



Moduli of cubic surfaces and their anticanonical divisors

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Abstract

We consider the moduli space of log smooth pairs formed by a cubic surface and an anticanonical divisor. We describe all compactifications of this moduli space which are constructed using geometric invariant theory and the anticanonical polarization. The construction depends on a weight on the divisor. For smaller weights the stable pairs consist of mildly singular surfaces and very singular divisors. Conversely, a larger weight allows more singular surfaces, but it restricts the singularities on the divisor. The one-dimensional space of stability conditions decomposes in a wall-chamber structure. We describe all the walls and relate their value to the worst singularities appearing in the compactification locus. Furthermore, we give a complete characterization of stable and polystable pairs in terms of their singularities for each of the compactifications considered.

Keywords Moduli of pairs · Fano varieties · Cubic surfaces · Geometric invariant theory · Classification of singularities

Mathematics Subject Classification 14L24 · 14J10 · 14J26 · 14J45 · 14Q10

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1 Introduction

The moduli space of cubic surfaces is a classic space in algebraic geometry. Indeed, its GIT compactification was first described by Hilbert in 1893 [14], and several alternative compactifications have followed it (see [12, 15, 20]). In this article, we enrich this moduli problem by parametrizing pairs (S, D) where $S \subset \mathbb{P}^3$ is a cubic surface, and $D \in |-K_S|$ is an anticanonical divisor. There are several motivations for our construction. Firstly, it was recently established that the GIT compactification of cubic surfaces corresponds to the moduli space of K -stable del Pezzo surfaces of degree three [21]. The concept of K -stability has a natural generalization to log- K -stability for pairs, and our GIT quotients are natural candidates to construct compactifications of log K -stable pairs of cubic surfaces and their anticanonical divisors. Therefore, our description is a first step toward a generalization of [21] to the log setting, an approach considered in the sequel to this article [11]. Secondly, a precise description of the GIT of cubic surfaces is important for describing the complex hyperbolic geometry of the moduli of cubic surfaces, and constructing new examples of ball quotients (see [1]). More specifically, Laza et al. [17] predicted a Hodge theoretical compactification of the moduli space of pairs (S, D) via a particular loci within the moduli space of cubic fourfolds. One may expect that such uniformization coincides with one of the compactifications of the moduli space of pairs (S, D) that we obtain in this article. Finally, our compactifications explore the setting of variations of GIT quotients for log pairs for which few examples exist (see [16] and [6, Theorem 11.2]).

The GIT quotients considered depend on a choice of a linearization \mathcal{L}_t of the parameter space \mathcal{H} of cubic forms and linear forms in \mathbb{P}^3 . We have that $\mathcal{H} \cong \mathbb{P}^{19} \times \mathbb{P}^3$. Every ample divisor in $\text{Pic}(\mathbb{P}^{19} \times \mathbb{P}^3) \cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle$ is of bidegree (a, b) for some positive integers a and b . Thus, the different GIT quotients arising by picking different polarizations of \mathcal{H} are controlled by the parameter $t = \frac{b}{a} \in \mathbb{Q}_{>0}$ (see Sect. 3 for a thorough treatment). For each value of t , there is a GIT compactification $\overline{M}(t)$ of the moduli space of pairs (S, D) where S is a cubic surface and $D \in |-K_S|$ is an anticanonical divisor. It follows from the general theory of variations of GIT (see [7, 24], c.f. [10, Theorem 1.1]) that $0 \leq t \leq 1$ and that there are only finitely many different GIT quotients associated to t . Indeed, there is a set of chambers (t_i, t_{i+1}) where the GIT quotients $\overline{M}(t)$ are isomorphic for all $t \in (t_i, t_{i+1})$, and there are finitely many GIT walls t_1, \dots, t_k where the GIT quotient is a birational modification of $\overline{M}(t)$ where $0 < |t - t_i| < \epsilon \ll 1$. Additionally there are initial and end walls $t_0 = 0$ and $t_{k+1} = 1$.

Lemma 1 *The GIT walls are*

$$t_0 = 0, \quad t_1 = \frac{1}{5}, \quad t_2 = \frac{1}{3}, \quad t_3 = \frac{3}{7}, \quad t_4 = \frac{5}{9}, \quad t_5 = \frac{9}{13}, \quad t_6 = 1.$$

Given $t \in \mathbb{Q}_{>0}$ we say that a pair (S, D) is t -stable (respectively t -semistable) if it is t -stable (respectively t -semistable) under the $\text{SL}(4, \mathbb{C})$ -action. A pair is *strictly t -semistable* if it is t -semistable but not t -stable. The space $M(t)$ parametrizes t -stable pairs and $\overline{M}(t)$ parametrizes closed t -semistable orbits.

The GIT walls can be interpreted geometrically as follows. Let T be one of the possible isolated singularities in a cubic surface (see Proposition 1), let $w(T)$ be the sum of its associated weights (see Definition 2). For example, the set of weights for the A_n singularity is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{n+1})$ and $w(A_n) = \frac{n+2}{n+1}$. We define $\text{Wall}(T) := \frac{4}{w(T)} - 3$.

Theorem 1 *There are 13 non-isomorphic GIT quotients $\overline{M}(t)$. Seven of these quotients correspond to the walls t_i in Lemma 1 and they can be recovered as $t_i = \text{Wall}(T)$ for each isolated ADE singularity T occurring at a point p of an irreducible cubic surface S such that*

- (i) *for some $D \in |-K_S|$, the log pair (S, D) is strictly t_i -semistable,*
- (ii) *for $t' < t_i$ (S, D) is t' -unstable, and*
- (iii) *$p \notin \text{Supp}(D)$ unless $t = 0$.*

Indeed, the values of the walls are:

$$\begin{aligned}
 t_0 &= \text{Wall}(A_2) = 0, & t_1 &= \text{Wall}(A_3) = \frac{1}{5}, & t_2 &= \text{Wall}(A_4) = \frac{1}{3}, \\
 t_3 &= \text{Wall}(A_5) = \text{Wall}(D_4) = \frac{3}{7}, & t_4 &= \text{Wall}(D_5) = \frac{5}{9}, \\
 t_5 &= \text{Wall}(E_6) = \frac{9}{13}, & t_6 &= \text{Wall}(\tilde{E}_6) = 1.
 \end{aligned}$$

The other six GIT quotients $\overline{M}(t)$ correspond to linearizations $t \in (t_i, t_{i+1})$, $i = 1, \dots, 6$. All the points in $\overline{M}(t_0)$ and $\overline{M}(t_6)$ correspond to strictly semi-stable pairs, while all other $\overline{M}(t)$ with $t \in (0, 1)$ have stable points. The GIT quotient is empty for any $t \notin [0, 1]$.

We will learn in Sect. 2 that the walls t_i and classification of the log pairs (S, D) parametrized by $\overline{M}(t)$ depend on both the singularities of the surface and the divisor D in a complementary way. Indeed, the singularities of the surfaces will be worse when t approaches 1 while the singularities of the hyperplane section will be worse when t approaches 0 (see Table 1).

Furthermore, we have a complete analysis of the stability of pairs (S, D) represented in $\overline{M}(t)$ and $M(t)$ for each t in the space of stability conditions $[0, 1] \cap \mathbb{Q}$. Specifically for each $t \in (0, 1) \cap \mathbb{Q}$, in Theorem 2 and Table 1 we give a list of all t -stable pairs represented in $M(t)$, and in Theorem 3 and Table 2, we classify all strictly t -semistable pairs with close orbits, which compactify $M(t)$ into $\overline{M}(t)$. The quotient $\overline{M}(0)$ is isomorphic to the GIT of cubic surfaces and the quotient $\overline{M}(1)$ is the GIT of plane cubic curves (see [10, Lemma 4.1]). These spaces are classical and they are described in [19, Sec 7.2 (b)] and [19, Example 7.12] respectively. Henceforth we will focus on the case $t \in (0, 1)$. As mentioned earlier, the following theorem gives a first approximation to the classification of log stable pairs of other stability theories, in particular for log K-stability (and the existence of Kähler-Einstein metrics with conical singularities along a boundary). This was first observed for cubic surfaces (no boundary) by Ding and Tian in [5].

Table 1 Worst possible singularities in a t -stable pair (S, D) for each $t \in (0, 1)$

t	$(0, \frac{1}{5})$	$\frac{1}{5}$	$(\frac{1}{5}, \frac{1}{3})$	$\frac{1}{3}$
Sing(S)	A_2	A_2	A_3	A_3
Sing(D)	On smooth or $A_1 \in S$	Isolated on smooth or $A_1 \in S$	Isolated on smooth or $A_1 \in S$	Isolated or cuspidal at $A_1 \in S$
t	$(\frac{1}{3}, \frac{3}{7})$	$\frac{3}{7}$	$(\frac{3}{7}, \frac{5}{9})$	$\frac{5}{9}$
Sing(S)	A_4	A_4	A_5, D_4	A_5, D_4
Sing(D)	Isolated or cuspidal at $A_1 \in S$	Tacnodal or normal crossings at $A_1 \in S$	Tacnodal or normal crossings at $A_1 \in S$	Cuspidal or normal crossings at $A_1 \in S$
t	$(\frac{5}{9}, \frac{9}{13})$	$\frac{9}{13}$	$(\frac{9}{13}, 1)$	
Sing(S)	A_5, D_5	A_5, D_5	E_6	
Sing(D)	Cuspidal or normal crossings at $A_1 \in S$	Normal crossings on smooth or $A_1 \in S$	Normal crossings on smooth or $A_1 \in S$	

Throughout the article a pair (S, D) consists of a cubic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ and an anticanonical section $D \in |-K_S| \cong \mathbb{P}(H^0(S, \mathcal{O}_S(1)))$. Hence, $D = S \cap H$ for some hyperplane $H = \{l(x_0, \dots, x_3) = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$. Whenever we consider a parameter $t \in (t_i, t_{i+1})$ we implicitly mean $t \in (t_i, t_{i+1}) \cap \mathbb{Q}$.

In Sect. 3 we describe in detail the GIT setting we consider. We introduce the required singularity theory in Sect. 4. GIT-stability depends on a finite list of geometric configurations characterized in Sect. 5. We prove Theorem 2 in Sect. 6. We prove Theorems 1 and 3 in Sect. 6.

Our article does an extensive use of J.W. Bruce and C.T.C. Wall’s elegant classification of singular cubic surfaces [4] in the modern language of Arnold. Our results use some computations done via software. The computations, together with full source code written in Python can be found in [9]. The code is based on the theory developed in our previous article [10] and a rough idea of the algorithm can be found there. The source code and data, but not the text of this article, are released under a Creative Commons CC BY-SA 4.0 license. See [9] for details. If you make use of the source code and/or data in an academic or commercial context, you should acknowledge this by including a reference or citation to [10]—in the case of the code—or to this article—in the case of the data.

2 Classification of stable orbits and compactification log pairs

A nice feature of $M(t)$ is that for each $t \in (0, 1)$ and each t -stable pair (S, D) , the surface S has isolated ADE singularities. The classification is simplified by using the notion of ‘worse singularity’. Roughly speaking, a singularity germ T_1 is worse than a singularity T_2 if the former can be partially deformed into the latter. See Definition 1

Table 2 Strictly semistable pairs with closed orbits appearing in each $t \in (0, 1)$

t	$(0, 1)$	$\frac{1}{5}$		$\frac{1}{3}$
Sing(S)	$3A_2$	$A_3 + 2A_1$		$A_4 + A_1$
D	Unique three lines in S	Double line containing $2A_1$ and unique line not containing surface singularities		Tacnodal curve at A_1
t	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{9}{13}$
Sing(S)	D_4 , S has an Eckardt point p	$A_5 + A_1$	$p = D_5$	$p = E_6$
D	Unique three coplanar lines through p	\mathbb{C}^* -invariant cuspidal curve at A_1 ,	\mathbb{C}^* -invariant tacnodal curve, $p \notin D$	\mathbb{C}^* -invariant cuspidal curve, $p \notin D$

and Fig. 2 for a formal definition. Table 1 gives a summary of the t -stable pairs (S, D) for each t in terms of their *worst* singularities and the intersection of the components of D .

See Table 3 to reinterpret D in the language of ADE singularities. Our first classification result describes the stable orbits of $M(t)$ in terms of their singularities:

Theorem 2 Consider a pair (S, D) formed by a cubic surface S and a hyperplane section $D \in |-K_S|$.

- (i) Let $t \in (0, \frac{1}{5})$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_2 and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S . In particular D may be non-reduced.
- (ii) Let $t = \frac{1}{5}$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_2 , D is reduced, and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S .
- (iii) Let $t \in (\frac{1}{5}, \frac{1}{3})$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_3 , D is reduced and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S .
- (iv) Let $t = \frac{1}{3}$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_3 , D is reduced and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a cuspidal singularity at P .
- (v) Let $t \in (\frac{1}{3}, \frac{3}{7})$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_4 , D is reduced and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a normal crossing singularity at P as a plane cubic curve.

- (vi) Let $t = \frac{3}{7}$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_4 , D has at worst a tacnodal singularity and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a normal crossing singularity at P as a plane cubic curve.
- (vii) Let $t \in (\frac{3}{7}, \frac{5}{9})$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_5 or D_4 , D has at worst a tacnodal singularity and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a normal crossing singularity at P as a plane cubic curve.
- (viii) Let $t = \frac{5}{9}$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_5 or D_4 , D has at worst an A_2 singularity and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a normal crossing singularity at P as a plane cubic curve.
- (ix) Let $t \in (\frac{5}{9}, \frac{9}{13})$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_5 or D_5 , D has at worst a cuspidal singularity and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S and D has at worst a normal crossing singularity at P as a plane cubic curve.
- (x) Let $t = \frac{9}{13}$. The pair (S, D) is t -stable if and only if S has finitely many singularities at worst of type A_5 or D_5 , D has at worst normal crossing singularities as a plane cubic curve and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S .
- (xi) Let $t \in (\frac{9}{13}, 1)$. The pair (S, D) is t -stable if and only if S has finitely many ADE singularities, D has at worst normal crossing singularities as a plane cubic curve and if $P \in D$ is a surface singularity, then P is at worst an A_1 singularity of S .

The next theorem gives a full of classification of the pairs (S, D) associated to each of the unique closed orbits in $\overline{M(t)} \setminus M(t)$ for each $t \in (0, 1)$. Normal cubic surfaces with a \mathbb{C}^* -action have been classified [8, Table 3]. They play a central role in our classification, as they are all realized as part of some strictly semistable log pair of some wall.

Figure 1 gives sketches of each of these pairs and Table 1 summarises these orbits. Recall that an Eckardt point of a cubic surface S is a point where three coplanar lines of S intersect.

Theorem 3 Let $t \in (0, 1)$. If $t \neq t_i$, then $\overline{M(t)}$ is the compactification of the stable loci $M(t)$ by the closed $SL(4, \mathbb{C})$ -orbit in $\overline{M(t)} \setminus M(t)$ represented by the pair (S_0, D_0) , where S_0 is the unique \mathbb{C}^* -invariant cubic surface with three A_2 singularities and D_0 is the union of the unique three lines in S_0 , each of them passing through two of those singularities.

If $t = t_i$, $i = 1, 2, 4, 5$, then $\overline{M(t_i)}$ is the compactification of the stable loci $M(t_i)$ by the two closed $SL(4, \mathbb{C})$ -orbits in $\overline{M(t_i)} \setminus M(t_i)$ represented by the uniquely defined pair (S_0, D_0) described above and the \mathbb{C}^* -invariant pair (S_i, D_i) uniquely defined as follows:

- (i) the cubic surface S_1 with an A_3 singularity and two A_1 singularities and the divisor $D_1 = 2L + L' \in |-K_S|$, where L and L' are lines such that L is the line containing both A_1 singularities and L' is the only line in S not containing any singularities;

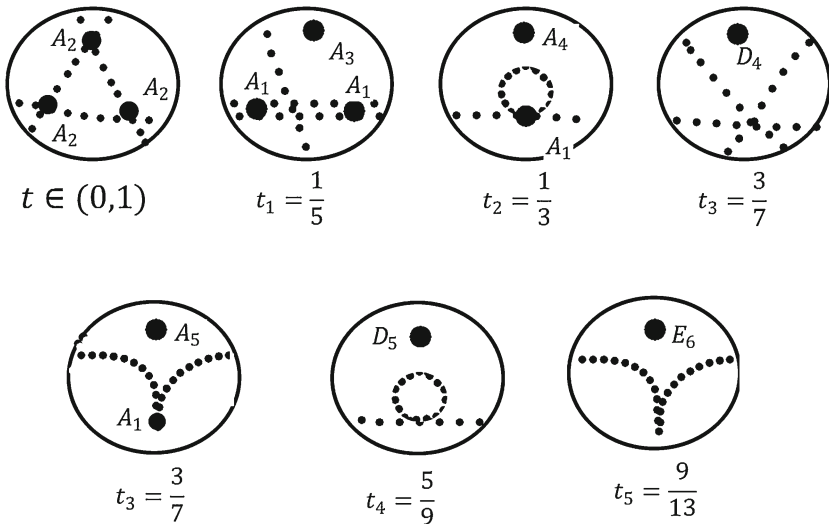


Fig. 1 Pairs in $\overline{M}(t) \setminus M(t)$ for each $t \in (0, 1)$. The dotted lines represent the divisor D . The bold points are singularities of the surface

- (ii) the cubic surface S_2 with an A_4 singularity and an A_1 singularity and the divisor $D_2 \in |-K_S|$, which is a tacnodal curve singular at the A_1 singularity of S ;
- (iii) the cubic surface S_4 with a D_5 singularity and the divisor $D_4 \in |-K_S|$, which is a tacnodal curve that does not contain the surface singularity;
- (v) the cubic surface S_5 with an E_6 singularity and the cuspidal rational curve $D_5 \in |-K_S|$, that does not contain the surface singularity.

The space $\overline{M}(t_3)$ is the compactification of the stable loci $M(t_3)$ by the three closed $SL(4, \mathbb{C})$ -orbits in $\overline{M}(t_3) \setminus M(t_3)$ represented by the \mathbb{C}^* -invariant pairs uniquely defined as follows:

- (i) the pair (S_0, D_0) described above;
- (ii) the pair (S_3, D_3) , where S_3 is the cubic surface with a D_4 singularity and an Eckardt point and D_3 consists of the unique three coplanar lines intersecting at the Eckardt point;
- (iii) the pair (S'_3, D'_3) , where S'_3 is the cubic surface with an A_5 and an A_1 singularity and the divisor D'_3 , which is an irreducible curve with a cuspidal point at the A_1 singularity of S'_3 .

The theory of variations of GIT quotients used to construct these quotients can be used to understand the birational maps among them. In particular, for $\varepsilon > 0$ sufficiently small, we have morphisms $\overline{M}(\varepsilon) \rightarrow \overline{M}(0)$ and $\overline{M}(1 - \varepsilon) \rightarrow \overline{M}(1)$.

By Pinkham’s theory on deformation of singularities with \mathbb{C}^* -action, the deformations of negative weight can be globalized and interpreted as a moduli space of pairs (see [22, Theorem 2.9]). In particular, the fiber of the map $M(1 - \varepsilon) \rightarrow M(1)$ over a point representing a smooth curve with trivial stabilizer is isomorphic to the deformation of the \tilde{E}_6 singularity in negative direction modulo the natural action

of \mathbb{C}^* (c.f. [16, Section 2.4] for an analogue situation with the N_{16} singularity). Such deformations of \tilde{E}_6 were determined by Looijenga [18, Theorem 3.4]. To make this explicit, let E be a smooth elliptic curve and $p_E \in M(1) \subset \overline{M}(1)$ be the point representing it. The fiber over p_E of $M(1 - \epsilon) \rightarrow M(1)$ is isomorphic to $(E \otimes E_6)/W(E_6) \cong \mathbb{P}(1, g_1, g_2, g_3, g_4, g_5, g_6)$ where g_i are the coefficients of the highest root of E_6 with respect to a set of simple roots, i.e the fiber is isomorphic to $\mathbb{P}(1, 1, 1, 2, 2, 2, 3)$.

3 GIT set-up and computational methods

In this section, we briefly describe the GIT setting for constructing our compact moduli spaces. We refer the reader to [10], where the problem is thoroughly discussed and solved for pairs formed by a hyperplane and a hypersurface of \mathbb{P}^{n+1} of a fixed degree. Our GIT quotients are given by

$$\overline{M}\left(\frac{b}{a}\right) := \left(\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))) \times \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)))\right)^{ss} //_{\mathcal{O}(a,b)} \text{SL}(4, \mathbb{C}),$$

and they depend only of one parameter $t := \frac{b}{a} \in \mathbb{Q}_{\geq 0}$. The use of GIT requires three initial combinatorial steps which are computed with the algorithm described in [10] and implemented in [9]. The first step is to find a set of *candidate GIT walls* which includes all GIT walls (see [10, Theorem 1.1]). Some of these walls may be redundant and they are removed by comparing if there is any geometric change to the t -(semi)stable pairs (S, D) for $t = t_i \pm \epsilon$ for $0 < \epsilon \ll 1$. The set of candidate GIT walls is precisely the one in Lemma 1 and once Theorem 3 is proven this proves Lemma 1.

The second step (see [10, Lemma 3.2]) is to find the finite set $S_{2,3}$ of one-parameter subgroups that determine the t -stability of all pairs (S, D) for all t . For convenience, given a one-parameter subgroup $\lambda = \text{Diag}(r_0, \dots, r_3)$, we define its dual one as $\bar{\lambda} = \text{Diag}(-r_3, \dots, -r_0)$.

Lemma 2 *The elements $S_{2,3}$ are λ_k and $\bar{\lambda}_k$ where λ_k is one of the following:*

$$\begin{aligned} \lambda_1 &= \text{Diag}(1, 0, 0, -1) & \lambda_2 &= \text{Diag}(2, 0, -1, -1) & \lambda_3 &= \text{Diag}(5, 1, -3, -3) \\ \lambda_4 &= \text{Diag}(13, 1, -3, -11) & \lambda_5 &= \text{Diag}(3, 1, -1, -3) & \lambda_6 &= \text{Diag}(9, 1, -3, -7) \\ \lambda_7 &= \text{Diag}(5, 5, -3, -7) & \lambda_8 &= \text{Diag}(1, 1, 1, -3) & \lambda_9 &= \text{Diag}(5, 1, 1, -7) \\ \lambda_{10} &= \text{Diag}(1, 1, -1, -1) \end{aligned}$$

Let \mathcal{E}_k be the set of all monomials in four variables of degree k . Let $g \in \text{SL}(4, \mathbb{C})$. Suppose $g \cdot S$ is given by the vanishing locus of a homogeneous polynomial F of degree 3 and $g \cdot D$ is given by the vanishing locus of F and a homogeneous polynomial l of degree 1. We say that F and l are *associated* to the pair $(g \cdot S, g \cdot D)$ and to the corresponding pair of sets of monomials. Let $\lambda = \text{Diag}(r_0, \dots, r_3)$. Denote by $\mathcal{S} \subseteq \mathcal{E}_3$ and $\mathcal{D} \subseteq \mathcal{E}_1$ the monomials with non-zero coefficients in F and l , respectively. There is a natural pairing $\langle v, \lambda \rangle \in \mathbb{Z}$ for any $v \in \mathcal{E}_k$, namely $\langle x_0^{i_0} \cdots x_3^{i_3}, \text{Diag}(r_0, \dots, r_3) \rangle = \sum i_j r_j$. We define

$$\mu_t(g \cdot S, g \cdot D, \lambda) := \min_{v \in S} \langle v, \lambda \rangle + t \min_{x_i \in D} \langle x_i, \lambda \rangle.$$

Lemma 3 (Hilbert-Mumford Criterion, see [10, Lemma 3.2]) *A pair (S, D) , where $D = S \cap H$, is not t -stable if and only if there is $g \in \text{SL}_n$ satisfying*

$$\mu_t(S, D) = \max_{\lambda \in S_{2,3}} \{\mu_t(g \cdot S, g \cdot D, \lambda)\} \geq 0.$$

Given $t \in (0, 1)$, and $\lambda \in S_{2,3}$ and $i \in \{0, \dots, 3\}$, the next step is to find the pairs of sets $N_t^\oplus(\lambda, x_i) := (V_t^\oplus(\lambda, x_i), B^\oplus(x_i))$ defined as:

$$V_t^\oplus(\lambda, x_i) = \{v \in \mathcal{E}_d \mid \langle v, \lambda \rangle + t \langle x_i, \lambda \rangle > 0\}, \quad B^\oplus(x_i) = \{x_k \in \mathcal{E}_1 \mid k \leq i\},$$

which are maximal with respect to the containment order. Since by [10, Lemma 3.2], we only need to consider the one-parameter subgroups in Lemma 2, which is a finite computation. Hence, they can be computed using computer software [9]. A more detailed algorithm can be found in [10].

Theorem 4 ([10, Theorem 1.4]) *Let $t \in (0, 1)$. A pair $(S, S \cap H)$ is not t -stable if and only if there exists $g \in \text{SL}(4, \mathbb{C})$ such that the set of monomials associated to $(g \cdot S, g \cdot H)$ is contained in a pair of sets $N_t^\oplus(\lambda, x_i)$.*

Given $N_t^\oplus(\lambda, x)$, define $N_t^0(\lambda, x_i) := (V_t^0(\lambda, x_i), B^0(x_i))$ (see [10, Proposition 5.3]) where

$$V_t^0(\lambda, x_i) \times B^0(x_i) = \{(v, m) \in V_t^\oplus(\lambda, x_i) \times B^\oplus(x_i) \mid \langle v, \lambda \rangle + t \langle m, \lambda \rangle = 0\}.$$

Theorem 5 ([10, Theorem 1.6]) *Let $t \in (0, 1)$. If a pair $(S, S \cap H)$ belongs to a closed strictly t -semistable orbit, then there exist $g \in \text{SL}(4, \mathbb{C})$, $\lambda \in S_{2,3}$ and x_i such that the set of monomials associated to $(g \cdot S, g \cdot D)$ corresponds to those in a pair of sets $N_t^0(\lambda, x_i)$.*

4 Preliminaries in singularity theory

We recall the admissible singularities in normal cubic surfaces.

Proposition 1 ([4]) *Let X be an irreducible and reduced cubic surface and $p \in X$ be an isolated singular point. Then, the singularity at p is either a Du val singularity (of type A_k, D_k with $k \leq 5$ or E_6), or a cone over a smooth elliptic curve (i.e. a simple elliptic singularity of type \tilde{E}_6).*

Definition 1 ([2, p.88]) A class of singularities T_2 is adjacent to a class T_1 , and one writes $T_1 \leftarrow T_2$ if every germ of $f \in T_2$ can be locally deformed into a germ in T_1 by an arbitrary small deformation. We say that the singularity T_2 is worse than T_1 ; or that T_2 is a *degeneration* of T_1 .

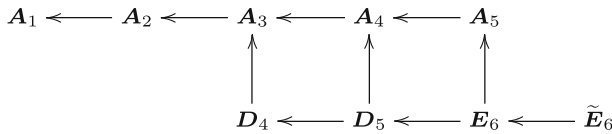


Fig. 2 Degeneration of germs of isolated singularities appearing in cubic surfaces

Table 3 Classification of plane cubic curves with isolated singularities

Non-singular	-	Cuspidal cubic	A_2
Nodal cubic	A_1	Three lines intersecting normally	$3A_1$
Line and conic intersecting normally	$2A_1$	Three lines intersecting at a point	D_4
Line and conic tangent at a point	A_3		

The degenerations of the isolated singularities that appear in a cubic surface (or in their anticanonical divisors, which are plane cubic curves) are described in Figure 2 (for details see [2, p. 88] and [3, §13]).

The above theory considers only local deformations of singularities. When we study degenerations in the GIT quotient we are interested in global deformations.

Lemma 4 ([23, Theorem 1], c.f. [13]) *Let $V(T_1, \dots, T_r)$ be the set of cubic hypersurfaces in \mathbb{P}^n for $n \leq 3$ with r isolated singular points of types T_1, \dots, T_r . The germ of the linear system $|\mathcal{O}_{\mathbb{P}^3}(3)|$ at any $X \in V(T_1, \dots, T_r)$ is a joint versal deformation of all singular points of X if $\sum_{i=1}^r \mu(T_i) \leq 9$ where $\mu(T_i)$ is the Milnor number of T_i .*

Recall that $\mu(A_k) = k$, $\mu(D_k) = k$ and $\mu(E_6) = 6$. By checking carefully how these singularities appear together in each cubic surface (see [4, p. 255]) we conclude that $\sum_{i=1}^r \mu(T_i) \leq 6$ for all cubic surfaces with ADE singularities. Furthermore, by looking at Table 3, we see that $\sum_{i=1}^r \mu(T_i) \leq 4$ for any plane cubic curve with isolated singularities. Hence, Lemma 4 implies that for cubic plane curves and cubic surfaces, any local deformation of isolated singularities is induced by a global deformation.

Definition 2 ([4]) A polynomial F in $n + 1$ variables is *semi-quasi-homogeneous (SQH)* with respect to the weights (w_1, w_2, \dots, w_n) if all the monomials of F have weight larger or equal than 1 and those monomials of weight 1 define a function with an isolated singularity. In particular, the weights associated to the ADE singularities A_k , D_k and E_6 are

$$\left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{k+1} \right), \quad \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(k-2)}{2(k-1)}, \frac{1}{k-1} \right), \quad \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right),$$

respectively. Furthermore, the weight of \tilde{E}_6 is $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. These weights are uniquely associated to their respective singularity.

Lemma 5 ([4, p. 246]) *If $F(x_0, x_1, x_2)$ is SQH with respect to one of the sets of weights in Definition 2 we can, by a locally analytic change of coordinates, reduce the terms of*

weight 1 to the normal forms for A_k, D_k, E_6 , which are locally analytically isomorphic to the following surface singularities:

$$A_k: x_1^{k+1} + x_2^2 + x_3^2 \ (k \geq 1), \quad D_k: x_1^{k-1} + x_1x_2^2 + x_3^2 \ (k \geq 4),$$

$$E_6: x_1^3 + x_2^4 + x_3^2, \quad \tilde{E}_6: x_1^3 + x_2^3 + x_3^3 + 3\lambda x_1x_2x_3, \quad \lambda^3 \neq -1.$$

and the resulting function will remain SQH.

Reduced plane cubic curves are completely characterized according to the number and type of their ADE singularities (see Table 3).

5 Geometric characterization of pairs

In this section we relate the classifications of pairs in terms of singularity theory and the equations defining them. We have divided our lemmas in four groups: classification of singular cubic surfaces, classification of pairs (S, D) with singular boundary D , classification of pairs (S, D) where S is singular at a point $P \in D$ and classification of pairs (S, D) invariant under a \mathbb{C}^* -action. We will denote homogenous polynomials of degree d in $n + 1$ variables as $f_d(x_0, \dots, x_n), g_d$, etc. Recall that pairs (S, D) and (S', D') are projectively equivalent if and only if they are conjugate to each others by elements of $\text{Aut}(\mathbb{P}^3)$.

Lemma 6 ([4, Lemma 3]) *Let $F = x_0x_1x_3 + f_3(x_0, x_1, x_2)$, $P = (0, 0, 0, 1)$, $Q = (0, 0, 1, 0)$, $H = \{x_3 = 0\} \cong \mathbb{P}^2_{(x_0, x_1, x_2)}$ and $H_i = \{x_i = x_3 = 0\} \subset H$ for $i = 0, 1$.*

1. *The singularities of $\{F = 0\}$ other than that at P correspond to the intersection of $C = \{x_0x_1 = 0\} \subset H$ and $C' = \{f_3 = 0\}$ at points R other than Q . Indeed, if $\text{mult}_R(C \cdot C') = k$, then R is an A_{k-1} singularity.*
2. *If $f_3(0, 0, 1) \neq 0$, then P is an A_2 singularity. Let $k_i = \text{mult}_Q(H_i \cdot C')$. If both k_0 and k_1 are both at least 2, then $\{F = 0\}$ has non-isolated singularities. Otherwise P is an $A_{k_0+k_1+1}$ singularity for $\{k_0, k_1\} = \{1, 1\}, \{1, 2\}, \{1, 3\}$.*

Lemma 7 *A pair (S, D) is such that S has an A_2 singularity at a point $P \in D$ or a degeneration of one if and only if P is conjugate to $(0, 0, 0, 1)$ and simultaneously (S, D) is projectively equivalent to the pair defined by equations*

$$x_3 f_2(x_0, x_1) + f_3(x_0, x_1, x_2) = 0, \quad l_1(x_0, x_1, x_2) = 0.$$

Proof Without loss of generality, we may assume $P = (0, 0, 0, 1)$. By Lemma 6, S has (a degeneration of) an A_2 singularity at P if and only if it is given by the equation $x_0x_1x_3 + f_3(x_0, x_1, x_2) = 0$. Any quadric $f_2(x_0, x_1)$ can be transformed to x_0x_1 or to a degeneration of x_0x_1 (e.g. x_0^2) by a change of coordinates preserving x_2 and x_3 . The lemma follows because a hyperplane section D contains P if and only if D is given by a linear form $l_1(x_0, x_1, x_2)$.

Lemma 8 *A surface S has an A_3 singularity or a degeneration of one if and only if it is projectively equivalent to:*

$$\{x_3 f_2(x_0, x_1) + x_2^2 f_1(x_0, x_1) + x_2 g_2(x_0, x_1) + g_3(x_0, x_1) = 0\}.$$

Proof By Lemma 6, we may assume $S = \{x_0 x_1 x_3 + f_3(x_0, x_1, x_2) = 0\}$ and $P = (0, 0, 0, 1)$. Moreover, the singularity is of type A_k with $k \geq 3$ if and only if $f_3(0, 0, 1) = 0$. Therefore $f_3(x_0, x_1, x_2) = x_2^2 f_1(x_0, x_1) + x_2 g_2(x_0, x_1) + g_3(x_0, x_1)$.

Lemma 9 *A surface S has an A_4 singularity or a degeneration of one if and only if it is projectively equivalent to $\{x_3 x_0 l_1(x_0, x_1) + x_0 x_2^2 + x_2 g_2(x_0, x_1) + g_3(x_0, x_1) = 0\}$.*

Proof By Lemma 6, the surface S is defined by the equation

$$x_0 x_1 x_3 + f_3(x_0, x_1, x_2) = 0,$$

where $f_3(x_0 x_1 x_2) = x_2^2 f_1(x_0, x_1) + x_2 g_2(x_0, x_1) + g_3(x_0, x_1)$, $k_0 = \text{mult}_Q(H_0 \cdot C') \geq 2$ and $k_1 = \text{mult}_Q(H_1 \cdot C') \geq 1$ if and only if P is (a degeneration of) an A_4 singularity, where C' is the curve given in Lemma 6. Notice that

$$k_i = \text{mult}_Q(H_i \cdot C') = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x_0, x_1]}{\langle x_i, f_1 + g_2 + g_3 \rangle} \right).$$

Therefore $k_0 \geq 2$ if and only if $f_1(0, 1) = 0$. Hence, $f_1 = x_0$. The lemma follows from noticing that $x_0 x_1 x_3$ is projectively equivalent to $x_0 x_3 l_1(x_0, x_1)$ by an element of $\text{Aut}(\mathbb{P}^3)$ fixing x_0, x_2, x_3 .

The proof of the next lemma is similar to the proof of Lemma 6, so we omit it.

Lemma 10 *A surface S has an A_5 singularity or a degeneration of one if and only if it is projectively equivalent to*

$$\{x_3 x_0 l_1(x_0, x_1) + x_0 x_2 f_1(x_0, x_1, x_2) + f_3(x_0, x_1) = 0\}.$$

In Figure 2 we see that the only non-trivial degenerations of a D_4 singularity in a cubic surface which are not a \tilde{E}_6 singularity are D_5 and E_6 singularities. Hence the next lemma follows at once from [4, Case C].

Lemma 11 *A surface S has a D_4 singularity or a degeneration of one if and only if it is projectively equivalent to*

$$\{x_3 x_0^2 + f_3(x_0, x_1, x_2) = 0\}.$$

Lemma 12 *A surface S has a D_5 singularity or a degeneration of one if and only if it is projectively equivalent to*

$$\{f_3(x_0, x_1) + x_2 g_2(x_0, x_1) + x_0 x_2^2 + x_0^2 x_3 = 0\}.$$

Proof By Lemma 11 and Figure 2, we may assume that S is given by $x_3x_0^2 + f_3(x_0, x_1, x_2)$ since D_5 is a degeneration of D_4 . Let $H = \{x_3 = 0\}$, $C = \{x_3 = f_3(x_0, x_1, x_2) = 0\} \subset H$ and $C' = \{x_3 = x_0 = 0\} \subset H$. We can rewrite $f_3 = x_2^2g_1(x_0, x_1) + x_2g_2(x_0, x_1) + g_3(x_0, x_1)$. By [4, Lemma 4], the point $P = (0, 0, 0, 1)$ is (a degeneration of) a D_5 singularity if and only if $C \cap C'$ consist of at most two points. The equation of $S \cap H \subset H$ localized at $Q = (0, 0, 1, 0)$ is $g_1(x_0, x_1) + g_2(x_0, x_1) + g_3(x_0, x_1) = 0$, and $C \cap C'$ consists of at most two points if and only if

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x_0, x_1]}{\langle x_0, g_1 + g_2 + g_3 \rangle} \right) \geq 2.$$

The latter is equivalent to taking $g_1 = ax_0$, which by rescaling x_2 gives the result.

Lemma 13 *The unique cubic surface S with a E_6 singularity or a degeneration of one such surface is projectively equivalent to*

$$\{x_3x_0^2 + x_0x_2l_1(x_0, x_1, x_2) + f_3(x_0, x_1) = 0\}.$$

Proof Using the same notation as in Lemma 12 and following [4, Lemma 4], S is defined by $x_3x_0^2 + x_2^2g_1(x_0, x_1) + x_2g_2(x_0, x_1) + g_3(x_0, x_1) = 0$, and has (a degeneration of) an E_6 singularity if and only if

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x_0, x_1]}{\langle x_0, g_1 + g_2 + g_3 \rangle} \right) \geq 3.$$

The latter is equivalent to take $g_1 = x_0$ and $g_2 = x_0l_1(x_0, x_1)$.

Remark 1 (see [4, Case E]) A surface S has an isolated \tilde{E}_6 singularity if and only if S is the cone over a smooth plane cubic curve given by $f_3(x_0, x_1, x_2) = 0$.

Consider a pair (S, D) and a point $P \in D \subset S$. By choosing coordinates appropriately we can suppose that $P = (0, 0, 0, 1)$ and $(S, D) = (\{F = 0\}, \{F = H = 0\})$ for F and H given as

$$F = x_0f_2(x_0, \dots, x_3) + x_3^2f_1(x_1, x_2) + x_3g_2(x_1, x_2) + f_3(x_1, x_2), \quad H = x_0. \tag{1}$$

Lemma 14 *A pair (S, D) has D with an A_2 singularity at a point P or a degeneration of one if and only if (S, D) is projectively equivalent to the pair defined by equations:*

$$x_0f_2(x_0, x_1, x_2, x_3) + x_3x_1^2 + f_3(x_1, x_2) = 0, \quad x_0 = 0. \tag{2}$$

Proof Without loss of generality we can suppose (S, D) is given by (1). The equation of (a degeneration of) a plane cubic curve in $\{x_0 = 0\}$ with an A_2 singularity at P is given by $x_2^2x_3 + f_3(x_1, x_2) = 0$, where the curve has an A_2 singularity at P if and only if x_2^3 has a non-zero coefficient in f_3 . Therefore D is as in the statement if and only if in (1) we take $f_1 = 0$ and $g_2 = x_1^2$.

Lemma 15 *A pair (S, D) has D with an A_3 singularity at P or a degeneration of one if and only if (S, D) is projectively equivalent to the pair defined by $x_0 f_2(x_0, x_1, x_2, x_3) + x_1(x_2^2 + x_1 l_1(x_1, x_2, x_3)) = 0$ and $x_0 = 0$.*

Proof We may assume that the equations of (S, D) are as in (1) and $P = (0, 0, 0, 1)$. By restricting to $\{x_0 = 0\} \cong \mathbb{P}^2$ and localizing at P , the equation for D is $f_1(x_1, x_2) + g_2(x_1, x_2) + f_3(x_1, x_2)$ and by choosing coordinates appropriately we may assume that $L = \{x_1 = 0\}$ and $C = \{x_2^2 + x_1 l_1(x_1, x_2) = 0\}$ are a line and a conic intersecting at P , where l is a polynomial of degree 1, not necessarily homogeneous. Therefore $D|_{x_0=0}$ has equation $x_1(x_2^2 + x_1 l_1(x_1, x_2, x_3))$ so $f_1 \equiv 0, g_2 \equiv ax_1^2, f_3 = x_1 x_2^2 + x_1 l_1(x_1, x_2, 0)$ and the result follows.

By similar arguments, one can prove the next two results:

Lemma 16 *A pair (S, D) has D with a D_4 singularity at P or a degeneration of one if and only if (S, D) is projectively equivalent to the pair defined by equations $x_0 f_2(x_0, x_1, x_2, x_3) + f_3(x_1, x_2) = 0$ and $x_0 = 0$.*

Lemma 17 *A pair (S, D) has D non-reduced if and only if it is projectively equivalent to the pair defined by equations:*

$$x_0 f_2(x_0, x_1, x_2, x_3) + x_1^2 f_1(x_1, x_2, x_3) = 0, \quad x_0 = 0.$$

Lemma 18 *A pair (S, D) has $D = L + C$ where L is a line and C is a conic such that $3L \in |-K_S|$ if and only if it is projectively equivalent to the pair defined by equations:*

$$x_0 f_2(x_0, x_1, x_2, x_3) + ax_1^3 = 0, \quad l_1(x_0, x_1) = 0.$$

where L and $3L$ are projectively equivalent to $\{x_0 = x_1 = 0\}$ and $= \{x_0 = 0\}|_S$, respectively. This surface has a point $Q \in L \subset \text{Supp}(D)$ such that S has a singularity at Q that is not of type A_1 .

Proof Suppose (S, D) as in the statement. Without loss of generality, we may suppose that the equation of S is as in (1), $D = \{x_0 + bx_1 = 0\}$ and let $D' := \{x_0 = 0\}$. Clearly $L \subset \text{Supp}(D') \cap \text{Supp}(D)$ and $D = D'$ if and only if $b = 0$. In this case, the equation of $D = D'$ in $\{x_0 = 0\} \cong \mathbb{P}^2$ is given by $x_3^2 f_1(x_1, x_2) + x_3 g_2(x_1, x_2) + f_3(x_1, x_2) = 0$ and $3L \in |-K_S|$ if and only if $f_1 = g_2 \equiv 0$ and $f_3 = ax_1^3$. If $b \neq 0$, then $x_1 = -\frac{x_0}{b}$. Take $x_0 = 0$ in (1). The equation of $D' = \{x_0 = 0\}|_S$ is $x_2^2 f_1 + x_3 g_2 + f_3 = 0$ and $D' \equiv 3L$ if and only if $f_1 = g_2 = 0$ and $f_3 = x_1^3$. But then, the equation of D in $\{x_0 + bx_1 = 0\}$ is $x_1(bf_2 + x_1^2)$ and $C = \{bf_2 + x_1^2 = x_0 + bx_1 = 0\}$. It is a well known fact that the line L contains a point Q at which S is singular and Q is not of type A_1 (see [19, p. 227]).

Lemma 19 *Given a pair (S, D) , S is singular at a point $P \in D$ and D is an A_2 singularity at P or a degeneration of one if and only if (S, D) is projectively equivalent to the pair defined by equations:*

$$x_3 x_0 l_1(x_0, x_1, x_2) + x_3 x_1^2 + f_3(x_1, x_2) + x_0 f_2(x_0, x_1, x_2) = 0, \quad x_0 = 0. \quad (3)$$

Table 4 Some pairs (S, D) invariant under a \mathbb{C}^* -action

$\text{Sing}(S)$	$\text{Sing}(D)$	F	H	λ
$P_i = A_2, i = 1, 2, 3$	A_1 at each P_i	$x_0x_1x_3 + x_2^3$	x_2	$\bar{\lambda}_2$
$P = A_3, Q_1 = A_1, Q_2 = A_1$	$D = 2L + L',$ $Q_1, Q_2 \in L,$ $\text{Sing}(S) \cap L' = \emptyset$	$x_0x_1x_3 + x_1x_2^2 + x_0x_2^2$	x_3	$\bar{\lambda}_3$
$P = A_4, Q = A_1$	A_3 at Q	$x_0x_1x_3 + x_0x_2^2 + x_1^2x_2$	x_3	λ_5
$P = A_5, Q = A_1$	A_2 at Q	$x_0x_2^2 + x_0x_1x_3 + x_1^3$	x_3	λ_6
$P = D_4$	D_4 not at P	$x_0^2x_3 + x_1^3 + x_2^3$	x_3	λ_9
$P = D_5$	A_3 not at P	$x_0^2x_3 + x_0x_2^2 + x_1^2x_2$	x_3	$\bar{\lambda}_6$
$P = E_6$	A_2 not at P	$x_0^2x_3 + x_0x_2^2 + x_1^3$	x_3	$\bar{\lambda}_4$

Proof Without loss of generality we can assume $P = (0, 0, 0, 1)$. Then, the equation of S can be written as (see [4, Section 2, pp. 247–252])

$$x_3h_2(x_0, x_1, x_2) + h_3(x_0, x_1, x_2) = a_0x_3x_1^2 + x_0f_2(x_0, x_1, x_2) + f_3(x_1, x_2) + x_1x_3g_1(x_0, x_2) + x_3g_2(x_0, x_2).$$

By comparing with the equation in Lemma 14, D has (a degeneration of) an A_2 singularity at P if and only if $g_1(x_0, x_2) = ax_0$ and $g_2(x_0, x_2) = bx_0^2 + cx_0x_2$. The lemma follows.

The proof of the next lemma is similar to that of Lemma 19.

Lemma 20 *Given a pair (S, D) , S is singular at a point $P \in D$ and D has an A_3 singularity at P or a degeneration of one if and only if (S, D) is projectively equivalent to the pair defined by equations:*

$$x_0^2l_1(x_0, x_1, x_2, x_3) + x_0f_2(x_1, x_2) + x_0x_3g_1(x_1, x_2) + x_1^2h_1(x_1, x_2, x_3) + x_1x_2^2 = 0,$$

$$x_0 = 0.$$

Lemma 21 *Let (S, D) be a pair that is invariant under a non-trivial \mathbb{C}^* -action. Suppose the singularities of S and D are given as in the first and second entries in one of the rows of Table 4, respectively. Then (S, D) is projectively equivalent to $(\{F = 0\}, \{F = H = 0\})$ for F and H as in the third and fourth entries in the same row of Table 4, respectively. In particular, any such pair (S, D) is unique. Conversely, if (S, D) is given by equations as in the third and fourth entries in a given row of Table 4, then (S, D) has singularities as in the first and second entries in the same row of Table 4 and (S, D) is \mathbb{C}^* -invariant. Furthermore the element $\lambda \in \text{SL}(4, \mathbb{C}^*)$, as defined in Lemma 2, given in the fifth entry of the corresponding row of Table 4 is a generator of the \mathbb{C}^* -action.*

Proof There is a unique surface S with three A_2 singularities [4, p. 255] which corresponds to the equation in Table 4. When a surface S has singularities $A_4 + A_1$, $A_5 + A_1$, D_4 , D_5 or E_6 , and a \mathbb{C}^* -action, the equation for F follows from [8, Table 3]. If S has singularities $A_3 + 2A_1$, then [8, Table 3] gives that S has equation $x_3 f_2(x_0, x_1) + x_2^2 l_1(x_0, x_1) = 0$, where $x_0 x_1$ has a non-zero coefficient in f_2 , since otherwise S is singular along a line. Hence, after a change of coordinates involving only variables x_0 and x_1 and rescaling x_3 , we obtain the desired result. It is trivial to check that each one-parameter subgroup λ in the corresponding row of Table 4 leaves S invariant, and therefore λ is a generator of the \mathbb{C}^* -action.

Given H , denote $D_H = \{F = H = 0\} \subset S$. We need to show that for (S, D) with prescribed singularities, $D_H = D$ if and only if H is as stated in Table 4. Verifying that for F and H as in the table, the pair (S, D) has the expected singularities is straight forward and we omit it. We verify the converse.

Suppose that S has three A_2 singularities. Then we may assume that $F = x_0 x_1 x_3 + x_2^3$ and the singularities correspond to $P_1 = (1, 0, 0, 0)$, $P_2 = (0, 1, 0, 0)$ and $P_3 = (1, 0, 0, 0)$. There are only three lines L_1, L_2, L_3 in S [4, p. 255], which correspond to $\{x_2 = x_i = 0\}$ for $i = 0, 1, 3$, respectively. Clearly any two of these intersect at each of the points P_j . Moreover $D_H = D = \sum L_i$ and D has an A_1 singularity at each P_i , as stated in Table 4.

Suppose that S has an E_6 singularity at a point P and D has an A_2 singularity at a point $Q \neq P$ and (S, D) is \mathbb{C}^* -invariant. Without loss of generality, we can now assume that $F = x_0^2 x_3 + x_0 x_2^2 + x_1^3$, $H = \sum a_i x_i$ for some parameters a_i and $P = (0, 0, 0, 1)$. Since $\bar{\lambda}_4$ is a generator of the \mathbb{C}^* -action, then $\bar{\lambda}_4(t) \cdot H = a_0 t^{11} x_0 + a_1 t^3 x_1 + a_2 t^{-1} x_2 + a_3 t^{-13} x_3$. Therefore D_H is \mathbb{C}^* -invariant if and only if $H = x_i$ for some $i = 0, \dots, 3$. Notice that this happens every time the entries of λ are distinct. If $H = x_0$, then D_H is a triple line. If $H = x_1$, then D_H is the union of a conic and a line, and therefore D_H does not have an A_2 singularity. If $H = x_2$, then D_H has an A_2 singularity at P . If $H = x_3$, then D_H has an A_2 singularity at $Q = (1, 0, 0, 0) \neq P$ and $D_H = D$.

Suppose S has a D_5 singularity at a point P , D has an A_3 singularity at a point $Q \neq P$ and (S, D) is \mathbb{C}^* -invariant. There is a unique pair satisfying these conditions. Reasoning as in the previous case, we may assume $F = x_0^2 x_3 + x_0 x_2^2 + x_1^2 x_2$, $H = x_i$ for some $i = 0, \dots, 3$ and $P = (0, 0, 0, 1)$. It follows from the equations that $\bar{\lambda}_6$ generates the \mathbb{C}^* -action. If $H = x_0$ or $H = x_2$, then the support of D_H contains a double line. If $H = x_2$, then D_H has an A_3 singularity at P . If $H = x_3$, then D_H has an A_3 singularity at $Q = (1, 0, 0, 0) \neq P$ and $D_H = D$.

Suppose S has an A_5 singularity at a point P and an A_1 singularity at a point Q , D has an A_2 singularity at Q and (S, D) is \mathbb{C}^* -invariant. We may assume λ_6 generates the \mathbb{C}^* -action, $F = x_0 x_2^2 + x_0 x_1 x_3 + x_1^3$, $H = x_i$ for some $i = 0, \dots, 3$, $P = (0, 0, 0, 1)$ and $Q = (1, 0, 0, 0)$. If $H = x_0$ then D_H is a triple line. If $H = x_1$, then D_H has a double line in its support. If $H = x_2$, then D_H has two A_1 singularities. If $H = x_3$, then D_H has an A_2 singularity at $Q = (1, 0, 0, 0) \neq P$ and $D_H = D$.

Suppose S has an A_4 singularity at a point P and an A_1 singularity at a point Q , D has an A_3 singularity at Q and (S, D) is \mathbb{C}^* -invariant. We may assume λ_5 generates the \mathbb{C}^* -action, $F = x_0 x_1 x_3 + x_0 x_2^2 + x_1^2 x_2$, $H = x_i$ for some $i = 0, \dots, 3$, $P = (0, 0, 0, 1)$

and $Q = (1, 0, 0, 0)$. If $H = x_0$ or $H = x_1$ then D_H contains a double line in its support. If $H = x_2$, then D_H has three A_2 singularities and if $H = x_3$, then D_H has an A_2 singularity at Q and $D_H = D$.

Suppose S has a D_4 singularity at a point P , D has a D_4 singularity at a point $Q \neq P$ and (S, D) is \mathbb{C}^* -invariant. We may assume the generator of the \mathbb{C}^* -action is λ_9 , $F = x_0^2x_3 + x_1^3 + x_2^3$ and $P = (0, 0, 0, 1)$. If D_H is λ_9 -invariant, either $H = x_i$ for some $i = 0, \dots, 3$ or $H = x_1 - ax_2$ for $a \neq 0$. If $H = x_0$, then D_H has a D_4 singularity at P . If $H = x_1$ or $H = x_2$, then D_H has an A_2 singularity. If $H = x_1 - ax_2$ with $a \neq 0$, then $D_H = \{x_0^2x_3 + (1 + \frac{1}{a})x_1^3 = 0, x_2 = \frac{x_1}{a}\}$ has an A_2 singularity. If $H = x_3$, then D_H has a D_4 singularity at $Q = (1, 0, 0, 0) \neq P$ and $D_H = D$.

Suppose S has an A_3 singularity at a point P , two A_1 singularities at points Q_1 and Q_2 , $D = 2L + L'$ where L is a line containing Q_1 and Q_2 and L' is a line such that $P, Q_1, Q_2 \notin L'$. Furthermore, suppose (S, D) is \mathbb{C}^* -invariant. We may assume that $\bar{\lambda}_3$ is the generator of the \mathbb{C}^* -action, $F = x_0x_1x_3 + x_1x_2^2 + x_0x_2^2$, $P = (0, 0, 0, 1)$, $Q_1 = (1, 0, 0, 0)$, $Q_2 = (0, 1, 0, 0)$ and $L = \{x_2 = x_3 = 0\}$. Moreover, if D_H is $\bar{\lambda}_3$ -invariant, either $H = x_i$ for some $i = 0, \dots, 3$ or $H = x_0 - ax_1$ for $a \neq 0$. If $H = x_0$ or $H = x_1$, then D_H does not contain L in its support. If $H = x_2$ or $H = x_0 - ax_1$, then D_H is reduced. If $H = x_3$, then $D_H = 2L + L'$, where $L' = \{x_1 + x_0 = x_3 = 0\}$. Since $P, Q_1, Q_2 \notin L$, then $D_H = D$.

6 Proof of main theorems

We present the proofs of theorems 2 and 3. First, we reduce the amount of pairs we need to consider to those with isolated singularities:

Lemma 22 *Let (S, D) be a pair.*

1. *If S is reducible or not normal, then (S, D) is t -unstable for $t \in [0, 1)$.*
2. *If D is not reduced, then, (S, D) is t -unstable for $t \in (1/5, 1]$.*

Proof The case where S is reducible follows from [10, Theorem 1.3]. By Serre’s criterion, any hypersurface of dimension 2 is non-normal if and only if it has non-isolated singularities. The latter are classified for cubic surfaces in [4, Case E], hence S is an irreducible non-normal cubic surface if and only if it is projectively equivalent to $\{x_3f_2(x_0, x_1) + f_3(x_0, x_1) + x_2g_2(x_0, x_1) = 0\}$. Then $\mu_t(S, D, \lambda_{10}) \geq 1 - t > 0$. If D is not reduced, we may assume (S, D) is as in Lemma 17. Then $\mu_t(S, D, \lambda_3) = -1 + 5t > 0$, if $t > \frac{1}{5}$.

Proof (Theorem 2) Let (S, D) be a pair defined by equations F and H . Notice that Lemma 22 tells us that S being normal is a necessary condition for (S, D) to be t -stable for any $t \in (0, 1)$. In particular S has a finite number of singularities, since it is a surface. By Theorem 4, the pair (S, D) is t -stable if and only if for any $g \in \text{SL}(4, \mathbb{C})$ the monomials with non-zero coefficients of $(g \cdot F, g \cdot H)$ are not contained in a pair of sets $N_t^\oplus(\lambda, x_i)$ —characterized geometrically in Sect. 4—which is maximal for every given t , as stated in Theorem 4. These maximal sets can be found algorithmically [9,10]. This is equivalent to the conditions in the statement. We verify the conditions for each $t \in (0, 1)$. We will refer to the singularities of D in terms of the ADE

classification as in Sects. 4 and 5. These will be equivalent to the global description used in the statement of Theorem 2 by Table 3.

Suppose $t \in (0, \frac{1}{5})$ and $(\lambda, x_i) = (\bar{\lambda}_3, x_3)$. Then S cannot have an A_3 singularity or a degeneration of one. When $(\lambda, x_i) = (\lambda_9, x_3)$, we deduce that S cannot have a D_4 singularity or a degeneration of one (this condition is redundant since D_4 is a degeneration of A_3). From $(\lambda, x_i) = (\lambda_1, x_2)$ or $(\lambda, x_i) = (\bar{\lambda}_2, x_2)$ we deduce that if $P \in D$ then P is a singular point of S of type at worst A_1 . We obtain the same condition if $(\lambda, x_i) = (\lambda_2, x_1)$. This completes the proof when $t \in (0, \frac{1}{5})$.

When $t = \frac{1}{5}$, the maximal sets $N_t^\oplus(\lambda, x_i)$ are the same as for $t \in (0, \frac{1}{5})$ with the addition of $N_t^\oplus(\lambda_3, x_0)$, which represents the monomials of the equations of any pair (S', D') such that D' is not reduced. Therefore (S, D) is $\frac{1}{5}$ -stable if and only if in addition to the conditions for t -stability when $t \in (0, \frac{1}{5})$, D is not reduced. Hence (ii) follows.

Let $t \in (\frac{1}{5}, \frac{1}{3})$. The maximal t -non-stable sets $N_t^\oplus(\lambda, x_i)$ are the same as for $t = \frac{1}{5}$ but replacing the set $N_t^\oplus(\bar{\lambda}_3, x_3)$ with both $N_t^\oplus(\lambda_7, x_3)$ and $N_t^\oplus(\lambda_5, x_3)$. A pair (S', D') whose defining equations have coefficients in one of $N_t^\oplus(\bar{\lambda}_3, x_3)$, $N_t^\oplus(\lambda_7, x_3)$ and $N_t^\oplus(\lambda_5, x_3)$ require that S' has (a degeneration of) an A_3 singularity, S' is not normal or S' has (a degeneration of) an A_4 singularity, respectively. The second condition is redundant by Lemma 22. Hence a t -stable pair (S, D) may now have A_3 singularities but not A_4 singularities. However, the coefficients of the equations of (S, D) cannot be in $N_t^\oplus(\lambda_9, x_3)$ and hence S cannot have (degenerations of) D_4 singularities. Therefore (S, D) is t -stable if and only if S has at worst A_3 singularities, D is reduced and if D supports a surface singularity P , then P must be an A_1 -singularity and (iii) follows.

Let $t = \frac{1}{3}$. The maximal sets $N_t^\oplus(\lambda, x_i)$ are the same as for $t \in (\frac{1}{5}, \frac{1}{3})$ with the addition of $N_t^\oplus(\lambda_5, x_0)$, which represents the monomials of the equations of any pair (S', D') such that D' has (a degeneration of) an A_3 singularity at a singular point P of S . Hence (S, D) is $\frac{1}{3}$ -stable if and only if it is t -stable for $t \in (\frac{1}{5}, \frac{1}{3})$ but D does not have (a degeneration of) an A_3 singularity at a singular point of P . Hence (iv) follows.

Let $t \in (\frac{1}{3}, \frac{3}{7})$. The maximal sets are $N_t^\oplus(\lambda, x_i)$ the same as for $t = \frac{1}{3}$ but replacing the set $N_t^\oplus(\lambda_5, x_3)$ — parametrizing pairs (S', D') where S' has (a degeneration of) an A_4 singularity—with the set $N_t^\oplus(\lambda_6, x_3)$ —parametrizing pairs (S', D') where S' has (a degeneration of) an A_5 singularity. Hence a t -stable pair (S, D) may now have A_4 singularities but not A_5 singularities. However, the coefficients of the equations of (S, D) cannot be in $N_t^\oplus(\lambda_9, x_3)$ and hence S cannot have (degenerations of) D_4 singularities. Furthermore the restrictions for $t = \frac{1}{3}$ regarding D still apply. Therefore a pair (S, D) is t -stable if and only if satisfies the conditions in (v).

Let $t = \frac{3}{7}$. The maximal sets $N_t^\oplus(\lambda, x_i)$ are the same as for $t \in (\frac{1}{3}, \frac{3}{7})$ but replacing the set $N_t^\oplus(\lambda_5, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_3 singularity at a surface singularity of S' —, for both the set $N_t^\oplus(\bar{\lambda}_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_2 singularity at a surface singularity of S' —and the set $N_t^\oplus(\bar{\lambda}_9, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_4 singularity. Hence (vi) follows.

Let $t \in (\frac{3}{7}, \frac{5}{9}]$. The difference between the maximal sets for $N_t^\oplus(\lambda, x_i)$ and for $N_{\frac{3}{7}}^\oplus(\lambda, x_i)$ consists of three new sets $(N_t^\oplus(\bar{\lambda}_6, x_3), N_t^\oplus(\lambda_8, x_3)$ and $N_t^\oplus(\lambda_{10}, x_3))$ and

three sets that do not appear for t anymore ($N_t^\oplus(\lambda_9, x_3), N_t^\oplus(\lambda_6, x_3), N_t^\oplus(\lambda_7, x_3)$). The three new sets parametrize pairs (S', D') such that S' has at least either (a degeneration of) one D_5 singularity, a degeneration of one \tilde{E}_6 singularity or one line of singularities, respectively. The three sets that are not maximal non-stable sets for t parametrize pairs (S', D') such that S' has (a degeneration of) a D_4 , an A_5 and a line of singularities, respectively. Hence, the only difference with respect to $t = \frac{3}{7}$ is that we include pairs (S, D) such that S has at worst A_5 or D_4 singularities and (vii) follows.

Let $t = \frac{5}{9}$. The difference between the maximal sets for $N_t^\oplus(\lambda, x_i)$ for $t \in (\frac{3}{7}, \frac{5}{9})$ and for $N_{\frac{5}{9}}^\oplus(\lambda, x_i)$ consists of replacing the set $N_t^\oplus(\lambda_3, x_0)$ —parametrizing pairs (S', D') such that D' is non-reduced—for the set $N_t^\oplus(\lambda_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_3 singularity. Hence a $\frac{5}{9}$ -stable pair (S, D) is a t -stable pair for $t \in (\frac{3}{7}, \frac{5}{9})$ such that D has at worst an A_2 singularity. Notice that D is still reduced by Lemma 22. Hence (viii) follows.

Let $t \in (\frac{5}{9}, \frac{9}{13})$. The difference between the maximal sets for $N_t^\oplus(\lambda, x_i)$ for $t \in (\frac{5}{9}, \frac{9}{13})$ and for $N_{\frac{5}{9}}^\oplus(\lambda, x_i)$ consists of replacing the set $N_t^\oplus(\bar{\lambda}_6, x_3)$ —parametrizing pairs (S', D') such that S' has (a degeneration of) a D_5 singularity—for the set $N_t^\oplus(\bar{\lambda}_4, x_3)$ —parametrizing pairs (S', D') such that S' has (a degeneration of) an E_6 singularity. Hence (ix) follows.

Let $t = \frac{9}{13}$. The difference between the maximal sets for $N_t^\oplus(\lambda, x_i)$ for $t \in (\frac{5}{9}, \frac{9}{13})$ and for $N_{\frac{9}{13}}^\oplus(\lambda, x_i)$ consists of replacing the set $N_t^\oplus(\bar{\lambda}_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_2 singularity at a singular point of S' —, the set $N_t^\oplus(\bar{\lambda}_9, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) a D_4 singularity—and the set $N_t^\oplus(\lambda_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_3 singularity—for the set $N_t^\oplus(\lambda_4, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an A_2 singularity. Hence (x) follows.

Let $t \in (\frac{9}{13}, 1)$. The maximal sets $N_t^\oplus(\lambda, x_i)$ are the same as for $N_{\frac{9}{13}}^\oplus(\lambda, x_i)$ but removing the set $N_t^\oplus(\bar{\lambda}_4, x_3)$, which parametrizes pairs (S', D') where S' has an E_6 singularities. Hence such surfaces are now t -stable providing they do not violate any other conditions. This concludes the proof of the theorem.

Theorem 3 Suppose (S, D) —defined by polynomials F and H —belongs to a closed strictly t -semistable orbit. By Lemma 21, they are generated by monomials in $N_t^0(\lambda, x_i)$ for some (λ, x_i) such that $N_t^\oplus(x_i, \lambda)$ is maximal with respect to the containment of order of sets. Since there is a finite number of λ to consider (those in Lemma 2), this is a finite computation which can be carried out by software [9,10]. For each pair (λ, x_i) , there is a change of coordinates that gives a natural bijection between $N^0(\lambda, x_i)$ and $N^0(\bar{\lambda}, x_{3-i})$. Therefore about half of the values are redundant and we have two possible choices for each F and H if $t \neq t_1, \dots, t_5$ three choices if $t = t_1, t_2, t_4, t_5$ and four if $t = t_3$.

Similarly, by [10, Lemma 3.2] and Lemma 2 we can check that the pair (\bar{S}, \bar{D}) corresponding to $\bar{F} = x_0x_3x_1 + x_2^3, \bar{H} = x_2$ is strictly t -semistable. Suppose that

$(\lambda, x_i) = (\lambda_1, x_2)$. Then $F = x_0x_3f_1(x_1, x_2) + f_3(x_1, x_2)$ and $H = g_1(x_1, x_2)$. After a change of variables involving only x_1 and x_2 , we may assume that $F = x_0x_3x_1 + f_3(x_1, x_2)$. We will show that the closure of (S, D) contains (\bar{S}, \bar{D}) . Let $\gamma = \text{Diag}(1, 1, 0, -2)$ be a one-parameter subgroup. Then

$$\lim_{t \rightarrow 0} \gamma(t) \cdot F = x_0x_1x_3 + bx_2^3 \text{ and } \lim_{t \rightarrow 0} \gamma(t) \cdot H = x_2.$$

If $b = 0$, then $\lim_{t \rightarrow 0} \gamma(t) \cdot S$ is reducible, which is impossible as it is not t -stable for any value of $t \in (0, 1)$ by Lemma 22. Therefore $b \neq 0$ and by rescaling we see that $\lim_{t \rightarrow 0} \gamma(t) \cdot (S, D) = (\bar{S}, \bar{D})$. Hence, the closure of the orbit of (S, D) contains (\bar{S}, \bar{D}) , which we tackle next.

Suppose that $(\lambda, x_i) = (\lambda_2, x_1)$. Then $F = x_1^3 + x_0f_2(x_2, x_3)$ and $H = x_1$. After a change of variables involving only x_2 and x_3 we may assume that $F = x_1^3 + x_0x_2x_3$. We can do similar changes of variables in the rest of the cases and end up with F and H not depending on any parameters. Observe that since (S, D) is strictly t -semistable, the stabilizer subgroup of (S, D) , namely $G_{(S,D)} \subset \text{SL}(4, \mathbb{C})$ is infinite (see [6, Remark 8.1 (5)]). In particular there is a \mathbb{C}^* -action on (S, D) . Lemma 21 classifies the singularities of (S, D) uniquely according to their equations. For each $t \in (0, 1)$, the proof of Theorem 3 follows once we recall the classification of plane cubic curves according to their isolated singularities (see Table 3).

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