Enhanced Linguistic Computational Models and Their similarity with Yager's Computing with Words

Prashant K. Gupta[1], Deepak Sharma[1], Javier Andreu-Perez*[1,2]

Highlights

- A formal discussion about the links and similarities between Yager's Computing with Words (CWW) framework, the linguistic computational models based on the extension principle and, the symbolic method.
- An augmented version of extension principle based linguistic computational model is proposed to solve the problem of unequally weighted linguistic information.
- Two new CWW methodologies are proposed: intuitionistic fuzzy sets based CWW methodology and rough sets based CWW methodology.
- Step-by-step numerical examples are provided for each proposed method to allow reproducibility of the presented methodologies.

Abstract- A generalized computational framework for Computing with Words (CWW) using linguistic information (LI) was proposed by Prof. Yager. This framework is based on three steps: translation, manipulation and retranslation. Other works have independently proposed the Linguistic Computational Model (LCM) to express the semantics of LI using Type-1 Fuzzy Sets and Ordinal term sets. The former is called the extension principle, and the latter, the symbolic method. We found that a high degree of similarity can be drawn between these methodologies and Yager's CWW framework, but no discussion exists in the literature of the similarity drawn between them. Further, the extension principle has a drawback: it considers LI to be equally weighted in the aggregation phase. Also, Intuitionistic fuzzy sets (IFSs) and rough sets have gained popularity to model semantics of LI, but no CWW methodologies have been proposed using them. Thus, the novel contributions of this work are twofold. Firstly, showing the similarity of the linguistic computational models based on extension principle and symbolic method, to the Yager's generalized CWW framework. Secondly, proposing a new augmented flexible weighting for LCM based on the extension principle and two novel CWW methodologies based on IFS and rough sets.

Keywords- Computing with Words, Extension principle, Intuitionistic Fuzzy Sets, Rough Sets, Symbolic method, Type-1 Fuzzy Sets.

1. Introduction

Computing with Words (CWW) [5-8], [13], [20-22], [30], was conceptualized by Prof. Zadeh in his seminal work [41]. According to Prof. Zadeh, CWW [41], [43], [45] is a novel data processing methodology that aims to impart the machines with a capability, such that machines can process the linguistic information (LI) seamlessly like human beings do. LI is subjective in nature as "words can mean different things to different people" [13], [15], [19], [23-26], [32-36], [45]. Perhaps the subjectivity of the LI is a reason for its imprecision and vagueness. However, humans communicate, process, and make decisions seamlessly using the LI. This rational process is possible because human cognition has a tolerance for imprecision and

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[1]Prashant K. Gupta, Deepak Sharma, Javier Andreu-Perez are with the Institute for Advancing Artificial Intelligence (I4AAI), Colchester, CO2 7GT, United Kingdom. e-mail: {*prashant.gupta, deepak.sharma, andreu-perez*}*@i4aai.org*
[2]School of Computer Science and Electronic Engineering, University of Essex, United Kingdom {*javier.andreu@essex.ac.uk*}.
*Corresponding author: Javier Andreu-Perez {*andreu-perez@i4aai.org*}

vagueness [42], [44]. Prof. Zadeh premised that the use of CWW is justified in situations where there is tolerance for imprecision and the use of LI, instead of numbers, is suitable in the circumstances.

Motivated by Prof. Zadeh's works on CWW, Prof. Yager proposed a generalized framework (for the CWW) in [37]. This generalized framework is shown in Fig. 1. It consists of three steps viz., translation, manipulation, and retranslation. In translation, the LI is converted to numeric form. In manipulation, the numeric information is aggregated, and in the retranslation step, the aggregated numeric information from the previous step is converted back to linguistic form. The input to Yager's generalized framework is the LI (as humans naturally express themselves linguistically). Further, humans also naturally understand the LI. Therefore, the output from this Yager's generalized framework is also the LI.

Apart from this Yager's work [37], several literary works have been published which propose methodologies to manipulate the LI. These methodologies represent the semantics of LI using T1 Fuzzy Sets (FSs) [39], [40] and Ordinal term sets [12]. Further, the methodology based on T1 FSs and ordinal term sets are called the *extension principle based linguistic computational model* (EPLCM) [12] and *symbolic method based linguistic computational model* (SMLCM) [12], respectively.

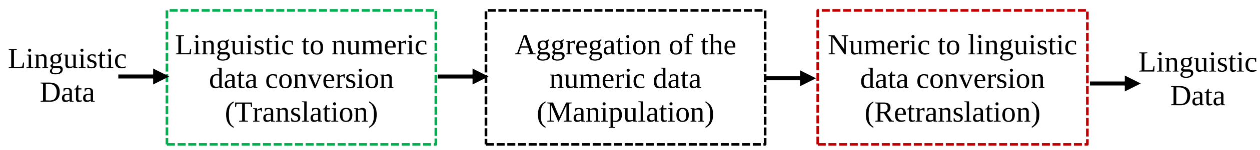

Fig 1. Computing with words methodology [37]

Apart from this Yager's work [37], a number of different literary works have been published which propose methodologies to manipulate the LI. These methodologies represent the semantics of LI using Type-1(T1) Fuzzy Sets (FSs) [39], [40] and Ordinal term sets [12]. The methodology based on T1 FSs and ordinal term sets is respectively called the *extension principle based linguistic computational model* (EPLCM) [12] and *symbolic method based linguistic computational model* (SMLCM) [12].

Through the study of the EPLCM and SMLCM, we found that a high degree of similarity can be established between these models and Yager's generalized CWW framework. By similarity, both linguistic computational models process the LI in three steps. In Step 1, a mapping is performed from the linguistic information to numeric. Step 2 involves aggregating the numeric information received from Step 1. Finally, the last step consists of converting the aggregated numeric information from Step 2 to the linguistic form. Thus, the three steps of the EPLCM and the SMLCM process the LI similar to the respective translation, manipulation and retranslation steps of Yager's generalized CWW framework. Thus, it can be inferred that both the EPLCM and SMLCM use the CWW as an underlying methodology to deal with the LI. However, no proof exists in the literature where an attempt has been made to draw a similarity between these two linguistic computational models and Yager's CWW framework. Thus, we have presented the similarity of the EPLCM and the SMLCM to the Yager's generalized CWW framework in this paper (a detailed discussion to draw the similarity of the EPLCM and the SMLCM to the Yager's generalized framework is given in Section 5).

Also, a limitation of the EPLCM is that it does not aggregate the LI, which is differentially weighted. Equally weighted LI may not always be present in an application domain. For example, in [46-48], authors have proposed methodologies to deal with differentially weighted LI through the use of unbalanced linguistic term sets (ULTS). In these works, either the nature of linguistic variables or higher information granularity justifies ULTS use. Information granularity has a direct relation to the amount of information conveyed by the linguistic term. However, certain situations demand that, though the information conveyed by all the linguistic terms is the same, some linguistic terms carry more importance than others. To exemplify, consider the investment judgement advisor presented in [25]. Here, it was considered that an investor had a moderate amount of capital to invest in a certain number of alternatives based on specific criteria. It could be assumed that the investment choice depended on the trade-off between the amount of profit to be earned and the risk of losing capital, one measure of which could be the amount of importance attached to the investment criteria. Thus, for modelling such situations, we need differentially weighted LI, and hence we propose an augmented LCM based on the extension principle that aggregates differentially weighted LI. Further discussions and important findings on the augmented EPLCM are given in Section 5.

Recently, novel uncertainty modelling concepts like intuitionistic fuzzy sets (IFSs) [1] and rough sets [29], have been proposed. However, no CWW methodologies have been proposed based on these concepts. There are various research problems or scenarios where the T1 FS based CWW methodology falls short in modelling the semantics of LI. In such cases, we can use the IFS or rough set based CWW methodology. We have explained these in greater detail in Section 5.

Therefore, all these above factors: 1) the non-existence of literary works on the similarity of the EPLCM and SMLCM to the Yager's CWW framework; 2) the inability of the EPLCM to process the differentially weighted LI; and 3) the absence of CWW methodologies on IFS and rough sets, motivated us to improve these shortcomings through the scientific contributions presented in this paper. Thus, the contributions of this work are threefold:

1) showing the similarity of the linguistic computational models based on extension principle and symbolic method, to the Yager's generalized CWW framework;
2) proposing an augmented linguistic computational model based on the extension principle that has the capability to compute using the differentially weighted LI and;
3) presenting novel CWW methodologies based on IFS and rough sets.

It is pertinent to mention that CWW methodology based on IFS is motivated by EPLCM. Further, the CWW methodology based on rough sets is inspired by the SMLCM.

The rest of the paper is organized as follows: Section II discusses the details of the EPLCM as well as SMLCM and shows their similarity to Yager's generalized CWW framework; Section III proposes an augmented version of the EPLCM to process unequally weighted LI and linguistic computational models based on IFS and rough sets; Section IV illustrates their working using a suitable example as well as experimental results; Section V provides essential discussions on the research work presented in this paper and finally, Section VI concludes this work as well as throws some light on its future scope.

2. Linguistic Computational Models based on Extension Principle and Symbolic Method

In this section, we discuss the detailed step by step methodology adopted by the EPLCM and SMLCM to process the LI. We will also show the similarity of these respective steps to those of translation, manipulation, and retranslation steps with respect to Yager's generalized CWW framework (please see Fig. 1). These similarities are listed in Table I.

The EPLCM or SMLCM has an associated linguistic term set containing a collection of linguistic terms or labels. Let the term set be denoted as T, and contains $g+1$ linguistic terms represented as:

$$T = \{t_0, t_1, \dots t_g\} \tag{1}$$

Here $t_0, t_1, \dots t_g$ are the linguistic terms.

The EPLCM represents the semantics of these linguistic terms using the T1 FSs. The linguistic terms $t_0, t_1, \dots t_g$ are assumed to be distributed equally on an information representation scale with bounds p and q, and represented as uniform triangular T1 FSs, called the T1 membership functions (MFs). This is shown in Fig. 2. Generally used values of the information representation scale bounds are $p = 0$ and $q = 10$.

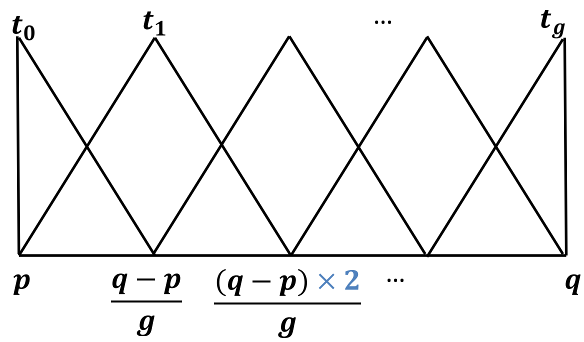

Fig 2. Individual terms of the term set represented on the scale [14], [16], [28]

The SMLCM, on the other hand, represents the semantics of the linguistic terms, constituting the term set T in Eq. (1), using the ordinal term sets. By ordinal term sets, we mean that there is a semantic order in the indices of the terms. The SMLCM can process the differentially weighted LI. However, an improvement for such a scenario is proposed for the EPLCM in Section 3.

TABLE I
SIMILARITIES OF YAGER'S GENERALIZED CWW FRAMEWORK'S STEPS TO EPLCM[a] AND SMLCM[a]

Steps of Yager's Generalized framework	EPLCM	SMLCM
Translation: *Convert linguistic information to numeric form*	Using tri-tuple representation of T1 FS membership functions	Using indices corresponding to linguistic terms
Manipulation: *Aggregate the numeric information*	Arithmetic mean of tri-tuples	Recursive combination of indices
Retranslation: *Convert numeric information to linguistic form*	Linguistic approximation to closest linguistic term	Index of the recommended linguistic term

[a] EPLCM= Extension principle based linguistic computational model, SMLCM= Symbolic method based linguistic computational model

2.1. Linguistic Computational model based on Extension Principle [10], [12]

Let us consider a scenario pertaining to feedback collection from i number of subjects, using linguistic terms taken from term set T of Eq. (1). Let the collective preference vector related to the feedbacks of these users be given as:

$$\{t_{j1}, t_{j2}, \dots t_{ji}\} \tag{2}$$

where $j = 0 \ to \ g$ and each $t_{jk} \in T; k = 1 \ to \ i$.

2.1.1. Translation

In EPLCM, the semantics of these linguistic terms are represented by T1 FSs (Please see Fig. 2). For the translation step, each of these linguistic terms given in Eq. (2), are represented in the tri-tuple form as (l_{ji}, m_{ji}, r_{ji}). Each value in the tuple corresponds to the left, middle and right end, respectively, of the triangular T1 MF. Therefore, the modified collective preference vector of Eq. (2) is given in Eq. (3) as:

$$\{(l_{j1}, m_{j1}, r_{j1}), (l_{j2}, m_{j2}, r_{j2}), \ldots (l_{ji}, m_{ji}, r_{ji})\} \tag{3}$$

where $j = 0\ to\ g$.

2.1.2. Manipulation

For the manipulation, the left, middle and right ends of the tri-tuples from Eq. (3) are aggregated by performing calculations as shown in Eq. (4), to obtain the collective preference vector C shown in Eq. (5) as:

$$l_c = \frac{l_{j1} + l_{j2} + \cdots + l_{ji}}{i}, \qquad m_c = \frac{m_{j1} + m_{j2} + \cdots + m_{ji}}{i}, \qquad r_c = \frac{r_{j1} + r_{j2} + \cdots + r_{ji}}{i} \tag{4}$$

$$C = (l_c, m_c, r_c) = \left(\frac{1}{i}\sum_{k=1}^{i} l_{jk}, \frac{1}{i}\sum_{k=1}^{i} m_{jk}, \frac{1}{i}\sum_{k=1}^{i} r_{jk}\right) \tag{5}$$

2.1.3. Retranslation

The collective preference vector in Eq. (5) is also a tri-tuple, and it generally does not match any linguistic terms in the term set T of Eq. (1). Thus, to generate a linguistic output for the collective preference vector given in Eq. (5), the Euclidian distance of this collective preference vector is calculated from the tri-tuples corresponding to each linguistic term in T of Eq. (1). Note, in this work; we adopt Euclidean distance [12], which is a symmetrical metric of equivalence. However, the choice of equivalence depends on the CWW problem to model, and more meticulous measures of equivalence between terms could be used, for example, based on support or cardinality.

Let each linguistic terms in T of Eq. (1) is represented in the tri-tuple form as $t_p = (l_q, m_q, r_q), q = 0\ to\ g$. Then the term with maximum similarity or minimum distance is recommended as the linguistic term corresponding to the collective preference vector, $C = (l_c, m_c, r_c)$, of Eq. (5), as:

$$d(t_q, C) = \sqrt{P_1(l_q - l_c)^2 + P_2(m_q - m_c)^2 + P_3(r_q - r_c)^2} \tag{6}$$

where the $P_i, i = 1,2,3$ are the weights, with values 0.2, 0.6 and 0.2 respectively. We have used the respective values 0.2, 0.6 and 0.2 of $P_i's, i = 1,2,3$, to emphasize a fact, stated in [12], that the centre value of tri-tuple from a triangular MF has more importance than the two other ends. To change this notion, different values for respective $P_i's$ can be used. Thus, the recommended linguistic term from Eq. (6) is $t_b^* \in T$, such that $d(t_b^*, C) \leq d(t_q, C), \forall t_q \in T$.

2.2. The linguistic computational model based on the Symbolic method [10], [12]

The linguistic computational model based on the Symbolic method uses ordinal sets to represent the semantics of LI. Further, each of these pieces of LI may have an associated weight.

2.2.1. Translation

A linguistic preference vector containing the user preferences is the starting point of the symbolic method based linguistic computational model. Consider the linguistic preference set from Eq. (2). Each of these linguistic preferences may have an associated weight, given in the form of a weight vector as:

$$W = [w_1, \dots, w_i] \tag{7}$$

where each $w_p \in [0,1]; p = 1\ to\ i$, is the associated weight of the respective linguistic preference t_{jk} given in Eq. (2). Further, a condition exists on all the w_p that $\sum_{p=1}^{i} w_p = 1$.

2.2.2. Manipulation

In this step, the linguistic preference vector from Eq. (2), is aggregated according to the weight vector given in Eq. (7). Initially, the linguistic preference vector from Eq. (2), is ordered according to the indices of the linguistic terms drawn from T of Eq. (1). Thus, after ordering, the linguistic preference vector may be given as:

$$\{T_{j1}, T_{j2}, \dots T_{ji}\} \tag{8}$$

where $T_{jk} \in T, k = 1, \dots, i$. Further, each of the T_{jk} may or may not be equal to respective t_{jk}. The linguistic preference vector from Eq. (8), are aggregated recursively, using the recursive function (AG^i), by performing the computations shown in Eq. (9) as:

$$AG^i \left\{w_p, I_{T_{jp}}, p = 1, \dots, i | i > 2, i \in \mathbb{Z}\right\} = \left(w_1 \odot I_{T_{j1}}\right) \oplus \left((1 - w_1) \odot AG^{i-1}\left\{\delta_h, I_{T_{jh}}, h = 2, \dots, i\right\}\right) \tag{9}$$

where $I_{T_{jp}}, p = 1, \dots, i$, $I_{T_{jh}}, h = 2, \dots, i$ are the indices[3] of the linguistic terms given in Eq. (8) and $\delta_h = {w_h}/{\sum_{l=2}^{i} w_l}; h = 2, 3, \dots, i$. It is pertinent to mention that Eq. (9) is used for aggregating the linguistic preference vector from Eq. (8), as long as the number of terms to be aggregated are more than 2. For the number of terms reduced to 2, the boundary condition is met. It is shown in Eq. (10) as:

$$AG^2 \left\{\{w_{i-1}, w_i\}, \left\{I_{T_{ji-1}}, I_{T_{ji}}\right\} | i = 2\right\} = (w_{i-1} \odot I_{T_{ji-1}}) \oplus (w_i \odot I_{T_{ji}}) \tag{10}$$

where $I_{T_{ji-1}}$ and $I_{T_{ji}}$ are the respective indices of the last two terms from the preference vector Eq. (8), and w_{i-1} and w_i are the respective weights.

2.2.3. Retranslation

The Eq. (10) recommends the index of a linguistic term. This index can be matched to one of the terms from Eq. (1), to find out the recommended linguistic term. This recommended index value is given as:

$$I_r = min\left\{i, I_{T_{ji}} + round\left(\frac{w_{i-1} - w_i + 1}{2} . \left(I_{T_{ji-1}} - I_{T_{ji}}\right)\right)\right\} \tag{11}$$

here $round()$ is the round function, given as $round\ (x) = \lfloor x + 0.5 \rfloor, x \in \mathbb{R}$, $\lfloor\quad\rfloor$ being the floor function.

Thus, before reaching Eq. (10) from Eq. (9) in a top-down manner, we get a series of $i - 2$ intermediate recursive equations. From Eq. (10), an index of the linguistic term is recommended using Eq. (11). This index

[3] The original equations in [12] use the linguistic labels. However the actual aggregation is performed on the indices of the linguistic labels. So, we have chosen to show directly the indices of the linguistic terms in the Eq. (9)-(10).

is then given as an input to the preceding recursive equation. Thus, the combination process then follows a bottom-up approach. Therefore, at each step of the bottom-up combination process, there will be two indices of the linguistic terms to combine. The index of the recommended linguistic term can be found by performing computations similar to Eq. (11).

Further, it has been observed that in an equation where only two indices need to be combined, similar to Eq. (10), $w_{i-1} = w$ and $w_i = 1 - w$, for some value of the weight w, obtained through weight aggregation in Eq. (9). Thus, putting indices also in a generalized form viz., $I_{T_{ji-1}} = I_l$ and $I_{T_{ji}} = I_q$, Eq. (10) and Eq. (11) can be written in generalized form in Eq. (12) and Eq. (13), respectively as:

$$AG^2\left\{\{w, 1-w\}, \{I_l, I_q\}\right\} = (w \odot I_l) \oplus ((1-w) \odot I_q) \tag{12}$$

$$I_r = min\,\{i, I_q + round(w.(I_l - I_q))\} \tag{13}$$

Finally, the recommended value is a unique index of the term belonging to the term set T of Eq. (1).

3. An Augmented Extension Principle based Linguistic Computational with Flexible Weighting Scheme and New CWW methodologies based on Intuitionistic and Rough Sets

Here, we propose an augmented EPLCM that permits aggregating differentially weighted linguistic terms as well as novel CWW methodologies based on IFS and rough sets. We will also show the similarity of the respective steps of augmented EPLCM and novel CWW methodologies based on IFS and rough sets to those of translation, manipulation and retranslation steps, respectively of Yager's generalized CWW framework, of Fig. 1. These similarities are listed in Table II. Further, we will demonstrate the working of these proposed methodologies using a suitable example in Section 4.

TABLE II
SIMILARITIES OF YAGER'S GENERALIZED CWW FRAMEWORK'S STEPS TO AUGMENTED EPLCM[a], IFS BASED CWW METHODOLOGY AND ROUGH SET BASED CWW METHODOLOGY

Steps of Yager's Generalized framework	Augmented EPLCM	IFS based CWW methodology	Rough Set based CWW methodology
Translation: Convert linguistic information to numeric form	Using tri-tuples of T1 FS MFs of linguistic terms and weights	Using tri-tuples of T1 FS membership and non-membership functions of linguistic terms and weights	Division of linguistic terms into equivalence classes and weight assignment
Manipulation: Aggregate the numeric information	Product of T1 MFs of respective linguistic terms and weights followed by the arithmetic mean of tri-tuples of products	Product of T1 MFs of membership and non-membership functions of respective linguistic terms and weights followed by the arithmetic mean of respective tri-tuples of products	Recursive combination of indices of linguistic terms and respective weights
Retranslation: Convert numeric information to linguistic form	Linguistic approximation to closest linguistic term	Linguistic approximation to the closest linguistic term for membership and non-membership functions	Index of the recommended linguistic term

[a] EPLCM= Extension principle based linguistic computational model

3.1. Proposed Augmented EPLCM Methodology

We now discuss the details of our proposed augmented EPLCM. The augmented EPLCM can process the linguistic preferences, which are differentially weighted. The existing EPLCM (discussed in Section 2.1) aggregates the LI, which is equally weighted, and equally weighted LI may not be present in a number of scenarios [11], [19], [25], [26], [34-36]. Thus, we felt the need to incorporate the differential weighting in the EPLCM and hence propose the augmented EPLCM.

As pointed out in Section 2.1, the EPLCM represents the semantics of LI using T1 MFs, as shown in Fig. 2. For processing, each piece of LI is represented as a tri-tuple $\{l, m, r\}$, where l, m, r correspond respectively to the left, middle and right end of the T1 MFs.

3.1.1. Translation

Consider the scenario of feedback collection from i number of subjects, using linguistic terms taken from term set T of Eq. (1). The collective preference vector about these users' feedback is given in Eq. (2). In the augmented EPLCM, each linguistic user' feedback is assigned a different linguistic weight. Let the weight matrix be given in Eq. (7). However, the semantics of each of these linguistic weights are also represented by T1 MFs (similar to Fig. 2) and hence also represented as tri-tuples (like the respective terms of LI from linguistic preference vector, please see Eq. (3)).

Thus, the vectors containing linguistic preference (given in Eq. (2)), and the associated weights (given in Eq. (7)), are initially converted to the tri-tuple form. These are given in Eq. (14) and Eq. (15) as:

$$\{(l_{j1}, m_{j1}, r_{j1}), (l_{j2}, m_{j2}, r_{j2}), \ldots (l_{ji}, m_{ji}, r_{ji})\} \tag{14}$$

$$\{(l_{w_1}, m_{w_1}, r_{w_1}), (l_{w_2}, m_{w_2}, r_{w_2}), \ldots (l_{w_i}, m_{w_i}, r_{w_i})\} \tag{15}$$

where $j = 0\ to\ g$. Each $(l_{jk}, m_{jk}, r_{jk}), k = 1, \ldots, i$ are left, middle and right end, respectively of the triangular MF pertaining to user preferences. $(l_{w_k}, m_{w_k}, r_{w_k}), k = 1, \ldots, i$ are the corresponding values for the respective linguistic weights.

3.1.2. Manipulation

Now, we perform the weighted aggregation of the linguistic preference vector and associated linguistic weights given in Eq. (14) and Eq. (15), using the concept of α-cuts. Consider, tri-tuples of any arbitrary linguistic preference from Eq. (14) and its associated weight from Eq. (15) are given respectively as $L_k = (l_{jk}, m_{jk}, r_{jk}), k = 1, \ldots, i$ and $W_k = (l_{w_k}, m_{w_k}, r_{w_k}), k = 1, \ldots, i$. For simplicity of notation, let's denote L_k by $\{a, b, c\}$ and W_k by $\{d, e, f\}$. The α-cuts of the T1 MFs of L_k and W_k are shown in Fig. 3. The α-cuts of L_k and W_k are intervals and respectively given as $(L_k)_\alpha$ and $(W_k)_\alpha$ in the following equations:

$$(L_k)_\alpha = [a + (b - a)\alpha, c - (c - b)\alpha] \tag{16}$$

$$(W_k)_\alpha = [d + (e - d)\alpha, f - (f - e)\alpha] \tag{17}$$

Thus, the product of $(L_k)_\alpha$ and $(W_k)_\alpha$ is obtained by using interval arithmetic as:

$$(L_k)_\alpha \otimes (W_k)_\alpha = [min(prod), max\ (prod)] \tag{18}$$

where the quantity $prod$ is given as:

$$prod = \begin{Bmatrix} \{a + (b - a)\alpha\} \times \{d + (e - d)\alpha\}, \\ \{a + (b - a)\alpha\} \times \{f - (f - e)\alpha\}, \\ \{c - (c - b)\alpha\} \times \{d + (e - d)\alpha\}, \\ \{c - (c - b)\alpha\} \times \{f - (f - e)\alpha\} \end{Bmatrix} \tag{19}$$

From Eq. (19), it can be seen that $prod$ has squared quantities, and so do the $min(prod)$ and $max(prod)$. Thus, the product of $(L_k)_\alpha$ and $(W_k)_\alpha$ is not a linear function, rather parabolic. Fig. 4 shows these two parabolic curves viz., the $min(prod)$ and $max\ (prod)$, respectively. The parabolic curves $min(prod)$ and $max(prod)$ can be linearly approximated to arrive at the values of the two legs for the T1 MF. Dashed lines in Fig. 4 show the linear approximations.

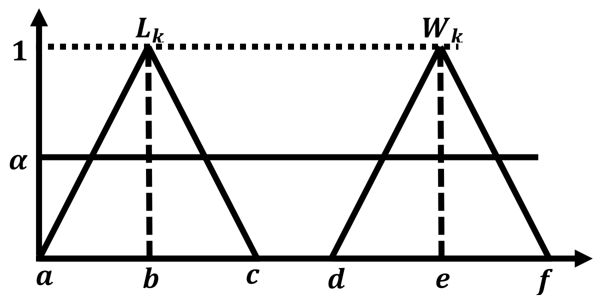

Fig. 3 α-cuts of T-1 MF of L_k and W_k

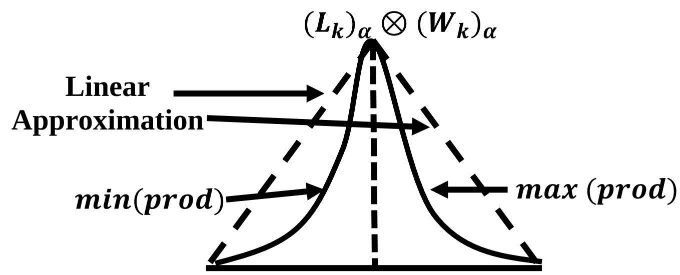

Fig. 4 Product of $(L_k)_\alpha$ and $(W_k)_\alpha$ and its approximation to T1 MFs

Theorem 1: *The product of the α-cuts of two triangular T1 MFs will always converge to the two legs of the resulting T1 MF.*

Proof: Let us consider the product of the α-cuts of the linguistic preferences and the associated weights shown in Fig. 4. The linguistic preferences and the associated weights are represented as triangular T1 MFs, as shown in Fig. 3. We will assume that the linguistic preferences and the associated weights are represented on a positive information representation scale, as this the most common form of data representation in almost all the research problems. Thus, it directly follows that $a < b < c$ and $d < e < f$. Consequently, from Eqs. (18) and (19) we get: $min(prod) = \{a + (b - a)\alpha\} \times \{d + (e - d)\alpha\}$ and $max(prod) = \{c - (c - b)\alpha\} \times \{f - (f - e)\alpha\}$.

Consider $min(prod) = \{a + (b - a)\alpha\} \times \{d + (e - d)\alpha\}$. As the minimum value of $\alpha = 0$ and maximum value of $\alpha = 1$, $\therefore min(prod) \epsilon [a \times d, b \times e]$. Therefore, looking at Fig. 4, it follows that the lower and upper end points of the left leg of the linearly approximated T1 MF are given as $(a \times d, 0)$ and $(b \times e, 1)$, respectively.

Thus, the equation of the line joining these two points is given as $x = ad + (be - ad)\alpha$. Here, for simplicity of notation, we have written $a \times d = ad$ and $b \times e = be$. Similarly, the equation of the curve $min(prod)$ is given as $x = ad + \{a(e - d) + d(b - a)\}\alpha + \{(b - a)(e - d)\}\alpha^2$. The distance between this line and the curve is given as $dist = \{(b - a)(e - d)\}\alpha - \{(b - a)(e - d)\}\alpha^2$. On differentiating $dist$ with respect to α and equating it to 0, gives that maximum value of $dist$ occurs at $\alpha = \frac{1}{2}$. Thus, the maximum value of $dist = \frac{(b-a)(e-d)}{4}$, which is the maximum distance between the $min(prod)$ and the left leg of T1 MF.

Proceeding similarly (and looking at Fig. 4), it follows that the lower and upper end points of the right leg of the linearly approximated T1 MF are given as $(c \times f, 0)$ and $(b \times e, 1)$, respectively. Thus, the maximum distance between the curve $max(prod)$ and the right leg of the linearly approximated T1 MF (shown in Fig. 4) is given as: $dist' = \frac{(c-b)(f-e)}{4}$.

As it can be seen that the products of the linguistic preferences with the respective weights using α-cuts involve lots of computations. These computations can be reduced by considering the particular case of theorem 1. We present this special case in Theorem 2.

Theorem 2: *The product of L_k and W_k can be approximately assumed to be a triangular MF, if the information representation scale for both the L_k and W_k is positive (as shown in Fig. 2).*

Proof: It directly follows from the proof of Theorem 1, that the product of the α-cuts of the linguistic preferences and their respective associated weights converges to the two legs of the T1 MF. Thus, if the information representation scale is positive, the triangular MFs for the L_k and W_k, as shown in Fig. 3, will have all the tri-tuples as positive. In such scenarios, the product of L_k and W_k can be approximately assumed to be a triangular MF, which can be represented in the tri-tuple form as (and pictorially shown in Fig. 5):

$$L_k \otimes W_k = \{l, m, r\} = \{a \times d, b \times e, c \times f\} \tag{20}$$

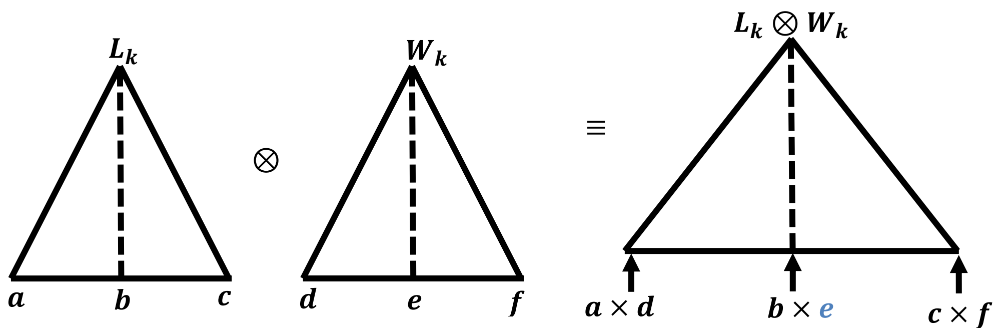

Fig. 5 Product of L_k and W_k and its resulting T1 MF

Algorithm 1: Augmented Extension principle based Linguistic Computational Model

Step 1. Input:

i. $T = \{t_0, t_1, \dots t_g\}$, Linguistic term set of cardinality $g + 1$

ii. $\{t_{j1}, t_{j2}, \dots t_{ji}\}, j = 0 \text{ to } g, \forall t_{jk} \in T; k = 1 \text{ to } i$, Linguistic preferences of i stakeholders

iii. $W = [w_1, \dots, w_i]$, Weight Matrix with respective weights for each linguistic preferences t_{jk}

Step 2. Compute Tri-tuples of each:

i. Linguistic preferences:$\{(l_{j1}, m_{j1}, r_{j1}), (l_{j2}, m_{j2}, r_{j2}), \dots (l_{ji}, m_{ji}, r_{ji})\}$

ii. Respective weights: $\{(l_{w_1}, m_{w_1}, r_{w_1}), (l_{w_2}, m_{w_2}, r_{w_2}), \dots (l_{w_i}, m_{w_i}, r_{w_i})\}$

Step 3. Obtain:

Tri-tuples by product of k^{th} linguistic term to its respective weight using α-cut decomposition: $L_k \otimes W_k = \{l, m, r\} = \{l_{jk}, m_{jk}, r_{jk}\}$.

Step 4. Aggregate:

Tri-tuples to obtain the collective preference vector: $C = (l_c, m_c, r_c) = \left(\frac{1}{i}\sum_{k=1}^{i} l_{jk}, \frac{1}{i}\sum_{k=1}^{i} m_{jk}, \frac{1}{i}\sum_{k=1}^{i} r_{jk}\right)$

Step 5. Perform:

Linguistic approximation: $d(t_q, C) = \sqrt{P_1(l_q - l_c)^2 + P_2(m_q - m_c)^2 + P_3(r_q - r_c)^2}, P_i, i = 1,2,3$ have values 0.2, 0.6 and 0.2 respectively, $t_p = (l_q, m_q, r_q) \in S, q = 0 \text{ to } g$ is the linguistic term from T.

Step 6. Recommended linguistic term is:

$t_b^* \in T$, such that $d(t_b^*, C) \leq d(t_q, C), \forall t_q \in T$.

Thus, in this manner, the products of all the linguistic preferences and their associated weights are performed. These products are then approximated to the triangular MFs and aggregated using the Eq. (5) to arrive at a collective preference vector.

3.1.3. Retranslation

To generate a linguistic term corresponding to the collective preference vector, resulting in the above step, (which generally does not match any linguistic terms in the term set T), we calculate the Euclidian distance of this collective preference vector from the tri-tuples corresponding to each of the linguistic terms given in T, using Eq. (7). Thus, the recommended linguistic term is the one with maximum similarity or minimum distance.

The working of the augmented extension principle based linguistic computational model is summarized in Algorithm 1.

3.2. IFS based CWW methodology

IFSs are a more generalized version of the FSs [1], [2], [4], [27]. Consider a T1 FS. In a T1 FS, each set element is a twin tuple with the values as ordered pair {(set element, its membership value)}. An IFS generalizes this concept by introducing a third dimension in each set element viz., the degree of non-membership. Thus, an IFS (A) is defined on a universe say X in the following manner:

$$A = \{(x, \mu_A(x), \nu_A(x) |\, x \in X\} \tag{21}$$

where $\mu_A(x)$ is called the degree of membership and $\nu_A(x)$ is called the degree of non-membership. The condition on $\mu_A(x), \nu_A(x)$ is as: $0 \le \mu_A(x), \nu_A(x) \le 1$, and $\mu_A(x) + \nu_A(x) \le 1$.

For a T1 FS, $\nu_A(x) = 1 - \mu_A(x)$ and it also satisfies the law of excluded middle. However, this is not true for an IFS. Graphically, the $\nu_A(x)$ and $\mu_A(x)$ are shown in Fig. 6.

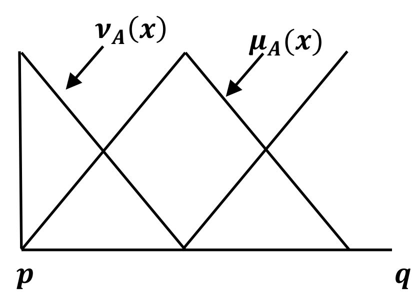

Fig. 6 Degree of membership and non-membership in IFS

3.2.1. Translation

In the design of CWW approach based on IFS, consider again the scenario where the linguistic preferences of the users are given by Eq. (2). Each linguistic preference of the user has an associated weight also, given in Eq. (7).

As considered in Section 2.1, the linguistic preferences and the weights are represented by the T1 MFs, as shown in Fig. 2. Now, in the IFS-based CWW methodology, which we build on the idea of EPLCM, difference comes in the fact that every value of the linguistic preference as well as the respective weight is represented by a membership value and the non-membership value. The non-membership values are taken as the average of the tri-tuples of multiple linguistic terms other than the respective linguistic term from the term set.

To explain this concept, consider the linguistic preference of the first user from term set of Eq. (2) and its associated weight from Eq. (7), given in tri-tuple form be as: $t_{j1} = (l_{j1}, m_{j1}, r_{j1})$ and $w_1 = (l_{w_1}, m_{w_1}, r_{w_1})$. It is pertinent to mention that we have assumed that $t_{j1}, w_1 \in T$, T being given in Eq. (1). Therefore, $\exists\ t_p \in T, p = 0, \dots, g,$ such that $t_{j1} = t_p$. Also $\exists t_q \in T, q = 0, \dots, g,$ such that $w_1 = t_q$. Further, t_p can be equal or unequal to t_q. Therefore, the degree of non-membership for t_{j1} is given by the average of tri-tuples of all linguistic terms from T other than t_p. Similarly, the degree of non-membership for w_1 is given by the average of tri-tuples of all linguistic terms from T other than t_q. This is given in Eq. (22) for t_{j1} and Eq. (23) for w_1 as:

$$(l'_{j1}, m'_{j1}, r'_{j1}) = \left(\frac{1}{g} \sum_{k=0, k \neq p}^{g} l_k, \frac{1}{g} \sum_{k=0, k \neq p}^{g} m_k, \frac{1}{g} \sum_{k=0, k \neq p}^{g} r_k \right) \tag{22}$$

$$(l'_{w_1}, m'_{w_1}, r'_{w_1}) = \left(\frac{1}{g} \sum_{k=0, k \neq q}^{g} l_k, \frac{1}{g} \sum_{k=0, k \neq q}^{g} m_k, \frac{1}{g} \sum_{k=0, k \neq q}^{g} r_k \right) \tag{23}$$

The degree of membership in tri-tuple form for the t_{j1} and w_1 being given by (l_{j1}, m_{j1}, r_{j1}) and $(l_{w_1}, m_{w_1}, r_{w_1})$, respectively.

Similarly, the degree of membership and non-membership can be found for all the Eq. (2) linguistic preferences and their associated weights in Eq. (7). When putting together the degrees of memberships and non-memberships for linguistic preferences and the associated weights for all the users together, similar to Eqs. (14) and (15), we obtain Eqs. (24) and (25), respectively as:

$$\{[(l_{j1}, m_{j1}, r_{j1}), (l'_{j1}, m'_{j1}, r'_{j1})], \dots, [(l_{ji}, m_{ji}, r_{ji}), (l'_{ji}, m'_{ji}, r'_{ji})]\} \tag{24}$$

$$\{[(l_{w_1}, m_{w_1}, r_{w_1}), (l'_{w_1}, m'_{w_1}, r'_{w_1})], \dots [(l_{w_i}, m_{w_i}, r_{w_i}), (l'_{w_i}, m'_{w_i}, r'_{w_i})]\} \tag{25}$$

where $j = 0\ to\ g$. Each $(l_{jk}, m_{jk}, r_{jk}), k = 1, \dots, i$ are left, middle and right end, respectively of the triangular MF about user preferences corresponding to the degree of membership and $(l'_{jk}, m'_{jk}, r'_{jk})$ are left, middle and right end, respectively, of the triangular MF pertaining to user preferences corresponding to the degree of non-membership. $(l_{w_k}, m_{w_k}, r_{w_k}), k = 1, \dots, i$ are the corresponding values for the membership degree of the respective associated weights and $(l'_{w_k}, m'_{w_k}, r'_{w_k})$ are the corresponding values for the non-membership degree of the respective associated weights.

3.2.2. Manipulation

Here, the aggregation is performed similarly to Section 2.1.2, using α-cuts or using the approximation techniques. In either case, we multiply the tri-tuples corresponding to the degree of memberships (non-memberships) of the linguistic preferences and the respective weights given in Eqs. (24)-(25), to obtain the respective tri-tuples for the associated degree of memberships (non-memberships) for the respective user.

Then we aggregate the weighted preferences of the users to obtain the twin tuple collective preference vector: one corresponding to the membership and one corresponding to the non-membership. This is given in Eq. (26) as:

$$C=\{(l_c,m_c,r_c),(l'_c,m'_c,r'_c)\}=\left\{\left(\frac{1}{i}\sum_{k=1}^{i}l_{jk},\frac{1}{i}\sum_{k=1}^{i}m_{jk},\frac{1}{i}\sum_{k=1}^{i}r_{jk}\right),\left(\frac{1}{i}\sum_{k=1}^{i}l'_{jk},\frac{1}{i}\sum_{k=1}^{i}m'_{jk},\frac{1}{i}\sum_{k=1}^{i}r'_{jk}\right)\right\} \quad (26)$$

Algorithm 2: IFS based CWW methodology

Step 1. Input:

i. $T=\{t_0,t_1,\ldots t_g\}$, Linguistic term set of cardinality $g+1$

ii. $\{t_{j1},t_{j2},\ldots t_{ji}\}, j=0\ to\ g, \forall t_{jk}\in T; k=1\ to\ i$, Linguistic preferences of i stakeholders

iii. $W=[w_1,\ldots,w_i]$, Weight Matrix with respective weights for each linguistic preferences t_{jk}

Step 2. Compute Tri-tuples of (here each value of set containing one term for membership and non-membership):

i. Linguistic preferences:$\{[(l_{j1},m_{j1},r_{j1}),(l'_{j1},m'_{j1},r'_{j1})],\ldots,[(l_{ji},m_{ji},r_{ji}),(l'_{ji},m'_{ji},r'_{ji})]\}$

ii. Respective weights: $\{[(l_{w_1},m_{w_1},r_{w_1}),(l'_{w_1},m'_{w_1},r'_{w_1})],\ldots[(l_{w_i},m_{w_i},r_{w_i}),(l'_{w_i},m'_{w_i},r'_{w_i})]\}$

iii. Linguistic preference non-memberships: $(\mathrm{l'}_{j1},\mathrm{m'}_{j1},\mathrm{r'}_{j1})=\left(\frac{1}{\mathrm{g}}\sum_{\mathrm{k=0,k\neq p}}^{\mathrm{g}}\mathrm{l_k},\frac{1}{\mathrm{g}}\sum_{\mathrm{k=0,k\neq p}}^{\mathrm{g}}\mathrm{m_k},\frac{1}{\mathrm{g}}\sum_{\mathrm{k=0,k\neq p}}^{\mathrm{g}}\mathrm{r_k}\right)$

iv. Respective weight non-memberships: $(l'_{w_1},m'_{w_1},r'_{w_1})=\left(\frac{1}{g}\sum_{k=0,k\neq q}^{g}l_k,\frac{1}{g}\sum_{k=0,k\neq q}^{g}m_k,\frac{1}{g}\sum_{k=0,k\neq q}^{g}r_k\right)$

Step 3. Obtain:

Tri-tuples by product of k^{th} linguistic term to its respective weight using α-cut decomposition for memberships and non-memberships: $\{(L_k\otimes W_k),(L'_k\otimes W'_k)\}=\{(l_{jk},m_{jk},r_{jk}),(l'_{jk},m'_{jk},r'_{jk})\}$.

Step 4. Obtain collective preference vector by aggregation of tri-tuples:

$$C=\{(l_c,m_c,r_c),(l'_c,m'_c,r'_c)\}=\left\{\left(\frac{1}{i}\sum_{k=1}^{i}l_{jk},\frac{1}{i}\sum_{k=1}^{i}m_{jk},\frac{1}{i}\sum_{k=1}^{i}r_{jk}\right),\left(\frac{1}{i}\sum_{k=1}^{i}l'_{jk},\frac{1}{i}\sum_{k=1}^{i}m'_{jk},\frac{1}{i}\sum_{k=1}^{i}r'_{jk}\right)\right\}$$

Step 5. Perform Linguistic approximation for membership and non-membership respectively as:

$$d(t_q,C)=\left\{\left(\sqrt{P_1(l_q-l_c)^2+P_2(m_q-m_c)^2+P_3(r_q-r_c)^2}\right),\left(\sqrt{P_1(l_q-l'_c)^2+P_2(m_q-m'_c)^2+P_3(r_q-r'_c)^2}\right)\right\}$$

$P_i, i=1,2,3$ have values 0.2, 0.6 and 0.2 respectively, $t_{\mathrm{p}}=(l_q,m_q,r_q)\in S, q=0\ to\ g$ is the linguistic term from T.

Step 6. Recommended linguistic term for membership and non-membership respectively are $t_{\mathrm{b}}^{*}\in T$ and $\mathrm{t^{*\prime}}_{\mathrm{b}}\in \mathrm{T}$: Such that $d(t_b^*,C)\le d(t_q,C),\forall t_q\in T$ and $d(t^{*\prime}_b,C)\le d(t_q,C),\forall t_q\in T$

3.2.3. Retranslation

Here, the retranslation enables the mapping of the numeric collective preference vector back to the linguistic form. The mapping is done similar to Section 2.1.3. The difference is that the mapping is done separately for membership and non-membership components of the collective preference vector given in Eq. (26). Therefore assuming, that each of the linguistic term in T, given in Eq. (1) is represented in the tri-tuple form as $t_p=(l_q,m_q,r_q)\in T, q=0\ to\ g$. The term with maximum similarity or minimum distance is recommended as the linguistic term corresponding to the membership degree in the collective preference vector of Eq. (26), as:

$$d(t_q,C)=\sqrt{P_1(l_q-l_c)^2+P_2(m_q-m_c)^2+P_3(r_q-r_c)^2} \quad (27)$$

where the $P_i, i = 1,2,3$ are the weights, with values 0.2, 0.6 and 0.2 respectively. The recommended linguistic term is $t_{\mathrm{b}}^{*} \in T$, such that $d(t_b^*, C) \leq d(t_q, C), \forall t_q \in T$.

Similarly, the term with maximum similarity or minimum distance is recommended as the linguistic term corresponding to the non-membership degree in the collective preference vector of Eq. (26), as:

$$d'(t_q, C) = \sqrt{P_1(l_q - l'_c)^2 + P_2(m_q - m'_c)^2 + P_3(r_q - r'_c)^2} \tag{28}$$

where the $P_i, i = 1,2,3$ are the weights, with values 0.2, 0.6 and 0.2 respectively. The recommended linguistic term is $t^{*\prime}{}_b \in T$, such that $d(t^{*\prime}{}_b, C) \leq d'(t_q, C), \forall t_q \in T$.

Therefore, we get two recommended linguistic terms, one corresponding to the membership and other corresponding to the non-membership.

The IFS based CWW methodology is summarized in the form of Algorithm 2.

3.3. Rough set based CWW methodology

Rough sets [9], [17], [18], [29], [38] are a formal approximation of a crisp set. This approximation is given in terms of sets called the lower and the upper approximation of the original set. We are using the indiscernibility property of rough sets and the concepts of SMLCM to build the rough set based CWW methodology. Let us briefly review some of the associated definitions of the rough sets, which will be required for building the rough set based CWW methodology.

Consider an information representation system I consisting of a collection of attributes given by the universe of discourse U and a non-empty finite set of attributes A is defined on U. Therefore, $I = (U, A)$ or more formally put, I is a mapping defined as follows: $I: U \to V_a$, where V_a is the set of values that the attribute a may take and $a \in A$. There exists an information table in the system that assigns a value $a(x)$ from V_a to each attribute a and object x in the universe U.

A P-indiscernibility relation or property, denoted as $IND(P)$, is defined on the rough set in the following manner:

$$IND(P) = \{(x, y) \in U^2 \mid \forall a \in P, a(x) = a(y)\} \tag{29}$$

where $P \subseteq A$, is called an associated equivalence relation. The universe U can be partitioned into a family of equivalence classes $IND(P)$ denoted by $U/IND(P)$ or U/P. We use the concept of P-indiscernibility to develop our rough set based CWW linguistic computational model.

3.3.1 Translation

In the design of CWW methodology based on rough sets, we consider again the linguistic preferences of the users, contained in the preference vector, in Eq. (2). The important step here is to partition these linguistic preferences into equivalence classes using the indiscernibility criteria. Thus, the linguistic preference vector of Eq. (2) changes to a collection of equivalence classes given in Eq. (30) as:

$$\{C_1, C_2, \ldots C_n\} \tag{30}$$

where $C_i, i = 1, \dots, n$ are the equivalence classes obtained by grouping the same linguistic preferences of users from Eq. (2) into one class. Thus, each equivalence class is a set given by a collection of linguistic preferences as $C_i = (t_{j1}, t_{j2}, \dots t_{jp})$, where each $t_{jq} \in T; q = 1\ to\ p, t_{j1} = t_{j2} = \dots = t_{jp}$. It is pertinent to mention that the number of linguistic preferences in any two equivalence classes can be the same or different. We can refer to the number of linguistic preferences in a class as the class cardinality, denoted as $|C_i|, i = 1, \dots, n$.

From Section 2.2.1, we have that the sum of all the weights for the corresponding linguistic preferences is 1. In Eq. (30), we have n equivalence classes. Thus, each class can be assigned a weight equal to $1/n$. Further, since each class contains $|C_i|, i = 1, \dots, n$ number of terms, therefore within each class, each linguistic term can be assigned a weight of $1/(n * |C_i|)$. In this way, we have decided the weights for each of the linguistic preferences given in Eq. (2) and arrived at a new weight matrix, which is given in Eq. (31) as:

$$W = [w'_1, \dots, w'_i] \tag{31}$$

where each $w'_p \in [0,1]; p = 1\ to\ i$, corresponds to the associated weight of the respective linguistic preference t_{jk} given in Eq. (2). Further, a condition exists on all the w'_p that $\sum_{p=1}^{i} w'_p = 1$.

3.3.2 Manipulation

Now we need to aggregate the linguistic preferences of the users according to the weight matrix given in Eq. (31). We first convert the linguistic preferences to a sorted order, similar to Eq. (8), which is given as:

$$\{T_1, T_2, \dots T_i\} \tag{32}$$

where $T_k \in T, k = 1, \dots, i$. Further, each of the T_k may or may not be equal to respective t_{jk}. These linguistic preferences from Eq. (32), are aggregated according to the weight vector given in Eq. (31), recursively, using the recursive function (AG'^{i}), by performing the computations shown in Eq. (33) as:

$$AG'^{i}\left\{w'_p, I_{T_p}, p = 1, \dots, i | i > 2, i \in \mathbb{Z}\right\} = \left(w'_1 \odot I_{T_1}\right) \oplus \left((1 - w'_1) \odot AG'^{i-1}\{\delta_h, I_{T_h}, h = 2, \dots, i\}\right) \tag{33}$$

where $I_{T_p}, p = 1, \dots, i$, $I_{T_h}, h = 2, \dots, i$ are the indices[4] of the linguistic terms given in Eq. (32) and $\delta_h = {w'_h}\Big/{\sum_{l=2}^{i} w'_l}\ ; h = 2, 3, \dots, i$.

Eq. (33) proceeds in a top-down fashion performing the successive aggregations until the number of terms is reduced to 2, referred to as the boundary condition. For the boundary condition, the aggregation is performed as:

$$AG'^{2}\left\{\{w'_{i-1}, w'_i\}, \{I_{T_{i-1}}, I_{T_i}\} | \ i = 2\right\} = (w'_{i-1} \odot I_{T_{i-1}}) \oplus (w'_i \odot I_{T_i}) \tag{34}$$

where $I_{T_{i-1}}$ and I_{T_i} are the respective indices of the last two terms from the preference vector Eq. (32), and w'_{i-1} and w'_i are the respective weights.

3.3.3 Retranslation

Eq. (34) recommends the index of a linguistic term. This recommended index value is given as:

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[4] The original equations in [12] use the linguistic labels. However, the actual aggregation is performed on the indices of the linguistic labels. So, we have chosen to show directly the indices of the linguistic terms in the Eq. (33)-(34).

$$I_r = min\left\{i, I_{T_i} + round\left(\frac{w'_{i-1} - w'_i + 1}{2}.\left(I_{T_{i-1}} - I_{T_i}\right)\right)\right\} \quad (35)$$

where $round()$is the round function.

Thus, before reaching Eq. (34) from Eq. (33) in a top-down manner, we get a series of $i-2$ intermediate recursive equations. From Eq. (34), an index of the linguistic term is recommended using Eq. (35). This index is then given as an input to the preceding recursive equation. Thus, the combination process then follows a bottom-up approach. Therefore, at each step of the bottom-up combination process, there will be two indices of the linguistic terms to combine. The index of the recommended linguistic term can be found by performing computations similar to Eq. (35).

Thus, finally, we reach Eq. (33), or the original equation, where the index of the recommended linguistic term is matched to one of the terms from Eq. (1), to find out the final recommended linguistic value.

Algorithm 3: Rough Set based CWW methodology

Step 1. Input:

i. $T = \{t_0, t_1, \dots t_g\}$, Linguistic term set of cardinality $g+1$

ii. $\{t_{j1}, t_{j2}, \dots t_{ji}\}, j = 0\ to\ g, \forall t_{jk} \in T; k = 1\ to\ i$, Linguistic preferences of i stakeholders

Step 2. Compute n Equivalence Classes by grouping same linguistic terms into one class:

$$\{C_1, C_2, \dots C_n\}$$

Step 3. Compute weight matrix, $W = [w'_1, \dots, w'_i]$ for each linguistic term in Equivalence Classes:

i. Each class can be assigned a weight equal to $1/n$.

ii. Each class contains $|C_i|$, number of terms, therefore weight assigned to each linguistic term is $1/(n * |C_i|)$.

Step 4. Sort the linguistic preferences to obtain a new preference vector as:

$$\{T_1, T_2, \dots T_i\}$$

Step 5. Aggregate linguistic preferences using weight matrix using recursive function as:

i. $AG'^i\left\{w'_p, I_{T_p}, p = 1, \dots, i | i > 2, i \in \mathbb{Z}\right\} = \left(w'_1 \odot I_{T_1}\right) \oplus \left((1 - w'_1) \odot AG'^{i-1}\{\delta_h, I_{T_h}, h = 2, \dots, i\}\right)$

ii. $AG'^2\left\{\{w'_{i-1}, w'_i\}, \{I_{T_{i-1}}, I_{T_i}\} | i = 2\right\} = (w'_{i-1} \odot I_{T_{i-1}}) \oplus (w'_i \odot I_{T_i})$

Step 6. Recommend the index of Linguistic term as:

$$I_r = min\left\{i, I_{T_i} + round\left(\frac{w'_{i-1} - w'_i + 1}{2}.\left(I_{T_{i-1}} - I_{T_i}\right)\right)\right\}$$

Step 7. Use Step 6 in a bottom up manner to recommend an index of the linguistic term from $AG'^i\left\{w'_p, I_{T_p}, p = 1, \dots, i\right\}$. Thus, the recommended linguistic term is the one whose index matched the term set T in Step 1.i.

It has been observed that in an equation where only two indices need to be combined, similar to Eq. (34), if $w'_{i-1} = w$ and $w'_i = 1 - w$, for some value of the weight w, then putting indices also in a generalized form viz., $I_{T_{i-1}} = I_l$ and $I_{T_i} = I_q$, Eq. (34) and Eq. (35) can be written (in generalized form) as in Eq. (36) and Eq. (37), respectively as:

$$AG'^2\left\{\{w, 1 - w\}, \{I_l, I_q\}\right\} = (w \odot I_l) \oplus ((1 - w) \odot I_q) \quad (36)$$

$$I_r = min\ \{i, I_q + round(w.(I_l - I_q))\} \quad (37)$$

Finally, the recommended value is a unique index of the term belonging to the term set T, in Eq. (1).

It is pertinent to mention that in case the inputs are weighted, the rough set based CWW methodology will behave exactly like the Symbolic method based linguistic computational model of Section 2.2.

The working of the rough set based CWW methodology is summarized in the form of Algorithm 3.

4. Demonstration by Numerical Example and Experimental Results

In this section, we demonstrate the working of the augmented extension principle based linguistic computational model, IFS based CWW methodology and rough sets based CWW methodology, using a suitable example, in the first three subsections. Then in the fourth subsection, we present the experimental results on 30 datasets of five users, obtained by the application of 1) EPLCM, 2) SMLCM, 3) augmented extension principle based linguistic computational model, 4) IFS based CWW methodology, and 5) rough sets based CWW methodology.

The problem definition of these users, linguistic inputs and weights are as follows: consider a group of five users (User 1…5), where we capture their linguistic preferences on a determined scenario. Their linguistic preferences are elicited using a term set T, containing three linguistic terms, and is given in Eq. (38) (similar to Eq. (1)) as:

$$T = \{t_0: Low\ (L), t_1: Medium\ (M), t_2: High\ (H)\} \tag{38}$$

T1 MFs represent the semantics of the linguistic terms in the term set of Eq. (38) on a scale of 0 to 10, shown in Fig. 7.

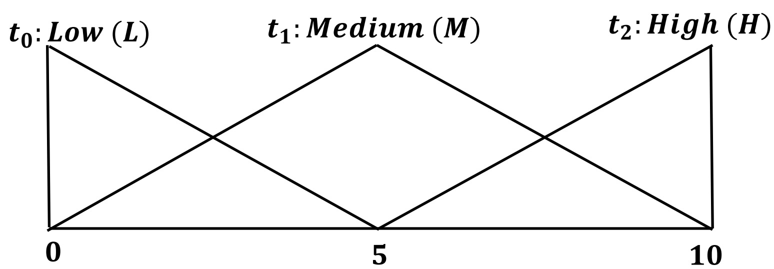

Fig 7. Individual terms of the linguistic term set

Let the users' preferences be given as User 1 and User 2: Low, User 3: $Medium$, User 4 and User 5: $High$. Thus, the preference vector containing the users' preferences is given in Eq. (39) (similar to Eq. (2)) as:

$$\{t_{01}: L, t_{02}: L, t_{13}: M, t_{24}: H, t_{25}: H\} \tag{39}$$

here $t_{jk}, j = 0,1,2, k = 1,2,3,4,5$ denotes the linguistic preference of the k^{th} user occurring at j^{th} index in the term set given in Eq. (38). Now, let us illustrate the working of each of the linguistic computational models proposed in Section 3 one by one.

4.1. Example using proposed augmented EPLCM

Let the weights to be assigned to the linguistic preferences of the users be taken from the vector given in Eq. (40), and its semantics be represented using the T1 MFs, shown in Fig 8.

$$WW = \{w0{:}\, Less\ (LW), w1{:}\, Average\ (AW), w2{:}\, More\ (MW)\} \tag{40}$$

Let the weight vector, similar to Eq. (7), corresponding to the linguistic preferences of the users from Eq. (39), is given as:

$$W = [w_1{:}\, LW, w_2{:}\, AW, w_3{:}\, MW, w_4{:}\, LW, w_5{:}\, AW] \tag{41}$$

Thus, now we aggregate the linguistic preferences of the users along with the respective weights.

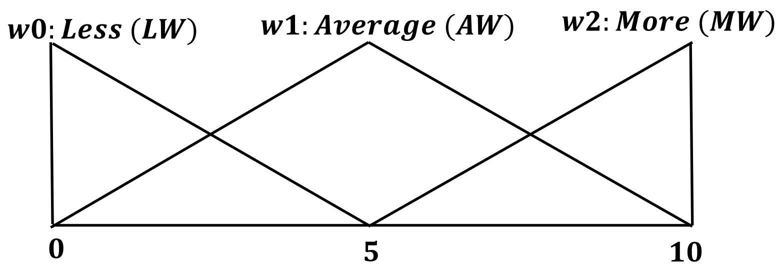

Fig 8. Individual terms of the weight term set

4.1.1. Translation

We convert the linguistic preferences of the users from Eq. (39) and respective weight vector from Eq. (41) to tri-tuple form using Fig. 7 and Fig. 8 respectively, to arrive at Eq. (42) and Eq. (43), similar to Eq. (14) and Eq. (15), respectively as:

$$\{(0,0,5), (0,0,5), (0,5,10), (5,10,10), (5,10,10)\} \tag{42}$$

$$\{(0,0,5), (0,5,10), (5,10,10), (0,0,5), (0,5,10)\} \tag{43}$$

4.1.2. Manipulation

Since the information representation scale is positive viz., 0 to 10, we multiply the respective linguistic preference of the user with its associated weight to derive the respective tri-tuples corresponding to the weighted preferences of individual users using Eq. (20). Thus, the new matrix corresponding to the weighted preferences of the users is given as:

$$\{(0,0,25), (0,0,50), (0,50,100), (0,0,50), (0,50,100)\} \tag{44}$$

Thus, the collective preference vector corresponding to weighted preferences of all the users is obtained by aggregating the terms of Eq. (44), using Eq. (5), as:

$$C = (0,20,65) \tag{45}$$

4.1.3. Retranslation

Now, we calculate the distance of the collective preference vector of Eq. (45) from all the linguistic terms of the term set in Eq. (38), using Eq. (6). The linguistic term with minimum distance is $High$ (H). Therefore, $High$ is the recommended linguistic term.

4.2. Example using IFS based CWW methodology

Let us illustrate the working of IFS based CWW methodology for the linguistic preferences of the users given in Eq. (39) and the associated weight matrix given in Eq. (41). Now the task here is to consider two quantities associated with each linguistic preference and weight, viz., degree of membership and degree of non-membership.

4.2.1. Translation

Consider linguistic preference of the first user from Eq. (39) as $t_{01}: Low\ (L)$. Therefore, the t_{01} belongs to $Low\ (L)$ and does not belong to $Moderate\ (M)$ and $High\ (H)$. Therefore, the tri-tuple corresponding to the degree of membership of t_{01} is (0,0,5) and tri-tuple corresponding to the degree of non-membership of t_{01} is $\left(\frac{0+5}{2}, \frac{5+10}{2}, \frac{10+10}{2}\right) = (2.5, 7.5, 10)$.

Similarly, the associated weight to the linguistic preference of the first user from Eq. (41) is given as: $w_1: LW$. Therefore, w_1 belongs to $Less\ (LW)$ and does not belong to $Average\ (AW)$ and $More\ (MW)$. Therefore, the tri-tuple corresponding to the degree of membership of w_1 is (0,0,5) and tri-tuple corresponding to the degree of non-membership of w_1 is $\left(\frac{0+5}{2}, \frac{5+10}{2}, \frac{10+10}{2}\right) = (2.5, 7.5, 10)$.

Proceeding similarly, we find out the tri-tuples corresponding to the memberships and non-memberships for linguistic preferences of the users and respectively associated weights. These are given in Eq. (46) and Eq. (47), similar to Eq. (24) and Eq. (25), respectively as:

$$\left\{\begin{matrix} [(0,0,5), (2.5, 7.5, 10)], [(0,0,5), (2.5, 7.5, 10)], [(0,5,10), (2.5, 5, 7.5)], \\ [(5,10,10), (0,2.5, 7.5)], [(5,10,10), (0,2.5, 7.5)] \end{matrix}\right\} \tag{46}$$

$$\left\{\begin{matrix} [(0,0,5), (2.5, 7.5, 10)], [(0,5,10), (2.5, 5, 7.5)], [(5,10,10), (0,2.5, 7.5)], \\ [(0,0,5), (2.5, 7.5, 10)], [(0,5,10), (2.5, 5, 7.5)] \end{matrix}\right\} \tag{47}$$

4.2.2. Manipulation

Here, again the information representation scale is positive (0 to 10); we multiply the tri-tuples corresponding to the membership (non-membership) degree of the respective linguistic preference of the user, with the tri-tuples corresponding to the membership (non-membership) degree of the associated weights. We finally derive the respective tri-tuples corresponding to the membership (non-membership) of the weighted preferences of individual users using Eq. (20). Thus, the new matrix corresponding to the tri-tuples for membership and non-membership of the users' weighted preferences is given as:

$$\left\{\begin{matrix} [(0,0,25), (6.25, 56.25, 100)], [(0,0,50), (6.25, 37.5, 75)], [(0,50,100), (0,12.5, 56.25)], \\ \left[(0,0,50), (0,18.75, 75), [(0,50,100), (0,12.5, 56.25)]\right] \end{matrix}\right\} \tag{48}$$

Therefore, the collective preference vector, containing the tri-tuples corresponding to the membership as well as non-membership similar to Eq. (26), is given as:

$$C = \{(0,20,65),(2.5,27.5,72.5)\} \tag{49}$$

4.2.3. Retranslation

Now, the components of the collective preference vector of Eq. (49), corresponding to the degree of membership as well as non-membership, are converted to linguistic form using Eq. (27) and Eq. (28), respectively. Thus, the linguistic output corresponding to both the membership and non-membership is found to be $High\ (H)$.

4.3. Example using Rough set based CWW methodology

With the rough set based CWW methodology, the first task is to convert the linguistic preferences of the users into equivalence classes.

4.3.1. Translation

Consider again the linguistic preferences of the users, given in Eq. (39). We partition them into equivalence classes. Thus, we obtain the matrix similar to Eq. (30) as:

$$\{C_1, C_2, C_3\} \tag{50}$$

where $C_1 = (t_{01}, t_{02}) = (L, L), C_2 = (t_{13}) = (M), C_3 = (t_{24}, t_{25}) = (H, H)$. Each of the C_1, C_2, C_3 are assigned a weight of $1/3$. Therefore, the weight of individual terms inside each of C_1, and C_3 is $1/6$, whereas that inside C_2 is 1/3. Therefore, the weight matrix similar to Eq. (31) is given as:

$$W = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right] \tag{51}$$

4.3.2. Manipulation

Now we order the linguistic preference term set of the users given in Eq. (39), to arrive at a term set similar to Eq. (32) as:

$$\{t_2, t_2, t_1, t_0, t_0\} \tag{52}$$

Now we combine the indices of the linguistic preferences of the Eq. (52) according to the weight matrix given in Eq. (51) using the recursive function given in Eq. (33) as:

$$AG'^5\left\{\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right], [2,2,1,0,0]\right\} = \left(\frac{1}{6} \odot 2\right) \oplus \left(\frac{5}{6} \odot AG'^4\left\{\left[\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right], [2,1,0,0]\right\}\right) \tag{53}$$

$$AG'^4\left\{\left[\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right], [2,1,0,0]\right\} = \left(\frac{1}{5} \odot 2\right) \oplus \left(\frac{4}{5} \odot AG'^3\left\{\left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right], [1,0,0]\right\}\right) \tag{54}$$

$$AG'^3\left\{\left[\frac{1}{2},\frac{1}{4},\frac{1}{4}\right],[1,0,0]\right\} = \left(\frac{1}{2}\odot 1\right) \oplus \left(\frac{1}{2} \odot AG'^2\left\{\left[\frac{1}{2},\frac{1}{2}\right],[0,0]\right\}\right) \tag{55}$$

4.3.3. Retranslation

Thus, for calculation of the $AG'^2\left\{\left[\frac{1}{2},\frac{1}{2}\right],[0,0]\right\}$, we have arrived at the boundary condition. Therefore, we use Eq. (34) and Eq. (35) (or Eq. (36) and Eq. (37)) to calculate the values as:

$$AG'^2\left\{\left[\frac{1}{2},\frac{1}{2}\right],[0,0]\right\} = \left(\frac{1}{2}\odot 0\right) \oplus \left(\frac{1}{2} \odot 0\right) \tag{56}$$

$$I_r = \min\left\{5, 0 + round\left(\frac{1}{2} \times (0-0)\right)\right\} = 0 \tag{57}$$

Thus, putting $I_r = 0$ from Eq. (57), in Eq. (55), we get Eq. (58) as:

$$AG'^3\left\{\left[\frac{1}{2},\frac{1}{4},\frac{1}{4}\right],[1,0,0]\right\} = \left(\frac{1}{2}\odot 1\right) \oplus \left(\frac{1}{2} \odot 0\right) \tag{58}$$

Here, the value of the R.H.S of Eq. (58) is found by performing computations similar to Eq. (56) as:

$$I_r = \min\left\{5, 0 + round\left(\frac{1}{2} \times (1-0)\right)\right\} = 1 \tag{59}$$

Thus, putting $I_r = 1$ from Eq. (59), in Eq. (54), we get Eq. (60) as:

$$AG'^4\left\{\left[\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right],[2,1,0,0]\right\} = \left(\frac{1}{5}\odot 2\right) \oplus \left(\frac{4}{5} \odot 1\right) \tag{60}$$

Here, the value of the R.H.S of Eq. (60) is found by performing computations similar to Eq. (56) as:

$$I_r = \min\left\{5, 1 + round\left(\frac{1}{5} \times (2-1)\right)\right\} = 1 \tag{61}$$

Thus, putting $I_r = 1$ from Eq. (61), in Eq. (53), we get Eq. (62) as:

$$AG'^5\left\{\left[\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right],[2,2,1,0,0]\right\} = \left(\frac{1}{6}\odot 2\right) \oplus \left(\frac{5}{6} \odot 1\right) \tag{62}$$

Here, the value of the R.H.S of Eq. (62) is found by performing computations similar to Eq. (56) as:

$$I_r = \min\left\{5, 1 + round\left(\frac{1}{6} \times (2-1)\right)\right\} = 1 \tag{63}$$

Therefore, the index of the recommended linguistic term is 1. Therefore, the recommended linguistic term is t_1 or $Medium\ (M)$ from Eq. (38).

TABLE III

LINGUISTIC INPUTS OF USERS, ASSOCIATED WEIGHTS AND GENERATED LINGUISTIC RECOMMENDATIONS

S.No.	Linguistic Inputs of Users[a]					Linguistic Weights associated to linguistic inputs of users[b]					Linguistic recommendation by					
Users→	1	2	3	4	5	1	2	3	4	5	EPLCM[c]	SMLCM[d]	A-EPLCM[e]	IFS[f]		RS[g]
														m[h]	nm[i]	
1	H	H	M	H	H	AW	LW	AW	HW	HW	H	H	H	H	H	H
2	M	L	L	M	M	AW	HW	HW	LW	LW	M	L	M	M	H	L
3	H	H	M	H	H	HW	HW	HW	HW	HW	H	H	H	H	M	H
4	M	H	H	M	H	AW	AW	AW	AW	AW	H	M	H	H	H	H
5	H	H	H	L	H	HW	LW	AW	AW	LW	H	M	H	H	H	M
6	L	H	H	M	L	AW	HW	AW	AW	HW	M	L	H	H	H	L
7	M	H	H	H	H	HW	HW	HW	HW	HW	H	H	H	H	M	H
8	L	H	H	M	H	AW	LW	LW	HW	HW	M	M	H	H	H	M
9	M	H	H	H	H	LW	LW	HW	LW	AW	H	M	H	H	H	H
10	H	H	H	H	H	AW	HW	HW	HW	HW	H	H	H	H	M	H
11	H	L	H	L	H	AW	LW	HW	LW	AW	M	L	H	H	H	L
12	M	L	M	H	L	LW	HW	LW	HW	LW	M	L	H	H	H	L
13	H	M	H	H	H	HW	HW	HW	HW	HW	H	H	H	H	M	H
14	H	H	H	M	H	LW	AW	LW	LW	AW	H	M	H	H	H	H
15	H	H	H	H	H	HW	HW	HW	HW	AW	H	H	H	H	M	H
16	L	H	H	M	L	LW	HW	AW	LW	AW	M	L	H	H	H	L
17	H	H	H	H	H	HW	AW	HW	HW	HW	H	H	H	H	M	H
18	M	L	M	L	M	AW	HW	HW	HW	HW	M	L	H	H	H	L
19	M	L	L	L	H	LW	HW	AW	AW	AW	M	L	H	H	H	L
20	H	H	H	M	H	HW	HW	HW	HW	HW	H	H	H	H	M	H
21	M	L	L	M	L	AW	AW	LW	HW	AW	L	L	H	H	H	L
22	H	H	H	H	H	HW	HW	HW	AW	HW	H	H	H	H	M	H
23	M	H	L	M	M	HW	HW	LW	AW	HW	M	L	H	H	H	M
24	L	M	M	L	H	HW	AW	HW	HW	AW	M	L	H	H	H	L
25	H	H	H	H	M	HW	HW	HW	HW	HW	H	H	H	H	M	H
26	M	H	H	L	M	LW	HW	HW	LW	AW	M	L	H	H	H	M
27	L	H	M	H	L	AW	HW	LW	LW	LW	M	L	H	H	H	L
28	H	H	H	H	H	HW	HW	AW	HW	HW	H	H	H	H	M	H
29	M	L	L	L	L	LW	HW	LW	HW	HW	L	L	M	M	H	L
30	L	M	M	H	H	LW	HW	LW	HW	AW	M	M	H	H	H	M

[a] L= Low, M= Medium, H= High, [b] LW= Less, AW= Average, MW= More, [c] EPLCM= Extension principle based linguistic computational model, [d] SMLCM= Symbolic method based linguistic computational model, [e] A-EPLCM= Augmented Extension principle based linguistic computational model, [f] IFS= IFSbased CWW methodology, [g] RS= Rough Set based CWW methodology, [h] m= Linguistic term for degree of membership (L= Low, M= Medium, H= High), [i] nm= Linguistic term for degree of non-membership (L= Low, M= Medium, H= High)

4.4. Experimental Results

We compare the results obtained by applying EPLCM, SMLCM, Augmented extension principle based linguistic computational model, IFS based CWW methodology, and rough sets based CWW methodology on 30 datasets of the five users, Users 1,..,5. Their linguistic preferences are described using the linguistic terms given in Eq. (38), and the semantics of these linguistic terms are represented by T1 MFs in Fig. 7. Further, the linguistic weights corresponding to the linguistic inputs in the respective data sets are described using linguistic terms given in Eq. (40) and the semantics of these linguistic terms are represented by T1 MFs in Fig. 8. These data sets are presented at S.No. 1 to 30, in the rows 4 to 33 in Table III.

In Table III, Columns 2 to 6 (rows 4 to 33) give the linguistic preferences of the five users, and columns 7 to 11 (rows 4 to 33) provide the corresponding linguistic weights of the respective linguistic preferences of the users. These linguistic preferences of the users are processed by using the EPLCM technique (discussed in Section 2.1) to give linguistic recommendations. The linguistic recommendations are given in column 12 (rows 4 to 33).

Further, the users' linguistic preferences and the associated linguistic weights are processed using the SMLCM, Augmented-EPLCM, IFS based CWW methodology, and the rough set based CWW methodology, which is discussed in Section 2.2, Section 3.1, Section 3.2 and Section 3.3, respectively. The linguistic recommendation obtained respectively by SMLCM, Augmented-EPLCM, IFS based CWW methodology and the rough set based CWW methodology are given respectively in column 13, 14, 15-16 and 17 (rows 4 to 33). The IFS based CWW methodology provides two linguistic recommendations, one corresponding to the degree of membership and another to the degree of non-membership, as seen in columns 15 and 16, rows 4 to 33, of Table III.

5. Discussions

In this Section, we will bring out important findings as well as throw light on various aspects that have emerged from the research work presented in this paper.

1). We have shown the similarity of the EPLCM and the SMLCM to Yager’s generalized CWW framework. In [49], authors stated that ever since Prof. Zadeh proposed the concept of CWW in his work [41], various literary works have presented discussions and extensions on it. Thus, according to the authors, all such research literature points out that the CWW itself has subjective interpretations, but all such interpretations require fuzzy logic to implement them.

Since EPLCM represents the semantics of LI using T1 FSs, therefore, showing the similarity of the EPLCM to Yager’s CWW framework affords us the use of T1 FSs for the CWW (in the form of the EPLCM). Further, by showing the similarity of the SMLCM to Yager’s generalized CWW framework, we put forth the idea that it is also possible to achieve CWW by methodologies other than fuzzy logic, viz., use of ordinal term sets (as done in the SMLCM). However, it is pertinent to mention that the emphasis lies on using the correct uncertainty model for achieving CWW. Higher order FSs have also been used to achieve the CWW [11], [19], [24], [26], [28], [33], [36], and they do perform better than the T1 FSs or the ordinal term sets. However, this improvement in the performance is accompanied by additional computational complexity. It may happen that the decision-making scenario at hand does not warrant the performance improvement at additional computational complexity, because the information granularity may be represented sufficiently by the T1 FSs or the ordinal term sets. Such scenarios can easily employ the EPLCM or the SMLCM.

2). We have proposed a novel augmented linguistic computational model based on the extension principle, which enables the processing of LI, which is differentially weighted. By processing, we mean that the LI is operated on by using the principles of CWW viz., translation, manipulation and retranslation. The differential weighting means that, out of the individual pieces of LI, which have been referred to as the linguistic preferences from individual stakeholders, one or more linguistic preferences may be assigned the same or different weights. These weights are also represented in the linguistic form (crisp or equal weights are a special case of linguistic weights). Numerous works in the literature support our claim that LI is seldom uniformly or equally weighted. More often than not, LI is differentially weighted (Please see [11], [19], [27], [26], [34-36] for details).

Further, we support our claims using the numerical example discussed in Section 4 above. Consider the case of 5 users discussed in the numerical example of Section 4. These users may belong to the same management department within an organization in a real-life scenario. This organization wants to decide with respect to the implementation of policies for changing its supply chain operations. These five users are in managerial executive positions and have been asked to provide their preferences (in linguistic form) on this decision. Based on the inputs received from the users, a collective decision will be taken for policy implementation for the change of supply chain operations. These users may have joined the organizations at different times. Their work experience within the organization is taken as a measure of their expertise in providing their linguistic inputs. Thus, their inputs are differentially weighted. Say, User 3 is the oldest in the organization, and therefore his preferences attract $More$ weight. Users 2 and 5 may have joined the organization at almost the same time, so their preferences are assigned a weight of $Average$. Further, Users 1 and 4 may have joined the organization recently; therefore, their preferences are assigned $Less$ weight.

3). We have also proposed a novel CWW methodology based on IFS. T1 FSs represent the semantics of LI as a collection of points (or elements) which are twin values viz., consider a T1 FS A, where each element is given as $A = \{(x, \mu_A(x)) | \forall x \in X\}$, x being the set element, $\mu_A(x)$ its degree of membership and X is the universe of discourse. Here, an element x belongs to T1 FS A with a degree of membership $\mu_A(x)$ and does not belong to A with a degree of non-membership as $1 - \mu_A(x)$.

IFS, on the other hand, represent each element as tri-tuples viz., $A = \{(x, \mu_A(x), \nu_A(x)) | \forall x \in X\}$, where the added term $\nu_A(x)$ is called the degree of non-membership. For an IFS, the sum of membership and non-membership degrees do not necessarily add to 1 viz., $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Further, $\nu_A(x)$ may or not be equal to $1 - \mu_A(x)$. IFS also define a term called hesitation given as $\pi(x) = 1 - \mu_A(x) - \nu_A(x)$.

In [3], the author stated that IFSs could be applied in all the scenarios where the T1 FSs, however not vice versa. The author described a scenario in [3] to show the usefulness of IFS over T1 FSs, regarding the direct election of governments in different countries. The support offered to the parliament in each of these countries was the ratio of the number of the Members of Parliament (MPs) from the ruling party to their total number. If this ratio does not change then, T1 FSs are useful for this case. However, suppose the attitudes of the MPs towards the parties keep changing. In that case, this will impact two ratios: (1) the number of MPs who strongly support the government compared to the total number of MPs, and (2) the number of MPs who are firmly against the government to the total number of MPs. Thus, IFS are useful in such scenarios. This is just one example of a scenario where IFS find more utility over the T1 FSs. More such applications can be found in [3].

4). We have also proposed a novel CWW methodology based on Rough Sets. Originally, the Rough Sets were defined as a formal overlap of two crisp sets viz., lower and upper. However, currently, the lower and upper sets of a Rough Set may be defined as the Fuzzy Sets [9], [17], [18], [29], [38]. The most significant advantage of the Rough Sets is that they do not require any a priori knowledge about the data like the grade of membership needed in the FS theory [31]. Rough Sets use the boundary of a set to represent the vagueness in concepts. Thus, the expert defining the research problem is free to choose any means to represent the semantics of LI while using rough sets, hence providing greater flexibility compared to T1 FSs.

Furthermore, we have proposed the Rough Sets based CWW methodology as an extension to the SMLCM. One of the greatest advantages of the Rough Set based CWW methodology compared to the SMLCM is how weights are assigned to the linguistic terms. In the Rough Set based CWW methodology, all the equivalence classes are assigned equal weights ($^1/_n$), n being the number of equivalence classes. Then all the linguistic terms within an equivalence class are assigned an equal weight of this $^1/_n$ value viz., $^1/_n\,|C_i|$, $|C_i|$ being the number of linguistic terms in i^{th} equivalence class (Please see Section 3). This is a more realistic approach in a real-life scenario on decision-making. Multiple users with the same level of expertise can be grouped into one equivalence class and assigned weights accordingly. This is quite difficult to achieve with SMLCM.

5). The similarity between the augmented EPLCM, IFS-based CWW methodology and Rough Set-based CWW methodology is that each of these approaches processes differentially weighted LI according to the steps in Yager's generalized CWW framework (Please see Fig. 1) and generates linguistic output. However, the difference lies in the way semantics of LI are represented and the application at hand. Thus, if the semantics of LI is known to be represented using T1 FSs and the preferences of users don't change with time, then the augmented EPLCM can be used. The representation of LI semantics using T1 FSs and changing users' preferences enable the use of IFS based CWW methodology. Further, if there is no clarity about the methodology used to represent the semantics of LI, then the Rough Set based CWW methodology can prove to be quite valuable.

6. Conclusions and Future Work

In this paper, we have studied the linguistic computational models based on extension principle and symbolic method. We have shown that the data processing steps (to process LI) adopted by these two linguistic computational models bear a lot of similarity to Yager's generalized CWW framework. However, no proof exists in the literature of the similarity being drawn between these linguistic computational models and Yager's generalized CWW framework. We have established this similarity. So, our finding is that these two linguistic computational models can be called the Extension principle based CWW methodology and Symbolic method based CWW methodology, respectively.

We have also found a limitation of the linguistic computational model based on the extension principle that it cannot process linguistic terms with differential weighting. Therefore, we have proposed an augmented linguistic computational model based on the extension principle, which can compute with differentially weighted LI.

Recently, novel concepts like IFSs and rough sets have been proposed. However, no CWW methodologies exist for these two. Therefore, we have proposed two novel CWW methodologies based on IFS and rough sets.

We have demonstrated the working of augmented extension principle based linguistic computational, CWW methodology based on IFS and CWW methodology based on rough sets using a suitable example.

We feel that the present work will enable future researchers to see the various linguistic computational methodologies in the framework of CWW. They will be able to improve the existing and develop novel CWW methodologies. Also, it is mentioned here that the two CWW methodologies proposed in this paper, viz., based on IFS and rough sets, are motivated by the linguistic models based on extension principle and symbolic method, respectively. Recent work has been proposed by Labella et al. [50], where both the extension principle and symbolic method are combined to handle the complex linguistic expressions. Therefore, connecting the extension principle and symbolic method can motivate a new CWW methodology based on the combination of IFS and rough sets.

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