
Hausdorff dimension and σ finiteness of p -harmonic measures in space when $p \geq n$

Murat Akman · John Lewis · Andrew Vogel

June 24, 2013

Abstract In this paper we study a measure, $\hat{\mu}$, associated with a positive p harmonic function \hat{u} defined in an open set $O \subset \mathbb{R}^n$ and vanishing on a portion Γ of ∂O . If $p > n$ we show $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure while if $p = n$ the same conclusion holds provided Γ is uniformly fat in the sense of n capacity. Our work nearly answers in the affirmative a conjecture in [14] and also appears to be the natural extension of [10, 23] to higher dimensions.

Keywords p harmonic function · p laplacian · p harmonic measure · Hausdorff measure

Mathematics Subject Classification (2010) 35J25 · 37F35

1 Introduction

Denote points in Euclidean n -space \mathbb{R}^n by $x = (x_1, \dots, x_n)$ and let $\bar{E}, \partial E, \text{diam } E$, be the closure, boundary, and diameter of the set $E \subset \mathbb{R}^n$. Let $d(E, F)$ be the distance between the sets E, F and $d(y, E) = d(\{y\}, E)$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^n and let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x . Set $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ whenever $x \in \mathbb{R}^n, r > 0$, and let dx denote Lebesgue n -measure on \mathbb{R}^n . If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O . Let $\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$ be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q norm in O . Next let $C_0^\infty(O)$ be the set of infinitely

Akman and Lewis were partially supported by NSF DMS-0900291

Murat Akman

Mathematics Department, University of Kentucky, Lexington, Kentucky, 40506

E-mail: murat.akman@uky.edu

John Lewis

Mathematics Department, University of Kentucky, Lexington, Kentucky, 40506

E-mail: johnl@uky.edu

Andrew Vogel

Department of Mathematics, Syracuse University, Syracuse, New York 13244

E-mail: alvogel@syracuse.edu

differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. If $K \subset \bar{B}(x, r)$ is a compact set let

$$C(K, B(x, 2r)) = \inf \int_{\mathbb{R}^n} |\nabla \phi|^n dx$$

where the infimum is taken over all $\phi \in W_0^{1,n}(B(x, 2r))$ with $\phi \equiv 1$ on K . We say that a compact set $E \subset \mathbb{R}^n$ is locally (n, r_0) uniformly fat or locally uniformly (n, r_0) thick provided there exists $r_0, \beta > 0$, such that whenever $x \in E, 0 < r \leq r_0$,

$$C(E \cap \bar{B}(x, r), B(x, 2r)) \geq \beta.$$

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$. Fix $p, 1 < p < \infty$, and suppose that \hat{u} is a positive weak solution to the p Laplace equation in $O \cap B(\hat{z}, \rho)$. That is, $\hat{u} \in W^{1,p}(O \cap B(\hat{z}, \rho))$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle dx = 0 \quad (1)$$

whenever $\theta \in W_0^{1,p}(O \cap B(\hat{z}, \rho))$. Equivalently we say that \hat{u} is p harmonic in $O \cap B(\hat{z}, \rho)$. Observe that if \hat{u} is smooth and $\nabla \hat{u} \neq 0$ in $O \cap B(\hat{z}, \rho)$, then $\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$, in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that \hat{u} has zero boundary values on $\partial O \cap B(\hat{z}, \rho)$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(B(\hat{z}, \rho))$, then $\hat{u} \zeta \in W_0^{1,p}(O \cap B(\hat{z}, \rho))$. Extend \hat{u} to $B(\hat{z}, \rho)$ by putting $\hat{u} \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then $\hat{u} \in W^{1,p}(B(\hat{z}, \rho))$ and it follows from (1), as in [9, Chapter 21], that there exists a positive Borel measure $\hat{\mu}$ on \mathbb{R}^n with support contained in $\partial O \cap \bar{B}(\hat{z}, \rho)$ and the property that

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle dx = - \int \phi d\hat{\mu} \quad (2)$$

whenever $\phi \in C_0^\infty(B(\hat{z}, \rho))$. We note that if ∂O is smooth enough, then $d\hat{\mu} = |\nabla \hat{u}|^{p-1} dH^{n-1}$ where H^{n-1} denotes Hausdorff $n - 1$ dimensional measure defined after Theorem 1.

In this paper we continue our study of $\hat{\mu}$ for $n \leq p < \infty$. We prove

Theorem 1 *Fix $p, n \leq p < \infty$ and let $\hat{z}, \rho, \hat{u}, \hat{\mu}$ be as in (2). If $p > n$, then $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure. If $p = n$ and $\partial O \cap B(\hat{z}, \rho)$ is locally (n, r_0) uniformly fat, then $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure.*

To define Hausdorff measure and outline previous work we shall need some more notation. If $\lambda > 0$ is a positive function on $(0, \hat{r}_0)$ with $\lim_{r \rightarrow 0} \lambda(r) = 0$ define H^λ Hausdorff measure on \mathbb{R}^n as follows: For fixed $0 < \delta < \hat{r}_0$ and $E \subseteq \mathbb{R}^2$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, i = 1, 2, \dots$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Then

$$H^\lambda(E) = \lim_{\delta \rightarrow 0} \phi_\delta^\lambda(E).$$

In case $\lambda(r) = r^\alpha$ we write H^α for H^λ .

Define the Hausdorff dimension of a Borel measure ν on \mathbb{R}^n by

$$\text{H-dim } \nu = \inf \{ \alpha : \exists E \text{ Borel with } H^\alpha(E) = 0 \text{ and } \nu(\mathbb{R}^n \setminus E) = 0 \}.$$

From Theorem 1 and the definition of H-dim ν it is easily seen that

Corollary 1 *Let $\hat{u}, \hat{\mu}$, be as in Theorem 1. Then $H\text{-dim } \hat{\mu} \leq n - 1$.*

For $n = 2, 1 < p < \infty$, Lewis proved in [14] the following theorem which generalized earlier results in [4, 13, 15].

Theorem 2 *Given $p, 1 < p < \infty, p \neq 2$, let $\hat{u}, \hat{\mu}$ be as in (1), (2), with $\rho = \infty$ and suppose O is a simply connected bounded domain. Put*

$$\lambda(r) = \lambda(r, A) = r \exp[A\sqrt{\log 1/r \log \log \log 1/r}], 0 < r < 10^{-6}.$$

Then the following is true.

- (a) *If $p > 2$, then $\hat{\mu}$ is concentrated on a set of σ finite H^1 measure.*
- (b) *If $1 < p < 2$, then $\hat{\mu}$ is absolutely continuous with respect to H^λ provided $A = A(p) \geq 1$ is large enough.*

Remark 1 Makarov in [18] (see also [8, 19, 21]), essentially proved Theorem 2 for harmonic measure, ω , with respect to a point in O (the $p = 2$ case). Moreover, [10] showed for any planar domain whose complement is a compact set and for which ω exists, that $H\text{-dim } \omega \leq 1$. Wolff [23] improved this result by showing that for any planar domain ω is concentrated on a set of σ finite H^1 measure.

In higher dimensions, $n \geq 3$, Bourgain [5] showed that $H\text{-dim } \omega < n$ for any open set O for which ω exists. Building on an idea of Carleson in [6], Wolff in [24] constructed in \mathbb{R}^3 , a *Wolff snowflake* for which $H\text{-dim } \omega > 2$ and also one for which $H\text{-dim } \omega < 2$. This was further generalized in [17] where it was shown that both sides of a Wolff snowflake in \mathbb{R}^n could have harmonic measures, say ω_1, ω_2 , with either $\min(H\text{-dim } \omega_1, H\text{-dim } \omega_2) > n - 1$ or $\max(H\text{-dim } \omega_1, H\text{-dim } \omega_2) < n - 1$.

Theorem 4 of [12] implies for fixed $p, 1 < p < \infty$, and $\hat{u}, \hat{\mu}$ as in (2) that $H\text{-dim } \hat{\mu} < n - \tau$ where $\tau = \tau(p, n) > 0$. Theorem 1 was proved in [16] when $\rho = \infty$ and O is a sufficiently flat Reifenberg domain. Also Wolff's method was extended to the p harmonic setting and produced examples of Wolff type snowflakes and p harmonic functions u_∞ vanishing on the boundary of these snowflakes for which the corresponding measures, say μ_∞ , had the following Hausdorff dimensions.

Theorem 3 *If $p \geq n$, then all examples produced by Wolff's method had*

$$H\text{-dim } \mu_\infty|_{B(0,1/2)} < n - 1.$$

Moreover for $p > 2$, near enough 2, there existed a Wolff snowflake for which

$$H\text{-dim } \mu_\infty|_{B(0,1/2)} > n - 1.$$

In view of Theorem 3 and the above results it is natural to conjecture that Theorem 1 remains valid for $p = n$ without the uniform fatness assumption on $\partial O \cap B(\hat{z}, \rho)$. A slightly wilder conjecture is that there exists $p_0, 2 < p_0 < n$, such that if $p_0 \leq p$ and $\hat{u}, \hat{\mu}$, are the p harmonic function-corresponding measure as in (2), then $H\text{-dim } \hat{\mu} \leq n - 1$.

As for our proof of Theorem 1, here we first remark that it is embarrassingly simple compared to the proof in Theorem 1(a) of [14]. Moreover the main idea for the proof comes from [23] where a simple proof for harmonic measure in planar domains, whose boundaries are uniformly fat in the sense of logarithmic capacity, is outlined. Our proof also makes important use of work in [14] and [16]. More specifically suppose for fixed $p, 1 < p < \infty$, that $\hat{u}, \hat{\mu}, O, \hat{z}, \rho$ are as in (2). Then from Lemma 4 we see that $\hat{u}_{x_k}, 1 \leq k \leq n$, are Hölder continuous in $O \cap B(\hat{z}, \rho)$. If also $\hat{x} \in O \cap B(\hat{z}, \rho)$

and $\nabla\hat{u}(\hat{x}) \neq 0$, then \hat{u} is infinitely differentiable in $B(\hat{x}, \delta)$ for some $\delta > 0$. Let $\xi \in \partial B(0, 1)$ differentiating the p Laplace equation, $\nabla \cdot (|\nabla\hat{u}|^{p-2}\nabla\hat{u}) = 0$ with respect to ξ it follows that both $\zeta = \hat{u}_\xi$ and $\zeta = \hat{u}$, satisfy the divergence form PDE for x in $B(\hat{x}, \delta)$:

$$L\zeta(x) = \sum_{i,k=1}^n \frac{\partial}{\partial x_i} [b_{ik}(x)\zeta_{x_k}(x)] = 0, \quad (3)$$

where at x

$$b_{ik}(x) = |\nabla\hat{u}|^{p-4}[(p-2)\hat{u}_{x_i}\hat{u}_{x_k} + \delta_{ik}|\nabla\hat{u}|^2](x), \quad 1 \leq i, k \leq n, \quad (4)$$

and δ_{ik} is the Kronecker δ . From smoothness of \hat{u} we see that b_{ik} are infinitely differentiable in $B(\hat{x}, \delta)$ and at $x \in B(\hat{x}, \delta)$,

$$\min\{p-1, 1\}|\xi|^2 |\nabla\hat{u}(x)|^{p-2} \leq \sum_{i,k=1}^n b_{ik} \xi_i \xi_k \leq \max\{1, p-1\} |\nabla\hat{u}(x)|^{p-2} |\xi|^2. \quad (5)$$

The PDE in (3) for $\hat{u}, \hat{u}_{x_k}, 1 \leq k \leq n$, was used in Lemma 5.1 of [15] to show that if $v = \log |\nabla\hat{u}|$ and $\nabla\hat{u}(\hat{x}) \neq 0$, then for $x \in B(\hat{x}, \delta)$,

$$Lv(x) \geq 0 \text{ when } p \geq n. \quad (6)$$

(3)-(6) are used throughout [4,13,15,16]. Another key inequality in these papers was called the fundamental inequality:

$$\frac{1}{c} |\nabla\hat{u}(x)| \leq \frac{\hat{u}(x)}{d(x, \partial\Omega)} \leq c |\nabla\hat{u}(x)|, \quad (7)$$

where $c = c(n, p)$. (7) was shown to hold for all x near ∂O in the special domains considered in Theorems 2, 3. Observe that if (7) holds, then from (5) it follows that L is locally a uniformly elliptic operator. Hence in these papers results from elliptic PDE were used.

The upper inequality in (7) follows from PDE type estimates and is true for O as in Theorem 1. However the lower estimate is easily seen to fail when ∂O is not connected. Thus we are not able to use either of the strategies in [14] or [15] in our proof of Theorem 1. The argument in section 3 essentially uses only (3) - (6) and the basic estimates for p harmonic functions in section 2.

As for the plan of this paper, in section 2 we list some basic estimates for p harmonic functions. In section 3 we use these estimates and (3)-(6) to prove Theorem 1. Finally in section 4 we make closing remarks and discuss future research.

2 Basic Estimates for p Harmonic Functions.

In the sequel c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n , unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1 , which may depend only on p, n, a_1, \dots, a_n not necessarily the same at each occurrence. $A \approx B$ means that A/B is bounded above and below by positive constants depending only on p, n . In this section, we will always assume that $2 \leq n \leq p < \infty$, and $r > 0$. Let Ω be an open set, $w \in \partial\Omega$, and suppose that \tilde{u} is p harmonic in $\Omega \cap B(w, 4r)$. If $p = n$ we also assume that $\partial\Omega \cap \bar{B}(w, 4r)$ is (n, r_0) uniformly fat as defined above (1).

We begin by stating some interior and boundary estimates for \tilde{u} , a positive weak solution to the p Laplacian in $\Omega \cap B(w, 4r)$ with $\tilde{u} \equiv 0$ on $\partial\Omega \cap B(w, 4r)$ in the Sobolev sense, as indicated

after (1). Extend \tilde{u} to $B(w, 4r)$ by putting $\tilde{u} \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then there exists a locally finite positive Borel measure $\tilde{\mu}$ with support $\subset \partial\Omega \cap \bar{B}(w, 4r)$ and for which (2) holds with \hat{u} replaced by \tilde{u} and $\phi \in C_0^\infty(B(w, 4r))$. Let $\max_{B(z,s)} \tilde{u}$, $\min_{B(z,s)} \tilde{u}$ be the essential supremum and infimum of \tilde{u} on $B(z, s)$ whenever $B(z, s) \subset B(w, 4r)$. For proofs of Lemmas 1 - 2 (see [9, Chapters 6 and 7]).

Lemma 1 Fix $p, 1 < p < \infty$, and let Ω, w, r, \tilde{u} , be as above. Then

$$\frac{1}{c} r^{p-n} \int_{B(w,r/2)} |\nabla \tilde{u}|^p dx \leq \max_{B(w,r)} \tilde{u}^p \leq \frac{c}{r^n} \int_{B(w,2r)} \tilde{u}^p dx.$$

If $B(z, 2s) \subset \Omega$, then

$$\max_{B(z,s)} \tilde{u} \leq c \min_{B(z,s)} \tilde{u}.$$

Lemma 2 Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 1. Then there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that \tilde{u} has a Hölder α continuous representative in $B(w, 4r)$ (also denoted \tilde{u}). Moreover if $z_1, z_2 \in B(w, r)$ then

$$|\tilde{u}(z_1) - \tilde{u}(z_2)| \leq c \left(\frac{|z_1 - z_2|}{r} \right)^\alpha \max_{B(w,2r)} \tilde{u}$$

Lemma 3 Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 1 and let $\tilde{\mu}$ be the measure associated with \tilde{u} as in (2). Then there exists $c, \gamma = \gamma(p, n) \geq 1$, such that

$$\frac{1}{c} r^{p-n} \tilde{\mu}[B(w, r/2)] \leq \max_{B(w,r)} \tilde{u}^{p-1} \leq c r^{p-n} \tilde{\mu}[B(w, 2r)].$$

For the proof of Lemma 3 see [11]. The left-hand side of the above inequality is true for any open Ω and $p \geq n$. However the right-hand side of this inequality requires uniform fatness when $p = n$ and is the main reason we have this assumption in Theorem 1. The reader is referred to [4] for references concerning the proof of the next lemma.

Lemma 4 Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 1. Then \tilde{u} has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular, there exists $\sigma \in (0, 1]$, depending only on p, n , such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$, then

$$\frac{1}{c} |\nabla \tilde{u}(x) - \nabla \tilde{u}(y)| \leq \left(\frac{|x - y|}{\hat{r}} \right)^\sigma \max_{B(\hat{w}, \hat{r})} |\nabla \tilde{u}| \leq \frac{c}{\hat{r}} \left(\frac{|x - y|}{\hat{r}} \right)^\sigma \max_{B(\hat{w}, 2\hat{r})} \tilde{u}.$$

If $x \in B(\hat{w}, 4\hat{r})$ and $\nabla \tilde{u}(x) \neq 0$, then \tilde{u} is infinitely differentiable in an open neighborhood of x . Moreover,

$$\int_{B(\hat{w}, \hat{r}) \cap \{|\nabla \tilde{u}| > 0\}} |\nabla \tilde{u}|^{p-2} \sum_{i,j=1}^n \tilde{u}_{x_i x_j}^2 dx \leq \frac{c}{\hat{r}^2} \int_{B(\hat{w}, 2\hat{r})} |\nabla \tilde{u}|^p dx.$$

Lemma 5 Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 1. Suppose for some $z \in \mathbb{R}^n, t \geq 100r$, that $w \in \partial B(z, t)$ and

$$B(w, 4r) \setminus \bar{B}(z, t) = B(w, 4r) \cap \Omega.$$

There exists $\sigma = \sigma(p, n) \in (0, 1)$ for which $\tilde{u}|_{\Omega \cap B(w, 3r)}$ has a $C^{1, \sigma} \cap W^{1, p}$ extension to $B(w, 3r)$ (denoted \bar{u}). If $x \in B(w, 3r) \setminus \partial B(z, t)$ and $\nabla \tilde{u}(x) \neq 0$, then \bar{u} is infinitely differentiable in an open neighborhood of x . Moreover,

$$\int_{\Omega \cap B(w, r/2) \cap \{|\nabla \bar{u}| > 0\}} |\nabla \bar{u}|^{p-2} \sum_{i, j=1}^n \bar{u}_{x_i x_j}^2 dx \leq \frac{c}{r^2} \int_{\Omega \cap B(w, 2r)} |\nabla \bar{u}|^p dx$$

and if $x, y \in \Omega \cap B(w, r/2)$, then

$$\begin{aligned} \frac{1}{c} |\nabla \bar{u}(x) - \nabla \bar{u}(y)| &\leq \left(\frac{|x-y|}{r} \right)^\sigma \max_{\Omega \cap B(w, r)} |\nabla \bar{u}| \\ &\leq \frac{c}{r} \left(\frac{|x-y|}{r} \right)^\sigma \max_{\Omega \cap B(w, 2r)} \bar{u}. \end{aligned}$$

Proof We assume as we may that $z = 0$ and $t = 1$ since otherwise we consider $u^*(x) = \tilde{u}(z + tx)$ and use translation - dilation invariance of the p Laplacian. Let

$$\bar{u}(x) = \begin{cases} \tilde{u}(x) & \text{when } x \in \bar{\Omega} \cap B(w, 3r) \\ -\tilde{u}(\frac{x}{|x|^2}) & \text{when } x \in B(0, 1) \cap B(w, 3r). \end{cases}$$

If $y = x/|x|^2 \in B(0, 1) \cap B(w, 3r)$ and $\nabla \tilde{u}(x) \neq 0$, one can use the chain rule to calculate at y that

$$\nabla \cdot (|y|^{2p-2n} |\nabla \bar{u}|^{p-2} \nabla \bar{u}) = \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(|y|^{2p-2n} |\nabla \bar{u}|^{p-2} \frac{\partial \bar{u}}{\partial y_i} \right) = 0. \quad (8)$$

Put

$$\gamma(x) = \begin{cases} |x|^{2p-2n} & \text{when } |x| \leq 1 \\ 1 & \text{when } |x| > 1. \end{cases}$$

We assert that \bar{u} is a weak solution in $B(w, 3r)$ to

$$\nabla \cdot (\gamma |\nabla \bar{u}|^{p-2} \nabla \bar{u}) = 0. \quad (9)$$

Indeed from the assumptions on \tilde{u} we see that $\bar{u} \in W^{1, p}(B(w, 3r))$. Let $\phi \in C_0^\infty(B(w, 3r))$ and put

$$\phi_1(x) = \frac{1}{2} (\phi(x) - \phi(\frac{x}{|x|^2}))$$

while

$$\phi_2(x) = \frac{1}{2} (\phi(x) + \phi(\frac{x}{|x|^2})).$$

Using the change of variables theorem and the knowledge garnered from (8) we see that

$$\int_{B(w, 4r)} \gamma |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \phi_2 dx = 0$$

and

$$\int_{B(w, 4r)} \gamma |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \phi_1 dx = 2 \int_{\Omega \cap B(w, 4r)} |\nabla \bar{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \phi_1 dx = 0$$

Since $\phi = \phi_1 + \phi_2$, we conclude from the above displays that \bar{u} is a weak solution to (9) in $B(w, 3r)$.

From our assertion we see that \bar{u} satisfies the hypotheses in [22], except for γ being continuously differentiable. However the argument in [22] and all constants use only Lipschitzness of γ , so is also valid in our situation. Applying the results in [22] (similar to Lemma 4) and using the definition of \bar{u} , we obtain the first and second displays in Lemma 5. \square

Lemma 6 *Let p, Ω, w, r, \bar{u} , be as in Lemma 1 and $-\infty < \eta \leq -1$. Let $L, (b_{ik})$ be as in (3), (4), when $x \in \Omega \cap B(w, 4r)$ and $\nabla \bar{u}(x) \neq 0$. Let $b_{ij} = \delta_{ij}$ when $\nabla \bar{u}(x) = 0$ and put $v = \max\{\log |\nabla \bar{u}|, \eta\}$. Then v is locally a weak sub solution to L in $\Omega \cap B(w, 4r)$.*

Proof From Lemma 4 we see that v is locally in $W^{1,2}(\Omega \cap B(w, 4r))$. Given $\epsilon, \delta, \sigma > 0$ small define g by

$$g(x) = (\max\{v - \eta - \epsilon, 0\} + \sigma)^\delta - \sigma^\delta, x \in \Omega \cap B(w, 4r).$$

As mentioned earlier in Lemma 5.1 of [15] we showed that $Lv \geq 0$ at $x \in \Omega \cap B(w, 4r)$ when $v(x) \neq \eta$. For the reader's convenience we repeat this calculation after the proof of Lemma 6.

From this fact we deduce that if $0 \leq \theta \in C_0^\infty(\Omega \cap B(w, 4r))$, then

$$\begin{aligned} 0 \leq \int_{\Omega \cap B(w, 4r)} \theta g L v dx &= - \sum_{i,k=1}^n \int_{\Omega \cap B(w, 4r)} b_{ik} (\theta g)_{x_i} v_{x_k} dx \\ &\leq - \sum_{i,k=1}^n \int_{\Omega \cap B(w, 4r)} g b_{ik} \theta_{x_i} v_{x_k} dx, \end{aligned}$$

where in the last inequality we have used (5). Using the above inequality, the bounded convergence theorem, and letting first ϵ , second σ , and third $\delta \rightarrow 0$, we get Lemma 6. \square

To show $Lv(x) \geq 0$ when $v(x) \neq \eta$, put $\tau(x) = 2v(x) = \log |\nabla \bar{u}|^2$. We calculate at x ,

$$\tau_{x_j} = \sum_{k=1}^n \frac{2\tilde{u}_{x_k} \tilde{u}_{x_k x_j}}{|\nabla \bar{u}|^2}.$$

Furthermore,

$$\begin{aligned} L\tau &= \sum_{i,j,k=1}^n \left(b_{ij} \frac{2\tilde{u}_{x_k} \tilde{u}_{x_k x_j}}{|\nabla \bar{u}|^2} \right)_{x_i} \\ &= \sum_{i,j,k=1}^n \frac{2\tilde{u}_{x_k}}{|\nabla \bar{u}|^2} (b_{ij} \tilde{u}_{x_k x_j})_{x_i} + \sum_{i,j,k=1}^n 2b_{ij} \tilde{u}_{x_k x_j} \left(\frac{\tilde{u}_{x_k}}{|\nabla \bar{u}|^2} \right)_{x_i}. \end{aligned}$$

The first term on the right is zero since $L\tilde{u}_{x_k} = 0$ (see (3)). We differentiate the second term to get

$$L\tau = \sum_{i,j,k=1}^n \left[2|\nabla \bar{u}|^{-2} b_{ij} \tilde{u}_{x_k x_j} \tilde{u}_{x_k x_i} - \sum_{i,j,k,l=1}^n 4|\nabla \bar{u}|^{-4} \tilde{u}_{x_k} \tilde{u}_{x_k x_j} b_{ij} \tilde{u}_{x_i} \tilde{u}_{x_l x_i} \right]. \quad (10)$$

We assume as we may that $\tilde{u}_{x_j} = 0$ for $j \neq 1$, since otherwise we rotate our coordinate system and use invariance of the p Laplace equation under rotations. Under this assumption we have

$$\begin{aligned} b_{11} &= (p-1)|\nabla\tilde{u}|^{p-2}, \\ b_{ii} &= |\nabla\tilde{u}|^{p-2} \quad i \neq 1, \\ b_{ij} &= 0 \quad i \neq j. \end{aligned}$$

Using these equalities in (10) we obtain, at x ,

$$L\tau = 2|\nabla\tilde{u}|^{p-4} \left((p-1) \sum_{k=1}^n \tilde{u}_{x_k x_1}^2 + \sum_{i=2, k=1}^n \tilde{u}_{x_k x_i}^2 - 2(p-1)\tilde{u}_{x_1 x_1}^2 - \sum_{i=2}^n 2\tilde{u}_{x_1 x_i}^2 \right).$$

Collecting the $x_1 x_1$ and $x_1 x_i$ ($i \neq 1$) derivatives yields

$$L\tau = 2|\nabla\tilde{u}|^{p-4} \left(-(p-1)\tilde{u}_{x_1 x_1}^2 + (p-2) \sum_{k=2}^n \tilde{u}_{x_k x_1}^2 + \sum_{k,i=2}^n \tilde{u}_{x_k x_i}^2 \right). \quad (11)$$

The last sum contains the pure second derivatives of \tilde{u} in the x_k direction when $k \neq 1$. These derivatives may be estimated using the p -Laplace equation for u at the point x , i.e., at x we have

$$(p-1)\tilde{u}_{x_1 x_1} + \sum_{k=2}^n \tilde{u}_{x_k x_k} = 0.$$

Solving for $\tilde{u}_{x_1 x_1}$, taking squares and using Hölder's inequality we see that

$$\sum_{k=2}^n \tilde{u}_{x_k x_k}^2 \geq \frac{(p-1)^2}{n-1} \tilde{u}_{x_1 x_1}^2.$$

Substituting this expression into (11) gives

$$L\tau \geq 2|\nabla\tilde{u}|^{p-4} \left(\left(\frac{(p-1)^2}{n-1} - (p-1) \right) \tilde{u}_{x_1 x_1}^2 + (p-2) \sum_{k=2}^n \tilde{u}_{x_k x_1}^2 + \sum_{k,i=2, k \neq i}^n \tilde{u}_{x_k x_i}^2 \right).$$

Thus, $L\tau \geq 0$ when $\frac{(p-1)^2}{n-1} - (p-1) = \frac{(p-1)(p-n)}{n-1} \geq 0$. In particular, $L\tau \geq 0$ if $p \geq n$. Note that when $p = n$ then $\tilde{u}(x) = \log|x|$ is n harmonic and $L(\log|\nabla\tilde{u}|) \equiv 0$ when $x \neq 0$.

3 Proof of Theorem 1.

Let $p, n, O, \hat{u}, \hat{\mu}, \rho, \hat{z}$, be as in Theorem 1 and suppose that λ is a positive nondecreasing function on $(0, 1]$ with $\lim_{t \rightarrow 0} t^{1-n} \lambda(t) = 0$. Theorem 1 follows easily from the next proposition (See section 3.2).

3.1 Proof of Proposition 1

Proposition 1 *There exists $c = c(p, n)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ with the following properties. $\hat{\mu}(\partial O \cap B(\hat{z}, \rho) \setminus Q) = 0$ and for every $w \in Q$ there are arbitrarily small $r = r(w), 0 < r \leq 10^{-10}$, such that*

$$(a) \bar{B}(w, 100r) \subset B(\hat{z}, \rho) \quad \text{and} \quad \hat{\mu}(B(w, 100r)) \leq c \hat{\mu}(B(w, r)).$$

Moreover there is a compact set $F = F(w, r) \subset \partial O \cap B(w, 20r)$ with

$$(b) H^\lambda(F) = 0 \quad \text{and} \quad \hat{\mu}(F) \geq \frac{1}{c} \hat{\mu}(B(w, 100r)).$$

Proof To prove (a) of Proposition 1 we note that $\hat{\mu}(B(x, t)) \neq 0$ whenever $x \in \partial O$ and $\partial O \cap B(x, t) \subset \partial O \cap B(\hat{z}, \rho)$ and $t > 0$ as follows from Lemma 3. Let

$$\Theta = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \liminf_{t \rightarrow 0} \frac{\hat{\mu}(B(x, 100t))}{\hat{\mu}(B(x, t))} \geq c \right\}$$

If $x \in \Theta$, then there exists $t_0(x) > 0$ for which

$$\hat{\mu}(B(x, 100t)) \geq \frac{c}{2} \hat{\mu}(B(x, t)) \quad \text{for } 0 < t < t_0(x).$$

Iterating this inequality it follows that if c is large enough then

$$\lim_{t \rightarrow 0} \frac{\hat{\mu}(B(x, t))}{t^{n+1}} = 0 \quad \text{whenever } x \in \Theta.$$

Since $H^{n+1}(\mathbb{R}^n) = 0$, we conclude that $\hat{\mu}(\Theta) = 0$. Thus we assume (a) holds for some $c' = c'(n), w \in \partial O \cap B(\hat{z}, \rho)$, and $r > 0$.

To prove (b) of Proposition 1 let

$$\gamma^{-1} = \max_{B(w, 10r)} \hat{u}$$

and put

$$u(x) = \gamma \hat{u}(w + rx) \quad \text{when } w + rx \in B(\hat{z}, \rho).$$

Let

$$\Omega = \{x : w + rx \in O \cap B(\hat{z}, \rho)\}.$$

Using translation and dilation invariance of the p Laplacian we find that u is p harmonic in Ω and if $\zeta = r^{-1}(\hat{z} - w)$, then u is continuous in $B(\zeta, \rho/r)$ with $u \equiv 0$ on $B(\zeta, \rho/r) \setminus \Omega$. Moreover there exists a measure μ on \mathbb{R}^n with support in $\partial \Omega \cap \bar{B}(\zeta, \rho/r)$ corresponding to u . In fact if E is a Borel set and $T(E) = \{w + rx : x \in E\}$ then $\mu(E) = r^{p-n} \gamma^{p-1} \hat{\mu}(T(E))$. From Lemma 3 and Proposition 1 (a), we obtain for some $c = c(p, n) \geq 1$ and $2 \leq t \leq 50$ that

$$\frac{1}{c} \leq \mu(B(0, 1)) \leq \max_{B(0, 2)} u \leq \max_{B(0, t)} u \leq c \mu(B(0, 100)) \leq c^2. \quad (12)$$

From (12) and the definition of u we observe that to prove Proposition 1 (b) it suffices to show that there exists a compact set $F' \subset B(0, 20)$ and $\hat{c} = \hat{c}(p, n) \geq 1$ with

$$\mu(F') \geq \frac{1}{\hat{c}} \quad \text{and} \quad H^\lambda(F') = 0. \quad (13)$$

To prove (13) we first show for given $\epsilon, \tau > 0$ that there exists a Borel set $E \subset B(0, 20)$ and $c = c(p, n) \geq 1$ with

$$\phi_\tau^\lambda(E) \leq \epsilon \quad \text{and} \quad \mu(E) \geq \frac{1}{c}. \quad (14)$$

(13) follows easily from (14). Indeed, choose E_m relative to $\tau = \epsilon = 2^{-m}, m = 1, 2, \dots$ and put

$$E = \bigcap_k \left(\bigcup_{m=k} E_m \right).$$

Then from measure theoretic arguments it follows that (13) is valid with F' replaced by E and \hat{c} by c'' . Using regularity of μ we then get (13) for a compact set $F' \subset E$. Thus to complete the proof of Proposition 1 we need only prove (14).

To prove (14) we note from the definition of u that $u(\tilde{z}) = 1$ for some $\tilde{z} \in \partial B(0, 10)$. This note, (12), and Lemma 2 imply for some $c_- = c_-(p, n) \geq 1$ that

$$d(\tilde{z}, \partial\Omega) \geq \frac{1}{c_-}. \quad (15)$$

In fact otherwise it would follow from Lemma 2 that $\max_{B(0, 20)} u$ is too large for (12) to hold.

Next let M be a large positive number and $0 < s < e^{-M}$. For the moment we allow M to vary but shall later fix it to satisfy several conditions. We then choose $s = s(M)$. First given $0 < \tilde{\tau} < \min(\tau, 10^{-5})$ choose M so large that if

$$z \in \partial\Omega \cap \bar{B}(0, 15) \quad \text{and} \quad \mu(B(z, t)) = Mt^{n-1} \quad \text{for some } t = t(z) \leq 1, \quad \text{then } t \leq \tilde{\tau}. \quad (16)$$

Existence of $1 \leq M = M(\tilde{\tau})$ follows from (12). Next following Wolff [23] we observe from (16) that for each $z \in \partial\Omega \cap \bar{B}(0, 15)$ there exists a largest $t = t(z), s \leq t \leq \tilde{\tau}$, with either

$$\begin{aligned} (\alpha) \quad & \mu(B(z, t)) = Mt^{n-1}, t > s, \\ \text{or} \\ (\beta) \quad & t = s. \end{aligned} \quad (17)$$

Using the Besicovitch covering theorem (see [20]) we now obtain a covering $\{B(z_j, t_j)\}_1^N$ of $\partial\Omega \cap \bar{B}(0, 15)$, where $t_j = t(z_j)$ is the maximal t for which either (17) (α) or (β) holds. Moreover each point of $\bigcup_{j=1}^N B(z_j, t_j)$ lies in at most $c = c(n)$ of $\{B(z_j, t_j)\}_1^N$. Let c_-, \tilde{z} , be as in (15) and set $r_1 = (8c_-)^{-1}$. Choosing $\tilde{\tau}$ smaller (so M larger) if necessary we may assume, thanks to (16), that

$$\bigcup_{j=1}^N \bar{B}(z_j, 6t_j) \cap B(\tilde{z}, 6r_1) = \emptyset. \quad (18)$$

Also put

$$\Omega' = \Omega \cap B(0, 15) \setminus \bigcup_{j=1}^N \bar{B}(z_j, t_j)$$

and

$$D = \Omega' \setminus \bar{B}(\tilde{z}, 2r_1).$$

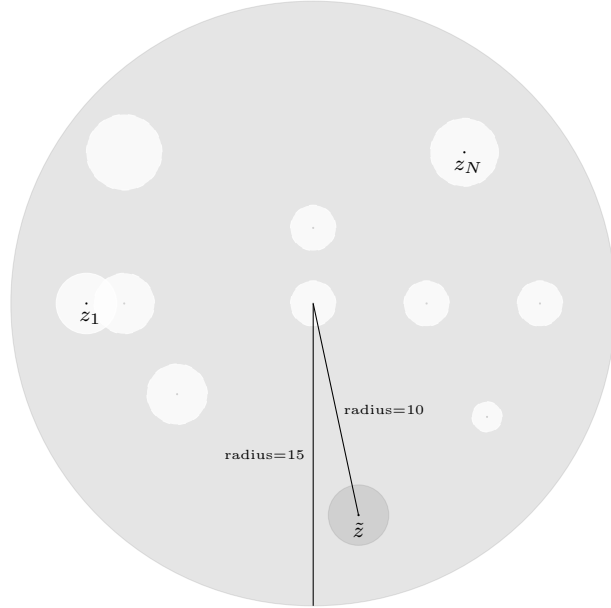


Fig. 1: An example of $\Omega' = \Omega \cap B(0, 15) \setminus \bigcup_{j=1}^N \bar{B}(z_j, t_j)$.

Let u' be the p harmonic function in D with continuous boundary values,

$$u'(x) \equiv \begin{cases} 0 & \text{when } x \in \partial\Omega' \\ \min_{\bar{B}(\tilde{z}, 2r_1)} u & \text{when } x \in \partial B(\tilde{z}, 2r_1). \end{cases}$$

Extend u' continuously to $\bar{B}(0, 15)$ (also denoted u') by putting

$$u'(x) \equiv \begin{cases} 0 & \text{when } x \in \bar{B}(0, 15) \setminus \Omega' \\ \min_{\bar{B}(\tilde{z}, 2r_1)} u & \text{when } x \in \bar{B}(\tilde{z}, 2r_1). \end{cases}$$

We note that $u' \leq u$ on ∂D so by the maximum principle for p harmonic functions $u' \leq u$ in D . Also, ∂D is locally (n, r'_0) uniformly fat where r'_0 depends only on p, n , and r_0 in Theorem 1.

To continue the proof of (14) we shall need several lemmas.

Lemma 7 *If $x \in D$, then*

$$|\nabla u'(x)| \leq c M^{\frac{1}{p-1}}.$$

Proof To prove Lemma 7 let $x \in D$ and choose $y \in \partial D$ with $|x-y| = d(x, \partial D) = d$. If $y \in \partial B(z_k, t_k)$ and $x \in B(z_k, 2t_k)$ we put

$$f(w) = A \left(|w - z_k|^{\frac{p-n}{p-1}} - t_k^{\frac{p-n}{p-1}} \right), w \in B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k),$$

when $p > n$ and

$$f(w) = A (\log |w - z_k| - \log t_k), w \in B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k)$$

when $p = n$. Then $f \equiv 0$ on $\partial B(z_k, t_k)$ and A is chosen so that

$$f \equiv \max_{B(z_k, 2t_k)} u \text{ on } \partial B(z_k, 2t_k).$$

Then from $u' \leq u$ and the maximum principle for p harmonic functions, $u' \leq f$ in $B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k)$. Using this inequality and applying Lemma 4 to u' we conclude that

$$|\nabla u'(x)| \leq \frac{c}{d} u'(x) \leq \frac{c}{d} f(x) \leq \frac{c^2}{t_k} \max_{B(z_k, 2t_k)} u. \quad (19)$$

Also from Lemma 3 and (16)-(18) we find that

$$t_k^{1-p} \max_{B(z_k, 2t_k)} u^{p-1} \leq c t_k^{1-n} \mu(B(z_k, 4t_k)) \leq c^2 M. \quad (20)$$

Taking $1/(p-1)$ powers of both sides of (20) and using the resulting inequality in (19) we get Lemma 7 when $y \in \partial B(z_k, t_k)$ and $x \in D \cap B(z_k, 2t_k)$. If $y \in \partial B(0, 15)$ or $\partial B(\tilde{z}, 2r_1)$ a similar argument applies. Thus there is an open neighborhood, say W , containing ∂D for which the conclusion of Lemma 7 is valid when $x \in W \cap D$. From this conclusion, Lemma 6 applied to u' , and a maximum principle for weak sub solutions to L , we conclude that Lemma 7 is valid in D . \square

Next we prove

Lemma 8 *The functions $|\nabla u'|^{p-2} |u'_{x_k x_i}|$ for $1 \leq i, k \leq n$ are all integrable on D*

$$\sum_{i,k=1}^n \int_D |\nabla u'|^{p-2} |u'_{x_k x_i}| dx < \infty$$

Proof Let $\Lambda \subset \partial \Omega'$ be the set of points where $\partial \Omega'$ is not smooth. Clearly $H^{n-1}(\Lambda) = 0$. If $\hat{x} \in \partial D \setminus \Lambda$, then \hat{x} lies in exactly one of the finite number of spheres which contain points of ∂D . Let $d'(\hat{x})$ denote the distance from \hat{x} to the union of spheres not containing \hat{x} but containing points of ∂D . If $d' = d'(\hat{x}) < s/100$, then from Lemma 5 applied to u' we see that each component of $\nabla u'$ has a Hölder continuous extension to $B(\hat{x}, 3d'/4)$. Also from Hölder, Lemma 5, and Lemma 7 we see that

$$\begin{aligned} \frac{1}{c} \sum_{i,k=1}^n \int_{D \cap B(\hat{x}, \frac{d'}{8})} |\nabla u'|^{p-2} |u'_{x_i x_k}| dx &\leq (d')^{\frac{n}{2}} M^{\frac{p-2}{2(p-1)}} \sum_{i,k=1}^n \left(\int_{D \cap B(\hat{x}, \frac{d'}{8})} |\nabla u'|^{p-2} |u'_{x_i x_k}|^2 dx \right)^{\frac{1}{2}} \\ &\leq c(d')^{\frac{(n-2)}{2}} M^{\frac{p-2}{2(p-1)}} \left(\int_{D \cap B(\hat{x}, \frac{d'}{2})} |\nabla u'|^p dx \right)^{\frac{1}{2}} \\ &\leq c^2 M (d')^{(n-1)}. \end{aligned} \quad (21)$$

To prove Lemma 8 we assume as we may that $B(z_l, t_l) \not\subset B(z_\nu, t_\nu)$ when $\nu \neq l$, since otherwise we discard one of these balls. Also from a well known covering theorem we get a covering $\{B(y_j, \frac{1}{20}d'(y_j))\}$ of $\partial D \setminus \Lambda$ with $\{B(y_j, \frac{1}{100}d'(y_j))\}$, pairwise disjoint. From (21) we find that

$$\begin{aligned} \sum_{i,j,k} \int_{D \cap B(y_j, \frac{1}{8}d'(y_j))} |\nabla u'|^{p-2} |u'_{x_k x_i}| dx &\leq cM \sum_j (d'(y_j))^{n-1} \\ &\leq c^2 M H^{n-1}(\partial D). \end{aligned} \quad (22)$$

For short we now write $d(x)$ for $d(x, \partial D)$ and choose a covering $\{B(x_m, \frac{1}{2}d(x_m))\}$ of D with $\{B(x_m, \frac{1}{20}d(x_m))\}$, pairwise disjoint. We note that if $x \in D$ and $y \in \partial D$ with $|y - x| = d(x)$, then $y \in \partial D \setminus \Lambda$. Indeed otherwise y would be on the boundary of at least two balls contained in the complement of D and so by the no containment assumption above, would have to intersect $B(x, d(x))$, which clearly is a contradiction. Also we note that if $d(x) \leq 1000s$, then $d(x) \leq \kappa d'(y)$ where κ can depend on various quantities including the configuration of the $B(z_k, t_k)$ balls but is independent of $x \in D$ with $d(x) \leq 1000s$. Indeed from the no containment assumption one just needs to consider $d(x)/d'(y)$ as $d(x), d'(y) \rightarrow 0$. To do this suppose $z \in \Lambda$ with $|y - z| = d'(y)$. Then one sees, from consideration of half planes containing z and tangent to two intersecting spheres, that x, y eventually lie in a truncated cone of height γ with vertex at z , and of angle opening $\leq \alpha < \pi/2$, where α, γ are independent of x, y, z . Moreover the complement of this truncated cone in a certain hemisphere of radius γ with center z lies outside of Ω' . Then a ballpark estimate using trigonometry gives $d'(y) \geq (1 - \sin \alpha)d(x)$ (See Figure 2).

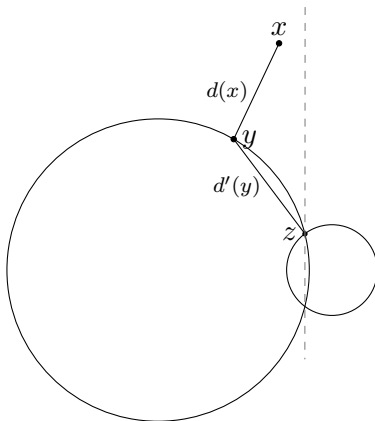


Fig. 2: $d'(y) \geq (1 - \sin \alpha)d(x)$.

From this analysis and our choice of covering of D we see that for a given $B(x_m, \frac{1}{2}d(x_m))$ with $d(x_m) < 1000s$, there exists $j = j(m)$ with $B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \kappa' d'(y_j))$ for some $0 < \kappa' < \infty$ independent of m .

Let $S_l, l = 1, 2, 3$, be disjoint sets of integers defined as follows.

$$\begin{cases} m \in S_1 & \text{if } d(x_m) \geq 1000s, \\ m \in S_2 & \text{if } m \notin S_1 \text{ and there does not exist } j \text{ with } B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \frac{1}{8}d'(y_j)), \\ m \in S_3 & \text{if } m \text{ not in either } S_1 \text{ or } S_2. \end{cases}$$

Let

$$K_l = \sum_{m \in S_l} \int_{D \cap B(x_m, \frac{1}{2}d(x_m))} |\nabla u'|^{p-2} |u'_{x_k x_i}| dx \text{ for } l = 1, 2, 3.$$

Then

$$\int_D |\nabla u'|^{p-2} |u'_{x_k x_i}| dx \leq K_1 + K_2 + K_3. \quad (23)$$

From Lemma 4 and the same argument as in (21) we see that

$$K_1 \leq cM \sum_{m \in S_1} d(x_m)^{n-1} \leq c^2 M s^{-1} \quad (24)$$

where we have used disjointness of our covering, $\{B(x_m, \frac{1}{20}d(x_m))\}$. Using disjointness of these balls and (22) we get

$$K_3 \leq cMH^{n-1}(\partial D). \quad (25)$$

Finally if $m \in S_2$, then as discussed earlier there exists $j = j(m)$ with $d(x_m) \approx d'(y_j)$, where proportionality constants are independent of m , so $B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \kappa' d'(y_j))$. From disjointness of $\{B(x_m, \frac{1}{20}d(x_m))\}$ and a volume type argument we deduce that each j corresponds to at most κ'' integers $m \in S_2$ where κ'' is independent of j . From this fact, (21), and disjointness of $\{B(y_j, \frac{1}{100}d'(y_j))\}$ we conclude that there is a $\tilde{\kappa}, 0 < \tilde{\kappa} < \infty$, with

$$K_2 \leq \tilde{\kappa}M \sum_{m \in S_2} d(x_m)^{n-1} \leq \tilde{\kappa}^2 M \sum_j d'(y_j)^{n-1} \leq \tilde{\kappa}^3 MH^{n-1}(\partial D). \quad (26)$$

Using (24)-(26) in (23) we find that Lemma 8 is valid. \square

Recall that $\nabla u'$ is Hölder continuous in $\bar{D} \setminus \Lambda$. We use this recollection and Lemmas 7, 8, to prove

Lemma 9 *There exists $c = c(p, n)$ such that*

$$\int_{\partial D} |\nabla u'|^{p-1} |\log |\nabla u'||| dH^{n-1} \leq c \log M.$$

Proof From smoothness of u' in $\bar{D} \setminus \Lambda$, (2), and integration by parts, we see that

$$d\mu'/dH^{n-1} = |\nabla u'|^{p-1} > 0 \text{ on } \partial\Omega' \setminus \Lambda. \quad (27)$$

We claim for some $c = c(p, n) \geq 1$ that

$$\frac{1}{c} \leq \mu'(\partial\Omega' \cap B(0, 10)) \leq \mu'(\partial\Omega') \leq c. \quad (28)$$

To prove the left hand inequality in (28) we first observe from $u(\bar{z}) = 1$ and Lemmas 1, 2, and (18) that $c^*u' \geq 1$ on $\partial B(\bar{z}, 4r_1)$ for some $c^* = c^*(p, n) \geq 1$. Let l denote the line from the origin

through \tilde{z} and let ζ_1 be the point on this line segment in $\partial B(\tilde{z}, 4r_1) \cap B(0, 10)$. Let ζ_2 be the point on the line segment from ζ_1 to the origin with $d(\zeta_2, \partial\Omega') = \frac{1}{20}r_1$ while $d(\zeta, \partial\Omega') > \frac{1}{20}r_1$ at every other point on the line segment from ζ_1 to ζ_2 . Then from (15), Lemma 1, and the above discussion we see that $c^{**}u(\zeta_2) \geq 1$ for some $c^{**}(p, n) \geq 1$. Also, $B(\zeta_2, \frac{1}{2}r_1) \subset B(0, 10)$. Let $\hat{\zeta}$ be the point in $\partial\Omega'$ with $|\hat{\zeta} - \zeta_2| = d(\zeta_2, \partial\Omega')$. Applying Lemma 3 with $w = \hat{\zeta}$, $r = 2d(\zeta_2, \partial\Omega')$, we deduce that the left hand inequality in (28) is valid. The right hand inequality in this claim follows once again from Lemma 3 and $u' \leq u$.

Let

$$\log^+ t = \max\{\log t, 0\}$$

and

$$\log^- t = \log^+(1/t)$$

for $t \in (0, \infty)$. From Lemma 7, (27), (28), and $H^{n-1}(A) = 0$ we obtain for some $c = c(p, n) \geq 1$,

$$\int_{\partial\Omega'} |\nabla u'|^{p-1} \log^+ |\nabla u'| dH^{n-1} \leq c \log M \mu'(\partial\Omega') \leq c^2 \log M. \quad (29)$$

To estimate $\log^- |\nabla u'|$, fix η , $-\infty \leq \eta \leq -1$, and let $v'(x) = \max\{\log |\nabla u'|, \eta\}$ when $x \in \bar{D} \setminus A$. Given a small $\theta > 0$ let

$$A(\theta) = \{x \in D : d(x, A) \leq \theta\} \text{ and } D(\theta) = D \setminus A(\theta).$$

From Lemma 4 and Lemmas 7, 8 we deduce that $|\nabla u'|^{p-2} u'_{x_i}$ has a $W^{1,2}(D(\theta))$ extension with distributional derivative $(|\nabla u'|^{p-2} u'_{x_i})_{x_j} = 0$ when $|\nabla u'| = 0$ and $1 \leq i, j \leq n$. Moreover these functions are continuous near $\partial D(\theta)$ thanks to Lemmas 4 and 5. Let $\{b_{ik}\}$, L , be as defined in (3), (4) relative to u' and note from the above discussion that

$$Lu'(x) = (p-1)\nabla \cdot (|\nabla u'|^{p-2} \nabla u')(x) = 0$$

exists pointwise for almost every $x \in D(\theta)$. Put

$$I(\theta) = \int_{D(\theta)} Lu' v' dx + \int_{D(\theta)} \sum_{i,k=1}^n b_{ik} u'_{x_k} v'_{x_i} dx = I_1(\theta) + I_2(\theta). \quad (30)$$

Clearly $I_1(\theta) = 0$. To handle $I_2(\theta)$ we first argue as in (19), i.e, use a barrier argument, and second use Lemma 5 to deduce for some $c = c(p, n) \geq 1$, that if $r_2 = (1 + c^{-1})r_1$, then

$$\frac{1}{c} \leq |\nabla u'| \leq c \text{ on } \bar{B}(\tilde{z}, 2r_2) \setminus B(\tilde{z}, 2r_1). \quad (31)$$

Let ψ be infinitely differentiable and $0 \leq \psi \leq 1$ on \mathbb{R}^n with $\psi \equiv 1$ on $\mathbb{R}^n \setminus B(\tilde{z}, 2r_2)$ and $|\nabla \psi| \leq cr_1^{-1} \leq c^2$, where the last inequality follows from (15) and the definition of r_1 . Suppose also that ψ vanishes in an open set containing $\bar{B}(\tilde{z}, 2r_1)$. Then

$$\begin{aligned} I_2(\theta) &= \int_{D(\theta)} \sum_{i,k=1}^n b_{ik} (\psi u')_{x_k} v'_{x_i} dx + \int_{D(\theta)} \sum_{i,k=1}^n b_{ik} ((1-\psi)u')_{x_k} v'_{x_i} dx \\ &= I_{21}(\theta) + I_{22}(\theta). \end{aligned} \quad (32)$$

From Lemmas 4, 5, (31), and an argument similar to the one in (21) we deduce for some $c = c(p, n) \geq 1$ that

$$|I_{22}| \leq c. \quad (33)$$

Turning to $I_{21}(\theta)$ we note from Lemmas 7 and 8 that the integrand in the integral defining $I_{21}(\theta)$ is dominated by an integrable function independent of θ . Thus from the Lebesgue dominated convergence theorem,

$$\lim_{\theta \rightarrow 0} I_{21}(\theta) = \int_D \sum_{i,k=1}^n b_{ik}(\psi u')_{x_k} v'_{x_i} dx = I'. \quad (34)$$

We assert that

$$I' \leq 0. \quad (35)$$

To verify this assertion let $u'' = u''(\delta) = \max(u' - \delta, 0)$. Using the convolution of $\psi u''$ with an approximate identity and taking limits we see from Lemma 6 that

$$\int_D \sum_{i,k=1}^n b_{ik}(\psi u'')_{x_k} v'_{x_i} dx \leq 0.$$

Now again from Lemmas 7 and 8, we observe that the above integrand is dominated by an integrable function independent of δ . Using this fact, the above inequality, and the Lebesgue dominated convergence theorem we get assertion (35). Using (30) - (35) we conclude (since $I_{22}(\theta)$ is independent of θ) that

$$\lim_{\theta \rightarrow 0} I(\theta) \leq c. \quad (36)$$

On the other hand from [7, Chapter 5] and the discussion above (30) we see that integration by parts can be used to get

$$I_1(\theta) = -I_2(\theta) + \int_{\partial D(\theta)} v' \sum_{i,k=1}^n b_{ik} u'_{x_k} \nu_i dH^{n-1} \quad (37)$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the outer unit normal to $\partial D(\theta)$. From (31) we see that

$$\left| \int_{\partial B(\bar{z}, 2r_1)} v' \sum_{i,k=1}^n b_{ik} u'_{x_k} \nu_i dH^{n-1} \right| \leq c = c(p, n). \quad (38)$$

From Lemma 7, dominated convergence, and the definition of $D(\theta)$, we have

$$\int_{\partial D(\theta) \setminus \partial B(\bar{z}, 2r_1)} v' \sum_{i,k=1}^n b_{ik} u'_{x_k} \nu_i dH^{n-1} \rightarrow \int_{\partial \Omega' \setminus \Lambda} v' \sum_{i,k=1}^n b_{ik} u'_{x_k} \nu_i dH^{n-1} \text{ as } \theta \rightarrow 0. \quad (39)$$

Observe that $\nu = -\frac{\nabla u'}{|\nabla u'|}$ on $\partial \Omega' \setminus \Lambda$. From this observation and (4) we calculate

$$\begin{aligned} \sum_{i,k=1}^n b_{ik} u'_{x_k} \nu_i &= - \sum_{i,k=1}^n |\nabla u'|^{p-5} [(p-2)(u')_{x_i}^2 (u')_{x_k}^2 + \delta_{ik} |\nabla u'|^2] u_{x_i} u_{x_k} \\ &= -(p-1) |\nabla u'|^{p-1}. \end{aligned} \quad (40)$$

From (30), (36)-(40) we find that

$$-(p-1) \int_{\partial\Omega'} v |\nabla u'|^{p-1} dH^{n-1} \leq \lim_{\theta \rightarrow 0} I(\theta) + c \leq 2c. \quad (41)$$

Letting $\eta \rightarrow -\infty$ in (41) and using the monotone convergence theorem we see that (41) holds with v replaced by $\log |\nabla u|$. Finally from (41) for $\log |\nabla u|$ and (29) we conclude the validity of Lemma 9. \square

With these lemmas in hand, we go back to the proof of (14) and Proposition 1b. We note from Lemma 3 and $u' \leq u$ that for given $j, 1 \leq j \leq N$,

$$t_j^{1-n} \mu'(\bar{B}(z_j, t_j)) \leq c t_j^{1-p} \max_{B(z_j, 2t_j)} u^{p-1} \leq c^2 t_j^{1-n} \mu(B(z_j, 4t_j)). \quad (42)$$

For given $A \gg 1$, we see from (17) that $\{1, 2, \dots, N\}$ can be divided into disjoint subsets Φ_1, Φ_2, Φ_3 , as follows.

$$\begin{cases} j \in \Phi_1 & \text{if } t_j > s, \\ j \in \Phi_2 & \text{if } t_j = s \text{ and } |\nabla u'|^{p-1}(x) \geq M^{-A}, \text{ for some } x \in \partial\Omega' \cap \partial B(z_j, t_j) \setminus \Lambda \\ j \in \Phi_3 & \text{if } j \text{ is not in } \Phi_1 \text{ or } \Phi_2. \end{cases}$$

Let $t'_j = t_j$ when $j \in \Phi_1$ and $t'_j = 4s$ when $j \in \Phi_2$. To prove (14) set

$$E = \partial\Omega \cap \bigcup_{j \in \Phi_1 \cup \Phi_2} B(z_j, t'_j).$$

To estimate $\phi_\tau^\lambda(E)$ we first observe that if

$$x \in \bigcup_{j \in \Phi_1 \cup \Phi_2} B(z_j, t'_j) \text{ then } x \text{ lies in at most } c = c(n) \text{ of } \{B(z_j, t'_j)\}. \quad (43)$$

This observation can be proved using $t_j \geq s, 1 \leq j \leq N$, a volume type argument, and the fact that $\{B(z_j, t_j)\}_1^N$ is a Besicovitch covering of $\partial\Omega \cap \bar{B}(0, 15)$. If $j \in \Phi_2$ we get from (19), (42), that for some $c = c(p, n) \geq 1$

$$M^{-A} \leq |\nabla u'(x)|^{p-1} \leq c s^{1-n} \mu(B(z_j, 4s)).$$

Rearranging this inequality, summing, and using (12), (43), we see that

$$\sum_{j \in \Phi_2} (t'_j)^{n-1} \leq \tilde{c} M^A \mu\left(\bigcup_{j \in \Phi_2} B(z_j, t'_j)\right) \leq (\tilde{c})^2 M^A$$

provided $\tilde{c} = \tilde{c}(p, n)$ is large enough. Now since $t'_j = s$ for all $j \in \Phi_2$ we may for given A, M, ϵ choose $s > 0$ so small that

$$s^{1-n} \lambda(s) \leq \frac{\epsilon}{2(\tilde{c})^2 M^A} \quad (44)$$

where we have used the definition of λ . Using this choice of s in the above display we get

$$\sum_{j \in \Phi_2} \lambda(t'_j) \leq \epsilon/2. \quad (45)$$

On the other hand we may suppose $\bar{\tau}$ in (16) is so small that $\lambda(t_j) \leq t_j^{n-1}$ for $1 \leq j \leq N$. Then from (12), (17), and (43), we see that

$$\begin{aligned} \sum_{j \in \Phi_1} \lambda(t'_j) &\leq \sum_{j \in \Phi_1} (t'_j)^{n-1} \\ &= M^{-1} \sum_{j \in \Phi_1} \mu(B(z_j, t_j)) \leq \epsilon/2 \end{aligned} \quad (46)$$

provided $M = M(\epsilon)$ is chosen large enough. Fix M satisfying all of the above requirements. In view of (45), (46), we have proved the left hand inequality in (14) for E as defined above, i.e. $\phi_\tau^\lambda(E) \leq \epsilon$.

To prove the right hand inequality in (14) we use Lemma 9 and the definition of Φ_3 to obtain

$$\begin{aligned} \mu' \left(\partial\Omega' \cap \bigcup_{j \in \Phi_3} \bar{B}(z_j, t_j) \right) &\leq \mu'(\{x \in \partial\Omega' : |\nabla u'(x)|^{p-1} \leq M^{-A}\}) \\ &\leq (p-1)(A \log M)^{-1} \int_{\partial\Omega'} |\nabla u'|^{p-1} |\log |\nabla u'||| dH^{n-1} \\ &\leq \frac{c}{A}. \end{aligned} \quad (47)$$

Choosing $A = A(n)$ large enough we have from (28), (47),

$$\mu' \left(\bigcup_{j \in \Phi_1 \cup \Phi_2} B(0, 10) \cap \bar{B}(z_j, t_j) \right) \geq \mu'(B(0, 10)) - \mu' \left(\bigcup_{j \in \Phi_3} \bar{B}(z_j, t_j) \right) \geq c_*^{-1} \quad (48)$$

for some $c_*(p, n)$. Finally from (42), (43), and (48), we get for some $c = c(p, n) \geq 1$ that

$$\mu(E) \geq c^{-1} \sum_{j \in \Phi_1 \cup \Phi_2} \mu(\bar{B}(z_j, t'_j)) \geq c^{-2} \sum_{j \in \Phi_1 \cup \Phi_2} \mu'(\bar{B}(z_j, t_j)) \geq c^{-3}. \quad (49)$$

For $j \in \Phi_1$ we have used the definition of t_j so that

$$\mu(B(z_j, 4t_j)) < M4^{n-1}t_j^{n-1} = 4^{n-1}\mu(B(z_j, t_j)) = 4^{n-1}\mu(B(z_j, t'_j))$$

Thus (14) is valid. Proposition 1 follows from (14) and our earlier remarks. \square

3.2 Proof of Theorem 1

Next we show for λ, Q as in Proposition 1 that there exists a Borel set Q_1 with

$$Q_1 \subset Q, \hat{\mu}(\partial O \cap B(\hat{z}, \rho) \setminus Q_1) = 0, \text{ and } H^\lambda(Q_1) = 0. \quad (50)$$

To prove (50) we assume, as we may, that $\hat{\mu}(\partial O \cap B(\hat{z}, \rho)) < \infty$ since otherwise we can write $\partial O \cap B(\hat{z}, \rho)$ as a countable union of Borel sets with finite $\hat{\mu}$ measure and apply the following argument in each set. Under this assumption we can use Proposition 1 and a Vitali type covering

argument (see [20]), as well as induction to get compact sets $\{F_l\}$, $F_l \subset Q$, with $F_k \cap F_j = \emptyset$, $k \neq j$, $\hat{\mu}(F_1) > 0$ and with

$$c' \hat{\mu}(F_{m+1}) \geq \hat{\mu}(Q \setminus \bigcup_{l=1}^m F_l), m = 1, 2, \dots,$$

for some $c' = c'(p, n) \geq 1$. Moreover $H^\lambda(F_l) = 0$ for all l . Then $Q_1 = \bigcup_{l=1}^\infty F_l$ has the desired properties as follows from measure theoretic arguments.

To prove Theorem 1 we first note from a covering argument as in [15] or [23] that if

$$P = \{x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \rightarrow 0} \frac{\hat{\mu}(B(x, t))}{t^{n-1}} > 0\},$$

then P has σ finite H^{n-1} measure. For completeness we prove this statement after finishing the proof of Theorem 1. Thus to prove Theorem 1 it suffices to show that

$$\hat{\mu}(Q_1 \setminus P) = 0. \quad (51)$$

Indeed otherwise from Egoroff's theorem there exists a compact set $K \subset Q_1 \setminus P$ with

$$\hat{\mu}(K) > 0 \text{ and } \lim_{t \rightarrow 0} \frac{\hat{\mu}(B(x, t))}{t^{n-1}} = 0 \text{ uniformly for } x \in K. \quad (52)$$

Choose $\alpha_k \in (0, 1)$, $k = 1, 2, \dots$, with $\alpha_{k+1} < \alpha_k/2$ and so that

$$\sup_{0 < t \leq \alpha_k} \frac{\hat{\mu}(B(x, t))}{t^{n-1}} \leq 2^{-2k} \text{ for all } x \in K.$$

Let $\alpha_0 = 1$. With $(\alpha_k)_0^\infty$ now chosen, define $\lambda(t)$ on $(0, 1]$ by $\lambda(\alpha_k) = 2^{-k}(\alpha_k)^{n-1}$, $k = 0, 1, \dots$, and $t^{1-n}\lambda(t)$ is linear for t in the intervals $[\alpha_{k+1}, \alpha_k]$ for $k = 0, 1, \dots$. Put $\lambda(0) = 0$. Clearly $t^{1-n}\lambda(t) \rightarrow 0$ as $t \rightarrow 0$. Also, if $\alpha_{k+1} \leq t \leq \alpha_k$, and $x \in K$, then

$$\frac{\hat{\mu}(B(x, t))}{\lambda(t)} \leq 2^{1-k}. \quad (53)$$

Given m a positive integer we note from (50) that there is a covering $\{B(x_j, r_j)\}$ of K with $r_j \leq \alpha_m/2$ for all j and

$$\sum_j \lambda(2r_j) \leq 1$$

We may assume that there is an $x'_j \in K \cap B(x_j, r_j)$ for each j since otherwise we discard $B(x_j, r_j)$. Moreover from (53) we see that

$$\hat{\mu}(K) \leq \sum_j \hat{\mu}(B(x'_j, 2r_j)) \leq 2^{1-m} \sum_j \lambda(2r_j) \leq 2^{1-m}.$$

Since m is arbitrary we have reached a contradiction to $\hat{\mu}(K) > 0$ in (52). From this contradiction we conclude first (51) and second Theorem 1. \square

To prove that P has σ finite H^{n-1} measure we once again may assume $\hat{\mu}(\partial O \cap B(\hat{z}, \hat{\rho})) < \infty$. Let

$$P_m = \{x \in P : \limsup_{t \rightarrow 0} t^{1-n} \hat{\mu}(B(x, t)) > \frac{1}{m}\}$$

for $m = 1, 2, \dots$. Given $\delta > 0$ we choose a Besicovitch covering $\{B(y_i, r_i)\}$ of P_m with $y_i \in P_m$, $r_i \leq \delta$, $B(y_i, r_i) \subset B(\hat{z}, \rho)$ and

$$\mu(B(y_i, r_i)) > \frac{r_i^{n-1}}{m}.$$

Thus

$$\sum_i r_i^{n-1} < m \sum_i \hat{\mu}(B(x_i, r_i)) \leq c m \hat{\mu}(\partial O \cap B(\hat{z}, \rho)) < \infty. \quad (54)$$

Letting $\delta \rightarrow 0$ and using the definition of H^{n-1} measure we conclude from (54) that $H^{n-1}(P_m) < \infty$. Hence P has σ finite H^{n-1} measure.

4 Closing Remarks

The existence of a measure, say μ , corresponding to a positive weak solution u in $O \cap B(\hat{z}, r)$ with vanishing boundary values, as in (2), can be shown for a large class of divergence form partial differential equations. What can be said about H-dim μ ? What can be said about analogues of Theorems 1, 2? Regarding these questions we note that Akman in [1] has considered PDE's whose Euler equations arise from minimization problems with integrands involving $f(\nabla v)$ and $v \in W^{1,p}$. More specifically for fixed $p, 1 < p < \infty$, the function $f : \mathbb{R}^2 \setminus \{0\} \rightarrow (0, \infty)$, is homogeneous of degree p on \mathbb{R}^2 . That is,

$$f(\eta) = |\eta|^p f\left(\frac{\eta}{|\eta|}\right) > 0 \text{ when } \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{0\}.$$

Also $\nabla f = (f_{\eta_1}, f_{\eta_2})$ is δ monotone on \mathbb{R}^2 for some $\delta > 0$ (see [3] for a definition of δ monotone). In [1], Akman considers weak solutions to the Euler-Lagrange equation,

$$\sum_{k=1}^2 \frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial \eta_k}(\nabla u(x)) \right) = 0 \text{ when } x = (x_1, x_2) \in \Omega \cap N, \quad (55)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain and N is a neighborhood of $\partial\Omega$. Assume also that $u > 0$ is continuous in N with $u \equiv 0$ in $N \setminus \Omega$. Under these assumptions it follows that there exists a unique finite positive Borel measure μ with support in $\partial\Omega$ satisfying

$$\int_{\mathbb{R}^2} \langle \nabla f(\nabla u), \nabla \phi \rangle dA = - \int_{\partial\Omega} \phi d\mu$$

whenever $\phi \in C_0^\infty(N)$. He proves

Theorem 4 *Let p, f, Ω, N, u, μ be as above and put*

$$\lambda(r) = r \exp \left[A \sqrt{\log \frac{1}{r} \log \log \frac{1}{r}} \right] \text{ for } 0 < r < 10^{-6}.$$

- (a) *If $p \geq 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of σ -finite H^λ Hausdorff measure.*
- (b) *If $1 < p \leq 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^λ Hausdorff measure.*

For $p = 2$ and $f(\eta) = |\eta|^p$ the above theorem is slightly weaker than Theorem 2. It is easily seen that Theorem 4 implies

$$\text{H-dim } \mu \leq 1 \text{ for } p \geq 2 \text{ and H-dim } \mu \geq 1 \text{ for } 1 < p \leq 2.$$

A key argument in the proof of Theorem 4 involves showing that $\zeta = \log f(\nabla u)$ is a weak subsolution, supersolution or solution to

$$L\zeta(x) = \sum_{k,j=1}^2 \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial \zeta(x)}{\partial x_j} \right) \text{ when } x \in \Omega \cap N$$

and $p > 2, 1 < p < 2, p = 2$, respectively. In [2] this was shown pointwise at $x \in \Omega \cap N$ when $\nabla u, f$, are sufficiently smooth and $\nabla u(x) \neq 0$. We plan to use this fact and the technique in Theorem 4 to prove analogues of Theorem 4 when $n = 2$ and also higher dimensional analogues. The case $p = n$ in Theorem 1 and $p = 2$ in the proposed generalization of Theorem 4 are particularly interesting. Can one for example do away with the uniform fatness assumption in Theorem 1 or the proposed generalization of Theorem 4 when $p = 2, n = 2$? The argument in [23] and [10] relies on a certain integral inequality (see Lemma 3.1 in [10]).

References

1. Akman, M.: On the dimension of a certain measure in the plane. arXiv:1301.5860 (submitted)
2. Akman, M., Lewis, J., Vogel, A.: On the logarithm of the minimizing integrand for certain variational problems in two dimensions. *Analysis and Mathematical Physics* **2**(1), 79–88 (2012)
3. Astala, K., Iwaniec, T., Martin, G.: Elliptic partial differential equations and quasiconformal mappings in the plane, *Princeton Mathematical Series*, vol. 48. Princeton University Press, Princeton, NJ (2009)
4. Bennewitz, B., Lewis, J.: On the dimension of p -harmonic measure. *Ann. Acad. Sci. Fenn. Math.* **30**(2), 459–505 (2005)
5. Bourgain, J.: On the Hausdorff dimension of harmonic measure in higher dimension. *Inventiones Mathematicae* **87**, 477–483 (1987)
6. Carleson, L.: On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.* **10**, 113–123 (1985)
7. Evans, L.C., Gariepy, R.F.: *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992)
8. Garnett, J.B., Marshall, D.E.: *Harmonic Measure, New Mathematical Monographs*, vol. 2. Cambridge University Press, Cambridge (2008)
9. Heinonen, J., Kilpeläinen, T., Martio, O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Dover Publications Inc. (2006)
10. Jones, P.W., Wolff, T.: Hausdorff dimension of harmonic measures in the plane. *Acta Math.* **161**(1-2), 131–144 (1988)
11. Kilpeläinen, T., Zhong, X.: Growth of entire A subharmonic functions. *Ann. Acad. Sci. Fenn. Math.* **28**, 181–192 (2003)
12. Lewis, J.: On a conditional theorem of Littlewood for quasiregular entire functions. *J. Anal. Math.* **62**, 169–198 (1994)
13. Lewis, J.: Note on p -harmonic measure. *Comput. Methods Funct. Theory* **6**(1), 109–144 (2006)
14. Lewis, J.: p -harmonic measure in simply connected domains revisited. *Transactions of the American Mathematical Society* (To appear)
15. Lewis, J., Nyström, K., Poggi-Corradini, P.: p -harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble* **61**(2), 689–715 (2011)
16. Lewis, J., Nyström, K., Vogel, A.: p -harmonic measure in space. *JEMS* (To appear)
17. Lewis, J., Verchota, G.C., Vogel, A.: On Wolff snowflakes. *Pacific Journal of Mathematics* **218**(1), 139–166 (2005)

-
18. Makarov, N.G.: On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.* (3) **51**(2), 369–384 (1985)
 19. Makarov, N.G.: Probability methods in conformal mappings. *Leningrad Math. J* **1**, 1–56 (1990)
 20. Mattila, P.: *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press (1995)
 21. Pommerenke, C.: *Boundary Behaviour of Conformal Maps*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag (1992)
 22. Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. *Journal of Differential Equations* **51**, 126–150 (1984)
 23. Wolff, T.: Plane harmonic measures live on sets of σ -finite length. *Ark. Mat.* **31**(1), 137–172 (1993)
 24. Wolff, T.: Counterexamples with harmonic gradients in \mathbf{R}^3 . In: *Essays on Fourier analysis in honor of Elias M. Stein* (Princeton, NJ, 1991), *Princeton Math. Ser.*, vol. 42, pp. 321–384. Princeton Univ. Press, Princeton, NJ (1995)