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Lie Symmetry Analysis of a Fractional Black-Scholes Equation

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Abstract. In 2000, Walter Wyss looked into the fractional version of the Black-Scholes equation for the first time. He gave a solution of the fractional Black-Scholes equation by using the Greens function [14]. In this paper, Lie symmetry analysis of a time fractional Black-Scholes equation with Riemann-Liouville derivative is performed. The operators admitted are obtained and finally an example of the invariant solution of the equation is discussed.

1. INTRODUCTION

The Black-Scholes equation is given by

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \tag{1}$$

where $u = u(t, x)$ is the price of a derivative, x is the price of a stock, σ is the constant volatility of the underlying asset, r is the constant risk-free interest rate and t is time in years, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$ and $u_{xx} = \partial^2 u/\partial x^2$, was first introduced by Fischer Black and Myron Scholes in 1973 [1]. Robert C. Merton then published his paper [2] expanding the mathematical understanding of the option pricing model. The main idea of the model is to hedge the option by buying and selling the underlying asset in the right moment to minimize risk.

The story of fractional calculus started on the day when Leibniz wrote a letter to LHopital in 1695. The idea of a derivative of order one half was brought out. His question was left unanswered for the next hundred years until Lacroix presented the first definition of fractional derivative based on the usual expression for the nth derivative of the power function in 1819. Fractional calculus had finally gained the attention it deserved when numerous fractional differential operators were introduced by the Grunwald-Letnikow, Riemann-Liouville, Hadamard, Caputo, Riesz.

In 2000, Walter Wyss gave a complete solution of a fractional Black-Scholes equation by using the Green's function [14]. In his work, he replaced the first derivative in time by a fractional derivative in time of order α, $0 < \alpha \leq 1$. His proof was solely done on the basis of mathematical content.

Lie group analysis was named after Sophus Lie, a Norwegian mathematician. It is one of the most powerful tool to find the analytical solution of partial or ordinary differential equation [3,4]. The application of Lie group analysis to fractional calculus, on the other hand, is relatively underrated. Gazizov et al, in 2007, formulated the prolongation formulae for fractional derivative[5]. Huang and Zhdanov then gave an explicit form of the finding in [5] in their paper later in 2014 [6]. The methods of Lie have been applied to linear and nonlinear partial differential equation with some success, for example see [11,12,13].

In this paper, we performed the Lie symmetry analysis upon the time fractional Black-Scholes equation

$$D_t^{\alpha} u + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \tag{2}$$

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where $0 < \alpha < 1$ and $D_t^\alpha u$ is the Riemann-Liouville fractional derivative of order α with respect to time, t.

The paper is organized as follows. In section 2, the definition and the elements of Lie symmetry analysis of fractional differential equations are discussed. In section 3, we determine the operators admitted by the equation (2) and finally the group-invariant solution is constructed in section 4. Section 5 contains concluding discussion.

2. PRELIMINARIES

Among all the fractional differential operators that were introduced, the Riemann-Liouville fractional partial derivative is most recognized and applied. Riemann and Liouville defined

$$D_t^\alpha u(t,x) = \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial t^m}\int_0^t \frac{u(\xi,x)}{(t-\xi)^{\alpha+1-m}}d\xi, \tag{3}$$

where $0 < m-1 < \alpha \leq m, m \in N$. $D_t^\alpha u(t,x)$ is the partial derivative of the function $u(t,x)$ respect to t of order α. Here $\Gamma(\alpha)$ denotes the Euler's Gamma function.

Consider a fractional differential equation (FDE) of the form

$$D_t^\alpha u = F(t,x,u,u_x,u_{xx}), \quad (0 < \alpha < 1). \tag{4}$$

Recall the point transformations of variables u, t, and x:

$$\bar{u} = f(u,t,x;\epsilon), \quad \bar{t} = g(u,t,x;\epsilon), \quad \bar{x} = h(u,t,x;\epsilon)$$

with a continuous parameter ϵ are said to be symmetry transformations of equation (4) if they satisfy the initial condition

$$\bar{u}|_{\epsilon=0} = u, \quad \bar{t}|_{\epsilon=0} = t, \quad \bar{x}|_{\epsilon=0} = x,$$

and leave the equation (4) invariant after the transformation.

The transformations generate a continuous group G that has the identity, inverse and the compositions of every transformations. Generating the symmetry group G is equivalent to determining its infinitesimal transformations

$$\bar{u} = u + \epsilon\eta(u,t,x) + o(\epsilon^2), \quad \bar{t} = t + \epsilon\tau(u,t,x) + o(\epsilon^2), \quad \bar{x} = x + \epsilon\xi(u,t,x) + o(\epsilon^2). \tag{5}$$

The symmetry group G is known as the group admitted by equation (4). The generator of the group G, which is also known as the infinitesimal operator of the group G, is introduced as

$$X = \eta(u,t,x)\partial_u + \tau(u,t,x)\partial_t + \xi(u,t,x)\partial_x, \tag{6}$$

where

$$\eta = \left.\frac{d\bar{u}}{d\epsilon}\right|_{\epsilon=0}, \quad \tau = \left.\frac{d\bar{t}}{d\epsilon}\right|_{\epsilon=0}, \quad \xi = \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0}.$$

When the infinitesimal transformations are applied to partial derivatives u_x, u_{xx}, equations (5) are extended to the derivative of u respect to x by some classical theory [7]:

$$\begin{aligned} \bar{u}_{\bar{x}}(\bar{t},\bar{x}) &= u_x(t,x) + \epsilon\eta^x + o(\epsilon^2), \\ \bar{u}_{\bar{x}\bar{x}}(\bar{t},\bar{x}) &= u_{xx}(t,x) + \epsilon\eta^{xx} + o(\epsilon^2), \end{aligned} \tag{7}$$

where η^x and η^{xx} are given by the prolongation formulae:

$$\begin{aligned} \eta^x &= D_x\eta - u_xD_x\xi - u_tD_x\tau, \\ \eta^{xx} &= D_x\eta^x - u_{xx}D_x\xi - u_{xt}D_x\tau. \end{aligned} \tag{8}$$

Here $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \ldots$ is the total derivative.

Gazizov et al in [5] examined the transformation of the form (5) that conserve the structure of the fractional derivative operator (3) and the infinitesimal transformation of fractional derivative is introduced:

$$D_{\bar{t}}^\alpha \bar{u}(\bar{t},\bar{x}) = D_t^\alpha u(t,x) + \epsilon\eta^\alpha + o(\epsilon^2), \tag{9}$$

where η^α is the prolongation formula:

$$\eta^\alpha = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u). \tag{10}$$

Huang and Zhdanov later gave the explicit form of (10) in their work [6] as

$$\eta^\alpha = \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\partial_t^{\alpha-n}(u_x) + \partial_t^\alpha\eta + (\eta_u - \alpha D_t(\tau))\partial_t^\alpha u - u\partial_t^\alpha\eta_u + \mu \tag{11}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\binom{\alpha}{n}\binom{n}{m}\frac{t^{n-\alpha}U_k}{k!\Gamma(n+1-\alpha)}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Here the infinitesimal η is linear in the variable u and $\mu = 0$ because $\dfrac{\partial^k\eta}{\partial u^k} = 0$ for $k \geq 2$. Using the above results, we derive the Lie symmetries admitted by the equation (2) in the next section.

3. LIE SYMMETRIES OF A FRACTIONAL BLACK-SCHOLES EQUATION

Recall the time fractional BS equation

$$D_t^\alpha u + \frac{1}{2}\sigma^2x^2u_{xx} + rxu_x - ru = 0$$

where $0 < \alpha < 1, \sigma$ and r are two different scalars that represent volatility and interest rate respectively. The invariance condition of the equation (2) is

$$\eta^\alpha + \frac{1}{2}\sigma^2x^2\eta^{xx} + \sigma^2xu_{xx}\xi + rx\eta^x + ru_x\xi - r\eta = 0. \tag{12}$$

Substituting the derivatives η^x, η^{xx}, which can be obtained in many text books, say [8], and η^α from (11) into the equation (12) and equating the coefficients of of $u_t, u_x, u_{xx}, u_{xt}, u_tu_{xx}, u_xuq_{xx}, u_x^2, \partial_t^{\alpha-n}u$ and $\partial_t^{\alpha-n}u_x$ to zero gives the following system of determining equations:

$$\begin{aligned}
\frac{1}{2}\sigma^2x^2\tau_{xx} + rx\tau_x &= 0,\\
\sigma^2x^2\eta_{xu} - \frac{1}{2}\sigma^2x^2\xi_{xx} + r\alpha x\tau_t - rx\xi_x + r\xi &= 0,\\
\frac{1}{2}\alpha x\tau_t - x\xi_x + \xi &= 0,\\
\tau_x = \tau_u = \xi_u = \xi_t &= 0\\
\frac{1}{2}\sigma^2x^2\eta_{uu} - \sigma^2x^2\xi_{xu} - rx\xi_u &= 0,\\
\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau) &= 0, \quad n = 1, 2, 3, \ldots.
\end{aligned}$$

The determining equations above are easily integrated to yield the following general solution

$$\eta = \left(\frac{c_1\alpha}{2\sigma^2}\left(\frac{1}{2}\sigma^2 - r\right)\ln x + c_3\right)u + B(x,t), \quad \xi = \frac{1}{2}c_1\alpha x\ln x + c_2x, \quad \tau = c_1t + c_4.$$

The transformations of variables should retain the structure of the Riemann-Liouville fractional derivative operator, thus

$$\tau(u, t, x)|_{t=0} = 0.$$

That is, $\tau = c_1t$. Hence, the symmetry group of the fractional Black-Scholes equation is spanned by the vector fields

$$X_1 = t\frac{\partial}{\partial t} + \frac{1}{2}\alpha x\ln x\frac{\partial}{\partial x} + \frac{\alpha}{2\sigma^2}\left(\frac{1}{2}\sigma^2 - r\right)u\ln x\frac{\partial}{\partial u}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_\infty = B(x,t)\frac{\partial}{\partial u}.$$

4. GROUP INVARIANT SOLUTIONS

The group invariant solutions of fractional differential equations is defined similairly as the partial differential equations. The function $u = u(x, t)$ is an invariant solution of a fractional differential equation corresponding to its infinitesimal operator (6) if and only if it fulfils the invariant surface condition

$$\tau(u, t, x)u_t + \xi(u, t, x)u_x = \eta(u, t, x).$$

Suppose ξ and τ are not both zero, then the above invariant surface condition can be solved by the method of characteristics:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{13}$$

If $p(u, t, x)$ and $q(u, t, x)$ (with $q_u \neq 0$) are two functionally independent first integral of (13), the general solution of the invariance surface condition is

$$q = F(p). \tag{14}$$

This solution is now substituted into the fractional differential equation (4) to determine the function of F.

Example

Consider the infinitesimal operator X_2 we obtained earlier, the characteristic equations are

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0}. \tag{15}$$

The equations (15) give the similarity variables t and u. It is more convenient if we write it as $u = g(t)$. Inserting it into the equation (2) yields the following fractional ordinary differential equation

$$D_t^\alpha g(t) = rg(t). \tag{16}$$

Using Laplace transform [9], the solution of equation (16) is

$$u(x, t) = g(t) = k_1 t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha), \tag{17}$$

where $k_1 = D^{-(1-\alpha)}g(0)$ and $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$ is the Mittag-Leffler function [10].

Here we should mention the solution (17) that we suggested in the example is a one-variable function in t. This solution is true in the mathematical perspective. However, the lack of variable x in the solution is trivial in the financial point of view. The comparison of the equation (17) with the classical solution of the Black-Scholes equation is superflous. The diagrams below show the differences between the classical solution of Black-Scholes and our solution (17).

Figure 1 shows the graphs of price of a call and put option against time to expirity using $S_0 = E = 30, r = 0.1, \sigma = 0.2$, where S_0 is the initial stock price and E is the strike price in the classical Black-Scholes equation. Figure 1 also shows the graphs of our suggested solution (17), using the same parameters, with $\alpha = 0.5$ and $\alpha = 0.8$ and $k_1 = 1$.

The inifinitesimal operator X_1 will generate a much more complicated invariant solution. We are still working on it and the result will be reported later somewhere.

5. CONCLUSION

In this paper, we performed a Lie group analysis on a fractional Black-Scholes equation. We gave the infinitesimal operators and a solution. Another solution admitted by the second generator will be studied later.

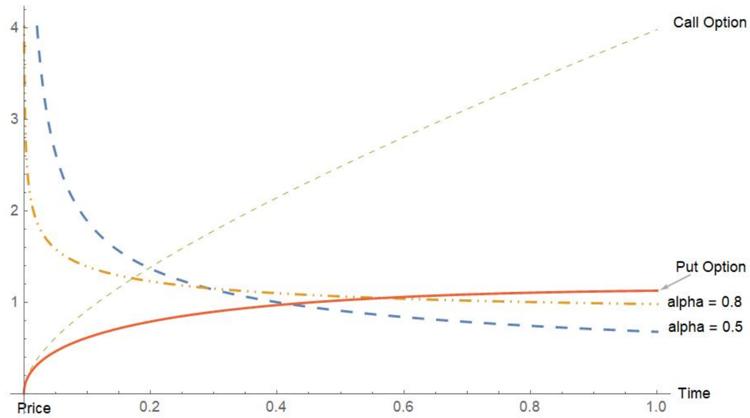

FIGURE 1. Price of a put option, call option, our suggestions with $\alpha = 0.8$ and $\alpha = 0.5$ against time to expirity using $S_0 = E = 30, r = 0.1, \sigma = 0.2$.

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