmas and propositions, and the statements of Propositions 4 and AlvarezJermann_equivalence. Additional technical material can be found in the online Appendix B.

Proof of Lemma 1. To prove this result, we ignore technicalities related to the continuum of states. Namely, we assume that there exist N states with associated positive probabilities, and view the continuum as an approximation of the discrete state space for $N \to \infty$. This is without loss of generality. The proof involves two steps.

1. We start by proving that, if $\phi_2 > \phi_1$, then, $\mu_{\phi_1} > \alpha - \phi_1 \Rightarrow \mu_{\phi_2} = \mu_{\phi_1} > \alpha - \phi_2$. To see why, recall that the planner's objective is to deliver the promised utility ν given by (2) in a profit-maximizing way. Since u, P, and $\bar{P}$ are strictly concave, and X is strictly convex, then, profit maximization is attained by setting $c_{\phi_2} = c_{\phi_1}$, $p_{\phi_2} = p_{\phi_1}$, $\omega_{\phi_2} = \omega_{\phi_1}$, and $\bar{\omega}_{\phi_2} = \bar{\omega}_{\phi_1}$. This planning choice implies that $\mu_{\phi_2}(\nu) = \mu_{\phi_1}(\nu)$. This is feasible since the assumption that the PC is slack in state ϕ_1 implies that *a fortiori* the PC is slack in state ϕ_2. Two cases are then possible: either $\exists \tilde{\phi}(\nu)$ such that $\mu_\phi(\nu) = \alpha - \tilde{\phi}(\nu)$ for all $\phi \geq \tilde{\phi}(\nu)$; or $\mu_\phi(\nu) = \mu(\nu) > \alpha - \phi_{\min}$ $\forall \phi \in [\phi_{\min}, \phi_{\max}]$. The latter case would imply that no PC ever binds and can be ignored.

2. Next, we prove that, if $\hat{\phi}_2 > \hat{\phi}_1$, then, $\mu_{\hat{\phi}_2} = \alpha - \hat{\phi}_2$ implies that $\mu_{\hat{\phi}_1} = \alpha - \hat{\phi}_1$. To derive a contradiction, suppose that this is not the case, and that $\mu_{\hat{\phi}_1} > \alpha - \hat{\phi}_1 > \mu_{\hat{\phi}_2}$ (the opposite inequality would violate the PC and is not feasible). Then, the planner could deliver to the agent the promised utility ν by uniformly increasing ω_2, and $\bar{\omega}_2$ and reducing ω_1, and $\bar{\omega}_1$ also uniformly, so as to keep ν unchanged, while leaving consumption and effort constant (it is easy to check that this is feasible). The strict concavity of P and $\bar{P}$ guarantees that this change increases profits. Let $\tilde{\phi}(\nu)$ denote the largest ϕ such that $\mu_{\tilde{\phi}(\nu)}(\nu) = \alpha - \tilde{\phi}(\nu)$. Then, $\mu_\phi(\nu) = \alpha - \phi$ for all $\phi \leq \tilde{\phi}(\nu)$.

Parts 1. and 2. above jointly establish that the threshold $\tilde{\phi}(\nu)$ is unique. ■

Proof of Proposition 3. This proof builds on Lemma 3.2 in Appendix B establishing that P is strictly decreasing and differentiable with $P'(\nu) = -1/u'(c(\nu)) < 0$ for all $\nu > \underline{\nu}$.

The planner solves (1) subject to (2)-(5), and (15). The Lagrangian yields

$$\begin{aligned}
\mathcal{L} \quad = \quad & \int_{\aleph} \left[\underline{w} - c_\phi + \beta \left(p_\phi \bar{P}(\bar{\omega}_\phi) + (1-p_\phi) P(\omega_\phi)\right)\right] f(\phi) d\phi \\
& + \vartheta \left(\int_{\aleph} \left(u(c_\phi) - X(p_\phi) + \beta\left(p_\phi \bar{\omega}_\phi + (1-p_\phi)\omega_\phi\right)\right) f(\phi) d\phi - \nu \right) \\
& + \int_{\aleph} \lambda_\phi \left(u(c_\phi) - X(p_\phi) + \beta\left(p_\phi \bar{\omega}_\phi + (1-p_\phi)\omega_\phi\right) - [\alpha - \phi]\right) d\phi \\
& + \int_{\aleph} \chi_\phi \left(-X'(p_\phi) + \beta\left(\bar{\omega}_\phi - \omega_\phi\right)\right) d\phi + \int_{\aleph} \theta_\phi \left(\omega_\phi - \underline{\nu}\right) d\phi, \qquad (43)
\end{aligned}$$

where $\lambda_\phi \geq 0$, $\theta_\phi \geq 0$, ϑ, and χ_ϕ denote the multipliers. The first-order conditions (FOCs) with respect to c_ϕ, $\bar{\omega}_\phi$, ω_ϕ, p_ϕ and χ_ϕ yield

$$\begin{aligned}
0 &= -f(\phi) + [\vartheta f(\phi) + \lambda_\phi]\, u'(c_\phi), \\
0 &= \beta p_\phi \bar{P}'(\bar{\omega}_\phi)\, f(\phi) + [\vartheta f(\phi) + \lambda_\phi]\, \beta p_\phi + \chi_\phi \beta \\
0 &= \beta(1-p_\phi) P'(\omega_\phi) f(\phi) + [\vartheta f(\phi) + \lambda_\phi]\, \beta(1-p_\phi) - \chi_\phi \beta, \quad \forall \omega_\phi > \underline{\nu} \\
0 &= \beta \left[\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)\right] f(\phi) + [\vartheta f(\phi) + \lambda_\phi] \left(-X'(p_\phi) + \beta(\bar{\omega}_\phi - \omega_\phi)\right) - \chi_\phi X''(p_\phi) \\
0 &= -X'(p_\phi) + \beta(\bar{\omega}_\phi - \omega_\phi)
\end{aligned}$$

Lemma 3.3 in Appendix B rules out corner solutions establishing that the FOCs are necessary optimality conditions. The envelope condition yields $-P'(\nu) = \vartheta$ for all $\nu > \underline{\nu}$ (note that P is only differentiable at the interior support of ν), while the slackness condition for θ_ϕ implies $0 = \theta_\phi(\omega_\phi - \underline{\nu})$. Combining the FOCs and the envelope condition, and noting that $-\bar{P}'(\bar{\omega}_\phi) = 1/u'(\bar{c}(\bar{\omega}_\phi))$, yields

$$\frac{1}{u'(c_\phi)} = -P'(\nu) + \frac{\lambda_\phi}{f(\phi)}, \qquad \forall \nu > \underline{\nu}, \tag{44}$$

and equations (16) and (17) in the text. Note that we have also used the facts that $p_\phi = \Upsilon(\bar{\omega}_\phi - \omega_\phi)$ (established in the text, see equation (15)) and $\Upsilon'(\bar{\omega}_\phi - \omega_\phi) = \beta / X''(\Upsilon(\bar{\omega}_\phi - \omega_\phi))$. When $\lambda_\phi = 0$, then (44) implies $-P'(\nu) = 1/u'(c(\nu)) = \vartheta > 0$ for all $\nu > \underline{\nu}$ (cf. part 3 of the proposition).

To prove that $\tilde{\phi}(\nu)$ is strictly decreasing, recall that, by Lemma 1, all PCs associated with $\phi < \tilde{\phi}(\nu)$ are binding, while all those with $\phi \geq \tilde{\phi}(\nu)$ are slack. Hence, the PK can be written as $\nu = \int_{\phi_{\min}}^{\tilde{\phi}(\nu)} (\alpha - \phi)\, dF(\phi) + \left(1 - F\left(\tilde{\phi}(\nu)\right)\right)\left(\alpha - \tilde{\phi}(\nu)\right)$, which can be rearranged to yield equation (13). Standard differentiation establishes then that $\tilde{\phi}(\nu)$ is strictly decreasing.

Consider part 1 of the proposition. For all $\phi < \tilde{\phi}(\nu)$, the PC is binding and holds with equality. Then, the optimal choice is independent of ν and c_ϕ, ω_ϕ, $\bar{\omega}_\phi$, and θ_ϕ are pinned down by the PC (4) holding with equality, and by the FOCs (16)–(18).

Consider, next, part 2. For all $\phi \geq \tilde{\phi}(\nu)$ the PC is slack, and Lemma 1 implies equation (19). The solution is history dependent: $c_\phi = c(\nu)$, $\omega_\phi = \omega(\nu)$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, $\theta_\phi = \theta(\nu)$, where $c(\nu)$, $\omega(\nu)$, $\bar{\omega}(\nu)$, and $\theta(\nu)$ are determined by equations (16)–(19). To prove that $c(\nu)$ is an increasing function, note that, since P and u are strictly concave, then (44) implies that $c(\nu)$ is strictly increasing for $\nu > \underline{\nu}$. Consumption is also increasing in ν at the lower bound, $c(\underline{\nu}) < c(\nu)\ \forall \nu > \underline{\nu}$, as the following argument proves by contradiction: Suppose that $\exists \nu > \underline{\nu}$ such that $c(\underline{\nu}) \geq c(\nu)$. The optimal allocation associated with $c(\nu)$ yields then utility $\nu > \underline{\nu}$. But, then, $c(\underline{\nu}) \geq c(\nu)$ would violate the PK. A contradiction.

Next, we prove that when the PC is slack and the recession lingers promised utility falls over time, i.e., $\omega_\phi = \omega(\nu) = \nu' \leq \nu$ if $\theta_\phi = 0$ (i.e., as long as ν is sufficiently large). To this aim, we establish, first, that $\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) > 0$ in the range $\nu > \nu^-$ and, second, that this implies that $\bar{c}(\bar{\omega}_\phi(\nu)) > c_\phi(\nu) \geq c(\nu) > c(\omega(\nu))$ and $\omega(\nu) = \nu' \leq \nu$. Suppose, to derive a contradiction, that $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) \leq 0$ for $\theta_\phi = 0$. Then (16), (17), and (18) imply the following consumption ordering

$$\bar{c}(\bar{\omega}_\phi) \leq c_\phi \leq c(\omega_\phi). \tag{45}$$

We show that this impossible. To see why, suppose first that $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) \leq 0$ and $\omega_\phi \geq \nu$. Since in normal state consumption, promised utility and profits are constant, we can write

$$\begin{aligned}
\bar{c}(\bar{\omega}_\phi) &= \bar{w} - \bar{P}(\bar{\omega}_\phi) + \beta \bar{P}(\bar{\omega}_\phi) = \bar{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}\bar{P}(\bar{\omega}_\phi) \\
&> \underline{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}\bar{P}(\bar{\omega}_\phi) \\
&\geq \underline{w} - P(\omega_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}P(\omega_\phi) \\
&\geq \underline{w} - P(\nu) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}P(\omega_\phi) = c_\phi.
\end{aligned}$$

The second inequality follows from the fact that $[(1-p_\phi)/R - 1]\,\bar{P}(\bar{\omega}_\phi) \geq [(1-p_\phi)/R - 1]\,P(\omega_\phi)$ since $[(1-p_\phi)/R - 1] < 0$ and, per hypothesis, $\bar{P}(\bar{\omega}_\phi) \leq P(\omega_\phi)$. The last inequality follows from $\omega_\phi \geq \nu$ (again, per hypothesis) and $P'(\nu) < 0$. The conclusion that $\bar{c}(\bar{\omega}_\phi) > c_\phi$ contradicts the ordering in (45). Thus, we can rule out that $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) \leq 0$ and $\omega_\phi \geq \nu$ for $\nu > \nu^-$. Nor can $\omega_\phi < \nu$, because consumption is strictly increasing in promised utility, hence, $\omega_\phi < \nu \Rightarrow c(\omega_\phi) < c(\nu) \leq c_\phi(\nu)$, again contradicting the ordering in (45). We conclude that $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) > 0$ for $\nu > \nu^-$.

Next, combining $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) > 0$, $\theta_\phi = 0$, (16), (17), and (18) yields

$$\bar{c}(\bar{\omega}_\phi(\nu)) > c_\phi(\nu) \geq c(\nu) > c(\omega(\nu)).$$

Since $c(\nu)$ is strictly increasing in ν, the last inequality also implies that $\omega(\nu) < \nu$. As soon as $\omega(\nu) = \underline{\nu}$, promised utility and consumption remain constant.

Finally, we prove that the effort function $p(\nu) = \Upsilon(\bar{\omega}(\nu) - \omega(\nu))$ is strictly increasing in a range of low ν that includes $\nu \leq \nu^-$. We do so by establishing that $\bar{\omega}(\nu)$ is strictly increasing and $\omega(\nu)$ constant on the interval $[\underline{\nu}, \nu^-]$. We invoke Topkis' theorem to show that $\bar{\omega}(\nu)$ is strictly increasing in ν when $\omega(\nu) = \underline{\nu}$, and also that $\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) > 0$ for $\nu \leq \nu^-$. For all $\phi \geq \tilde{\phi}(\nu)$, the planner solves the following maximization problem

$$\langle \omega(\nu), \bar{\omega}(\nu) \rangle = \max_{\omega \in [\underline{\nu}, \tilde{\omega}], \bar{\omega} \in [\underline{\omega}, \tilde{\omega}]} \underline{w} - u^{-1}(x(\nu, \omega, \bar{\omega})) + \beta \left[\begin{array}{c} \Upsilon(\bar{\omega} - \omega)\bar{P}(\bar{\omega}) \\ +(1 - \Upsilon(\bar{\omega} - \omega))P(\omega) \end{array} \right],$$

where $x(\nu, \omega, \bar{\omega}) = \alpha - \tilde{\phi}(\nu) + X(\Upsilon(\bar{\omega} - \omega)) - \beta\left[\Upsilon(\bar{\omega} - \omega)\bar{\omega} + (1 - \Upsilon(\bar{\omega} - \omega))\omega\right]$. Since

$$\frac{u''(x(\nu, \omega, \bar{\omega}))\tilde{\phi}'(\nu)}{u'(x(\nu, \omega, \bar{\omega}))^2} \times \beta \Upsilon(\bar{\omega} - \omega) > 0,$$

then, for given ω, the objective function is supermodular in $(\nu, \bar{\omega})$. Topkis' theorem implies then that $\bar{\omega}(\nu)$ is strictly increasing in ν when $\omega(\nu) = \underline{\nu}$. This also implies that $\bar{P}(\bar{\omega}(\nu)) - P(\underline{\nu}) \geq \bar{P}(\bar{\omega}(\nu^-)) - P(\underline{\nu}) > 0$, for $\nu \leq \nu^-$, thereby establishing that, $\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) > 0$ for all ν (part 3 of the proposition). Since $\bar{\omega}(\nu)$ is strictly increasing in ν and $\omega(\nu)$ is constant for $\nu \leq \nu^-$, then $\bar{\omega}(\nu) - \omega(\nu)$, and hence effort, are strictly increasing in the range $\nu \in [\nu^-, \underline{\nu}]$. By continuity, the result extends to a contiguous range of $\nu > \nu^-$. This concludes the proof of the proposition. ■

Proposition 4 *The value function of the primal COA satisfies equation (26). Given π and ϕ, the optimal choice of consumption is $c_\phi = C^{pl}(b_\phi(\pi))$, where*

$$C^{pl}(b_\phi(\pi)) = \underline{w} - b_\phi(\pi) + \beta \left[\begin{array}{c} \Psi^{pl}\left[\Pi^{pl}(b_\phi(\pi)), \bar{\Pi}^{pl}(b_\phi(\pi))\right] \cdot \bar{\Pi}^{pl}(b_\phi(\pi)) \\ +\left(1 - \Psi^{pl}\left[\Pi^{pl}(b_\phi(\pi)), \bar{\Pi}^{pl}(b_\phi(\pi))\right]\right) \cdot \Pi^{pl}(b_\phi(\pi)) \end{array} \right], \qquad (46)$$

$\Psi^{pl}(\pi', \bar{\pi}') = \Upsilon(\bar{\mu}^{pl}(\bar{\pi}') - \mu^{pl}(\pi'))$, $b_\phi(\pi)$ *is as in (27)–(28), and*

$$\begin{aligned}\left\langle \bar{\Pi}^{pl}\left(b_\phi\left(\pi\right)\right), \Pi^{pl}\left(b_\phi\left(\pi\right)\right)\right\rangle &= \arg \max_{\bar{\pi}' \in [\underline{\pi}, \tilde{\pi}], \pi' \in [\underline{\pi}, P(\alpha - \mathbb{E}[\phi])]} u\left(\underline{w} - b_\phi + \beta \Psi^{pl}(\pi', \bar{\pi}')\bar{\pi}' + \beta(1 - \Psi^{pl}(\pi', \bar{\pi}'))\pi'\right) \\ &\quad - X(\Psi^{pl}(\pi', \bar{\pi}')) + \beta \Psi^{pl}(\pi', \bar{\pi}')\bar{\mu}^{pl}(\bar{\pi}') + \beta(1 - \Psi^{pl}(\pi', \bar{\pi}'))\mu^{pl}(\pi').\end{aligned}$$

Moreover,

$$\left\langle \bar{\Pi}^{pl}\left(b_\phi\left(\pi\right)\right), \Pi^{pl}\left(b_\phi\left(\pi\right)\right)\right\rangle = \left\langle \bar{P}(\bar{\omega}_\phi(\nu)), P(\omega_\phi(\nu))\right\rangle. \tag{47}$$

Proof of Proposition 4. We start by proving that the proposed allocation is feasible, i.e., satisfies the set of constraints of the primal problem. Since $c_\phi = \underline{w} - b_\phi + \beta p_\phi \bar{\pi}'_\phi + \beta(1 - p_\phi)\pi'_\phi$, we can rewrite (22) as $\mu^{pl}(\pi) = \max_{b_\phi} \int_{\aleph} W^{pl}(b_\phi) dF(\phi)$, where:

$$\begin{aligned}W^{pl}(b_\phi) &= \max_{\bar{\pi}' \in [\underline{b}, \tilde{b}], \pi' \in [\underline{\pi}, P(\alpha - \mathbb{E}[\phi])]} u\left(\underline{w} - b_\phi + \beta \Psi^{pl}(\pi', \bar{\pi}')\bar{\pi}' + \beta(1 - \Psi^{pl}(\pi', \bar{\pi}'))\pi'\right) \\ &\quad - X(\Psi^{pl}(\pi', \bar{\pi}')) + \beta \Psi^{pl}(\pi', \bar{\pi}')\bar{\mu}^{pl}(\bar{\pi}') + \beta(1 - \Psi^{pl}(\pi', \bar{\pi}'))\mu^{pl}(\pi').\end{aligned} \tag{48}$$

Here, $\Psi^{pl}(\pi', \bar{\pi}') = \Upsilon(\bar{\mu}^{pl}(\bar{\pi}') - \mu^{pl}(\pi'))$ is the incentive compatible effort. Note that the duality property $\mu^{pl}(\pi_{\max}) = P^{-1}(\pi_{\max}) = \alpha - \mathbb{E}[\phi]$ allows us to write the upper bound for π' in (28) explicitly as $\pi_{\max} = P(\alpha - \mathbb{E}[\phi])$. The problem is subject to the constraints (23) and (24). It is immediate to verify that the proposed solution satisfies both constraints. The fact that (23) is satisfied is proved in the text. That (24) is satisfied follows from the observation that (27) implies that $W^{pl}(b_\phi(\pi)) \geq \alpha - \phi$, $\forall \phi \in \aleph$, thus the set of PCs are satisfied.

Next, we verify that the proposed primal solution yields the same utility as the dual COA, i.e., $\mu^{pl}(\pi) = \nu$. The expression of $\mu^{pl}(\pi)$ in (26) yields

$$\begin{aligned}\mu^{pl}(\pi) &= \left[1 - F(\Phi^{pl}(\pi))\right]\left(\alpha - \Phi^{pl}(\pi)\right) + \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} (\alpha - \phi)\, dF(\phi) \\ &= \left[1 - F(\tilde{\phi}(\nu))\right]\left(\alpha - \tilde{\phi}(\nu)\right) + \int_{\phi_{\min}}^{\tilde{\phi}(\nu)} (\alpha - \phi)\, dF(\phi) = \nu\end{aligned}$$

Finally, we prove that (27) implies the optimal solution for consumption. The fact that the optimal choices of future promised profits are such that $\bar{\pi}' = \bar{\Pi}^{pl}(b_\phi(\pi)) = \bar{P}(\bar{\omega}_\phi(\nu))$ and $\pi' = \Pi^{pl}(b_\phi(\pi)) = P(\omega_\phi(\nu))$ follow, respectively, from full commitment under normal times and from the duality property that $\pi = P(\nu)$ for all feasible ν's. Moreover, we have already established that setting $b_\phi = b_\phi(\pi)$ as in (27) yields the expected utility ν of the COA. Since the utility function is concave, the consumption plan maximizing expected utility must be unique. Given the definition of c_ϕ, it must therefore be optimal to set $c_\phi = C^{pl}(b_\phi(\pi))$ as in (46), and consumption must be the same as in the dual allocation in Proposition 3. ■

Proof of Proposition 5. PART (A): The proof establishes that $\left\langle W, \hat{b}\right\rangle$ is the unique fixed-point of two contracting operators of the functional equations (31) and (41).

Road Map: We proceed in two steps: **Step 1**: We define an inner operator T_δ that maps value functions into value functions conditional on an *arbitrary* debt threshold function $\delta(\phi)$. We show that this operator has a unique fixed point $W_\delta = \lim_{n \to \infty} T_\delta^n(\gamma)$ that satisfies (31). **Step 2**: We define an outer operator S that maps debt threshold functions into debt threshold functions. We show that this operator has a unique fixed point $\hat{b}(\phi) = \lim_{n\to\infty} S^n(\delta)$ that satisfies (41) when W_δ is evaluated at $\delta(\phi) = \hat{b}(\phi)$, i.e., $W_{\hat{b}}(\hat{b}(\phi)) = \alpha - \phi$. Let $W = W_{\hat{b}}$. Then, the fixed point $\left\langle W, \hat{b} \right\rangle$ must be unique. The uniqueness of V follows then from (30).

Step 1: Let Γ be the space of bounded, continuous, and non-increasing functions $\gamma : [\underline{b}, \tilde{b}] \to [\alpha - \phi_{\max}, u(\bar{w} + (1-\beta)\underline{b})/(1-\beta)]$ with $\gamma(\tilde{b}) = \alpha - \phi_{\max}$. Moreover, let Λ be the space of bounded and continuous functions $\delta : \aleph \to [\underline{b}, \tilde{b}]$. Define $d_\infty(y, z) \equiv \sup_{x\in X} |y(x) - z(x)|$ such that (Γ, d_∞) and (Λ, d_∞) are complete metric spaces. Let $\gamma \in \Gamma$ and $\delta \in \Lambda$ and define the mapping $\widetilde{T}$,

$$\begin{aligned} \widetilde{T}_\delta(\gamma)(b) &= \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \le \bar{b}'} u \begin{pmatrix} \underline{w} - b + \beta \Psi^*\left(\gamma(b'), \bar{b}'\right) \bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right) \Pi_\delta(b') \end{pmatrix} + Z^*\left(\gamma(b'), \bar{b}'\right), \\ &\equiv \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \le \bar{b}'} O\left(b, \bar{b}', b', \gamma(b'); \delta\right), \end{aligned} \tag{49}$$

where $\Pi_\delta(b') = \int_\aleph \min\{b', \delta(\phi')\} dF(\phi)$, Z^* and Ψ^* are defined as Z and Ψ in Equations (32) and (40), respectively, after replacing b' by $\gamma(b')$ and $\mu(b')$ by $\mu^*(\gamma(b'))$. In turn, $\mu^*(\gamma(b'))$ is defined in analogy with $\mu(b'))$ in Definition 1, i.e., $\mu^*(\gamma(b')) = (1 - F(\Phi^*(\gamma(b')))) \times \gamma(b') + \int_{\phi_{\min}}^{\Phi^*(\gamma(b'))} [\alpha - \phi'] \, dF(\phi')$, where $\Phi^*(\gamma(b')) = \alpha - \gamma(b')$. Note that $\widetilde{T}_\delta(\gamma)$ maps γ for a given debt threshold function $\delta(\phi)$.

Define, next,

$$T_\delta(\gamma)(b) = \begin{cases} \max\left\{\widetilde{T}_\delta(\gamma)(b), \alpha - \phi_{\max}\right\} & \text{if} \quad b \in [\underline{b}, b_0(\delta)] \\ \alpha - \phi_{\max} & \text{if} \quad b \in (b_0(\delta), \tilde{b}] \end{cases}, \tag{50}$$

where

$$\begin{aligned} b_0(\delta) &\equiv \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \le \bar{b}'} \left\{ \underline{w} + \beta \begin{pmatrix} \Psi^*\left(\gamma(b'), \bar{b}'\right) \bar{b}' \\ +\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right) \Pi_\delta(b') \end{pmatrix} \right\} \\ &= \max_{\bar{b}' \in [\underline{b}, \tilde{b}], \Pi_\delta(\tilde{b}) \le \bar{b}'} \left\{ \underline{w} + \beta \begin{pmatrix} \Psi^*\left(\gamma\left(\tilde{b}\right), \bar{b}'\right) \bar{b}' \\ +\left(1 - \Psi^*\left(\gamma\left(\tilde{b}\right), \bar{b}'\right)\right) \Pi_\delta(\tilde{b}) \end{pmatrix} \right\} < \tilde{b} = \bar{w}/(1-\beta), \end{aligned}$$

denotes the largest b for which non-negative consumption is feasible. Note that b_0 is independent of γ because the maximum debt revenue always involves setting $b' = \tilde{b}$ and since $\gamma(\tilde{b}) = \alpha - \phi_{\max}$ for all $\gamma \in \Gamma$. The definition of T_δ extends the mapping to a possible range in which $\widetilde{T}_\delta$ is not well-defined and ensures that $T_\delta(\gamma)(\tilde{b}) = \alpha - \phi_{\max}$.

Lemma 5.1. *Suppose the LSS constraint $\Pi_\delta(b') \le \bar{b}'$ holds. Then for any $\delta \in \Lambda$, $T_\delta(\gamma)$ is a contraction mapping with a unique fixed point $W_\delta = \lim_{n\to\infty} T_\delta^n(\gamma) \in \Gamma$.*

The detailed proof of Lemma 5.1 is deferred to Appendix B. Here, we sketch its argument. The proof establishes that T_δ maps Γ into Γ and is strictly decreasing in b for all b such that $T_\delta(\gamma)(b) > \alpha - \phi_{\max}$, otherwise $T_\delta = \alpha - \phi_{\max}$. Moreover, T_δ satisfies Blackwell's sufficient conditions (monotonicity and discounting) on the complete metric space (Γ, d_∞), thereby being a contraction mapping. Therefore, $W_\delta = \lim_{n\to\infty} T_\delta^n(\gamma)$ exists and is unique (see Stokey *et al.* (1989), Theorem 3.3).

Step 2: We establish that there exists a unique threshold function $\hat{b} \in \Lambda$ such that $\left\langle W_{\hat{b}}, \hat{b} \right\rangle$ satisfies equation (41). Let $\delta \in \Lambda$. Define the mapping:

$$S(\delta)(\phi) = \begin{cases} \min\left\{b \in [\underline{b}, \tilde{b}] : W_\delta(b) = \alpha - \phi\right\} & \text{if} \quad W_\delta(\underline{b}) \ge \alpha - \phi \\ \underline{b} & \text{if} \quad W_\delta(\underline{b}) < \alpha - \phi \end{cases}, \quad \forall \phi \in \aleph.$$

The following lemma establishes that S has a unique fixed point in Λ:

Lemma 5.2. *$S(\delta)$ has a unique fixed point $\hat{b} \equiv \lim_{n\to\infty} S^n(\delta) \in \Lambda$ and $W_{\hat{b}}(\hat{b}(\phi)) = \alpha - \phi$, $\forall \phi \in \aleph$.*

The detailed proof of Lemma 5.2 is deferred to Appendix B. Here, we sketch the argument. $S(\delta)$ is bounded by $[\underline{b}, \tilde{b}]$, continuous and non-decreasing in ϕ since $W_\delta(b)$ is continuous and non-increasing in b. Thus, S is an operator on the complete metric space (d_∞, Λ). The proof establishes that the operator satisfies Blackwell's sufficient conditions (monotonicity and discounting), implying that S is a contraction mapping.

Taking stock: Since the fixed point $\hat{b}$ meets the indifference condition (41) and $W_{\hat{b}}$ satisfies the Bellman equation of the market equilibrium, Lemmas **5.1** and **5.2** imply that $\left\langle W_{\hat{b}}, \hat{b} \right\rangle$ is the unique pair of value and threshold functions satisfying the market equilibrium conditions. This concludes the proof of Proposition 5 (A).

PART (B): The strategy of the proof is to show that, given the initial condition $\pi = \Pi(b)$, the primal COA in Proposition 4 solves the same problem as the market equilibrium. Then, it follows from part (A) that the unique market equilibrium constraint decentralizes the COA.

The primal COA is characterized by the value functions μ^{pl} and W^{pl}, where W^{pl} is a fixed point to the following functional equation

$$\begin{aligned} W^{pl}(b^{pl}(\pi)) \quad = \quad & \max_{\bar{\pi}' \in [\underline{\pi}, \tilde{\pi}], \pi' \in [\underline{\pi}, P(\alpha - \mathbb{E}[\phi])], \bar{\pi}' \geq \pi'} u \left(\begin{array}{c} \underline{w} - b^{pl}(\pi) \\ +\beta \left[\Psi^{pl}(\pi', \bar{\pi}')\bar{\pi}' + (1 - \Psi^{pl}(\pi', \bar{\pi}'))\pi' \right] \end{array} \right) \\ & - X(\Psi^{pl}(\pi', \bar{\pi}')) + \beta \left[\begin{array}{c} \Psi^{pl}(\pi', \bar{\pi}')\bar{\mu}^{pl}(\bar{\pi}') + \\ (1 - \Psi^{pl}(\pi', \bar{\pi}'))\mu^{pl}(\pi') \end{array} \right], \end{aligned} \tag{51}$$

and μ^{pl} and b^{pl} are defined as follows

$$\begin{aligned} \mu^{pl}(\pi) \quad &= \quad \int_{\aleph} \max \left\{ W^{pl}(b^{pl}(\pi)), \alpha - \phi \right\} dF(\phi) \\ \pi \quad &= \quad \int_{\aleph} \min \left\{ b^{pl}(\pi), \hat{b}^{pl}(\phi) \right\} dF(\phi) \equiv \hat{\Pi}(b^{pl}(\pi)) \\ \hat{b}^{pl}(\phi) \quad &= \quad \left(W^{pl} \right)^{-1} (\alpha - \phi) \ \ \forall \phi \in \aleph. \end{aligned} \tag{52}$$

Note also that $\hat{\Pi}$ is increasing in b^{pl} for $b^{pl} \in \left[\underline{b}, \hat{b}^{pl}(\phi_{\max})\right)$ and constant at $\hat{\Pi}(b^{pl}) = P(\alpha - \mathbb{E}[\phi])$ for $b^{pl} \geq \hat{b}^{pl}(\phi_{\max})$. The definition of $\hat{\Pi}$ implies that $\hat{\Pi}(b^{pl}(\pi')) = \pi'$, since $\pi' \leq P(\alpha - \mathbb{E}[\phi])$.

To establish equivalence, we must impose the LSS constraint to the primal planning problem – recall that LSS guarantees uniqueness of the equilibrium. Namely, $\bar{\pi}' \geq \pi'$. This condition is always satisfied in the COA (cf. Proposition 3, which shows that in the dual COA $\bar{P}(\bar{\omega}_\phi) > P(\omega_\phi)$).

We can then redefine the program (51) in terms of the state variable b and of the choice variables b' and $\bar{b}'$, after setting $b = b^{pl}(\pi)$, $b' = b^{pl}(\pi')$, $\bar{b}' = \bar{\pi}'$, $\underline{b} = \underline{\pi}$, and $\tilde{b} \equiv \tilde{\pi}$. Furthermore, $\hat{\Pi}(\tilde{b}) = P(\alpha - \mathbb{E}[\phi])$. More formally, we rewrite (51) as

$$\begin{aligned} W^{pl}(b) \quad = \quad & \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \bar{b}' \geq \hat{\Pi}(b')} u \left(\underline{w} - b + \beta \left[\begin{array}{c} \Psi^{pl}(\hat{\Pi}(b'), \bar{b}')\bar{b}' \\ +(1 - \Psi^{pl}(\hat{\Pi}(b'), \bar{b}'))\hat{\Pi}(b') \end{array} \right] \right) \\ & - X(\Psi^{pl}(\hat{\Pi}(b'), \bar{b}')) + \beta \left[\begin{array}{c} \Psi^{pl}(\hat{\Pi}(b'), \bar{b}')\bar{\mu}^{pl}(\bar{b}') + \\ (1 - \Psi^{pl}(\hat{\Pi}(b'), \bar{b}'))\mu^{pl}\left(\hat{\Pi}(b')\right) \end{array} \right], \end{aligned}$$

where $\mu^{pl}\left(\hat{\Pi}(b)\right) = \int_{\aleph} \max\left\{W^{pl}(b), \alpha - \phi\right\} dF(\phi)$, $\pi = \hat{\Pi}(b)$, and $\hat{b}^{pl}$ is defined as in (52). In addition, the program must satisfy the LSS constraint $\bar{b}' \geq \hat{\Pi}(b')$. This program is identical to the market equilibrium and must share the same properties.

Taking stock: We conclude that, if $\pi = \hat{\Pi}(b) = \Pi(b)$, the primal COA and the unique market equilibrium subject to the LSS constraint are a fixed point of the same functional equation with a unique solution. In particular, $\mu^{pl}(\pi) = \mu(b)$, $\bar{\mu}^{pl}(\bar{b}') = \bar{\mu}(\bar{b}')$, $W^{pl}(b_\phi(\pi)) = W(\mathbb{B}(b,\phi))$, $\Psi^{pl}(\pi', \bar{b}') = \Psi(b', \bar{b}')$, $\Phi^{pl}(\pi) = \Phi(b)$, and $b_\phi(\pi) = \mathbb{B}(b,\phi)$. This concludes the proof of part (B) of the proposition. ■

Proof of Proposition 6. With slight abuse of notation we write $x(0;\alpha)$ instead of just $x(0)$ to make explicit the dependence of (a function) $x(0)$ on α. We show first that $\Phi(0;\alpha)$ is strictly increasing in α. Suppose to the opposite that $\partial\Phi(0;\alpha)/\partial\alpha \leq 0$. Then, the indifference condition implies that $\partial W(0;\alpha)/\partial\alpha > 0$ since $\partial\Phi(0;\alpha)/\partial\alpha = 1 - \partial W(0;\alpha)/\partial\alpha$. This also implies that, $\partial\mu(0;\alpha)/\partial\alpha > 0$ since $\mu(0;\alpha) = \int_{\aleph} \max\{W(0;\alpha), \alpha - \phi\} dF(\phi)$. At the same time, $\Pi(0;\alpha) = 0$ $\forall\alpha$ due to the assumptions that the sovereign can always honor the contract and that α is sufficiently low to ensure that default is never efficient. However, we know from Proposition 5 that $P(\nu;\alpha) = \Pi(0;\alpha)$ implies that $\nu = \mu(0;\alpha)$. Thus, for any $\alpha' > \alpha$ there exists an allocation that yields the same profits $\Pi(0;\alpha) = \Pi(0;\alpha')$ but higher promised-utility $\mu(0;\alpha') > \mu(0;\alpha)$ than the COA for α. This must be a contradiction, since – for a given α – it is always feasible for the planner to replicate the allocation with outside option α' by relaxing all the PCs by $\alpha' - \alpha$. However, the planner chooses not to do so because its not optimal. Thus, $\partial\Phi(0;\alpha)/\partial\alpha > 0$.

Next, by the Theorem of the Maximum $W(0;\alpha)$ (and therefore $\Phi(0;\alpha)$) is continuous in α. Moreover, at the fixed point $W(0) = W(0;W(0)) = W(0) - \Phi(0;W(0))$ the threshold must be zero. Then, since $\Phi(0;\alpha)$ is continuous and strictly increasing in α on the interval $[u(\underline{w})/(1-\beta), \bar{\mu}(0)]$ and $u(\underline{w})/(1-\beta) < W(0) < \bar{\mu}(0)$, it is sufficient to state that $\Phi(0;u(\underline{w})/(1-\beta)) < \Phi(0,W(0)) = 0$ and $\Phi(0;\bar{\mu}(0)) > \Phi(0,W(0)) = 0$ to establish that the fixed-point $W(0)$ exists in the interior of this interval and is unique. ■

Proposition 7 *Assume the economy has an AJ market structure, where the sovereign can issue non-renegotiable debt contingent on the realization of w and ϕ, subject to borrowing constraints. Then, the COA can be decentralized by an AJ equilibrium with borrowing constraints $b'_{AJ,\phi} \leq \hat{b}(\phi)$. The borrowing constraint is binding for $\phi \leq \Phi(B(b))$ and slack otherwise, where b denotes the current debt repayment and B and Φ are defined in Definition 1.*

Proof of Proposition 7. Let ϕ-specific recession debt be denoted $b'_{AJ,\phi} = b'_\phi$ and let the recession and normal time value functions be given by W_{AJ} and $\bar{W}_{AJ}$, respectively. Since debt is non-renegotiable, the debt prices in recession and normal time are $(1-p_{AJ})f(\phi)R^{-1}$ and $p_{AJ}R^{-1}$, respectively, where p_{AJ} is expected effort. The problem for a sovereign who owes b in recession is then

$$\begin{aligned} W_{AJ}(b) &= \max_{\{b'_\phi\}_{\phi\in\aleph}, \bar{b}'_{AJ}} \left\{ u\left(\underline{w} - b + p_{AJ}R^{-1} \times \bar{b}'_{AJ} + (1-p_{AJ})R^{-1} \times \int_{\aleph} b'_\phi f(\phi)\, d\phi\right) \right. \\ &\left. -X(p_{AJ}) + \beta p_{AJ}\bar{W}_{AJ}\left(\bar{b}'_{AJ}\right) + \beta(1-p_{AJ}) \int_{\aleph} W_{AJ}\left(b'_\phi\right) dF(\phi) \right\}, \end{aligned}$$

where $p_{AJ} = \arg\max_{p\in[\underline{p},\bar{p}]}\{-X(p) + \beta p\bar{W}_{AJ}\left(\bar{b}'_{AJ}\right) + \beta(1-p)\int_{\aleph} W_{AJ}\left(b'_\phi\right) F(\phi)\}$, subject to a set of no-default borrowing constraints $b'_\phi \leq J(\phi)$ $\forall\phi\in\aleph$, and $b'_\phi \geq \underline{b}$, $\bar{b}'_{AJ} \in [\underline{b}, \tilde{b}]$.

The proof strategy is to show that the market allocation of Definition 1 is feasible, rules out default, is consistent with the FOCs, and yields the same expected utility in the AJ economy. Let the policy functions $\hat{b}(\phi)$, $\Phi(b)$, $\Pi(b)$, $\bar{B}\left(\bar{b}\right)$, $B(b)$, $\mathbb{B}(b,\phi)$, $C(b)$, $\bar{C}(\bar{b})$, $W(b)$, $\Psi(b,\bar{b})$, and $\bar{\mu}\left(\bar{b}\right)$ be given by the market allocation.

Guess that (i) the borrowing constraints are $J(\phi) = \hat{b}(\phi)$, (ii) optimal debt issuance is given by:

$$b'_{\phi} = \begin{cases} B(b) & \text{for } \phi \geq \Phi(B(b)) \\ \hat{b}(\phi) & \text{for } \phi < \Phi(B(b)) \end{cases},$$

and $\bar{b}'_{AJ} = \bar{B}(b)$, and (iii) optimal consumption is $C_{AJ}(b) = C(b)$ and $\bar{C}_{AJ}(\bar{b}) = \bar{C}(\bar{b})$. Since $b'_{\phi} = \mathbb{B}(B(b), \phi)$, the realized debt payments are equivalent to the market allocation. Equilibrium effort p_{AJ} must therefore be as in the market economy. Recession debt revenue is identical in the two economies, since

$$\begin{aligned}(1-p_{AJ})R^{-1} \times \int_{\aleph} b'_{\phi} f(\phi)\, d\phi &= \left(1-\Psi(B(b),\bar{B}(b))\right) R^{-1} \times \begin{pmatrix} (1-F(\Phi(B(b)))) \cdot B(b) \\ +\int_{\phi_{\min}}^{\Phi(B(b))} \hat{b}(\phi)\ f(\phi)\, d\phi \end{pmatrix} \\ &= \left(1-\Psi(B(b),\bar{B}(b))\right) R^{-1} \times \Pi(B(b)).\end{aligned}$$

Hence, the proposed allocation $\left\{b'_{\phi}, \bar{b}'_{AJ}, C_{AJ}, \bar{C}_{AJ}, p_{AJ}\right\}$ satisfies the budget constraint and is therefore feasible. It follows that discounted value must be identical; $\bar{W}_{AJ}(b) = \bar{\mu}(b)$ and $W_{AJ}(b) = W(b)$. Since $W_{AJ}(J(\phi)) = W(\hat{b}(\phi)) = \alpha - \phi\ \forall \phi \in \aleph$, it follows that $J(\phi)$ is not too tight, in the sense that the PC holds with equality at the borrowing constraint. Hence, J rules out default.

Finally, it is straightforward to verify that the proposed allocation satisfies the FOCs in the AJ economy. Since the proposed allocation is feasible, satisfies the AJ optimality conditions, and at the same time attains the same utility as in the COA, it must represent an equilibrium allocation in the AJ economy. ■

B Appendix B (Online)

B.1 Analysis of the One-Asset Economy (Section 4)

In this section, we provide the technical analysis of the results summarized in Section 4 and one related figure. We consider the (Markov) market equilibrium for an economy where the sovereign can issue only a one-period renegotiable non-contingent bond.[34]

We provide (i) a definition of the market equilibrium for the one-asset economy; (ii) a proof of existence and uniqueness of W^R and the associated equilibrium functions when the effort is exogenously given; (iii) a derivation of the CEE in equation (42). All proofs are in a separate section of this appendix below.

Definition 2 *A market equilibrium with non-contingent renegotiable debt is a set of value functions* $\left\{V^R, W^R\right\}$*, a threshold renegotiation function* Φ^R*, an equilibrium debt price function* Q^R*, and a set of optimal decision rules* $\{\mathbb{B}^R, B^R, C^R, \Psi^R\}$ *such that, conditional on the state vector* $(b,\phi) \in \left([\underline{b},\tilde{b}] \times [\phi_{\min},\phi_{\max}]\right)$*, the sovereign maximizes utility, the creditors maximize profits, and markets clear. More formally:*

- *The value function* V^R *satisfies*

$$V^R(b,\phi) = \max\left\{W^R(b), \alpha - \phi\right\},$$

where $W^R(b)$ *is the value function conditional on the debt level* b *being honored,*

$$W^R(b) = \max_{b' \in [\underline{b},\tilde{b}]} u\left(Q^R(b') \times b' + \underline{w} - b\right) + Z^R(b'), \tag{53}$$

continuation utility Z^R *is defined as*

$$Z^R(b') = \max_{p \in [\underline{p},\bar{p}]} \left\{-X(p) + \beta\left(p \times \bar{\mu}^R(b') + (1-p) \times \mu^R(b')\right)\right\}, \tag{54}$$

the value of starting in recession with debt b *and in normal time with debt* $\bar{b}$ *are* $\mu^R(b) = \int_{\aleph} V^R(b,\phi)\, dF(\phi)$ *and* $\bar{\mu}^R(\bar{b}) = u\left(\bar{w} - \left(1 - R^{-1}\right)\bar{b}\right)/(1-\beta)$*, respectively.*

- *The threshold renegotiation function* Φ^R *satisfies*

$$\Phi^R(b) = \alpha - W^R(b).$$

- *The debt price function satisfies the following arbitrage condition:*

$$Q^R(b') \times b' = R^{-1}\left(\Psi^R(b') \times b' + \left[1 - \Psi^R(b')\right] \times \Pi^R(b')\right) \tag{55}$$

[34] The extension in which we rule out renegotiation is qualitatively similar. The only difference is that the bond price when renegotiation is allowed, given by equation (55) in the below Definition 2, becomes

$$Q^{NR}(b') = R^{-1}\left(\Psi^{NR}(b') + \left[1 - \Psi^{NR}(b')\right] \times \left[1 - F\left(\Phi^{NR}(b')\right)\right]\right)$$

when renegotiation is ruled out.

where $\Pi^R(b')$ *is the expected repayment of the non-contingent bond conditional on next period being a recession,*

$$\Pi^R(b) = \left(1 - F\left(\Phi^R(b)\right)\right) b + \int_{\phi_{\min}}^{\Phi^R(b)} \hat{b}^R(\phi) \times dF(\phi), \tag{56}$$

and where $\hat{b}^R(\phi) = \left(\Phi^R\right)^{-1}(\phi)$ *is the new post-renegotiation debt after a realization* ϕ*.*

- *The set of optimal decision rules comprises:*

  1. *A take-it-or-leave-it debt renegotiation offer:*

  $$\mathbb{B}^R(b,\phi) = \begin{cases} \hat{b}^R(\phi) & \text{if} \quad \phi \leq \Phi^R(b), \\ b & \text{if} \quad \phi > \Phi^R(b). \end{cases}$$

  2. *An optimal debt accumulation and an associated consumption decision rule:*

  $$\begin{aligned} B^R\left(\mathbb{B}^R(b,\phi)\right) &= \arg\max_{b' \in [\underline{b},\tilde{b}]} \left\{u\left(Q^R(b') \times b' + \underline{w} - \mathbb{B}^R(b,\phi)\right) + Z^R(b')\right\} \\ C^R\left(\mathbb{B}^R(b,\phi)\right) &= Q^R\left(B^R\left(\mathbb{B}^R(b,\phi)\right)\right) \times B^R\left(\mathbb{B}^R(b,\phi)\right) + \underline{w} - \mathbb{B}^R(b,\phi). \end{aligned}$$

  3. *An optimal effort decision rule:*

  $$\Psi^R(b') = \arg\max_{p \in [\underline{p},\bar{p}]} \left\{-X(p) + \beta\left(p \times \bar{\mu}^R(b') + (1-p) \times \mu^R(b')\right)\right\}.$$

- *The equilibrium law of motion of debt is* $b' = B^R\left(\mathbb{B}^R(b,\phi)\right)$*.*
- *The probability that the recession ends is* $p = \Psi^R(b')$*.*

Since the haircut $\hat{b}^R(\phi)$ keeps the sovereign indifferent between accepting the creditors' offer and defaulting, this implies the following indifference condition,

$$W^R(\hat{b}^R(\phi)) = \alpha - \phi. \tag{57}$$

With some abuse of notation, let $W^R(b;\alpha)$ denote the value function conditional on honoring debt b in an economy with exogenous outside option α as defined above. In the analysis of Section 4, we assume that in each economy (with and without renegotiation) the outside option is given by the market equilibrium with a zero debt position. Namely, $\alpha_R = W^R(0;\alpha_R)$ in the case with renegotiation, and $\alpha_{NR} = W^{NR}(0;\alpha_{NR})$, when we rule out renegotiation.

We prove, next, that for an exogenously given effort $\Psi^R = p$ a market equilibrium satisfying Definition 2 exists and that the set of equilibrium functions $\{V^R, W^R, \Phi^R, Q^R, \mathbb{B}^R\}$ is unique.

Proposition 8 *Assume* $\Psi^R = p \in [\underline{p},\bar{p}] \subset \mathbb{R}^+$*. Then the Markov equilibrium with non-contingent debt exists and is unique: (A) The equilibrium functions* $\{V^R, W^R, \Phi^R, Q^R, \mathbb{B}^R\}$ *satisfying Definition 2 exist and are unique. The value functions* $\{V^R, W^R\}$ *are continuous,* W^R *is strictly decreasing in* b*, and* V^R *is non-increasing in* b*; (B) There exists a unique* α_R *satisfying the fixed point* $\alpha_R = W^R(0;\alpha_R)$*.*

Finally, we derive formally the CEE of equation (42). Since the CEE is derived from first-order conditions, the proposition must first establish appropriate differentiability properties that ensures that the first-order conditions are necessary conditions for an equilibrium. It turns out that in the one-asset economy with renegotiation the equilibrium functions are not continuous and differentiable everywhere. However, we can prove that they are differentiable at all interior level of debt that can be the result of an optimal choice. Moreover, the discontinuities in the policy functions do not invalidate the fact that the FOCs are necessary conditions for an equilibrium. It is then useful to define $\hat{B}^R$ as the set of debt levels b' that can be the result of an optimal interior choice given debt b.

Definition 3 $\hat{B}^R = \{b' \in (\underline{b}, \tilde{b}) | B^R \left(\mathbb{B}^R(b,\phi)\right) = b'$, *for* $b \in [\underline{b}, \tilde{b}]\}$.

Proposition 9 *Let $\bar{C}^R(b)$ denote the consumption function in normal time. The equilibrium functions $W^R(b')$, $\Phi^R(b')$, $Q^R(b')$, and $\Psi^R(b')$ are differentiable for all $b' \in \hat{B}^R$. Moreover, for any $b' \in \hat{B}^R$, the FOC $(\partial/\partial b')u(Q^R(b')b' + \underline{w} - b) + (\partial/\partial b')Z^R(b') = 0$ and the envelope condition $\partial W^R(b')/\partial b' = -u'(C^R(b'))$ holds true, such that the conditional Euler equation (CEE)*

$$\begin{aligned}
& \frac{\Psi^R(b')}{(1-\Psi^R(b'))\left[1-F(\Phi^R(b'))\right]+\Psi^R(b')}\frac{u'(\bar{C}^R(b'))}{u'(C^R(b))} + \frac{(1-\Psi^R(b'))\left[1-F(\Phi^R(b'))\right]}{(1-\Psi^R(b'))\left[1-F(\Phi^R(b'))\right]+\Psi^R(b')}\frac{u'(C^R(b'))}{u'(C^R(b))} \\
= \;& 1 + \frac{\left(\Psi^R\right)'(b')\left[b'-\Pi^R(b')\right]}{(1-\Psi^R(b'))\left[1-F(\Phi^R(b'))\right]+\Psi^R(b')}, \qquad (58)
\end{aligned}$$

is a necessary condition for an interior optimum.

Note that the left-hand side of (58) is the expected ratio between next-period and current-period marginal utility conditional on debt being honored in the next period. More precisely, the term $u'(C^R(b'))/u'(C^R(b))$ is the ratio of marginal utilities if the recession continues, whereas the term $u'(\bar{C}^R(b'))/u'(C^R(b))$ is the ratio of marginal utilities if the recession ends. Therefore, equation (58) is identical to equation (42).

B.1.1 Figure 4

Figure 4 illustrates the properties of the one-asset economy with and without renegotiation. This figure is discussed in Section 4.1 in the text.

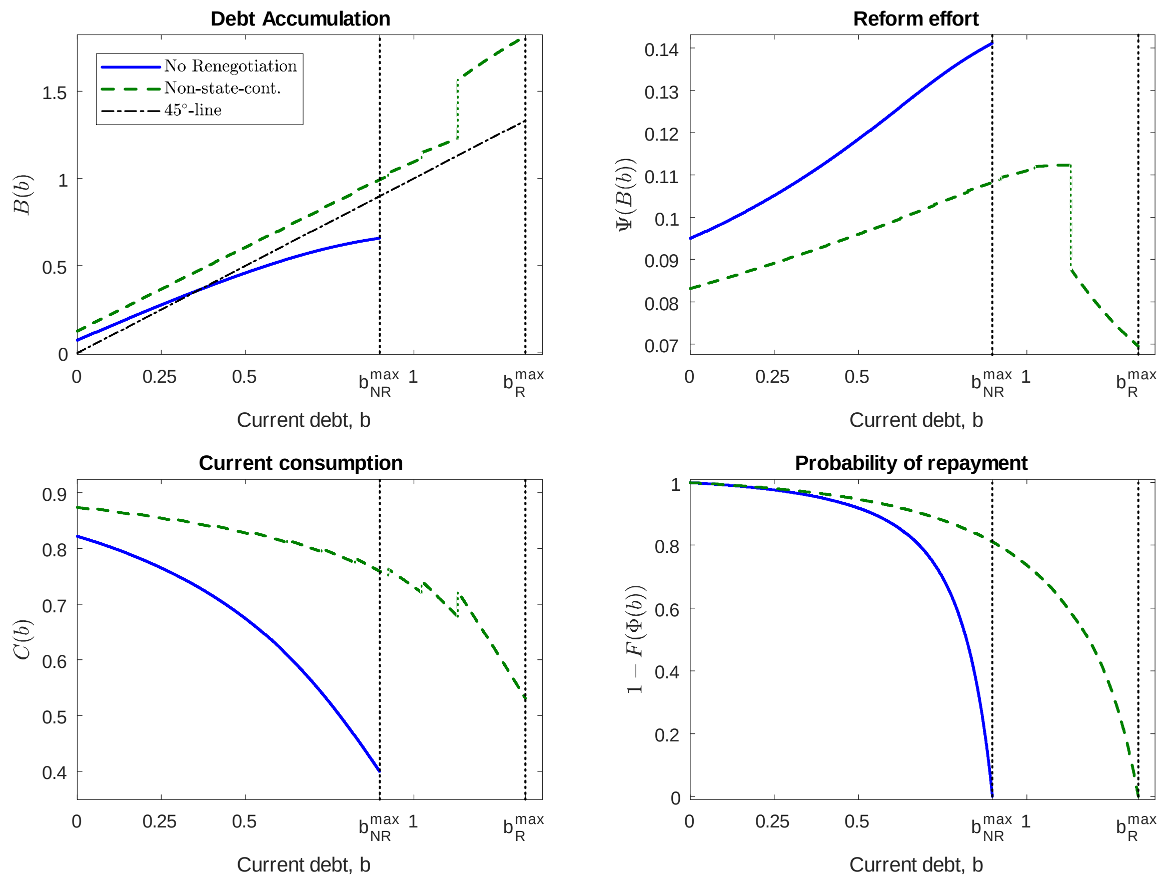

Figure 4: Policy functions of the one-asset economy, conditional on the maximum cost realization $\phi_{\max}$. The dashed lines show the Markov equilibrium with renegotiation, the solid lines the equilibrium where renegotiation is ruled out. The calibration of these economies is described in Section B.3.

B.2 Proofs

B.2.1 First Best (Section 2.1)

In this section, we provide the proof of Proposition 1.

Proof of Proposition 1. In the first part of the proof we take as given that the profit function $\bar{P}$ has the solution (9) and verify this below. Moreover, we also take as given that P is strictly decreasing, strictly concave, and differentiable in ν. We defer the formal proof of these properties to Lemma 2.1 further below in this appendix.

The Lagrangian of the planner's problem in recession reads as

$$\begin{aligned}\mathcal{L} &= \int_{\aleph}\left[\underline{w} - c_\phi + \beta\left(p_\phi \bar{P}\left(\bar{\omega}_\phi\right) + \left(1-p_\phi\right)P\left(\omega_\phi\right)\right)\right]f(\phi)d\phi \\ &\quad +\vartheta\left(\int_{\aleph}\left[u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left(p_\phi\bar{\omega}_\phi + \left(1-p_\phi\right)\omega_\phi\right)\right]f(\phi)d\phi - \nu\right)\end{aligned}$$

where the Lagrange multiplier on the PK is given by ϑ. The FOCs with respect to the controls c_ϕ, ω_ϕ, $\bar{\omega}_\phi$, and p_ϕ yield:

$$\begin{aligned} f\left(\phi\right) &= u'\left(c_\phi\right)\vartheta f\left(\phi\right), & (59)\\ \vartheta f\left(\phi\right) &= -P'\left(\omega_\phi\right) f\left(\phi\right), & (60)\\ \vartheta f\left(\phi\right) &= -\bar{P}'\left(\bar{\omega}_\phi\right) f\left(\phi\right), & (61)\\ \beta\left(\bar{P}\left(\bar{\omega}_\phi\right) - P\left(\omega_\phi\right)\right) f\left(\phi\right) &= \vartheta f\left(\phi\right)\left(X'\left(p_\phi\right) - \beta\left(\bar{\omega}_\phi - \omega_\phi\right)\right), & (62)\end{aligned}$$

while the envelope condition is given by

$$-P'\left(\nu\right) = \vartheta. \tag{63}$$

First, since $f(\phi) > 0$ over the relevant support of ϕ the optimal allocation is independent of the default cost realization. Thus, the planner fully insures the agent against the risk in ϕ. The optimality condition in (61) implies that $\vartheta > 0$, since $-\bar{P}'\left(\bar{\omega}_\phi\right) > 0$. The optimality conditions (60) and (63) imply $\omega^{FB}(\nu) = \nu$ such that promised utility, consumption, and reform effort stay constant during recessions. Equations (59)-(61) together with (9) imply that the planner provides the agent with full consumption insurance across the income states, $u'\left(c^{FB}(\nu)\right) = u'\left(\bar{c}(\bar{\omega}^{FB}(\nu))\right) \Leftrightarrow c^{FB}(\nu) = \bar{c}(\bar{\omega}^{FB}(\nu)) = u^{-1}\left[(1-\beta)\bar{\omega}^{FB}(\nu)\right]$. Given the constant allocation, equation (8) in Proposition 1 follows immediately from the PK (2). Moreover, since in normal time the agent gets the same consumption as in recession (but reform effort is absent), equation (8) implies that promised utility in normal time can be expressed as $\bar{\omega}^{FB}(\nu) = \nu + X\left(p^{FB}\left(\nu\right)\right)/\left(1-\beta\left(1-p^{FB}\left(\nu\right)\right)\right) = u(c^{FB}(\nu))/(1-\beta)$. The FOC with respect to effort (62) can then be expressed as

$$\beta\left(\bar{P}\left(\bar{\omega}^{FB}(\nu)\right) - P(\nu)\right) = u'(c^{FB}(\nu))^{-1}\left(X'\left(p^{FB}(\nu)\right) - \beta\left(\bar{\omega}^{FB}(\nu) - \nu\right)\right), \tag{64}$$

where

$$\begin{aligned}\bar{P}\left(\bar{\omega}^{FB}(\nu)\right) - P(\nu) &= \bar{w} - \underline{w} + \beta\left(1-p^{FB}\left(\nu\right)\right)\left(\bar{P}\left(\bar{\omega}^{FB}(\nu)\right) - P(\nu)\right)\\ &= \frac{1}{1-\beta\left(1-p^{FB}\left(\nu\right)\right)}\left(\bar{w} - \underline{w}\right)\\ \bar{\omega}^{FB}(\nu) - \nu &= \frac{X\left(p^{FB}\left(\nu\right)\right)}{1-\beta\left(1-p^{FB}\left(\nu\right)\right)}.\end{aligned}$$

Substituting for $\bar{P}\left(\bar{\omega}^{FB}(\nu)\right) - P(\nu)$ and $\bar{\omega}^{FB}(\nu) - \nu$ in (64) yields equation (7) in Proposition 1. Note that the profit function in recession

$$\begin{aligned} P(\nu) &= \frac{\underline{w} - c^{FB}(\nu)}{1-\beta\left(1-p^{FB}(\nu)\right)} + \frac{\beta p^{FB}(\nu)}{1-\beta}\frac{\bar{w} - c^{FB}(\nu)}{1-\beta\left(1-p^{FB}(\nu)\right)} \\ &= \frac{\bar{w} - c^{FB}(\nu)}{1-\beta} - \frac{\bar{w} - \underline{w}}{1-\beta\left(1-p^{FB}(\nu)\right)}, \end{aligned} \tag{65}$$

defines a positively sloped locus in the plane (p, c), while equation (7) defines a negatively sloped locus in the same plane. The two equations pin down a unique interior solution for $p^{FB}(\nu)$ and $c^{FB}(\nu)$. Now, consider the comparative statics with respect to ν. An increase in ν yields a strict increase in consumption $c^{FB}(\nu)$ according to (59) and (63) since P is strictly concave, while (7) is independent of ν such that the increase in $c^{FB}(\nu)$ must come with a strict decrease in $p^{FB}(\nu)$. Finally, set $\underline{w} = \bar{w}$ in (65) to see that the profit function in normal time is indeed given by (9). This concludes the proof of the proposition. ■

B.2.2 Constrained Optimum without Moral Hazard (Section 2.2.1)

In this section, we provide the proof of Proposition 2. As a preliminary step, we state Lemma 2.1 that is used in the proof of Proposition 2. Since the proof of Lemma 2.1 is long and uses standard methods, we defer it to a separate section below.

Lemma 2.1. *The profit functions P and $\bar{P}$ that solve the Bellman equation (1) subject to (2)-(3), or, subject to (2)-(5), are strictly decreasing, strictly concave, and differentiable at the interior of their support. The FOCs of the planning problem are necessary and sufficient to characterize the COA.* ■

Proof of Proposition 2. In this proof we take as given Lemma 2.1, that is proved separately. We limit the proof to the arguments that do not overlap with those in the proof of Proposition 3 in Appendix A.

The Lagrangian of the planner's problem is the same as in (43), except that we can drop all terms that involve the incentive constraint ($\chi_\phi = 0$). The FOCs yield, then:

$$f(\phi) = u'(c_\phi)\left(\vartheta f(\phi) + \lambda_\phi\right), \tag{66}$$

$$\vartheta f(\phi) + \lambda_\phi = -P'(\omega_\phi)\, f(\phi), \quad \forall \omega_\phi > \underline{\nu}, \tag{67}$$

$$\vartheta f(\phi) + \lambda_\phi = -\bar{P}'(\bar{\omega}_\phi)\, f(\phi), \tag{68}$$

$$\beta\left(\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)\right) f(\phi) = \left(\vartheta f(\phi) + \lambda_\phi\right)\left(X'(p_\phi) - \beta\left(\bar{\omega}_\phi - \omega_\phi\right)\right). \tag{69}$$

The envelope condition yields $-P'(\nu) = \vartheta > 0$, $\forall \nu > \underline{\nu}$, and the slackness condition for θ_ϕ reads $0 = \theta_\phi\left(\omega_\phi - \underline{\nu}\right)$.

Since P is strictly concave, Lemma 1 implies that the solution is characterized by the unique threshold $\tilde{\phi}(\nu)$ in (13). The PC binds only for $\phi < \tilde{\phi}(\nu)$.

The FOCs (66)–(69) imply equations (10)-(12) in the text. Combine equation (67) and the envelope condition to yield $P'(\nu) = P'(\omega_\phi) + \lambda_\phi / f(\phi) \geq P'(\omega_\phi)$ for $\nu, \omega_\phi > \underline{\nu}$. Since P is strictly concave this implies that promised utility is weakly increasing conditional on staying in recession, $\omega_\phi \geq \nu > \underline{\nu}$. Note that this property extends to the lower bound where $\nu = \underline{\nu} = \omega(\nu)$ if the PC is slack. This can be proved by a contradiction argument: Suppose that $\exists \nu > \underline{\nu}$ such that $\omega(\underline{\nu}) \geq \omega(\nu)$. The optimal allocation associated with $\omega(\nu)$ yields utility $\nu > \underline{\nu}$. Thus, $\omega(\underline{\nu}) \geq \omega(\nu)$ would violate the PK, which is not feasible. In summary, promised utility is weakly increasing $\omega_\phi \geq \nu$, such that its lower bound is never relevant and we can drop the multiplier θ_ϕ from the further analysis.

Since λ_ϕ enters the optimality conditions, the solution will depend on whether the PC is slack or binding:

1. When the PC is binding and the recession continues, $\phi < \tilde{\phi}(\nu)$, $\lambda_\phi > 0$, $\omega_\phi > \nu$, and

$$u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left[p_\phi \bar{\omega}_\phi + \left(1-p_\phi\right)\omega_\phi\right] = \alpha - \phi. \tag{70}$$

   Then, (10), (12), (11) and (70) determine jointly the solution for $(c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi)$. In this case, there is no history dependence, i.e., ν does not matter.

2. When the PC is not binding, $\phi \geq \tilde{\phi}(\nu)$ and $\lambda_\phi = 0$. Then, $\omega_\phi = \nu$, and $c_\phi = c\left(\nu\right)$, $p_\phi = p(\nu)$, and $\bar{\omega}_\phi = \bar{\omega}_\phi(\nu)$ are determined by (19), (10), and (12), respectively. The solution is history dependent.

By the same argument made in the proof of Proposition 3, $c\left(\nu\right)$ must be strictly increasing in ν. In turn $1/u'(c\left(\nu\right)) = 1/u'(\bar{c}\left(\bar{\omega}(\nu)\right))$ implies that also $\bar{\omega}(\nu)$ is strictly increasing in ν. Finally, equation (12) implies that

$$u'\left(c\left(\nu\right)\right)\left[\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - P\left(\nu\right)\right] + \left[\bar{\omega}\left(\nu\right) - \nu\right] = \beta^{-1}X'\left(p\left(\nu\right)\right).$$

For $\nu > \underline{\nu}$, differentiating the left-hand side yields

$$\begin{aligned} & \underbrace{u''\left(c\left(\nu\right)\right)c'\left(\nu\right) \times \left[\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - P\left(\nu\right)\right]}_{<0} + \left[u'\left(c\left(\nu\right)\right)P'\left(\nu\right) + 1\right]\left(\bar{\omega}'\left(\nu\right) - 1\right) \\ = \; & u''\left(c\left(\nu\right)\right)c'\left(\nu\right) \times \left(\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - P\left(\nu\right)\right) < 0 \end{aligned}$$

since (11) implies that $P'\left(\nu\right) = -1/u'\left(c\left(\nu\right)\right)$, and we establish below that $\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - P(\omega(\nu)) = \bar{P}\left(\bar{\omega}\left(\nu\right)\right) - P\left(\nu\right) > 0$. This implies that the right-hand side must also be strictly decreasing in ν. Since X is convex and increasing, this implies in turn that $p\left(\nu\right)$ must be strictly decreasing in $\nu > \underline{\nu}$. Note that this property extends to the lower bound, $p(\underline{\nu}) > p(\nu)\ \forall \nu > \underline{\nu}$. Suppose not, $p(\underline{\nu}) \leq p(\nu)$, then $\nu = \omega(\nu) > \omega(\underline{\nu}) = \underline{\nu}$ which contradicts the fact that $\omega(\nu)$ and $p(\nu)$ are optimal given ν and the same ϕ. Thus, effort $p(\nu)$ must be strictly decreasing.

Finally, we must establish that $\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) > 0$, $\forall \omega_\phi(\nu) > \underline{\nu}$. Suppose, to derive a contradiction, that $\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) \leq 0$. For simplicity, we write ω_ϕ for $\omega_\phi(\nu)$. The FOCs of the planner problem in equations (10) and (11) imply

$$\bar{c}(\bar{\omega}_\phi) = c_\phi = c(\omega_\phi).$$

Recall that $\omega_\phi \geq \nu$. Then, once the economy recovers, promised-utility and profits remains constant such that consumption can be written as

$$\begin{aligned} \bar{c}(\bar{\omega}_\phi) &= \bar{w} - \bar{P}(\omega_\phi) + \beta\bar{P}(\omega_\phi) = \bar{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}\bar{P}(\bar{\omega}_\phi) \\ &> \underline{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}\bar{P}(\bar{\omega}_\phi) \\ &\geq \underline{w} - P(\omega_\phi) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}P(\omega_\phi) \\ &\geq \underline{w} - P(\nu) + \frac{p_\phi}{R}\bar{P}(\bar{\omega}_\phi) + \frac{1-p_\phi}{R}P(\omega_\phi) = c_\phi. \end{aligned}$$

To see why, note that $[(1-p_\phi)/R-1]\,\bar{P}(\bar{\omega}_\phi) \geq [(1-p_\phi)/R-1]\,P(\omega_\phi)$ since $[(1-p_\phi)/R-1] < 0$ and (by assumption) $\bar{P}(\bar{\omega}_\phi) \leq P(\omega_\phi)$. The last inequality follows from $\omega_\phi \geq \nu$ and $P'(\nu) < 0$. The conclusion that $\bar{c}(\bar{\omega}_\phi) > c_\phi$ contradicts $\bar{c}(\bar{\omega}_\phi) = c_\phi$ which was derived above. Thus, we have proven that $\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) > 0$.

This concludes the proof of Proposition 2. ■

B.2.3 Constrained Optimum with Moral Hazard (Section 2.2.2)

This section contains three lemmas that are used in the analysis of the planner problem with moral hazard in Section 2.2.2. Lemma 3.1 provides a sufficient condition for the effort function to be falling in promised utility when ν is sufficiently large. Lemmas 3.2 and 3.3 are instrumental to prove Proposition 3 in Appendix A.

Lemma 3.1. *Suppose* $\lim_{c\to\infty} u'(c) = 0$ *and that* $\lim_{p\to\underline{p}} X''(p) > 0$. *Then* $\lim_{\nu\to\infty} p(\nu) = \underline{p}$.

Proof. We conjecture that in the limit the COA is given by $\lim_{\nu\to\infty}\{\bar{\omega}(\nu)-\omega(\nu)\} = 0$, $\lim_{\nu\to\infty} c(\nu) = \lim_{\nu\to\infty} c(\omega(\nu)) = \lim_{\nu\to\infty} \bar{c}(\bar{\omega}(\nu)) = \infty$, and $\lim_{\nu\to\infty} p(\nu) = \underline{p}$, where $p(\nu) \equiv \Upsilon(\bar{\omega}-\omega)$. We verify that this allocation satisfies the necessary FOCs of the COA. First, eq. (2) implies $\lim_{\nu\to\infty}\omega(\nu) = \lim_{\nu\to\infty}\bar{\omega}(\nu) = \infty$. Second, $\lim_{\nu\to\infty} X'(p(\nu)) = 0$ satisfies eq. (15). Third, the lower bound on ω and the PC (4) become irrelevant when ν is sufficiently large. Fourth, note that $\Upsilon'(\bar{\omega}-\omega) = \beta/X''(\Upsilon(\bar{\omega}-\omega))$. Equations (16)-(17) can then be rewritten as

$$1-p(\nu) = (1-p(\nu))\frac{u'(c(\nu))}{u'(c(\omega(\nu)))} + u'(c(\nu))\frac{\beta}{X''(p(\nu))}\left[\bar{P}(\bar{\omega}(\nu)) - P(\omega(\nu))\right] \tag{71}$$

$$p(\nu) = p(\nu)\frac{u'(c(\nu))}{u'(\bar{c}(\bar{\omega}(\nu)))} - u'(c(\nu))\frac{\beta}{X''(p(\nu))}\left[\bar{P}(\bar{\omega}(\nu)) - P(\omega(\nu))\right]. \tag{72}$$

Consider the limit when $\nu\to\infty$. Equations (71)-(72) hold as $\nu\to\infty$ since $\lim_{\nu\to\infty} p(\nu) = \underline{p}$, $\lim_{\nu\to\infty}\left\{\bar{P}(\bar{\omega}(\nu)) - P(\omega(\nu))\right\} = (\bar{w}-\underline{w})/\left[1-\beta(1-\underline{p})\right]$, $\lim_{\nu\to\infty} u'(c(\nu)) = 0$, $\lim_{\nu\to\infty}\{u'[c(\nu)]/u'[c(\omega(\nu))]\}$ $= \lim_{\nu\to\infty}\{u'[c(\nu)]/u'[\bar{c}(\bar{\omega}(\nu))]\} = 1$, and $X''(p)$ is bounded away from zero by assumption. Finally, note that the conjectured COA coincides with the FB in the limit, since $\lim_{\nu\to\infty} u'(c^{FB}(\nu))(\bar{w}-\underline{w})$ $= 0$ in (7). Thus, it must yield the maximal profits given ν. This implies that the conjectured limiting allocation is indeed the COA. ■

Lemma 3.2. *Assume P is strictly concave. Then, P is differentiable at the interior of its support with* $P'(\nu) = -1/u'(c(\nu)) < 0$.

Proof. The proof is an application of Benveniste and Scheinkman (1979, Lemma 1). Consider the profit of a pseudo planner that is committed to deliver the initial promise $\tilde{\nu}$, but suboptimally chooses effort and future promised-utility like in the optimal contract given an initial promise ν

$$\begin{aligned}\widetilde{P}(\tilde{\nu},\nu) \;\equiv\; & \int_{\phi_{\min}}^{\tilde{\phi}(\tilde{\nu})}\left[\underline{w} - x(\phi, p_\phi(\nu), \bar{\omega}_\phi(\nu), \omega_\phi(\nu)) + \beta\left[\begin{array}{c} p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) \\ +(1-p_\phi(\nu))P(\omega_\phi(\nu))\end{array}\right]\right]dF(\phi) \\ & + \int_{\tilde{\phi}(\tilde{\nu})}^{\infty}\left[\underline{w} - x(\phi(\tilde{\nu}), p_\phi(\nu), \bar{\omega}_\phi(\nu), \omega_\phi(\nu)) + \beta\left[\begin{array}{c} p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) \\ +(1-p_\phi(\nu))P(\omega_\phi(\nu))\end{array}\right]\right]dF(\phi),\end{aligned}$$

where consumption provided by the pseudo planner is determined by

$$x(\phi,p,\bar{\omega},\omega) = u^{-1}\left(\alpha - \phi + X(p) - \beta\left[p\bar{\omega} + (1-p)\omega\right]\right)$$

and $p_\phi(\nu) = \Upsilon(\bar{\omega}_\phi(\nu) - \omega_\phi(\nu))$. Note that for $\tilde{\nu} = \nu$, the pseudo planner achieves the same profit as in the optimal contract, $\widetilde{P}(\tilde{\nu}, \tilde{\nu}) = P(\tilde{\nu})$, but profits must be weakly lower otherwise, $\widetilde{P}(\tilde{\nu}, \nu) \leq P(\tilde{\nu})$. Furthermore, $\widetilde{P}(\tilde{\nu}, \nu)$ is twice differentiable in $\tilde{\nu}$ and strictly concave. Then, Lemma 1 in Benveniste and Scheinkman (1979) applies and the profit function $P(\nu)$ is differentiable at the interior of its support ν with derivative

$$P'(\nu) = \widetilde{P}_1(\nu, \nu) = -1/u'(c(\nu)) < 0.$$

This concludes the proof of the lemma. ■

Lemma 3.3. *The FOCs of the planning problem are necessary for optimality.*

Proof. That the optimal effort is interior follows from the assumed properties of the X function ($X'(\underline{p}) = 0$, $X'(p) > 0$ for $p > \underline{p}$, and $\lim_{p \to \bar{p}} X'(\bar{p}) = +\infty$). The optimality condition for effort $X'(p_\phi(\nu)) = \beta(\bar{\omega}_\phi(\nu) - \omega_\phi(\nu))$ implies then $\bar{\omega}_\phi(\nu) > \omega_\phi(\nu)$, and that there exists an interior maximum effort level $p^+ = \max\{p_\phi(\nu)\} < \bar{p}$. Since $\omega_\phi(\nu) \geq \underline{\nu}$, then $\bar{\omega}_\phi(\nu) > \underline{\nu}$. In conclusion, the optimal choice of p_ϕ is interior and $\bar{\omega}_\phi(\nu)$ will never be at the lower bound $\underline{\omega}$. Note that the possibility $\omega_\phi(\nu) = \underline{\nu}$ is taken into account by the stated FOCs.

Next, consider the upper bound $\tilde{\omega}$ for $\omega_\phi(\nu)$ which is sufficiently high that none of the PCs will bind if the economy starts at $\tilde{\omega}$, i.e., $\omega_\phi(\tilde{\omega}) = \omega(\tilde{\omega}) > \alpha - \phi_{\min}$. Then, the FOC with respect to ω_ϕ (17) implies that - if the planner was not constrained by $\omega_\phi \leq \tilde{\omega}$ - profits are maximized when $\omega_\phi(\tilde{\omega}) < \tilde{\omega}$. This allocation is feasible in the constrained problem thus it must also be the optimal choice when $\omega_\phi(\nu)$ is bounded by $\tilde{\omega}$. The same applies to any level of promised-utility below $\tilde{\omega}$ when the PC is slack. Finally, in states where the PC binds, $\omega_\phi(\nu)$ always remains below $\alpha - \phi_{\min} < \tilde{\omega}$. Thus, $\omega_\phi(\nu)$ always remains strictly below $\tilde{\omega}$. In turn, the optimality condition for effort then implies that the $\bar{\omega}_\phi(\nu)$ can never be higher than $X'(p^+)/\beta + \tilde{\omega} < \tilde{\bar{\omega}}$, where $X'(p^+) < +\infty$. In summary, the optimal choices of $\bar{\omega}_\phi$ and ω_ϕ are also interior (apart from the corner solution, $\omega_\phi = \underline{\nu}$). Finally, consumption must always be positive since $\lim_{c \to 0} u(c) = -\infty$ and it is interior because promised-utility and effort is interior. Thus, the solution to the planner problem must be interior and the stated FOCs are necessary. ■

B.2.4 Decentralization (Section 3.2)

This section contains the complete proofs of two lemmas that are stated in the proof of Proposition 5 in Appendix A.

Proof of Lemma 5.1. We prove that the mapping T_δ defined in eq. (50), satisfies Blackwell's sufficient conditions on the complete metric space (Γ, d_∞), thereby being a contraction mapping. Therefore, $W_\delta = \lim_{n \to \infty} T_\delta^n(\gamma)$ exists and is unique (see Stokey *et al.* (1989), Theorem 3.3).

Claim (i): T_δ is strictly decreasing in b for all b such that $T_\delta(\gamma)(b) > \alpha - \phi_{\max}$, otherwise $T_\delta = \alpha - \phi_{\max}$, in particular, $T_\delta(\gamma)(\tilde{b}) = \alpha - \phi_{\max}$. Proof of the claim: For any $\varepsilon > 0$, $\widetilde{T}_\delta(\gamma)(b + \varepsilon) < \widetilde{T}_\delta(\gamma)(b)$ since

$$\begin{aligned}
\widetilde{T}_\delta(\gamma)(b + \varepsilon) &= O(b + \varepsilon, B_\gamma^*(b + \varepsilon), \bar{B}_\gamma^*(b + \varepsilon), \gamma(B_\gamma^*(b + \varepsilon)); \delta) \\
&< O(b, B_\gamma^*(b + \varepsilon), \bar{B}_\gamma^*(b + \varepsilon), \gamma(B_\gamma^*(b + \varepsilon)); \delta) \\
&\leq O(b, B_\gamma^*(b), \bar{B}_\gamma^*(b), \gamma(B_\gamma^*(b)); \delta) = \widetilde{T}_\delta(\gamma)(b),
\end{aligned}$$

where $\langle \bar{B}_\gamma^*(b), B_\gamma^*(b) \rangle = \arg\max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \leq \bar{b}'} O(b, b', \bar{b}', \gamma(b'); \delta)$. The strict inequality follows from $b + \varepsilon > b$ and $u'(\cdot) > 0$. The weak inequality follows from the fact that $\langle \bar{B}_\gamma^*(b), B_\gamma^*(b) \rangle$ is the optimal policy for the debt level b. Since $\widetilde{T}_\delta$ is strictly decreasing in $b \in (\underline{b}, b_0(\delta))$, and $\lim_{b \to b_0(\delta)} \widetilde{T}_\delta(\gamma)(b) =$

$-\infty < \alpha - \phi_{\max}$, then, $T_\delta(\gamma)(b)$ is strictly decreasing in b for all b such that $T_\delta(\gamma)(b) > \alpha - \phi_{\max}$, being constant at $\alpha - \phi_{\max}$ otherwise. Finally, since $b_0(\delta) < \tilde{b}$ (49) implies that $T_\delta(\gamma)(\tilde{b}) = \alpha - \phi_{\max}$.

Claim (*ii*): T_δ maps Γ into Γ. Proof of the claim: Recall that T_δ is bounded from below by $\alpha - \phi_{\max}$. T_δ is also bounded from above because consumption, reform effort, the support of the default cost, and the elements of Γ and Λ are bounded. Continuity of $\widetilde{T}_\delta$ in $b \in [\underline{b}, b_0(\delta))$ follows by the Theorem of the Maximum. Since $\lim_{b \to b_0(\delta)} \widetilde{T}_\delta(\gamma)(b) = -\infty < \alpha - \phi_{\max}$, then $T_\delta(\gamma)(b) = \max\left\{\widetilde{T}_\delta(\gamma)(b), \alpha - \phi_{\max}\right\}$ is also continuous in b. Finally, we have already established above that T_δ is non-increasing in b and that $T_\delta(\gamma)(\tilde{b}) = \alpha - \phi_{\max}$. Thus, $T_\delta(\gamma) \in \Gamma$.

Claim (*iii*): T_δ discounts: for any scalar $a \geq 0$ and $\gamma \in \Gamma$, $T_\delta(\gamma + a)(b) \leq T_\delta(\gamma)(b) + \beta a$. Proof of the claim: Let $a \geq 0$ be a real constant. Then

$$\begin{aligned}
\widetilde{T}_\delta(\gamma + a)(b) &\leq \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \leq \bar{b}'} u\begin{pmatrix} \underline{w} - b + \beta \Psi^*\left(\gamma(b') + a, \bar{b}'\right)\bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b') + a, \bar{b}'\right)\right)\Pi_\delta(b') \end{pmatrix} + Z^*\left(\gamma(b'), \bar{b}'\right) + \beta a \quad (73) \\
&\leq \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \leq \bar{b}'} u\begin{pmatrix} \underline{w} - b + \beta \Psi^*\left(\gamma(b'), \bar{b}'\right)\bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right)\Pi_\delta(b') \end{pmatrix} + Z^*\left(\gamma(b'), \bar{b}'\right) + \beta a \quad (74) \\
&= \widetilde{T}_\delta(\gamma)(b) + \beta a.
\end{aligned}$$

The first inequality follows from an envelope argument implying that $dZ^*(\gamma(b')+a, \bar{b}')/da = \beta\left(1 - \Psi^*\left(\gamma(b') + a, \bar{b}'\right)\right)$ $[0, \beta]$. Therefore, a linear expansion yields $Z^*\left(\gamma(b') + a, \bar{b}'\right) \leq Z^*\left(\gamma(b'), \bar{b}'\right) + \beta a$. The second inequality follows from observing that $\Psi^*\left(\gamma(b') + a, \bar{b}'\right) \leq \Psi^*\left(\gamma(b'), \bar{b}'\right)$ and $\bar{b}' \geq \Pi_\delta(b')$, implying that debt revenue and utility is (weakly) higher in (74) than in (73). So, $\widetilde{T}_\delta$ discounts. The definition of T_δ implies that if $\widetilde{T}_\delta$ discounts, so does T_δ.

Claim (*iv*): T_δ is a monotone mapping, i.e., $\forall \gamma, \gamma^+ \in \Gamma$ such that $\gamma^+(b) \geq \gamma(b)$, $T_\delta(\gamma^+)(b) \geq T_\delta(\gamma)(b)$ $\forall b \in [\underline{b}, \tilde{b}]$. Proof of the claim: Let $\gamma, \gamma^+ \in \Gamma$ with $\gamma^+(b) \geq \gamma(b)$, $\forall b \in [\underline{b}, \tilde{b}]$. We first establish that $\gamma^+(b) \geq \gamma(b) \Rightarrow \widetilde{T}_\delta(\gamma^+)(b) \geq \widetilde{T}_\delta(\gamma)(b)$. To this aim, let $\left(b', \bar{b}'\right) = \left(B^*_{\gamma^+}(b), \bar{B}^*_{\gamma^+}(b)\right)$ denote the optimal debt issuance under the function γ^+. Let $\xi(b) \geq 0$ be such that $\gamma^+\left(B^*_\gamma(b) + \xi(b)\right) = \gamma(B^*_\gamma(b))$. (That such a positive function exists follows immediately from the properties of γ^+ and γ.) Suppose, first, that b is such that $\Pi_\delta\left(B^*_\gamma(b) + \xi(b)\right) \leq \bar{B}^*_\gamma(b)$, so that the LSS constraint is satisfied. Then, the following sequence of inequalities holds true:

$$\begin{aligned}
\widetilde{T}_\delta(\gamma^+)(b) &= O\left(b, B^*_{\gamma^+}(b), \bar{B}^*_{\gamma^+}(b), \gamma^+\left(B^*_{\gamma^+}(b)\right); \delta\right) \\
&\geq O\left(b, B^*_\gamma(b) + \xi(b), \bar{B}^*_\gamma(b), \gamma^+\left(B^*_\gamma(b) + \xi(b)\right); \delta\right) \\
&= O\left(b, B^*_\gamma(b) + \xi(b), \bar{B}^*_\gamma(b), \gamma\left(B^*_\gamma(b)\right); \delta\right) \\
&\geq O\left(b, B^*_\gamma(b), \bar{B}^*_\gamma(b), \gamma\left(B^*_\gamma(b)\right); \delta\right) = \widetilde{T}_\delta(\gamma)(b).
\end{aligned}$$

The first inequality follows from the fact that, under γ^+, the choice $\left(b', \bar{b}'\right) = \left(B^*_\gamma(b) + \xi(b), \bar{B}^*_{\gamma^+}(b)\right)$ is feasible and suboptimal. The second inequality follows from the fact that the expression in the third line has a larger b' than that in the fourth line, while effort is held constant across the two expressions. Thus, the former grants (weakly) higher consumption than the latter.

Consider, next, the case when the LSS binds, i.e., $\Pi_\delta\left(B^*_\gamma(b) + \xi(b)\right) > \bar{B}^*_\gamma(b)$. Define $\tilde{\xi}(b)$ such that

$\Pi_\delta(B^*_\gamma(b)+\tilde{\xi}(b)) = \bar{B}^*_\gamma(b)$ and note that $0 \leq \tilde{\xi}(b) < \xi(b)$. Then,

$$\begin{aligned}
\widetilde{T}_\delta(\gamma^+)(b) &= O\left(b, B^*_{\gamma^+}(b), \bar{B}^*_{\gamma^+}(b), \gamma^+\left(B^*_{\gamma^+}(b)\right); \delta\right) \\
&\geq O\left(b, B^*_\gamma(b)+\tilde{\xi}(b), \bar{B}^*_\gamma(b), \gamma^+\left(B^*_\gamma(b)+\tilde{\xi}(b)\right); \delta\right) \\
&= u\left(\underline{w} - b + \beta \bar{B}^*_\gamma(b)\right) + Z^*\left(\gamma^+\left(B^*_\gamma(b)+\tilde{\xi}(b)\right), \bar{B}^*_\gamma(b)\right) \\
&\geq O\left(b, B^*_\gamma(b), \bar{B}^*_\gamma(b), \gamma\left(B^*_\gamma(b)\right); \delta\right) = \widetilde{T}_\delta(\gamma)(b).
\end{aligned}$$

The first inequality follows from the fact that, under γ^+, the choice $(b', \bar{b}') = \left(B^*_\gamma(b)+\tilde{\xi}(b), \bar{B}^*_{\gamma^+}(b)\right)$ is suboptimal. The second equality follows from the fact that $p\Pi_\delta(B^*_\gamma(b)+\tilde{\xi}(b)) + (1-p)\bar{B}^*_\gamma(b) = \bar{B}^*_\gamma(b)$. The second inequality follows from the observation that both consumption and continuation utility are higher in the expression in the third line than in that in the fourth line. Consumption is higher because (by the LSS) $\bar{B}^*_\gamma(b) \geq \Pi_\delta(B^*_\gamma(b))$. Continuation utility is higher because (i) $\tilde{\xi}(b) \leq \xi(b) \Rightarrow \gamma^+\left(B^*_\gamma(b)+\tilde{\xi}(b)\right) \geq \gamma^+\left(B^*_\gamma(b)+\xi(b)\right) = \gamma\left(B^*_\gamma(b)\right)$ and (ii) $Z^*(y^+, x) \geq Z^*(y, x)$. This establishes that $\tilde{T}_\delta$ is a monotone mapping. Next, observe that $T_\delta(\gamma^+) = \max\{\widetilde{T}_\delta(\gamma^+), \alpha - \phi_{\max}\} \geq \max\{\widetilde{T}_\delta(\gamma), \alpha - \phi_{\max}\} = T_\delta(\gamma)$. Thus, T_δ is also a monotone mapping.

By Blackwell's theorem, Claims (i)–(iv) jointly imply that $T_\delta(\gamma)$ is a contraction operator on the complete metric space (Γ, d_∞), thus its fixed point $W_\delta = \lim_{n\to\infty} T^n_\delta(\gamma)$ exists in Γ and is unique. ■

Proof of Lemma 5.2. First, note that $S(\delta)$ is bounded by $[\underline{b}, \tilde{b}]$, continuous and non-decreasing in ϕ since $W_\delta(b)$ is continuous and non-increasing in b. Thus, S is an operator on the complete metric space (d_∞, Λ). We now verify Blackwell's sufficient conditions for S being a contraction mapping.

Monotonicity: Let $\delta^+, \delta \in \Lambda$ and assume $\delta^+ \geq \delta$. We claim that $S(\delta^+) \geq S(\delta)$. Corollary 5.1 below establishes that $\delta^+ \geq \delta \Rightarrow W_{\delta^+}(b) \geq W_\delta(b)$, $\forall b \in [\underline{b}, \tilde{b}]$. This property must also hold true for $b = S(\delta)(\phi)$, so $W_{\delta^+}(S(\delta)(\phi)) \geq W_\delta(S(\delta)(\phi))$. To prove the claim, we distinguish two cases:
(i) $W_\delta(\underline{b}) \geq \alpha - \phi$: The definition of S implies that $W_{\delta^+}(S(\delta^+)(\phi)) = \alpha - \phi$ and $W_\delta(S(\delta)(\phi)) = \alpha - \phi$, implying that $W_{\delta^+}(S(\delta^+)(\phi)) = W_\delta(S(\delta)(\phi))$. Joint with the above inequality this yields $W_{\delta^+}(S(\delta)(\phi)) \geq W_{\delta^+}(S(\delta^+)(\phi))$. Since $W_{\delta^+}(b)$ is monotone decreasing in b, then, $S(\delta)(\phi) \leq S(\delta^+)(\phi)$.
(ii) $W_\delta(\underline{b}) < \alpha - \phi$: The definition of S yields $S(\delta)(\phi) = \underline{b}$, implying $S(\delta^+)(\phi) \geq S(\delta)(\phi) = \underline{b}$.

Discounting: Let $a \geq 0$. We claim that $S(\delta+a) \leq S(\delta) + \beta a$. Corollary 5.1 below establishes that for any $b \in [\underline{b}, \tilde{b}]$ $\exists \tilde{\beta}(b) \in [0, \beta]$ such that $W_\delta(b) = W_{\delta+a}(b + \tilde{\beta}(b)a)$. This property must also hold for $b = S(\delta)(\phi)$. Thus, $\exists \tilde{\beta}_\phi \in [0, \beta]$ such that $W_\delta(S(\delta)(\phi)) = W_{\delta+a}\left(S(\delta)(\phi) + \tilde{\beta}_\phi a\right)$. We now distinguish three cases to prove the claim.
(i) $W_\delta(\underline{b}) \geq \alpha - \phi$: The definition of S implies $W_\delta(S(\delta)(\phi)) = \alpha - \phi$ and $W_{\delta+a}(S(\delta+a)(\phi)) = \alpha - \phi$. Thus, $W_{\delta+a}(\delta+a)(S(\delta+a)(\phi)) = W_\delta(S(\delta)(\phi)) = W_{\delta+a}\left(S(\delta)(\phi) + \tilde{\beta}_\phi a\right)$, implying that $S(\delta+a)(\phi) = S(\delta)(\phi) + \tilde{\beta}_\phi a \leq S(\delta)(\phi) + \beta a$.
(ii) $W_\delta(\underline{b}) < \alpha - \phi$ and $W_{\delta+a}(\underline{b}) \geq \alpha - \phi$: The definition of S implies $S(\delta)(\phi) = \underline{b}$ and $W_{\delta+a}(S(\delta+a)(\phi)) = \alpha - \phi$. The above equality implies that $W_\delta(\underline{b}) = W_{\delta+a}(\underline{b} + \tilde{\beta}(\underline{b})a) = \alpha - \phi$. It follows that $W_{\delta+a}(S(\delta+a)(\phi)) = \alpha - \phi = W_{\delta+a}(\underline{b} + \tilde{\beta}(\underline{b})a)$ and therefore $S(\delta+a)(\phi) = \underline{b} + \tilde{\beta}(\underline{b})a \leq S(\delta) + \beta a$.
(iii) $W_{\delta+a}(\underline{b}) < \alpha - \phi$: The definition of S implies that $S(\delta)(\phi) = \underline{b}$ and $S(\delta+a)(\phi) = \underline{b}$, implying that $S(\delta+a)(\phi) = S(\delta)(\phi) \leq S(\delta)(\phi) + \beta a$ is necessarily satisfied.

Thus, S is a contracting operator with a unique fixed point $\hat{\delta}(\phi) = \lim_{n\to\infty} S^n(\delta)$ in Λ. This concludes the proof of the lemma. ■

The following corollary of Lemma 5.1 was used in the proof of Lemma 5.2.

Corollary 5.1.. *W_δ has the following properties: (a) W_δ is monotone in δ: if $\delta^+ \geq \delta$, then $W_{\delta^+}(b) \geq W_\delta(b)\ \forall b \in [\underline{b}, \tilde{b}]$; (b) for any $b \in [\underline{b}, \tilde{b}]$ and $a \geq 0$, $\exists \tilde{\beta}(b) \in [0, \beta]$: $W_{\delta+a}(b + \tilde{\beta}(b)a) = W_\delta(b)$.* ■

Proof of Corollary 5.1.1. **Part (a)**: First, note that $T_{\delta^+}(\gamma)(b) \geq T_\delta(\gamma)(b)$. The reason is that $\delta^+ \geq \delta \Rightarrow \Pi_{\delta^+}(b') \geq \Pi_\delta(b')$. Moreover, any feasible debt revenue under δ $\left(\beta\left[p\bar{b}' + (1-p)\Pi_\delta(b')\right]\right)$ is also feasible under δ^+ while yielding a weakly higher continuation value than $Z^*\left(\gamma(b'), \bar{b}'\right)$. Since T_δ is a contraction mapping, it follows that $W_{\delta^+}(b) \geq W_\delta(b)$. **Part (b)**: Part (a) implies that $\widetilde{T}_\delta(\gamma)(b)$ is bounded from above by $\widetilde{T}_{\delta+a}(\gamma)(b)$. $\widetilde{T}_\delta(\gamma)(b)$ is also bounded from below by

$$\begin{aligned}
\widetilde{T}_\delta(\gamma)(b) &= \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \leq \bar{b}'} u\left(\begin{array}{c} \underline{w} - b + \beta\Psi^*\left(\gamma(b'), \bar{b}'\right)\bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right)\Pi_\delta(b') \end{array}\right) + Z^*\left(\gamma(b'), \bar{b}'\right) \\
&\geq \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_\delta(b') \leq \bar{b}'} u\left(\begin{array}{c} \underline{w} - b + \beta\Psi^*\left(\gamma(b'), \bar{b}'\right)\bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right)\left[\Pi_{\delta+a}(b') - a\right] \end{array}\right) + Z^*\left(\gamma(b'), \bar{b}'\right) \\
&\geq \max_{\bar{b}', b' \in [\underline{b}, \tilde{b}], \Pi_{\delta+a}(b') \leq \bar{b}'} u\left(\begin{array}{c} \underline{w} - b - \beta a + \beta\Psi^*\left(\gamma(b'), \bar{b}'\right)\bar{b}' \\ +\beta\left(1 - \Psi^*\left(\gamma(b'), \bar{b}'\right)\right)\Pi_{\delta+a}(b') \end{array}\right) + Z^*\left(\gamma(b'), \bar{b}'\right) \\
&= \widetilde{T}_{\delta+a}(\gamma)(b + \beta a).
\end{aligned}$$

The first inequality follows from the fact that $\Pi_{\delta+a}(b) - a \leq \Pi_\delta(b)$ for all $b \in [\underline{b}, \tilde{b}]$. The second inequality follows from $1 - \Psi^*\left(\gamma(b'), \bar{b}'\right) \leq 1$ and the fact that $\Pi_{\delta+a}(b') \leq \bar{b}'$ is a tighter constraint than $\Pi_\delta(b') \leq \bar{b}'$. Since the function $\widetilde{T}_{\delta+a}(\gamma)(x)$ is continuous in x, there must exist a $\tilde{\beta}(b) \in [0, \beta]$ such that $\widetilde{T}_{\delta+a}(\gamma)(b + \tilde{\beta}(b)a) = \widetilde{T}_\delta(\gamma)(b)$. This implies that $T_{\delta+a}(\gamma)(b + \tilde{\beta}(b)a) = T_\delta(\gamma)(b)$. Since T_δ is a contraction mapping, the same holds true at the fixed point. ■

B.2.5 Less Complete Markets (Section 3.2)

Proof of Proposition 8. **Part (A):** We follow the same strategy used to prove Proposition 5. First (**step 1**), we define an inner operator T_δ that maps value functions into value functions conditional on an *arbitrary* debt threshold function $\delta(\phi)$. We show that this operator has a unique fixed point $W_\delta = \lim_{n\to\infty} T_\delta^n(\gamma)$ that satisfies (53). Second (**step 2**), we define an outer operator S that maps debt threshold functions into debt threshold functions. We show that this operator has a unique fixed point $\hat{b}^R(\phi) = \lim_{n\to\infty} S^n(\delta)$ that satisfies (57) when W_δ is evaluated at $\delta(\phi) = \hat{b}^R(\phi)$, i.e., $W_{\hat{b}^R}(\hat{b}^R(\phi)) = \alpha - \phi$. Let $W^R = W_{\hat{b}^R}$. Then, the fixed point $\left\langle W^R, \hat{b}^R \right\rangle$ must be unique. The uniqueness of the remaining equilibrium functions follows then from Definition (2).

Step 1: Let Γ be the space of bounded, continuous, and non-increasing functions $\gamma : [\underline{b}, \tilde{b}] \to [\alpha - \phi_{\max}, u(\bar{w} + (1-\beta)\underline{b})/(1-\beta)]$. Moreover, let Λ be the space of bounded and continuous functions $\delta : \aleph \to [\underline{b}, \tilde{b}]$. Define $d_\infty(y, z) \equiv \sup_{x\in X} |y(x) - z(x)|$ such that (Γ, d_∞) and (Λ, d_∞) are complete metric spaces. Let $\gamma \in \Gamma$ and $\delta \in \Lambda$ and define the mapping:

$$\begin{aligned}
\widetilde{T}_\delta(\gamma)(b) &= \max_{b' \in [\underline{b}, \tilde{b}]} u\left(\begin{array}{c} \underline{w} - b + R^{-1}pb' \\ +R^{-1}(1-p)\Pi_\delta(b') \end{array}\right) + Z^*\left(\gamma(b'), b'\right), \\
&\equiv \max_{b' \in [\underline{b}, \tilde{b}]} O\left(b, b', \gamma(b'); \delta\right), \qquad (75)
\end{aligned}$$

where $\Pi_\delta(b') = \int_\aleph \min\{b', \delta(\phi')\} dF(\phi)$ and $Z^*(\gamma(b'), b') = -X(p) + \beta\left[p\bar{\mu}(b') + (1-p)\mu^*(b')\right]$ In turn, $\mu^*(\gamma(b'))$ is defined in analogy with $\mu(b')$ in Definition 2, i.e., $\mu^*(\gamma(b')) = (1 - F(\Phi^*(\gamma(b')))) \times \gamma(b')$

$+\int_{\phi_{\min}}^{\Phi^*(\gamma(b'))}\left[\alpha-\phi'\right]dF(\phi')$, where $\Phi^*\left(\gamma(b')\right)=\alpha-\gamma(b')$. Note that $\widetilde{T}_\delta(\gamma)$ maps γ for a given debt threshold function $\delta(\phi)$ and effort is exogenously given by p. Define, next,

$$T_\delta(\gamma)(b)=\begin{cases}\max\left\{\widetilde{T}_\delta(\gamma)(b),\alpha-\phi_{\max}\right\} & \text{if} \quad b\in[\underline{b},b_0(\delta)]\\ \alpha-\phi_{\max} & \text{if} \quad b\in(b_0(\delta),\tilde{b}]\end{cases}, \tag{76}$$

where

$$b_0(\delta)\equiv\underline{w}-b+R^{-1}p\tilde{b}+R^{-1}\left(1-p\right)\Pi_\delta(\tilde{b})<\tilde{b}=\bar{w}/(1-\beta),$$

denotes the largest b for which non-negative consumption is feasible. The definition of T_δ extends the mapping to a possible range in which $\widetilde{T}_\delta$ is not well-defined.

Lemma 8.1. For any $\delta\in\Lambda$ the mapping $T_\delta(\gamma)$ has a unique fixed point $W_\delta=\lim_{n\to\infty}T_\delta^n(\gamma)\in\Gamma$.

Proof of Lemma 8.1. The same argument used in the proof of Lemma 5.1 establishes that T_δ maps Γ into Γ and is strictly decreasing in b for all b such that $T_\delta(\gamma)(b)>\alpha-\phi_{\max}$, otherwise $T_\delta=\alpha-\phi_{\max}$. Moreover T_δ discounts by the same argument used in the proof of Lemma 5.2 – noting that effort is exogenous. Finally, T_δ is also monotone since $\gamma^+(b)\geq\gamma(b)\Rightarrow Z^*(\gamma^+(b'),b')\geq Z^*(\gamma(b'),b')$ such that

$$\begin{aligned}\widetilde{T}_\delta(\gamma^+)(b) &\geq O(b,B^*_\gamma(b),\gamma^+(B^*_\gamma(b));\delta)\\ &\geq O(b,B^*_\gamma(b),\gamma(B^*_\gamma(b));\delta)=\widetilde{T}_\delta(\gamma)(b),\end{aligned}$$

where $B^*_\gamma(b)=\arg\max_{b'\in[\underline{b},\tilde{b}]}O(b,b',\gamma(b');\delta)$. Then, $\widetilde{T}_\delta$ satisfies Blackwell's sufficient conditions (monotonicity and discounting) on the complete metric space (Γ,d_∞), thereby being a contraction mapping. Therefore, $W_\delta=\lim_{n\to\infty}T_\delta^n(\gamma)$ exists and is unique (see Stokey *et al.* (1989), Theorem 3.3).

Step 2: We establish that there exists a unique threshold function $\hat{b}^R\in\Lambda$ such that $\left\langle W_{\hat{b}^R},\hat{b}^R\right\rangle$ satisfies equation (57). Let $\delta\in\Lambda$. Define the mapping:

$$S(\delta)(\phi)=\begin{cases}\min\left\{b\in[\underline{b},\tilde{b}]:W_\delta(b)=\alpha-\phi\right\} & \text{if} \quad W_\delta(\underline{b})\geq\alpha-\phi\\ \underline{b} & \text{if} \quad W_\delta(\underline{b})<\alpha-\phi\end{cases}, \quad \forall\phi\in\aleph.$$

Lemma 5.2 then establishes that S has a unique fixed point $\hat{\delta}\equiv\lim_{n\to\infty}S^n\left(\delta\right)\in\Lambda$ and $W_{\hat{\delta}}(\hat{\delta}(\phi))=\alpha-\phi$, $\forall\phi\in\aleph$.

Since the fixed point $\hat{\delta}$ meets the indifference condition (57) and $W_{\hat{\delta}}$ satisfies the Bellman equation of the market equilibrium, Lemma 5.1 and Lemma 5.2 imply that $\left\langle W_{\hat{b}^R},\hat{b}^R\right\rangle$ is the unique pair of value and threshold functions satisfying the market equilibrium conditions.

Then, the uniqueness of V^R, Φ^R, Q^R, and $\mathbb{B}^R$ follows from Definition 2. Note that we do not claim uniqueness of B^R and C^R. The continuity of the value function $W^R(b)$ in b follows from the Theorem of the Maximum, and implies that also the equilibrium functions V^R, Φ^R, and Q^R are continuous in b. Since $\widetilde{T}_{\hat{b}^R}(\gamma)$ maps decreasing functions into strictly decreasing functions, it follows that the fixed-point W^R is strictly decreasing in b and, hence, $V^R(b,\phi)=\max\left\{W^R(b),\alpha-\phi\right\}$ is non-increasing in b.

Part (B): Having proved the existence and uniqueness of a fixed point W^R conditional on α, we now show that there exists a unique $\alpha_R\in[W_{MIN},W_{MAX}]\equiv[\alpha-\phi_{\max},u(\bar{w}+(1-\beta)\underline{b})/(1-\beta)]$ such that $W^R\left(0;\alpha_R\right)=\alpha_R\in(W_{MIN},W_{MAX})$ (with slight abuse of notation we add α as a function argument in $W^R(b;\alpha)$ and $\Phi^*(\gamma(b);\alpha))$. To see why, note that, by the Theorem of the Maximum, $W^R\left(b;\alpha\right)$ is continuous in α. Moreover, since $W^R\in[W_{MIN},W_{MAX}]$, then, Brouwer's fixed-point theorem ensures that there exists an $\alpha\in[W_{MIN},W_{MAX}]$ such that $W^R\left(0;\alpha\right)=\alpha$. Since $W^R\left(0;\alpha\right)=$

$\alpha - \Phi^*\left(W^R(0;\alpha);\alpha\right)$, then $\Phi^*\left(W^R(0;\alpha);\alpha\right) = 0$. To prove that such an α is unique, we note that Φ^* is monotone increasing in α (the set of states of nature in which the outside option is preferred expands as α increases). Therefore, there exists a unique fixed point α_R such that $\Phi^*\left(W^R(0;\alpha_R),\alpha_R\right) = 0$. In particular, $\alpha_R = W^R(0;\alpha_R) \in (W_{MIN}, W_{MAX})$.

This concludes the proof of the proposition ■

Proof of Proposition 9. The proof is an application of the generalized envelope theorem in Clausen and Strub (2016) which allows for discrete choices (i.e., repayment or renegotiation) and non-concave value functions. Consider the program $W^R(b) = \max_{b' \in [\underline{b},\tilde{b}]} O(b')$, $O(b') \equiv u(Q^R(b')b' - b + \underline{w}) + \beta Z^R(b')$. Theorem 1 in Clausen and Strub (2016) ensures that if we can find a *differentiable lower support function* (DLSF) for O, then O is differentiable at all interior optimal debt choices $b' \in \hat{B}^R$ where $\hat{B}^R$ was defined in Definition 3 above.

To construct a DLSF for O, we follow the strategy of Benveniste and Scheinkman (1979), and consider the value function of a pseudo borrower with post-renegotiation debt b that chooses debt issuance $b' = B^R(x)$ instead of the optimal $b' = B^R(b)$,

$$\widetilde{W}(b,x) \equiv u\left(Q^R\left(B^R(x)\right)B^R(x) - b + \underline{w}\right) + Z^R\left(B^R(x)\right).$$

Note that $\widetilde{W}$ is differentiable and strictly decreasing in b. Since debt issuance is chosen suboptimally, it must be that $\widetilde{W}(b,x) \le W^R(b)$ with equality holding at $x = b$. Furthermore, let the pseudo borrower set the default threshold at the level $\widetilde{\Phi}(b,x) = W^R(0) - \widetilde{W}(b,x)$, where $\widetilde{\Phi}(b,x) \ge \Phi^R(b)$. Thus, the pseudo borrower renegotiates even for some ϕ larger than $\Phi^R(b)$. Note that $\widetilde{\Phi}(b,x)$ is differentiable and strictly increasing in b. Thus, the inverse function exists and is such that $\widetilde{\Phi}_x^{-1}(\phi) \le \hat{b}^R(\phi)$ (where we define $\widetilde{\Phi}_x(b) \equiv \widetilde{\Phi}(b,x)$).

Let

$$\widetilde{O}(b',x) = u\left(\widetilde{Q}(b',x)b' - b + \underline{w}\right) + \widetilde{Z}(b',x),$$

where $\widetilde{Q}(b',x)b'$ and $\widetilde{Z}(b',x)$ are given by

$$\begin{aligned}
\widetilde{Q}(b',x)\,b' &= R^{-1}\left[\left(1-\widetilde{\Psi}(b',x)\right)\left(\left[1-F(\widetilde{\Phi}(b',x))\right]b' + \int_{\phi_{\min}}^{\widetilde{\Phi}(b',x)} \widetilde{\Phi}_x^{-1}(\phi)dF(\phi)\right) + \widetilde{\Psi}(b',x)b'\right], \\
\widetilde{Z}(b',x) &= -X(\widetilde{\Psi}(b',x)) + \beta\left[\begin{array}{c}\widetilde{\Psi}(b',x)\bar{\mu}^R(b') + (1-\widetilde{\Psi}(b',x)) \\ \times\left(\left[1-F(\widetilde{\Phi}(b',x))\right]\widetilde{W}(b',x) + \int_{\phi_{\min}}^{\widetilde{\Phi}(b',x)}\left[W^R(0)-\phi\right]dF(\phi)\right)\end{array}\right],
\end{aligned}$$

having defined $\widetilde{\Psi}(b',x)$ as

$$\widetilde{\Psi}(b',x) = \left(X'\right)^{-1}\left(\beta\left[\bar{\mu}^R(b') - \left(\left[1-F(\widetilde{\Phi}(b',x))\right]\widetilde{W}(b',x) + \int_{\phi_{\min}}^{\widetilde{\Phi}(b',x)}\left[W^R(0)-\phi\right]dF(\phi)\right)\right]\right).$$

Note that $\widetilde{Q}$, $\widetilde{Z}$ and $\widetilde{\Psi}$ are differentiable in b' since we established above that $\widetilde{W}$ and $\widetilde{\Phi}$ are differentiable.

Then, $\widetilde{O}$ is a DLSF for O such that $\widetilde{O}(b',x) \le O(b')$ with equality (only) at $b' = x$. Thus, Theorem 1 in Clausen and Strub (2016) ensures that $O(b')$ is differentiable at all optimal interior choices $b' \in \hat{B}^R$ and that $\partial O(B^R(b))/\partial B^R(b) = \partial\widetilde{O}(B^R(b), B^R(b))/\partial B^R(b) = 0$. In this case, a standard FOC holds

$$\frac{\partial u\left(Q^R(B^R(b))B^R(b) - b + \underline{w}\right)}{\partial B^R(b)} + \frac{\partial Z^R(B^R(b))}{\partial B^R(b)} = 0.$$

Moreover, Lemma 3 in Clausen and Strub (2016) ensures that also the functions $W^R(b')$, $Z^R(b')$, $\Phi^R(b')$, $Q^R(b')$, and $\Psi^R(b')$ are differentiable in $b' \in \hat{B}^R$ and that a standard envelope condition applies, namely,

$$\begin{aligned}
\frac{\partial Z^R(B^R(b))}{\partial B^R(b)} &= \beta \left[(1-\Psi^R(B^R(b))) \left[1-F(\Phi^R(B^R(b)))\right] \frac{\partial W^R(B^R(b))}{\partial B^R(b)} + \Psi^R(B^R(b)) \frac{\partial \bar{\mu}^R(B(b))}{\partial B^R(b)} \right], \\
\frac{\partial W^R(B^R(b))}{\partial B^R(b)} &= -u'\left(C^R(B^R(b))\right) < 0.
\end{aligned}$$

This proves that the FOC stated in Proposition 9 is necessary for an interior optimum. ■

B.2.6 Proof of Lemma 2.1

In this section, we prove Lemma 2.1. The proof strategy follows Thomas and Worrall (1990, Proof of Proposition 1). We show first that the planner's problem is a contraction mapping with a strictly concave fixed-point P. The differentiability of P follows from Benveniste and Scheinkman (1979, Lemma 1). Note that $\bar{P}$ is given by (9) and has the same properties. Finally, we prove that P and $\bar{P}$ pin down uniquely interior promised utilities, effort and consumption.

We prove the results in the form of five claims. Each of them has a separate prove below. We demonstrate the proof for the COA. The properties of the first-best planning problem follow immediately by dropping the PC and adjusting the boundary conditions appropriately.

Define, first, the mapping $T(\gamma)(\nu)$ as the right-hand side of the planner's functional equation

$$T(\gamma)(\nu) = \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \omega_\phi\}_{\phi\in\aleph}) \in \Lambda(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)\gamma(\omega_\phi) \end{array} \right] \right] dF(\phi)$$

where maximization is constrained by the set $\Lambda(\nu)$ defined by

$$\begin{aligned}
\int_\aleph \left[u(c_\phi) - X(p_\phi) + \beta\left[p_\phi \bar{\omega}_\phi + (1-p_\phi)\omega_\phi\right]\right] dF(\phi) &= \nu \\
u(c_\phi) - X(p_\phi) + \beta\left[p_\phi \bar{\omega}_\phi + (1-p_\phi)\omega_\phi\right] &\geq \alpha - \phi, \quad \forall \phi \in \aleph, \\
c_\phi \in [0, \tilde{c}],\ p_\phi \in [\underline{p}, \bar{p}],\ \nu, \omega_\phi \in [\underline{\nu}, \tilde{\omega}],\ \bar{\omega}_\phi &\in [\underline{\omega}, \tilde{\bar{\omega}}].
\end{aligned}$$

We take as given that $\bar{P}$ is strictly concave and bounded between $\bar{P}_{MIN}$ and $\bar{P}_{MAX}$.

Claim 1 *$T(\gamma)$ maps concave functions into strictly concave functions.*

Proof. Let $\nu' \neq \nu'' \in [\underline{\nu}, \tilde{\omega}]$, $\delta \in (0,1)$, $\nu = \delta\nu' + (1-\delta)\nu''$, $P_k(\nu) = T(P_{k-1})(\nu)$, and P_{k-1} be concave. Then,

$$P_{k-1}(\delta\nu' + (1-\delta)\nu'') \geq \delta P_{k-1}(\nu') + (1-\delta)P_{k-1}(\nu'').$$

We follow the strategy of Thomas and Worrall (1990, Proof of Proposition 1), i.e., we construct a feasible but (weakly) suboptimal contract, $\left\{c^o_\phi(\nu), p^o_\phi(\nu), \bar{\omega}^o_\phi(\nu), \omega^o_\phi(\nu)\right\}_{\phi\in\aleph}$, such that even the profit generated by the suboptimal contract $P^o_k(\delta\nu' + (1-\delta)\nu'') \leq P_k(\delta\nu' + (1-\delta)\nu'')$ is higher than the linear combination of maximal profits $\delta P_k(\nu') + (1-\delta)P_k(\nu'')$. Define the weights $\underline{\delta}, \bar{\delta} \in (0,1)$ and the 4-tuple $(c^o_\phi(\nu), p^o_\phi(\nu), \omega^o_\phi(\nu), \bar{\omega}^o_\phi(\nu))$ such that

$$\begin{aligned}
\underline{\delta} &\equiv \frac{\delta\left[1-p_\phi(\nu')\right]}{\delta(1-p_\phi(\nu')) + (1-\delta)(1-p_\phi(\nu''))} \equiv \delta \frac{1-p_\phi(\nu')}{1-p^o_\phi(\nu)} \\
\bar{\delta} &\equiv \frac{\delta p_\phi(\nu')}{\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')} \equiv \delta \frac{p_\phi(\nu')}{p^o_\phi(\nu)} \\
\omega^o_\phi(\nu) &= \underline{\delta}\omega_\phi(\nu') + (1-\underline{\delta})\omega_\phi(\nu'') \\
\bar{\omega}^o_\phi(\nu) &= \bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'') \\
c^o_\phi(\nu) &= u^{-1}\left[\begin{array}{c} \delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu'')) \\ -\left[\delta X(p_\phi(\nu')) + (1-\delta)X(p_\phi(\nu''))\right] + X\left(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right) \end{array}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-p^o_\phi(\nu))\omega^o_\phi(\nu) &= \delta\left(1-p_\phi(\nu')\right)\omega_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\omega_\phi(\nu'') \\
p^o_\phi(\nu)\bar{\omega}^o_\phi(\nu) &= \delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu'')\bar{\omega}_\phi(\nu'') \\
u\left(c^o_\phi(\nu)\right) - X(p^o_\phi(\nu)) &= \delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu'')) - \left[\delta X(p_\phi(\nu')) + (1-\delta)X(p_\phi(\nu''))\right]
\end{aligned}$$

By construction the suboptimal allocation satisfies

$$c^o_\phi(\nu) \in [0, \tilde{c}],\ p^o_\phi(\nu) \in [\underline{p}, \bar{p}],\ \omega^o_\phi(\nu) \in [\underline{\nu}, \tilde{\omega}],\ \bar{\omega}^o_\phi(\nu) \in [\underline{\omega}, \tilde{\bar{\omega}}],$$

and, given the promised-utility ν, is also consistent with the PK

$$\begin{aligned}
&\int_\aleph \left[u\left(c^o_\phi(\nu)\right) - X(p^o_\phi(\nu)) + \beta\left[p^o_\phi(\nu)\bar{\omega}^o_\phi(\nu) + (1-p^o_\phi(\nu))\omega^o_\phi(\nu)\right]\right] dF(\phi) \\
= &\int_\aleph \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)u\left(c_\phi(\nu'')\right) - \left[\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))\omega_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\omega_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] dF(\phi) \\
= &\ \delta\nu' + (1-\delta)\nu'' = \nu.
\end{aligned}$$

Moreover, the PC for any ϕ yields

$$\begin{aligned}
&u\left(c^o_\phi(\nu)\right) - X(p^o_\phi(\nu)) + \beta\left[p^o_\phi(\nu)\bar{\omega}^o_\phi(\nu) + (1-p^o_\phi(\nu))\omega^o_\phi(\nu)\right] \\
= &\left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right) + (1-\delta)u\left(c_\phi(\nu'')\right) - \left[\delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu') + (1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))\omega_\phi(\nu') + (1-\delta)(1-p_\phi(\nu''))\omega_\phi(\nu'')\right] \end{array}\right] \\
= &\ \delta\left[u\left(c_\phi(\nu')\right) - X(p_\phi(\nu') + \beta p_\phi(\nu')\bar{\omega}_\phi(\nu') + \beta(1-p_\phi(\nu'))\omega_\phi(\nu')\right] \\
&+(1-\delta)\left[u(c_\phi(\nu'')) - X(p_\phi(\nu'')) + \beta p_\phi(\nu''))\bar{\omega}_\phi(\nu'') + \beta(1-p_\phi(\nu''))\omega_\phi(\nu'')\right] \\
\geq &\ \delta\left(\alpha-\phi\right) + (1-\delta)\left(\alpha-\phi\right) = \alpha-\phi,
\end{aligned}$$

Thus, we have proven that the suboptimal allocation $\left\{c^o_\phi(\nu), p^o_\phi(\nu), \omega^o_\phi(\nu), \bar{\omega}^o_\phi(\nu)\right\}_{\phi\in\aleph}$ is feasible. Namely, it satisfies the PCs and delivers promised utility ν. The profit function evaluated at the optimal contract

$\{c_\phi(\nu), p_\phi(\nu), \omega_\phi(\nu), \bar{\omega}_\phi(\nu)\}_{\phi\in\aleph}$ then implies the following inequality,

$$\begin{aligned}
& \delta P_k(\nu') + (1-\delta)P_k(\nu'') \\
= \;& \delta T(P_{k-1})(\nu') + (1-\delta)T(P_{k-1})(\nu'') \\
= \;& \int_\aleph \begin{bmatrix} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] \\ +\beta\left[\delta p_\phi(\nu')\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\delta)p_\phi(\nu'')\bar{P}(\bar{\omega}_\phi(\nu''))\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))P_{k-1}(\omega_\phi(\nu')) + (1-\delta)(1-p_\phi(\nu''))P_{k-1}(\omega_\phi(\nu''))\right] \end{bmatrix} dF(\phi) \\
= \;& \int_\aleph \begin{bmatrix} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] \\ +\beta p^o_\phi(\nu)\left[\bar{\delta}\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\bar{\delta})\bar{P}(\bar{\omega}_\phi(\nu''))\right] \\ +\beta(1-p^o_\phi(\nu))\left[\underline{\delta}P_{k-1}(\omega_\phi(\nu')) + (1-\underline{\delta})P_{k-1}(\omega_\phi(\nu''))\right] \end{bmatrix} dF(\phi) \\
< \;& \int_\aleph \begin{bmatrix} \underline{w} - u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right) \\ +\beta p^o_\phi(\nu)\bar{P}(\bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'')) \\ +\beta(1-p^o_\phi(\nu))P_{k-1}(\underline{\delta}\omega_\phi(\nu') + (1-\underline{\delta})\omega_\phi(\nu'')) \end{bmatrix} dF(\phi) \\
< \;& \int_\aleph \begin{bmatrix} \underline{w} - u^{-1}\begin{pmatrix} \delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu'')) \\ -[\delta X(p_\phi(\nu')) + (1-\delta)X(p_\phi(\nu''))] + X\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'') \end{pmatrix} \\ +\beta p^o_\phi(\nu)\bar{P}(\bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'')) \\ +\beta(1-p^o_\phi(\nu))P_{k-1}(\underline{\delta}\omega_\phi(\nu') + (1-\underline{\delta})\omega_\phi(\nu'')) \end{bmatrix} dF(\phi) \\
= \;& \int_\aleph \left[\underline{w} - c^o_\phi(\nu) + \beta\left[p^o_\phi(\nu)\bar{P}(\bar{\omega}^o_\phi(\nu)) + (1-p^o_\phi(\nu))P_{k-1}(\omega^o_\phi(\nu))\right]\right] dF(\phi) \\
\equiv \;& P^o_k(\nu) \leq P_k(\nu) = P_k(\delta\nu' + (1-\delta)\nu'').
\end{aligned}$$

The first inequality follows from the strict concavity of u and $\bar{P}$, along with the concavity of P_{k-1}. The second inequality follows from $0 \leq X\left(\delta p_\phi(\nu') + (1-\delta)p_\phi(\nu'')\right) < \delta X(p_\phi(\nu') + (1-\delta)X(p_\phi(\nu''))$ since X is strictly convex. The third inequality, $P^o_k(\nu) \leq P_k(\nu)$ follows from the fact that the optimal allocation delivers (weakly) larger profits than the suboptimal one. We conclude that $P_k(\delta\nu' + (1-\delta)\nu'') > \delta P_k(\nu') + (1-\delta)P_k(\nu'')$, i.e., P_k is strictly concave. This concludes the proof of the lemma. ■

Let Γ denote the space of continuous functions defined over the interval $[\underline{\nu}, \tilde{\omega}]$ and bounded between $P_{MIN} = \left(\underline{w} - \tilde{c} + \beta\underline{p}\bar{P}_{MIN}\right)/(1-\beta(1-\underline{p}))$ and $P_{MAX} = \bar{w}/(1-\beta)$. Moreover, let d_∞ denote the supremum norm, such that (Γ, d_∞) is a complete metric space.

Claim 2 *The mapping $T(\gamma)$ is an operator on the complete metric space (Γ, d_∞), $T(\gamma)$ is a contraction mapping with a unique fixed-point $P \in \Gamma$.*

Proof. By the Theorem of the Maximum $T(\gamma)(\nu)$ is continuous in ν. Moreover, $T(\gamma)(\nu)$ is bounded between P_{MIN} and P_{MAX} since even choosing zero consumption for any realization of ϕ would induce profits not exceeding P_{MAX}

$$\begin{aligned}
\underline{w} + \beta\int_\aleph \left[p_\phi\bar{P}(\bar{\omega}_\phi) + (1-p_\phi)\gamma(\omega_\phi)\right] dF(\phi) &< \bar{w} + \beta/(1-\beta)\bar{w} \\
&= \bar{w}/(1-\beta) = P_{MAX},
\end{aligned}$$

and choosing the maximal consumption $\tilde{c}$ and promised utility $\tilde{\omega}$ and $\tilde{\bar{\omega}}$ for any ϕ would induce profits no lower than P_{MIN}. Thus, $T(\gamma)(\nu)$ is indeed an operator on (Γ, d_∞).

According to Blackwell's sufficient conditions $T(\gamma)$ is a contraction mapping (see Stokey *et al.* (1989, Theorem 3.3) if: (i) T is monotone, (ii) T discounts.

1. Monotonicity: Let $\gamma^+, \gamma \in \Gamma$ with $\gamma^+(\nu) \geq \gamma(\nu)$, $\forall \nu \in [\underline{\nu}, \tilde{\omega}]$. Then

$$\begin{aligned}
T(\gamma^+)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \omega_\phi\}_{\phi \in \aleph}) \in \Lambda(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)\gamma^+(\omega_\phi) \end{array} \right] \right] dF(\phi) \\
&\geq \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \omega_\phi\}_{\phi \in \aleph}) \in \Lambda(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)\gamma(\omega_\phi) \end{array} \right] \right] dF(\phi) \\
&= T(\gamma)(\nu).
\end{aligned}$$

2. Discounting: Let $\gamma \in \Gamma$ and $a \geq 0$ be a real constant. Then

$$\begin{aligned}
T(\gamma + a)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \omega_\phi\}_{\phi \in \aleph}) \in \Lambda(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)\left(\gamma(\omega_\phi) + a\right) \end{array} \right] \right] dF(\phi) \\
&= T(\gamma)(\nu) + \beta a \int_\aleph (1-p_\phi) dF(\phi) \\
&\leq T(\gamma)(\nu) + \beta a
\end{aligned}$$

   and $\beta \in (0, 1)$.

Thus, $T(\gamma)$ is indeed a contraction mapping and according to Banach's fixed-point theorem (see Stokey *et al.* (1989), Theorem 3.2) there exists a unique fixed-point $P \in \Gamma$ satisfying the stationary functional equation,

$$P(\nu) = T(P)(\nu).$$

■

Claim 3 *The profit function P is strictly concave.*

This claim follows immediately from Stokey *et al.* (1989, Corollary 1). Since the unique fixed-point of $T(\gamma)$ is the limit of applying the operator n times $T^n(\gamma)(\nu)$ starting from any (and, in particular the concave ones) element γ in Γ, and the operator $T(\gamma)$ maps concave into strictly concave functions the fixed-point P must be strictly concave.

Claim 4 *The profit function P differentiable at its interior support with $P'(\nu) = 1/u'(c(\nu)) < 0$.*

Proof. Given the strict concavity of the profit function, the proof is the same as for Lemma 3.2. The only difference is that $p_\phi(\nu)$ denotes the optimal effort stated in Proposition 2 instead of Proposition 3. ■

We can now establish that the FOCs of the COA are necessary and sufficient.

Claim 5 *The FOCs of the planner problem without moral hazard are necessary and sufficient for optimality.*

Proof. Lemma 1 implies that there cannot be two optimal contracts with distinct ω_ϕ and $\bar{\omega}_\phi$. Suppose not, so that there exists a 4-tuple of promised utilities $\left\{\omega'_\phi, \omega''_\phi, \bar{\omega}'_\phi, \bar{\omega}''_\phi\right\}$ such that either $\omega'_\phi(\nu) \neq \omega''_\phi(\nu)$ or $\bar{\omega}'_\phi(\nu) \neq \bar{\omega}''_\phi(\nu)$ (or both). Then, from the strict concavity of P and $\bar{P}$, it would be possible to construct a feasible allocation that dominates the continuation profit implied by the proposed optimal allocations, i.e., either $P(\underline{\delta}\omega'_{\phi} + (1-\underline{\delta})\omega''_{\phi}) > \underline{\delta}P(\omega'_{\phi}) + (1-\underline{\delta})P(\omega''_{\phi})$, or $\bar{P}(\bar{\delta}\bar{\omega}'_{\phi} + (1-\bar{\delta})\bar{\omega}''_{\phi}) > \bar{\delta}\bar{P}(\bar{\omega}'_{\phi}) + (1-\bar{\delta})\bar{P}(\bar{\omega}''_{\phi})$ (or both). This contradicts the assumption that the proposed allocations are optimal, establishing that the optimal contract pins down a unique pair of promised utilities, $\left\{\omega_{\phi}, \bar{\omega}'_{\phi}\right\}$.

Finally, we show that a unique pair of promised utilities pins down uniquely effort and consumption. The assumptions on X rule out corner solutions for effort, the assumptions on u and the fact that promised utility is interior (ω_{ϕ} and $\bar{\omega}_{\phi}$ remain constant for $\nu \geq \alpha - \phi_{\min}$) implies that also consumption is interior. Then the FOCs in (10) and (12) imply that

$$\begin{aligned} -\bar{P}'\left(\bar{\omega}_{\phi}(\nu)\right)^{-1} &= u'(c_{\phi}(\nu)), \\ X'\left(p_{\phi}(\nu)\right) &= \beta\left(-\bar{P}'\left(\bar{\omega}_{\phi}(\nu)\right)^{-1}\left(\bar{P}\left(\bar{\omega}_{\phi}(\nu)\right) - P\left(\omega_{\phi}(\nu)\right)\right) + \left(\bar{\omega}_{\phi}(\nu) - \omega_{\phi}(\nu)\right)\right), \end{aligned}$$

which shows that, given ν and ϕ, effort and consumption are uniquely determined as well. ■

This concludes the proof of the proposition.

B.3 Parameterization

In this section, we provide details of the parameterization underlying the numerical examples shown in the figures of the paper. We focus on the quantitative properties of the one-asset economy (with renegotiation) of Section 4 since this is a more realistic positive representation of the world. We choose parameters so as to match salient moments observed for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) during the Great Recession.

A model period corresponds to one year. We normalize the GDP during normal time to $\bar{w} = 1$ and assume that the recession causes a drop in income of 25%, i.e., $\underline{w} = 0.75 \times \bar{w}$. This corresponds to the fall of real GDP per capita for Greece between 2007 and 2016.[35] The annual real gross interest rate is set to $R = 1.02$. The utility function is assumed to be CRRA with a relative risk aversion of 2. We assume an isoelastic effort cost function, $X(p) = \frac{\xi}{1+1/10}(p)^{1+1/10}$, and calibrate $\xi = 14.371$ so that a country starting with a 100% debt-output ratio in recession recovers in expectation after one decade (we have Greece in mind). Finally, we parameterize $f(\phi)$ and its support. The maximum default cost realization $\phi_{\max} = 2.275$ is calibrated to target a debt limit during recession of $b^{\max}/\underline{w} = 178\%$ in line with Collard *et al.* (2015, Table 3, Column 1).[36] Finally, we assume that $\phi_{\max} - \phi$ is distributed exponential with rate parameter $\eta = 1.625$ and truncation point $\phi_{\max}$.[37] The model then generates an average default premium of 4.04% for a country with a debt-output ratio of 100% in recession. This overlaps with the average debt and average default premium for the GIIPS during 2008-2012 (Eurostat).

B.4 References for Appendix B

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[35] Real GDP per capita of Greece fell from 22'700 to 17'100 Euro between 2007 and 2016 (Eurostat, nama_10_pc series). Where the years 2007 and 2016 correspond to the peak and the trough, respectively, of real GDP per capita relative to a 2% growth trend with base year 1995.

[36] We ignore the value of 282% for Korea which is a clear outlier.

[37] More formally, ϕ has the p.d.f.

$$f(\phi) = \frac{\eta e^{-\eta(\phi_{\max}-\phi)}}{1 - e^{-\eta\phi_{\max}}}, \ \phi \in [0, \phi_{\max}].$$

This also implies that $\phi_{\min} = 0$.

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