On modelling volatility and mortality for pension schemes

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A thesis presented for the degree of

Doctor of Philosophy

at the

Department of Mathematical Sciences

University of Essex

May 2020

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. The work in this thesis is all my own work, unless referenced, to the contrary, in the text. The research has been carried out at the Department of Mathematical Sciences, University of Essex, United Kingdom, and supervised by Dr Spyridon Vrontos and Dr Haslifah Hashim.

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May 2020

Acknowledgements

First and foremost, I would like to express my sincere gratitude to both of my supervisors Dr. Spyridon Vrontos and Dr. Haslifah Hashim for the continuous support during my PhD study, for their guidance, valuable experience, knowledge, patience, and endless support to me. My PhD has been an amazing experience and I thank both of them wholeheartedly, not only for their tremendous academic support, but also for giving me so many wonderful opportunities. I appreciate all their contributions of time, ideas, and support to make my PhD experience productive and stimulating. I am also very grateful to my Chair of Supervisory Board, Prof. Peter Higgins, for his insightful comments and encouragement during the board meetings, which helped me to be on track on my research. Thank you to my internal examiner, Dr Hongsheng Dai for the useful comments and suggestions. I could not have imagined having a better supervisors and mentors for my PhD study, and I am really thankful for that.

The most heartfelt gratitude is to my beloved husband, Mohd Khairul Aiman and my lovely daughter Eleia Ayesya for their sacrifice, understanding, and patience throughout my PhD study.

I would also like to thank my fellow PhD mates that include Lana, Rahmi, Kak Ilen, Rudy, Marius, Awatf, Amal, Junaid, Jonathan, Hajem and others (too many to list here but you know who you are!) for making my experience in the department exciting and fun. Thank you for providing me support and friendship that I needed and I am indebted to them for their help. I pay my sincere thanks to all the esteemed department staff for their help and support during my PhD study.

Most importantly, none of this would have been possible without the unconditional love and prayers from my family. I especially thank my parents: Abdul Aziz Mat Aris and Zabidah Kamaruddin for supporting me spiritually throughout my life, for always believing in me and providing unconditional love and care. I love them so much, and I would not have made it this far without them. Also to my siblings, my brother Syahril and my sister Farradila for their love and motivation. I would like also to thank my parents in law (Hamdan & Hayati) for their love and encouragement. And also, to all my friends (Nazrul, Ayu, Tasha, Khaleeda, Indah, Syaff, Ain, Akma, Syikin, Kak Syahida, Christina and others) and family in Malaysia that I know I always can count on when times are rough, and when I need someone to talk to, thank you for always being there!

Thank you to my financial sponsor, the Malaysia government and Majlis Amanah Rakyat (MARA) for providing me financial support. And thank you to my employer, Universiti Teknologi MARA (UiTM), for approving my study leaves and giving me time to finish my PhD.

Above all, to the Almighty, the most gracious and most merciful because of his favours and for giving me the strength and resources to complete my thesis.

This thesis marks the end of my PhD journey but it represent the beginning of another life-long learning journey.

Abstract

The purpose of this research is to develop volatility and mortality models that could be used in asset liability management in pension schemes. This study provides a comprehensive study of various advance multivariate DCC GARCH models which are used for construction of optimal portfolios in modelling asset return covariances. The effectiveness of using parametric copula in estimating portfolio risk measures are evaluated such that the DCC models are found to have better performance than any other parametric copula models. Several models were developed as extensions to existing mortality models in a single and multiple population, in particular the Lee Carter (LC) mortality model and the Common Age Effect (CAE) model by proposing a modification of singular value decomposition (SVD) and principal component analysis (PCA) methods. Complementing this, a further study on mortality model by applying a range of multivariate DCC GARCH models in modelling the mortality dependence across multiple populations is evaluated.

Finally, the proposed models of volatility and mortality are applied to the pension schemes. The volatility models were fitted using multivariate DCC GARCH model to obtain the investment returns and the cohort actuarial tables were produced based on LC approach for the out-of-sample period in the UK population. The fits from the modelling of volatility and mortality were analysed on defined benefit (DB), defined contribution (DC) and hybrid schemes to evaluate the fund value and actuarial liabilities. This research underlined the important role that econometric volatility modelling and stochastic mortality modelling can play in managing pension schemes to ensure that future liabilities can be meet.

Publications

Most of the work of this thesis has been published or is going to appear for publication.

1. Evaluation of multivariate GARCH models in an optimal asset allocation framework (Aziz, N.S.A, Vrontos, S. and Hasim, H). The article is published at The North American Journal of Economics and Finance on 14 July 2018 at https://www.sciencedirect.com/science/article/pii/S106294081830038X.

2. Mortality modelling for a single and multiple population (Aziz, N.S.A, Vrontos, S. and Hasim, H). Target journal: TBA (To submit).

3. Multiple population mortality models: A DCC GARCH and Copula approach (Aziz, N.S.A, Vrontos, S. and Hasim, H). Target journal: TBA (To submit).

The above works were presented in several conferences including the 7th International Conference of Mathematics and Statistical Methods for Actuarial Sciences and Finance, MAF, in April 2016 at University of Dauphine, Paris, France, at the 4th Young Finance Scholars' Conference at University of Sussex on 12-13 June 2017 and at the Actuarial Research Conference, 26-29 July 2017, at Georgia State University, Atlanta. The second work above has received a grant by the Scientific Research Network (WOG) for a presentation in Actuarial and Financial Mathematics Conference in Brussels, Belgium on 07-08 February 2019.

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Chapter 1

Introduction

Retirement may be viewed as a time for joy with family and friends after working for years but over time it may create a burden due to financial uncertainty. When we retire, we need a secure source of income to ensure that it is sufficient to maintain the standard of living just as before retirement or some may settle for a adequate standard of living, that may not be the same standard as when they were working. A pension is a regular payment made during a person's retirement from an investment fund to which the person and/or their employer have contributed during their working life. Commonly, it is made by the employer to people reaching the retirement age as one of the ways to provide the retirees with a fixed income when they are no longer working, due to advanced age or disability or loss of income (death of wage earner in the family). Pensions seek to give protection and smoothing the consumption in old age, since the pension payment corresponds to a lifetime benefit for the employees. Before retirement, people normally work for 40 to 50 years. They are no longer employed after, and thus rely on the pension given by the employer and/or personal savings during the pre-retirement period. Almost half of pre-retirees admit they save nothing for retirement whereas two-thirds are counting on pension for income, government benefits and the rest plan to work part-time [72]. It is worrying when people do not save for retirement, yet expect income for when they retire.

Prior to retirement, each individual should have their own saving apart from monthly contributions for the pension. For example, individuals may invest voluntarily in financial markets or save in the form of buying properties or jewellery. This is important since many studies have estimated that 70% of pre-retirement income is needed to maintain current standard of living in retirement [94], [54], [108] and [6]. Saving for retirement is essential for everyone due to increasing life expectancy, reducing employer benefits, lower market returns and increasing costs of living (due to expensive medical costs, long term care and expensive hobbies such as golf, travelling, etc.). Pre-retirees must save accordingly, based on their future consumption plan, so that they would be able to cover their living expenses in retirement or else they have to extend their working years or live in poverty. The burden of pre-retirement savings is now shifted from the sponsoring employer to the employees. The pension received from the employer may not be sufficient to cover future consumption. Therefore, each individual must plan, save and invest wisely before approaching retirement.

A pension system tries to smooth consumption between the pre-retirement and post-retirement years so that individuals would be able to maintain their lifestyle in retirement. The government is playing its roles to make sure everyone has basic standard of living in retirement. Frequently, governments support pension benefits directly or require employers' participation in pension schemes. The government may have laws to ensure that compulsory contributions are made by individuals and/or sponsoring employers prior to retirement. The law is made to reduce the increasing cost of pension and protect the pension benefits provided. During the pre-retirement period, individuals often consume as much as possible while forgetting future needs. The government enforcement to participate in the pension scheme can prepare the employee for the retirement.

In the United Kingdom (UK), the government has enforced *auto enrolment* as an initiative to help people save for retirement. The employer is required by law to make contributions and automatically enrol its workers into a pension scheme. This means that, for every employee, a portion of their salary will be taken as a way to save money for their retirement. All of the contributions will be invested and the member will get the money when they retire. This is beneficial to the employees since many people do not save enough for retirement or take advantage of private pension schemes when the life expectancy keeps increasing. It is an effective way to increase employee participation and savings for pensions [68].

In general, there are three main types of funded pension schemes; the defined benefit (DB), the defined contribution(DC) and hybrid schemes. DB schemes promised a pension based on a fixed formula, involving the numbers of years in the scheme, the member's age and the member's final salary. Individuals who belong to DB pension schemes can plan more easily for their retirement, as they know in advance how much income they will receive every month until they die. Employers have turned to DC pension schemes as a way of offering pension savings to their employees while controlling costs. DC pension schemes offer no retirement income security in advance of retirement. Contributions to the DC scheme are made by both the employee and the employer. The employer's contribution is guaranteed but the future benefit is based on future investment return. A financial market crash at the time of retirement can result in unanticipated poverty, as the value of accumulated savings plummet. Whereas, a hybrid pension scheme is a combination of the two schemes with a guarantee that the benefit's would not fall below a certain threshold value. Therefore, the members will have protection against a bad investment performance.

Individuals should understand the basic investment principles so that they are aware of the various investment options available. The shift from DB to DC schemes has seen the responsibility for retirement planning transfer to the employee [20]. Individuals must decide when to start save for retirement, the amount to save and how they should invest. Whereas for DB members, they must decide whether their current pension scheme is sufficient to provide retirement income or should they contribute to another retirement saving plan. Individuals who lack knowledge in financial markets and pension schemes will not know how to plan properly and will not know the best pension scheme that suits their retirement goals. With insufficient savings, the employees are unable to achieve a balance between the accumulation phase during their working period and future consumption in post-retirement.

It is challenging to manage retirees' financial resources with the increasing life expectancy, decline in the DB scheme and unanticipated events (i.e. health issues) [36]. The medical costs are rising which can cause a big hole in the retirees' pocket while the income received in retirement is lower compared to during their working period [116]. The need for a professional financial adviser is crucial to ensure that the standard of living just before retirement can be maintained. Practitioners or pension fund managers, who are advising pension schemes, are responsible for, making decisions relating to investment and mortality experience of the fund. An appropriate measure of mortality improvement is essential as it will affects the premiums and reserves of annuity and pension products which eventually can cause financial distress to pension providers.

In this thesis, the study of volatility and mortality modelling will enable the pension scheme to choose the best models that will help them to meet the pension liabilities. This study analyses the plethora of advance multivariate econometric models which forecast the mean and variance-covariance of asset returns to create optimal asset allocation models for pension schemes. Different mortality models involving single and multiple populations are used in determining the future life expectancy.

Good volatility and mortality modelling in a pension scheme is very important as it will help the pension providers in making investment decisions and projecting future mortality as close as possible to the actual ones to ensure the pension liabilities can be meet and avoiding the schemes closing. The on-going economic instability worldwide has made pension schemes face losses, and therefore, this study will provide insights to the pension providers on how to re-examine their asset allocation strategies to provide the best pension schemes to retirees. This study will provide such an overview and comparison which is an important contribution to the pension literature.

1.1 Problem Statement

Asset Liability Management (hereafter, ALM) is an approach to managing assets and liabilities through examining pension risks for funding, benefit pay out, and asset allocation in a pension scheme. It is a form of risk management, whereby the sponsors try to hedge the risk of failing to meet the pension obligations. It involves a comprehensive approach in analysing risk and return in terms of the overall pension scheme's impact. Pension schemes involve a long term investment decision to meet the pension liabilities and therefore strategic asset liability management is essential. However, the current pension scheme's system does not seem to be utopian at all. Individuals should understand the basic investment principle so that they are aware of the various investment options available and know how to allocate their portfolio's assets.

While maximising wealth, in ALM, investors are also trying to limit its exposures to risks and possible future losses. To fund the pension schemes, a reasonable asset allocation strategy must be chosen by investing a proportion of the fund in different asset classes to diversify the risks.

In general, there are two types of risks involved which are intrinsic and non-systematic risks. Intrinsic risk refers to uncertainty in share value as the price fluctuates according to the market and this type of risk is not easily eliminated through diversification strategies. Whereas, the non systematic risks can be eliminated through asset classes diversification. For instance, currency risk can be eliminated by diversifying through assets in different countries in the world.

There are three main asset classes, each with different levels of risk and return, which are equities, fixed income (bonds) and cash equivalents (money market instruments). Each asset in the pension's portfolio behaves differently, depending on the market and economic conditions, hence playing an important role in determining the portfolio's overall risk and return from investment.

Every individual have different specific goals which can be an early retirement, paying children's university education or buying properties. A retirement goal might be different in terms of when to retire, their consumption preferences and future lifestyle. From an investors' view, goals generally involve achieving the target benefit in providing pension to its members. Once the goals are clearly specified, then the investment objectives can be formulated. As a result, ALM studies, especially relating to asset pricing, pension liabilities, portfolio selection and risk management are growing in importance.

Pension schemes depend on steady stock market returns to ensure that they will be able to pay its members. Investing in the stock market may help in achieving a retiree's financial objectives in a short period of time as long as we buy the right stock at the right time. But, no one knows which stock is the best or even the best time to invest. And, if the stock price falls (lower realised return), it is hard for the pension schemes to meet their financial obligations. In extreme cases, the pension benefits may be reduced or the scheme winds up. Moreover, the ongoing financial crisis has made many corporate pension schemes record losses and made the sponsors re-examine their asset allocation strategies in the pension scheme by considering the risk exposures to the schemes. As a result, the pension management studies especially relating to portfolio's investment are growing significantly. The investment strategy in a portfolio (i.e., the decision on how to invest the accumulated fund across different asset classes) will influence the volatility of the pension fund and consumption in different time periods [10], and hence determine the scheme's capabilities in providing pension benefits to its members. Therefore, a flexible decision-making tool for pension schemes is needed which can appropriately describe the individual's preferences and helping the members to make an optimal decision that maximises their expected utility.

This study makes several contributions to the literature. First, this study involves different ways of modelling volatility and mortality. Different model specifications using multivariate GARCH processes are studied in modelling the mean returns and variance-covariance matrices. From the volatility modelling, optimal portfolio is constructed using different optimisation models which are able to capture different characteristics in the data and therefore improving the portfolio performance. This study, therefore, will provide useful insights for those wishing to explore the different GARCH models that are available and use a dynamic approach for asset allocation and portfolio construction purposes. This study develops alternative estimation methodologies with the modification of Singular Value Decomposition (SVD) and Principal Component Analysis (PCA) in modelling the mortality rates in individual and multiple population. This study also introduces the use of DCC GARCH models to fit and forecast mortality rates. Finally, this study evaluates the pension schemes with regard to the proposed volatility and mortality models.

1.2 Research Objectives

1. To analyse multivariate GARCH models which forecast the mean and variance-covariance of the returns and construct optimal portfolios based on different optimisation models and comparing the out-of-sample performance.

2. To examine the effectiveness of using parametric copula GARCH in estimating portfolio risk measures over the sophisticated DCC model.

3. To evaluate various mortality models in single and multiple populations that produces a better mortality forecasts using different approaches.

4. To evaluate the pension schemes' funding level with regard to the management of assets and liabilities.

1.3 Structure of the Thesis

The remainder of the thesis is organised as follows: Chapter 2 reviews the relevant literature review including financial econometric models, portfolio optimisation strategies, mortality forecasting and various types of pension schemes that have been used widely in the literature and its empirical application. This chapter will discuss how practitioners and academics have traditionally addressed the main problems of asset liability modelling.

Chapter 3 reviews the econometric models used to model the asset returns covariances and describe the portfolio optimisation strategies to construct optimal asset portfolios. This chapter will look at how the model evolves and give details on the data used for the study, empirical analysis and results of the proposed models and methods.

In Chapter 4, the feasibility of using stochastic mortality modelling methodology in constructing the mortality forecasts for the single and multiple population is discussed. This chapter discusses different approaches relating to the modelling of mortality rates and the possible extensions of this problems that could arise when considering realistic scenarios.

Further investigation of the mortality modelling using multivariate econometric modelling is discussed in Chapter 5. The DCC GARCH and copula approach are used to model the mortality rates in a multiple population. Chapter 6 combines the research analysis by matching the asset with the liabilities. In here, the application of volatility and mortality models will be evaluated on the pension fund.

Finally, in Chapter 7, some concluding remarks are presented and potential future research is discussed.

Chapter 2

Literature Review

2.1 Background

Pension schemes are among the largest institutional investors. In 2017 the OECD countries had their highest level of pension assets amounting to over $40 trillion with liabilities several times larger [96]. As the pension schemes continue to account for a steadily growing share in the market, it is essential to understand their behaviour in determining the allocation of asset for retirement savings' investment. It is widely known that the ageing population issue is threatening the global pension systems' sustainability.

Previously, ALM has never been an issue for pension schemes but with the current environment mentioned earlier, it becomes a great tool in managing the assets and liabilities. Moreover, pension schemes face a variety of risks including investment risk, longevity risk, financial risk, salary risk, mortality risk, interest rate risk and inflation risk. Therefore, pension scheme's need to solve the portfolio selection problem which maximises the expected utility of a member's wealth with the given risks. Through

ALM, the investors are able to match future liabilities with future cash flow streams of assets in finding investment strategy which maximises or minimises the objectives of the retirees.

Asset allocation is a way to describe how much should be invested into each class of asset within a portfolio investment. An asset class is a broad group of securities or investments that have similar characteristics that tend to react similarly in the marketplace and are subject to the same laws, forces and regulations. Typical asset classes are equities, bonds, cash and property which have different levels of risk and return. The risk and return in each of the asset class is different which can be illustrated in Figure 2.1.

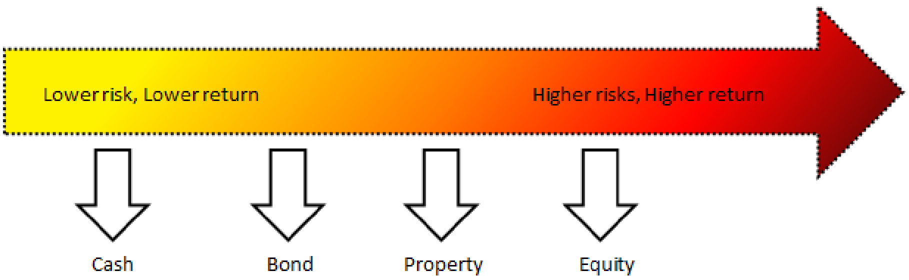


Figure 2.1. The risk return spectrum

Investments can go up or down depending on the asset allocation in the portfolio and how the markets are performing. It is a time-honoured investment strategy whereby it shows how the investor of pension schemes pursues their objective of outperforming the market by allocating each asset in an investment portfolio to meet the expected future liabilities. The goal is to create a balanced mix of assets which have the growth potential that satisfies the investor's risk preferences and investment objectives.

Asset diversification is about reducing risk by spreading the portfolio investments across different assets class. This is true because by diversifying assets in each portfolio, the volatility within the portfolio is reduced. If one of the asset classes does not perform well in the market, the other asset class might perform better. A young investor with a greater risk preference may want to invest his assets in a higher potential growth asset as he would have benefited from the long term growth. In contrast, an investor approaching retirement will choose more conservative investments that have a steady return and risk. A conservative portfolio normally has a higher percentage of risk free bonds and cash, while a risky portfolio, aiming for a higher return, will have more stocks and risky bonds. A risky portfolio tends to produce higher returns over time but have a significantly higher volatility as compared to the conservative portfolios.

Through asset diversification, the volatility in the portfolio may be reduced. Each asset behaves differently in different market and economic conditions and hence asset allocation is essential in determining the investment portfolio's returns. The overall risk can be reducing in terms of the return's variability for a given level of expected return. The asset allocation in global pension schemes is changing remarkably. The evolving landscape for pensions schemes that has been influenced by a combination of financial market performance, regulation, changes to accounting standards and increasing longevity has driven many schemes to de-risk and fundamentally change their investment strategies [95].

A recent global pension asset study by Towers Watson has analysed 22 major global pension markets including Australia, India, South Africa, Spain, Switzerland, UK and US in terms of their asset allocation, asset size and DB and DC share of pension's assets [136]. At the end of 2017, the total pension assets were estimated around USD 37.78

billion, representing a 3.1% increase as compared to USD 36.6 billion at the end of 2016. The seven largest markets (Australia, Canada, Japan, UK, Netherlands, Switzerland and United States) has allocated its assets about 46% in equities, 27% in bonds, 2% in cash and 25% in other assets (including property and other alternatives). According to the consultancy's European asset allocation survey, since 1996, the pension funds across Europe have continued to reduce or maintain their exposure to equities, bond and cash as they bid to limit funding level volatility [89]. The investment's portfolios are diversified into other asset classes such as hedge funds or commodities but equities and bonds still remain as the main key investment. In 2015, Australia, the UK and the US have continued to have above average equity allocations, whilst Japan, Netherlands and Switzerland have more conservative investment with higher allocation to bonds. Over the past 10 years, DC assets have increased at a rate of 5.6% per annum while DB assets have grown at a slower pace of 3.1% per annum.

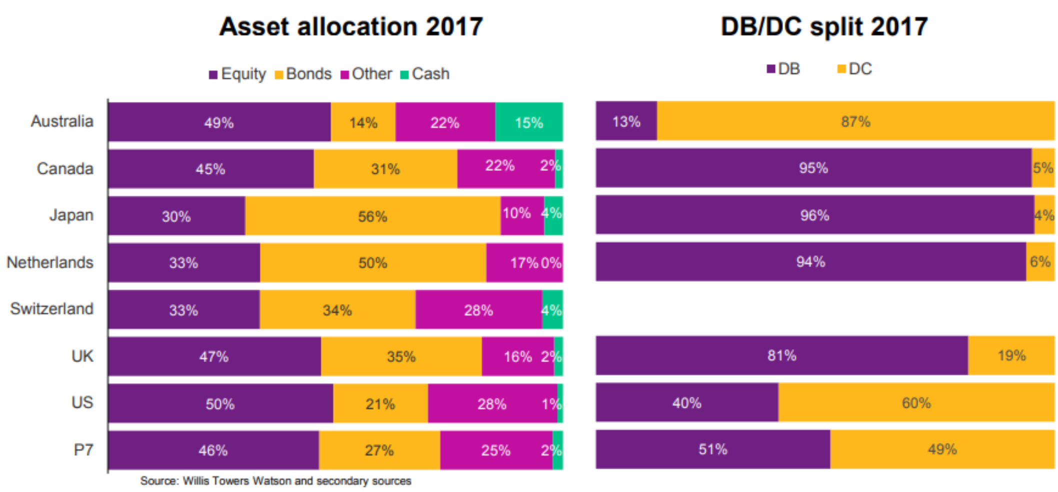


Figure 2.2. The global pension asset allocation and DB/DC asset split. Source: [136].

2.2 Existing knowledge base

2.2.1 Modelling asset returns covariances

Over the past years, several studies have developed methods and approaches to examine the dynamics of covariance of assets. Previous studies on asset allocation, mainly in asset management, focus on a limited or specific econometric model without comparing the different models to model its asset return covariances, see for example; [55], [63], and [77]. One of the widely used models is the Vector Autoregressive (VAR) model which is an extension of the univariate autoregressive model to dynamic multivariate time series. Another model which is the Vector Equilibrium Correction (VEqC) has been obtained from the VAR model for differences by adding an 'equilibrium correction term' to the right hand side of the equations. The VAR model is proven to be useful for describing the dynamic behaviour of economic and financial time series and for forecasting, but it has limitations. The standard VAR model may give misleading results if some of the variables are highly persistent, and without modifications, the model misses non-linearities, conditional heteroskedasticity and drifts in its parameters [123]. It is useful to consider different multivariate econometric models that can capture different characteristics of the data in selecting the best model to create optimal portfolios.

To deal with a large number of parameters in multivariate models, [12] suggest a Constant Conditional Correlation model (CCC) such that the conditional correlations are assumed to be constant. This model reduces the number of parameters and thus simplifies the estimations considerably. However, the assumption of a CCC model may not be realistic in empirical applications of multivariate GARCH models because the conditional shocks are correlated only in the same market, and not across markets [30]. [50] proposes a generalisation of CCC model, by allowing the correlations to change over time, known as the Dynamic Conditional Correlation model (DCC). This model is estimated using a two-steps approach - the estimation of mean and variance by a series of univariate GARCH models and the correlation estimation. It has the flexibility of univariate GARCH but not the complexity of conventional multivariate GARCH. The study by [81] has found that DCC GARCH with Student-t distributions outperforms other models in measuring the value at risk on portfolio stock returns. However, the limits of DCC model is constrained by the equal dynamics for the correlations of all the assets [8]. It is not effective to say that the correlations dynamic of, for example, UK stock indices, to be similar to the US stock indices. [132] undertake a comparative study of Asset-Liability Management (ALM) for pension funds in a time varying volatility environment and find that CCC and DCC models reduce portfolio risk and improve the out-of-sample risk-adjusted realised returns.

Alternative DCC models also have been proposed in the literature which are aiming to solve problems associated with the basic DCC model. The limits of the DCC model are constrained by the equal dynamics for the correlations of all the assets [8]. To avoid this problem, [8] propose the Flexible DCC (FDCC) model such that the correlation dynamic is constrained to be equal only between w groups of variables, providing flexible dynamics. Another study by [98] examines the performance of optimal asset allocation strategies using FDCC models with regime switching as compared to alternative models. Recently, [1] suggests a more tractable dynamic conditional correlation model, known as a corrected DCC model or cDCC model, which involves the three-step approach that is feasible with large systems and provides unbiased estimations. He proposes the model to obtain a consistent Quasi-Maximum Likelihood (QML) estimator of the parameters and modifies the form of the correlation driving process of the DCC model so that it has martingale difference innovations. He found that the cDCC correlation forecasts perform equally or significantly better than the DCC correlation forecasts. Other relevant works using a cDCC model include; [58], [2], [18], [62] and [51]. Recent proposals of multivariate GARCH models include the asymmetric DCC model (aDCC) of [29], the dynamic equicorrelation (DECO) model of [51] and the smooth transition conditional correlation (STCC-GARCH) of [118]. A DCC model can always be considered as a filter for estimating and forecasting conditional correlations despite its limitations [28]. Although a DCC model is severely biased for the case of large dimensions, it is still appealing to use the DCC family for asset modelling involving a not-so-large number of assets due to the fact that the model is easy to implement and widely used in the literature [58].

The copula theory was introduced by [121]. It states that any multivariate distribution function can be decomposed into its marginal distributions and a copula function. The application of copula function was only being introduced in the late 1990's in actuarial science and finance by [47], [84], and more recently by [66] and [139]. Over the years, copula function has been popular in the financial research, especially because of its application to risk management and asset allocation. However, there are very few studies assessing the out-of-sample performance of a portfolio based on the copula model (see, for example, [101], [110], [139] and [79]). The study by [71] presents a way to measure the conditional dependency in a multivariate GARCH using the copula functions when only marginal distributions are known. This method allows the marginal time series to follow a univariate GARCH process and the dependence structure between them is specified by a copula function [100]. [139] propose copula-based GARCH models to describe the time varying dependence structure of stock-bond returns. The out-of-sample performance is compared with other models which includes the passive, CCC GARCH and DCC GARCH models such that they find that a dynamic strategy based on the GJR-GARCH model with Student-*t* copula yields larger economic gains than passive and other dynamic strategies.

2.2.2 Portfolio optimisation and asset allocation

There is considerable literature on asset modelling and optimisation of portfolio allocation strategies using different empirical approaches; see, for instance; [132], [77], [8], [17] and [73]. One of the widely used and earliest approaches of the portfolio theory was developed by [90] - an approach known as Mean-variance optimisation. This theory introduces the concept of efficient frontier, which is a portfolio of investments with a set of optimal risk-return combinations that maximises the expected utility of wealth. The optimal portfolio diversifies the risk without reducing the expected return and enhancing the portfolio construction strategy [130]. It is a myopic strategy which assumes that the decision maker has a mean-variance criterion defined over the single period rate of return on the portfolio. Other studies that are related are [117], [52], and [105]. Other studies that uses myopic strategy are [117] and [105]. The seminal paper by [90] and [125] has become the foundation of the modern financial theory and inspired many to extend its theory such as Capital Asset Pricing Model and Lintner's model.

Despite its theoretical reputation, researchers find it very hard to solve the large scale portfolio model which requires solving quadratic optimisation problems. There is a lack of studies on the multi-period and continuous-time setting due to the difficulty in the extension from the single-period to the multi-period or continuous time framework. The mean-variance analysis are inefficient for a longer period as the expected rates of return that are inputs to the model are typically not expected rates of return in a single period, but rather estimated internal rates of return over long holding periods. Only recently, many authors have managed to extend the theory by taking into account the multi-period portfolio selection using different approaches. Optimal portfolio in a multi-period approach has been studied by [91], [114], [85], [33], and [117].

A new method to create optimal portfolios in a stochastic environment has been first developed by [92]. [33] and [143] solved the mean-variance problem by using the stochastic control theory. This method provides insight into questions of market behaviour and arbitrage, and can be used to construct portfolios with controlled behaviour. This method can be applied to a wide range of assumptions and conditions that may hold in actual equity markets. It uses the logarithmic representation for stocks and portfolios rather than the arithmetic representation used in 'classical' mathematical finance.

Some have used the stochastic programming optimisation models to evaluate the long term investment strategies whereby they have put the original problem into a stochastic linear quadratic problem and solving it using the standard methods (see [33] and [135]). The martingale approach has been used by [38], [111], and [7]. An advantage of using the martingale approach is that it is continuous in time and hence the investor is able to consume or to redistribute his portfolio at any time instance during his investment.

Another approach to solve the asset allocation problem is the dynamic programming techniques used in [37], [120] and [99]. The study by [120] considered the problem of multi-asset and multi-period portfolio optimisation over a finite horizon with a self-financing budget constraint and arbitrary distribution of asset returns, with objective to minimise the mean-square deviation in achieving a desired final wealth. When there are no additional constraints, the asset allocation problem can be solved using a standard dynamic linear programming.

Some authors aims to maximise the return while reducing the downside risk using Value at Risk (VaR) [15] or some using Conditional Value at Risk (CVaR) [11]. [4], [112] and [11] have approached the optimisation problem using the minimisation of mean CVaR such that the covariance risk is replaced by the CVaR as the risk measure. [112] showed that CVaR can be efficiently minimised using the linear programming and non-smooth optimisation techniques. CVaR is also known as expected shortfall, excess loss or tail VaR. By minimising CVaR, the VaR is also reduced since CVaR is greater or equal than VaR.

Alternative optimisation portfolio strategies includes the maximisation of Sharpe ratio, maximisation of the expected utility and the minimisation of mean absolute deviation (MAD). [79] analysed the application of Copula GARCH model and bootstrapping method in maximising Sharpe ratio to construct optimal portfolio using moving window approach. See the recent paper by [17] which propose minimising the CVaR by assuming that the dependence structure is modelled using the copula parameter. The study considers the classical mean-variance, mean-variance-copula, and mean-CVaR-copula analysis such that the empirical results show that the efficient frontier is influenced by the existence of long memory behaviour and the choice of the measure dependence. Whereas [64] illustrates the advantages of using GARCH-EVT-Copula-CVaR modelling in enhancing portfolio performance under short interval time period.

A study by [88] compares the performance of each optimal portfolio in an out-of-sample period and found that orthogonal GARCH (OGARCH) model outperformed the other models, such as Markov switching and the Exponentially Weighted Moving Average (EWMA) model, in producing an optimal portfolio. While [75] considers asset allocation problems under higher moments with GARCH effects using the expected utility maximisation, and uses a bootstrap method to measure the performance of the portfolio. Recently, [59] proposes a bi-objective portfolio optimisation model involving efficient portfolios of a disutility-based risk measure (DCVaR), known as Mean-DCVaR that constitutes an improvement over Mean-CVaR or Mean-Variance model.

The MAD portfolio optimisation model has been proposed by [78] in which the model replaces the variance in the mean-variance objective function with the MAD and transforms the portfolio selection problem from quadratic programming to linear programming. The study claims that the MAD model does not require the covariance matrix and consequently its estimation is not needed, it is easier to solve linear programming as compared to quadratic and MAD portfolios have fewer assets. However, a study by [119] finds that by ignoring the covariance matrix in a MAD model gives results with a greater estimation risk, such that the estimation errors are very severe, especially in small samples and for investors with high risk tolerance. We need to develop the optimisation model for supporting the decision-making concerning the allocation of assets so that the investment goals of the members can be achieved.

2.2.3 Mortality modelling

The development of stochastic mortality models has been very rapid in terms of both structure and statistical techniques used to fit the models. This section reviews the modelling of mortality that exists in the literature.

Mortality risk is the risk of having a higher probabilities of death than expected. A higher mortality rate may occur due to unexpected events such as wars or infectious disease. On the other hand, the longevity risk is defined as the risk of people living longer than expected which may, due to advances in medical sciences or lifestyle improvements, lead to a lower number of deaths. The declining in mortality rates have a significant impact to the pension schemes and insurers. In terms of financial planning, getting an accurate assessment of mortality rates is crucial for pension providers and to an individual as well, for future investments and pensions planning purposes.

In recent years, different stochastic models for mortality rates have been proposed. Early work on stochastic models was pioneered by [82] with the introduction of the Lee Carter (LC) model. This model is the most widely used in mortality forecasting which assumes that the dynamic of the logarithm of central death rates is driven by an age specific constant plus the speed of change at each age, multiplied by the overall period trend of mortality rates. This model has then motivated various extensions (see for example; [83], [109], [25], [23], [14], [27], [41], and [104]).

The Renshaw and Haberman (RH) model of [109] is a generalised model of the LC model by including a cohort effect. The study investigated the feasibility of extending the model and projection of age-period-cohort effects. [127] extends the LC model by including multiple bi-linear age-period components in their model. The other widely used mortality model is the CBD model by [25] which was proposed to solve the problem of projected mortality rates being perfectly correlated in a single age or period term models. It is designed for modelling mortality at higher ages and suitable for modelling longevity risk in pension and annuities. Whereas study by [23] estimates the parameter embedded in the LC model into a Poisson regression setting.

The original CBD model has then been complemented by [27] to include the cohort effect and a quadratic age effect which are known as the M6, M7 and M8 models. M6 model is the original CBD model with added cohort effects, while M7 is the extension of M6 with additional quadratic term in age. M8 determines the impact of cohort effect for any specific cohort that diminishes over time. Another model that combines the features of the LC model and CBD model is the PLAT model that is suitable for all range of ages and also captures the cohort effect [104]. This model has a predictor structure that includes a static age function, three age-period terms with pre-specified age modulating parameters and a cohort effect. In general, these models are different to one another based on different bases such as assumptions on the ease of implementation, age and period dimensions, incorporation of cohort effects, forecasting properties and method of estimations.

The first generation of mortality models, are mostly univariate, which model a single population at a time. Other recent models which have been proposed for a single population include studies by [39], [21] and [31]. In a single population, when a mortality model is fitted to a number of population individually, the age effect is obtained individually for each population which makes it difficult to compare the period effects observed in different populations as they are actually fitted to different age effects.

To deal with multi-country longevity risk, we need a mortality model which is able to forecast future mortality rates for different countries at the same time. A multiple population mortality model considers more than one population in a joint mortality model. There have been a very few contributions on the study of mortality forecasting in a multiple population settings.

[131] provide a comparative study of two population models for the assessment of basis risk in longevity hedges and overview of existing multiple population mortality modelling methodologies in the actuarial and statistical literature. Models that have been proposed for a multiple population includes [76] which proposes a common age effect (CAE) that allows estimation of period effects in different countries. [49] extends the study by reviewing a number of different multiple population mortality models and found that CAE model fits best.

There are few recent studies that attempts to model mortality rates using econometric models such as [133] which introduces mortality dependence using a dynamic copula approach and [32] which uses a factor copula approach. Whereas, [134] uses a DCC GARCH model to capture the evolution of the aggregate mortality rates for different countries jointly and applied to pricing catastrophic mortality bonds. This model prevent the pricing inaccuracy that may potentially arise from the independence assumption as cross population is explicitly captured. The mortality data set is bigger in a multiple population as it combines data from different sources which can be from different countries or regions or genders, allowing a robust mortality modelling by identifying "similar" characteristics within the sub populations.

2.3 Pension Schemes

Although there exist various designs of pension schemes, it can be divided into two broad types: Defined Contribution (DC) and Defined Benefit (DB). Each of the schemes have significantly different characteristics with respect to the risks faced by employers and employees, the types of benefits, the structure of the scheme, and the funding flexibility for the benefits which will be discussed further below.

2.3.1 Defined Benefit

A DB scheme is a type of pension scheme that promises to pay a defined benefit in retirement. Generally, employee/member pension benefits' accrue independently of the contributions payable and investment returns. They may be a function of length of service in the company and salary history, or a fixed amount of money. Individuals who belong to a DB scheme can plan more easily for their retirement, as the retirement income they will receive every month until they die is known in advance. For instance, retiring in the midst of a financial market crisis does not result in a benefit reduction as it is a consequence of the investment risk sharing in which the investment returns are smoothed across time. The cost for the pension benefits are normally shared between the employer and the employee where the employee pays a fixed percentage of salary and the employer pays the rest. On retirement, which is normally between age 60 and 75, the employee will typically receive a pension benefit in the form of periodic payments which end when they die. The amount of benefit distributed is usually in the form of annuities and not guaranteed however, as it is dependent on the scheme's underlying performance. Some countries allow only one form of retirement benefit while others allow several forms including lump sum payments, life annuities or a combination of both. In some DB schemes, a person may elect to choose a one-off lump sum payment reflecting the value of the annuity benefits rather than the traditional life annuity payments.

The contributions to the pension fund may be made by the employees, the employers or both. Generally, the employees make a periodic predetermined contribution amount whereas the employers make a contribution to meet the balance of the cost of the promised pension. The periodic payments generally begin when the member retires which is normally at 65 years old and ends when the member dies. Each DB scheme has a different way of defining the pension benefits. However, typically, it is a formula that incorporates the length of time that the member contributed to the scheme and the salary. A simple example is a *dollars times service* plan that pays £100 per month in retirement for each year that member has contributed to the scheme. Thus a member who has contributed for 20 years would receive a monthly pension of £2,000 in retirement. Another type of pension benefit is the *final salary* benefit. For example, a member with a salary rate of £8,000 per annum at retirement and who has contributed for the last 30 years, will receive a pension of $30 \times £8,000 \times (1/60) = £4,000$ per annum. The fraction $1/60$ is called the *accrual rate*. Regardless of funding method used for the pension benefits, the result is a series of monthly payments that the scheme is required to pay until the member dies. Nevertheless, there is no full guarantee that a given level of contributions will be enough to meet the pension benefits as the future return on investments and the future pension benefits are not known with certainty. The pension benefits may be increased while in payment. The motivation is to maintain the living standards of the retiree. The increases may be fixed or linked to a inflation index, and they may be guaranteed or given at the discretion of the trustees.

A funding surplus arises when the value of the assets exceeds the calculated value of the liability. During favourable investment periods, most DB schemes encounter substantial surpluses where the assets increase by more than expected. When there is a surplus, the employer may reduce their contribution rate or even stop contributing to the scheme. The latter is called as *contribution holiday*. The contribution holiday may not be really 'reasonable' to the employee as the contribution holiday can be taken by the employer and not to the employees when actually the surplus is coming from the employees' contributions. However, it is debatable as the employer has made most of the contributions and they bear most of the "normal" risk. During unfavourable market conditions, the asset values decline and impact the employer since they have to inject extra money from company's current profit into the pension fund which consequently increasing its costs. Hence, when markets rise, the employer gets its benefit from the rising investment values and the cost of funding the pension fund decreases which allow the employer to contribute less while the members continues to receive the same promised benefit.

On the other hand, a funding deficit arises when the value of the assets is less than the expected cost of promised benefits. This means that there are insufficient funds to pay the entire pension benefits promised to the members. The employer and trustee must come up with a recovery plan to eliminate the funding deficit. Generally, the employer is financially responsible for funding any deficit. However, the member contribution rates may also be increased over a sustained period. In extreme cases, the pension benefits may be reduced or the scheme wound up.

DB schemes used to be a popular type of pension scheme. This is because the members in the scheme know, with a reasonable degree of certainty, the amount of pension benefits they will get in the future. This helps them to plan for their retirement. A survey by [107], found that by knowing earlier the amount of pension benefits received in the future, a member is able to make better decisions relating to their retirement such as to know i) when is the best time for them to retire, ii) what they should do with retirement plan and savings accumulations at retirement, iii) the amount they can spend each year, and iv) the need for insurance protection. Hence, this will help the members to make financial decisions relating to their money in retirement.

However, not many people are participating in the risk-sharing advantages of DB schemes. In 2011, only 3% of workers in the US private sector participated in a DB pension scheme, compared to 28% in 1979 [57]. The employers face difficulties in estimating the DB scheme liabilities to evaluate the retirement benefit expenditures. Thus, over the past few years, employers reacted to the risks by decreasing the pension benefits/increasing contribution rates and made a significant shift from DB to DC schemes. Consequently, through DC schemes, the primary responsibilities for funding retirement have been shifted from employer to employees.

In literature, there are only few papers dealing with DB pension schemes for portfolio optimisation and asset allocation. Previous studies by [43] deal with contribution rates and asset allocation strategies. They investigated a DB scheme during the pre-retirement period for a continuous time economy in which the accumulated funds for retirements are converted into annuities by assuming that the asset return is driven by a Levy process. The paper considers two mean-variance optimisation problems, which are quadratic control problems with an additional constraints on the expected value of the funding surplus. While the latter studied a DB pension scheme in infinite time such that the benefits are modelled as a geometric Brownian motion and introduces another state variable in a control problem. [11] and [132] examine ALM for pension funds using CVaR constraints which is solved using linear programming techniques.

2.3.2 Defined Contribution

Whereas the DB scheme focuses on the amount of pension benefits the member will receive at retirement, the DC plan focuses on the value of the accumulated contributions in the member's account. In a DC plan, there is a fixed predetermined contribution that must be made monthly during the member's working life which are invested, and the returns on investment are credited to the member's account. The sponsoring employer is obliged to pay a specified amount into a pension fund for each member and the members may also contribute. The employer's contribution is guaranteed but the future benefit is based on future investment return. Normally, the member's contributions are deducted from certain percentage of member's salary and the employer will add up a certain amount to match it. The member and the sponsoring employer will contribute periodically where the contributions are pooled and invested.

This scheme does not guarantee the amount of pension benefits on retirement. The member's pension benefit is dependent on how well the contributions are invested and investment earnings received from the employer, less any charges. During a good investment performance period, DC members will receive substantially higher pension benefits than a typical DB scheme. This is true because the DB scheme is invested in a way to meet the targeted benefit levels and not to maximise the benefit amount. It is very difficult to predict the final amount of the pension benefit for the DC scheme as it is depending on the investment performance. The total accumulated value in the member's fund is used to provide pension benefits on retirement but we would not know in advance (during the contribution period) the value of the pension benefit we will get.

Traditionally, in most countries, DB schemes used to be the most popular type of pension but now gradually shifted towards DC scheme. Many countries are now using DC schemes including United Kingdom, United States, Germany, Australia, Singapore, Malaysia and New Zealand. Basically, the shifts are driven due to the desire for risk mitigation and reduce pension costs [20]. In a typical DB scheme, the sponsoring employer covers most of the risk whereas in DC, most of the risk is passed to the scheme members but they reap most of the benefits in the scheme. The shifts are due to many factors including increasing DB scheme costs due to increasing longevity risks, additional burden by pension accountants and regulators, declining long term interest rates, and underfunded DB pension fund.

The basic difference between DC and DB schemes is the risk allocation. Many employers are switching from DB to DC plans as a way to reduce the investment risk as it is assumed by each retiree and not by the employer [34]. The shift passes the risk away to the employees, who lack resources and knowledge in achieving the same results of investment returns as in larger DB plans which are managed by the employers. The DC plans also appear to be riskier because, at retirement, there is no annuity provided and the members have to seek for themselves the best annuity products available from the insurance company. Pension life annuity is a form of longevity insurance that converts the amount of pension benefits received on retirement with a regular income payment throughout a person's life. This will protect the member from outliving his DC savings in the retirement period. The pension life annuity can be purchased using the amount of funds available in the pension pot. When buying a life annuity, the member bears inflation risk, the risk of losses in the real value of the pension due to unanticipated inflation. However, there is no obligation for the DC's members to purchase the annuities with their savings upon retirement though it is compulsory in some countries. It is important for the employees to realise the implications of not having a secure pension income and the possible consequences that might occur. The members are also exposed to a higher longevity risk as compared in the DB scheme due to pooling of individual risk.

DC schemes have found to be extremely higher risk compared to DB schemes by [9] and the value at risk estimates are sensitive to asset allocation strategies and asset return models. In addition, the study found that static strategies with a higher equity weight on the portfolio will deliver better results than the dynamic strategies.

2.3.3 Hybrid Pension Schemes

Even within the shift from DB to DC scheme, some of the pension sponsors have converted to hybrid plans that combine the characteristics of both DB and DC schemes. Under hybrid schemes, the fund is treated as DB plans for tax, accounting and regulatory purposes but the benefits are expressed similar to DC scheme which are often payable as a lump sum on retirement. Plan sponsors are trying to share the pension risks more evenly with their employees while giving the best benefit available by providing alternative pension schemes which can satisfies both parties. There are different types of hybrid schemes available in the literature and in real life.

One of the popular types is the cash balance (CB) scheme which was introduced in the United States but has become popular in some other countries such as the United Kingdom and Japan. The features of CB plans are similar to DC schemes but the scheme has some DB scheme features. It works like a DC scheme such that the employer decides the contribution levels (which are normally expressed as a percentage of salary) and the benefit is expressed as a lump sum payment upon retirement just like a DB scheme. The employer is responsible in contributing to the scheme but the employee may also contribute at the minimum level. The trustees are responsible in all aspects of the scheme including investment and paying out benefits to its members. If the investment return is less than the minimum guaranteed benefit level, the employer will make further contributions so that the minimum guaranteed level can be achieve. But if the return on investment is higher than the minimum guaranteed level, the employer may hold back some, or all, of the excess investment return and use this amount for the future when the minimum guaranteed investment return is not achieved. Employers bear the investment risk prior to retirement which is similar to DB schemes. Upon retirement, the retirees can buy an annuity to fund for the retirement.

Another type of hybrid scheme found in the literature is the group self-annuitisation (hereafter, GSA) which provides annuity benefit payments adjustments from a longevity risk pooling fund. GSA takes advantage of economies of scale by allowing the members to pool mortality risk together and protects against the longevity risk. In return, the member will receive a regular benefit annuity payment, calculated based on chosen mortality and investment basis. The scheme mixes characteristics of annuities and the mutual fund. It acts like a mutual fund in terms of pooling the wealth of the members but the members cannot withdraw their money; and acts like annuities such that the funds of dying members will be redistributed among other surviving members in the pool fund [122]. The benefit is payable as an ordinary annuity such that each member who survived in each period will receive a constant benefit payment known as a mortality credit or survivor credit. If the actual value in the investment return is higher than expected, then the benefit payment will be larger. Whereas, a lower actual rate of mortality than expected will gives lower benefit payments to the GSA members. Basically, when a member dies over the short time interval $(t-, t)$, their wealth is shared equally between other surviving members. By assuming that we have 100 people in the scheme, and suddenly one of them dies, leaving wealth of £99,000, then every survivor in the fund will each receive a mortality credit of £$99,000 \div 99$ people = £$1,000$. There is no fixed time to receive the benefit as it is dependent on the death of the fund members in which we don't know the exact time of the death. However, according to [45] and [122], as the pool size become infinite, the mortality risk is eliminated and the deaths will occur at a constant rate. Therefore, surviving members will receive a continuous mortality credit. The more members in the fund, the lower is the mortality credit but the higher probability to earn a frequent mortality credit. And, the bigger wealth of the dying member has, the larger mortality credit or benefit the members will receive. When death occurs more frequently, the surviving GSA members get more frequent benefit from it. So, in general, the benefit is adjusted, based on the previous period benefit multiplied with two factors which are mortality experience and interest rate adjustment.

There is a possibility of moral hazard for this model since the benefits are coming from the death of other members. Moral hazard is the risk that one party entered a transaction with an agenda to earn profit from it. In addition, the members may also use their survival probabilities information in deciding to purchase the scheme. The adverse selection can be determined by looking at the changes in annuity demand with changes to the probability of survival. The research by [128], has found that the adverse selection problem does not exist at all in the GSA scheme. Even a member has a high survival rate; the purchasing behaviour towards GSA remains unchanged. This is true because if there are increases in the GSA purchases, the return on investment does not change much due to the fact that the longevity risk is absorbed by the member themselves. Systematic longevity risk refers to the portion of the risk that is non-diversifiable. [122] also stated that the GSA members have to bear idiosyncratic and systematic mortality risk in which the mortality credit is directly linked to the population development. Therefore, across time, the member will receive less and less benefit payments even though he survives the other annuitants.

There are several authors who have studied GSA. [122] studied the impact of the pool size on the risk to which the members are exposed. It shows the mortality benefits obtained from pooling the mortality risk in the scheme. In a similar vein, [129] compares people's preferences over GSA schemes and standard annuity schemes and examines how much people were willing to spend to protect themselves from aggregate mortality risk. Recently, [45] has analysed GSA from the member's view by comparing it to a mortality-linked fund. The study has found that the GSA has a higher expected return and members are willing to bear mortality risk in the fund. While [103] has studied the implications of pooling longevity risk through GSA. [106] has expand the study by [103] and proposed a method for collective risk pooling of systematic mortality in GSA rather than individual bearing the risk. So far, there is no literature relating to ALM studies discussing on GSA scheme. Our study will contribute to the literature by including GSA along with other types of pension schemes.

2.4 Conclusion

To sum up, the main aim of this chapter was to provide a comprehensive literature review to the reader in three main areas covered in the thesis – volatility and mortality modelling with applications to pension schemes. In particular, the volatility modelling are discussed in Chapter 3, where a review of econometric models is used to model the asset returns covariances in constructing optimal asset portfolios. Chapter 4 is mainly related to mortality modelling where a novel approach is used to propose alternative estimation methodologies for mortality modelling in a single and multiple population settings. Furthermore, Chapter 5 extends the study of mortality modelling by attempting to model mortality rates using econometric approaches. Chapter 6 combines elements from both the volatility and mortality modelling and evaluate their performance in pension schemes.

The objective of this study is to develop volatility and mortality models that could be used in asset liability management in pension schemes. This study provides a comprehensive study of various advance multivariate DCC GARCH models which are used for construction of optimal portfolios in modelling asset return covariances. Then, several models were proposed as alternative estimation methods to the existing mortality models in a single and multiple population. This research underlined the important role that econometric volatility modelling and stochastic mortality modelling can play in managing pension schemes to ensure that future liabilities can be meet.

Chapter 3

Evaluation of Multivariate GARCH Models in an Optimal Asset Allocation Framework

This chapter proposes different approaches for modelling volatility using Dynamic Conditional Correlation (DCC) models with multivariate GARCH and evaluating the copula functions' performance to the model.

3.1 Introduction

With today's challenging environments and highly volatile markets, modelling volatility has become important for organisations in managing their assets to obtain the best possible returns. Asset allocation aims to balance the risk and return by adjusting the percentage of each asset in an investment portfolio to meet the investor's goals, investment objectives, risk tolerance and investment horizon.

The future forecast is highly dependent on the choice of the volatility modelling. It is known that volatility is not directly observable, which makes it important to have a good model to predict future volatilities. Obtaining an optimal portfolio requires estimating and forecasting very large conditional covariance matrices of the asset returns which depend on many parameters [8], [81].

This study involves a large class of different advance multivariate DCC GARCH models in modelling the mean returns and variance covariance matrices. Specifically, a symmetric GARCH model and an asymmetric version of it (GJR-GARCH) are used. These models are implemented with the multivariate normal and Student-t distributions. For the conditional mean dynamics, this study allows a constant, univariate autoregressive (AR), autoregressive-moving average (ARMA) or vector autoregression (VAR) model to be fit. In general, the model specifications to model covariances include DCC models, aDCC models, FDCC models, Generalised Orthogonal GARCH (GO-GARCH) and copula GARCH models. Different model specifications are applied to obtain out-of-sample forecasts of the mean returns and the conditional covariance matrix of all assets. The estimated asset return and covariances are used to construct optimal short selling constrained and unconstrained portfolios. The portfolio is constructed by using different optimisation strategies including minimum-variance, mean-variance, maximisation of the Sharpe ratio, maximisation of Sortino ratio, for a given risk, and minimisation of mean-CVaR.

This chapter provides a comprehensive comparative study involving various multivariate GARCH models in an optimal asset allocation setting such that different characteristics in the data can be captured. With that, the portfolio performance can be improved and useful for the construction of optimal portfolios. Most of the existing literature compares the performance of limited GARCH models, and they are based on a specific optimisation model. Several studies have tried to examine the effectiveness of using parametric copula in estimating portfolio risk measures, but the results have been inconclusive. It is unclear whether it is optimal to use copula GARCH over the sophisticated DCC model. This study, therefore, provides useful insights for those wishing to explore the different GARCH models that are available, from the simplest autoregressive models up to more complex models and use a dynamic approach for asset allocation and portfolio construction purposes.

3.2 Notation

The variables used in the models are defined below:

x_t: $n \times 1$ vector of log returns of n assets at time t.

ϵ_t: $n \times 1$ vector of mean-corrected returns of n assets at time t.

μ_t: $n \times 1$ mean conditional expectation of x_t, conditional on Φ_{t-1}, the information set up to time $t-1$.

H_t: $n \times n$ matrix of conditional variances of ϵ_t, conditional on Φ_{t-1}, the information set up to time $t-1$.

$H_t^{1/2}$: $n \times n$ matrix which is obtained from Cholesky factorisation of the time varying conditional covariance matrix of ϵ_t.

D_t: $n \times n$ diagonal matrix of time-varying standard deviations of ϵ_t at time t.

R_t: $n \times n$ positive definite conditional correlation matrix of ϵ_t at time t.

z_t: $n \times 1$ vector of independent and identically distributed random errors.

Φ_{t-1} : information set up to time $t-1$.

Q_t: $n \times n$ symmetric positive definite matrix.

$\bar{Q}$: $n \times n$ unconditional correlation matrix of the standardised error ζ_t.

3.3 Econometric models for asset returns

Volatility has some characteristics that are not directly noticeable but are commonly seen in asset returns; i.e., the presence of volatility clustering in the data, a volatility jump is rare, stationary, and has a leverage effect [126]. Various volatility models were introduced, particularly to correct the weaknesses of their inability to capture the characteristics mentioned previously.

Consider the vector stochastic process $\boldsymbol{x_t}$ which is the $n \times 1$ vector of financial returns and $\boldsymbol{\mu_t}$ is the mean vector. The multivariate GARCH model can be written as,

$$\boldsymbol{x_t} = \boldsymbol{\mu_t} + \boldsymbol{\epsilon_t}, \qquad \boldsymbol{\epsilon_t}|\boldsymbol{\Phi}_{t-1} \sim N(0, \boldsymbol{H_t}), \tag{3.1}$$

where $\boldsymbol{\epsilon_t}$ is the $n \times 1$ residuals of the process which follows a conditionally multivariate normal distribution with mean 0 and time-varying conditional covariance matrix $\boldsymbol{H_t}$. $\boldsymbol{\Phi}_{t-1}$ is the information set at time $t-1$. The residuals are modelled as,

$$\boldsymbol{\epsilon_t} = \boldsymbol{H_t^{\frac{1}{2}}} \boldsymbol{z_t}, \tag{3.2}$$

where $\boldsymbol{H_t}$ is $n \times n$ positive definite matrix of conditional variances of $\boldsymbol{\epsilon_t}$ at time t. $\boldsymbol{H_t^{\frac{1}{2}}}$ is the Cholesky factorisation of the time varying conditional covariance matrix of $\boldsymbol{H_t}$. The symbol $\boldsymbol{z_t}$ is $n \times 1$ vector of independent and identically distributed random errors such that $E[\boldsymbol{z_t}] = 0$ and $E[\boldsymbol{z_t z_t'}] = \boldsymbol{I_n}$, whereby $\boldsymbol{I_n}$ denotes the identity matrix.

By using different econometric models, different characteristics in the data can be captured which improve the portfolio performance and are useful for the construction of optimal portfolios. The results of each model are compared to see their ability in optimising portfolio.

3.3.1 Modelling mean returns

There are various models for time series which are divided into the AR and MA models. The AR(p) model can be written as

$$\boldsymbol{x_t} = \sum_{i=1}^{p} \boldsymbol{\phi}_i \boldsymbol{x}_{t-i} + \boldsymbol{\epsilon_t}. \tag{3.3}$$

For the moving average, MA (q) refers to

$$\boldsymbol{x_t} = \boldsymbol{\epsilon}_t - \sum_{j=1}^{q} \boldsymbol{\theta}_j \boldsymbol{\epsilon}_{t-j}. \tag{3.4}$$

These models are commonly used to generate new models, i.e., ARMA or VAR models. The mean returns are modelled using different estimation processes, either using a constant mean, AR, ARMA or VAR models.

The ARMA model provides a parsimonious parametrisation and further simplification in modelling multivariate time series. It has both stationary stochastic processes of the autoregression and moving average methods, which are applied to a multivariate time series data. The ARMA (p,q) model refers to the p autoregressive terms and q moving average terms, which includes the AR (p) and MA (q) models. Thus, the mean of the process modelled by ARMA (p,q),

$$\boldsymbol{x_t} = \boldsymbol{\phi_0} + \sum_{i=1}^{p} \boldsymbol{\phi_i} \boldsymbol{x}_{t-i} - \sum_{j=1}^{q} \boldsymbol{\theta_j} \boldsymbol{\epsilon}_{t-j} + \boldsymbol{\epsilon_t}, \tag{3.5}$$

where $\boldsymbol{\phi_0} \in \mathbb{R}^n$. The autoregressive coefficients is denoted by $\boldsymbol{\phi_i}$ and moving average coefficients is denoted by $\boldsymbol{\theta_j}$ if there exist real coefficients $\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_p}$ and $\boldsymbol{\theta_1}, \ldots, \boldsymbol{\theta_q}$, such that $\boldsymbol{\epsilon_t}$ is the linear innovation process of $\boldsymbol{x_t}$. $\boldsymbol{\phi_i}$ and $\boldsymbol{\theta_j}$ are $n \times n$ matrices with $\boldsymbol{\phi_i} \neq 0$ and $\boldsymbol{\theta_j} \neq 0$.

The VAR model is one of the most commonly used multivariate econometric models. We allow the conditional mean to follow a VAR structure, where the model can be represented by,

$$\boldsymbol{x_t} = \boldsymbol{c} + \sum_{i=1}^{p} \boldsymbol{\phi_i} \boldsymbol{x}_{t-i} + \boldsymbol{\epsilon_t}, \tag{3.6}$$

where $\boldsymbol{x_t}$ is vector and $\boldsymbol{c}$ denotes $n \times 1$ dimensional vector of constants. $\boldsymbol{\phi_1}, \ldots\ldots\ldots, \boldsymbol{\phi_p}$ is parameter matrices. This model has an important characteristic which is its stability. It generates stationary time series with time invariant means, variances and covariances. If the process satisfying this equation is stationary we say that the VAR is stationary. For instance, the process is defined as a weak stationarity when it satisfies two conditions which are, (i) $E(x_t) = a$ and $\mathrm{Cov}(x_t, x_s) = r(t-s)$, where a is a constant and r is a appropriate function. If one of the elements in the vector is not stationary, then the whole vector is not stationary too. Stationarity refers to the statistical properties of a process generating a time series that does not change over time. The estimated model can have stationarity ensured by using regularisation of coefficient matrices ϕ_i is one way. Another is factor modelling of x_t first, and use VAR model on the estimated factor series.

3.3.2 Modelling covariances matrix

To model the covariance matrix, different specifications for multivariate GARCH processes are used, i.e., (i) dynamic conditional correlation (DCC), (ii) asymmetric dynamic conditional correlation (aDCC), (iii) flexible dynamic condition correlation (FDCC), (iv) generalised orthogonal GARCH (GOGARCH), and (v) copula GARCH (C-GARCH).

3.3.2.1 Dynamic Conditional Correlation (DCC)

The DCC model was proposed by [50], which is a generalisation of the Constant Conditional Correlation (CCC) model from [12], to allow for the time-varying correlation matrix of multiple asset returns. In the CCC model, the conditional covariance matrix is decomposed into conditional standard deviations and a constant correlation as

$$\boldsymbol{H_t} = \boldsymbol{D_t R D_t}, \tag{3.7}$$

where $\boldsymbol{D_t}$ is the $n \times n$ diagonal matrix of time-varying standard deviations from univariate GARCH models, $\boldsymbol{D_t} = diag(\sqrt{h_1 1, t}, \ldots, \sqrt{h_{nn,t}})$, and $\boldsymbol{R} = \rho_{ij}$ is the positive definite constant conditional correlation matrix with $\rho_{ii} = 1$ for $i = 1, \ldots, n$. The dimension for h_t is $n \times 1$. The off-diagonal elements of $\boldsymbol{H_t}$, are given by

$$[\boldsymbol{H_t}]_{ij} = \rho_{ij} \sqrt{h_{ii,t} h_{jj,t}} \quad i \neq j. \tag{3.8}$$

This model is computationally attractive and simple because of the constant correlation. However, the assumptions of constant conditional correlations may be unrealistic in practice and it may be too restrictive in some cases. The DCC model allows the time-varying correlation dynamics, $\boldsymbol{R} = \boldsymbol{R_t}$. It is defined as in [60],

$$\begin{aligned}
\boldsymbol{x_t} &= \boldsymbol{\mu_t} + \boldsymbol{\epsilon_t}, \\
\boldsymbol{\epsilon_t} &= \boldsymbol{H_t^{\frac{1}{2}} z_t}, \qquad (3.9) \\
\boldsymbol{H_t} &= \boldsymbol{D_t R_t D_t},
\end{aligned}$$

such that $\boldsymbol{x_t}$ is a $n \times 1$ vector of log returns of n assets at time t. $\boldsymbol{\epsilon_t}$ is a $n \times 1$ vector of mean corrected returns of n assets at time t such that $E[\boldsymbol{\epsilon_t}] = 0$ and $Cov[\boldsymbol{\epsilon_t}] = \boldsymbol{H_t}$. $\boldsymbol{\mu_t}$ is a $n \times 1$ vector of the expected value of the conditional $\boldsymbol{x_t}$. $\boldsymbol{H_t}$ is a $n \times n$ matrix of conditional variances of $\boldsymbol{\epsilon_t}$ at time t, and $\boldsymbol{D_t}$ is a $n \times n$ diagonal matrix of conditional standard deviation of $\boldsymbol{\epsilon_t}$ at time t. Note that the conditional correlation matrix of $\boldsymbol{\epsilon_t}$ is now time varying and denoted by a symbol $\boldsymbol{R_t}$. $\boldsymbol{R_t}$ is a positive definite conditional correlation matrix and $\boldsymbol{z_t}$ is $n \times 1$ vector of independent and identically distributed random errors with $E[\boldsymbol{z_t}] = 0$ and $E[\boldsymbol{z_t z_t'}] = \boldsymbol{I_n}$. The conditional variances, $\boldsymbol{h}_t, i = 1, \ldots, n$ can be estimated separately by a simple univariate GARCH (P,Q) specification of [50],

$$\boldsymbol{h}_t = \boldsymbol{g} + \sum_{i=1}^{P} \boldsymbol{\alpha_i} \boldsymbol{\epsilon}_{t-i}^{(2)} + \sum_{i=1}^{Q} \boldsymbol{\beta_i} \boldsymbol{h}_{t-i} \qquad , \qquad (3.10)$$

where $\boldsymbol{g} \in \mathbb{R}^n$, with a dimension of $n \times 1$, and $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are $n \times n$ diagonal matrices. $\boldsymbol{\epsilon}_{t-i}^{(2)} = \boldsymbol{\epsilon}_{t-i} \bigodot \boldsymbol{\epsilon}_{t-i}$ is the Hadamard product, which is the element by element product. The elements of g and the diagonal elements of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are positive so that the conditional covariance matrix of H_t is positive definite. $\boldsymbol{R_t}$ is the conditional matrix of the standardised disturbances ζ_t, that is,

$$\boldsymbol{\zeta_t} = D_t^{-1}\epsilon_t \sim N(0, \boldsymbol{R_t}). \tag{3.11}$$

The elements of $\boldsymbol{H_t} = \boldsymbol{D_t R_t D_t}$ with $\rho_{ii} = 1$ can be written as,

$$[\boldsymbol{H_t}]_{ij} = \rho_{ij,t}\sqrt{h_{ii,t}h_{jj,t}}. \tag{3.12}$$

The conditions for the positivity of the covariance matrix $\boldsymbol{H_t}$ requires $\boldsymbol{R_t}$ to be positive definite, $\boldsymbol{g}$ and all diagonal elements of matrices $\boldsymbol{\beta}_i$ and $\boldsymbol{\alpha}_i$ are all positive. Therefore, we need to decompose $\boldsymbol{R_t}$ into,

$$\begin{aligned} \boldsymbol{R_t} &= (\boldsymbol{Q_t^*})^{-1}\boldsymbol{Q_t}(\boldsymbol{Q_t^*})^{-1}, \\ \boldsymbol{Q_t} &= (1-a-b)\bar{\boldsymbol{Q}} + a\boldsymbol{\zeta}_{t-1}\boldsymbol{\zeta}_{t-1}^{'} + b\boldsymbol{Q}_{t-1}. \end{aligned} \tag{3.13}$$

$\bar{\boldsymbol{Q}}$ =Cov$[\boldsymbol{\zeta_t}] = E[\boldsymbol{\zeta_t}\boldsymbol{\zeta_t}^{'}]$ is a $n \times n$ unconditional matrix of the standardised errors $\boldsymbol{\zeta_t}$. $\boldsymbol{Q_t^*} = diag(\sqrt{q_{1t}}, \sqrt{q_{2t}}, \ldots, \sqrt{q_{nt}})$; a and b are non-negative parameters to be estimated such that $a+b<1$ to ensure stationarity and positive definiteness of $\boldsymbol{Q_t}$. $\bar{\boldsymbol{Q}}_t$ can be estimated by,

$$\bar{\boldsymbol{Q}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\zeta_t}\boldsymbol{\zeta_t}^{'}. \tag{3.14}$$

In general, as in [50], the DCC(M,N) GARCH model is given by

$$\boldsymbol{Q}_t = \left(1 - \sum_{m=1}^{M} a_m - \sum_{n=1}^{N} b_n\right)\bar{\boldsymbol{Q}}_t + \sum_{m=1}^{M} a_m \boldsymbol{\zeta}_{t-1}\boldsymbol{\zeta}_{t-1}^{'} + \sum_{n=1}^{N} b_n \boldsymbol{Q}_{t-1}. \tag{3.15}$$

This model is estimated using a two-step approach: the first implies the estimation of univariate GARCH, and the second step is the correlation estimation. The number of the estimated parameters in the correlation process of DCC-GARCH is independent of the number of series correlated, hence allowing for a potentially large correlation matrices to be feasibly estimated. The limitation of this model is to hypothesise the same correlation dynamics of all the assets [8], [98].

3.3.2.2 Asymmetric Dynamic Conditional Correlation (aDCC)

[29] introduced an aDCC model to investigate whether conditional variances, covariances, and correlations of assets in a portfolio are sensitive to the sign of past innovations. Compared to the DCC model, this model further explores whether the positive and negative shocks are of the same magnitude or have different impacts. As in Equation 3.9, the matrix $\boldsymbol{H_t}$ is decompose into $\boldsymbol{H_t} = \boldsymbol{D_t R_t D_t}$. $\boldsymbol{H_t}$ is the conditional covariance matrix whereas $\boldsymbol{h_t}$ is the conditional variance. In aDCC, $\boldsymbol{h_t}$ is assumed to follow the GJR-GARCH models,

$$\boldsymbol{h}_t = \boldsymbol{g} + \sum_{q=1}^{Q} \boldsymbol{\beta}_i \boldsymbol{h}_{i,t-q} + \sum_{p=1}^{P} \boldsymbol{\alpha_i} \boldsymbol{\epsilon}_{i,t-p}^{(2)} + \sum_{p=1}^{P} \boldsymbol{\gamma}_i \boldsymbol{\Psi}(\boldsymbol{\epsilon}_{i,t-p} < 0) \boldsymbol{\epsilon}_{i,t-p}^{(2)} \quad , \tag{3.16}$$

where $\Psi(.) = 1$ when $\boldsymbol{\epsilon}_{i,t-p}$ is negative, otherwise $\Psi(.) = 0$. The elements of g and the diagonal elements of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are positive so that the conditional covariance matrix of H_t is positive definite. If $\boldsymbol{\gamma_i} > 0$, then the asymmetric effects exist in the conditional variance that is a negative correlation is found in the asset return and volatilities.

3.3.2.3 Flexible Dynamic Condition Correlation (FDCC)

[8] proposed the FDCC model that allows equal correlation dynamics between w groups of assets, providing a flexible parameterisation of correlation dynamics. The FDCC model is an extension of DCC model, introduces a block diagonal structure to solve the problem of equal correlations dynamics in the assets. The study by [8] found that this model provides a better portfolio allocation with a lesser risk and a greater yield when they applied to the Italian sectorial stock indexes and on a sectorial asset allocation problem. As in [8], this model can parsimoniously be written as

$$\begin{aligned}
\boldsymbol{x_t} &= \boldsymbol{\mu_t} + \boldsymbol{\epsilon_t}, \\
\boldsymbol{H_t} &= \boldsymbol{D_t R_t D_t}, \\
\boldsymbol{R_t} &= (\boldsymbol{Q_t^*})^{-1} \boldsymbol{Q_t} (\boldsymbol{Q_t^*})^{-1}, \\
\boldsymbol{Q_t} &= cc' + aa' \bigodot \boldsymbol{\zeta}_t \boldsymbol{\zeta}_t' + bb' \bigodot \boldsymbol{Q}_{t-1},
\end{aligned} \tag{3.17}$$

where $\boldsymbol{\zeta}_t$ is the standardised residuals and $\boldsymbol{\zeta}_t = \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t$. The correlations dynamics are described as $\boldsymbol{Q}_t^* = diag(\sqrt{q_{1t}}, \sqrt{q_{2t}}, \ldots, \sqrt{q_{nt}})$ such that $\boldsymbol{Q}_t^*$ rescales the elements in Q_t to ensure the R_t is a correlation matrix at any t. The variables c,a and b are partitioned n-dimensional vectors of groups of assets,

$$a = [a_1 \times i'_{m_1}, a_2 \times i'_{m_2}, \ldots, a_k \times i'_{m_k}]'.$$

$m_i (i = 1, \ldots, k)$ is the number of assets in the group i; and similarly for b and c. i_h is an h-dimensional vector of ones. The coefficients must satisfy these constraints; $a_i a_j + b_i b_j < 1 (i, j = 1, \ldots, k)$, such that k is the number of blocks or asset classes. The GARCH type parameter restriction is needed as to avoid the explosive patterns of the correlations. a and b are non negative scalars to ensure stationarity and positive definiteness of matrix $\boldsymbol{Q}_t$. Furthermore, given a suitable $\boldsymbol{Q}_0$, the starting value of Q_t, is the sum of positive definite and semidefinite matrices. This is again to ensure that $\boldsymbol{H}_t$ and $\boldsymbol{R}_t$ are also positive definite.

In this study, we divided the assets into two blocks: stock and bond indices groups. This is reasonable since the correlation dynamics within the stock group is almost similar from one asset to another, and this is also the same for bond indices.

3.3.2.4 Generalised Orthogonal GARCH (GO-GARCH)

OGARCH was introduced by [44] and [3]. The observed time series can be linearly transformed to a set of uncorrelated time series using a principal component analysis. This model has commonly been used in much research to model the conditional covariance of financial time series due to its feasibility in estimating large covariance matrices [138], [88]. For non-Gaussian data, the independent component analysis (ICA) is used to perform the orthogonal transformation. [138] applies the concept of ICA to propose the generalised OGARCH model for volatility modelling. It consists of a set of conditionally uncorrelated univariate GARCH and a linear map that allows the linkage between these components and the observed data [16]. The matrix in OGARCH model is assumed to be orthogonal with a very small subset of all possible invertible matrices, whereas in the GOGARCH model, the orthogonal requirement is relaxed.

The GOGARCH model are based on the assumption that returns are generated by a set of unobserved underlying factors that are conditionally heteroscedastic. The dependence structure of the factors are used to determine the type of factor it belongs to, such that the correlated factors make up the Factor ARCH models, whereas the uncorrelated and independent factors make the OGARCH and GOGARCH model respectively. The unobserved independent factors can be determine by using statistical transformation. The GOGARCH models use the statistical transformations to place the unobserved independent factors in an independence framework with separability and weighted density convolution which give a large scale and feasible estimation. This model is considered as the suitable model to model the stock markets since it supports the random vectors with probability distributions that are asymmetric and heavy tailed.

Most of the factor models may be represented as a special case of the BEKK model. The GOGARCH model has a restricted BEKK as following,

$$H_t = \omega + \sum_{i=1}^{m} A_i x_{t-1} x_{t-1}^{'} A_i^{'} + BH_{t-1}B', \tag{3.18}$$

such that A_i and B are restricted to have same eigenvector Z. The eigenvalues of A_i are same set of eigenvectors, which is stored in the matrix Z. The linear map links the unobserved components with the observed variables is assumed to be constant over time and invertible. The unobserved independent factors can be determine by using statistical transformation to place the unobserved independent factors in an independence framework with separability and weighted density convolution which gives a large scale and feasible estimation.

Consider a set of n assets with returns $\boldsymbol{x_t}$, which are observed for T periods, with conditional mean of $E[\boldsymbol{x}_t|\boldsymbol{\Phi}_{t-1}] = \boldsymbol{m}_t$. $\boldsymbol{\Phi}_{t-1}$ is the information set at time t, which is the σ-algebra generated by the lagged values of the outcome process of $\boldsymbol{x_t}$, i.e., $\boldsymbol{\Phi}_t = \sigma(x_{t-1}, x_{t-2}, \ldots)$. The GOGARCH model of [138] maps a vector observed process of $\boldsymbol{x}_t - \boldsymbol{m}_t$ onto a set of n-vector of a linear combination of n conditionally unobserved independent factors of $\boldsymbol{f}_t = (\boldsymbol{f}_{1t}, \ldots, \boldsymbol{f}_{nt})'$. The process $\boldsymbol{x}_t$ satisfies the representation

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{m}_t + \boldsymbol{\epsilon}_t, \\ \boldsymbol{\epsilon}_t &= \boldsymbol{C}\boldsymbol{f}_t, \end{aligned} \tag{3.19}$$

$\boldsymbol{C}$ is a non singular matrix which is invertible and constant over time. It may be decomposed into the de-whitening matrix $\boldsymbol{\Sigma}^{1/2}$, i.e., the square root of unconditional covariance and an orthogonal matrix $\boldsymbol{U}_0$. $\boldsymbol{U}_0$ can be estimated using a computational method for separating multivariate mixed signals into additive statistically independent and non-Gaussian components using ICA [22], [141].

Let $\boldsymbol{C}$ be the map that links the uncorrelated components with the unobserved process, so that

$$\boldsymbol{C} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{U}_{\boldsymbol{0}}. \tag{3.20}$$

The factors are represented as

$$\boldsymbol{f}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{z}_t, \tag{3.21}$$

where $\boldsymbol{H}_t = E[\boldsymbol{f}_t\boldsymbol{f}_t'|\boldsymbol{\Phi}_{t-1}]$ is a diagonal matrix for all t with elements $(h_{1t}, \ldots, h_{nt})$, which are the conditional variances of $\boldsymbol{f}_t$. The symbol $\boldsymbol{z}_t$ is $n \times 1$ vector of independent and identically distributed random errors, such that $E[\boldsymbol{z}_t] = 0$ and $E[\boldsymbol{z}_t\boldsymbol{z}_t'] = 1$ implies that $E[\boldsymbol{f}_t|\boldsymbol{\Phi}_{t-1}] = 0$ and $E[\boldsymbol{\epsilon}_t|\boldsymbol{\Phi}_{t-1}] = 0$. The returns for the GO-GARCH model may be expressed as

$$\boldsymbol{x}_t = \boldsymbol{m}_t + \boldsymbol{C}\boldsymbol{H}_t^{1/2}\boldsymbol{z}_t. \tag{3.22}$$

For the conditional covariance matrix, it may be written as

$$\boldsymbol{\Sigma_t} = E[(\boldsymbol{x}_t - \boldsymbol{m}_t)(\boldsymbol{x_t} - \boldsymbol{m}_t)'|\Phi_{t-1}] = \boldsymbol{CH_tC'}. \tag{3.23}$$

3.3.2.5 Copula GARCH (C-GARCH)

The multivariate random variables using a copula function, is used to capture and model non-linear relationships between the asset returns. In the multivariate GARCH, the model assumes the stock and bond returns follow a multivariate normal or Student-t distribution with linear correlation, and these assumptions are normally disregarded in many empirical finance studies. This study proposes various copula GARCH based models considering the static version of copulas and dynamic copulas (more realistic in describing time-varying dependence structure between assets returns). The out-of-sample performance is evaluated and compared with other models discussed earlier.

Let $F_1(x_1),\ldots,F_n(x_n)$ be the marginal distributions with a random vector $\boldsymbol{X} = (x_1,\ldots,x_n)$. The random vector has uniform marginal distributions when we apply the probability integral transform to each of the component $(U_1, U_2,\ldots,U_n) = F_1(x_1),\ldots,F_n(x_n)$. [121] showed that the copula can be depicted as

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n)), \tag{3.24}$$

such that n-dimensional copula $C(u_1,\ldots,u_n)$ is an n-dimensional distribution in $[0,1]^d$ with uniform marginals. The copula can be deduced from Equation 3.24 as

$$C(u_1,\ldots,u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2),\ldots,F_n^{-1}(u_n)). \tag{3.25}$$

The copula $(X_1, X_2, \ldots, X_n)$ is defined as the joint cumulative distribution function of the continuous marginal distributions, which may be written as

$$C(u_1, \ldots, u_n) = P[F_1(x_1) \leq u_1, F_2(X_2) \leq u_2, \ldots, F_n(x_n) \leq u_n]. \tag{3.26}$$

The density function may be obtained as

$$f(x_1, \ldots, x_n) = c(F_1(x_1), \ldots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i), \tag{3.27}$$

such that f_i are marginal densities and F_i^{-1} is the quantile function of the margins. The density function of a copula is given by

$$c(u_1, \ldots, u_n) =$$

$\prod_{i=1}^{n} f_i\left(F_i^{-1}(u_i)\right)$.(3.28)

This study uses elliptical copulas which have been widely used in the literature for multivariate volatility modelling, i.e., Gaussian and Student-t copulas which is explained further in Appendix A.1. In this study, the copula GARCH models are implemented using the multivariate Gaussian and Student-t distributions, with static and dynamic (DCC) estimation of the correlation. The dynamic copula models are following [102], such that the static copula approach is extended to dynamic models for the conditional case. In an elliptical distribution setting, the dynamics is added to the correlation matrix of the copula by allowing the estimation of a Student copula with disparate shape parameters for the first stage. Let the vector stochastic process of financial returns $\boldsymbol{x_t} = x_{it},\ldots,x_{nt}$ follows a copula GARCH model with μ_t modelled with joint distribution given by,

$$F(x_t|\mu_t,h_t) = C(F_1(x_{1t}|\mu_{1t},h_{1t}),\ldots,F_n(x_{nt}|\mu_{nt},h_{nt})). \tag{3.29}$$

where $F_i, i = 1,\ldots,n$ is the conditional distribution of the i^{th} marginal series density and C is the n-dimensional Copula. The conditional mean $E[x_{it}|\mathcal{F}_{t-1}] = \mu_{it}$ such that $\mathcal{F}_{t-1}$ is the information set up to time $t-1$ and the conditional variance rh_{it} follows a GARCH(1,1) process of,

$$\begin{aligned} \boldsymbol{x}_{it} &= \boldsymbol{\mu}_{it} + \boldsymbol{\epsilon}_{it}, \\ \boldsymbol{\epsilon}_{it} &= \boldsymbol{H}_{it}^{\frac{1}{2}} \boldsymbol{z}_{it}, \end{aligned} \tag{3.30}$$

where z_{it} are identically independent distributed random variables which conditionally follow a standardised skew Student distribution, $z_{it} \sim f_i(0,1,\xi_i,\nu_i)$ such that ξ_i is the skew parameter and ν_i is the shape parameter. Similar to other DCC models, the conditional variances $\boldsymbol{H_t}$ can be estimated separately by a simple univariate GARCH specification. The dependence structure of the margins is assumed to follow a Student copula with conditional correlation R_t and constant shape parameter For the conditional density, it is given by,

$$C_t\left(u_{it},\ldots,u_{nt}|R_t,\eta\right) = \frac{f_t\left(F_i^{-1}\left(u_{it}|\eta\right),\ldots,F_n^{-1}\left(u_{nt}\right)|R_t,\eta\right)}{\prod_{i=1}^{n} f_i\left(F_i^{-1}\left(u_{it}|\eta\right)|\eta\right)}, \tag{3.31}$$

where $u_{it} = F_{it}(r_{it}|\mu_{it},h_{it},\xi_i,\nu_i)$ is the Probability Integral Transformation (PIT) of each series, which are estimated using parametric approach via the first stage GARCH process. R represents the standardised correlation matrix and the symbol $F_i^{-1}(u_{it}|\eta)$

represents the quantile function of the margins subject to the common shape parameter η of the multivariate density function.

The margins and PIT estimations are performed using a parametric density approach following [69]. This approach involves estimating univariate parameters from separately maximising univariate likelihoods, and then estimating the dependence parameters from a multivariate likelihood. The marginal distributions are estimated separately from the copula model such that the conditional distribution is treated parametrically. In general, to estimate the model, it requires the estimation of the parameters of the marginal distributions separately, and then estimate the copula parameters conditioning on the estimated marginal distribution parameters which simplify the estimation problem. There are other methods that can be consider, for example, the semiparametric, non parametric, bayesian and other estimation methods such as method of moments. In this study, the parametric density approach is chosen as it provide a computational feasibility in carrying out inference with multivariate models [70].

For the dependence measures, Kendall's τ is used, as this method is based on order statistics of the sample which makes no assumption about the marginal distribution but depends only on copula C. The pearson's product moment correlation R, which measures linear correlation between two variables only characterizes the dependence structure in the multivariate normal case which can only characterize the ellipses of equal density when the distribution belongs to the elliptical class. Whereas, for the multivariate Student distribution, the correlation cannot capture the tail dependence determined by the shape parameter. The use of Kendall's τ can overcome these problem since it is based on correlations that makes no assumption on the marginal distributions but depends only on the copula C. Therefore, the use of Kendall's τ are able to translate into the correlation matrix as in (3.33), providing a method of moments type of estimator. Kendall's τ is defined as,

$$\tau(X_i, X_j) = Pr[(X_i - X_j) - (Y_i - Y_j) > 0] - Pr[(X_i - X_j) - (Y_i - Y_j) < 0], \tag{3.32}$$

where $(X_i, Y_i)'$ and $(X_j, Y_j)'$ are vectors of random variables. Kendall's τ measures the difference between the probability of concordant and discordant pairs. The pairwise measure of concordance may be represented in terms of copula functions as,

$$\begin{aligned} \tau(X_i, X_j) &= 4E[C(F_i(X_i), F_j(X_j))] - 1, \\ &= 4 \int_0^1 \int_0^1 C(u_i, u_j) dC(u_i, u_j) - 1. \end{aligned} \tag{3.33}$$

The inputs in this study are efficient (since a large number of models are employed for the estimation) and necessary as they are needed to perform the optimisation procedure.

3.4 Application to portfolio optimisation

In the previous section, a variety of econometric models which are used to estimate the variance-covariance matrix in modelling the asset return covariances has been discussed. Different models are used to determine which econometric model produces the best estimates for the asset return and covariances in constructing optimal portfolio. The use of the multivariate DCC GARCH family is particularly appealing as it preserves the ease of estimation with a small number of parameters involved, thus solving the problems of dealing with a large number of parameters in other multivariate models. There are many approaches in constructing optimal portfolios, which can be found in Section 2.2.2.

One of the earliest approaches of the portfolio theory was developed by [90], a well-known approach known as a mean-variance optimisation. It is a myopic strategy which assumes that the decision maker has a mean-variance criterion defined over the single period rate of return on the portfolio. Other related studies by [117], [52], [105], [115], [112], [11] and [4] have approached the optimisation problem using the minimisation of mean Conditional Value at Risk (CVaR). Recently, [17] and [64] proposed minimising the CVaR assuming that the dependence structure is modelled by the copula parameter.

As mentioned earlier, there are many portfolio optimisation models available in the literature, but this study chooses five of the most commonly found in the literature. In this study, five different optimisation strategies are employed: minimum-variance, mean-variance, maximising Sharpe ratio, minimising mean-CVaR and maximising Sortino ratio. The purpose of employing different optimisation strategies is to find the best optimisation strategies among the most common ones, in constructing the optimal portfolio. This is essential as to ensure that our portfolios are giving the best possible return from the investment.

Here, efficient portfolios with and without short sales are constructed, which sets box and group constraints on the weights such that the weights for each asset and the weights of groups of selected assets are restricted by lower and upper bounds, i.e., (i) portfolio without short sales, where no more than 30% is invested in each asset (i.e.,

$0 \le w_i \le 0.3, i = 1,\ldots,n$), and (ii) portfolio with short sales, where a percentage between -30% to 30% is invested in each asset (i.e., $-0.3 \le w_i \le 0.3, i = 1,\ldots,n$), and 50% is the maximum percentage to invest in bonds and stocks.

3.4.1 Minimum-variance

The asset returns are assumed to be normally distributed and each investor wants to maximise their portfolio return at a minimal risk. This portfolio problem involves quadratic programming with linear constraints such that we want to construct an efficient portfolio with the lowest possible risk. The optimal proportion allocation w_i to the ith asset is written as,

$$\sum_{i=1}^{n} w_i = 1, \tag{3.34}$$

with the returns x_i such that $i = 1,2,\ldots,N$. To characterise the portfolio, $R_p = \sum_{i=1}^{n} x_i w_i$ and the expected return of a portfolio is written as

$$E[R_p] = \sum_{i=1}^{n} \mu_i w_i,$$

such that μ_i is the mean return on asset i and the variance of the portfolio is given by

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j = w^T Q w,$$

where Q is the covariance matrix of the assets returns. The vector w denotes the weight of the asset, subject to the condition of $0 \leq w_i \leq 1$ for portfolio without short sales. To solve the optimisation model, we choose to construct a portfolio of minimal risk,

$$\text{Minimise } w^T Q w, \tag{3.35}$$

subject to $R_p = \sum_{i=1}^{n} \mu_i w_i$ and $\sum_{i=1}^{n} w_i = 1$.

3.4.2 Mean-variance

In a mean-variance optimisation model, an efficient portfolio with the lowest possible risk is constructed such that the return of the portfolio is greater than the target return. The optimal proportion allocation w_i to the ith asset is $\sum_{i=1}^{n} w_i = 1$. The expected return of a portfolio is similar as minimum variance, which is written as,

$$E[R_p] = \sum_{i=1}^{n} \mu_i w_i.$$

The variance of the portfolio is given by,

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j = w^T Q w,$$

The mean-variance optimisation problem has two equivalent formulations: maximisation of the expected return or minimisation of the variance, of the followings,

1. To maximise R_p subject to $\sigma_p^2 \leq w$ and $\sum_{i=1}^{n} w_i = 1$. [1]

[1] The variance of the portfolio is a quadratic function of the vector w which has nonlinear constraints which make it difficult to solve.

2. To minimise $\frac{1}{2}\sigma_p^2$ subject to $R_p \geq r_{target}$ and $\sum_{i=1}^{n} w_i = 1$. [2]

It is possible to generate several portfolios by varying r_{target} in the variance minimisation model. If r_{target} is very large, an infeasible model may be obtained when there is no asset with expected return greater or equal to the target return. As r_{target} decreases, portfolios of assets with expected return equal to the target return may be obtained.

3.4.3 Maximising the Sharpe ratio

Now, instead of minimising the risk, the objective function is to maximise the Sharpe ratio for a given risk free rate r_f. Consider,

$$\text{Maximise } \frac{\hat{\mu}^T w - r_f}{w^T Q w}, \tag{3.36}$$

subject to $R_p = \sum_{i=1}^{n} x_i w_i$ and $\sum_{i=1}^{n} w_i = 1$. The p-dimensional vector $\hat{\mu}$ is the estimates of the expected mean of the assets.

3.4.4 Minimising mean-CVaR

Given a confidence level β and a fixed $x \in X$, VaR is defined as the smallest number l such that the probability of a loss L is not more than $1-\beta$ for losses greater than l:

$$VaR_\beta(x) = inf\{l \in \mathbb{R} : P(L > l) \leq 1-\beta\} = inf\{l \in \mathbb{R} : F_L(l) \geq \beta\}, \tag{3.37}$$

[2]This is a quadratic program with linear constraints which can be solve using standard optimisation software. r_{target} is the target expected return and the scaling factor $\frac{1}{2}$ is introduced in the objective function to simplify the calculation of derivatives.

where F_L is the distribution function of the losses. The minimisation of mean-CVaR optimisation model is formulated as

$$\text{Minimise CVaR}_{\beta}(x) = \text{Minimise } E[L|L \leq VaR_{\beta}(x)], \tag{3.38}$$

such that $R \geq r_{target}$.

3.4.5 Maximisation of the Sortino ratio

The last optimisation strategy is the maximisation of Sortino ratio, which is a prominent variant of the Sharpe ratio. This ratio uses downside deviation rather than standard deviation of the returns distribution as the measure of risk. Sortino ratio is ratio of the target return lowered by the risk-free rate and the CVaR risk, which can be written as,

$$\text{Maximise } \frac{\hat{\mu}^T w - r_f}{(w^T Q w)_{DR}}, \tag{3.39}$$

subject to $R_p = \sum_{i=1}^{n} x_i w_i$ and $\sum_{i=1}^{n} w_i = 1$. The risk-free rate is set at $r_f = 0$. In the Sortino ratio, the denominator is replaced with $(w^T Q w)_{DR}$ such that the total risk has been replaced by downside risk; portfolio managers will not be penalised for upside variability but will be penalised for variability below the minimum target return. The Sharpe ratio uses a symmetric risk concept by equally penalising downside and upside deviations from the sample mean return, whereas the Sortino ratio only consider the negative performance in the calculation of the squared returns. This performance measure provides a better capture of risk if returns are not normally distributed, for example in the case of hedge fund returns. This method is one of the most popular portfolio performance measure in the asset management industry. But, a poor estimation of the semideviation may result in a biased Sortino ratio. Normally, it is a standard practice to use the square-root-of-time rule to annualise the semideviation when it is computed on daily or weekly returns [97].

3.5 Data and statistical characteristics

This study considers 12 assets consisting of eight stocks and four bond indices of 10 years maturity, in the United States (US), United Kingdom (UK), Germany, Japan, Netherlands, Canada and Hong Kong. In particular, the data set consists of monthly observations on eight stock indices: FTSE100, MSCI Europe Excluding UK (MSEXUK), S&P 500 composite (S.PCOMP), DAX30 (DAXINDX), AEX (AMSTEOE), TOPIX (TOKYOSE), Hang Seng (HNGKNGI) and TSX composite (TTOCOMP). Four bond indices were also included: UK Benchmark 10-Year Government (BMUK10Y), US Benchmark 10-Year Government (BMUS10Y), Germany Benchmark 10-Year Government (BMBD10Y) and FTSE Britain Government Linked Bond (BGILALL). All time series data were collected from Datastream for 30 years from January 1985 to December 2014, yielding to 360 observations. The estimation was performed using R software. The following R packages are used, "rmgarch" in combination with "rugarch" and "CCgarch" for the covariance modeling. And the package "fPortfolio" was used for the optimisation part.

The historical monthly returns of these stocks from the preceding 240 months (from January 1985 to December 2004) are used as the in-sample period to estimate the models. The recursive forecast approach is employed (expanding-window), where the data from January 1985 to December 2004 are used to make the first estimation in

January 2005, data from January 1985 to January 2005 are used to make the second estimation in February 2005, and so on.

The monthly returns for the considered time period are shown in Figure 3.1, from which can be clearly seen the presence of volatility clustering. It is also obvious that the volatility of the series changes over time.

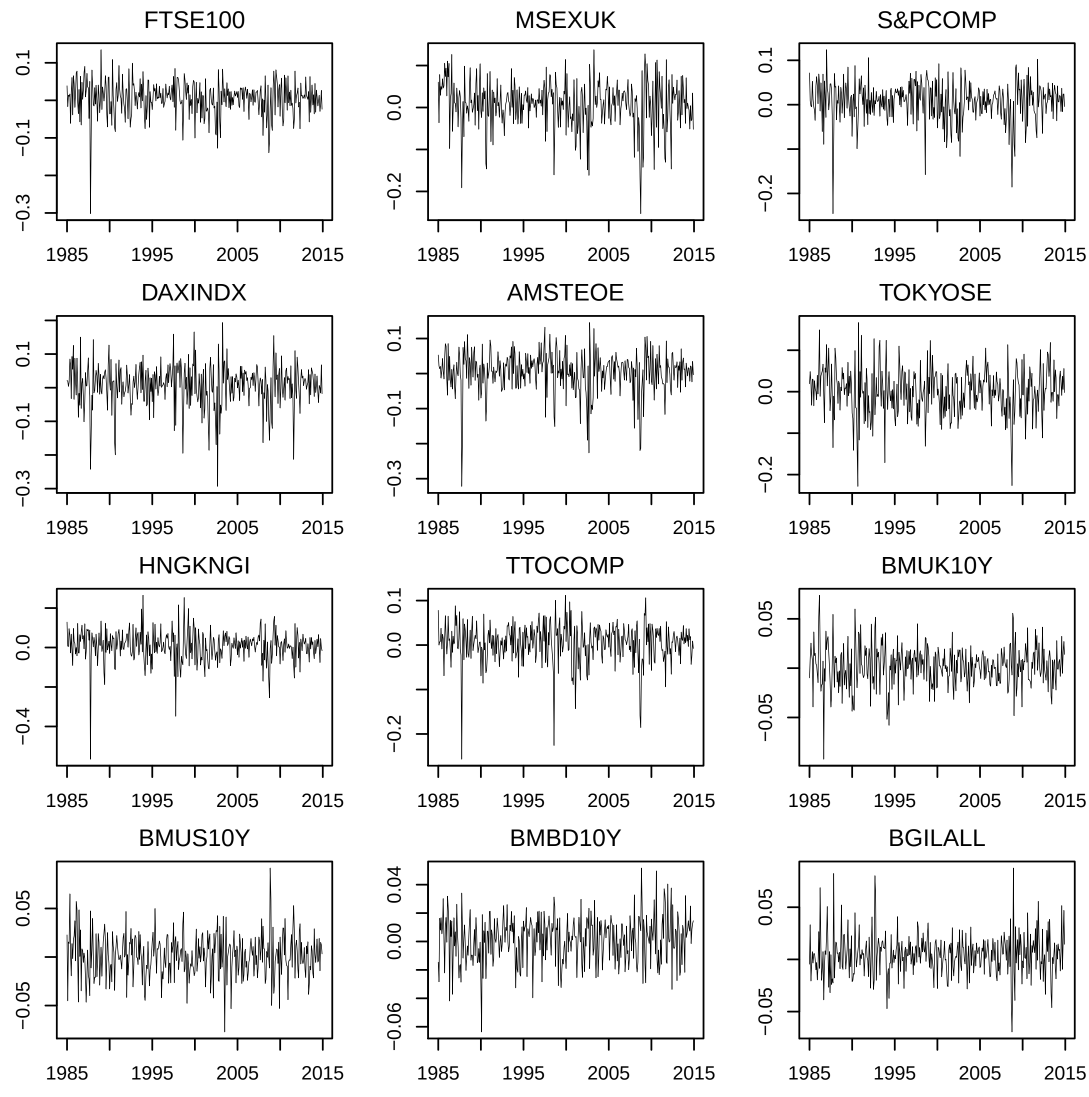


Figure 3.1. Plots of the monthly analysed asset returns series from January 1985 to December 2014.

Table 3.1 presents the basic statistical characteristics of the time series. There are substantial differences in the characteristics for each of the analysed asset return series; the stock indices have high average returns with a high volatility, while the bond returns have relatively lower average returns and volatilities.

Table 3.1: Statistical characteristics and Ljung-Box of historical monthly returns for the analysed assets from January 1985 to December 2014. The Ljung-box test is computed using 12 lags.

Assets	Rate of Returns, R_t					Sq. returns, R_t^2	Abs. returns, $\|R_t\|$
	Mean	Stdev	Skewness	Kurtosis	Q(12)	Q(12)	Q(12)
FTSE100	0.0046	0.0455	-1.1387	5.3324	7.1110	8.1196	34.587**
MSEXUK	0.0073	0.0563	-0.8630	1.8616	14.1310	55.498**	72.312**
S.PCOMP	0.0070	0.0444	-1.1008	3.6350	7.7062	20.4750	51.275**
DAXINDX	0.0069	0.0631	-0.9101	2.6948	7.3127	23.453**	35.758**
AMSTEOE	0.0044	0.0581	-1.3154	4.2702	15.0920	43.91**	57.092**
TOKYOSE	0.0012	0.0564	-0.3667	1.2278	16.2010	30.392**	11.320
HNGKNGI	0.0083	0.0782	-1.3354	8.8759	22.723**	3.5307	31.804**
TTOCOMP	0.0050	0.0437	-1.4879	6.1225	15.1270	7.1863	30.113**
BMUK10Y	0.0021	0.0206	-0.1586	1.3692	19.9310	40.983**	37.633**
BMUS10Y	0.0015	0.0218	0.0523	0.9303	16.3820	15.2260	9.5042
BMBD10Y	0.0015	0.0162	-0.2619	0.4050	27.532**	8.5521	11.2980
BGILALL	0.0044	0.0196	0.5009	2.4422	15.7770	23.109**	19.5850

Note: ** represents the p-value of < 0.05, indicating that the null hypothesis of no autocorrelation is rejected at 95% confidence level (critical value of 21.026).

The kurtosis for all of the assets ranges from 0.4 to 8.9, indicating fat tails in the asset return distributions. For the skewness, most of the assets are negatively skewed, indicating a distribution with an asymmetric tail extending toward more negative values.

The results indicate that the asset returns exhibit skewed distributions, large variance and they are not normally distributed, which means that a time-varying conditional volatility exists. A normal distribution must be symmetric with excess kurtosis of zero.

The evidence for serial correlations in the asset returns is examined using the Ljung-Box Q(m) test statistics for the rates of return R_t, absolute rates of return $|R_t|$ and squared rates of return R_t^2 (see Table 3.1).

There is evidence for a high level of autocorrelation in the absolute returns and the squared returns in which the null hypothesis of no autocorrelation is rejected at 5% level of significance for almost all of the assets series.

To test for normality, the Shapiro test is conducted and a p-value of $2.2e^{-16}$ is obtained, which again rejects the null hypothesis, indicating that the distribution is not normal. Statistical tests on these data indicate that the hypotheses for normality cannot be accepted for the majority of the assets.

The covariance matrix of the portfolio is used to quantify the deviation from the expected return and to capture the investment risk, given the standard deviations and the covariance, the correlation can be determined from $\rho_{ij} = \frac{\sigma_i \sigma_j}{\sigma ij}$.

Table 3.2 reports the pairwise correlation coefficients for the analysed asset returns. It is evident that the correlations within the stock and bond indices are relatively high. However, the pairwise correlation coefficients for the remaining pair asset returns exhibit low to medium correlations, indicating a potential for risk diversification in the constructed portfolio. The results confirm the presence of the stylised facts such as heavy tails, volatility clustering and heteroskedasticity in the asset returns distributions. The multivariate GARCH models are suitable to use to deal with these kinds of data.

Table 3.2: Correlations for the analysed asset returns series from January 1985 to December 2014.

Assets	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
FTSE100	1.0000	0.6990	0.7950	0.6840	0.7560	0.4300	0.6300	0.6850	0.1480	0.0736	0.0709	0.2200
MSEXUK	-	1.0000	0.7360	0.7830	0.7270	0.4920	0.5770	0.6530	0.0008	0.0586	0.0772	0.1450
S&PCOMP	-	-	1.0000	0.6910	0.7320	0.4630	0.6100	0.7820	0.0080	0.0259	0.1140	0.1410
DAXINDX	-	-	-	1.0000	0.8140	0.4220	0.5240	0.5970	0.0499	0.1820	0.0997	0.0579
AMSTEOE	-	-	-	-	1.0000	0.4430	0.5680	0.6660	0.0806	0.2210	0.1580	0.0808
TOKYOSE	-	-	-	-	-	1.0000	0.3570	0.4480	0.0409	0.1120	0.0836	0.0668
HNGKNGI	-	-	-	-	-	-	1.0000	0.6500	0.0210	0.0918	0.1180	0.1040
TTOCOMP	-	-	-	-	-	-	-	1.0000	0.0375	0.0776	0.1290	0.1590
BMUK10Y	-	-	-	-	-	-	-	-	1.0000	0.5820	0.7090	0.5820
BMUS10Y	-	-	-	-	-	-	-	-	-	1.0000	0.6930	0.3010
BMBD10Y	-	-	-	-	-	-	-	-	-	-	1.0000	0.3920
BGILALL	-	-	-	-	-	-	-	-	-	-	-	1.0000

3.6 Empirical analysis

The objective of this study is to investigate which econometric model produces the best estimates for the asset return and covariances in constructing optimal portfolio. As mentioned in Section 3.5, the data set of 360 observations period were split into 240 initial in sample estimation period, for the parameter estimation and model selection to get the forecast for asset return and covariance matrix. Then, an initial out-of-sample period of 120 evaluation period were used to evaluate the performance of the portfolio over the period. Each new forecast was generated by adding new observations and re-estimating the model with the new observations as the data become available. The steps were repeated for each of the optimisation strategies.

3.6.1 Evaluating the out-of-sample portfolio performance

To measure the performance of the portfolio, 120 out-of-sample periods were evaluated and perform the optimal asset allocation with a difference covariance estimator depending on the time period for each of the optimisation models.

The exercise is repeated several times using 26 different model specifications based on different combination of mean and covariance modelling discussed earlier. This is done while considering efficient portfolios with and without short sales. The models are (the abbreviation used are given in the bracket),

- DCC GARCH Normal (*DCC-MVN*)
- DCC GARCH Student (*DCC-MVT*)
- Asymmetric DCC GJR-GARCH Normal (*aDCC-MVN*)
- Asymmetric DCC GJR-GARCH Student (*aDCC-MVT*)

- FDCC GARCH Normal (*FDCC*)
- VAR DCC GARCH Normal (*VAR-MVN*)
- VAR DCC GARCH Student (*VAR-MVT*)
- ARMA DCC GARCH Normal (*ARMA-MVN*)
- ARMA Student (*ARMA-MVT*)
- GOGARCH Normal (*GG-MVN*)
- ARMA GOGARCH Normal (*ARMA-GG-MVN*)
- VAR GOGARCH Normal (*VAR-GG-MVN*)
- Static Copula Normal (*SCop-MVN*)
- Static Copula Student (*SCop-MVT*)
- Static ARMA Copula Normal (*ARMA-SCop-MVN*)
- Static ARMA Copula Student (*ARMA-SCop-MVT*)
- Static VAR Copula Normal (*VAR-SCop-MVN*)
- Static Asymmetric Copula Normal (*a-SCop-MVN*)
- Static Asymmetric Copula Student (*a-SCop-MVT*)
- Dynamic Copula Normal (*DCop-MVN*)
- Dynamic Copula Student (*DCop-MVT*)
- Dynamic ARMA Copula Normal (*ARMA-DCop-MVN*)
- Dynamic ARMA Copula Student (*ARMA-DCop-MVT*)
- Dynamic VAR Copula Normal (*VAR-DCop-MVN*)
- Dynamic Asymmetric Copula Normal (*a-DCop-MVN*)
- Dynamic Asymmetric Copula Student (*a-DCop-M*VT)

3.6.1.1 Efficient minimum-variance portfolio without short sales

Table 3.3 presents the out-of-sample portfolio performance of a minimum-variance portfolio without short sales and restricting the portfolio weights to be $0 \leq x_i \leq 0.3, i = 1, \ldots, n$, and the investment of 50% in stocks and bonds. In general, the empirical results suggest that the dynamic models are able to deliver performance gains over the static models. The best model is the aDCC-MVT, which accrues significant average monthly return of 0.38%, cumulative return of 45.48% and Sharpe ratio of 0.2013. The next best model is the VAR-SCop-MVN, which recorded an average monthly return of 0.33%, cumulative return of 39.36% and Sharpe ratio of 0.1766, but with a lower risk (1.88%) as compared to aDCC-MVT (1.91%).

Table 3.3: Descriptive statistics and out-of-sample performance of minimum-variance efficient portfolio without short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0027	0.3272	0.0191	0.1447	-0.0141	-0.0245	-0.0508
DCC-MVT	0.0031	0.3732	0.0191	0.1643	-0.0159	-0.0215	-0.0625
aDCC-MVN	0.0033	0.4007	0.0192	0.1723	-0.0137	-0.0238	-0.0627
aDCC-MVT	0.0038	0.4548	0.0192	0.2013	-0.0151	-0.0239	-0.0663
FDCC-MVN	0.0027	0.3223	0.0192	0.1420	-0.0142	-0.0244	-0.0507
VAR-MVN	0.0026	0.3081	0.0191	0.1329	-0.0136	-0.0246	-0.0681
VAR-MVT	0.0023	0.2780	0.0188	0.1207	-0.0141	-0.0252	-0.0611
ARMA-MVN	0.0027	0.3228	0.0194	0.1430	-0.0140	-0.0229	-0.0513
ARMA-MVT	0.0027	0.3286	0.0188	0.1452	-0.0138	-0.0235	-0.0673
GG-MVN	0.0031	0.3665	0.0188	0.1653	-0.0155	-0.0255	-0.0652
ARMA-GG-MVN	0.0026	0.3119	0.0188	0.1371	-0.0165	-0.0275	-0.0656
VAR-GG-MVN	0.0026	0.3104	0.0184	0.1428	-0.0160	-0.0299	-0.0612
SCop-MVN	0.0022	0.2696	0.0191	0.1209	-0.0195	-0.0331	-0.0605
SCop-MVT	0.0021	0.2526	0.0201	0.1012	-0.0212	-0.0320	-0.0663
ARMA-SCop-MVN	0.0020	0.2388	0.0189	0.1060	-0.0191	-0.0315	-0.0656
ARMA-SCop-MVT	0.0021	0.2482	0.0187	0.1109	-0.0187	-0.0285	-0.0689
VAR-SCop-MVN	0.0033	0.3936	0.0188	0.1766	-0.0184	-0.0299	-0.0623
a-SCop-MVN	0.0022	0.2639	0.0191	0.1148	-0.0183	-0.0333	-0.0613
a-SCop-MVT	0.0020	0.2453	0.0201	0.0975	-0.0209	-0.0303	-0.0673
DCop-MVN	0.0024	0.2848	0.0191	0.1253	-0.0180	-0.0333	-0.0645
DCop-MVT	0.0024	0.2915	0.0203	0.1196	-0.0207	-0.0334	-0.0718
ARMA-DCop-MVN	0.0025	0.2968	0.0193	0.1298	-0.0177	-0.0334	-0.0777
ARMA-DCop-MVT	0.0027	0.3265	0.0192	0.1379	-0.0163	-0.0314	-0.0707
VAR-DCop-MVN	0.0026	0.3147	0.0189	0.1373	-0.0194	-0.0311	-0.0670
a-DCop-MVN	0.0024	0.2841	0.0191	0.1248	-0.0118	-0.0333	-0.0644
a-DCop-MVT	0.0024	0.2917	0.0203	0.1196	-0.0207	-0.0334	-0.0717

Table A1 presents the average weight of an efficient portfolio without short sales using the minimum-variance. All models invested mainly in BMBD10Y which is a bond index, ranging from 20% to 30% out of the overall investment. The next biggest allocation is invested in stock indices, i.e., FTSE100, S&PCOMP and TTOCOMP with around 10% to 22% of total investment. Note that, most of the models allocate only a very small percentage to DAXINDX with only 0 to 1% out of total investments.

3.6.1.2 Efficient minimum-variance portfolio with short sales

The results presented in Table 3.4 are based on the out-of-sample performance of the constructed minimum-variance efficient portfolio with short sales based on the constraints of the portfolio weights to be $-0.3 \leq x_i \leq 0.3, i = 1, \ldots, n$ and the investment of 50% in stocks and bonds.

By using these constraints, the multivariate aDCC-MVT once again outperforms the other models. The aDCC-MVT has the highest cumulative return of 57.34% with the Sharpe ratio of 0.2603, but at a higher risk of 1.89%. On the other hand, the GOGARCH-MVN and DCC-MVT models produce a high average return at a lower risk. By incorporating the copula function, it does not bring any improvement to the model performance. For mean-variance portfolios with short sales, the GOGARCH, DCC and aDCC models perform well compared to other models.

Similar to the portfolio without short sales, all models have invested mainly in bonds, i.e., the highest allocation is to asset BMBD10Y with about 30% of the investment, while a substantial fraction of the investment is allocated to stock indices (see Table A2). This is expected since bond indices are known to be 'safer' as they have medium volatilities with lower average returns. If the investor is risk-averse, then he will take

Table 3.4: Descriptive statistics and out-of-sample performance of minimum-variance efficient portfolio with short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0029	0.3444	0.0187	0.1611	-0.0138	-0.0219	-0.0317
DCC-MVT	0.0039	0.4670	0.0187	0.2148	-0.0107	-0.0160	-0.0374
aDCC-MVN	0.0038	0.4567	0.0187	0.2100	-0.0114	-0.0167	-0.0462
aDCC-MVT	0.0048	0.5734	0.0189	0.2603	-0.0098	-0.0163	-0.0432
FDCC-MVN	0.0029	0.3474	0.0189	0.1596	-0.0139	-0.0205	-0.0317
VAR-MVN	0.0029	0.3439	0.0185	0.1569	-0.0139	-0.0199	-0.0409
VAR-MVT	0.0027	0.3269	0.0181	0.1508	-0.0118	-0.0188	-0.0404
ARMA-MVN	0.0030	0.3588	0.0191	0.1644	-0.0116	-0.0186	-0.0281
ARMA-MVT	0.0034	0.4025	0.0184	0.1870	-0.0114	-0.0192	-0.0392
GG-MVN	0.0039	0.4676	0.0186	0.2127	-0.0134	-0.0228	-0.0379
AR-GG-MVN	0.0038	0.4513	0.0189	0.1980	-0.0164	-0.0226	-0.0428
VAR-GG-MVN	0.0035	0.4213	0.0183	0.1915	-0.0125	-0.0242	-0.0364
SCop-MVN	0.0020	0.2380	0.0184	0.1133	-0.0199	-0.0242	-0.0534
SCop-MVT	0.0018	0.2147	0.0205	0.0817	-0.0194	-0.0311	-0.0624
ARMA-SCop-MVN	0.0021	0.2525	0.0181	0.1174	-0.0161	-0.0289	-0.0532
ARMA-SCop-MVT	0.0021	0.2482	0.0178	0.1170	-0.0169	-0.0248	-0.0550
VAR-SCop-MVN	0.0035	0.4162	0.0172	0.2002	-0.0167	-0.0274	-0.0610
a-SCop-MVN	0.0019	0.2251	0.0184	0.1069	-0.0193	-0.0237	-0.0559
a-SCop-MVT	0.0018	0.2156	0.0205	0.0806	-0.0203	-0.0289	-0.0645
DCop-MVN	0.0017	0.2088	0.0187	0.0988	-0.0166	-0.0243	-0.0630
DCop-MVT	0.0016	0.1917	0.0211	0.0761	-0.0205	-0.0352	-0.0793
ARMA-DCop-MVN	0.0022	0.2689	0.0195	0.1188	-0.0152	-0.0206	-0.0804
ARMA-DCop-MVT	0.0024	0.2921	0.0188	0.1248	-0.0164	-0.0224	-0.0642
VAR-Cop-MVN	0.0027	0.3261	0.0186	0.1389	-0.0201	-0.0262	-0.0487
a-DCop-MVN	0.0017	0.2082	0.0187	0.0984	-0.0166	-0.0245	-0.0624
a-DCop-MVT	0.0016	0.1906	0.0211	0.0755	-0.0205	-0.0353	-0.0794

the risk of investing in a higher risk portfolio. Note that the allocation to DAXINDX, AMSTEOE, HNGKNGI, BMUK10Y, and BGILALL are different across the models.

3.6.1.3 Efficient mean-variance portfolio without short sales

By using the same constraints, the results for the out-of-sample performance of mean-variance optimisation without short sales is presented in Table 3.5.

Given a target return r_{target} = 0.33%, the multivariate DCC-MVT model has the highest cumulative return of 40.99% and achieved a higher Sharpe ratio of 0.1528. Next, the average weight of an efficient portfolio without short sales is evaluated using the mean-variance optimisation. On the basis of the results in Table A3, similar to

Table 3.5: Descriptive statistics and out-of-sample performance of mean-variance efficient portfolio without short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0031	0.3672	0.0203	0.1530	-0.0199	-0.0240	-0.0867
DCC-MVT	0.0034	0.4099	0.0213	0.1528	-0.0192	-0.0338	-0.0742
aDCC-MVN	0.0033	0.4007	0.0192	0.1723	-0.0137	-0.0238	-0.0627
aDCC-MVT	0.0030	0.3629	0.0210	0.1356	-0.0182	-0.0288	-0.0801
FDCC-MVN	0.0031	0.1533	0.0203	0.1533	-0.0199	-0.0239	-0.0867
VAR-MVN	0.0038	0.4610	0.0200	0.1852	-0.0195	-0.0240	-0.0723
VAR-MVT	0.0036	0.4379	0.0205	0.1708	-0.0183	-0.0311	-0.0745
ARMA-MVN	0.0028	0.3400	0.0205	0.1457	-0.0193	-0.0258	-0.0857
ARMA-MVT	0.0022	0.2656	0.0189	0.1159	-0.0174	-0.0317	-0.0681
GG-MVN	0.0023	0.2816	0.0188	0.1282	-0.0175	-0.0295	-0.0743
ARMA-GG-MVN	0.0029	0.3440	0.0187	0.1534	-0.0171	-0.0293	-0.0719
VAR-GG-MVN	0.0021	0.2535	0.0184	0.1194	-0.0202	-0.0334	-0.0697
SCop-MVN	0.0022	0.2690	0.0191	0.1222	-0.0189	-0.0332	-0.0610
SCop-MVT	0.0022	0.2593	0.0201	0.1038	-0.0202	-0.0326	-0.0666
ARMA-SCop-MVN	0.0018	0.2157	0.0192	0.0991	-0.0186	-0.0332	-0.0737
ARMA-SCop-MVT	0.0021	0.2519	0.0186	0.1135	-0.0189	-0.0301	-0.0699
VAR-SCop-MVN	0.0033	0.3919	0.0201	0.1624	-0.0211	-0.0302	-0.0589
a-SCop-MVN	0.0022	0.2611	0.0191	0.1176	-0.0194	-0.0336	-0.0598
a-SCop-MVT	0.0022	0.2609	0.0201	0.1039	-0.0205	-0.0516	-0.0645
DCop-MVN	0.0026	0.3142	0.0194	0.1293	-0.0160	-0.0263	-0.0674
DCop-MVT	0.0024	0.2824	0.0202	0.1167	-0.0208	-0.0329	-0.0717
ARMA-DCop-MVN	0.0026	0.3075	0.0195	0.1210	-0.0164	-0.0270	-0.0659
ARMA-DCop-MVT	0.0027	0.3265	0.0192	0.1379	-0.0163	-0.0314	-0.0707
VAR-DCop-MVN	0.0023	0.2804	0.0189	0.1238	-0.0180	-0.0320	-0.0671
a-DCop-MVN	0.0026	0.3133	0.0194	0.1283	-0.0164	-0.0266	-0.0672
a-DCop-MVT	0.0024	0.2823	0.0202	0.1167	-0.0207	-0.0329	-0.0717

minimum-variance portfolios, all models invested mainly in the bond index BMBD10Y, ranging from 26% to 30% out of the overall investment. FTSE100 also has a high allocation at around 10% to 22% across all models. Most of the models do not invest in stock HNGKNGI, which has high volatility, while some models allocate only a small proportion, i.e., around 1%.

3.6.1.4 Efficient mean-variance portfolio with short sales

The results for the optimal mean-variance portfolios with the weights constraints of $-0.3 \leq x_i \leq 0.3, i = 1, \ldots, n$, and a target expected return of 0.34%, are presented in Table 3.6. The multivariate aDCC-MVT, VAR-MVT and VAR-MVN models outperform the other competing models. In general, these models have higher Sharpe ratios, i.e., 0.2100 (aDCC-MVN), 0.1716 (VAR-MVN) and 0.1715 (VAR-MVT). These models also have lower values of the corresponding risk measures such as the volatility and the value at risk values. In general, when short sales is allowed, all models short sell the stock HNGKNGI, ranging from -1% to -6% (refer Table A4). On the other hand, all models take long positions of their portfolios to FTSE100, S&PCOMP, TOKYOSE, TTOCOMP, BMUK10Y, and BMUS10Y.

Table 3.6: Descriptive statistics and out-of-sample performance of mean-variance efficient portfolio with short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0026	0.3081	0.0207	0.1286	-0.0184	-0.0287	-0.1004
DCC-MVT	0.0032	0.3837	0.0220	0.1336	-0.0193	-0.0370	-0.0583
aDCC-MVN	0.0038	0.4567	0.0187	0.2100	-0.0114	-0.0167	-0.0462
aDCC-MVT	0.0026	0.3095	0.0215	0.1095	-0.0220	-0.0270	-0.0767
FDCC-MVN	0.0026	0.3070	0.0207	0.1281	-0.0184	-0.0286	-0.1004
VAR-MVN	0.0038	0.4517	0.0208	0.1716	-0.0148	-0.0268	-0.0789
VAR-MVT	0.0038	0.4580	0.0209	0.1715	-0.0169	-0.0327	-0.0646
ARMA-MVN	0.0026	0.3123	0.0211	0.1568	-0.0180	-0.0279	-0.0942
ARMA-MVT	0.0021	0.2491	0.0184	0.1170	-0.0160	-0.0288	-0.0555
GG-MVN	0.0029	0.3497	0.0186	0.1558	-0.0192	-0.0303	-0.0682
AR-GG-MVN	0.0036	0.4297	0.0189	0.1916	-0.0189	-0.0278	-0.0634
VAR-GG-MVN	0.0030	0.3610	0.0183	0.1613	-0.0191	-0.0294	-0.0646
SCop-MVN	0.0020	0.2379	0.0184	0.1155	-0.0197	-0.0240	-0.0550
SCop-MVT	0.0018	0.2175	0.0206	0.0819	-0.0198	-0.0290	-0.0640
ARMA-SCop-MVN	0.0016	0.1861	0.0193	0.0871	-0.0169	-0.0263	-0.0751
ARMA-SCop-MVT	0.0023	0.2739	0.0178	0.1305	-0.0176	-0.0272	-0.0544
VAR-SCop-MVN	0.0042	0.5008	0.0192	0.2159	-0.0206	-0.0313	-0.0634
a-SCop-MVN	0.0019	0.2310	0.0184	0.1092	-0.0200	-0.0244	-0.0522
a-SCop-MVT	0.0019	0.2241	0.0205	0.0851	-0.0197	-0.0290	-0.0629
DCop-MVN	0.0023	0.2703	0.0189	0.1148	-0.0146	-0.0246	-0.0604
DCop-MVT	0.0015	0.1806	0.0211	0.0713	-0.0210	-0.0352	-0.0791
ARMA-DCop-MVN	0.0022	0.2652	0.0191	0.1112	-0.0160	-0.0225	-0.0606
ARMA-DCop-MVT	0.0024	0.2921	0.0188	0.1248	-0.0164	-0.0224	-0.0642
VAR-Cop-MVN	0.0024	0.2898	0.0181	0.1356	-0.0171	-0.0255	-0.0677
a-DCop-MVN	0.0023	0.2735	0.0189	0.1155	-0.0145	-0.0245	-0.0610
a-DCop-MVT	0.0015	0.1806	0.0211	0.0713	-0.0210	-0.0352	-0.0791

3.6.1.5 Efficient portfolio based on maximising Sharpe ratio without short sales

In this section, an optimal portfolio based on maximising the Sharpe ratio is constructed for a portfolio without short sales. The results are presented in Table 3.7 assuming the risk-free rate r_f to be 0%. From the results, the multivariate AR-GG-MVN model outperforms the other models, by having the highest value in three out of five portfolio measurements, with a high risk of 2.49%. The other best optimal models are ARMA-MVT and ARMA-MVN. The results for the maximisation of Sharpe ratio show that it has a better portfolio in terms of higher return and Sharpe ratio as compared to the mean-variance optimisation. However, this optimisation strategy has a higher risk in all of the models when compared to the mean-variance strategy.

Table A5 presents the average weight of an efficient portfolio without short sales based on maximising the Sharpe ratio. In the mean-variance optimisation, the biggest proportion of investment goes to BMBD10Y, while in maximising the Sharpe ratio, all models invested mainly in a bond index BGILALL for about 20% to 30% of the overall investment.

3.6.1.6 Efficient portfolio based on maximising Sharpe ratio with short sales

When short selling is allowed with constraints of the portfolio weights to be between $-0.3 \leq x_i \leq 0.3, i = 1, \ldots, n$, a more attractive portfolio is obtained as the results have higher monthly return, cumulative return and Sharpe ratio in most of the models as compared to the portfolio without short sales. The risk-free rate r_f is assumed to be 0%.

Table 3.8 presents the results obtained from constructing an efficient Sharpe ratio portfolio with short sales.

Table 3.7: Descriptive statistics and out-of-sample performance based on maximising Sharpe ratio without short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0041	0.4979	0.0235	0.1843	-0.0181	-0.0298	-0.0625
DCC-MVT	0.0049	0.5863	0.0239	0.1507	-0.0169	-0.0273	-0.0698
aDCC-MVN	0.0042	0.5091	0.0246	0.1796	-0.0156	-0.0320	-0.0706
aDCC-MVT	0.0053	0.6311	0.0248	0.2111	-0.0205	-0.0274	-0.0796
FDCC-MVN	0.0042	0.5047	0.0236	0.1864	-0.0181	-0.0300	-0.0625
VAR-MVN	0.0021	0.2469	0.0241	0.0783	-0.0212	-0.0400	-0.0861
VAR-MVT	0.0023	0.2744	0.0240	0.0870	-0.0229	-0.0400	-0.0846
ARMA-MVN	0.0075	0.9058	0.0243	0.3078	-0.0194	-0.0275	-0.0735
ARMA-MVT	0.0086	1.0312	0.0230	0.3474	-0.0153	-0.0215	-0.0518
GG-MVN	0.0047	0.5622	0.0256	0.1801	-0.0258	-0.0351	-0.0807
AR-GG-MVN	0.0132	1.5850	0.0249	0.5195	-0.0155	-0.0263	-0.0667
VAR-GG-MVN	0.0024	0.2938	0.0239	0.0934	-0.0247	-0.0400	-0.0835
SCop-MVN	0.0035	0.4151	0.0191	0.1842	-0.0191	-0.0437	-0.0689
SCop-MVT	0.0034	0.4034	0.0251	0.1354	-0.0220	-0.0428	-0.0674
ARMA-SCop-MVN	0.0049	0.5856	0.0236	0.2066	-0.0202	-0.0417	-0.0728
ARMA-SCop-MVT	0.0046	0.5551	0.0226	0.2088	-0.0202	-0.0373	-0.0826
VAR-SCop-MVN	0.0047	0.5588	0.0243	0.1890	-0.0247	-0.0354	-0.0551
a-SCop-MVN	0.0033	0.4020	0.0249	0.1326	-0.0177	-0.0360	-0.0875
a-SCop-MVT	0.0035	0.4236	0.0251	0.1409	-0.0212	-0.0413	-0.0686
DCop-MVN	0.0038	0.4571	0.0248	0.1508	-0.0181	-0.0426	-0.0915
DCop-MVT	0.0032	0.3844	0.0252	0.1318	-0.0189	-0.0406	-0.0730
ARMA-DCop-MVN	0.0035	0.4221	0.0236	0.1465	-0.0193	-0.0416	-0.0794
ARMA-DCop-MVT	0.0040	0.4763	0.0242	0.1608	-0.0250	-0.0436	-0.0700
VAR-Cop-MVN	0.0043	0.5184	0.0244	0.1906	-0.0223	-0.0341	-0.0763
a-DCop-MVN	0.0040	0.4768	0.0248	0.1567	-0.0166	-0.0385	-0.0982
a-DCop-MVT	0.0030	0.3566	0.0253	0.1245	-0.0196	-0.0403	-0.0700

Table 3.8: Descriptive statistics and out-of-sample performance based on maximising Sharpe ratio with short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0047	0.5592	0.0249	0.1953	-0.0240	-0.0414	-0.0734
DCC-MVT	0.0064	0.7621	0.0252	0.2483	-0.0137	-0.0290	-0.0420
aDCC-MVN	0.0055	0.6627	0.0285	0.1932	-0.0179	-0.0321	-0.0545
aDCC-MVT	0.0072	0.8675	0.0270	0.2707	-0.0161	-0.0254	-0.0519
FDCC-MVN	0.0046	0.5510	0.0247	0.1945	-0.0130	-0.0221	-0.0554
VAR-MVN	0.0042	0.5033	0.0263	0.1608	-0.0196	-0.0306	-0.0713
VAR-MVT	0.0043	0.5216	0.0263	0.1681	-0.0181	-0.0312	-0.0702
ARMA-MVN	0.0159	1.9050	0.0263	0.5867	-0.0073	-0.0144	-0.0362
ARMA-MVT	0.0159	1.9046	0.0259	0.5961	-0.0039	-0.0145	-0.0292
GG-MVN	0.0061	0.7293	0.0279	0.2096	-0.0247	-0.0402	-0.0566
AR-GG-MVN	0.0244	2.9290	0.0263	0.9171	-0.0032	-0.0101	-0.0289
VAR-GG-MVN	0.0046	0.5485	0.0264	0.1748	-0.0238	-0.0364	-0.0734
SCop-MVN	0.0027	0.3229	0.0184	0.1508	-0.0211	-0.0524	-0.0764
SCop-MVT	0.0025	0.2992	0.0277	0.0965	-0.0257	-0.0513	-0.0758
ARMA-SCop-MVN	0.0053	0.6332	0.0258	0.2224	-0.0219	-0.0383	-0.0698
ARMA-SCop-MVT	0.0064	0.7665	0.0254	0.2668	-0.0223	-0.0423	-0.0632
VAR-SCop-MVN	0.0075	0.8961	0.0247	0.2939	-0.0177	-0.0246	-0.0496
a-SCop-MVN	0.0026	0.3094	0.0272	0.1013	-0.0188	-0.0497	-0.0953
a-SCop-MVT	0.0030	0.3620	0.0278	0.1098	-0.0246	-0.0471	-0.0755
DCop-MVN	0.0034	0.4033	0.0277	0.1143	-0.0174	-0.0452	-0.1005
DCop-MVT	0.0029	0.3462	0.0284	0.1089	-0.0281	-0.0496	-0.0810
ARMA-DCop-MVN	0.0045	0.5428	0.0262	0.1604	-0.0210	-0.0434	-0.0705
ARMA-DCop-MVT	0.0044	0.5320	0.0281	0.1516	-0.0221	-0.0489	-0.0718
VAR-Cop-MVN	0.0042	0.5087	0.0260	0.1802	-0.0262	-0.0344	-0.0748
a-DCop-MVN	0.0033	0.4000	0.0277	0.1180	-0.0222	-0.0403	-0.1063
a-DCop-MVT	0.0026	0.3175	0.0284	0.0997	-0.0279	-0.0500	-0.0816

From the results, once again, the multivariate AR-GG-MVN, ARMA-MVN and ARMA-MVT models outperform the other models. The AR-GG-MVN and ARMA-MVN models have the highest risk of 2.63% among all models, whereas ARMA-MVT has a slightly lower risk of 2.59%. The average weights of the efficient portfolios with short sales based on maximising the Sharpe ratio are presented in Table A6. Similarly as with the portfolio without short sales, the biggest asset allocation goes to BGILALL for about 18% to 30% out of the total investment. Note that some assets have different allocations across the models.

3.6.1.7 Efficient portfolio based on minimising mean-CVaR portfolio without short sales

In the mean-CVaR optimisation, the optimised efficient portfolio which has the lowest risk for a given return is computed by minimising the conditional value at risk using a 95% probability level, with similar restrictions as in other models. When the portfolio is in the long position, the results are presented in Table 3.9.

Table 3.9: Descriptive statistics and out-of-sample performance based on minimising mean-CVaR without short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0158	1.8981	0.0226	0.6754	-0.0090	-0.0184	-0.0495
DCC-MVT	0.0020	0.2420	0.0197	0.1071	-0.0214	-0.0319	-0.0645
aDCC-MVN	0.0175	2.1035	0.0227	0.7545	-0.0068	-0.0156	-0.0492
aDCC-MVT	0.0022	0.2663	0.0181	0.1147	-0.0181	-0.0256	-0.0622
FDCC-MVN	0.0104	1.2471	0.0210	0.4620	-0.0144	-0.0224	-0.0581
VAR-MVN	0.0147	1.7618	0.0226	0.6176	-0.0110	-0.0288	-0.0683
VAR-MVT	0.0102	1.2266	0.0230	0.4409	-0.0011	-0.0208	-0.0766
ARMA-MVN	0.0173	2.0812	0.0227	0.7163	-0.0086	-0.0160	-0.0439
ARMA-MVT	0.0118	1.4152	0.0214	0.4847	-0.0114	-0.0208	-0.0621
GOGARCH-MVN	0.0183	2.1956	0.0232	0.7568	-0.0047	-0.0137	-0.0504
AR-GOGARCH-MVN	0.0150	1.7944	0.0230	0.6033	-0.0124	-0.0207	-0.0513
VAR-GOGARCH-MVN	0.0167	1.9996	0.0234	0.6979	-0.0099	-0.0245	-0.0554
SCop-MVN	0.0023	0.2766	0.0233	0.1025	-0.0262	-0.0396	-0.0886
SCop-MVT	0.0057	0.6888	0.0207	0.2661	-0.0174	-0.0306	-0.0648
ARMA-SCop-MVN	0.0159	1.9050	0.0229	0.6513	-0.0113	-0.0188	-0.0520
ARMA-SCop-MVT	0.0099	1.1857	0.0207	0.4368	-0.0104	-0.0221	-0.0510
VAR-SCop-MVN	0.0150	1.7993	0.0230	0.6125	-0.0091	-0.0191	-0.0728
a-SCop-MVN	0.0142	1.7008	0.0221	0.6080	-0.0111	-0.0173	-0.0514
a-SCop-MVT	0.0062	0.7433	0.0208	0.2889	-0.0170	-0.0312	-0.0671
DCop-MVN	0.0033	0.4001	0.0225	0.1482	-0.0223	-0.0310	-0.0771
DCop-MVT	0.0032	0.3847	0.0210	0.1481	-0.0193	-0.0291	-0.0634
ARMA-DCop-MVN	0.0021	0.2483	0.0222	0.0953	-0.0196	-0.0309	-0.0867
ARMA-DCop-MVT	0.0033	0.3910	0.0236	0.1452	-0.0209	-0.0408	-0.0833
VAR-Cop-MVN	0.0033	0.4009	0.0242	0.1444	-0.0289	-0.0407	-0.0840
a-DCop-MVN	0.0031	0.3684	0.0225	0.1388	-0.0216	-0.0303	-0.0802
a-DCop-MVT	0.0030	0.3550	0.0211	0.1403	-0.0210	-0.0262	-0.0810

The aDCC-MVN model once again outperforms other models with the highest average return of 1.75%, followed by ARMA-MVN, which recorded average return of 1.73% with a similar volatility of 2.27%.

The highest asset allocation of this model mainly goes to stock TOKYOSE, which dominating the portfolio, for about 10% to 30% of total investment. The other allocations are invested in bond indices, i.e., BMBD10Y and BMUS10Y with around 8% to 30%. The average weights for the analysed assets are presented in Table A7.

3.6.1.8 Efficient portfolio based on minimising mean-CVaR portfolio with short sales

Table 3.10 reports the monthly average returns of the constructed portfolios, the cumulative returns, risk and value at risk at 90%, 95%, and 99% probability levels. The portfolio allows short selling, that is the portfolio is constructed when we restrict the portfolio weights to be $-0.3 \leq x_i \leq 0.3, i = 1, \ldots, n$, with a target expected return of 0.34%. The results show that the multivariate ARMA-SCop-MVN model has the highest average return of 4.81% with a higher risk of 3.20%. Note that, models such as a-SCop-MVN, have a higher average return of 4.62% and Sharpe ratio of 1.5803 with a much lower risk of 2.92%.

Table A8 indicates that the allocation of assets to MSEXUK, S&PCOMP, DAXINDX, AMSTEOE, HNGKNGI, BMUS10Y and BGILALL are different across models. It is also interesting to note that, only the model a-DCop-MVT short sell the asset TOKYOSE when the other models remain in long position. In particular, all models invested heavily in bond indices, i.e., BMUK10Y and BMBD10Y, ranging from 15% to 22%, and 11% to 30%, respectively.

3.6.1.9 Efficient portfolio based on maximising Sortino ratio without short sales

Now, we consider the case of the efficient Sortino ratio portfolio without short sales. This portfolio is constructed by maximising the Sortino ratio for a given risk-free rate with the same weight restrictions as in the previous models. Table 3.11 reports the performance results of the portfolios. The DCC-MVT model has the highest average return of 1.94%, as well as the highest cumulative return and a Sharpe ratio. The other best models are aDCC-MVT, a-Dcop-MVT and SCop-MVT.

Table 3.10: Descriptive statistics and out-of-sample performance based on minimising mean-CVaR with short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0465	5.5828	0.0294	1.5766	0.0200	0.0159	0.0027
DCC-MVT	0.0399	4.7859	0.0266	1.4885	0.0135	0.0051	0.004
aDCC-MVN	0.0463	5.5511	0.0293	1.5780	0.0187	0.0091	0.0028
aDCC-MVT	0.0389	4.6720	0.0267	1.4563	0.0100	0.0045	-0.0006
FDCC-MVN	0.0446	5.3541	0.0287	1.5596	0.0187	0.0073	0.0000
VAR-MVN	0.0467	5.5993	0.0314	1.4776	0.0197	0.0051	-0.0056
VAR-MVT	0.0341	4.0925	0.0296	1.1401	0.0000	0.0000	0.0000
ARMA-MVN	0.0470	5.6432	0.0303	1.5548	0.0225	0.0123	-0.0054
ARMA-MVT	0.0443	5.3178	0.0293	1.5102	0.0194	0.0082	0.0014
GOGARCH-MVN	0.0468	5.6163	0.0296	1.5741	0.0202	0.0096	-0.0050
AR-GOGARCH-MVN	0.0464	5.5669	0.0296	1.5603	0.0197	0.0081	-0.0025
VAR-GOGARCH-MVN	0.0473	5.6746	0.0315	1.4939	0.0228	0.0076	-0.0068
SCop-MVN	0.0037	0.4487	0.0293	0.1294	-0.0350	-0.0443	-0.0739
SCop-MVT	0.0408	4.8908	0.0271	1.5015	0.0177	0.0046	-0.0019
ARMA-SCop-MVN	0.0481	5.7749	0.0320	1.5049	0.0242	0.0088	0.0001
ARMA-SCop-MVT	0.0428	5.1337	0.0285	1.4939	0.0171	0.0125	-0.0027
VAR-SCop-MVN	0.0477	5.7228	0.0305	1.5573	0.0249	0.0094	0.0019
a-SCop-MVN	0.0462	5.5412	0.0292	1.5803	0.0233	0.0105	-0.0019
a-SCop-MVT	0.0410	4.9226	0.0273	1.4946	0.0171	0.0062	-0.0068
DCop-MVN	0.0041	0.4881	0.0292	0.1343	-0.0334	-0.0405	-0.0771
DCop-MVT	0.0045	0.5391	0.0307	0.1388	-0.0350	-0.0500	-0.0806
ARMA-DCop-MVN	0.0008	0.0926	0.0299	0.0144	-0.0364	-0.0450	-0.1046
ARMA-DCop-MVT	0.0033	0.3964	0.0317	0.1126	-0.0363	-0.0547	-0.1073
VAR-Cop-MVN	0.0041	0.4975	0.0292	0.1368	-0.0353	-0.0391	-0.0625
a-DCop-MVN	0.0365	4.3811	0.0263	1.4062	0.0121	0.0072	0.0000
a-DCop-MVT	0.0045	0.5382	0.0306	0.1472	-0.0325	-0.0469	-0.1015

Table 3.11: Descriptive statistics and out-of-sample performance based on maximising Sortino ratio without short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0171	2.0530	0.0233	0.7480	0.0000	-0.0131	-0.0440
DCC-MVT	0.0194	2.3319	0.0231	0.8408	0.0000	-0.0119	-0.0440
aDCC-MVN	0.0170	2.0407	0.0230	0.7624	0.0000	-0.0129	-0.0450
aDCC-MVT	0.0191	2.2893	0.0228	0.8322	0.0000	-0.0119	-0.0397
FDCC-MVN	0.0172	2.0604	0.0232	0.7497	0.0000	-0.0131	-0.0440
VAR-MVN	0.0087	1.0493	0.0229	0.3914	-0.0161	-0.0420	-0.0755
VAR-MVT	0.0031	0.3677	0.0242	0.1460	-0.0178	-0.0466	-0.0902
ARMA-MVN	0.0110	1.3225	0.0228	0.4989	0.0000	-0.0156	-0.0647
ARMA-MVT	0.0137	1.6495	0.0225	0.6074	0.0000	-0.0139	-0.0438
GOGARCH-MVN	0.0155	1.8607	0.0232	0.6801	0.0000	-0.0128	-0.0448
AR-GOGARCH-MVN	0.0149	1.7886	0.0229	0.6515	0.0000	-0.0144	-0.0576
VAR-GOGARCH-MVN	0.0080	0.9571	0.0236	0.3316	0.0000	-0.0254	-0.0629
SCop-MVN	0.0174	2.0886	0.0227	0.7775	0.0000	-0.0128	-0.0440
SCop-MVT	0.0186	2.2354	0.0228	0.8038	0.0000	-0.0119	-0.0440
ARMA-SCop-MVN	0.0118	1.4117	0.0226	0.5306	-0.0011	-0.0167	-0.0775
ARMA-SCop-MVT	0.0148	1.7713	0.0228	0.6398	0.0000	-0.0128	-0.0479
VAR-SCop-MVN	0.0083	0.9946	0.0238	0.3338	0.0000	-0.0200	-0.0538
a-SCop-MVN	0.0168	2.0194	0.0226	0.7585	0.0000	-0.0115	-0.0440
a-SCop-MVT	0.0190	2.2784	0.0231	0.8197	0.0000	-0.0119	-0.0440
DCop-MVN	0.0168	2.0131	0.0229	0.7536	0.0000	-0.0128	-0.0440
DCop-MVT	0.0187	2.2452	0.0229	0.8165	0.0000	-0.0119	-0.0440
ARMA-DCop-MVN	0.0104	1.2483	0.0221	0.4692	0.0000	-0.0142	-0.0618
ARMA-DCop-MVT	0.0103	1.2419	0.0212	0.4978	0.0000	-0.0156	-0.0446
VAR-Cop-MVN	0.0087	1.0434	0.0234	0.3681	-0.0012	-0.0173	-0.0643
a-DCop-MVN	0.0170	2.0385	0.0230	0.7669	0.0000	-0.0128	-0.0440
a-DCop-MVT	0.0188	2.2518	0.0229	0.8177	0.0000	-0.0119	-0.0440

The highest asset allocation of this model mainly goes to bond indices, i.e., BMUK10Y, BMUS10Y, and BMBD10Y dominates the portfolio with about 7% to 20% of total investment. This is true for the entire model in mean-CVaR efficient portfolios without short sales. The average weights for the analysed assets are presented in Table A9.

3.6.1.10 Efficient portfolio based on maximising Sortino ratio with short sales

We construct an optimal portfolio based on maximising Sortino ratio, with short sales such that the portfolio weight is restricted to be between $-0.3 \leq x_i \leq 0.3, i = 1, \ldots, n$ at 95% probability level. Table 3.12 reports the performance measure for this optimisation strategy.

Table 3.12: Descriptive statistics and out-of-sample performance based on maximising Sortino ratio with short sale for the econometric models under study from January 2005 to December 2014.

Model	Return	Cumulative Return	Risk	Sharpe Ratio	VaR@90%	VaR@95%	VaR@99%
DCC-MVN	0.0362	4.3470	0.0272	1.3446	0.0014	0.0007	0.0000
DCC-MVT	0.0540	6.4811	0.0325	1.6599	0.0285	0.0172	0.0052
aDCC-MVN	0.0333	3.9977	0.0263	1.2776	0.0090	0.0000	0.0000
aDCC-MVT	0.0283	3.3954	0.0252	1.1447	0.0091	0.0000	0.0000
FDCC-MVN	0.0369	4.4271	0.0274	1.3605	0.0140	0.0074	0.0000
VAR-MVN	0.0268	3.2168	0.0292	0.9214	0.0000	0.0000	0.0000
VAR-MVT	0.0164	1.9721	0.0304	0.5471	0.0000	0.0000	0.0000
ARMA-MVN	0.0229	2.7496	0.0276	0.8444	0.0000	0.0000	0.0000
ARMA-MVT	0.0149	1.7829	0.0259	0.5799	0.0000	0.0000	0.0000
GOGARCH-MVN	0.0326	3.9096	0.0256	1.2733	0.0095	0.0000	0.0000
AR-GOGARCH-MVN	0.0125	1.4958	0.0254	0.4953	0.0000	0.0000	0.0000
VAR-GOGARCH-MVN	0.0294	3.5317	0.0303	0.9507	0.0000	0.0000	0.0000
SCop-MVN	0.0359	4.3089	0.0262	1.3844	0.0114	0.0000	0.0000
SCop-MVT	0.0339	4.0714	0.0257	1.3242	0.0122	0.0000	0.0000
ARMA-SCop-MVN	0.0275	3.2972	0.0283	0.9672	0.0000	0.0000	0.0000
ARMA-SCop-MVT	0.0231	2.7711	0.0265	0.8740	0.0000	0.0000	0.0000
VAR-SCop-MVN	0.0317	3.8009	0.0301	1.0673	0.0000	0.0000	0.0000
a-SCop-MVN	0.0362	4.3458	0.0263	1.3909	0.0123	0.0000	0.0000
a-SCop-MVT	0.0339	4.0639	0.0259	1.3170	0.0142	0.0000	0.0000
DCop-MVN	0.0358	4.2960	0.0261	1.3823	-0.0117	0.0000	0.0000
DCop-MVT	0.0333	3.9993	0.0257	1.3110	-0.0130	0.0000	0.0000
ARMA-DCop-MVN	0.0254	3.0434	0.0287	0.8573	0.0000	0.0000	0.0000
ARMA-DCop-MVT	0.0197	2.3607	0.0253	0.7462	0.0000	0.0000	0.0000
VAR-Cop-MVN	0.0283	3.3995	0.0300	0.9370	0.0000	0.0000	0.0000
a-DCop-MVN	0.0365	4.3811	0.0263	1.4062	0.0072	0.0000	0.0000
a-DCop-MVT	0.0329	3.9475	0.0259	1.2847	0.0000	0.0000	0.0000

The results show that the DCC-MVT model has the highest monthly average return, cumulative return and Sharpe ratio. However, this model has the highest risk as compared to the other models. On the other hand, models like FDCC-MVN, aDCC-MVN, SCop-MVN also have a high return with a high Sharpe ratio at a much lower risk.

The average weights of asset portfolios constructed based on maximising Sortino ratio are reported in Table A10. The asset allocation for assets MSEXUK, DAXINDX, AMSTEOE, HNGKNGI, and BGILALL are different across models. The allocation to TOKYOSE asset is high; around 10% to 25% of total investment in most of the models. It is interesting to note that only the ARMA-DCop-MVN model involves short sell for this optimisation strategy.

3.7 Conclusion

This chapter provides a comparative analysis of various multivariate DCC GARCH models using a portfolio allocation strategy as the loss function. The portfolio's performance is examined in terms of the returns, variance of returns, Sharpe ratio, and the value-at-risk. Different multivariate GARCH models are applied, i.e., VAR, ARMA, DCC, aDCC, FDCC, GOGARCH and Copula GARCH models by using the normal and Student-*t* distributions involving static and dynamic copulas. The use of the multivariate DCC GARCH family is particularly appealing as it preserves the ease of estimation with a small number of parameters involved, thus solving the problems of dealing with a large number of parameters in other multivariate models.

For asset allocation process and also for risk management purposes, different models are used to see which is the best model in optimising the portfolio return. The analysis focuses on 12 assets consisting of four bonds and eight stock indices in the US, UK, Germany, Japan, Netherlands, Canada and Hong Kong. The results confirm the presence of heteroskedasticity, fat tails and volatility clustering in the asset returns of the data. The multivariate GARCH models are fitted to the portfolio returns to get the estimation of the variance-covariance matrix. Then, for the asset allocation, the box-group constrained portfolio are used, whereby the weights for each asset and the weights of groups of selected assets are constrained by lower and upper bounds.

Specifically, for the minimum-variance portfolios, the multivariate aDCC-MVT model outperformed the other models, by having the highest average monthly return, cumulative return and Sharpe ratio for both portfolios, with or without short sales. Portfolios using the mean-variance optimisation strategy have good performance measures as compared to the other models such as the multivariate DCC-MVT, aDCC-MVN, VAR-MVN and VAR-SCop-MVN models. As for maximising the Sharpe ratio, the AR-GOGARCH-MVN and ARMA-MVN models outperformed the other models. For the mean-CVaR optimisation, the models DCC-MVT, aDCC-MVT, FDCC and a-DCop-MVT are comparatively the best models. Finally, when maximising the Sortino ratio, once again the models DCC-MVT, aDCC-MVT, a-DCop-MVT and FDCC provide the best out-of-sample performance. Among the optimisation strategies, the maximisation of the Sortino ratio provides the best out-of-sample performance, but this strategy is rarely used in the literature.

Two important results emerged from this study. Firstly, the dynamic models are more capable of delivering better performance gains than the static models. These models reduce portfolio risk and improve the realised return in the out-of-sample period. Secondly, the results show that adding copula functions to the models does not give a better performance when compared to the dynamic correlation models. While adding copula may capture additional characteristics in the data, it does not improve model selection outcomes in the performance of the portfolio. The results from this study are consistent with [137] and [110]; the DCC models are not easily outperformed by any of the parametric copula models. That is, the copula model may not perform well when involving many related assets since only stock and bond indices are considered in the portfolio. Overall, these findings are useful for practitioners who are involved in portfolio optimisation and risk measurement in pension schemes. It will help the fund manager to decide the best investment strategies to ensure maximum benefit for its members.

The analysis in this chapter can be extended in several ways. The problem has been solved using limited restrictions on the weights of the constructed portfolios. Future studies could incorporate different constraints to examine the robustness of the constructed portfolio and make a comparison. There are also other possible ways to extend the study by looking at different situations in which the copula model may perform better, i.e., by using other copula models, by looking at different asset allocation horizons, using multiple asset class in the portfolio or considering other frequency of asset returns. Future studies could also consider incorporating exogenous factors variables in the models, to see the effects it may have to the estimation of mean and variance-covariance matrix. It is interesting to note that, so far in this chapter, this study has evaluated the asset management in a portfolio, the research could be extended by including the liabilities which will be discuss in the next few chapters. In this context, the methodology proposed could be useful to pension managers and asset managers in determining amount of assets that is sufficient to support pension liabilities. This study, therefore, can be a good benchmark for future related research.

Chapter 4

Mortality modelling for single and multiple population

This chapter presents a novel approach for modelling mortality in single and multiple population. Existing mortality models are mostly dealing with single population based on the famous Lee Carter model and its extension. While in this study, a novel modification of singular value decomposition (SVD) and principal component analysis (PCA) methodology are proposed to different mortality models in a single and multiple population. It is known that the mortality rates are declining quite rapidly in most developed countries which presents challenges to the pension providers. As mortality declines, it becomes essential to plan and save for retirement. A study on mortality modelling is vital in projecting the future mortality as close as possible to the actual rates to avoid future systematic losses. This study provides a novel estimation methods in modelling the mortality rates for a single and also multiple population in order to project future life expectancy across populations.

4.1 Introduction

During the past twenty years, the life expectancy in most countries has increased significantly. For example, the Human Mortality Database shows that the UK life expectancy at birth from 1922 to 2016 rose from 58.85 years to 82.84 years for females and from 55.18 years to 79.18 years for males [65]. The increase in life expectancy, demonstrating improvements in health and social welfare, presents challenges to insurers and governments which clearly give major threats to the planning of public retirement systems as well as for the private life annuities. The ageing population and rising longevity has contributed to the risk of underestimating the survival probability, thus determining inappropriate premiums. Insurance companies charge premiums to their members for covering risks in which the premium is collected in advance but the claim amounts are not known beforehand. Most of the time the amount of premium is fixed to prevent the insurer from charging additional premiums due to adverse mortality trends over time.

There is a risk for the pension provider underestimating the survival risk which may lead to inaccurate prediction of future mortality and facing difficulties in meeting pension liabilities in the future. The pension provider has to estimate the future mortality rates in advance as close as possible to the actual ones to ensure that they are capable to meet future liabilities. In a pension scheme, a good asset and liability management is crucial to plan a good investment strategy, accurate mortality rates projection and applying these results to the pension scheme. Reasonable asset modelling and good mortality modelling are essential to ensure accurate values (funding level of the scheme, actuarial liabilities and future benefit commitments) can be determined to avoid future systematic losses.

increases in life expectancy have emphasis the importance of mortality modelling and forecasting. There are many models have been proposed since the Law of Gompertz in 1825. The past two decades have seen many sophisticated models are proposed and introducing the use of stochastic modelling in predicting future mortality.

One of the most influential stochastic models of mortality rates is the Lee Carter (LC) model [82]. The LC model is robust, has a good fit over wide age ranges but lacks the smoothness of age effect, especially in small populations, and has no cohort effect. It applies the use of singular value decomposition (SVD) to decompose the age-time vector into a bilinear combination of age and period parameters with minimised modelling error. Other related methods like SVD include the principal component analysis (PCA) which have been widely used to forecast the time series of mortality and it is also a popular method when dealing with multivariate data. As an example, the paper of [24] uses PCA to estimate the mortality rates for different age groups in the United States. [140] and [67] have applied the PCA approach to the logarithm of central mortality rates in estimating the age and period effects.

The LC model has inspired various extensions (see for example; [83], [23], [14], [41] and [104]). They are different to one another based on different basis such as assumptions on the ease of implementation, age and period dimensions, incorporation of cohort effects, forecasting properties and methods of estimation. [109] generalised the LC model by incorporating the cohort effect while [127] include the multiple bilinear age-period components in the equation. A study by [23] estimates the parameter embedded in the LC model into a Poisson regression setting.

The most famous extension of the LC model is the two-factor Cairns-Blake-Dowd (CBD) model proposed by [25]. This model assumes a linear relationship between age and the logit of initial mortality rates. It assumes that each of the two parameters follows a random walk with drift (RWD), such that the rate of drift is constant and changes in the parameters are correlated. This suggests that this approach is suitable for the pricing of mortality related derivatives for a short time period. A later study by [124] has found that for a long term period, the CBD model does not necessarily resemble a random walk with drift process but for most of the periods, each of the factors can actually be modelled as a random fluctuation around a trend that are changing periodically. [14] have also found that the level of drift is not constant and allowing the fitting to their model only for the period that shows apparent linear trend of mortality only.

The CBD model has then been extended to include combinations of a quadratic age term and a cohort effect [27]. A model by [104] combined the CBD model with some features in the LC model to propose another model that is applicable to a full age range and captures the cohort effect and has a non-trivial correlation structure. Some other papers have carried out the comparisons between a wider range of models derived from the famous LC model (see, [26], [13] and [49]). The CBD model is designed for modelling mortality at higher ages, suitable for annuities and pensions.

Most of the mortality models work well for a single population. In the academic literature, there has been little study of mortality forecasting in multiple population settings. A multi population mortality model considers more than one population at one time in a joint mortality model. For example, these models may be used to model both male and female UK mortality data into one model as well as the interactions between both populations. Or, we can model the mortality across different populations at one time. In a multiple population, the mortality data set is bigger which combines the data from different sources (which can be from different countries, regions or genders), allowing a robust mortality modelling by identifying "similar" characteristics within the sub-populations. When the dimension of data increases, it gets more difficult to summarise these data. PCA provides a way to condense multivariate data by extracting the components that can best describe the data properties. It is interesting to be able to model mortality simultaneously in a multiple population to determine the relationship between the populations. Given the rapid increase in the number of mortality models introduced in the literature, there has been some recent attempts to extend these models in a multiple population setting. [131] has provided a comparative study of two population models for the assessment of basis risk in longevity hedges and gave an overview of existing multiple population mortality modelling methodologies in the actuarial and statistical literature. Generally, these stochastic models are a variation of mortality modelling from age, period and cohort effects.

The purpose of this chapter is threefold. First, this study gives structured overview of existing mortality modelling in a single and multiple population, scattered within the actuarial and demographic literature. The study provides evaluation of using different estimation methodologies in estimating the parameters in the mortality modelling. The novelty of this study, is that alternative estimation methods are proposed in modelling the mortality rates for a single and multiple population. As mentioned earlier, the mortality rates are declining rapidly which presents challenges to insurer and pension providers. Therefore, it is essential to find the best estimation methods for the mortality modelling so that the future liabilities can be meet. Also, to the best of our knowledge, such a comprehensive analysis covering different methodologies for both types of populations has not been performed before. Finally, the third goal is to analyse the performance of individual models as compared to multiple population models using the United Kingdom data. We believe that providing such an overview and comparison is an important contribution to the mortality modelling literature.

The remainder of the chapter is organised as follows: Section 2 displays the notation used in the chapter. In Section 3, the base model for the mortality modelling which is the Lee Carter model is discussed. Section 4 represents the individual population models to model the mortality rates. Section 5 describes the multiple population mortality modelling. Section 6 presents the data, empirical analysis and results of the proposed models and methods, while Section 6 provides concluding remarks and identifies the shortcomings and future implications of the study.

4.2 Notation

The variables used in the models are defined below:

$D(x,t)$: The number of deaths in a population at age x during calendar year t.

$E(x,t)$: The number of exposure to risk in a population at age x during calendar year t.

$d(x,t)$: The observed number of deaths at age x during calendar year t.

$E^0(x,t)$: The initial exposure to risk at age x during calendar year t.

$E^c(x,t)$: The central exposure to risk at age x during calendar year t. [1]

$q(x,t)$: The one-year death rate for an individual age x during calendar year t can be

[1]If only central exposures are available, the initial exposures can be approximated by adding half of the reported number of deaths to the central exposures $E^0(x,t) \approx E^c(x,t) + \frac{1}{2}d(x,t)$. When the context is clear, $E(x,t)$ is used to refer $E^0(x,t)$ or $E^c(x,t)$.

estimated as $\hat{q}(x,t) = \frac{d(x,t)}{E^0(x,t)}$.

$\mu(x,t)$: The force of mortality at age x during calendar year t.

$m(x,t)$: The central death rate at age x during calendar year t. [2]

$\bar{m}(x,t)$: The average mortality rate at age x during calendar year t.

$\alpha(x)$: The average age specific pattern by age of mortality.

$\beta(x)$: The sensitivity of the logarithm of the hazard rate at age x to the time trend represented by $\kappa(t)$.

$\kappa(t)$: The underlying time trend for the general mortality during calendar year t.

$\epsilon(x,t)$: The effects not captured by the model, which are Gaussian distributed $N(0,\sigma^2)$ randoms effects by age and time.

4.3 Lee Carter model : The base model

The first mortality model known as Lee Carter model is proposed by [82] which has been used extensively in the literature. This model describes the log of a time series of age specific death rates $m(x,t)$ as the sum of an age specific component $\alpha(x)$ which is independent of time with $\kappa(t)$ and $\beta(x)$. The Lee Carter model proposes the following mortality model,

$$\ln(m(x,t)) = \alpha(x) + \beta(x)\kappa(t) + \epsilon(x,t), \tag{4.1}$$

with the term $\epsilon(x,t)$ is the error term which is independent and identically distributed with $N(0,\sigma^2)$. This term reflects the age-specific influences that are not captured by the

[2]When the force of mortality is assumed to be constant over each year, then the force of mortality $\mu(x,t)$ and the death rate $m(x,t)$ correspond each other.

model. α_x describes the average age specific pattern by age of mortality, β_x describes which rates decline rapidly and slowly in response to changes in the index κ_t and κ_t denotes the underlying time trend for the general mortality in year t. The term $\beta_x \kappa_t$ in this model capture the joint tendency of age-specific mortality rates to evolve over time. In general, β_x can be negative for some ages implying that mortality at those ages tends to rise when falling at other ages; but in practise this does not seem to happening over the long run. Whereas when κ_t is linear in time, mortality at each age changes as its own constant exponential rate, and when κ_t goes to negative infinity, each age-specific rate goes to 0 (but negative death rates cannot occur in this model, which gives advantage for forecasting) [83].

This model is a regression framework with no observed variable on the right hand side. It is based on two-stage process: (i) estimation of parameters using a close approximation of the classical singular value decomposition (SVD) method by assuming that errors are homoscedastic such that the $\sum_x \beta(x) = 1$ and $\sum_t \kappa(t) = 0$ are imposed to distinguish a unique solution and (ii) $\alpha(x)$ is estimated as the logarithm of the geometric mean of the mortality rates, average values over time of the $\ln(m(x,t))$ such that, we want to minimise the $\epsilon_{x,t}$ for a given t value. $\kappa(t)$ is refitted so that the observed number of deaths coincide with those estimated. $\kappa(t)$ is usually modeled by an ARIMA(0,1,0) process such that,

$$\kappa(t) = \delta + \kappa_(t-1) + \xi_t \quad \xi(t) \sim N(0, \sigma_\kappa^2), \tag{4.2}$$

where δ is the drift parameter and ξ_t is a Gaussian white noise process which are normally distributed with mean zero and variance σ_κ^2. The Equation 4.1 states the mortality rate at age x in the particular year t. To predict the mortality rate in the following year, all parameters in the LC model need to be estimated. To obtain the estimation for α_x, we have,

$$\begin{aligned}
\sum_{t=1}^{T}\ln(m(x,t)) &= \sum_{t=1}^{T}(\alpha(x)+\beta(x)\kappa(t)+\epsilon(x,t)) \\
\sum_{t=1}^{T}\ln(m(x,t)) &= \sum_{t=1}^{T}\alpha(x)+\sum_{t=1}^{T}\beta(x)\kappa(t)+\sum_{t=1}^{T}\epsilon(x,t) \\
\sum_{t=1}^{T}\epsilon(x,t) &= \sum_{t=1}^{T}\ln(m(x,t))-\sum_{t=1}^{T}\alpha(x)-\sum_{t=1}^{T}\beta(x)\kappa(t) \\
\sum_{t=1}^{T}\epsilon(x,t) &= \sum_{t=1}^{T}\ln(m(x,t))-T\alpha_x-\beta_x\sum_{t=1}^{T}\kappa_t
\end{aligned} \tag{4.3}$$

As mentioned above, we have the following constraints $\sum_{t=1}\epsilon(x,t)=0$, $\sum_{t=1}\kappa(t)=0$ and $\sum_{t=1}\beta(x)=1$ then,

$$\begin{aligned}
\sum_{t=1}^{T}\ln(m(x,t))-T\alpha_x &= 0 \\
T\alpha_x &= \sum_{t=1}^{T}\ln(m(x,t)).
\end{aligned} \tag{4.4}$$

The parameter value for $\alpha(x)$ can be then estimated as, $\hat{\alpha}(x)$, that is,

$$\hat{\alpha}(x)=\frac{1}{T}\sum_{t=1}^{T}\ln m(x,t). \tag{4.5}$$

The mortality rates are observed for T number of available time periods. Once the estimated values, $\hat{\alpha}_x$ is obtained, the estimated values for $\hat{\beta}_x$ and $\hat{\kappa}_t$ can be found using SVD methods which will be explained further in the next section.

4.4 Single population model

In this section, we will discuss the variation of the single population stochastic mortality models which are based on modification of different singular value decomposition (SVD) estimation methods. In general, the predictor (systematic component), $m(x,t)$ is able to capture the effects of age x, year t and cohort effect (year of birth), $c = t - x$ to the model.

Consider the mortality rates in k populations. The realised mortality rates $\tilde{m}_i(x,t)$ at age x in year $t = 1,\ldots,T$ for a life aged $x = x_1,\ldots,x_n$ in population $i = 1,\ldots,k$, is observe. In general, for a single population, we have $i = 1$, then,

$$\tilde{m}_i(x,t) = \frac{d_i(x,t)}{E_i(x,t)}. \tag{4.6}$$

The number of deaths in a population can follow either a Poisson or a Binomial distribution such that $d_i(x,t) \sim \text{Poisson}(E_i^c(x,t)\mu_i(x,t))$ or $d_i(x,t) \sim \text{Binomial}(E_i^0(x,t), q_i(x,t))$. The average mortality rate, $\bar{m}_i(x)$, which is actually equivalent to $\alpha(x)$ for a life aged x in population i is written as,

$$\alpha(x) = \bar{m}_i(x) = \frac{1}{T}\sum_{t=1}^{T} \ln \tilde{m}_i(x,t). \tag{4.7}$$

The centralised log mortality rates is defined as follows,

$$m_i(x,t) = \tilde{m}_i(x,t) - \alpha(x). \tag{4.8}$$

m_i represents the matrix of the observed centralised log mortality rates in a particular population, which is,

$$m_i = \begin{pmatrix} m_i(x_1,1) & \dots & m_i(x_1,T) \\ \vdots & \ddots & \vdots \\ m_i(x_n,1) & \dots & m_i(x_n,T) \end{pmatrix}. \tag{4.9}$$

The mortality rates are observed for n different ages, for a total of T years. The ages $x_1,\dots,x_n$ and the years $1,\dots,T$ are the same for all populations and it is assumed that $T > n$. The individual models of order p for the centralised mortality rates m_i in a population i, is an extension of the LC model to p age and period effects, that is,

$$\ln m_i(x,t) = \beta_i^{(1)}(x)\kappa_i^{(1)}(t) + \cdots + \beta_i^{(p)}(x)\kappa_i^{(p)}(t) + \epsilon_i(x,t). \tag{4.10}$$

This can be written in matrix form as,

$$\ln m_i =_p \beta_i \, _p\kappa_i + \epsilon_i \tag{4.11}$$

such that,

$$_p\beta_i = \begin{pmatrix} \beta_i^{(1)}(x_1) & \dots & \beta_i^{(p)}(x_1) \\ \vdots & \ddots & \vdots \\ \beta_i^{(1)}(x_n) & \dots & \beta_i^{(p)}(x_n) \end{pmatrix}. \tag{4.12}$$

$$_p\kappa_i = \begin{pmatrix} \kappa_i^{(1)}(1) & \dots & \kappa_i^{(1)}(T) \\ \vdots & \ddots & \vdots \\ \kappa_i^{(p)}(1) & \dots & \kappa_i^{(p)}(T) \end{pmatrix}. \tag{4.13}$$

The age effect ${}_p\beta_i$ is a $n \times p$ and the period effect ${}_p\kappa_i$ is a $p \times T$ matrix. The residuals of ϵ_i is a $n \times T$ matrix with $E[\epsilon_i(x,t)] = 0$ for population i. The extended form of the matrix multiplication can be found in Appendix A.3.

The maximum number of age effects is $p = n$ since there are only n ages. For simplification, we have,

$$\beta_i = {}_n\beta_i \tag{4.14}$$

It is expected that by adding a second order, third or higher order age effect, the mortality model will be improved [76]. To tackle the identifiability issue, we assume $||\beta_i^{(j)}|| = 1$ (column j in matrix β_i) for all i and j, where $||.||$ refers to the Frobenius norm, which can be written as, $||\beta_i^{(j)}|| = \sqrt{\beta_i^2 + \cdots + {\beta_n}^2} = 1$. The Frobenius norm is calculated as square root of the sum of the absolute squares of its elements which measures the length of the vector $\beta_i^{(j)}$.

There are different ways in fitting the single population model. One of the way is by using the maximum likelihood estimation. Or, other methods based on generalised linear models can also be used. These methods are based on models for the number of deaths rather than models for the mortality rates, and therefore the obtained estimates for the age and period effects are strongly dependent on those ages and periods in which large number of deaths have been observed and less dependent on ages and periods in which relatively few deaths have been observed [76]. But, since this study extend the individual modelling to a model considering multiple population setting with different sizes of populations, a model that can attach the same weight to all observed mortality rates is desirable.

In the next subsection, we will review the different estimation methodologies in estimating the parameters in the mortality modelling for a single population model.

4.4.1 Singular value decomposition (SVD)

The Lee Carter model cannot be fitted by ordinary regression methods, since there are no given regressors. This is true because on the right hand side of the Equation 4.1, there are only parameters to be estimated and the unknown index κ_t. One way to estimate the parameters of the Lee Carter model is by using the singular value decomposition fitting. Once the estimation of parameters α_x is obtained, the value of β_x and κ_t are estimated by using the SVD of the residuals, essentially approximating a matrix as the product of the two vectors. The SVD method can be used to find a least squares solution when applied to the matrix of the logarithms of the rates after the averages over time of the log age-specific rates have been deducted. The first right and left vectors and the leading value of the SVD after the normalisation will give a unique solution.

Specifically, first we estimate the $\hat{\alpha}(x)$ as the Equation 4.5. Then, the parameters $\hat{\kappa}(t)$ and $\hat{\beta}(x)$ in Equation 4.1 are estimated as the first right and first singular vectors in the SVD of the matrix log m_i. Using SVD, we have a factorisation of the matrix, that is a $n \times T$ matrix of $m_i = U_i D_i V_i^T$. U_i is a $n \times n$ orthogonal matrix, D_i is a $n \times T$ diagonal matrix and V_i is a $T \times T$ orthogonal matrix which is known as the SVD of the matrix m_i. The σ_i values of D_i are known as the singular values of m_i, such that

$D_i = diag(\sigma_i(1), \sigma_i(2), \ldots, \sigma_i(n))$ with $\sigma_i(1) \geq \sigma_i(2) \geq \cdots \geq \sigma_i(n)$. The rank of matrix m_i is assume to have full rank, which is equivalent to n since we assumed $T > n$. In general,

$$\log(m_i) = \beta(x)\kappa(t), \tag{4.15}$$

The matrix U, D and V are constructed based on the definition of SVD which gives us,

$$\begin{aligned} SVD(m_i(x,t)) &= UDV^T \\ &= D_1 U_{x,1} V_{t,1} + \cdots + D_r U_{x,r} V_{t,r} \\ &= \sum_{i=1}^{r} D_i U_{x,i} V_{t,i} \end{aligned} \tag{4.16}$$

such that D_i, U_i, V_i are the ordered singular values and vectors and $\beta(x)\kappa(t) = D_1 U_{x,1} V_{t,1}$ subject to the constraints on $\kappa(t)$ and $\beta(x)$. U is a value that is dependent to age x, V is a value that is dependent to time t. Therefore, we have,

$$\begin{aligned} SVD(m_i(x,t)) &= UDV^T \\ &= \sigma_1 U_{x,1} V_{t,1} + \cdots + \sigma_r U_{x,r} V_{t,r} \\ &= \sum_{i=1}^{r} \sigma_i U_{x,i} V_{t,i}. \end{aligned} \tag{4.17}$$

The Lee Carter model use rank $r = 1$ to estimate the parameters $\beta(x)$ and $\kappa(t)$, where $m_i(x,t) = \sigma_1 U_{x,1} V_{t,1} = \beta(x)\kappa(t)$. The estimated parameter $\hat{\beta}$ can be obtained as the result in the first column of matrix U in which $\beta(x) = (u_{1,1} u_{2,1} \ldots, u_{x,1})^T$. Whereas the parameter of $\hat{\kappa}_t$ can be obtained by the multiplication of first singular value with the first column of the matrix V, where $\kappa_t = \sigma_1 \times (v_{1,1}, v_{2,1}, \ldots, v_{t,1})$. With the constraints of $\sum_t \kappa(t) = 0$ and $\sum_x \beta(x) = 1$, the estimated value of $\hat{\beta}(x)$ and $\hat{\kappa}(t)$ can be obtained as following,

$$\hat{\beta}(x) = \frac{1}{\sum_x u_{x,1}} (u_{1,1} u_{2,1} \ldots, u_{x,1})^T \tag{4.18}$$

and

$$\hat{\kappa}(t) = \sum_x u_{x,1} \times \sigma_1 \times (v_{1,1} v_{2,1} \ldots, v_{t,1}) \tag{4.19}$$

$\hat{\kappa}(t)$ are adjusted for all t due to the fact that the result for the fitted and the actual date of mortality rate may not be the same. Therefore, to ensure that $\sum_{all,x} d_{xt} = \sum_{all,x} \hat{d}_{xt}$ for all t, the parameter $\hat{\kappa}(t)$ is reestimated for all t which satisfies,

$$\hat{d}_{xt} = E_{xt} \exp(\hat{\alpha}(x) + \hat{\beta}(x)\hat{\kappa}(t)). \tag{4.20}$$

Subsequently, the Lee Carter model has been extended in the mortality forecasting literature (see [23]; [109]). The model can also be formulated within a generalised linear model framework such that the parameters in the LC model can be estimated using the maximum likelihood (ML) methods based on the choice of error distribution. The SVD is a valuable tool in estimating the parameters in the Lee Carter model. However, this method requires a full data matrix (no missing values). If there is even one missing value, then the computation is implausible. The SVD also uses a least squares method which is very sensitive to outliers. This study proposes a novel and robust modification of the SVD methodologies in estimating the parameters in the LC model by using the robust SVD, regularised SVD and also robust regularised SVD which are explained in more details in the next few subsections.

4.4.2 Robust Singular Value Decomposition (Robust SVD)

An alternative estimation for these parameters is to apply the individual mortality modeling using a robust SVD approach. The standard SVD are useful in multivariate methods because it provides the most reliable method to estimate the rank of the matrix, finding the eigenvalues and eigenvectors, define principal components and perform principal component factor analysis, and many more [5]. The robust SVD uses alternating iteratively re-weighted least squares methods. The values of eigenvalues and eigenvectors are obtained from the standard SVD, and eventually the covariance matrix is found by solving the minimisation problems using a series of weighted QR decompositions. The robust SVD is obtained by replacing the least squares regression fit in the standard SVD with a robust regression fit, which can then be applied to robust location and covariance matrix estimation.

The robust SVD uses projection pursuit method to obtain robust eigenvector estimates by explicitly solving the minimisation problem such that it finds the direction of eigenvector v_n which minimises $\rho(m_i v_n)$, where $\rho(\cdot)$ is known as the projection index [113]. This method estimates each eigenvector and eigenvalue separately which provides a better estimate and more robust as compared to estimation of all parameters simultaneously as in the minimum volume ellipsoid (MVE) estimation. The projection pursuit approach is used to estimate the eigenvectors of the covariance matrix, such that we find v_n to minimise,

$$\sum_{i=1}^{n} \rho(m_i v_n), \tag{4.21}$$

subject to the condition of $||v_n||^2 = 1$, such that m_i, is the data matrix of the observed centralised log mortality rates in a particular population.

The above equation defines a generalised estimation problem for the estimation of the eigenvectors. For the ordinary eigenvectors, it can be obtained by setting $\rho(\cdot) = ||\cdot||^2$, while the projection pursuit approach uses other functions. The iterative methods are used due to no closed form solution for both methods of the eigenvectors to solve the minimisation problem. For the ordinary eigenvectors, the QR iteration can be used, while the projection pursuit approach uses the iterative optimisation methods as in [5].

The last eigenvector v_n defines the hyperplane that minimises the sum of squared of the orthogonal distances between the data points and the hyperplane such that v_n represents the residual subspace. Let D_n denotes the projection matrix onto the subspace spanned by $v_n, \ldots, v_r$, $2 \leq n \leq r$, we have the following,

$$P_n = [v_r \ldots v_n][v_r \ldots v_n]^T, \tag{4.22}$$

After determining the last eigenvector estimate, the data is projected to the subspace orthogonal to the eigenvector by removing the last column of the current data matrix m_i. The steps are repeated until all eigenvectors are estimated such that the first eigenvector is determined by the orthogonality constraints. The final weights for each of the columns are then stored in a matrix. The covariance matrix D can then be estimated such that the ordinary SVD$(m_i(x,t)) = UDV^T$. The estimated value of $\hat{\beta}_x$ and $\hat{\kappa}_t$ can then be obtained and similarly as previous section, the value of $\hat{\kappa}_t$ are adjusted to make sure the distance of the fitted and the actual values to be as close as possible. Therefore,

$$\begin{aligned} SVD(m_i(x,t)) &= UDV^T \\ &= \tau_1 U_{x,1} V_{t,1} + \cdots + \tau_r U_{x,r} V_{t,r} \\ &= \sum_{i=1}^{r} \tau_i U_{x,i} V_{t,i}. \end{aligned} \tag{4.23}$$

where τ now represents the adjusted eigenvalues. The details of the methods are well discussed in the literature; see [5].

4.4.3 Regularised Singular Value Decomposition (SSVD)

In this section, the regularised SVD method is used to estimate the parameters using two-way functional data as in [142]. The two-way functional data works in two ways; ie. both index x and t of data matrix $m_i(x,t)$ are structured in a way that both rows and columns of the matrix are structured with some underlying smooth functions [24]. For example, in our empirical analysis in Section 4.6, the UK mortality data were obtained from Human Mortality Database [40]. The mortality data set were gathered such that, every column exhibits an age group between 18 to 87 years, with every row exhibits a year between 1948-2007 and every cell shows the mortality rate for a particular age group during that year. The data are essentially two-way functional data because each row vector is a mortality curve of different age groups in a specific year and every column is a time series of mortality rate of a given age group. Therefore, the mortality rate is consider as a smooth function of both age and time period.

The regularised SVD method was proposed by [24] by introducing the left and right singular vectors in the SVD of the data matrix, involving two-way functional data. The method is based on minimisation of a regularised sum of squared reconstruction errors of a low rank matrix approximation.

m_i is a $n \times T$ matrix of the observed centralised log mortality rates in a particular population such that both effects x and t are structured with an underlying smooth function. The element of mortality rates $m_i(x,t)$ of the data matrix m_i is viewed as evaluation of an underlying function $m_i(\cdot,\cdot)$ on a rectangular grid of sampling points (x,t), where x represents age, $x = x_1,\ldots,x_n$ and t is the period of $t = 1,\ldots,T$. The SSVD for the two-way functional data is a fitting of a smooth rank r approximation model of,

$$\ln m_i(x,t) = \beta_i^{(1)}(x)\kappa_i^{(1)}(t) + \cdots + \beta_i^{(p)}(x)\kappa_i^{(p)}(t) + \epsilon_i(x,t). \tag{4.24}$$

where $\beta_i(x)$ and $\kappa_i(t)$ are smooth on their respective domains which has been incorporated in two-way regularised SVDs. According to [24], this model is consider as a regularisation with penalisation.

We denote $\beta_1 \equiv (\beta_1(x_1),\ldots,\beta_1(x_n))^T$ and $\kappa_1 \equiv (\kappa_1(1),\ldots,\kappa_1(T))^T$ as the discretised realisations in extracting the first pair of the components in $\beta_1(x)$ and $\kappa_1(t)$ respectively. This is known as sequential approach where subsequent pairs are extracted sequentially after removing the effects of the preceding pairs which allows different pairs of components to have different smoothing.

Generally, by solving a least square problem, the first pair of singular vectors of a data matrix m_i can be found, such that,

$$\left(\hat{\beta},\hat{\kappa}\right) = \underset{(\beta,\kappa)}{\operatorname{argmin}} ||m_i - (\beta\kappa)^T||^2. \tag{4.25}$$

β is a $n\times 1$, κ is a $1\times T$ matrix and $||\cdot||$ if the Frobenius norm of a matrix. The SSVD for a two-functional data as following [24], can be solve as,

$$\left(\hat{\beta},\hat{\kappa}\right) = \underset{(\beta,\kappa)}{\operatorname{argmin}}\{||X-(\beta\kappa)^T||^2 + \mathcal{P}_\lambda(\beta,\kappa)\}. \tag{4.26}$$

$\mathcal{P}_\lambda(\beta,\kappa)$ is a regularisation penalty to ensure the smoothness of β and κ and λ is a vector of regularisation parameters. The penalty function, $\mathcal{P}_\lambda(\beta,\kappa)$ has the following form,

$$\mathcal{P}_\lambda(\beta,\kappa) = \lambda_\beta\beta^T\Omega_\beta\beta\cdot||\kappa||^2 + \lambda_\kappa\kappa^T\Omega_\kappa\kappa\cdot||\beta||^2 + \lambda_\beta\beta^T\Omega_\beta\beta\lambda_\kappa\kappa^T\Omega_\kappa\kappa. \tag{4.27}$$

such that the Ω_β and Ω_κ are symmetric and nonnegative penalty matrices with $||\cdot||$ is the Frobenius norm. The singular vectors $(\hat{\beta},\hat{\kappa})$ measured the smoothness as measured by the penalties $\beta^T\Omega_\beta\beta$ and $\kappa^T\Omega_\kappa\kappa$.

4.4.4 Robust Regularised Singular Value Decomposition (RobRSVD)

[142] have proposed the Robust regularised SVD (RobRSVD) based on the two-way roughness penalty function introduced in [24] to ensure smoothness along each of the two functional domain for two-way functional data. This model was developed as a penalised loss minimisation problem where a robust loss function is used to measure the reconstruction error of a low-rank matrix approximation of the data.

To achieve robustness, the squared error is removed from Equation 4.26. Let $\rho(t)$ be a non-negative, symmetric function and let $\rho(\cdot)$ denotes the summation over elementwise applications when the scalar function $\rho(\cdot)$ is applied to a matrix. Then, we can write the loss function for the rank-one approximation of the matrix m_i as,

$$\rho\Big(\frac{m-\beta\kappa^T}{\sigma}\Big) = \sum_{n=1}^{x_n}\sum_{t=1}^{T}\rho\Big(\frac{m_i-\beta_i\kappa_t}{\sigma}\Big), \tag{4.28}$$

such that σ is a scale parameter that measures the variability in the approximation errors. Following [142], we define the first pair of singular vectors as,

$$\big(\hat{\beta},\hat{\kappa}\big) = \underset{(\beta,\kappa)}{\operatorname{argmin}}\,\rho\Big(\frac{m_i-\beta\kappa^T}{\sigma}\Big) + \mathcal{P}_\lambda(\beta,\kappa). \tag{4.29}$$

Similarly as in SSVD model, we have $\mathcal{P}_\lambda(\beta,\kappa)$ as the penalty function. The parameter σ can be estimated from the data using residuals from the preliminary rank-one approximation of m_i. The standard SVD is sufficient to use in estimating the scale parameter [142]. To define the loss function $\rho\big(\frac{m_i-\beta\kappa^T}{\sigma}\big)$, we use the Huber's function as follows,

$$\rho_\theta(x) = \begin{cases} \mathrm{x}^2, & \text{if } |x| \leq \theta\ , \\ 2\theta|x| - \theta^2, & \text{if } |x| > \theta, \end{cases}$$

θ is the parameter which controls the robustness level such that a lower value of θ will give a more robut estimation. In this study, $\theta = 1.345$ is used, following a study by [113], which is a value that has been commonly used in robust regression that produces 95% efficiency for normal errors. The Huber's function is used due to its simplicity and faster computation.

For the single population model, we used the normal SVD, the robust SVD (RobSVD), the regularised SVD (SSVD) and also the robust regularised SVD (RobRSVD) to estimate the parameters in modelling the mortality. When we use a robust loss function, the framework generally robustifies the regularised SVD method to becoming a robust regularised SVD method. Whereas, when we remove the penalty term in the framework, the loss function generally becomes a robust SVD. In general, the robust regularised SVD method can be viewed as a smoothing of a robust SVD method. We used different modification of SVD methods in estimating the parameters for the mortality modeling to obtain the best forecasting performance which have not been studied previously.

4.5 Multiple population model

The multiple population mortality models are structured in a way assuming that the forecasted mortality experiences of different populations are related together and do not diverge in a long run. This is justified by the long term mortality co-movements and applicable to longevity risk modelling. There are not many multiple population stochastic mortality models proposed in the literature. We will be discussing different estimation methods based on the principal component methods to estimate the parameters in the multiple population data.

PCA method is a statistical analysis that uses orthogonal transformation to convert a set of observations of possibly correlated variables into a set of uncorrelated variables known as principal components. This method is widely used in dealing with multivariate data as it is suitable for data reduction and for making predictive modelling. As the dimensions of data increase, it becomes more difficult to analyse these data. PCA is able to condense the data by extracting the components that can best describe the data properties. The components extracted are linear combinations of the original variables. The PCA can be done by using eigenvalue decomposition of a data covariance matrix or using a singular value decomposition of a data matrix. Suppose, we have $x_1,\ldots,x_k$ and the components are denoted by $y_1,\ldots,y_k$, we have,

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k,\\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k,\\
&\vdots\\
y_k &= a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k,
\end{aligned} \tag{4.30}$$

such that we have uncorrelated principal components. The PCA method involves the uses of covariance and its eigenvalues. Generally, we have the following mortality model,

$$\ln(m(x,t)) = \alpha(x) + \beta(x)\kappa(t) + \epsilon_(x,t). \tag{4.31}$$

For the generalisation of the PCA, we can express the mortality model as follows,

$$\ln(m(x,t)) = \alpha(x) + \sum_{j=1}^{J} \beta(x)(j)\kappa(t)(j) + \epsilon(x,t), \tag{4.32}$$

such that $\kappa(t)(j)$ is treated as the principal components. There are many variation of the PCA decomposition that can be used in estimating the parameters in the mortality model. We will be looking at the common age effect model of [76] and the multi group PCA models. Similarly, as in the previous section, the age and period effects are evaluated simultaneously for multiple population of $i = 1,\ldots,k$.

4.5.1 Common Age Effect model

[76] introduce a model in which age has the same effect on the centralised log mortality rates for all countries. The common age effect (CAE) model of order p has the same structure as the individual model, but now assuming that the impact of age is independent of the population i, that is,

$$m_i = {}_p\beta \; {}_p\kappa_i^C + \epsilon_i \tag{4.33}$$

where ${}_p\beta$ is a $n \times p$ matrix and the ${}_p\kappa_i^C$ is a $p \times T$ matrix for all populations i. These matrices are defined similarly as in Section 4.4. The period effects ${}_p\kappa_i^C$ are dependent on the specific population. Note that we use the notation ${}_p\kappa_i^C$ to denote the period effects in the CAE model, whilst in the single population model, we use ${}_p\kappa_i$ to represent the period effects. Whereas, for ${}_p\beta_i$ in the single population, some of their columns can be used in the multiple population for all i. This is true because, all period effects are population specific, but some age effects are the same for all populations.

To estimate the common age effect ${}_p\beta$ in Equation 4.33, we applied the common principal component analysis (cPCA) method as in [56], but instead of using the maximum likelihood estimation, we use a modification on the least squares estimation. As in Equation 4.14, we simplify the notation $\beta = {}_n\beta$.

Assume that we have a CAE model with order p, we are looking to find the orthogonal matrix $\beta = {}_n\beta$ and diagonal matrices $\mathcal{L}$ such that, the β_i can be computed from calculating the eigenvectors of $m_i m_i^T$, where

$$Q_i = m_i m_i^T = \beta \mathcal{L} \beta^T \qquad \text{for all} \quad i = 1, \ldots, k. \tag{4.34}$$

We have the following,

$$m_i = \beta_i L_i U_i^T. \tag{4.35}$$

Therefore,

$$\begin{aligned} m_i m_i^T &= \beta L_i U_i^T \beta_i^T L_i^T U_i, \\ &= \beta_i L_i L_i^T \beta_i^T, \\ &= \beta_i \mathcal{L} \beta_i^T \qquad \text{where} \quad \mathcal{L} = L_i L_i^T. \end{aligned} \tag{4.36}$$

Therefore, we have the diagonal matrix $\mathcal{L} = \beta^T Q_i \beta$ for all $i = 1, \ldots, k$. But, in general, it is difficult to find β using this way. Alternatively, the estimate of $\hat{\beta}$ for the CAE model is estimated as the orthogonal matrix which gives $\beta^T Q_i \beta$ as close as possible to the diagonal matrices. Therefore, β can be estimated by minimising $T(\beta)$ which is the sum of squares of the off diagonal elements of $\beta^T Q_i \beta$, as,

$$T(\beta) = \sum_{i=1}^{k} ||\beta^T Q_i \beta - diag(\beta^T Q_i \beta)||^2. \tag{4.37}$$

We denote $||A|| = \sqrt{\sum_i \sum_j a_{ij}^2}$ as the Frobenius-norm of the matrix $A = (a_{ij})_{i=1,\ldots,I, j=1,\ldots,J}$ such that a_{ij} is the element in row i and column j, and β is the $n \times n$ orthogonal matrices. So, we have the estimate $\hat{\beta}$ as following,

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \, T(\beta) \tag{4.38}$$

Then, we can now find the estimates for the diagonal matrices $\mathcal{L}$, as,

$$\hat{\mathcal{L}} = diag(\hat{\beta}^T Q_i \hat{\beta}). \tag{4.39}$$

These estimation method is known as the F-G algorithm which was proposed by [35]. We can now estimate the ${}_p\kappa_i^C$, similarly, such that,

$$ {}_p\hat{\kappa}_i^C = {}_p\,\hat{\beta}^T m_i, \tag{4.40} $$

and the residuals are given as,

$$ \epsilon_i = m_i - {}_p\hat{\beta}\ {}_p\hat{\kappa}_i^C \tag{4.41} $$

Other estimation methods such as Maximum Likelihood estimation can be used to find the estimates for the parameters in the CAE model. But, the value of the estimators obtained, will strongly depend on the mortality of a larger populations. Since we are only interested in common features such as the age effects across mortality rates in a number of populations that are of different sizes, we used the modification of the SVD and PCA methods in estimating the parameters in the mortality models.

The PCA method is used extensively for dimensionality reduction in multivariate analysis by performing linear mapping of the data to a lower dimensional space. It is common to measure the same variables for each individual. The individuals are divided into few groups so that same variables can be measured on a set of individuals rather than measuring it on its own. This partitioning is known as multi-group datasets which might cause instability to the solution, so rather than estimating separates PCA on the particular group, several methods based on a parsimonious models have been introduced.

To determine the common vector in the group is not easy and may lead to a complicated algorithm. For the multi-group PCA, this study discusses a general framework in determining the common variance covariance matrix using the cPCA model of [56]. Then, the multi-group principal components analysis (MGPCA) proposed by [80] and the dual generalised procrustes model are also discuss further.

4.5.2 Common PCA model of Flury(1984)

The common principal components analysis (cPCA) was first introduced by [56]. The previous CAE model by [76] also are using cPCA in the model estimation but a modification on least square estimation is applied. While for the cPCA by [56], the estimation of the parameters were rather based on the maximum likelihood estimation.

Similarly as CAE model, we consider a multiple population of $i,\ldots,k$. The covariance matrix $Q_1,\ldots,Q_k$ is expressed by $\mathcal{L} := m_i m_i^T = \beta Q_i \beta^T$, where β is an orthogonal $n \times n$ matrix. Assume that we have a simultaneously transformed variables $U_i = \beta^T m_i$ which is known as the common principal components. Note that the rank order of the diagonal elements of the $\mathcal{L}$ is probably not the same for all groups in the populations.

By assuming that the approach is similar to the previous section, we want to find the orthogonal matrix $\beta = {}_n\beta$ and diagonal matrices $\mathcal{L}$ such that,

$$Q_i = \beta \mathcal{L} \beta^T \qquad \text{for all} \quad i = 1,\ldots,k. \tag{4.42}$$

where β is an orthogonal $n \times n$ matrix. The steps are similar as previous CAE model, apart that in here, the estimation are based on maximum likelihood estimation which will gives us the estimate of β and $\mathcal{L}$. Once we have the estimate of ${}_p\beta$, the ${}_p\kappa_i$, can be found similarly as,

$$
{}_p\hat{\kappa}_i =_p \hat{\beta}^T m_i, \tag{4.43}
$$

and the observed residuals are given as $\epsilon_i = m_i$ - ${}_p\hat{\beta}$ ${}_p\hat{\kappa}_i$. By using the maximum likelihood estimation method, the estimated parameter would strongly depend on the mortality in larger populations. While, in CAE, the estimation are based on least square estimation which allow us to see the common age effects across mortality rates in a number of populations with different sizes.

In general, the PCA of [56] considers the variance covariance matrices involving the multi group populations which are looking for a common orthogonal vector of loadings associated with the components in the groups. The determination of the common vectors of loadings are based on the maximum likelihood estimation.

4.5.3 Multi Group PCA model of Krzanowski(1984)

Another PCA method for determining the common vector of loadings involving multiple population dataset was proposed by [80]. The dataset m_i is the observed centralised log mortality rates in a particular population. This dataset is described as when both common effects β_i and κ_t are measured on a set of N individuals. This dataset is a priori partitioned into G groups, $(m_1, \ldots, m_G)$ groups. Note that $m_g = m_i$ and $m_G = m_k$. Each group represents N_g individuals such that $\sum_{g=1}^{G} N_g = N$. We assume that each group m_g is a column centered in order to have mean 0, and therefore the variance covariance matrix of group g is given be $Q_g = \frac{1}{N_g} m_g^T m_g$.

The PCA method is used as a tool to reduce the dimensionality in multivariate analysis which may be performed on each group separately. However, this strategy involves a very large number of parameters which may cause instability problem due to insufficient data. Therefore, as in [53], a more parsimonious model such as the common PCA (cPCA) method can be applied to the multi group datasets. The cPCA model is expressed in terms of the variance covariance matrices associated with the group m, that is,

$$Q_g = m_g m_g^T = \beta \mathcal{L} \beta^T \qquad \text{for all} \quad g = 1, \ldots, G. \tag{4.44}$$

The linear combination of the variance covariance matrices $(Q_1, \ldots, Q_G)$, is as follows,

$$\sum_{g=1}^{G} \frac{N_g}{N} Q_g = \sum_{g=1}^{G} \frac{N_g}{N} \beta \mathcal{L} \beta^T = \beta \Big(\sum_{g=1}^{G} \frac{N_g}{N} \mathcal{L} \Big) \beta^T \tag{4.45}$$

The matrix of common loadings β can be obtained from the eigenanalysis of the matrix $Q_W = \sum_{g=1}^{G} \frac{N_g}{N} Q_g$ which is the within groups variance covariance matrix. The Q_W is a common variance covariance matrix to the various groups which is the closest matrix to $Q_1, \ldots, Q_G$ that minimises the following,

$$\min_{Q_c} \sum_{g=1}^{G} N_g ||Q_g - Q_c||^2. \tag{4.46}$$

$Q_c = \sum_{g=1}^{G} \alpha_g Q_g$ such that $\alpha = (\alpha_1, \ldots, \alpha_G)^T$ is the eigenvector of the matrix R which is associated to the largest eigenvalue. This strategy computes the Q_W, the within group variance covariance matrix and also for any linear combination of Q_g, as

$$\sum_{g=1}^{G} \alpha_g Q_g = \sum_{g=1}^{G} \alpha_g \beta \mathcal{L} \beta^T = \beta (\sum_{g=1}^{G} \alpha_g \mathcal{L}) \beta^T, \tag{4.47}$$

such that $\alpha_g \geq 0$. Therefore, the estimate values of β and $\mathcal{L}$ can be found which is then can be used to find the ${}_p\beta$ and ${}_p\kappa_g$. Finally,

$${}_p\hat{\kappa}_g =_p \hat{\beta}^T m_g, \tag{4.48}$$

and the observed residuals are given as $\epsilon_g = m_g$ - ${}_p\hat{\beta}\ {}_p\hat{\kappa}_g$.

4.5.4 Dual generalised procrustes model

The previous model aimed at computing a variance covariance matrix common to the various groups. Another estimation method which can be used is the dual generalised procrustes method. In this model, instead of computing a common variance-covariance matrix of Q_i, a dataset that would be an average of groups $\frac{1}{\sqrt{N_i}} m_i^T$ can be computed through orthogonal transforms. Similarly as before, the dataset m_i is the observed centralised log mortality rates in a particular group.

Suppose that m_1 and m_2 are two centered datasets with $n \times T$ matrix refering to the same β_i and κ_t variables but not necessarily to the same number of individuals. We assume that these two datasets have the same number of individuals. If this is not true, then the dataset with the smallest number of rows can be augmented with the necessary number of rows containing zeroes. The equality $m_1 m_1^T = m_2 m_2^T$ holds if and only if $m_1^T = m_2^T\beta$ where β is an orthogonal matrix. This method is similar as in [61], but the method used in this chapter is known as dual GPA as it is based on m_i^T instead of m_i.

If $m_1^T = m_2^T\beta$ then $m_1 m_1^T = m_2\beta\beta^T m_2^T = m_2 m_2^T$. Suppose that we have $m_1^T m_1 = m_2^T m_2$, then the SVD of m_1^T and m_2^T can be written as $m_1^T = UDV_1^T$ and $m_2^T = UDV_2^T$ where U is the matrix of eigenvectors of $m_1^T m_1 = m_2^T m_2$ associated with the eigenvalues in the diagonal matrix D. Then, we will have $m_1^T = UDV_1^T = UDV_2^T V_2 V_1^T = m_2^T\beta$ such that $\beta = V_2 V_1^T$ is an orthogonal matrix.

As mentioned earlier, instead of looking at the common variance covariance matrix Q_i, one could look for a dataset which gives the average of groups ($\frac{1}{\sqrt{N_i}} m_i^T$) through orthogonal transforms. This strategy is based on m_i^T instead of m_i and hence it is known as the dual generalised procrustes model. For this, we want to minimise the followings,

$$\sum_{i=1}^{k} \| \frac{1}{\sqrt{N_i}} m_i^T \beta_i - C \|^2 \tag{4.49}$$

such that $\frac{1}{\sqrt{N_i}} m_i^T$ is orthogonally transformed towards the common matrix of C by the orthogonal matrix associated with group i, β_i. When C is calculated, the common vector of loadings β can be calculated as the left singular vectors of the C and ${}_p\kappa_i$ can be find as,

$${}_p\hat{\kappa}_i = {}_p\,\hat{\beta}^T m_i. \tag{4.50}$$

Finally, the observed residuals can be calculated as $\epsilon_i = m_i$ - ${}_p\hat{\beta}\ {}_p\hat{\kappa}_i$.

4.6 Empirical analysis & data descriptions

This study is split into three areas. First, looking at the single population model; i.e. by doing a comparison of the performance for each model in UK and US. Second, to investigate the area modelling in which the sub-population data is obtained for the UK (England & Wales, Scotland and Northern Ireland) from the Human Mortality Database and carried out the multiple population modelling. The objective is to see whether the multiple population models will perform better as compared to the individual modelling in the UK. To further extend the analysis, the population is grouped together consisting of seven different countries with similar socio-characteristics and carried out comparison to see which type of modelling performs better.

This empirical study is based on three age groups with $n = 35$ for young or old age group or $n = 70$ for mix age group, with 60 years of observations, from 1948-2007 ($T = 60$). The analysis was done using several R packages including 'StMoMo', 'RobRSVD', 'robustSvd', 'demography', 'cpca', 'multigroup' and 'rpca'. The obtained results are shown in the next few tables.

4.6.1 Single population modelling: UK & US data

This analysis is carried out for a single population modelling for the data in UK and US.

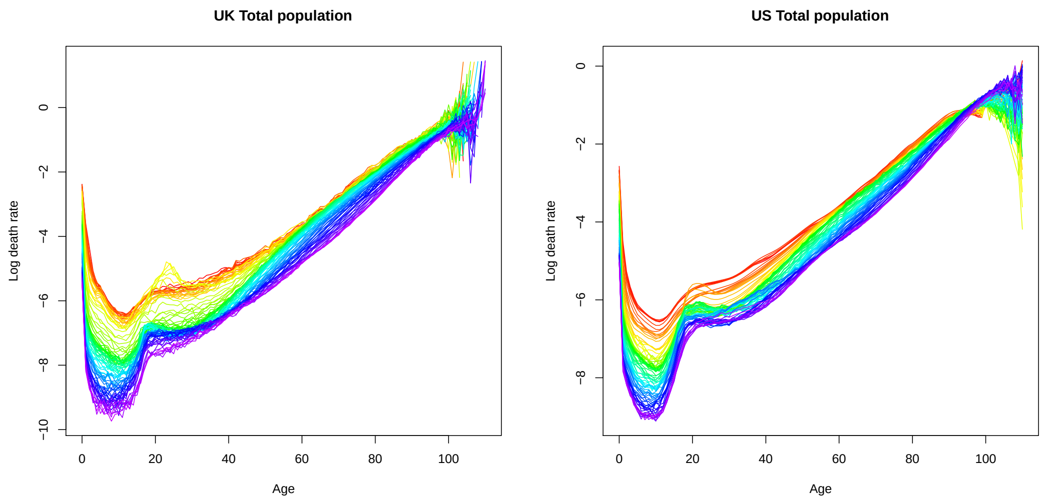


Figure 4.1. Logarithm of death rates according to age and time in UK and US total population.

From the above figure, both populations show that mortality is falling for all years with different behaviour of log death rates according to different ages. They are observed values and each colour is representing a different time period (most recent ones in violet, earliest in red). The aim of these graphs is not to differentiate each year but to give a rough indication of how the mortality moved over time in both populations. There were significantly bigger reductions in mortality rates for men over 100 years old in US than in the UK total population.

For the single population modelling, the overall mean squared error is evaluated as a function of the number of the age period effects p for the individual model, that is

$$\hat{m}_i(x,t) ={}_p\hat{\beta}_i\;{}_p\hat{\kappa}_i. \tag{4.51}$$

The mean square error (MSE) for the individual model is defined as,

$$MSE(p) = \frac{1}{nT}\sum_{j=1}^{n}\sum_{t=1}^{T}\Big(m_i(x_j,t) - \hat{m}_i(x,t)\Big)^2. \tag{4.52}$$

Table 4.1: MSE(p)$\times 10^3$ & BIC for the different models and different age group for United Kingdom as an individual modelling.

Model	Young: 18-52 years old		Old: 53-87 years old		Mix: 18-87 years old	
	MSE (p=1)	BIC (p=1)	MSE (p=1)	BIC (p=1)	MSE (p=1)	BIC (p=1)
SVD	8.25676	-9,346	1.23078	-13,344	7.20499	-19,634
Robust SVD	8.47942	-9,291	1.25033	-13,310	7.58473	-19,418
Regularised SVD	8.31406	-9,332	1.46773	-12,974	7.32621	-19,564
Robust Reg SVD	8.54535	-9,274	1.50837	-12,916	7.67804	-19,367
LC	9.44140	-9,065	1.31702	-13,201	8.25393	-19,063
LC Binomial	9.44126	-9,065	1.56367	-12,841	8.36236	-19,008

Table 4.2: MSE(p)$\times 10^3$ & BIC for the different models and different age group for United States as an individual modelling.

Model	Young: 18-52 years old		Old: 53-87 years old		Mix: 18-87 years old	
	MSE (p=1)	BIC (p=1)	MSE (p=1)	BIC (p=1)	MSE (p=1)	BIC (p=1)
SVD	4.79298	-10,489	0.84850	-14,125	3.42694	-22,755
Robust SVD	4.82233	-10,476	0.90251	-13,995	3.50152	-22,665
Regularised SVD	4.83064	-10,472	0.91578	-13,964	3.48138	-22,689
Robust Reg SVD	4.86374	-10,458	0.96758	-13,849	3.55045	-22,606
LC	5.11038	-10,354	0.88387	-14,039	3.65847	-22,480
LC Binomial	5.11340	-10,353	1.00958	-13,760	3.73713	-22,398

The results obtained for UK and US are shown in Table 4.1 and 4.2, respectively.

From the observation, as expected, the individual model using SVD fits the data better for all age groups. This can be seen from the lowest MSE obtained using the SVD model for all age groups. This is true for both UK and US data.

4.6.2 Area population modelling: UK data

The analysis is conducted specifically on the UK's data where the sub-area modelling is based on the data in the UK (England & Wales, Scotland and Northern Ireland). Figure 4.2 present the pattern of logarithm of death rates based on age and time for the UK's sub populations. From the plots, the mortality rates are falling at all ages with a different behaviour according to different ages.

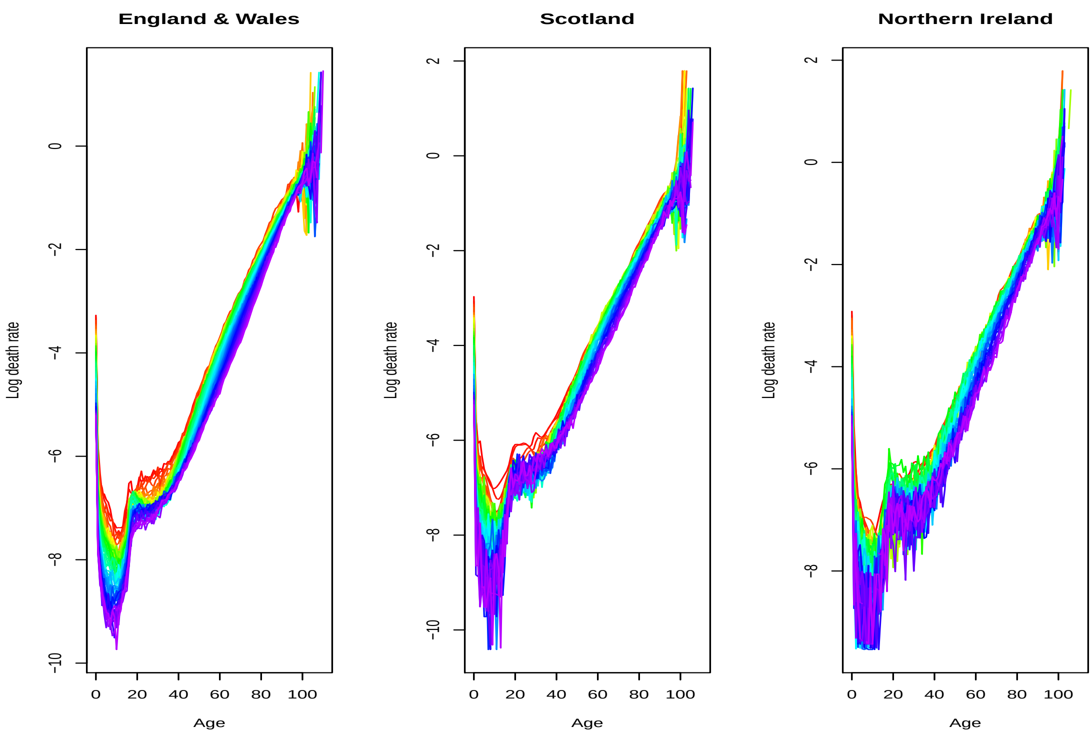


Figure 4.2. Logarithm of death rates according to age and time in UK's sub-populations (England & Wales, Scotland and Northern Ireland).

The area modelling performance is evaluated as a multiple population ($p = 1$ and $p = 2$) and also as a single population ($p = 1$). These are observed values and each colour is representing a different time period (most recent ones in violet, earliest in red). The results are presented in Table 4.3, 4.4 and 4.5 for young age group of 18-52 years old, old age group of 53-87 years old and mix age group 18-87 years old, respectively.

Table 4.3: MSE(p) $\times 10^3$ for the different models for 18-52 years old, for each population in the UK.

Panel A: Individual model, p=1

	Eng & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.) [b]
SVD	8.74606	25.3410	73.7986	35.96187	12.1918
Robust SVD	8.92738	25.6968	68.5813	34.40184	12.2262
Regularised SVD	8.80409	25.8124	71.0572	35.22455	12.1920
Robust Reg SVD	8.99008	26.1651	71.6453	35.60018	12.4155
LC	9.91362	27.21338	71.0316	36.05288	13.3041
LC Binomial	9.91408	27.21425	71.0343	36.05421	13.3047

Panel B: Multi model, p=1

	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.) [b]
CAE	9.07726	25.5433	68.6930	34.43784	12.3477
Multi	9.17237	25.6363	68.4419	34.41687	12.4322
Flury's	8.95021	25.8002	69.0059	34.58220	12.2684
Dual Generalised	9.12262	25.5927	68.5483	34.42119	12.3877

Panel B.1: Multi model, p=2

	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
CAE	4.76636	15.37384	57.04255	25.72758	7.2893
Multi	5.94245	15.96928	54.35336	25.42170	8.2975
Flury's	3.28989	17.51678	63.33848	28.04668	6.3718
Dual Generalised	5.25979	15.50713	55.80462	25.52384	7.6984

[a] Equal weighted.
[b] Exposure weighted.

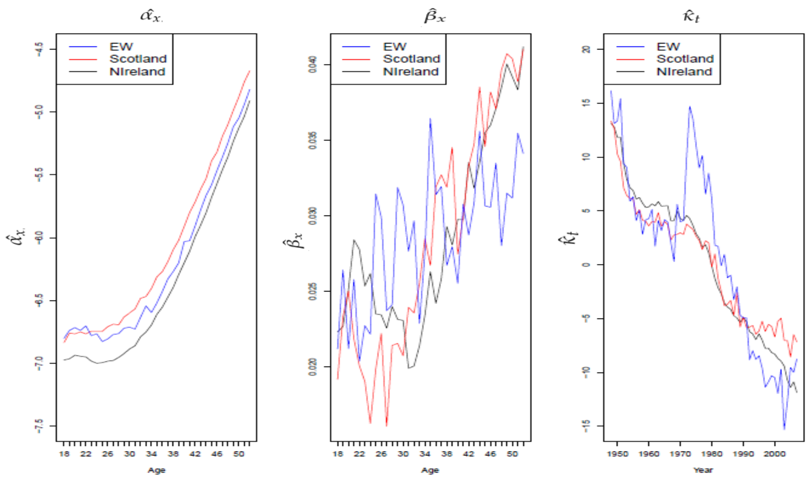


Figure 4.3. First age $\hat{\alpha}_x$, $\hat{\beta}_x$ and period effects $\hat{\kappa}_t$ estimated using individual SVD model for UK's sub-populations (England & Wales, Scotland and Northern Ireland) for 18-52 younger age group.

Table 4.4: The table shows MSE(p) $\times 10^3$ for the different models for 53-87 years old, for each population in the UK.

Panel A: Individual model, p=1					
	Eng & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
SVD	1.30168	2.38565	7.54079	3.74271	1.5864
Robust SVD	1.32209	2.39526	7.28779	3.66838	1.5976
Regularised SVD	1.54672	2.80231	8.22138	4.19013	1.8599
Robust Reg SVD	1.58831	2.81734	8.25687	4.22090	1.8989
LC	1.39599	2.72134	7.1032	3.32137	1.6865
LC Binomial	1.64231	2.72143	7.1034	3.32969	1.9033
Panel B: Multi model, p=1					
	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
CAE	1.36658	2.48932	7.43298	3.76296	1.6496
Multi	1.35741	2.49816	7.43159	3.76239	1.6423
Flury's	1.31442	2.47546	7.56836	3.78609	1.6065
Dual Generalised	1.36193	2.49394	7.43172	3.76253	1.6459
Panel B.1: Multi model, p=2					
	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
CAE	0.87848	1.87862	6.65933	3.13881	1.1419
Multi	0.88474	1.90240	6.62369	3.13695	1.1485
Flury's	0.87767	1.78844	6.99687	3.22051	1.1432
Dual Generalised	0.88374	1.87841	6.65437	3.13884	1.1464

[a] Equal weighted.
[b] Exposure weighted.

A similar behaviour of parameters is observed across the population in the UK. As expected the average mortality rate increases as age increases which is clearly seen from $\hat{\alpha(x)}$ pattern. For $\hat{b}(x)$, there is a greater value as approaching retirement. In addition, it is clearly from Figure 4.3, there is young mortality hump for males in the range of 30-36 years old which may be due to accidental deaths. For $\hat{\kappa}(t)$, as expected, it has a decreasing trend across the time.

Similarly as in previous section, the individual models fit the data better. It is interesting to note that when $p = 2$, the multiple population model has a lower MSE than the individual models with $p = 1$ for all age groups. This shows that the fit of a multiple population model with $p = 1$ can be improved by adding a second age or

Table 4.5: The table shows MSE(p)$\times 10^3$ for the different models for 18-87 years old, for each population in the UK.

Panel A: Individual model, p=1					
	Eng & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
SVD	7.34371	16.82684	44.6435	22.9380	9.3162
Robust SVD	7.68275	17.0796	43.5833	22.7819	9.6055
Regularised SVD	7.47085	17.1483	44.2182	22.9458	9.4442
Robust Reg SVD	7.78039	17.3429	44.6976	23.2736	9.7485
LC	8.36119	17.59083	46.0339	23.9953	10.3220
LC Binomial	8.46724	17.70442	46.1593	24.1103	10.4293
Panel B: Multi model, p=1					
	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
CAE	7.75406	17.6065	44.2387	23.1997	9.7353
Multi	7.73358	17.8005	44.0179	23.1840	9.7281
Flury's	7.42099	17.93856	44.6212	23.3269	9.4836
Dual Generalised	7.74195	17.71467	44.1045	23.1871	9.7304
Panel B.1: Multi model, p=2					
	England & Wales	Scotland	NI	UK(Eq.)[a]	UK(Exp.)[b]
CAE	3.37444	9.29570	34.28899	15.6530	4.8348
Multi	3.63672	9.38247	33.76336	15.5942	5.0576
Flury's	2.72446	9.85404	35.78046	16.1197	4.3578
Dual Generalised	3.49523	9.33526	34.01061	15.6137	4.9363

[a] Equal weighted.
[b] Exposure weighted.

period effect. The multiple population models tend to have a better fit as compared to the individual age effects for each country.

By adding age/period effects to the model, the multiple population models will have more parameters than an individual model with $p = 1$. Therefore, the approximation of the Bayesian Information Criterion (BIC) is considered to penalise the MSE for the number of parameters used. The BIC is estimated as,

$$BIC(p) = 3nT \log (MSE(p)) + k \log (3nT). \tag{4.53}$$

Following the BIC definition, a model is a better model when it has a smaller BIC value. BIC(p) corrects MSE(p) for the number of parameters and therefore can be used to provide a good measure for the goodness of fit for all models, following [76].

The total number of observed mortality rates in the area modelling is, $3nT = 6,300$. The number of parameters in the models for the centralised log rates is denoted by k. For the individual, $k = 3p(n+T)$ and for multiple population model, we have $k = p(3T+n)$. The results are tabulated in Table 4.6. All multiple population models have the best fitting qualities as compared to individual models except the Robust SVD model which also has a good BIC value. This is true for both young and old age groups.

Table 4.6: Approximate value of BIC for the different models in each age group for area population in the UK.

Model	Young: 18-52 years old		Old: 53-87 years old	
	BIC (p=1)	BIC (p=2)	BIC (p=1)	BIC (p=2)
SVD	-18,456	-	-32,711	-
Robust SVD	-18,736	-	-32,837	-
Regularised SVD	-18,587	-	-31,999	-
Robust Reg SVD	-18,520	-	-31,953	-
CAE	-18,729	-18,073	-32,677	-31,326
Flury's	-18,703	-17,529	-32,638	-31,164
Multi	-18,733	-18,148	-32,678	-31,330
Dual Generalised	-18,732	-18,123	-32,678	-31,326

The projection is carried out based on ARIMA extrapolation for 10 years period where the predicted values of κ_t is rescaled to zero in the last observed year (2007). The model is fitted based on the in-sample period of 1948-2007 and applying the expanding forecast window approach to obtain forecasts for the period 2008-2017. The recursive forecast (expanding-window) approach, is where data from 1948 to 2007 was used to make the first estimation in 2008, data from 1948 to 2008 to make the second estimation in 2009, and so on.

4.6.3 Multiple population modelling: seven countries analysis

So far in the previous sections, we have look at the mortality modelling for a single population model and also the sub-area population modelling. Now, in this section, the study has been extended to evaluate the multiple population modelling for a bigger population.

The model is applied to the mortality rates observed for males aged 18-87 in the following $k = 7$ countries: Austria, Australia, Canada, France, the United Kingdom, New Zealand and the United States. These countries are chosen since they are all well developed countries with similar socio-economic characteristics. Therefore, from this study, it is expected that a mortality model with common factors will allow us to jointly model mortality rates in those countries. The empirical results are based on observed mortality rates for the calendar years 1948-2007 (T=60). Similarly, the ages are split into three groups of $n = 35$, that is the young age group of 18-52 and older age group of 53-87 and $n = 70$ for the mix age group of 18-87 years old. The data are obtained from Human Mortality Database.

The estimated first age effects $\beta_i^{(1)}$ for the seven countries in our study are shown in Figure 4.4. It can be seen in this figure that the age effects for older age groups are indeed rather similar for different countries and might therefore be replaced by an age effect that is the same for all countries. For younger ages, this is less obvious.

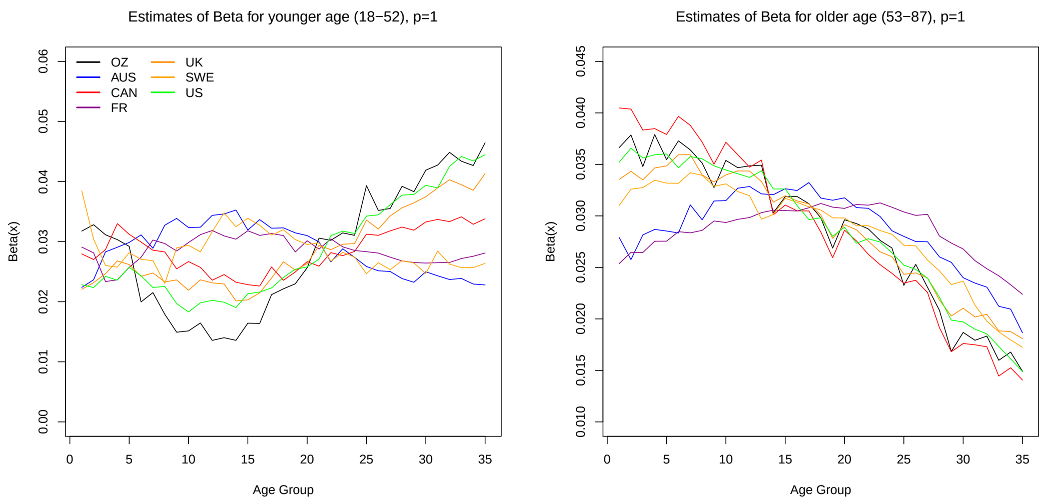


Figure 4.4. First order age effects for the period of 1948 to 2007 in seven countries.

In the plot, it appears that it is rather close at least for high ages, but it cannot directly conclude and suggest that the differences do not matter. To decide whether the individual β_i can be replaced with a common β, the impact of common age effect on the estimated period effects is examined and the goodness of fit of individual models with the goodness of fit in the multiple population models are compared.

Figure 4.5 shows the estimated first period effect $\hat{\kappa}_i$ and $\hat{\kappa}_i{}^c$ for the UK (solid line) and the US (dotted line) using individual SVD model and CAE model. These models were chosen as to see whether the individual β_i can be replace with a common β by looking at the impact of a common age effect on the estimated period effects. The models are fitted to the young (18-52) and old (53-87) age group.

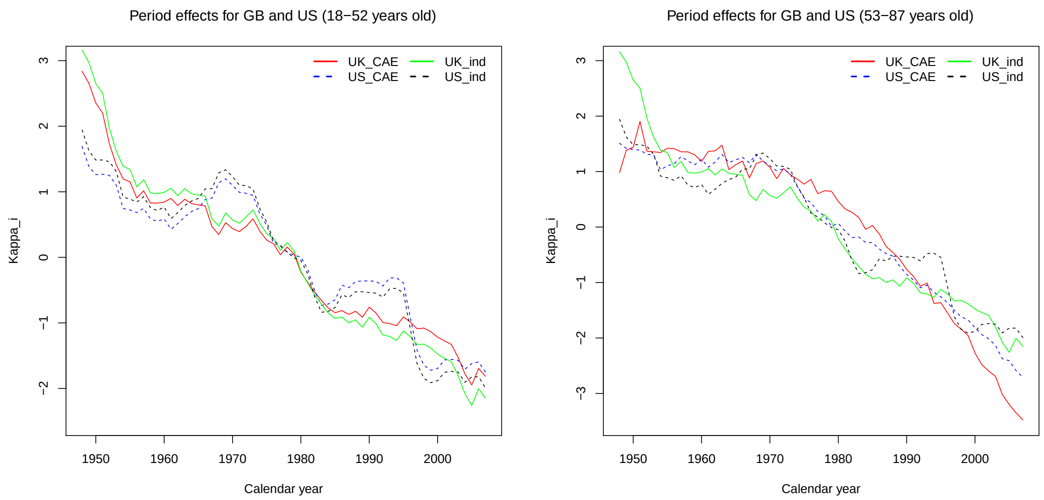


Figure 4.5. First period effects estimated in a single population model and in the CAE model for the UK and the US

From the plot, it looks like the first period for these two countries do not really change when individual age effects are replace by the common age effect. A very similar picture is observed for all countries. The result is also true when other models are fitted to the mix group (18-87) age range. To investigate further, the MSE is calculated using,

$$MSE(p) = \frac{1}{7nT}\sum_{i=1}^{7}\sum_{j=1}^{n}\sum_{t=1}^{T}\Big(m_i(x_j,t) - \hat{m}_i(x,t)\Big)^2, \tag{4.54}$$

such that the log mortality rates for the multiple population is defined as,

$$\hat{m}_i(x,t) =_p \hat{\beta}_p \hat{\kappa}_i^{multi}, \tag{4.55}$$

and similarly as in previous section, the BIC is approximated as,

$$BIC(p) = 7nTlog(MSE(p)) + klog(7nT). \tag{4.56}$$

The results for young age group are presented in Table 4.7.

Table 4.7: MSE(p) $\times 10^3$ & BIC for the different models for 18-52 years old for 7 countries.

Country	Young: 18-52 years old			
	MSE (p=1)	BIC (p=1)	MSE (p=2)	BIC (p=2)
SVD	9.61019	-67,369	-	-
Rob SVD	9.69728	-67,236	-	-
Regularised SVD	9.75974	-67,142	-	-
Robust Reg SVD	9.86671	-66,982	-	-
CAE	11.51560	-61,256	7.56869	-63,059
Multi	11.48878	-64,744	7.50389	-71,006
Flury's PCA	11.78839	-64,366	7.56137	-70,894
Dual Generalised	11.49305	-64,739	7.66325	-70,697

From Table 4.7, the individual SVD fits the data better as compared to the other models. But, similarly as explained earlier, when $p = 2$, better results are obtained for the multiple population models which resulting in a lower MSE values as compared to the individual models with $p = 1$.

The result of the MSE for an individual country are synthesise within the multiple population using the following,

$$MSE_i = \frac{1}{nT}\sum_{j=1}^{n}\sum_{t=1}^{T}\Big(m_i(x_j,t) - \hat{m}_i(x,t)\Big)^2. \tag{4.57}$$

The results are tabulated in Table 4.8.

When modelling the multiple population for an older age group, the MSE for the multiple population model using Dual Generalised PCA method is found to be lower as compared to all other models. Based on the BIC values, this model outperforms the other models for the age group 53-87 years old. When $p = 2$, all multiple population models have a better fit as compared to the single population model of $p = 1$. As seen in Figure 4.4, the age effects for each countries are close to each other for older age group which consequently will makes the multiple population models works better

Table 4.8: MSE(p)$\times 10^3$ for the different models for 18-52 years old, for each country in the multiple population.

Panel A: Individual model, p=1							
	Australia	Austria	Canada	France	UK	Sweden	US
SVD	11.35293	12.20227	6.41170	8.90545	8.25676	15.34902	4.79298
Robust SVD	11.40452	12.2473	6.44639	9.02207	8.47942	15.45890	4.82230
Regularised SVD	11.56147	12.42571	6.63033	9.11379	8.31406	15.44222	4.82230
Robust Reg SVD	11.69342	12.44628	6.67585	9.22756	8.54535	15.61475	4.86374
Panel B: Multi model, p=1							
	Australia	Austria	Canada	France	UK	Sweden	US
CAE	15.78582	16.3798	6.76642	9.8517	8.79951	17.3676	5.65798
Multi	16.42700	15.69730	6.79103	9.64716	8.99515	16.98684	5.87696
Flury's	19.34534	13.77723	7.46741	9.19394	9.67125	16.45238	6.61199
Dual Generalised	16.17206	15.95471	6.77862	9.71909	8.91467	17.12405	5.78814
Panel B.1: Multi model, p=2							
	Australia	Austria	Canada	France	UK	Sweden	US
CAE	8.59401	11.26091	5.37859	6.14444	3.57303	15.32668	2.70321
Multi	8.40226	10.85556	5.52205	6.24993	3.34798	15.33326	2.81621
Flury's	8.47691	10.8691	5.6361	6.45294	3.19049	15.5938	2.71034
Dual Generalised	8.80249	11.30025	5.15387	5.77628	3.78470	15.52792	3.29727

for this age group. This can be seen through the smaller MSE and BIC for older age group in Table 4.9 and Table 4.10.

Table 4.9: MSE(p)$\times 10^3$ & BIC for the different models for 53-87 years old for 7 countries.

Country	Old:53-87 years old			
	MSE(p=1)	BIC (p=1)	MSE (p=2)	BIC (p=2)
SVD	1.62558	-93,490	-	-
Rob SVD	1.64763	-93,292	-	-
Regularised SVD	1.88067	-91,347	-	-
Robust Reg SVD	1.90712	-91,142	-	-
CAE	2.15990	-85,858	1.32926	-88,628
Flury's PCA	2.19228	-89,094	1.33227	-95,504
Multi	2.15793	-89,326	1.32402	-95,595
Dual Generalised	1.62358	-93,596	1.46287	-94,129

The result for the mix age group ranging from 18 to 87 years old is presented in Table 4.11.

Looking at the results in Table 4.7, 4.8, 4.9, 4.10 and 4.11 all considered multiple population models fit the mortality rates poorly for the younger and mix age group with large mean squared errors as compared to the older age group. The models'

Table 4.10: MSE(p)$\times 10^3$ for the different models for 53-87 years old

Panel A: Individual model, p=1							
	Australia	Austria	Canada	France	UK	Sweden	US
SVD	11.35293	12.20227	6.41170	8.90545	8.25676	15.34902	4.79298
Robust SVD	11.40452	12.2473	6.44639	9.02207	8.47942	15.45890	4.82230
Regularised SVD	11.56147	12.42571	6.63033	9.11379	8.31406	15.44222	4.82230
Robust Reg SVD	11.69342	12.44628	6.67585	9.22756	8.54535	15.61475	4.86374
Panel B: Multi model, p=1							
	Australia	Austria	Canada	France	UK	Sweden	US
CAE	15.78582	16.3798	6.76642	9.8517	8.79951	17.3676	5.65798
Multi	16.42700	15.69730	6.79103	9.64716	8.99515	16.98684	5.87696
Flury's	19.34534	13.77723	7.46741	9.19394	9.67125	16.45238	6.61199
Dual Generalised	16.17206	15.95471	6.77862	9.71909	8.91467	17.12405	5.78814
Panel B.1: Multi model, p=2							
	Australia	Austria	Canada	France	UK	Sweden	US
CAE	8.59401	11.26091	5.37859	6.14444	3.57303	15.32668	2.70321
Multi	8.40226	10.85556	5.52205	6.24993	3.34798	15.33326	2.81621
Flury's	8.47691	10.8691	5.6361	6.45294	3.19049	15.5938	2.71034
Dual Generalised	8.80249	11.30025	5.15387	5.77628	3.78470	15.52792	3.29727

Table 4.11: MSE(p)$\times 10^3$ & BIC for the different models for 18-87 years old for 7 countries.

Country	Mix:18-87 years old			
	MSE(p=1)	BIC (p=1)	MSE (p=2)	BIC (p=2)
SVD	6.68621	-146,249	-	-
Rob SVD	6.81952	-145,669	-	-
Regularised SVD	6.85465	-145,158	-	-
Robust Reg SVD	6.97824	-144,992	-	-
CAE	8.60937	-135,113	5.51292	-143,537
Multi	8.60629	-138,827	5.49603	-151,035
Flury's PCA	8.90777	-137,815	5.53699	-150,817
Dual Generalised	8.60746	-138,823	5.57700	-150,605

performance may be improved by including more age/period effects to the models. For the older age group, the multiple population models performed better than other models. The performance gets even better when additional parameters are added to the model. These models allows estimation of age and period effects in different countries simultaneously which are better than comparing age and period effects that are country specific.

4.7 Conclusion

This chapter proposes various approaches to estimate the parameters in the mortality models in a single and multiple populations to obtain better and more consistent mortality forecasts. The age effects are assume to be the same for all populations while the period effects are country specific. By doing this, the estimation of the period effects can be done for different countries which are better as compared to estimation of period effects that are the same for all populations. The empirical results obtained in Section 4.5, suggest that by adding a second factor in the mortality model is more relevant to the fitting than looking at differences in the age effects in the estimation methods in the single populations.

The LC and CAE models are extended by using SVD and PCA method respectively. The LC model is modified by estimating the parameters using SVD rather than maximum likelihood methods. This study also concentrated on the use of PCA by applying the use of the common PCA model of [56], the multi group of [80] and the dual generalised PCA to obtain estimates of the age effects. In addition, other different methods such as Robust SVD, Regularised SVD and Robust Regularised SVD are also included in modelling the mortality for a single population instead of the normal SVD in the classical LC model.

This study contributes to existing literature by proposing alternative estimation methods in estimating the parameters from using the modification of the normal SVD and PCA for the single and multiple population groups and applying it to the mortality models in order to get the best model for forecasting life expectancy across populations.

The proposed models allow us to estimate the period effects in multiple population simultaneously which provides a better comparison rather than looking at period effects in a single population. It is learned that these proposed modification gives a better mortality models for multiple population such that these model outperforms the individual models which is reflected by the smaller MSE and BIC for the older age group. However, the proposed models fit the mortality rates at young ages poorly with larger MSE values as compared to the older age groups. This shows that the mortality rates for younger age group are more difficult to model, and this result might be improved by including more age period effects to the model. But, this is beyond the scope of this chapter since we are interested to look at the older age group for the liability calculation in the pension schemes.

This study is extended to examine the area modelling of the sub-populations in the UK (Scotland, Northern Ireland and England & Wales) and compared the performance of the multiple population with the individual population of the UK. The multiple population model performs better as compared to the other models for the older age groups. The model can be further improved, when the additional common age period effect is added to the model.

For future studies, the proposed model can be extended in different ways, for example, by including a cohort effect that is country specific or common to all population which may improve the goodness of fit. It would also be interesting to examining common factors for mortality rates in different sets of populations (i.e. looking at the mortality rates for males and females in different countries). The robustness of these multiple population models may be examined by looking at different perspectives, such as employing exposure weightings to the countries in the population rather than equal weighting, as it may be appropriate for a certain group of population but not for others. Further studies also may incorporate models where age terms are given by smooth parametric functions. In addition, other estimation method which are based on the observed rates may be consider rather than looking at the observed numbers of deaths and exposures. Different estimation methodologies would provide a further potential development in the mortality modeling studies as it might gives a better model.

Chapter 5

Multiple population mortality models: A DCC GARCH and Copula approach

In Chapter 3, the DCC GARCH models has been used for asset modelling purposes. This chapter extends the analysis by proposing the use of DCC GARCH models in modelling mortality for a multiple population to jointly capture the evolution of the aggregate mortality rates for different countries.

5.1 Introduction

The previous chapter used different types of SVD and PCA analysis based on the Lee Carter model to a single and multiple population groups to get a model for forecasting life expectancy across populations. The Lee Carter model has been used widely in mortality fitting and forecasting. This model assumes that the dynamic of the logarithm of central death rates is driven by an age specific constant plus the speed of change at each age, multiplied by the overall period trend of mortality rates. LC model is robust, has a good fit over wide age ranges but lacks the smoothness of age effect, especially in small populations, and has no cohort effect. This model has motivated various extensions (see for example; [83], [109], [25], [23], [14], [27], [41], and [104]). Another model known as the Cairns-Blake-Dowd (CBD) model proposed by [25] assumes a linear relationship between age and the logit of initial mortality rates. It is designed for modelling mortality at higher ages and suitable for modelling longevity risk in pension and annuities. There are many extensions introduced from these two models which are useful for mortality forecasting. But, all of the models, however, focus on modelling the mean level of mortality rates and ignoring the variance level of mortality rates and the temporal dependence structure between inter-age mortality rates [87].

Other methods have been introduced to tackle the problem in Lee Carter or CBD based models. A study by [93] used the modification of the LC model and expresses the change in the logarithm of mortality rates for each age group rather than looking at the logarithm of mortality rates as an age dependent linear transformation of mortality index. This study uses SVD to calibrate the parameters and to capture the dependence structure between mortality rates and ages but ended up giving a limited dependence structure.

Some methods of modelling mortality using econometric models have also been proposed in the literature. Studies by [31], [39], [21] and [86] apply the modelling of mortality in a single population at a time. These models are able to capture the characteristics of the data such as skewness and volatility clustering. In addition, the GARCH copula model proposed by [121] provides alternative to the use of multivariate normal distribution which constructs a highly flexible and non standard multivariate dependence structure. However, these models focus on modelling mortality in a single population and therefore are not capable to incorporate the cross correlations among the mortality dynamic in different population at a same time. Hence, the models may not be appropriate to do a pricing for mortality bonds or pension products which are linked to the mortality experience across different populations.

To date, there are only a few research attempts to model the mortality dependence across countries. [134] address the issue of pricing the catastrophic mortality bonds using DCC GARCH model, while [133] uses a dynamic copula approach to pricing the survivor index swaps. A study by [32] applied a factor copula approach to model mortality co-movements for multiple populations. Since these models are able to capture the cross population dependency, it prevents the pricing inaccuracy which may potentially occurs due to independence assumption in the population.

This chapter contributes to the existing literature by proposing a full multivariate DCC GARCH models to modelling the mortality dependence across multiple populations. In general, this study uses the variations of the DCC GARCH models such that the correlations are captured within the model structure and are allowed to vary over time. This chapter also employs the copula method in the mortality model considering the multiple population which are able to capture the inter-age mortality dependence structure. To the best of our knowledge, such comprehensive analysis in the multiple population has not been performed before.

5.2 Notation

The variables used in this chapter are defined below:

$D(x,t)$: The number of deaths in a population at age x during calendar year t.

$E(x,t)$: The number of exposure to risk in a population at age x during calendar year t.

$d(x,t)$: The observed number of deaths at age x during calendar year t.

$E^0(x,t)$: The initial exposure to risk at age x during calendar year t.

$E^c(x,t)$: The central exposure to risk at age x during calendar year t. [1]

$q(x,t)$: The one-year death rate for an individual age x during calendar year t can be estimated as $\hat{q}(x,t) = \frac{d(x,t)}{E^0(x,t)}$.

$\mu(x,t)$: Force of mortality at age x during calendar year t.

$m(x,t)$: Central death rate at age x during calendar year t. [2]

$\alpha(x)$: Describes the average age specific pattern by age of mortality.

$\beta(x)$: Describes the sensitivity of the logarithm of the hazard rate at age x to the time trend represented by $\kappa(t)$.

$\kappa(t)$: Describes the underlying time trend during calendar year t.

$\epsilon(x,t)$: Describes the effects not captured by the model, which are Gaussian distributed $N(0,\sigma^2)$ randoms effects by age and time.

5.3 Multi-country mortality modelling using multivariate DCC GARCH models

In this section, different econometric methodologies are used to model mean asset return and covariances for the mortality modelling. As discussed in Chapter 4, the

[1] If only central exposures are available, the initial exposures can be approximate by adding half of the reported number of deaths to the central exposures $E^0(x,t) \approx E^c(x,t) + \frac{1}{2}d(x,t)$. When the context is clear, $E(x,t)$ is used to refer $E^0(x,t)$ or $E^c(x,t)$.

[2] When the force of mortality is assumed to be constant over each year, then the force of mortality $\mu(x,t)$ and the death rate $m(x,t)$ correspond each other.

classical LC model expresses the logarithm of age specific death rates $m_{x,t}$ as the sum of an age specific component α_x which is independent of time with κ_t and β_x written as,

$$\log(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{5.1}$$

where κ_t represents the general mortality level and β_x is the age-specific reaction to the time varying factor and $\epsilon_{x,t}$ is the error term with $N(0,\sigma^2)$. The LC model represents the log central deaths as a linear function of mortality index which misrepresents the age-specific dependence structure or variance of mortality rates [87]. This study proposes DCC GARCH mortality models and employing copula function to the models to look at their performance in forecasting mortality rates.

Consider the vector stochastic process $\boldsymbol{x_t} = (x_{1,t},\ldots,x_{k,t})'$ which is the $n\times 1$ vector of log mortality rates at time t such that k is the total countries in the population, $i = 1,\ldots,k$. Each country is represented as i and ς_i is the $n\times 1$ vector of constant. The conditional mean of $x_{i,t}$ of the process is modelled by ARMA (M_i, N_i),

$$\boldsymbol{x}_{i,t} = \boldsymbol{\varsigma_i} + \sum_{m=1}^{M_i} \boldsymbol{\phi}_{i,m}\boldsymbol{x}_{i,t-m} - \sum_{n=1}^{N_i} \boldsymbol{\theta}_{i,n}\boldsymbol{\epsilon}_{i,t-n} + \boldsymbol{\epsilon}_{i,t}, \tag{5.2}$$

where the autoregressive coefficients are denoted by $\phi_{i,m}, m = 1,\ldots,M_i$, and moving average coefficients are denoted by $\theta_{i,n}, n = 1,\ldots,N_i$. The parameter $\epsilon_{i,t}$ is the linear innovation process of $x_{i,t}$, following a certain distribution with mean 0 and time-varying conditional covariance matrix $\boldsymbol{H_t}$. $\phi_{i,m}$ and $\theta_{i,n}$ are $n\times n$ matrices with $\phi_{i,m} \neq 0$ and

$\theta_{i,n} \neq 0$. As discussed in Chapter 3, the conditional covariance matrix $\boldsymbol{H_t}$ of $x_{i,t}$ may be defined following the DCC specification of [50],

$$\boldsymbol{H_t} = \boldsymbol{D_t R_t D_t}. \tag{5.3}$$

$\boldsymbol{R_t}$ is the time-varying correlation dynamics. $\boldsymbol{D_t}$ is the $n \times n$ diagonal matrix of time-varying standard deviations from univariate GARCH models,

$$\boldsymbol{D_t} = diag(\sqrt{h_{1,t}}, \ldots, \sqrt{h_k, t}),$$

The conditional variances, $h_{i,t}$ can be estimated separately by a GARCH(P_i, Q_i) specification of,

$$h_{i,t} = g_i + \sum_{p=1}^{P_i} \beta_{i,p} h_{i,t-p}^2 + \sum_{q=1}^{Q_i} \alpha_{i,q} a_{i,t-q}^{(2)}, \qquad i = 1, \ldots, k, \tag{5.4}$$

where $g_i > 0$, and $\alpha_{i,q}$, $\beta_{i,p} \geq 0$. $a_{i,t-q}^{(2)}$ is the Hadamard product and $\boldsymbol{H_t}$ is a positive definite matrix. $\boldsymbol{R_t}$ is also a positive definite conditional correlation matrix of the standardised disturbances of $\boldsymbol{z_t}$. Therefore, the elements of $\boldsymbol{H_t} = \boldsymbol{D_t R_t D_t}$ is modelled as

$$[\boldsymbol{R_t}]_{mn} = [\boldsymbol{Q_t}]_{mn} \sqrt{[\boldsymbol{Q_t}]_{mm} [\boldsymbol{Q_t}]_{nn}}. \tag{5.5}$$

The conditions for the positivity of the covariance matrix $\boldsymbol{H_t}$ requires $\boldsymbol{R_t}$ to be positive definite, g_i, all diagonal elements of matrices $\beta_{i,p}$ and $\alpha_{i,q}$ are non-negative parameters to be estimated such that $a+b<1$ to ensure stationarity and positive definiteness of $\boldsymbol{Q_t}$. $\boldsymbol{R_t}$ is decomposed into,

$$\boldsymbol{R_t} = (\boldsymbol{Q^{*t}})^{-1}\boldsymbol{Q_t}(\boldsymbol{Q^{*t}})^{-1}. \tag{5.6}$$

As in [50], $\bar{\boldsymbol{Q}}_{\boldsymbol{t}}$ is estimated as,

$$\bar{\boldsymbol{Q}}_{\boldsymbol{t}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{z_t z_t'}, \tag{5.7}$$

such that $\bar{\boldsymbol{Q}} = Cov\left[\boldsymbol{z_t z_t'}\right] = E\left[\boldsymbol{z_t z_t'}\right]$ is a $n \times n$ unconditional matrix of the standardised errors, where $\boldsymbol{z_t} = \boldsymbol{D_t^{-1}\epsilon_t}$ and $\boldsymbol{Q^{*t}}$=diag$\left(\sqrt{q_{1t}}, \sqrt{q_{2t}}, \ldots, \sqrt{q_{nt}}\right)$. In general, the DCC GARCH model is given by,

$$\boldsymbol{Q_t} = \left(1 - \sum_{m=1}^{M} a_m - \sum_{n=1}^{N} b_n\right)\bar{\boldsymbol{Q}}_{\boldsymbol{t}} + \sum_{m=1}^{M} a_m \boldsymbol{z}_{t-m}\boldsymbol{z}_{t-m}' + \sum_{n=1}^{N} b_n \boldsymbol{Q}_{t-n}. \tag{5.8}$$

M is the unconditional correlation matrix of $\boldsymbol{\epsilon_t}$, and $a_m, m = 1,\ldots,M$, and $b_n, n = 1\ldots,N$, are constant scalars.

5.3.1 DCC GARCH with copula

This study also proposes DCC GARCH with copula models to modelling the covariances for the mortality rates in multiple population. We examine the performance of these models and compare with other models discussed earlier.

Let $F_1(x_{i,1}),\ldots,F_n(x_{i,n})$ be the marginal distributions with a random vector $X = (x_{i,1},\ldots,x_{i,n})$ for country i, following [121]. The random vector has uniform marginal distributions when we apply the probability integral transform to each of the component

$(U_{i,1}, U_{i,2}, \ldots, U_{i,n}) = F_1(x_{i,1}), \ldots, F_n(x_{i,n})$. To capture the dependence between inter age mortality rates, the copula can be depicted as,

$$F(x_{i,1}, \ldots, x_{i,n}) = C(F_1(x_{i,1}), \ldots, F_n(x_{i_n})), \tag{5.9}$$

such that n-dimensional copula $C(u_{i,1}, \ldots, u_{i,n})$ is an n-dimensional random vector on $[0,1]^d$ with uniform marginals. The copula can be deduced from Eq. 5.9 as,

$$C(u_{i,1}, \ldots, u_{i,n}) = F(F_1^{-1}(u_{i,1}), F_2^{-1}(u_{i,2}), \ldots, F_n^{-1}(u_{i,n})). \tag{5.10}$$

5.4 Data & model checking

In this study, the model is applied to the mortality rates observed for males in the following $k = 10$ countries: Belgium, Denmark, Finland, France, Iceland, Italy, Netherlands, Norway, Sweden and Switzerland which are respectively represented by $i = 1, \ldots, 10$. These countries are chosen since they are all well developed countries with similar population characteristics. Therefore, a mortality model with common factors is expected which allows a joint model mortality rates in these countries. The empirical results are based on observed mortality rates for the calendar years 1900-2011, $t = 112$. The data are obtained from Human Mortality Database.

Let $\tilde{m}_i(x,t)$ be the aggregate death rate which is the ratio of the total number of deaths to the total number of exposures for a life aged $x \in x_1, \ldots, x_n$ in population $i = 1, \ldots, k$, is observe, then,

$$\tilde{m}_i(x,t) = \frac{d_i(x,t)}{E_i(x,t)}. \tag{5.11}$$

Figure 5.1 refers to the aggregate death rates for each of the country in time t. From the figure, the trend appears to be non-stationary.

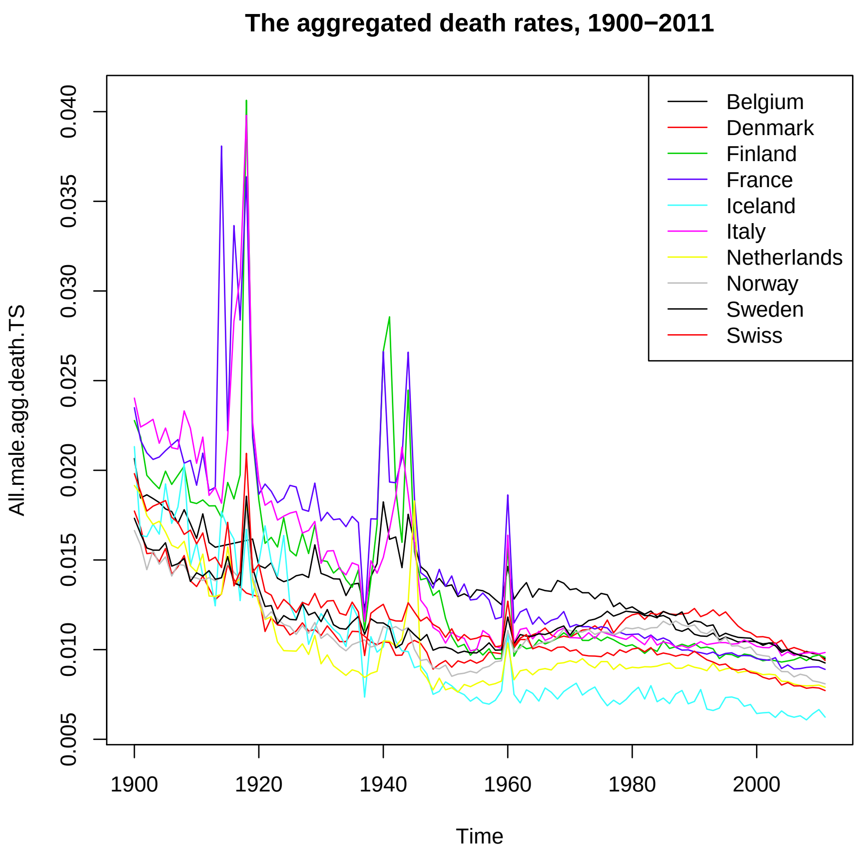


Figure 5.1. The aggregated death rates for ten countries under consideration, 1900-2011.

To confirm the observation, we evaluated the descriptive statistics of the time series which is presented in Table 5.1.

The kurtosis for all of the assets ranges from -0.3 to 5.97, indicating fat tails in the mortality distributions for all countries. The results show that the log mortality rates exhibit positive skewed distribution and a time-varying conditional volatility exists. From the results, it can be confirmed the presence of the stylised facts such as heavy tails, volatility clustering and heteroskedasticity in the mortality distributions data.

Table 5.1: Descriptive statistics of historical log mortality rates for the ten countries from 1900 to 2011.

	Mean	Std dev	Skewness	Kurtosis
Belgium	0.01337	0.00243	0.41243	-0.30286
Denmark	0.01145	0.00177	1.24737	1.54405
Finland	0.01366	0.00507	1.93386	5.97155
France	0.01509	0.00588	1.40299	2.43035
Iceland	0.00994	0.00383	1.09394	0.03086
Italy	0.01422	0.00539	1.65908	3.56573
Netherlands	0.01032	0.00288	1.58523	1.34233
Norway	0.01113	0.00198	0.98744	0.64434
Sweden	0.01177	0.00181	1.32447	1.60563
Switzerland	0.01161	0.00291	1.13827	0.70993

As in Chapter 3, the multivariate GARCH models are suitable to use to deal with these kinds of data. Therefore, we model instead

$$m_i(x,t) = \tilde{m}_i(x,t) - \alpha(x), \tag{5.12}$$

such that,

$$\alpha(x) = \frac{1}{T}\sum_{t=1}^{T} \ln \tilde{m}_i(x,t). \tag{5.13}$$

After differencing T = 111 observations remain. It is desirable for the mortality model to be as parsimonious as possible. Hence, the order of the models are set to be the smallest values so that the resulting model can capture the serial correlations in the conditional mean and variances.

To examine the null hypothesis of a unit root, the Augmented Dickey Fuller (ADF) test is used (see Table 5.2). There is evidence for a high level of autocorrelation in which the null hypothesis of no autocorrelation is rejected, at 5% level of significance, for almost all of the countries for all models, indicating that the mortality improvement rates for all country is stationary.

Table 5.2: The value of the ADF test statistics(lag 1-5) for the log mortality improvement rates for each country. The corresponding p-values are shown in parenthesis.

	lag 0	lag 1	lag 2	lag 3	lag 4	lag 5
Belgium	-16.10	-9.42	-7.71	-6.01	-4.80	-4.41
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Denmark	-13.54	-9.84	-7.44	-5.46	-3.77	-4.17
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Finland	-13.67	-10.11	-7.86	-6.79	-5.73	-5.11
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
France	-17.52	-9.5	-8.51	-6.33	-4.93	-5.45
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Iceland	-16.83	-12.81	-8.58	-7.10	-7.62	-6.29
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Italy	-12.76	-7.87	-6.53	-6.41	-6.16	-6.49
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Netherlands	-13.94	-9.81	-8.29	-6.00	-4.96	-5.20
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Norway	-14.39	-8.75	-6.12	-4.65	-4.11	-4.06
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Sweden	-15.48	-11.60	-7.74	-5.90	-5.05	-4.87
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
Switzerland	-17.50	-13.73	-9.59	-7.37	-6.00	-5.65
	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)

$m_i(x,t)$ is the log mortality improvement rate which can be illustrated as in Figure 5.2.

5.5 Model estimation and diagnostic checks

The DCC GARCH models provide a further simplification in modelling multivariate time series. There are various models for time series which are commonly used, such as ARMA or VAR models. As in Chapter 3, the mean returns are modelled using different estimation processes, either using a constant mean, ARMA or VAR models. ARMA model has both stationary stochastic processes of the autoregression and moving average methods. VAR model is one of the most commonly used multivariate econometric models which generalise the AR model and useful for describing the dynamic behaviour of financial time series and for forecasting. For the covariances

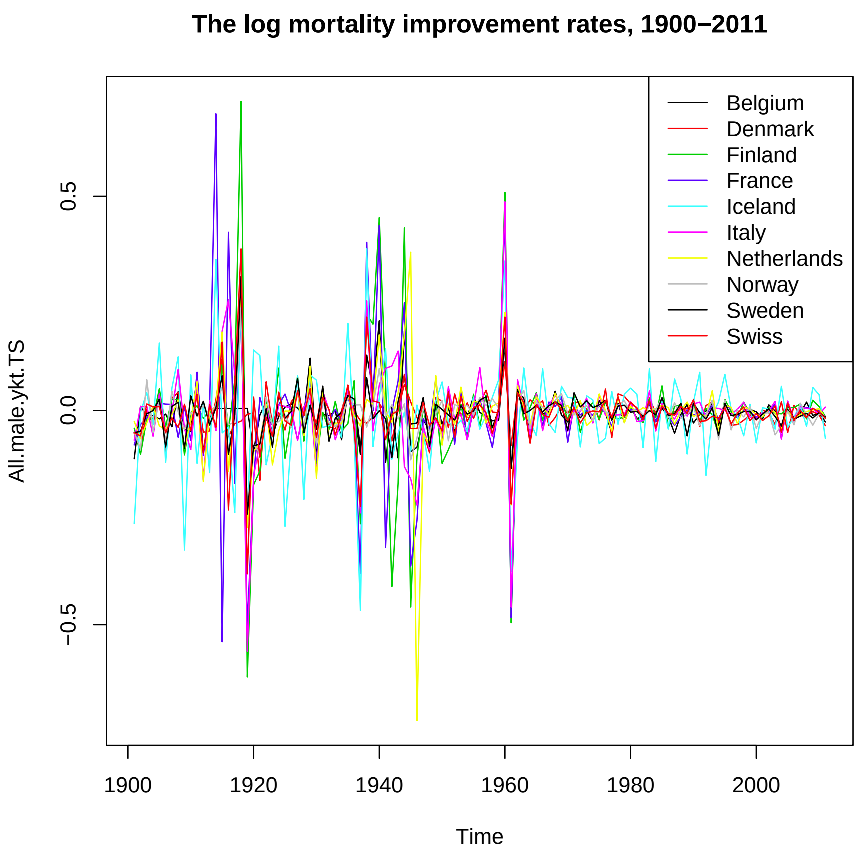


Figure 5.2. The log mortality improvement rates for ten countries under consideration, 1900-2011.

modelling, different specifications of DCC GARCH with and without copulas are examined.

This study evaluated 16 different model specifications of the mortality modelling for the multiple population. The models include,

- DCC GARCH Normal (*DCC-MVN*)
- DCC GARCH Student (*DCC-MVT*)
- Asymmetric DCC GJR-GARCH Normal (*aDCC-MVN*)
- Asymmetric DCC GJR-GARCH Student (*aDCC-MVT*)
- FDCC GARCH Normal (*FDCC*)
- VAR DCC GARCH Normal (*VAR-MVN*)

- VAR DCC GARCH Student (*VAR-MVT*)
- ARMA DCC GARCH Normal (*ARMA-MVN*)
- ARMA Student (*ARMA-MVT*)
- Copula Normal (*Cop-MVN*)
- Copula Student (*Cop-MVT*)
- ARMA Copula Normal (*ARMA-Cop-MVN*)
- ARMA Copula Student (*ARMA-Cop-MVT*)
- VAR Copula Normal (*VAR-Cop-MVN*)
- Asymmetric Copula Normal (*a-Cop-MVN*)
- Asymmetric Copula Student (*a-Cop-MVT*)

The resulting model's adequacy is assessed using the Ljung-Box test, which tests the null hypothesis of no auto correlations. The adequacy in modelling the conditional means and the conditional variances is tested on standardised residuals and squared standardised residuals respectively. The lag parameter is set to $m = 5$ with 5% significance level, following [134].

The Bayesian information criteria (BIC) is defined as:

$$BIC = -2\ ln(L) + m\ ln(n), \tag{5.14}$$

where $L = p(x|\hat{\theta}, M)$ is the maximised value of the likelihood function of the model M, $\hat{\theta}$ is the parameter values that maximise the likelihood function, n is the sample size and m is the number of parameters estimated by the model. The best model is the one that provides the minimum BIC. So based on the AIC and BIC, ARMA Copula with t distributed innovation model outperformed the other models since it has the

Table 5.3: The AIC,BIC, Loglikelihood and number of parameters values per observation for all models.

Model	AIC	BIC	Loglik	No. of Parameters [VAR GARCH DCC UncQ]
DCC-MVN	-29.383	-27.260	1717.78	87 [0+40+2+45]
DCC-MVT	-32.775	-30.383	1917.01	98 [0+50+3+45]
aDCC-MVN	-29.829	-27.680	1743.49	88 [0+40+3+45]
aDCC-MVT	-33.660	-31.512	1956.12	88 [0+40+3+45]
FDCC-MVN	-29.337	-27.165	1717.21	89 [0+40+4+45]
VAR-MVN	-30.242	-25.677	1865.43	187 [110+30+2+45]
VAR-MVT	-32.330	-27.741	1982.31	188 [110+30+3+45]
ARMA-MVN	-30.583	-27.971	1804.35	107 [0+60+2+45]
ARMA-MVT	-33.658	-30.778	1986.05	118 [0+70+3+45]
Cop-MVN	-30.175	-29.149	1716.70	87 [0+40+2+45]
Cop-MVT	-35.159	-33.865	2004.30	98 [0+50+3+45]
ARMA-Cop-MVN	-31.288	-29.775	1798.16	107 [0+60+2+45]
ARMA-Cop-MVT	-36.199	-34.417	2082.04	118 [0+70+3+45]
VAR-Cop-MVN	-35.403	-34.622	1996.89	287 [210+30+2+45]
a-Cop-MVN	-30.246	-29.197	1721.68	88 [0+40+3+45]
a-Cop-MVT	-35.143	-33.825	2004.43	99 [0+50+4+45]

lowest value as compared to the rest of the models. The parameter estimates for ARMA-Cop-MVT is presented in Table 5.4.

Table 5.4: Estimates of the parameters in the VAR-COP-MVT model.

	ς_k	$\phi_{k,1}$	$\theta_{k,1}$	ω	$\alpha_{k,1}$	$\beta_{k,1}$
Belgium	-0.0075	-0.1096	-0.3736	0.0001	0.2138	0.7774
Denmark	-0.0047	-0.1565	-0.1495	0.0000	0.0205	0.9661
Finland	-0.0036	-0.2938	-0.10000	0.0005	0.6612	0.6678
France	-0.0058	-0.2623	-0.2658	0.0002	0.4142	0.5848
Iceland	-0.0080	0.1066	-0.7108	0.0000	0.0405	0.9397
Italy	-0.0026	-0.1659	-0.1952	0.0002	0.4582	0.5408
Netherlands	-0.0033	-0.2116	-0.1200	0.0004	0.5710	0.4280
Norway	-0.0046	-0.5346	0.2521	0.0015	0.4431	0.0000
Sweden	-0.0067	-0.2438	-0.1927	0.0000	0.0607	0.9155
Switzerland	-0.0086	-0.1616	-0.3920	0.0000	0.0521	0.9168
a_1	0.0000		v	0.9051		
St.errors	(0.0000)			(0.0393)		

5.6 Conclusion

As mortality rates have been improving dramatically, the longevity risk has become important to annuity and pension providers. In this chapter, DCC GARCH mortality models were proposed. This study built DCC GARCH models for modelling the mortality rates in multiple population simultaneously. Further, DCC GARCH models with copulas were also evaluated.

Using mortality data from 10 different countries with similar socio-characteristics, this study demonstrate that the DCC GARCH with copulas gives better performance than the basic DCC GARCH models. This study has contributed to the literature by attempting to model mortality in multiple population using full multivariate DCC GARCH with and without copulas which to our knowledge has not been done before. Most of the mortality models in the literature are using the standard LC model or its extension. These models are useful for mortality forecasting but all of them focuses on modelling the mean level of mortality rates and ignoring the variance level of mortality rates and the temporal dependence structure between the inter-age mortality rates. Therefore, by proposing the DCC GARCH mortality models, it can help to capture the correlations between the inter age mortality rates, coupled with the copula methods which other LC based models are not capable of.

This chapter extends the mortality modelling in the previous chapters by using the econometric models in capturing the evolution of the aggregate mortality rates for different countries simultaneously. From the analysis, it is learnt that these models are able to capture the correlations between the age parameter in mortality rates function with the copula methods which other mortality models are not capable with. It will be much interesting to explicitly compare and stated its capabilities as opposed to the traditional Lee Carter model. However, the use of econometric models in modelling mortality is particularly new, especially involving multiple population models. There were very few references, which means a longer research time is anticipated. The whole purpose of this thesis is to do the asset and liability management and therefore to compare the performance of econometric on mortality modelling as opposed to traditional models, would need a further depth research on mortality modelling. This chapter proposed a short idea on the mortality modelling using the DCC GARCH based models which may be extend in the future to compare its performance opposed to the traditional LC or CBD models.

Future studies may also consider other improved version of DCC GARCH models such as the corrected DCC model by [18] or GOGARCH models by [138]. The study can also be applied to price mortality bonds, survivor index swaps or other annuity products involving a multiple population to see its performance. However, due to the limitation of data and timing constraint, this study is unable to provide application of the DCC GARCH mortality modelling as yet. This will be leave for future studies. In addition, the forecasting performance can be evaluated and compare among the models.

Chapter 6

Asset & liability valuation in pension schemes

This chapter combines volatility and mortality modelling obtained in Chapter 3 and 4 to design a tool for the asset and liability management in modelling different pension schemes. The funding value and actuarial liability in the pension schemes are evaluated. This study provides understanding to the source of risks to which pension schemes are exposed, enabling a better management of these risks.

6.1 Background

A pension scheme exists to provide income to its members in retirement. However, pension schemes are exposed to different types of risks, primarily financial risk and longevity risks. Financial risks are those associated with the performance of the portfolio of assets underlying in the relevant pension schemes. While longevity risks refer to the uncertainty over the average future mortality in a population which consequently, if a person lives longer than expected, the pension assets may be insufficient. Some pension providers have not explicitly integrated these risks to the pension liability's evaluation wisely, which may result in underfunded pension schemes. Proper risk management is important for pension schemes, to ensure sustainability and sufficient pension benefits are available.

ALM modelling is widely used by financial institutions in managing their assets and liabilities to ensure their pension objectives can be achieved. The duration of the liabilities is normally very long (more than 20 years), which means realistic models for long term investment returns and liability flows are crucial. A pension scheme model consists of three integral parts: investment returns projection, mortality rates projection and application to pension schemes to examine the funding level of the schemes, actuarial liabilities and outgoing benefit payment.

There are different pension schemes available with the most commonly used being Defined Contribution (DC) and Defined Benefit (DB) schemes. Each scheme has significantly different characteristics with respect to the risks faced by employers and employees, benefits provided and the structure of the schemes which have been discussed in Chapter 2. This study proposes a structure for the pension schemes that is explicitly defined by parameters that control the variability of benefits and contributions using the model developed by [46] and [74]. The mathematical equations derived by these studies has been modified to suit the structure of our proposed pension schemes.

6.1.1 The population model

Assume that there are $l_x \in \{1,2,3,\dots\}$ individuals participating in the fund which are independent copies of one another: same age x, same wealth amount, same risk preferences and same deterministic force of mortality. For simplicity, we further assume that no new entrants are allowed in the scheme after time 0 and, once an individual joins one of the schemes, they are not permitted to leave the scheme unless they die.

First, the following indicator for each life is assumed as,

$$N^k(t) = \begin{cases} 0 & \text{if } k_{th} \text{ person is alive at time } t, \\ 1 & \text{otherwise.} \end{cases}$$

d_t is the total number of deaths occurred at time t, which can be defined as,

$$d_t = \sum_{k=1}^{l_x} N_t^k. \tag{6.1}$$

In terms of the future lifetime random variables, if T_k is the future lifetime of the k^{th} member, then using $I[\cdot]$ to denote the zero-one indicator function,

$$N_t^k = I[T_k \leq t]. \tag{6.2}$$

The distribution of T_k is defined by,

$$P(T_k > t) = \exp^{-\int_0^t \lambda_u du} \equiv \ {}_tp_x. \tag{6.3}$$

$T_1, T_2, \ldots, T_{l_0}$ are independent random variables. l_t represents the number of individuals alive at time t such that

$$l_t = l_0 - d_t. \tag{6.4}$$

Hence, d_t is a Poisson process with rate $\lambda_{x+t} l_{t-}$, following the definition of N^k.

6.2 Investment strategy

Based on Chapter 3, this study applies the best performing multivariate DCC GARCH model that minimises the mean-CVAR to obtain the investment return for the portfolios. As discussed previously, the DCC GARCH model is given by

$$\boldsymbol{Q}_t = \left(1 - \sum_{m=1}^{M} a_m - \sum_{n=1}^{N} b_n\right)\bar{\boldsymbol{Q}}_{\boldsymbol{t}} + \sum_{m=1}^{M} a_m \boldsymbol{z}_{t-1}\boldsymbol{z}_{t-1}^{\prime} + \sum_{n=1}^{N} b_n \boldsymbol{Q}_{t-1}. \tag{6.5}$$

Given a confidence level β and a fixed $x \in X$, VaR is defined as the smallest number l such that the probability of a loss L is not more than $1-\beta$ for losses greater than l and F_L is the distribution function of the losses,

$$VaR_\beta(x) = inf\{l \in \mathbb{R} : P(L > l) \leq 1-\beta\} = inf\{l \in \mathbb{R} : F_L(l) \geq \beta\}, \tag{6.6}$$

The objective function for the optimisation model is to minimise the mean-CVaR as

$$\text{Minimise CVaR}_\beta(x) = \text{Minimise } E[L|L \leq VaR_\beta(x)], \tag{6.7}$$

such that $R \geq r_{target}$.

Similar to Chapter 3, this study considers 12 assets consisting of 8 stocks which are FTSE100, MSCI Europe Excluding UK (MSEXUK), S&P 500 composite (S.PCOMP), DAX30 (DAXINDX), AEX (AMSTEOE), TOPIX (TOKYOSE), Hang Seng (HNGKNGI), and TSX composite (TTOCOMP). The bond indices that were included: UK Benchmark 10-Year Government (BMUK10Y), US Benchmark 10-Year Government (BMUS10Y), Germany Benchmark 10-Year Government (BMBD10Y), and FTSE Britain Government Linked Bond (BGILALL).

The time series data were examined for 30 years from January 1985 to December 2014, yielding 360 observations. The in-sample period used to estimate the model is from January 1985 to December 2004, yielding 240 observations. The recursive forecast approach is employed (expanding-window), where the data from January 1985 to December 2004 are used to make the first estimation in January 2005, data from January 1985 to January 2005 are used to make the second estimation in February 2005, and so on.

To measure the performance of the portfolio, 120 out-of-sample periods were evaluated from January 2005 to December 2014 where the optimal asset allocation is performed with a different mean returns and covariances depending on the time period. These are monthly observations and therefore it is essential to calculate AR, which is the annualised return for the investments to see how the investment performed annually. To annualise the monthly returns,

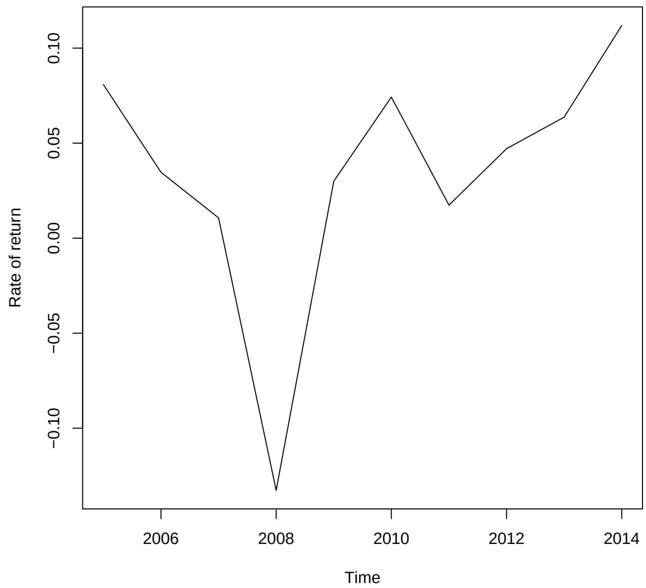
$$AR = \sum_{m=1}^{12} (1 + i_m). \tag{6.8}$$

The annualised return for the out-of-sample period appear in the Figure 6.1.

6.3 Mortality model

The mortality model used in this study follows the LC model by [82]. In general, the LC model is fitted to the matrix of UK death rates, from year 1985 to 2004. The data for the UK population are obtained from the Human Mortality Database.

Figure 6.1. Annualised return on investment for a minimised mean-CVaR portfolio using DCC GARCH with Student-t distribution for the out-of-sample period, from 2005 to 2014.



The realised log mortality rates $\tilde{m}_{x,t}$ at age x in year $t = 1, \ldots, T$ for a life aged x in population i is observed, that is,

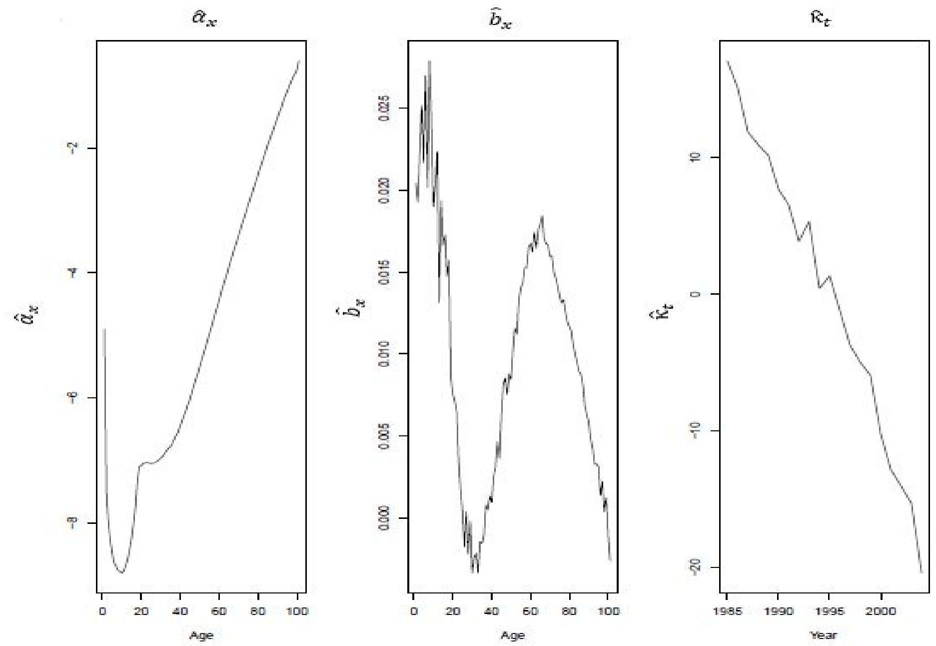
$$\tilde{m}_i(x,t) = \frac{d_i(x,t)}{E_i(x,t)}, \tag{6.9}$$

such that $d_{x,t}$ denotes the number of deaths in a population and $E_{x,t}$ denotes the number of exposure to risk in a population at age x during calendar year t. The LC model describes the log of a time series of age specific death rates $m_{x,t}$ as the sum of an age specific component α_x which is independent of time with κ_t and β_x as below,

$$\log(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{6.10}$$

α_x is the base mortality rate for age x, β_x describes the different of time t at each age, explaining the sensitivity of age x in response to κ_t which is an index that describes the variation in the level of mortality to t. $\epsilon_{x,t}$ is an independent and identical distributed random variables $N(0,\sigma^2)$ considering the age and time specific trends that is not captured by the model. These parameters are estimated using the LC model with a maximum age equal to 100. The diagram is presented in Figure 6.2.

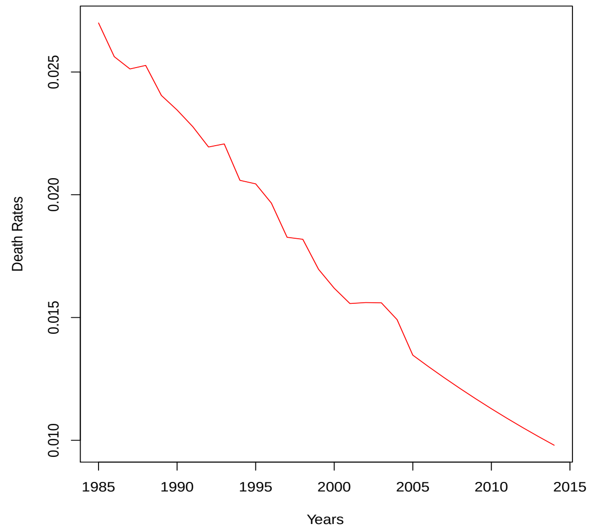
Figure 6.2. First age $\hat{\alpha}_x$, $\hat{\beta}_x$ and period effects $\hat{\kappa}_t$ estimated using LC model for UK total population.



As expected, the average mortality rate increases as age increases which is clearly seen from $\hat{\alpha}_x$ pattern. For $\hat{\beta}_x$, there is a greater value at younger age and greatest improvement in the age range (40-100 years old) showing the death rate at age x varies significantly when the general level of mortality changes. $\hat{b}_x$ gets smaller as it approaches older ages. For $\hat{\kappa}_t$, as expected, it has a decreasing trend across the time.

The forecast for the parameter $\hat{\kappa}_t$ is then evaluated for the out-of-sample period from 2005 to 2014 based on ARIMA extrapolation.

Figure 6.3. Past and forecasted mortality rates for individuals aged 65 in UK total population.



After obtaining the projected $\hat{\kappa}_t$, the past and forecasted rates are binded in the same matrix. This is essential to produce the cohort life table using the results from the LC model based on a person aged 65 at 2005 (which means he was born in 1940).

Cohort refers to a group of people with the same year of birth. The values obtained in the cohort table is an appropriate measure of how long a person, of a given age, would be expected to live on average. On the other hand, the life table represents mortality rates for a fixed period in time of a certain population.

As mentioned above, the cohort life table is depending on the year of birth which is created using the results from the Lee Carter model. To get a projected life table for a given cohort of birth, it is essential to blend the fitted rates from historical data to the projected rates are within the same cohort. Then, the life table is produced for ages $0, 1, \ldots, t$ based on the Lee Carter model as in the following equation,

$$\begin{aligned} \log(\hat{m}_{x,t}) &= \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t, \\ {}_t p_x &= \exp^{-\int_0^t \lambda_u du} \end{aligned} \tag{6.11}$$

The actuarial cohort life tables are presented for the out-of-sample period of 10 years (cohort 1940 to cohort 1949) in Appendix A.4. By producing the life table, the number of people alive, ${}_tp_x$ for a particular age group in a particular year can be determine, showing the relationship between the two equations above. The table is essential to perform the actuarial projection in calculating the actuarial present value, APV of $\ddot{a}_{65}^{(12)}$ for the selected cohorts.

6.4 Description of the schemes

This section discusses five pension schemes that are studied and looks at how the schemes evolves. The schemes are listed below:-

- Defined Benefit with no risk sharing (*DBnrs*)
- Defined Benefit with risk sharing (*DBrs*)
- Defined Contribution Scheme with draw down of assets at retirement (*DCasset*)
- Defined Contribution Scheme with life annuity purchase at retirement (*DCannuity*)
- Group Self Annuitisation Scheme (*GSA*)

6.5 Notation

In this study, the model used in [74] are improvised. In his study, a stationary population is assumed which simplifies the calculations. Whereas, in here, we are looking at more realistic assumptions, with a non stationary population assumption in which if a member dies, no new entrant is allowed to replace and no further benefit accrued. To simplify, all members are assume to enter the fund at the same age with same target benefit.

The variables used in the study are described below:

l^*_{x+t} = number of people alive at time t

B_t = benefit payment made at time t

C_t = contribution payment paid by the employer at time t

F_t = fund value at time t

d_t = total number of death occurred at time t

i_t = investment return obtained from the fund value between time *t-1* and t

r_f = risk free rate

σ^2 = the variance of i_t

TB = target benefit

AL_t = actuarial liability at time t

$\ddot{a}_x$ = annuity due whose payments are made at the beginning of each period.

k_c = spread parameter for contribution income

k_b = spread parameter for benefit income

In this study, there is no member contributions and the target benefit is defined as,

$$TB_t = TB \times l^*_{x+t}. \quad \text{s.t.,} \quad TB = 1. \tag{6.12}$$

6.6 Scheme Dynamics

The scheme dynamics for each pension scheme will be describe to see how each scheme evolves over time. In general,

$$F_{t+1} = (F_t + C_t - B_t)(1 + i_{t+1}), \tag{6.13}$$

$$C_t = k_c(AL_t - F_t). \tag{6.14}$$

k_c represents the spread parameter in the contribution income. For example, a value of $k_c = 0.2$ means that the employer will make up 20% of the difference from surplus or deficit to the scheme. During unfavourable market conditions, the asset values decline and impact the employer since they have to give extra contribution to cover the deficit. Whereas, a funding surplus arises when the value of the assets exceeds the calculated value of the liability, allowing the employer to reduce their contribution rate, or even stop contributing to the scheme altogether. Such action is known as *contribution holiday*.

The parameter k_b is the spread parameter for benefit payment. It indicates that when the scheme is in deficit, the benefit payout will be less then the target benefit. If the scheme has surpluses, the benefit payout is greater than the target benefit such that the surplus is used to provide additional benefits to the members.

$$B_t = TB_t\Big[1 - k_b\Big(\frac{AL_t - F_t}{AL_t}\Big)\Big]. \tag{6.15}$$

By substituting for C_t and B_t in 6.13,

$$F_{t+1} = \Big[(1-k_c-k_b\frac{TB_t}{AL_t})F_t + k_cAL_t - TB_t(1-k_b)\Big](1+i_{t+1}). \tag{6.16}$$

From the cohort actuarial life table, the number of people alive at each period of time is obtained. To examine the funding level after retirement, the total number of people alive aged 65 to 85 is calculated as below.

Table 6.1: Total number of people aged 65 to 85 alive for the out-of-sample period, from 2005 to 2014 obtained from the cohort tables (refer Appendix A.3).

	$l_{65,\ldots,85+t}$
2005	1187311
2006	1200466
2007	1235931
2008	1240628
2009	1248268
2010	1246470
2011	1235220
2012	1220068
2013	1192203

The benefit payment is adjusted based on the actual investment and mortality experience using the models described in Section 6.2 and Section 6.3.

Let π be the proportion of wealth invested in the equity and the rest $(1-\pi)$ invested in the bond. No withdrawal is allowed in the fund. The risk free rate, r_f is set at 5% and i_t is the investment return obtained from the DCC GARCH modelling as in Table 6.2.

Therefore, Equation 6.16 is modified to consider the investment experience, and hence the fund value is given by,

$$F_{t+1} = \Big[(1-k_c-k_b\frac{TB_t}{AL_t})F_t + k_cAL_t - TB_t(1-k_b)\Big]\Big[\pi(1+i_{t+1}) + (1-\pi)(1+r_f)\Big]. \tag{6.17}$$

Table 6.2: Investment return on fund for the out-of-sample period 2005 to 2014 based on multivariate DCC GARCH with Student-t innovation with a minimisation mean-CVaR portfolio.

Returns, i_{t+1}	
2005	0.0809
2006	0.0347
2007	0.0107
2008	-0.1327
2009	0.0299
2010	0.0742
2011	0.0174
2012	0.0471
2013	0.0637
2014	0.1120

6.6.1 DB scheme with no risk sharing (*DBnrs*)

In the DBnrs scheme, k_b is set to be zero, which means there is no benefit risk involved and the benefit payout will be equal to the total target benefit times the number of people alive at time t. At time 0, the fund value is assumed to be equal as actuarial liability such that,

$$F_0 = AL_0 = TB_0 \times \ddot{a}_x, \tag{6.18}$$

where $\ddot{a}_x$ is an annuity due whose payments are made at the beginning of each period. And from Equation 6.17, we have:

$$F_{t+1} = \Big[(1-k_c)F_t + k_c AL_t - TB_t\Big]\Big[\pi(1+i_{t+1}) + (1-\pi)(1+r_f)\Big]. \tag{6.19}$$

6.6.2 DB scheme with risk sharing (*DBrs*)

A DB scheme with risk sharing works similarly to a DB scheme with no risk sharing scheme, except that a value for k_b is set to allow for benefit risk sharing among members in the scheme. Different choices of k_b will give impact to the members' benefit payout such that if the scheme is in surplus, a greater value of k_b will give the member a higher benefit income. In contrast, if the scheme is in deficit, a greater value of k_b will give the member a lower benefit income payment. The formula for the fund value is as in Equation 6.16.

6.6.3 DC scheme with life annuity purchase at retirement (*DCannuity*)

DC schemes appear to be riskier, compared to DB schemes, because there is no guaranteed income at retirement. Each retiree must decide how much money to withdraw from a pension fund and how to invest the remaining funds. DC members may choose to get the best annuity products available from the insurance company to provide retirement income. The pension life annuity can be purchased using the amount of funds available in the pension pot. When buying a life annuity, the member bears inflation risk, the risk of losses in the real value of the pension due to unanticipated inflation, and they are also exposed to a higher longevity risk due to pooling of individual risk.

Each DC member needs to pay for the annuity cost plus loading cost, k. Loading cost is a charge imposed by the insurer or pension provider as a operating cost of administering the pension scheme. The life insurance company charges the DC members £$(1+k)\ddot{a}_{65}$ for £1 per annum paid continuously until death. So, assuming that everyone is aged 65 at time 0, the initial fund value will be,

$$F_0 = \ddot{a}_{65} \times (1+k).$$

And in general, the benefit payment can be written as:-

$$B_t = \frac{F_0}{(1+k)\ddot{a}_x}. \tag{6.20}$$

For example, assuming that a DC member retires at the age of 65 and each of them have £100 to pay to the life insurance company. Each DC member will get a level benefit payment of B_t per annum from age 65. With £100, the DC member can buy an income of:

$$\frac{100}{(1+k)\ddot{a}_{65}}.$$

per annum paid until death.

6.6.4 DC scheme with draw down of asset at retirement (*DCasset*)

Another option for the DC member to manage their retirement savings is by investing it. This will protect the member from outliving his DC savings in the retirement period. It is important for the employees to realise the implications of not having a secure pension income and the possible consequences that might occur.

In some countries, such as Malaysia and Singapore, the retirees are required to buy annuity to provide retirement income. However, in some countries like the UK, the retiree has an option to delay the purchase of an annuity until a certain age and receive income directly from the pension fund. This option is known as drawdown of asset. In this scheme, the individual's pension fund remains invested and members are able to draw income from it, as and when required. The member of the scheme bears the financial risk as there is no guarantee of a fixed benefit level at retirement because it depends on the investment returns.

Assuming at retirement, a DC member with drawdown of asset, who is aged 65 and have £100, the first benefit payout, B_0 is

$$B_0 = \frac{100}{\ddot{a}_{65}}.$$

Even though this is the drawdown situation, we use $\ddot{a}_{65}$ to calculate the withdrawal amount to simplify the calculations.

After paying out the first benefit, the balance in the fund will be $100(1 - \frac{1}{\ddot{a}_{65}})$. The fund value and retirement income will then be depending on the investment returns obtained each year. In general, the fund value is:

$$F_{t+1} = (F_t - B_t) \times (1 + i_{t+1}). \tag{6.21}$$

6.6.5 Group self annuitisation scheme (*GSA*)

GSA is a scheme that protects against the longevity risk that is the risk that the member will live longer than expected. This is true because, for example, there are ten members in a GSA scheme, each investing £100 in the scheme which gives 5% investment return over the year. At the end of the year, if no one dies, each person will get $\frac{100\times 10}{10} \times 1.05 = £105$ and if nine members out of ten dies, the last member alive will receive $\frac{100\times 10}{1} \times 1.05 = £1050$, based on actual mortality experience. Meaning that the longer a person lives with a reducing number of people in the scheme, the higher benefit they will get. The member has benefited from the longevity risk and the scheme allows the members to pool mortality risk together. In return, the member will receive a regular benefit annuity payment, calculated based on chosen mortality and investment basis. In this study, everyone is assumed to have the same investment strategy to avoid complicated calculations.

Now, assuming the initial fund value is £100 with $l^*_{65} = 10$, the benefit paid to each person at time 0 is,

$$B_0 = \frac{F_0}{l^*_{65} \times \ddot{a}_{65}} = \frac{100}{10} \times \frac{1}{\ddot{a}_{65}}.$$

The scheme will pay out to each of its member $B_0 \times l^*_{65} = £10$, leaving the fund value to be $100 - 10 = £90$. Therefore, at time 1, the total fund is $F_1 = £90 \times (1 + i_1)$ such that i_1 is the investment return over the year. And at time 1, when $l^*_{66} = 9$, the benefit payout to each survivor at time 1,

$$B_1 = \frac{F_1}{l^*_{66} \times \ddot{a}_{66}} = \frac{90(1+i_1)}{9 \times \ddot{a}_{66}} = \frac{10 \times (1+i_1)}{\ddot{a}_{66}}.$$

In general, for GSA, the fund value is evaluated as,

$$F_{t+1} = (F_t - l^*_{x+t} B_t) \times (1 + i_{t+1}). \tag{6.22}$$

$C_t = 0$, then,

$$B_t = \frac{F_t}{l^*_{x+t} \times \ddot{a}_{x+t}}. \tag{6.23}$$

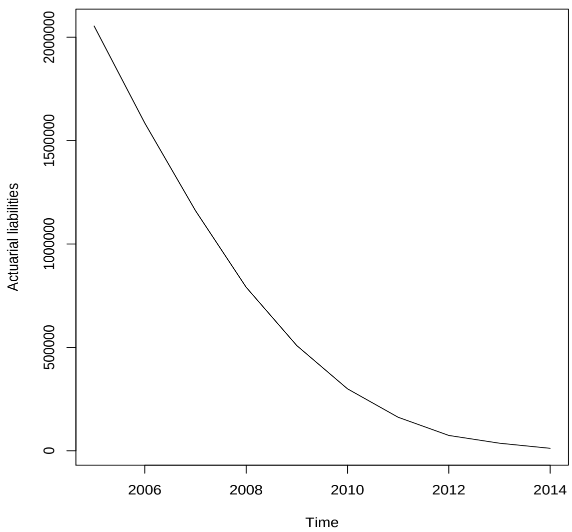
There is not much literature studying GSA. [103] has studied the implications of pooling longevity risk through GSA. [106] has expand the study by [103] and proposed a method for collective risk pooling of systematic mortality in GSA rather than individual bearing the risk.

6.7 Results

In this section, based on the assumptions made in previous sections, the valuation of the liabilities can be evaluated. The actuarial liabilities are obtained as,

$$AL_t = TB_t \times \ddot{a}_{x+t}. \tag{6.24}$$

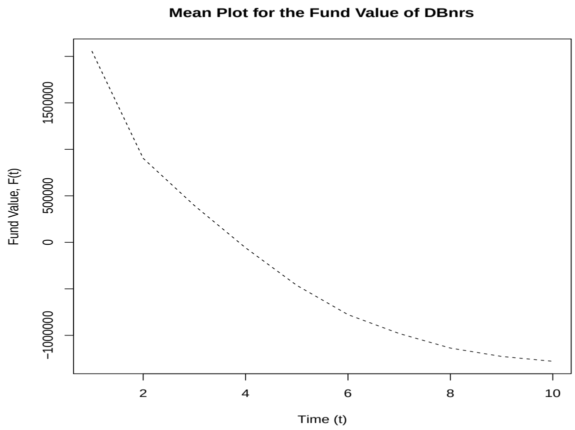
Figure 6.4. Liabilities value for the DB schemes from 2005 to 2014.

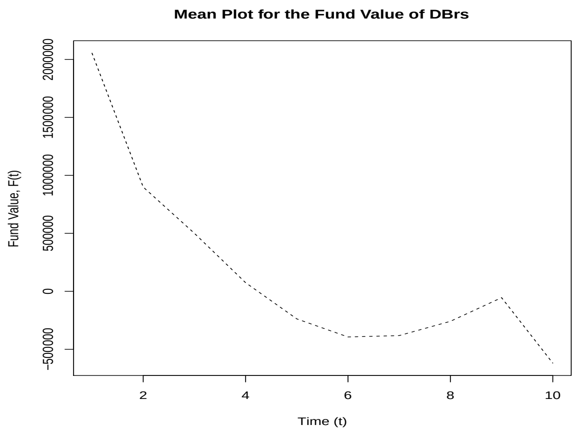


Note that the liabilities value for *DBnrs* and *DBrs* scheme are similar since the smoothing factor, k_b does not involve in the actuarial liabilities formula. As expected, the liabilities value for the DB schemes decreasing as time increases (refer Figure 6.4). This is due to the fact that many members have left the fund, meaning lesser liabilities to the pension providers.

The fund value in defined benefit schemes are illustrated in Figure 6.5. Initially, the *DBrs* performed better as expected than the *DBnrs*. This is because the *DBrs* benefited in risk sharing among members in the scheme, while in *DBnrs* there is no benefit

Figure 6.5. Fund value for the DB scheme with no risk sharing and DB scheme with risk sharing scheme from 2005 to 2014.





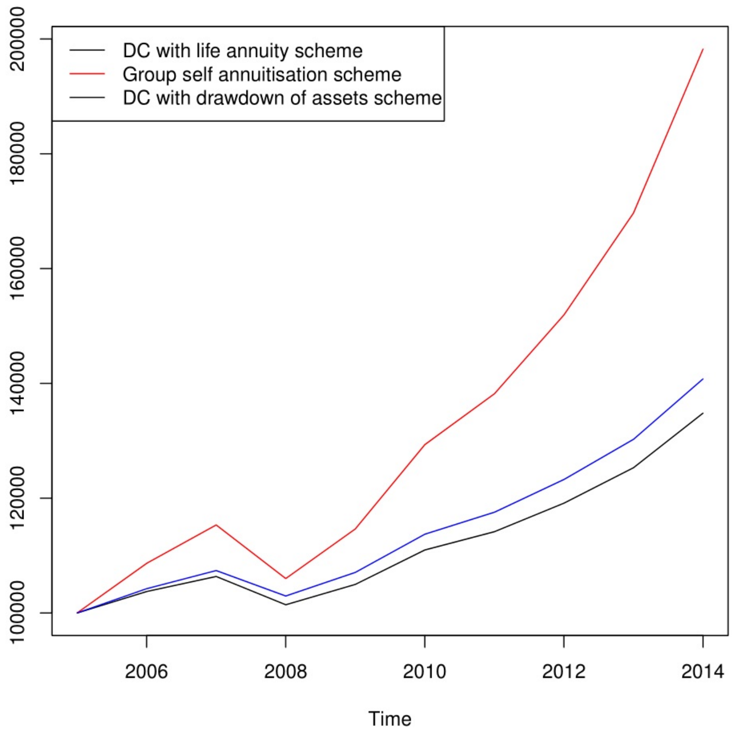
risk involved and the benefit payout will be equal to the total target benefit times the number of people alive at time t. As the *DBrs* approaches the last period, the fund value deteriorated extremely, which may be due to fewer people in the fund. According to Deloitte, during 2014 and 2015, pension deficits have been on the rise for most of the DB schemes which was driven by long term UK government gilt yields and falling expectation of future inflation [42].

A funding deficit arises in both schemes where the value of the assets is less than the expected cost of promised benefits. This means that there are insufficient funds to pay the pension benefits promised to the members. When there is a deficit, the sponsoring employer pays most, if not all of the additional contributions, over a sustained period to ensure that the funding level can be increased. The employer is responsible for the contribution to ensure the funding level can be improved during economic downturn while allowing contribution holidays during good times. This may affect investment strategies as when equity prices boom, pension schemes may not always rebalance the portfolio, leading to growing equity allocations in portfolios. Whereas during the downturn, pension scheme may sell some of their equities, which eventually drive markets down further. The funding level of these schemes can be improved by setting a minimum funding levels that are consistent with the pension goals.

As explained earlier, there are not many people participating in the DB schemes anymore. In 2011, only 3% of workers participated solely in a DB scheme compared to 28% in 1979 [48]. This is because there are not many employers providing pension schemes since the employers face difficulties in estimating the DB scheme liabilities to evaluate the retirement benefit expenditures. Consequently, employers made a significant shift from DB to DC schemes whereby through DC schemes, since these are more favourable to pension providers due to the benefits paid out being linked to investment performance.

Figure 6.6 shows the funding level for the other three schemes which are *DCannuity*, *DCasset* and *GSA* scheme. The fund values in these three schemes are in surplus with *GSA* scheme having the highest fund values among all schemes at the end of 2014. In *GSA* scheme, surviving members will receive a continuous mortality credit and the bigger wealth of the dying members have, the larger mortality credit will the surviving members receive. This scheme protects against the longevity risk such that a higher benefit will be given to those surviving longest. *GSA* is a hybrid pension scheme in which the members bear the pool's systematic risk but share idiosyncratic risk in

Figure 6.6. Fund value for the DC scheme with life annuity purchase at retirement, DC scheme with draw down of asset at retirement and Group self annuitisation scheme from 2005 to 2014.



providing its members with a more evenly pension risks while giving the best benefit available.

6.8 Conclusion

In this chapter, we have presented an application of the volatility modelling and mortality modelling underpinning the multivariate DCC GARCH and LC methodology for forecasting mortality rates.

In particular, this study has focused on forecasting the mean and variance covariance matrix using the best performing model in modelling volatility based on the finding in Chapter 3 which is the DCC GARCH with Student-t distribution. The rates obtained were used to get the investment returns which is based on a portfolio that minimises mean-CVaR. From our previous finding, the multivariate DCC GARCH with Student-t

has proven to be the best model that has a better performance as compared to the other models. The purpose of this asset modelling is to produce an asset model that would be useful to investors or pension managers in managing the pension assets.

While the asset modelling is done using mean-CVAR and DCC-GARCH volatility modelling for the return data, which is the best performing strategy in Chapter 3, it uses the traditional LC model for liabilities modelling. This is a potential limitation of this study. It would have been interesting to explore this aspect. However, this will be left for our future research. As mentioned in the thesis, modelling mortality in multiple population is particularly new and there were very few studies on this subject. This study produced the cohort actuarial table for the out-of-sample period, from 2005 to 2014, which is used to calculate the actuarial present value for the selected cohorts. The number of people alive in the pension fund can be projected which were essential to calculate the actuarial liability. But, to do this based on multiple population are not straightforward. Therefore, for now, this study uses the traditional LC modelling to estimate the parameters and project future liabilities. We will extend this study in the future to incorporate the multiple population in the mortality modeling.

From the modellings, this study has found that *GSA* scheme has a better funding level as compared to the other pension schemes. The defined benefit schemes is underfunded which may be improved by adjusting the contribution of the employers to the scheme.

The model built for this study was created for asset and liability modelling in UK pension funds, which can also be used for applications in life and longer term general insurance products.

This study has only applied the mortality modelling for a single population which is the UK total population using the classical LC model to simplify the calculation. As mentioned above, it would be interesting to extend the study to include the multiple population model in the mortality modelling for pension schemes as the results may be improved. Similarly, it would be interesting to allow other assumptions in the study, for example allowing a variable or increasing salary which may change the values of various parameters in the pension schemes.

Chapter 7

Conclusion & proposed future work

7.1 Conclusions

The aim of this thesis was to examine three different but interconnected problems, (which were presented in chapters 3, 4, 5 and 6) which lie in the broad area of asset liability management for long term financial investors, with the focus on pension schemes. Several new models were proposed, mainly as extension of existing models.

Chapter 1 (Introduction) described the problem statement of the thesis, namely, the challenging environments due to increasing life expectancy and highly volatile markets, which is faced by governments and financial institutions worldwide. The study of ALM has become important for organisations in managing their assets and liabilities to ensure the sustainability in their funding responsibilities. The research objectives, structure of the thesis and the scientific contributions were presented in this chapter.

Chapter 2 was a comprehensive literature review presented to help the reader understand and get the idea of the fundamental knowledge relating to asset liability management, mortality modelling and pension schemes structure, which are the most important elements for the next chapters in the thesis. The importance of asset allocations and diversification were discussed in detail explaining the purpose of ALM study, especially for long term investors including pension providers, insurers and governments. This chapter provided a review of the latest asset modelling, used widely in the literature, to forecast the mean and covariances used as input parameters to construct portfolios with the best out-of-sample performance. It described the most widely used models to solve the ALM problems such as DCC models and its extensions including copula models. The discussion continued with the most popular portfolio optimisation model of [90] and other related studies. Chapter 2 also reviewed the mortality modelling that exists in the literature including the famous LC model and other recent studies involving a single and multiple population mortality models. The structure of pension schemes were also discussed in detailed, which includes the defined benefit, defined contribution and hybrid pension schemes.

In chapter 3, the multivariate GARCH models were evaluated for volatility modelling which is based on different optimisation strategies. This study examined a large class of different advance multivariate DCC GARCH models in modelling the mean returns and variance covariance matrices, using five different optimisation strategies: the minimum-variance, mean-variance, maximising Sharpe ratio, minimising mean-CVaR and maximising Sortino ratio for the portfolio optimisation. The best performing model was the multivariate DCC GARCH with Student-t distribution which clearly had a better out-of-sample in most of the optimisation strategies. The use of multivariate GARCH models in the analysis were appealing as it kept the estimation to be as parsimonious as possible, with a small number of parameters involved, even dealing with a larger number of parameters than any other multivariate volatility modelling. From the study, it is found that the dynamic models have better performance than the static models which reduces portfolio risk and improve the realised return in the out-of-sample period. The addition of copula functions to the models may capture additional characteristics in the data but has been found that it does not give any better performance than the existing dynamic models. The study would be beneficial to practitioners working in the portfolio optimisation for investment in pension schemes and other financial options involving a long term investment decision.

Chapter 4 analysed the empirical mortality modelling using a single and multiple population data. This study has modified the existing mortality models available in the literature and proposed a novel alternative estimation methods using the modification of singular value decomposition (SVD) and principal component analysis (PCA) in modelling the mortality models. It is found that the mortality rates are decreasing worldwide, which shows an improvement in healthcare but clearly presents challenges as the longevity increases meaning people may live longer than expected, and the pension providers may underestimate the survival probabilities and, therefore, suggest inaccurate premiums. The insurer or pension provider charge premiums initially without knowing, for certain the liabilities that they have to meet in the future. Therefore, a study of mortality modelling is very important as it will help the insurers and pension providers to plan properly and be able to project future mortality as accurate as possible to avoid systematic losses. This chapter analysed different types of SVD and PCA estimation methods to propose different mortality modelling which are suitable for a single and multiple population in order to obtain a model for projecting life expectancy across populations. The area modelling of the sub-population in the UK data is also carried out in which the multiple population modelling has better out-of-sample performance than the single population modelling. The proposed models allow us to estimate the period effects in multiple population simultaneously which provides a better comparison rather than looking at period effects in a single population. It is learned that these proposed modification gives a better mortality models for multiple population such that these model outperforms the individual models which is reflected by the smaller MSE and BIC for the older age group. As for the younger age group, the result might be improved by including more age period effects to the model. The results indicate that the proposed models are suitable for modelling mortality for a older age group and suitable to apply in pension schemes.

For Chapter 5, the study has been extended to test econometric models in modelling mortality data to capture the evolution of the aggregate mortality rates for different countries simultaneously. This study analysed the use of multivariate DCC GARCH models and also adding copula function to forecast the life expectancy across multiple population. The use of econometric models in modelling mortality is particularly new. These models are able to capture the correlations between the age parameter in mortality rates function with the copula methods which other mortality models are not capable with. The models are evaluated based on the BIC values such that it is found that the ARMA DCC GARCH with Copula using Student-t distribution has the best performance as compared with the other models. It will be much interesting to explicitly compare and stated its capabilities as opposed to the traditional Lee Carter or CBD model. However, the use of econometric models in modelling mortality is particularly new, especially involving multiple population models. There were very few references available, and therefore we will leave this for our future study.

Finally, chapter 6 demonstrated the use of volatility and mortality modelling on how these models can be used in managing asset and liability in pension schemes. This study use the DCC GARCH with Student-*t* distribution to forecast the mean and covariances to obtained the best investment return for the pension schemes. For the mortality modelling, the LC model is used to project the future mortality for the out of sample period using a UK total population data. This study also incorporates significant feature not previously mentioned in the relevant pension literature such as using a non-stationary population in the analysis. In addition, this study produces cohort life table which are used to determine the number of people alive in the scheme to estimate the actuarial present value which is needed to calculate the actuarial liabilities and to determine funding level in each scheme. This chapter has examined a wide range of different pension schemes which include the defined benefit schemes, defined contribution schemes and the hybrid scheme known as the Group self annuitisation scheme .

The choice of volatility and mortality modelling in a pension scheme is very important as it will help the pension providers in making investment decisions and projecting future mortality to ensure the pension liabilities can be meet and avoiding the schemes closing. In addition, the on-going financial instability worldwide has made pension schemes face losses, and this study will contribute to society by giving the insurers and pension providers ideas on how to re-examine their asset allocation strategies in providing the best pension schemes to retirees. To conclude, the new models introduced in this thesis are flexible models that can be used to analyse and forecast the investment returns and human mortality. Also, as discussed earlier, the multivariate DCC GARCH models is a powerful tool for analysing uncertainty and investment risk.

7.2 Suggestions for future research

Undoubtedly, any research is subject to further improvement and extensions. This section will provide some ideas for potential future research.

To further increase the understanding in modelling volatility, one could consider different forecasting methodologies by looking at different forecast horizons and consider more complex conditional mean and covariances models, using other distributions or considering different constraints in examining the robustness of the portfolios. It is interesting to see the performance of other recent multivariate DCC GARCH models in modelling volatility such as the Bayesian GARCH, the dynamic equicorrelation (DECO) model or the smooth transition conditional correlation (STCC GARCH). Furthermore, the use of copulas in the study may be adjusted to involve a diversified portfolio in the asset modelling as this study has only uses equities and bonds in the portfolio and the results may be improved in the future. Other asset classes, such as property, alternative investments, such as commodities and hedge funds, may be considered. The use of futures and options may enhance the risk return trade off by hedging risk on the liability side and future research could explore the benefits of investing in this area.

In chapter 4, modification of SVD and PCA methods was applied to the mortality modelling. Future research could investigate other methods such as the use of maximum likelihood methods in estimating parameters in the mortality modelling.

The proposed model could also be improved and extended in a number of ways. The extension of the model may include a cohort effect which is common to all countries or may be country dependent. It is also interesting to model mortality rates in other sets of populations; i.e. by looking at mortality rates for males and females in a different populations rather than just looking at only males in the populations as in this thesis.

Chapter 5 produced mortality modelling which is based on multivariate DCC GARCH models. Again, it is interesting to look at other recent multivariate GARCH models such as the corrected DCC or GOGARCH models in modelling the mortality rates. However, due to the limitation of data and timing constraint, this study is unable to provide application of the DCC GARCH mortality modelling. Future studies may use the finding obtained in this thesis to price mortality bonds, survivor index swaps or any other annuity products which involving multiple populations. The performance of the DCC GARCH mortality modelling may also be examined by comparing with other mortality modelling and evaluate them in a single population. It will also be much interesting to explicitly compare and stated its capabilities as opposed to the traditional Lee Carter model. But, the use of econometric models in modelling mortality is particularly new, especially involving multiple population models. There were very limited references available in the literature. The study may be extend in the future to compare its performance as compared to the traditional LC or CBD models in multiple population settings.

Chapter 6 focused on using DCC GARCH with Student-*t* distribution for the asset modelling and using LC model in modelling the mortality rates. Currently, this model was based on a single population. While the asset modelling is done using the best performing strategy in Chapter 3, it uses the traditional LC model for the liabilities modelling. This is a potential limitation of this study. It would have been interesting to extend the study to multiple population settings. However, this will be left for our future research. As mentioned in many part of the thesis, modelling mortality in multiple population is particularly new and there were very few studies on this subject. Therefore, for now, this study uses the traditional LC modelling to estimate the parameters and project future liabilities. This study will be extended in the future to incorporate the multiple population in the mortality modeling.

In further research, the study can be extended to a multiple populations and including other mortality model specifications can be explored as well. A more complex model for the population modelling can be considered, for example, by allowing a salary increment and withdrawal in the fund. This study considered the decumulation phase in the pension scheme (retirement until death). Future study could incorporate a more detailed study by involving the accumulation phase as well (working period), which could add value to, and improve the reliability of, this study.

Appendix

A.1 Copula GARCH

The d-dimensional Gaussian copula $C^{GA}(u_1, u_2, \ldots, u_n)$ is an n-dimensional distribution over the unit hypercube $[0,1]^n$ with uniform margins. The dependence structure is determined by the standardised correlation matrix $\boldsymbol{R}$, such that the dispersion parameter $\rho_{1,n}$ is estimated using Kendall's τ method. The Gaussian copula is represented by

$$C^{GA}(u_1, u_2, \ldots, u_n; R) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{2\pi^{n/2}|R|^{1/2}} \cdot e^{\left\{-\frac{1}{2}x'R^{-1}x\right\}} dx_1, \ldots, dx_n. \tag{1}$$

The density of the Gaussian copula, of the d-dimensional random vector X may be written as in [19]:

$$C^{GA}(u_1, u_2, \ldots, u_n; R) = \frac{1}{|R|^{1/2}} e^{\left\{-\frac{1}{2}\varsigma'(R^{-1}-I)\varsigma\right\}}, \tag{2}$$

where $\varsigma = (\phi^{-1}(u_1), \ldots, \phi^{-1}(u_n))'$ represents the quantile of the Probability Integral Transformation (PIT) values of X and I is the identity matrix. The Gaussian copula cannot account for tail dependence.

For the Student-t copula model, it is possible to joint fat tails and an increased probability of joint extreme events. This copula may be represented as

$$C^{T}(u_1, u_2, \ldots, u_n; R, \nu) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma(\frac{\nu+n}{2})|R|^{-1/2}}{\Gamma(\frac{\nu}{2})(\nu\pi)^{n/2}} \cdot (1 + \frac{1}{\nu}x'R^{-1}x)^{-\frac{\nu+n}{2}} dx_1 \ldots dx_n, \tag{3}$$

and the density of the Student-t copula as

$$C^{T}(u_1, u_2, \ldots, u_n) = |R|^{-1/2} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} \left(\frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})}\right)^n \frac{(1 + \frac{1}{\nu}\varsigma' R^{-1}\varsigma)^{-\frac{\nu+n}{2}}}{\prod_{j=1}^{n}(1 + \frac{\varsigma_j^2}{\nu})^{-\frac{\nu-n}{2}}}. \tag{4}$$

where $\varsigma = (t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n))'$. In Student-$t$ copula, the dependence structure introduces an additional parameter which is the degree of freedom ν. As the value of ν increases, the tendency to exhibit extreme co-movements decreases.

A.2 Average weights of the analysed assets

Table A1: Average weights of the analysed assets of minimum-variance efficient portfolio without short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1218	0.0041	0.1466	0.0098	0.0055	0.0795	0.0055	0.1271	0.0758	0.1074	0.2860	0.0309
DCC-MVT	0.0951	0.0043	0.1465	0.0132	0.0113	0.0660	0.0056	0.1580	0.0829	0.1067	0.2701	0.0403
aDCC-MVN	0.1360	0.0027	0.1400	0.0045	0.0109	0.0636	0.0124	0.1300	0.0980	0.1039	0.2695	0.0285
aDCC-MVT	0.0832	0.0097	0.1081	0.0107	0.0104	0.0815	0.0232	0.1732	0.1226	0.1111	0.2537	0.0125
FDCC-MVN	0.1225	0.0047	0.1468	0.0100	0.0063	0.0817	0.0058	0.1222	0.0706	0.1065	0.2868	0.0361
VAR-MVN	0.0869	0.0094	0.1553	0.0084	0.0046	0.0884	0.0042	0.1428	0.0565	0.1151	0.2974	0.0309
VAR-MVT	0.0970	0.0101	0.1571	0.0083	0.0039	0.0856	0.0013	0.1365	0.0712	0.1277	0.2916	0.0096
ARMA-MVN	0.1153	0.0051	0.1480	0.0101	0.0050	0.0786	0.0123	0.1256	0.0676	0.1016	0.2843	0.0464
ARMA-MVT	0.1078	0.0020	0.1463	0.0081	0.0061	0.0786	0.0045	0.1466	0.0873	0.1090	0.2906	0.0131
GG-MVN	0.2214	0.0009	0.0161	0.0001	0.0007	0.1070	0.0018	0.1519	0.0202	0.1395	0.2615	0.0789
AR-GG-MVN	0.2111	0.0016	0.0293	0.0000	0.0002	0.1104	0.0006	0.1469	0.0155	0.1254	0.2720	0.0871
VAR-GG-MVN	0.2117	0.0012	0.0211	0.0000	0.0002	0.0925	0.0000	0.1733	0.0012	0.1481	0.2972	0.0535
SCop-MVN	0.1292	0.0031	0.1407	0.0066	0.0185	0.0909	0.0066	0.1047	0.1186	0.0828	0.2875	0.0111
SCop-MVT	0.0897	0.0137	0.1913	0.0049	0.0461	0.1065	0.0008	0.0469	0.0890	0.0591	0.2834	0.0685
ARMA-SCop-MVN	0.1139	0.0000	0.1321	0.0083	0.0070	0.0901	0.0091	0.1393	0.0485	0.1367	0.2992	0.0155
ARMA-SCop-MVT	0.1071	0.0012	0.1238	0.0089	0.0100	0.0910	0.0006	0.1575	0.0475	0.1446	0.2993	0.0086
VAR-SCop-MVN	0.1271	0.0000	0.1939	0.0000	0.0049	0.0555	0.0000	0.1191	0.0270	0.0937	0.3000	0.0793
a-SCop-MVN	0.1297	0.0032	0.1407	0.0061	0.0184	0.0899	0.0061	0.1058	0.1218	0.0780	0.2902	0.0100
a-SCop-MVT	0.0896	0.0153	0.1913	0.0056	0.0443	0.1061	0.0005	0.0472	0.0896	0.0581	0.2837	0.0686
DCop-MVN	0.1410	0.0053	0.1225	0.0047	0.0226	0.0961	0.0055	0.1024	0.1053	0.0875	0.2962	0.0110
DCop-MVT	0.0917	0.0200	0.1720	0.0040	0.0532	0.1099	0.0014	0.0479	0.0732	0.0621	0.2903	0.0744
ARMA-DCop-MVN	0.1578	0.0056	0.1049	0.0058	0.0156	0.0947	0.0044	0.1112	0.0684	0.0798	0.2959	0.0559
ARMA-DCop-MVT	0.1318	0.0035	0.1081	0.0077	0.0245	0.0880	0.0024	0.1341	0.0734	0.1817	0.2111	0.0338
VAR-Cop-MVN	0.2021	0.0007	0.1151	0.0099	0.0000	0.1042	0.0000	0.0676	0.0567	0.1048	0.3000	0.0384
a-DCop-MVN	0.1409	0.0052	0.1233	0.0047	0.0221	0.0948	0.0054	0.1035	0.1078	0.0850	0.2958	0.0114
a-DCop-MVT	0.0919	0.0198	0.1723	0.0040	0.0532	0.1095	0.0014	0.0479	0.0737	0.0613	0.2903	0.0746

Table A2: Average weights of the analysed assets of minimum-variance efficient portfolio with short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1792	-0.0460	0.1446	0.0246	-0.0100	0.0860	-0.0188	0.1403	0.0618	0.1063	0.2954	0.0364
DCC-MVT	0.1409	-0.0582	0.1484	0.0371	0.0064	0.0726	-0.0179	0.1706	0.0567	0.1110	0.2784	0.0540
aDCC-MVN	0.1958	-0.0378	0.1504	-0.0013	-0.0185	0.0726	-0.0015	0.1402	0.0887	0.0864	0.2933	0.0316
aDCC-MVT	0.1155	-0.0346	0.1121	0.0211	0.0022	0.0866	0.0153	0.1819	0.1239	0.1004	0.2752	0.0005
FDCC-MVN	0.1810	-0.0389	0.1403	0.0267	-0.0134	0.0867	-0.0188	0.1363	0.0349	0.1096	0.2989	0.0566
VAR-MVN	0.1599	-0.0501	0.1360	0.0398	-0.0171	0.0920	-0.0241	0.1636	0.0401	0.1320	0.2982	0.0297
VAR-MVT	0.1573	-0.0530	0.1504	0.0359	-0.0129	0.0896	-0.0282	0.1608	0.0710	0.1379	0.2956	-0.0045
ARMA-MVN	0.1719	-0.0482	0.1441	0.0306	-0.0185	0.0879	0.0031	0.1292	0.0435	0.1037	0.2911	0.0616
ARMA-MVT	0.1610	-0.0552	0.1505	0.0208	0.0059	0.0860	-0.0191	0.1502	0.0767	0.1128	0.2969	0.0136
GG-MVN	0.2742	-0.0370	0.0238	-0.0377	-0.0072	0.1223	-0.0259	0.1875	-0.0827	0.1590	0.2930	0.1306
AR-GG-MVN	0.2648	-0.0192	0.0330	-0.0491	0.0034	0.1246	-0.0328	0.1753	-0.1040	0.1643	0.2940	0.1457
VAR-GG-MVN	0.2664	-0.0590	0.0402	-0.0299	0.0148	0.1027	-0.0295	0.1943	-0.1511	0.2207	0.3000	0.1304
SCop-MVN	0.1806	-0.0415	0.1245	0.0190	0.0148	0.0929	-0.0296	0.1395	0.1388	0.0876	0.2987	-0.0251
SCop-MVT	0.1367	0.0030	0.1753	-0.0122	0.0778	0.1059	-0.0306	0.0441	0.0117	0.0847	0.2866	0.1169
ARMA-SCop-MVN	0.1756	-0.0734	0.1250	0.0210	-0.0003	0.0998	-0.0040	0.1562	0.0332	0.1534	0.2998	0.0136
ARMA-SCop-MVT	0.1635	-0.0658	0.1181	0.0156	0.0221	0.0980	-0.0245	0.1731	0.0371	0.1693	0.2987	-0.0051
VAR-SCop-MVN	0.1845	-0.1332	0.2236	0.0108	0.0511	0.0700	-0.0443	0.1376	-0.0050	0.1143	0.3000	0.0908
a-SCop-MVN	0.1808	-0.0404	0.1261	0.0200	0.0127	0.0923	-0.0305	0.1390	0.1421	0.0820	0.2996	-0.0237
a-SCop-MVT	0.1366	0.0047	0.1758	-0.0116	0.0746	0.1057	-0.0308	0.0449	0.0108	0.0834	0.2872	0.1186
DCop-MVN	0.1936	-0.017	0.0956	-0.0103	0.0372	0.0988	-0.0301	0.1324	0.1460	0.0990	0.3000	-0.0450
DCop-MVT	0.1433	0.0325	0.1451	-0.0452	0.1045	0.1101	-0.0345	0.0441	-0.0079	0.0874	0.2917	0.1288
ARMA-DCop-MVN	0.2015	-0.0124	0.0771	0.0191	-0.0072	0.1017	-0.0074	0.1275	0.0503	0.0815	0.2986	0.0696
ARMA-DCop-MVT	0.1809	-0.0456	0.0755	0.0010	0.0584	0.0941	-0.0115	0.1471	0.0666	0.2071	0.2143	0.0120
VAR-Cop-MVN	0.2464	0.0008	0.0674	0.0491	-0.0675	0.1051	-0.0494	0.1482	0.0423	0.1007	0.3000	0.0570
a-DCop-MVN	0.1938	-0.0198	0.0965	-0.0091	0.0371	0.0981	-0.0308	0.1343	0.1493	0.0968	0.3000	-0.0461
a-DCop-MVT	0.1439	0.0321	0.1453	-0.0454	0.1048	0.1099	-0.0350	0.0444	-0.0072	0.0864	0.2918	0.1290

Table A3: Average weights of the analysed assets of mean-variance efficient portfolio without short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1044	0.0554	0.0666	0.0034	0.0284	0.0924	0.0000	0.1495	0.0312	0.0942	0.2948	0.0799
DCC-MVT	0.0996	0.0715	0.1185	0.1185	0.0237	0.0681	0.0001	0.1162	0.0085	0.0567	0.2646	0.1701
aDCC-MVN	0.1360	0.0027	0.1400	0.1400	0.0109	0.0636	0.0124	0.1300	0.0980	0.1039	0.2695	0.0285
aDCC-MVT	0.1224	0.0806	0.0739	0.0739	0.0176	0.0726	0.0005	0.1302	0.0202	0.1224	0.2402	0.1171
FDCC-MVN	0.1044	0.0553	0.0670	0.0670	0.0288	0.0921	0.0000	0.1490	0.0326	0.0937	0.2957	0.0779
VAR-MVN	0.1688	0.0470	0.0657	0.0657	0.0020	0.0914	0.0000	0.1237	0.0026	0.1059	0.2897	0.1017
VAR-MVT	0.1038	0.0640	0.0792	0.0792	0.0113	0.1033	0.0000	0.1312	0.0087	0.0953	0.2872	0.1088
ARMA-MVN	0.1102	0.0564	0.0666	0.0666	0.0249	0.0921	0.0008	0.1457	0.0270	0.0833	0.2894	0.1003
ARMA-MVT	0.1078	0.0020	0.1463	0.1463	0.0063	0.0788	0.0045	0.1461	0.0884	0.1076	0.2906	0.0135
GG-MVN	0.2235	0.0004	0.0146	0.0146	0.0004	0.1094	0.0011	0.1506	0.0224	0.1340	0.2611	0.0825
AR-GG-MVN	0.2087	0.0022	0.0290	0.0290	0.0003	0.1076	0.0007	0.1474	0.0147	0.1279	0.2697	0.0836
VAR-GG-MVN	0.2108	0.0011	0.0222	0.0222	0.0001	0.0924	0.0000	0.1733	0.1733	0.1459	0.2976	0.0551
SCop-MVN	0.1284	0.0035	0.1417	0.0071	0.0186	0.0908	0.0064	0.1037	0.1202	0.0824	0.2871	0.0103
SCop-MVT	0.0884	0.0152	0.1897	0.0059	0.0473	0.1074	0.0007	0.0454	0.0878	0.0594	0.2841	0.0687
ARMA-SCop-MVN	0.1503	0.0055	0.1071	0.0087	0.0174	0.0918	0.0049	0.1144	0.0821	0.0857	0.2879	0.0444
ARMA-SCop-MVT	0.1082	0.0013	0.1241	0.0082	0.0112	0.0908	0.0004	0.1558	0.0451	0.1483	0.2979	0.0087
VAR-SCop-MVN	0.0822	0.0000	0.2220	0.0451	0.0000	0.1436	0.0000	0.0072	0.0554	0.0525	0.3000	0.0922
a-SCop-MVN	0.1294	0.0031	0.1430	0.0067	0.0178	0.0895	0.0064	0.1042	0.1209	0.0783	0.2894	0.0114
a-SCop-MVT	0.0905	0.0145	0.1908	0.0046	0.0455	0.1064	0.0006	0.0471	0.0890	0.0572	0.2837	0.0702
DCop-MVN	0.1358	0.0027	0.1165	0.0083	0.0170	0.0918	0.0083	0.1197	0.0831	0.1763	0.2062	0.0344
DCop-MVT	0.0950	0.0187	0.1695	0.0044	0.0513	0.1095	0.0007	0.0509	0.0745	0.0633	0.2906	0.0715
ARMA-DCop-MVN	0.1342	0.0024	0.1174	0.0079	0.0166	0.0901	0.0141	0.1173	0.0777	0.1744	0.2073	0.0406
ARMA-DCop-MVT	0.1318	0.0035	0.1080	0.0077	0.0245	0.0880	0.0024	0.1341	0.0732	0.1815	0.2118	0.0335
VAR-Cop-MVN	0.0986	0.0001	0.1379	0.0137	0.0115	0.0838	0.0026	0.1519	0.0448	0.1274	0.3000	0.0278
a-DCop-MVN	0.1360	0.0027	0.1187	0.0084	0.0161	0.0868	0.0081	0.1232	0.0911	0.1694	0.2030	0.0364
a-DCop-MVT	0.0950	0.0187	0.1695	0.0044	0.0513	0.1095	0.0007	0.0509	0.0745	0.0633	0.2906	0.0715

Table A4: Average weights of the analysed assets of mean-variance efficient portfolio with short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1518	0.0625	0.0109	-0.0276	0.0640	0.0894	-0.0563	0.2054	-0.0810	0.1408	0.2959	0.1444
DCC-MVT	0.1564	0.0841	0.0793	-0.0355	0.0484	0.0652	-0.0555	0.1576	-0.1407	0.0960	0.2902	0.2546
aDCC-MVN	0.1958	-0.0378	0.1504	-0.0013	-0.0185	0.0726	-0.0015	0.1402	0.0887	0.0864	0.2933	0.0316
aDCC-MVT	0.1836	0.1064	0.0358	-0.0361	0.0187	0.0682	-0.0453	0.1687	-0.1181	0.1524	0.2781	0.1876
FDCC-MVN	0.1514	0.0626	0.0108	-0.0262	0.0638	0.0892	-0.0562	0.2047	-0.0814	0.1412	0.2971	0.1431
VAR-MVN	0.1931	0.0598	0.0103	-0.0088	0.0018	0.0899	-0.0557	0.2096	-0.1840	0.1611	0.2983	0.2246
VAR-MVT	0.1567	0.0733	0.0421	-0.0118	0.0019	0.0978	-0.0557	0.1959	-0.1375	0.1353	0.2993	0.2029
ARMA-MVN	0.1359	0.0660	0.0190	-0.0299	0.0542	0.0938	-0.0283	0.1893	-0.0778	0.1246	0.2953	0.1579
ARMA-MVT	0.1610	-0.0546	0.1505	0.0210	0.0059	0.0862	-0.0190	0.1489	0.0773	0.1113	0.2969	0.0146
GG-MVN	0.2784	-0.0376	0.0164	-0.0375	-0.0063	0.1251	-0.0263	0.1879	-0.0823	0.1581	0.2919	0.1323
AR-GG-MVN	0.2654	-0.0229	0.0347	-0.0506	0.0049	0.1225	-0.0331	0.1791	-0.1040	0.1661	0.2957	0.1422
VAR-GG-MVN	0.2653	-0.0566	0.0407	-0.0306	0.0150	0.1015	-0.0293	0.1941	-0.1553	0.2202	0.3000	0.1351
SCop-MVN	0.1790	-0.0407	0.1252	0.0199	0.0137	0.0927	-0.0297	0.1400	0.1390	0.0862	0.2990	-0.0242
SCop-MVT	0.1341	0.0060	0.1756	-0.0118	0.0775	0.1064	-0.0310	0.0422	0.0118	0.0840	0.2866	0.1176
ARMA-SCop-MVN	0.1980	-0.0307	0.0735	0.0363	-0.0019	0.0971	-0.0115	0.1393	0.0555	0.0916	0.2901	0.00629
ARMA-SCop-MVT	0.1647	-0.0650	0.1170	0.0152	0.0228	0.0977	-0.0252	0.1727	0.0349	0.1728	0.2981	-0.0059
VAR-SCop-MVN	0.1464	-0.0608	0.2402	0.1622	-0.1309	0.1350	-0.0489	0.0567	0.0210	0.0389	0.3000	0.1401
a-SCop-MVN	0.1798	-0.0400	0.1260	0.0188	0.0140	0.0916	-0.0311	0.1409	0.1406	0.0826	0.2999	-0.0231
a-SCop-MVT	0.1379	0.0039	0.1745	-0.0138	0.0788	0.1061	-0.0318	0.0444	0.0104	0.0834	0.2852	0.1210
DCop-MVN	0.1949	-0.0575	0.0868	0.0136	0.0364	0.0987	-0.0032	0.1302	0.0757	0.2006	0.2096	0.0141
DCop-MVT	0.1414	0.0269	0.1450	-0.0444	0.1025	0.1125	-0.0184	0.0343	-0.0063	0.0897	0.2945	0.1221
ARMA-DCop-MVN	0.1911	-0.0588	0.0842	0.0150	0.0330	0.0995	0.0131	0.1229	0.0712	0.1980	0.2122	0.0186
ARMA-DCop-MVT	0.1809	-0.0456	0.0755	0.0010	0.0584	0.0941	-0.0115	0.1471	0.0666	0.2071	0.2143	0.0120
VAR-Cop-MVN	0.1654	-0.0792	0.1199	0.0492	0.0055	0.0887	-0.0301	0.1805	0.0112	0.1494	0.3000	0.0394
a-DCop-MVN	0.1974	-0.0657	0.0898	0.0164	0.0366	0.0959	-0.0057	0.1354	0.0863	0.1944	0.2071	0.0123
a-DCop-MVT	0.1414	0.0269	0.1451	-0.0444	0.1025	0.1125	-0.0184	0.0343	-0.0063	0.0897	0.2945	0.1221

Table A5: Average weights of the analysed assets based on maximising Sharpe ratio without short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.0231	0.0677	0.1381	0.0941	0.0119	0.0000	0.0034	0.1617	0.0569	0.0648	0.0798	0.2985
DCC-MVT	0.0127	0.0412	0.1650	0.1063	0.0091	0.0000	0.0795	0.0862	0.0079	0.0130	0.1937	0.2853
aDCC-MVN	0.0206	0.0899	0.1829	0.0628	0.0191	0.0000	0.0313	0.0934	0.0898	0.1176	0.0000	0.2926
aDCC-MVT	0.0113	0.0551	0.1241	0.0770	0.0070	0.0001	0.1462	0.0791	0.0087	0.0000	0.2229	0.2684
FDCC-MVN	0.0224	0.0686	0.1402	0.0942	0.0110	0.0001	0.0037	0.1597	0.0608	0.0515	0.0891	0.2986
VAR-MVN	0.0736	0.0668	0.1135	0.0311	0.0650	0.0366	0.0518	0.0615	0.0922	0.1084	0.0906	0.2088
VAR-MVT	0.0779	0.0663	0.1085	0.0313	0.0684	0.0345	0.0519	0.0613	0.0933	0.1106	0.0932	0.2029
ARMA-MVN	0.0572	0.0295	0.1342	0.0398	0.0116	0.0340	0.1314	0.0623	0.0704	0.0871	0.1134	0.2292
ARMA-MVT	0.0556	0.0238	0.1329	0.0907	0.0160	0.0282	0.0592	0.0936	0.0774	0.0945	0.1178	0.2103
GG-MVN	0.0307	0.1390	0.2296	0.0014	0.0000	0.0000	0.0716	0.0277	0.0881	0.0605	0.0514	0.3000
AR-GG-MVN	0.0392	0.1275	0.1200	0.0085	0.0323	0.0249	0.0565	0.0913	0.0805	0.0718	0.0890	0.2587
VAR-GG-MVN	0.0953	0.0737	0.0909	0.0141	0.0704	0.0327	0.0509	0.0720	0.0756	0.1109	0.0978	0.2157
SCop-MVN	0.0068	0.0548	0.1782	0.0594	0.1200	0.0009	0.0257	0.0542	0.1053	0.0845	0.0103	0.2999
SCop-MVT	0.0165	0.0938	0.2418	0.0501	0.0531	0.0000	0.0226	0.0221	0.1189	0.0479	0.0332	0.3000
ARMA-SCop-MVN	0.0494	0.0189	0.1256	0.0473	0.0090	0.0492	0.1022	0.0984	0.0402	0.0817	0.1212	0.2569
ARMA-SCop-MVT	0.0392	0.0196	0.1559	0.0938	0.0055	0.0356	0.0524	0.0981	0.0362	0.0961	0.1491	0.2186
VAR-SCop-MVN	0.0927	0.1253	0.0681	0.0264	0.0466	0.0416	0.0384	0.0609	0.0859	0.0984	0.1385	0.1772
a-SCop-MVN	0.0047	0.0608	0.1772	0.0570	0.1186	0.0006	0.0253	0.0447	0.1038	0.0824	0.0138	0.3000
a-SCop-MVT	0.0163	0.0964	0.2376	0.0520	0.0519	0.0000	0.0228	0.0230	0.1179	0.0442	0.0379	0.3000
DCop-MVN	0.0061	0.0657	0.1590	0.0437	0.1456	0.0000	0.0249	0.0549	0.1074	0.0795	0.0461	0.2670
DCop-MVT	0.0205	0.1108	0.2228	0.0308	0.0660	0.0000	0.0193	0.0298	0.0894	0.0315	0.0792	0.3000
ARMA-DCop-MVN	0.0582	0.0461	0.1185	0.0101	0.0242	0.1986	0.0280	0.0164	0.0737	0.0784	0.1290	0.2189
ARMA-DCop-MVT	0.0394	0.0661	0.1476	0.1001	0.0276	0.0532	0.0609	0.0052	0.0558	0.0743	0.1593	0.2106
VAR-Cop-MVN	0.1005	0.0715	0.0797	0.0713	0.0238	0.0425	0.0751	0.0356	0.1141	0.1141	0.0944	0.1774
a-DCop-MVN	0.0066	0.0650	0.1538	0.0448	0.1442	0.0000	0.0289	0.0566	0.1116	0.0826	0.0428	0.2630
a-DCop-MVT	0.0170	0.1143	0.2208	0.0319	0.0672	0.0000	0.0159	0.0328	0.1002	0.0292	0.0706	0.3000

Table A6: Average weights of the analysed assets based on maximising Sharpe ratio with short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.0185	0.0804	0.1548	0.1518	-0.0054	-0.0712	-0.0244	0.1954	0.0210	0.0627	0.1164	0.2999
DCC-MVT	0.0154	0.0245	0.1926	0.1771	-0.0440	-0.0891	0.1052	0.1183	-0.0643	-0.0074	0.2764	0.2953
aDCC-MVN	0.0367	0.1400	0.1783	0.1003	0.0221	-0.0766	0.0393	0.0599	0.2462	0.2258	-0.2691	0.2971
aDCC-MVT	-0.0552	0.0710	0.2029	0.1432	-0.0929	-0.0647	0.1783	0.1174	0.1170	-0.1962	0.3000	0.2792
FDCC-MVN	0.0226	0.0845	0.1551	0.1487	-0.0181	-0.0651	-0.0228	0.1951	0.0267	0.0297	0.1439	0.2996
VAR-MVN	0.0768	0.0411	0.1174	0.0900	0.0133	0.0109	0.0362	0.1143	0.0946	0.0518	0.1625	0.1910
VAR-MVT	0.0687	0.0403	0.1268	0.0888	0.0197	0.0079	0.0376	0.1102	0.0974	0.0561	0.1638	0.1827
ARMA-MVN	0.0556	-0.0205	0.1442	0.1400	-0.0481	0.0004	0.1373	0.0910	0.0470	0.0789	0.1354	0.2388
ARMA-MVT	0.0547	0.0007	0.1852	0.1779	-0.0443	-0.0519	0.0871	0.0905	0.0611	0.0809	0.1297	0.2283
GG-MVN	0.1135	0.2255	0.2815	0.0139	-0.1769	-0.1445	0.0778	0.1092	0.0302	-0.0229	0.1927	0.3000
AR-GG-MVN	0.1113	0.1326	0.1965	-0.0286	-0.0416	-0.0187	0.0420	0.1065	0.0073	0.0614	0.1668	0.2645
VAR-GG-MVN	0.1062	0.0746	0.1000	-0.0023	0.0355	0.0100	0.0394	0.1365	0.0243	0.0673	0.1988	0.2096
SCop-MVN	-0.1040	0.0657	0.1906	0.0932	0.1703	-0.0504	0.0290	0.1055	0.2410	0.1255	-0.1665	0.3000
SCop-MVT	0.0275	0.1408	0.2498	0.0608	0.0919	-0.0605	0.0305	-0.0408	0.1545	0.0832	-0.0376	0.3000
ARMA-SCop-MVN	-0.0068	-0.0230	0.1539	0.1603	-0.0611	0.0218	0.1010	0.1538	0.0234	0.0241	0.1752	0.2773
ARMA-SCop-MVT	-0.0090	-0.0151	0.2127	0.1914	-0.0399	-0.0395	0.0879	0.1114	0.0263	0.0462	0.1726	0.2548
VAR-SCop-MVN	0.0207	0.0146	0.1508	0.0593	0.1014	0.0136	-0.0118	0.1512	0.0183	0.0822	0.2258	0.1736
a-SCop-MVN	-0.0987	0.0682	0.1887	0.0957	0.1622	-0.0547	0.0274	0.1114	0.2329	0.1228	-0.1557	0.3000
a-SCop-MVT	0.0242	0.1447	0.2462	0.0608	0.0945	-0.0652	0.0295	-0.0346	0.1573	0.0778	-0.0351	0.3000
DCop-MVN	-0.1128	0.0938	0.1758	0.0703	0.2126	-0.0739	0.0357	0.0985	0.2158	0.0819	-0.0678	0.2701
DCop-MVT	0.0291	0.1860	0.2295	0.0137	0.1332	-0.0863	0.0272	-0.0325	0.1012	0.0415	0.0576	0.3000
ARMA-DCop-MVN	0.0783	0.0565	0.1603	0.0558	-0.0482	0.2113	0.0514	-0.0654	0.0722	0.0087	0.1641	0.2549
ARMA-DCop-MVT	0.0779	0.0689	0.1821	0.1506	-0.0038	0.0731	0.1182	-0.1669	0.0081	0.0663	0.1750	0.2506
VAR-Cop-MVN	0.1497	0.0790	0.1099	0.0246	-0.0335	0.0561	0.0344	0.0797	0.0837	0.0578	0.1864	0.1721
a-DCop-MVN	-0.1072	0.0984	0.1742	0.0633	0.2088	-0.0753	0.0373	0.1004	0.2038	0.1031	-0.0737	0.2668
a-DCop-MVT	0.0273	0.1901	0.2317	0.0155	0.1291	-0.0880	0.0220	-0.0277	0.1148	0.0199	0.0653	0.3000

Table A7: Average weights of the analysed assets based on minimising mean-CVaR without short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.0528	0.0205	0.0457	0.0137	0.0304	0.2093	0.0775	0.0500	0.1175	0.1552	0.1860	0.0412
DCC-MVT	0.0984	0.0004	0.0013	0.0000	0.0002	0.2998	0.0000	0.0602	0.0802	0.2983	0.1190	0.0000
aDCC-MVN	0.0469	0.0281	0.0405	0.0222	0.0294	0.1664	0.0707	0.0957	0.0840	0.1222	0.2147	0.0790
aDCC-MVT	0.0035	0.0004	0.0003	0.0003	0.0000	0.2833	0.0000	0.1237	0.0508	0.2532	0.1905	0.0055
FDCC-MVN	0.0620	0.0190	0.0171	0.0127	0.0164	0.2184	0.0175	0.1343	0.0513	0.1535	0.2605	0.0346
VAR-MVN	0.0415	0.0628	0.0618	0.0470	0.0687	0.0676	0.0506	0.0655	0.0769	0.1415	0.1454	0.1289
VAR-MVT	0.0542	0.0285	0.0440	0.0484	0.0615	0.1027	0.0662	0.0943	0.1084	0.1535	0.1623	0.0758
ARMA-MVN	0.0504	0.0520	0.0639	0.0477	0.0502	0.1109	0.0242	0.0946	0.0983	0.1170	0.1640	0.1204
ARMA-MVT	0.0411	0.0247	0.0302	0.0259	0.0411	0.1737	0.0288	0.1095	0.0927	0.1268	0.1858	0.0946
GOGARCH-MVN	0.0302	0.0297	0.0285	0.0540	0.0803	0.1581	0.0500	0.0694	0.0923	0.1291	0.1772	0.1015
AR-GOGARCH-MVN	0.0743	0.0281	0.0338	0.0578	0.0403	0.1258	0.0732	0.0610	0.1020	0.1448	0.1570	0.0962
VAR-GOGARCH-MVN	0.0389	0.0527	0.0536	0.0564	0.0576	0.0905	0.0656	0.0639	0.0992	0.1293	0.1292	0.1380
SCop-MVN	0.0014	0.0000	0.0281	0.0000	0.1685	0.3000	0.0000	0.0021	0.1439	0.0000	0.0000	0.2993
SCop-MVT	0.0719	0.0038	0.0030	0.0041	0.0105	0.2941	0.0021	0.1041	0.0476	0.2077	0.2077	0.2338
ARMA-SCop-MVN	0.0671	0.0557	0.0080	0.0424	0.0777	0.1152	0.0405	0.0869	0.1118	0.1494	0.1494	0.1514
ARMA-SCop-MVT	0.0795	0.0321	0.0297	0.0085	0.0344	0.1645	0.0335	0.0784	0.1234	0.1712	0.1712	0.1521
VAR-SCop-MVN	0.0468	0.0455	0.0807	0.0510	0.0294	0.0860	0.0672	0.0761	0.0982	0.1619	0.1619	0.0930
a-SCop-MVN	0.0758	0.0334	0.0271	0.0186	0.0186	0.2282	0.0394	0.0590	0.1067	0.1513	0.1513	0.1995
a-SCop-MVT	0.0719	0.0060	0.0046	0.0043	0.0110	0.2925	0.0036	0.0991	0.0536	0.1954	0.1954	0.2336
DCop-MVN	0.0042	0.0000	0.0000	0.0000	0.0089	0.3000	0.0000	0.1869	0.0194	0.0050	0.2600	0.2157
DCop-MVT	0.0304	0.0000	0.0000	0.0008	0.0024	0.3000	0.0000	0.1583	0.0079	0.1240	0.2818	0.0863
ARMA-DCop-MVN	0.0645	0.0051	0.0017	0.0060	0.0836	0.2017	0.0102	0.1248	0.1206	0.1194	0.1186	0.1396
ARMA-DCop-MVT	0.0029	0.0041	0.0032	0.0270	0.0663	0.2752	0.0000	0.1213	0.1262	0.0431	0.1068	0.2229
VAR-Cop-MVN	0.0241	0.0487	0.0036	0.0533	0.0620	0.2430	0.0102	0.0374	0.0492	0.0841	0.1260	0.2349
a-DCop-MVN	0.0014	0.0000	0.0000	0.0000	0.0082	0.2992	0.0000	0.1912	0.0149	0.0030	0.2650	0.2171
a-DCop-MVT	0.0315	0.0000	0.0000	0.0000	0.0041	0.3000	0.0000	0.1574	0.0112	0.1238	0.2770	0.0881

Table A8: Average weights of the analysed assets based on minimising mean-CVaR with short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1293	-0.0114	0.1295	0.0172	0.0064	0.1174	0.0609	0.0508	0.1880	0.1391	0.1977	-0.0248
DCC-MVT	0.1636	-0.0582	0.1244	-0.0360	0.0561	0.2303	-0.1198	0.1394	0.2131	0.1557	0.1700	-0.0388
aDCC-MVN	0.1049	-0.0068	0.0966	0.0522	0.0298	0.1044	0.0265	0.0924	0.1702	0.1040	0.2103	0.0155
aDCC-MVT	0.1615	0.0035	0.0433	0.0606	0.0111	0.2030	-0.1608	0.1778	0.1550	0.1627	0.1793	0.0030
FDCC-MVN	0.1250	0.0336	0.1174	0.0924	-0.0293	0.1208	-0.0567	0.0969	0.1907	0.1445	0.2077	-0.0429
VAR-MVN	0.0565	0.0742	0.1030	0.1039	0.0336	0.0204	0.0174	0.0911	0.1274	0.1112	0.1499	0.1115
VAR-MVT	0.0775	0.0211	0.0334	0.1011	0.0295	0.1398	-0.0243	0.1220	0.1544	0.1076	0.1911	0.0468
ARMA-MVN	0.1012	0.0343	0.1149	0.0972	0.0504	0.0384	-0.0122	0.0758	0.1716	0.0932	0.1275	0.1078
ARMA-MVT	0.0988	0.0190	0.0794	0.0488	0.0221	0.1276	-0.0146	0.1188	0.1700	0.1091	0.1500	0.0708
GOGARCH-MVN	0.0706	0.0399	0.0826	0.0914	0.0490	0.1019	-0.0012	0.0659	0.1691	0.1095	0.1981	0.0233
AR-GOGARCH-MVN	0.1349	-0.0295	0.0977	0.1169	0.0167	0.0837	0.0150	0.0646	0.1492	0.1126	0.1479	0.0904
VAR-GOGARCH-MVN	0.0598	0.0708	0.1054	0.0965	0.0521	0.0419	0.0168	0.0566	0.1520	0.0976	0.1479	0.1009
SCop-MVN	0.2188	-0.2358	-0.3000	0.2218	0.2952	0.3000	-0.3000	0.3000	0.1928	-0.2874	0.3000	0.2946
SCop-MVT	0.1753	-0.0567	0.0515	0.0221	0.0488	0.2391	-0.0942	0.1141	0.1689	0.1434	0.1978	-0.0101
ARMA-SCop-MVN	0.1227	0.0680	-0.0883	0.1255	0.0670	0.0628	0.0646	0.0778	0.1808	0.0933	0.1268	0.0990
ARMA-SCop-MVT	0.1559	0.0071	0.1128	-0.0281	0.0594	0.1564	-0.0705	0.1069	0.1912	0.1450	0.1430	0.0208
VAR-SCop-MVN	0.1107	0.0022	0.1251	0.0720	0.0342	0.0660	0.0356	0.0541	0.1593	0.1342	0.1086	0.0979
a-SCop-MVN	0.1546	0.0036	0.0984	0.0623	-0.0217	0.1356	0.0037	0.0634	0.1916	0.1331	0.1908	-0.0156
a-SCop-MVT	0.1548	-0.0284	0.0525	0.0133	0.0284	0.2385	-0.0799	0.1208	0.1821	0.1349	0.2064	-0.0234
DCop-MVN	0.2190	-0.2200	-0.3000	0.2058	0.2953	0.3000	-0.3000	0.3000	0.2005	-0.2927	0.3000	0.2922
DCop-MVT	-0.0496	-0.426	-0.3000	0.2943	0.3000	0.2997	-0.3000	0.2982	0.2002	-0.2994	0.2992	0.3000
ARMA-DCop-MVN	0.0041	0.0412	-0.2949	0.2735	0.2309	0.2693	-0.2855	0.2613	0.0880	0.0201	0.1860	0.2059
ARMA-DCop-MVT	-0.1246	0.0721	-0.2962	0.2896	0.2987	0.2953	-0.3000	0.2651	0.1356	-0.1072	0.1782	0.2934
VAR-Cop-MVN	0.2273	-0.2441	-0.3000	0.2246	0.2930	0.3000	-0.3000	0.2992	0.1819	-0.2752	0.2992	0.2941
a-DCop-MVN	0.2250	-0.0129	0.1350	-0.0342	-0.1136	-0.1136	-0.0451	0.1162	0.2221	0.2073	0.2583	-0.1877
a-DCop-MVT	-0.0361	-0.0542	-0.3000	0.2939	0.3000	0.3000	-0.3000	0.2966	0.1959	-0.2942	0.2983	0.3000

Table A9: Average weights of the analysed assets based on maximising Sortino ratio without short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.0661	0.0279	0.0417	0.0500	0.0322	0.1457	0.0810	0.0553	0.1250	0.1472	0.1665	0.0613
DCC-MVT	0.0522	0.0299	0.0403	0.0404	0.0456	0.1597	0.0522	0.0797	0.1142	0.1589	0.1433	0.0836
aDCC-MVN	0.0477	0.0328	0.0419	0.0520	0.0287	0.1406	0.0720	0.0844	0.0994	0.1300	0.2094	0.0611
aDCC-MVT	0.0510	0.0271	0.0452	0.0350	0.0597	0.1555	0.0418	0.0847	0.0992	0.1991	0.1300	0.0717
FDCC-MVN	0.0685	0.0272	0.0413	0.0486	0.0299	0.1472	0.0818	0.0553	0.1211	0.1524	0.1648	0.0617
VAR-MVN	0.0812	0.0491	0.0726	0.0199	0.0605	0.1177	0.0528	0.0462	0.1229	0.1731	0.1498	0.0541
VAR-MVT	0.0753	0.0586	0.0584	0.0376	0.0603	0.1047	0.0708	0.0343	0.1291	0.1319	0.1106	0.1284
ARMA-MVN	0.0890	0.0247	0.0620	0.0217	0.0358	0.1496	0.0759	0.0413	0.1294	0.1539	0.1705	0.0462
ARMA-MVT	0.0580	0.0292	0.0510	0.0253	0.0333	0.1711	0.0429	0.0892	0.1161	0.1453	0.1668	0.0718
GOGARCH-MVN	0.0410	0.0293	0.0280	0.0566	0.0770	0.1554	0.0502	0.0625	0.0945	0.1553	0.1820	0.0681
AR-GOGARCH-MVN	0.0679	0.0110	0.0303	0.0455	0.0522	0.1818	0.0658	0.0454	0.0733	0.1451	0.2070	0.0746
VAR-GOGARCH-MVN	0.0782	0.0543	0.0450	0.0700	0.0503	0.1050	0.0617	0.0354	0.1260	0.1476	0.1494	0.0769
SCop-MVN	0.0833	0.0346	0.0287	0.0331	0.0347	0.1542	0.0601	0.0712	0.1056	0.1531	0.1892	0.0521
SCop-MVT	0.0523	0.0308	0.0262	0.0345	0.0402	0.1760	0.0483	0.0917	0.0925	0.1505	0.1767	0.0803
ARMA-SCop-MVN	0.0713	0.0337	0.0550	0.0264	0.0379	0.1501	0.0560	0.0696	0.1289	0.1546	0.1633	0.0532
ARMA-SCop-MVT	0.0501	0.0438	0.0420	0.0265	0.0523	0.1477	0.0368	0.1006	0.1257	0.1463	0.1523	0.0758
VAR-SCop-MVN	0.0898	0.0876	0.0427	0.0414	0.0391	0.1112	0.0522	0.0360	0.1132	0.1525	0.1397	0.0947
a-SCop-MVN	0.0830	0.0297	0.0375	0.0425	0.0326	0.1541	0.0523	0.0683	0.1084	0.1547	0.1839	0.0530
a-SCop-MVT	0.0483	0.0368	0.0220	0.0396	0.0425	0.1738	0.0545	0.0825	0.0917	0.1590	0.1725	0.0530
DCop-MVN	0.0792	0.0357	0.0316	0.0368	0.0325	0.1498	0.0633	0.0712	0.1122	0.1561	0.1756	0.0561
DCop-MVT	0.0530	0.0300	0.0253	0.0380	0.0408	0.1716	0.0529	0.0883	0.0992	0.1448	0.1825	0.0735
ARMA-DCop-MVN	0.0829	0.0202	0.0387	0.0297	0.0926	0.0332	0.0524	0.1504	0.1432	0.1616	0.1306	0.0646
ARMA-DCop-MVT	0.0401	0.0076	0.0249	0.0180	0.0568	0.1170	0.0283	0.2073	0.1667	0.1742	0.1020	0.0571
VAR-Cop-MVN	0.0673	0.0401	0.0466	0.0801	0.0577	0.0977	0.0593	0.0513	0.1149	0.1508	0.1526	0.0818
a-DCop-MVN	0.0777	0.0326	0.0378	0.0501	0.0314	0.1429	0.0627	0.0648	0.1121	0.1455	0.1722	0.0702
a-DCop-MVT	0.0520	0.0298	0.0258	0.0421	0.0417	0.1746	0.0522	0.0818	0.0971	0.1517	0.1775	0.0737

Table A10: Average weights of the analysed assets based on maximising Sortino ratio with short sale for the econometric models under study from January 2005 to December 2014.

Model	FTSE100	MSEXUK	S&PCOMP	DAXINDX	AMSTEOE	TOKYOSE	HNGKNGI	TTOCOMP	BMUK10Y	BMUS10Y	BMBD10Y	BGILALL
DCC-MVN	0.1991	-0.0698	0.1564	-0.1126	-0.0068	0.2242	0.0680	0.0415	0.2354	0.2144	0.2539	-0.2037
DCC-MVT	0.0650	0.0433	0.1025	0.1075	0.0458	0.0225	0.0492	0.0642	0.1700	0.0942	0.1175	0.1183
aDCC-MVN	0.2131	-0.0979	0.1168	-0.1198	0.0107	0.2320	0.0021	0.1431	0.1948	0.1918	0.2800	-0.1666
aDCC-MVT	0.2462	-0.1167	0.1683	-0.1229	0.1099	0.2332	-0.1738	0.1558	0.2186	0.2624	0.1680	-0.1490
FDCC-MVN	0.1956	-0.0657	0.1536	-0.1112	-0.0011	0.2214	0.0660	0.0414	0.2317	0.2056	0.2531	-0.1904
VAR-MVN	0.0834	-0.0175	0.0734	0.0852	0.0356	0.1517	0.0077	0.0804	0.1650	0.1499	0.2140	-0.0289
VAR-MVT	0.0946	-0.0102	0.0522	0.1068	0.0461	0.0804	0.0694	0.0609	0.1833	0.0904	0.1756	0.0507
ARMA-MVN	0.1484	0.0351	0.0832	-0.0912	0.0595	0.2114	-0.0769	0.1305	0.2032	0.1973	0.2166	-0.1172
ARMA-MVT	0.1744	-0.0289	0.1341	-0.1509	0.1392	0.2364	-0.1649	0.1606	0.1885	0.2226	0.2248	-0.1360
GOGARCH-MVN	0.2068	-0.1463	0.0232	0.0168	0.1782	0.2391	-0.1600	0.1420	0.2014	0.1976	0.2718	-0.1708
AR-GOGARCH-MVN	0.1908	-0.1606	0.0165	0.0226	0.1341	0.2823	-0.1366	0.1510	0.1482	0.2364	0.2811	-0.1657
VAR-GOGARCH-MVN	0.0469	0.0322	0.0588	0.0725	0.0728	0.1162	0.0149	0.0857	0.1333	0.1221	0.1866	0.0580
SCop-MVN	0.2333	0.0016	0.1204	-0.0124	-0.1207	0.2284	-0.0702	0.1197	0.2215	0.2065	0.2586	-0.1865
SCop-MVT	0.2394	-0.0848	0.0298	-0.0546	0.0619	0.2381	-0.1284	0.1987	0.1959	0.2142	0.2476	-0.1578
ARMA-SCop-MVN	0.1553	0.0029	0.0955	-0.0235	0.0677	0.1316	-0.0155	0.0860	0.2334	0.1834	0.2092	-0.1260
ARMA-SCop-MVT	0.1806	-0.0136	0.0823	-0.1054	0.1270	0.2089	-0.1459	0.1660	0.2363	0.1676	0.2149	-0.1187
VAR-SCop-MVN	0.1010	-0.0067	0.0737	0.1073	0.0400	0.0730	0.0565	0.0552	0.1541	0.1320	0.1584	0.0555
a-SCop-MVN	0.2163	-0.0190	0.1268	-0.0339	-0.1074	0.2291	-0.0399	-0.0399	0.2198	0.2051	0.2649	-0.1899
a-SCop-MVT	0.2427	-0.0864	0.0337	-0.0573	0.0459	0.2352	-0.1189	0.2051	0.2001	0.2078	0.2565	-0.1644
DCop-MVN	0.2199	-0.0151	0.1426	-0.0364	-0.1089	0.2285	-0.0530	0.1224	0.2190	0.2059	0.2606	-0.1856
DCop-MVT	0.2504	-0.0868	0.0070	-0.0568	0.0635	0.2318	-0.1292	0.2201	0.2024	0.2026	0.2546	-0.1596
ARMA-DCop-MVN	0.1846	-0.0243	0.0593	0.0979	0.1117	-0.1358	-0.0393	0.2459	0.2052	0.2050	0.1494	-0.0596
ARMA-DCop-MVT	0.2256	-0.1252	0.0913	-0.1775	0.1283	0.1973	-0.1197	0.2799	0.2400	0.2318	0.1102	-0.0820
VAR-Cop-MVN	0.0554	0.0485	0.0415	0.0555	0.0731	0.1356	0.0054	0.0849	0.1614	0.1157	0.2006	0.0223
a-DCop-MVN	0.2250	-0.0129	0.1350	-0.0342	-0.1136	0.2295	-0.0451	0.1162	0.2221	0.2073	0.2583	-0.1877
a-DCop-MVT	0.2447	-0.0760	0.0084	0.0084	0.0608	0.2395	-0.1160	0.2197	0.2160	0.1955	0.2560	-0.1676

A.3 Lee Carter model for a single population

The individual models of order p for the centralised mortality rates m_i in a population i, is an extension of the LC model to p age and period effects, that is,

$$m_i(x,t) = \beta_i^{(1)}(x)\kappa_i^{(1)}(t) + \cdots + \beta_i^{(p)}(x)\kappa_i^{(p)}(t) + \epsilon_i(x,t). \tag{5}$$

This can be written in matrix form as,

$$m_i =_p \beta_i \,_p\kappa_i + \epsilon_i \tag{6}$$

such that,

$$m_i = \begin{pmatrix} \beta_i^{(1)}(x_1) & \dots & \beta_i^{(p)}(x_1) \\ \vdots & \ddots & \vdots \\ \beta_i^{(1)}(x_n) & \dots & \beta_i^{(p)}(x_n) \end{pmatrix} \begin{pmatrix} \kappa_i^{(1)}(1) & \dots & \kappa_i^{(1)}(T) \\ \vdots & \ddots & \vdots \\ \kappa_i^{(p)}(1) & \dots & \kappa_i^{(p)}(T) \end{pmatrix} + \begin{pmatrix} \epsilon_i(x_1,1) & \dots & \epsilon_i(x_1,T) \\ \vdots & \ddots & \vdots \\ \epsilon_i(x_n,1) & \dots & \epsilon_i(x_n,T) \end{pmatrix} \tag{7}$$

A.4 Cohort life tables

Actuarial table cohort 1940.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3634892.00	6630.06	30098.23	1550247.00
1	93239.94	91446.87	3534892.00	875.81	23468.17	1520149.00
2	92329.45	88812.47	3443445.00	305.98	22592.37	1496681.00
3	92005.12	86798.55	3354633.00	232.43	22286.38	1474088.00
4	91753.92	84896.92	3267834.00	175.59	22053.96	1451802.00
5	91560.43	83088.70	3182937.00	145.41	21878.37	1429748.00
6	91397.05	81345.43	3099848.00	96.40	21732.96	1407870.00
7	91286.62	79684.70	3018503.00	83.11	21636.56	1386137.00
8	91189.54	78069.19	2938818.00	55.95	21553.45	1364500.00
9	91122.90	76511.91	2860749.00	49.21	21497.50	1342947.00
10	91063.15	74991.32	2784237.00	38.31	21448.29	1321449.00
11	91015.72	73510.87	2709246.00	36.62	21409.98	1300001.00
12	90969.49	72060.58	2635735.00	36.32	21373.37	1278591.00
13	90922.74	70638.48	2563674.00	37.96	21337.05	1257218.00
14	90872.93	69242.09	2493036.00	35.17	21299.09	1235880.00
15	90825.87	67875.35	2423794.00	48.51	21263.93	1214581.00
16	90759.68	66521.54	2355919.00	47.02	21215.41	1193317.00
17	90694.27	65195.26	2289397.00	55.92	21168.39	1172102.00
18	90614.95	63885.58	2224202.00	60.37	21112.47	1150934.00
19	90527.64	62596.64	2160316.00	73.26	21052.09	1129821.00
20	90419.61	61319.60	2097720.00	73.03	20978.84	1108769.00
21	90309.82	60067.35	2036400.00	64.30	20905.81	1087790.00
22	90211.25	58847.91	1976333.00	67.20	20841.51	1066884.00

23	90106.22	57649.02	1917485.00	58.21	20774.31	1046043.00
24	90013.46	56482.18	1859836.00	56.53	20716.11	1025269.00
25	89921.60	55339.45	1803354.00	55.06	20659.58	1004553.00
26	89830.38	54220.17	1748014.00	53.58	20604.51	983892.90
27	89739.87	53123.90	1693794.00	49.99	20550.94	963288.40
28	89653.76	52052.29	1640670.00	47.30	20500.94	942737.50
29	89570.69	51003.98	1588618.00	51.45	20453.64	922236.50
30	89478.57	49971.69	1537614.00	50.26	20402.19	901782.90
31	89386.81	48960.44	1487642.00	49.91	20351.93	881380.70
32	89293.90	47968.97	1438682.00	48.62	20302.02	861028.80
33	89201.62	46997.87	1390713.00	52.52	20253.40	840726.70
34	89099.99	46041.55	1343715.00	60.11	20200.88	820473.30
35	88981.38	45096.03	1297673.00	59.58	20140.77	800272.50
36	88861.52	44169.22	1252577.00	64.33	20081.19	780131.70
37	88729.57	43255.48	1208408.00	66.00	20016.87	760050.50
38	88591.52	42357.64	1165152.00	75.33	19950.87	740033.60
39	88430.88	41467.74	1122795.00	78.17	19875.53	720082.80
40	88260.90	40592.11	1081327.00	81.85	19797.36	700207.20
41	88079.44	39729.65	1040735.00	86.72	19715.51	680409.90

Table A11: Actuarial table cohort 1940.

Actuarial table cohort 1940 [cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	87883.42	38878.90	1001005.00	89.62	19628.79	660694.30
43	87676.88	38041.61	962126.40	98.89	19539.18	641065.60
44	87444.49	37211.15	924084.80	104.30	19440.29	621526.40
45	87194.58	36391.25	886873.70	119.47	19335.99	602086.10
46	86902.71	35571.94	850482.40	127.66	19216.51	582750.10
47	86584.71	34760.20	814910.50	130.56	19088.85	563533.60
48	86253.12	33961.18	780150.30	143.58	18958.29	544444.70
49	85881.31	33164.50	746189.10	146.82	18814.71	525486.50
50	85493.66	32379.90	713024.60	158.80	18667.89	506671.70
51	85066.15	31598.41	680644.70	167.85	18509.09	488003.90
52	84605.41	30822.89	649046.30	176.99	18341.23	469494.80
53	84110.06	30053.15	618223.40	195.48	18164.24	451153.50
54	83552.25	29279.73	588170.20	191.53	17968.77	432989.30
55	82994.99	28525.13	558890.50	212.87	17777.24	415020.50
56	82363.48	27763.70	530365.40	224.42	17564.36	397243.30
57	81684.66	27005.36	502601.70	244.21	17339.94	379678.90
58	80931.51	26241.82	475596.30	257.41	17095.73	362339.00
59	80122.06	25479.75	449354.50	258.99	16838.32	345243.20
60	79291.68	24730.77	423874.70	277.91	16579.33	328404.90
61	78383.17	23977.26	399144.00	281.79	16301.42	311825.60
62	77443.93	23234.37	375166.70	298.14	16019.63	295524.20
63	76430.68	22489.42	351932.30	310.34	15721.49	279504.60
64	75355.29	21746.58	329442.90	315.73	15411.14	263783.10
65	74239.78	21012.65	307696.30	320.52	15095.41	248371.90
66	73085.15	20288.04	286683.70	330.89	14774.89	233276.50

67	71869.80	19567.00	266395.70	336.99	14444.01	218501.60
68	70607.77	18853.72	246828.70	362.05	14107.02	204057.60
69	69225.28	18129.10	227974.90	363.23	13744.96	189950.60
70	67811.12	17417.23	209845.80	369.08	13381.74	176205.60
71	66345.98	16713.20	192428.60	390.21	13012.65	162823.90
72	64766.61	16001.59	175715.40	401.05	12622.45	149811.20
73	63111.51	15292.81	159713.80	414.87	12221.39	137188.80
74	61365.83	14583.85	144421.00	429.79	11806.52	124967.40
75	59521.89	13873.60	129837.10	455.92	11376.73	113160.90
76	57527.51	13150.88	115963.50	460.88	10920.81	101784.10
77	55471.88	12437.10	102812.70	648.28	10459.93	90863.34
78	52523.72	11549.64	90375.57	652.34	9811.65	80403.41
79	49498.93	10675.19	78825.93	652.02	9159.31	70591.76
80	46416.35	9817.88	68150.74	654.03	8507.29	61432.45
81	43263.64	8975.04	58332.87	653.32	7853.26	52925.17
82	40052.60	8149.13	49357.82	649.32	7199.94	45071.91
83	36798.66	7343.09	41208.70	639.49	6550.62	37871.98
84	33531.14	6562.39	33865.61	624.96	5911.13	31321.36
85	30275.26	5811.24	27303.21	595.86	5286.18	25410.23

Actuarial table cohort 1940 [cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	27110.10	5103.62	21491.97	575.24	4690.32	20124.05
87	23994.56	4430.24	16388.35	701.75	4115.08	15433.73
88	20119.32	3643.30	11958.11	743.55	3413.33	11318.65
89	15932.74	2829.69	8314.81	706.78	2669.79	7905.32
90	11875.13	2068.49	5485.13	611.16	1963.01	5235.53
91	8297.70	1417.55	3416.64	483.60	1351.85	3272.53
92	5411.45	906.70	1999.08	350.74	868.25	1920.68
93	3277.07	538.52	1092.39	232.92	517.51	1052.42
94	1831.87	295.24	553.87	141.19	284.59	534.91
95	938.68	148.38	258.63	77.73	143.40	250.32
96	437.26	67.79	110.25	38.61	65.67	106.92
97	183.32	27.87	42.46	17.15	27.06	41.25
98	68.32	10.19	14.59	6.73	9.91	14.19
99	22.28	3.26	4.40	2.30	3.17	4.29
100	6.23	0.89	1.14	0.67	0.87	1.11
101	1.45	0.20	0.25	0.16	0.20	0.24
102	0.27	0.04	0.04	0.03	0.04	0.04
103	0.04	0.01	0.01	0.00	0.01	0.01
104	0.00	0.00	0.00	0.00	0.00	0.00
105	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1941.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3639981.00	7003.77	30000.36	1550596.00
1	92858.90	91073.16	3539981.00	583.41	22996.59	1520596.00
2	92252.39	88738.34	3448908.00	280.14	22413.18	1497599.00
3	91955.45	86751.69	3360170.00	200.39	22133.04	1475186.00
4	91738.87	84883.00	3273418.00	161.68	21932.65	1453053.00
5	91560.70	83088.95	3188535.00	101.15	21770.97	1431120.00
6	91447.06	81389.94	3105446.00	93.90	21669.82	1409349.00
7	91339.49	79730.85	3024056.00	70.50	21575.92	1387679.00
8	91257.14	78127.06	2944325.00	58.52	21505.42	1366103.00
9	91187.44	76566.10	2866198.00	37.69	21446.90	1344598.00
10	91141.68	75055.99	2789632.00	41.43	21409.21	1323151.00
11	91090.38	73571.17	2714576.00	33.47	21367.78	1301742.00
12	91048.12	72122.87	2641005.00	29.42	21334.31	1280374.00
13	91010.25	70706.47	2568882.00	32.65	21304.89	1259040.00
14	90967.40	69314.07	2498176.00	37.65	21272.23	1237735.00
15	90917.02	67943.46	2428862.00	38.64	21234.58	1216463.00
16	90864.30	66598.22	2360918.00	52.04	21195.94	1195228.00
17	90791.91	65265.45	2294320.00	58.73	21143.91	1174032.00
18	90708.60	63951.61	2229055.00	69.83	21085.17	1152888.00
19	90607.61	62651.93	2165103.00	76.02	21015.34	1131803.00
20	90495.51	61371.06	2102451.00	73.51	20939.31	1110788.00
21	90384.99	60117.34	2041080.00	61.46	20865.81	1089848.00
22	90290.77	58899.78	1980963.00	62.41	20804.34	1068983.00
23	90193.21	57704.68	1922063.00	63.75	20741.93	1048178.00
24	90091.62	56531.22	1864358.00	60.18	20678.18	1027436.00
25	89993.84	55383.91	1807827.00	49.35	20618.00	1006758.00
26	89912.07	54269.48	1752443.00	56.02	20568.65	986140.10
27	89817.44	53169.82	1698174.00	48.06	20512.63	965571.50
28	89734.67	52099.26	1645004.00	46.02	20464.57	945058.80
29	89653.85	51051.34	1592905.00	47.79	20418.56	924594.30
30	89568.28	50021.78	1541853.00	48.55	20370.76	904175.70
31	89479.65	49011.28	1491831.00	53.76	20322.22	883804.90
32	89379.57	48015.00	1442820.00	54.22	20268.46	863482.70
33	89276.67	47037.42	1394805.00	54.87	20214.24	843214.30
34	89170.49	46077.98	1347768.00	54.74	20159.37	823000.00
35	89062.48	45137.13	1301690.00	58.35	20104.64	802840.70
36	88945.09	44210.76	1256553.00	60.40	20046.28	782736.00
37	88821.18	43300.15	1212342.00	67.60	19985.88	762689.70
38	88679.81	42399.85	1169042.00	67.40	19918.28	742703.90
39	88536.08	41517.08	1126642.00	72.13	19850.89	722785.60
40	88379.25	40646.54	1085125.00	86.69	19778.76	702934.70
41	88187.06	39778.19	1044478.00	88.34	19692.07	683155.90

Table A14: Actuarial table cohort 1941.

Actuarial table cohort 1941[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	87987.36	38924.88	1004700.00	90.83	19603.72	663463.90
43	87778.03	38085.49	965775.10	93.04	19512.90	643860.10
44	87559.38	37260.04	927689.60	100.90	19419.85	624347.20
45	87317.61	36442.60	890429.60	114.80	19318.95	604927.40
46	87037.16	35626.98	853987.00	119.82	19204.15	585608.40
47	86738.70	34822.03	818360.00	129.12	19084.33	566404.30
48	86410.76	34023.25	783538.00	134.21	18955.21	547319.90
49	86063.22	33234.75	749514.70	140.93	18821.00	528364.70
50	85691.12	32454.69	716280.00	153.56	18680.07	509543.70
51	85277.72	31677.00	683825.30	157.21	18526.51	490863.70
52	84846.20	30910.62	652148.30	175.78	18369.30	472337.10
53	84354.25	30140.41	621237.70	179.24	18193.53	453967.80
54	83842.76	29381.54	591097.30	192.54	18014.28	435774.30
55	83282.56	28623.97	561715.80	209.23	17821.74	417760.00
56	82661.85	27864.27	533091.80	216.64	17612.51	399938.30
57	82006.56	27111.78	505227.50	226.01	17395.87	382325.80
58	81309.53	26364.39	478115.70	240.74	17169.86	364929.90
59	80552.50	25616.64	451751.30	251.26	16929.11	347760.10
60	79746.93	24872.76	426134.70	260.59	16677.86	330830.90
61	78895.03	24133.84	401261.90	271.77	16417.26	314153.10
62	77989.19	23397.96	377128.10	292.27	16145.50	297735.80
63	76995.90	22655.73	353730.10	281.95	15853.23	281590.30
64	76018.91	21938.10	331074.40	306.55	15571.28	265737.10
65	74935.83	21209.66	309136.30	314.49	15264.73	250165.80
66	73802.91	20487.29	287926.70	325.68	14950.24	234901.10
67	72606.70	19767.62	267439.40	330.90	14624.56	219950.90
68	71367.47	19056.58	247671.80	346.70	14293.66	205326.30
69	70043.62	18343.41	228615.20	356.15	13946.96	191032.60
70	68657.01	17634.50	210271.80	374.85	13590.81	177085.70
71	67168.96	16920.52	192637.30	376.83	13215.96	163494.90
72	65643.74	16218.30	175716.70	402.27	12839.13	150278.90
73	63983.62	15504.14	159498.50	397.80	12436.86	137439.80
74	62309.77	14808.18	143994.30	442.32	12039.06	125002.90
75	60412.09	14081.09	129186.10	450.31	11596.74	112963.90
76	58442.26	13359.99	115105.00	626.33	11146.43	101367.10
77	55648.71	12476.74	101745.00	634.86	10520.11	90220.69
78	52761.59	11601.95	89268.31	640.42	9885.25	79700.58
79	49792.06	10738.41	77666.36	641.41	9244.82	69815.33
80	46759.67	9890.50	66927.95	644.72	8603.42	60570.51
81	43651.85	9055.58	57037.46	646.16	7958.70	51967.09
82	40475.99	8235.27	47981.88	643.94	7312.54	44008.39
83	37248.99	7432.96	39746.61	635.87	6668.60	36695.85
84	33999.95	6654.15	32313.66	623.19	6032.73	30027.25
85	30753.27	5902.99	25659.51	595.44	5409.54	23994.52

Actuarial table cohort 1941[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	27590.33	5194.03	19756.52	764.47	4814.10	18584.98
87	23449.92	4329.68	14562.49	837.76	4049.63	13770.89
88	18823.58	3408.66	10232.81	817.40	3211.87	9721.26
89	14221.13	2525.70	6824.16	722.64	2394.47	6509.38
90	10072.50	1754.50	4298.45	583.24	1671.83	4114.91
91	6658.52	1137.52	2543.96	430.82	1088.60	2443.08
92	4087.27	684.83	1406.44	291.08	657.78	1354.48
93	2315.93	380.58	721.61	179.39	366.70	696.70
94	1202.90	193.87	341.03	100.36	187.31	330.00
95	567.99	89.78	147.16	50.64	86.95	142.69
96	241.38	37.42	57.38	22.84	36.32	55.74
97	91.18	13.86	19.96	9.10	13.48	19.42
98	30.14	4.49	6.09	3.16	4.38	5.94
99	8.54	1.25	1.60	0.94	1.22	1.56
100	2.02	0.29	0.35	0.23	0.28	0.34
101	0.38	0.05	0.06	0.05	0.05	0.06
102	0.05	0.01	0.01	0.01	0.01	0.01
103	0.01	0.00	0.00	0.00	0.00	0.00
104	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1942.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3680047.00	6164.33	29229.86	1566783.00
1	93714.81	91912.60	3580047.00	564.70	23065.53	1537553.00
2	93127.75	89580.35	3488135.00	250.65	22500.84	1514488.00
3	92862.06	87607.00	3398554.00	187.36	22250.18	1491987.00
4	92659.57	85734.89	3310947.00	136.53	22062.82	1469737.00
5	92509.12	83949.61	3225213.00	114.61	21926.29	1447674.00
6	92380.34	82220.58	3141263.00	83.10	21811.68	1425748.00
7	92285.15	80556.32	3059042.00	65.39	21728.58	1403936.00
8	92208.76	78941.77	2978486.00	56.89	21663.19	1382208.00
9	92141.02	77366.77	2899544.00	41.49	21606.31	1360544.00
10	92090.63	75837.46	2822178.00	34.06	21564.81	1338938.00
11	92048.46	74344.99	2746340.00	30.84	21530.75	1317373.00
12	92009.53	72884.44	2671995.00	29.59	21499.92	1295842.00
13	91971.45	71453.23	2599111.00	32.30	21470.33	1274343.00
14	91929.06	70046.83	2527657.00	34.41	21438.03	1252872.00
15	91883.01	68665.36	2457611.00	43.02	21403.62	1231434.00
16	91824.32	67301.86	2388945.00	42.23	21360.60	1210031.00
17	91765.57	65965.36	2321643.00	54.19	21318.37	1188670.00
18	91688.70	64642.60	2255678.00	62.54	21264.18	1167352.00
19	91598.25	63336.93	2191035.00	68.23	21201.63	1146087.00
20	91497.64	62050.68	2127698.00	69.03	21133.40	1124886.00
21	91393.85	60788.36	2065648.00	68.52	21064.37	1103752.00
22	91288.81	59550.83	2004859.00	59.13	20995.84	1082688.00
23	91196.38	58346.49	1945309.00	57.71	20936.71	1061692.00
24	91104.41	57166.73	1886962.00	54.70	20879.00	1040755.00
25	91015.53	56012.68	1829795.00	55.90	20824.30	1019876.00
26	90922.93	54879.61	1773783.00	48.26	20768.41	999052.20
27	90841.41	53775.98	1718903.00	47.71	20720.15	978283.80
28	90759.23	52694.12	1665127.00	48.25	20672.44	957563.60
29	90674.50	51632.52	1612433.00	46.01	20624.19	936891.20
30	90592.12	50593.58	1560800.00	50.09	20578.18	916267.00
31	90500.66	49570.53	1510207.00	45.14	20528.09	895688.80
32	90416.63	48572.11	1460636.00	50.47	20482.95	875160.70
33	90320.84	47587.56	1412064.00	53.55	20432.48	854677.80
34	90217.21	46618.86	1364477.00	52.82	20378.93	834245.30
35	90112.98	45669.52	1317858.00	51.79	20326.10	813866.40
36	90008.78	44739.47	1272188.00	58.80	20274.31	793540.30
37	89888.16	43820.29	1227449.00	65.96	20215.51	773265.90
38	89750.19	42911.63	1183629.00	67.58	20149.54	753050.40
39	89606.08	42018.83	1140717.00	66.34	20081.97	732900.90
40	89461.84	41144.44	1098698.00	76.56	20015.63	712818.90
41	89292.11	40276.64	1057554.00	80.50	19939.07	692803.30

Table A17: Actuarial table cohort 1942.

Actuarial table cohort 1942[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	89110.14	39421.59	1017277.00	83.73	19858.57	672864.20
43	88917.16	38579.75	977855.40	93.61	19774.84	653005.70
44	88697.19	37744.22	939275.70	100.26	19681.23	633230.80
45	88456.97	36918.11	901531.50	106.33	19580.97	613549.60
46	88197.20	36101.82	864613.40	117.01	19474.64	593968.60
47	87905.75	35290.55	828511.50	129.55	19357.63	574494.00
48	87576.72	34482.33	793221.00	126.79	19228.08	555136.40
49	87248.40	33692.42	758738.70	143.04	19101.29	535908.30
50	86870.73	32901.46	725046.20	150.79	18958.26	516807.00
51	86464.78	32117.94	692144.80	165.23	18807.47	497848.70
52	86011.26	31335.06	660026.80	171.77	18642.24	479041.30
53	85530.51	30560.69	628691.80	186.00	18470.47	460399.00
54	84999.74	29786.98	598131.10	193.89	18284.46	441928.50
55	84435.61	29020.27	568344.10	198.85	18090.57	423644.10
56	83845.70	28263.33	539323.80	214.38	17891.72	405553.50
57	83197.24	27505.43	511060.50	245.82	17677.34	387661.80
58	82439.13	26730.66	483555.10	225.17	17431.52	369984.50
59	81731.07	25991.44	456824.40	245.85	17206.35	352552.90
60	80942.84	25245.76	430833.00	266.31	16960.51	335346.60
61	80072.24	24493.95	405587.20	274.63	16694.19	318386.10
62	79156.84	23748.27	381093.30	293.63	16419.56	301691.90
63	78158.92	22997.94	357345.00	293.51	16125.92	285272.30
64	77141.87	22262.17	334347.10	302.90	15832.42	269146.40
65	76071.69	21531.15	312084.90	314.75	15529.51	253314.00
66	74937.86	20802.34	290553.70	320.97	15214.77	237784.50
67	73758.94	20081.33	269751.40	322.34	14893.80	222569.70
68	72551.76	19372.81	249670.10	336.57	14571.46	207675.90
69	71266.56	18663.68	230297.30	336.78	14234.88	193104.40
70	69955.35	17967.98	211633.60	357.44	13898.10	178869.60
71	68536.41	17265.00	193665.60	366.69	13540.66	164971.50
72	67052.25	16566.29	176400.60	397.99	13173.97	151430.80
73	65409.81	15849.72	159834.30	413.57	12775.98	138256.80
74	63669.60	15131.35	143984.60	429.70	12362.42	125480.80
75	61826.07	14410.67	128853.30	616.23	11932.72	113118.40
76	59130.44	13517.31	114442.60	617.68	11316.49	101185.70
77	56375.48	12639.69	100925.30	627.82	10698.82	89869.21
78	53520.36	11768.79	88285.59	634.88	10070.99	79170.39
79	50576.54	10907.59	76516.79	637.12	9436.11	69099.40
80	47564.39	10060.71	65609.20	641.72	8798.99	59663.28
81	44471.04	9225.52	55548.49	645.27	8157.28	50864.29
82	41299.55	8402.83	46322.98	644.78	7512.00	42707.01
83	38068.36	7596.46	37920.15	638.36	6867.23	35195.01
84	34806.59	6812.01	30323.68	627.39	6228.87	28327.78
85	31538.02	6053.62	23511.67	840.68	5601.47	22098.92

Actuarial table cohort 1942[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	27072.36	5096.52	17458.05	946.14	4760.79	16497.44
87	21947.96	4052.37	12361.53	941.84	3814.64	11736.66
88	16746.86	3032.60	8309.17	846.67	2872.81	7922.01
89	11979.64	2127.61	5276.57	693.52	2026.14	5049.21
90	7998.18	1393.18	3148.96	519.28	1332.62	3023.07
91	4958.54	847.10	1755.78	355.36	813.34	1690.45
92	2837.62	475.45	908.68	221.69	457.97	877.11
93	1488.56	244.61	433.23	125.50	236.28	419.14
94	709.88	114.41	188.62	64.05	110.78	182.86
95	304.68	48.16	74.21	29.21	46.73	72.07
96	116.24	18.02	26.05	11.77	17.52	25.34
97	38.80	5.90	8.03	4.13	5.75	7.82
98	11.10	1.66	2.13	1.24	1.61	2.07
99	2.65	0.39	0.47	0.31	0.38	0.46
100	0.51	0.07	0.08	0.06	0.07	0.08
101	0.07	0.01	0.01	0.01	0.01	0.01
102	0.01	0.00	0.00	0.00	0.00	0.00
103	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1943.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3700878.00	5833.49	28829.28	1576780.00
1	94052.13	92243.43	3600878.00	461.93	22995.79	1547951.00
2	93571.91	90007.59	3508634.00	230.63	22533.86	1524955.00
3	93327.44	88046.05	3418627.00	154.61	22303.23	1502421.00
4	93160.35	86198.25	3330580.00	119.29	22148.62	1480118.00
5	93028.90	84421.30	3244382.00	91.03	22029.34	1457969.00
6	92926.62	82706.79	3159961.00	71.68	21938.31	1435940.00
7	92844.51	81044.60	3077254.00	57.21	21866.63	1414002.00
8	92777.69	79428.84	2996210.00	44.70	21809.42	1392135.00
9	92724.45	77856.66	2916781.00	38.63	21764.72	1370326.00
10	92677.54	76320.79	2838924.00	30.83	21726.09	1348561.00
11	92639.37	74822.25	2762603.00	25.90	21695.26	1326835.00
12	92606.67	73357.46	2687781.00	33.74	21669.36	1305140.00
13	92563.25	71913.00	2614424.00	28.56	21635.62	1283470.00
14	92525.77	70501.50	2542511.00	36.15	21607.07	1261835.00
15	92477.39	69109.55	2472009.00	38.56	21570.91	1240228.00
16	92424.79	67741.96	2402900.00	53.06	21532.36	1218657.00
17	92350.97	66386.17	2335158.00	61.50	21479.29	1197124.00
18	92263.74	65048.01	2268771.00	67.78	21417.79	1175645.00
19	92165.71	63729.31	2203723.00	72.28	21350.01	1154227.00
20	92059.14	62431.47	2139994.00	73.00	21277.74	1132877.00
21	91949.38	61157.86	2077563.00	68.76	21204.73	1111599.00
22	91843.97	59912.99	2016405.00	59.55	21135.97	1090395.00
23	91750.89	58701.26	1956492.00	59.61	21076.42	1069259.00
24	91655.89	57512.78	1897790.00	50.29	21016.81	1048182.00
25	91574.16	56356.47	1840278.00	46.19	20966.51	1027165.00
26	91497.64	55226.50	1783921.00	50.94	20920.32	1006199.00
27	91411.58	54113.51	1728695.00	51.19	20869.38	985278.70
28	91323.41	53021.67	1674581.00	47.14	20818.19	964409.30
29	91240.62	51954.88	1621560.00	47.72	20771.04	943591.10
30	91155.17	50908.03	1569605.00	53.50	20723.32	922820.00
31	91057.50	49875.53	1518697.00	43.66	20669.83	902096.70
32	90976.22	48872.72	1468821.00	54.61	20626.16	881426.90
33	90872.57	47878.25	1419948.00	50.22	20571.55	860800.70
34	90775.39	46907.30	1372070.00	53.29	20521.33	840229.20
35	90670.24	45951.95	1325163.00	63.19	20468.04	819707.80
36	90543.12	45005.07	1279211.00	60.91	20404.86	799239.80
37	90418.16	44078.67	1234206.00	63.89	20343.94	778834.90
38	90284.54	43167.11	1190127.00	67.85	20280.05	758491.00
39	90139.83	42269.12	1146960.00	67.85	20212.20	738210.90
40	89992.31	41388.40	1104691.00	74.18	20144.35	717998.70
41	89827.86	40518.30	1063303.00	77.34	20070.17	697854.40

Table A20: Actuarial table cohort 1943.

Actuarial table cohort 1943[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	89653.05	39661.76	1022784.00	84.16	19992.84	677784.20
43	89459.07	38814.87	983122.50	93.34	19908.67	657791.40
44	89239.72	37975.09	944307.60	101.73	19815.33	637882.70
45	88995.98	37143.07	906332.50	109.05	19713.60	618067.40
46	88729.57	36319.73	869189.40	114.23	19604.55	598353.80
47	88445.03	35507.05	832869.70	126.29	19490.32	578749.20
48	88124.29	34697.93	797362.60	133.31	19364.03	559258.90
49	87779.08	33897.35	762664.70	137.72	19230.73	539894.90
50	87415.46	33107.77	728767.40	150.77	19093.01	520664.20
51	87009.58	32320.31	695659.60	155.76	18942.24	501571.10
52	86582.02	31543.00	663339.30	166.98	18786.48	482628.90
53	86114.70	30769.43	631796.30	177.91	18619.50	463842.40
54	85607.01	29999.80	601026.90	188.99	18441.59	445222.90
55	85057.13	29233.88	571027.10	204.19	18252.59	426781.30
56	84451.40	28467.51	541793.20	215.18	18048.41	408528.70
57	83800.52	27704.87	513325.70	218.96	17833.22	390480.30
58	83125.22	26953.12	485620.80	235.38	17614.26	372647.10
59	82385.07	26199.42	458667.70	243.21	17378.88	355032.90
60	81605.30	25452.37	432468.30	268.07	17135.68	337654.00
61	80728.95	24694.83	407015.90	264.60	16867.60	320518.30
62	79846.99	23955.33	382321.00	278.60	16603.00	303650.70
63	78900.15	23216.05	358365.70	286.02	16324.40	287047.70
64	77909.06	22483.57	335149.70	298.00	16038.38	270723.30
65	76856.20	21753.19	312666.10	304.89	15740.38	254684.90
66	75757.86	21029.97	290912.90	307.34	15435.49	238944.50
67	74629.00	20318.21	269882.90	315.64	15128.15	223509.00
68	73446.94	19611.84	249564.70	315.58	14812.52	208380.90
69	72241.90	18919.11	229952.90	339.92	14496.93	193568.40
70	70918.49	18215.36	211033.80	354.01	14157.02	179071.40
71	69513.17	17511.05	192818.40	366.94	13803.00	164914.40
72	68027.99	16807.36	175307.40	394.27	13436.07	151111.40
73	66400.87	16089.87	158500.00	403.60	13041.79	137675.30
74	64702.59	15376.85	142410.10	596.91	12638.19	124633.50
75	62141.68	14484.23	127033.30	603.87	12041.28	111995.40
76	59500.10	13601.82	112549.10	605.81	11437.41	99954.08
77	56798.07	12734.43	98947.25	617.45	10831.60	88516.67
78	53990.14	11872.10	86212.82	625.90	10214.16	77685.06
79	51087.97	11017.89	74340.72	629.34	9588.26	67470.91
80	48112.62	10176.67	63322.83	635.15	8958.92	57882.65
81	45050.92	9345.81	53146.17	640.76	8323.77	48923.73
82	41901.64	8525.33	43800.35	641.95	7683.02	40599.96
83	38684.59	7719.43	35275.02	637.21	7041.06	32916.94
84	35428.71	6933.77	27555.59	900.15	6403.85	25875.88
85	30739.13	5900.28	20621.83	1044.92	5503.70	19472.02

Actuarial table cohort 1943[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	25188.61	4741.89	14721.55	1063.94	4458.79	13968.32
87	19426.21	3586.76	9979.66	974.33	3394.84	9509.53
88	14045.69	2543.46	6392.90	811.16	2420.52	6114.69
89	9478.39	1683.38	3849.44	616.45	1609.36	3694.17
90	5939.39	1034.56	2166.06	427.76	992.91	2084.82
91	3435.48	586.91	1131.49	270.41	565.15	1091.91
92	1821.56	305.21	544.59	155.05	294.73	526.76
93	878.03	144.29	239.38	80.12	139.68	232.03
94	380.90	61.39	95.10	36.99	59.56	92.35
95	146.88	23.22	33.71	15.09	22.57	32.79
96	49.56	7.68	10.49	5.36	7.48	10.22
97	14.33	2.18	2.81	1.62	2.13	2.74
98	3.45	0.52	0.63	0.41	0.50	0.61
99	0.67	0.10	0.11	0.08	0.10	0.11
100	0.10	0.01	0.02	0.01	0.01	0.01
101	0.01	0.00	0.00	0.00	0.00	0.00
102	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1944.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3713512.00	5544.87	28586.30	1585083.00
1	94346.41	92532.05	3613512.00	428.70	23041.43	1556497.00
2	93900.73	90323.89	3520980.00	194.85	22612.74	1533455.00
3	93694.19	88392.04	3430656.00	153.74	22417.88	1510842.00
4	93528.03	86538.45	3342264.00	106.62	22264.14	1488425.00
5	93410.54	84767.63	3255726.00	84.51	22157.52	1466160.00
6	93315.59	83052.98	3170958.00	65.95	22073.01	1444003.00
7	93240.03	81389.85	3087905.00	54.26	22007.06	1421930.00
8	93176.65	79770.40	3006515.00	39.89	21952.80	1399923.00
9	93129.15	78196.46	2926745.00	35.73	21912.91	1377970.00
10	93085.76	76656.95	2848548.00	32.92	21877.18	1356057.00
11	93044.99	75149.86	2771892.00	29.62	21844.25	1334180.00
12	93007.60	73675.05	2696742.00	28.25	21814.63	1312336.00
13	92971.24	72229.97	2623067.00	35.98	21786.38	1290521.00
14	92924.02	70804.95	2550837.00	32.42	21750.40	1268735.00
15	92880.64	69410.90	2480032.00	42.94	21717.98	1246984.00
16	92822.05	68033.13	2410621.00	53.23	21675.04	1225266.00
17	92748.00	66671.58	2342588.00	69.54	21621.81	1203591.00
18	92649.37	65319.89	2275916.00	73.82	21552.28	1181969.00
19	92542.61	63989.92	2210596.00	75.08	21478.45	1160417.00
20	92431.90	62684.27	2146606.00	71.58	21403.38	1138939.00
21	92324.28	61407.22	2083922.00	64.41	21331.79	1117535.00
22	92225.55	60161.90	2022515.00	67.29	21267.39	1096204.00
23	92120.38	58937.66	1962353.00	53.39	21200.10	1074936.00
24	92035.30	57750.85	1903415.00	50.56	21146.71	1053736.00
25	91953.15	56589.70	1845664.00	54.48	21096.16	1032589.00
26	91862.89	55446.96	1789075.00	48.38	21041.68	1011493.00
27	91781.17	54332.30	1733628.00	47.72	20993.30	990451.60
28	91698.97	53239.72	1679295.00	43.90	20945.58	969458.30
29	91621.89	52171.99	1626056.00	52.98	20901.69	948512.70
30	91527.02	51115.70	1573884.00	54.66	20848.70	927611.00
31	91427.21	50078.04	1522768.00	52.97	20794.04	906762.30
32	91328.62	49062.03	1472690.00	53.24	20741.07	885968.20
33	91227.57	48065.29	1423628.00	50.41	20687.83	865227.20
34	91130.01	47090.55	1375563.00	54.79	20637.42	844539.30
35	91021.90	46130.17	1328472.00	57.92	20582.63	823901.90
36	90905.38	45185.13	1282342.00	58.02	20524.71	803319.30
37	90786.37	44258.17	1237157.00	59.95	20466.69	782794.60
38	90660.99	43347.11	1192899.00	61.68	20406.75	762327.90
39	90529.45	42451.82	1149552.00	70.55	20345.06	741921.10
40	90376.04	41564.89	1107100.00	72.13	20274.51	721576.10
41	90216.13	40693.43	1065535.00	80.38	20202.38	701301.60

Table A23: Actuarial table cohort 1944.

Actuarial table cohort 1944[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	90034.43	39830.49	1024841.00	84.21	20122.00	681099.20
43	89840.35	38980.30	985010.90	91.99	20037.79	660977.20
44	89624.18	38138.70	946030.60	101.38	19945.80	640939.40
45	89381.27	37303.88	907891.90	107.01	19844.42	620993.60
46	89119.86	36479.49	870588.00	117.77	19737.41	601149.20
47	88826.51	35660.20	834108.50	113.48	19619.65	581411.80
48	88538.30	34860.94	798448.30	129.06	19506.17	561792.10
49	88204.08	34061.47	763587.40	134.79	19377.10	542286.00
50	87848.19	33271.66	729525.90	140.24	19242.31	522908.90
51	87470.64	32491.57	696254.20	153.45	19102.07	503666.50
52	87049.44	31713.29	663762.70	173.20	18948.62	484564.50
53	86564.71	30930.22	632049.40	180.26	18775.42	465615.90
54	86050.32	30155.14	601119.20	194.08	18595.16	446840.40
55	85485.62	29381.15	570964.00	198.03	18401.08	428245.30
56	84898.14	28618.10	541582.90	214.96	18203.04	409844.20
57	84247.95	27852.79	512964.80	218.97	17988.09	391641.20
58	83572.64	27098.20	485112.00	230.90	17769.12	373653.10
59	82846.56	26346.18	458013.80	238.09	17538.22	355883.90
60	82083.19	25601.43	431667.60	249.22	17300.13	338345.70
61	81268.48	24859.87	406066.20	255.27	17050.91	321045.60
62	80417.61	24126.53	381206.30	272.85	16795.63	303994.70
63	79490.31	23389.70	357079.80	281.57	16522.78	287199.10
64	78514.65	22658.33	333690.10	289.85	16241.22	270676.30
65	77490.56	21932.74	311031.70	288.96	15951.36	254435.10
66	76449.64	21222.00	289099.00	309.39	15662.41	238483.70
67	75313.26	20504.50	267877.00	310.55	15353.02	222821.30
68	74150.23	19799.63	247372.50	318.37	15042.47	207468.30
69	72934.53	19100.50	227572.90	339.53	14724.09	192425.80
70	71612.61	18393.64	208472.40	355.00	14384.56	177701.70
71	70203.36	17684.92	190078.70	382.49	14029.56	163317.10
72	68655.22	16962.33	172393.80	394.53	13647.06	149287.60
73	67027.04	16241.60	155431.50	563.23	13252.53	135640.50
74	64657.09	15366.03	139189.90	581.08	12689.30	122388.00
75	62164.06	14489.45	123823.80	588.96	12108.22	109698.70
76	59587.72	13621.85	109334.40	591.34	11519.26	97590.47
77	56950.22	12768.55	95712.55	604.33	10927.92	86071.21
78	54201.95	11918.67	82944.00	614.06	10323.60	75143.29
79	51354.65	11075.40	71025.33	618.63	9709.53	64819.69
80	48429.92	10243.78	59949.93	625.57	9090.90	55110.16
81	45414.38	9421.21	49706.15	633.14	8465.33	46019.26
82	42302.54	8606.90	40284.93	635.98	7832.19	37553.93
83	39115.45	7805.41	31678.03	949.31	7196.21	29721.74
84	34264.87	6705.99	23872.63	1135.68	6246.90	22525.53
85	28348.26	5441.36	17166.64	1181.22	5111.23	16278.63

Actuarial table cohort 1944[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	22073.67	4155.49	11725.28	1100.43	3930.00	11167.40
87	16113.66	2975.15	7569.79	929.86	2829.58	7237.40
88	10978.69	1988.07	4594.64	716.25	1899.71	4407.82
89	6945.80	1233.59	2606.57	503.31	1183.46	2508.11
90	4056.32	706.56	1372.98	322.00	680.15	1324.65
91	2171.46	370.97	666.43	186.77	358.15	644.50
92	1056.77	177.06	295.46	97.60	171.38	286.35
93	462.86	76.06	118.40	45.56	73.79	114.96
94	180.21	29.04	42.34	18.78	28.23	41.18
95	61.39	9.70	13.29	6.74	9.45	12.95
96	17.93	2.78	3.59	2.06	2.71	3.50
97	4.36	0.66	0.81	0.52	0.65	0.79
98	0.85	0.13	0.15	0.11	0.12	0.14
99	0.12	0.02	0.02	0.02	0.02	0.02
100	0.01	0.00	0.00	0.00	0.00	0.00
101	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1945.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3735473.00	5110.34	28163.98	1596508.00
1	94789.46	92966.58	3635473.00	359.26	23053.64	1568345.00
2	94415.97	90819.50	3542506.00	204.63	22694.38	1545291.00
3	94199.06	88868.34	3451687.00	116.45	22489.74	1522596.00
4	94073.20	87042.88	3362819.00	94.45	22373.29	1500107.00
5	93969.12	85274.52	3275776.00	81.09	22278.84	1477733.00
6	93878.01	83553.54	3190501.00	60.29	22197.75	1455455.00
7	93808.94	81886.45	3106948.00	48.89	22137.46	1433257.00
8	93751.83	80262.82	3025061.00	37.78	22088.57	1411119.00
9	93706.84	78681.53	2944798.00	30.32	22050.79	1389031.00
10	93670.02	77138.10	2866117.00	28.52	22020.47	1366980.00
11	93634.71	75626.16	2788979.00	25.88	21991.95	1344960.00
12	93602.04	74145.93	2713353.00	33.59	21966.07	1322968.00
13	93558.81	72686.46	2639207.00	30.65	21932.48	1301002.00
14	93518.58	71257.99	2566520.00	33.47	21901.83	1279069.00
15	93473.80	69854.18	2495262.00	32.74	21868.37	1257167.00
16	93429.13	68478.09	2425408.00	51.63	21835.63	1235299.00
17	93357.31	67109.57	2356930.00	65.92	21784.00	1213463.00
18	93263.81	65753.09	2289820.00	71.74	21718.08	1191679.00
19	93160.07	64416.87	2224067.00	77.79	21646.35	1169961.00
20	93045.37	63100.30	2159650.00	67.98	21568.56	1148315.00
21	92943.17	61818.85	2096550.00	63.63	21500.58	1126746.00
22	92845.63	60566.40	2034731.00	56.23	21436.95	1105246.00
23	92757.74	59345.44	1974165.00	53.52	21380.73	1083809.00
24	92672.45	58150.65	1914819.00	53.41	21327.20	1062428.00
25	92585.65	56978.96	1856669.00	49.77	21273.79	1041101.00
26	92503.20	55833.44	1799690.00	49.92	21224.02	1019827.00
27	92418.87	54709.80	1743856.00	47.41	21174.10	998603.00
28	92337.21	53610.28	1689147.00	53.66	21126.69	977428.90
29	92242.98	52525.65	1635536.00	49.95	21073.03	956302.20
30	92153.55	51465.60	1583011.00	48.79	21023.09	935229.20
31	92064.48	50427.09	1531545.00	49.38	20974.30	914206.10
32	91972.55	49407.95	1481118.00	52.21	20924.92	893231.80
33	91873.46	48405.59	1431710.00	51.06	20872.71	872306.90
34	91774.66	47423.66	1383304.00	53.60	20821.65	851434.20
35	91668.90	46458.07	1335881.00	59.29	20768.06	830612.50
36	91549.63	45505.36	1289423.00	57.76	20708.77	809844.40
37	91431.15	44572.50	1243917.00	61.99	20651.01	789135.70
38	91301.50	43653.35	1199345.00	62.08	20589.02	768484.70
39	91169.11	42751.78	1155691.00	65.65	20526.94	747895.60
40	91026.36	41863.98	1112940.00	82.20	20461.29	727368.70
41	90844.13	40976.70	1071076.00	76.61	20379.09	706907.40

Table A26: Actuarial table cohort 1945.

Actuarial table cohort 1945[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	90670.97	40112.08	1030099.00	90.85	20302.49	686528.30
43	90461.58	39249.84	989986.90	92.16	20211.64	666225.80
44	90245.00	38402.88	950737.10	97.27	20119.47	646014.20
45	90011.93	37567.09	912334.20	108.72	20022.20	625894.70
46	89746.33	36735.93	874767.10	121.07	19913.48	605872.50
47	89444.76	35908.40	838031.20	116.76	19792.41	585959.00
48	89148.20	35101.09	802122.80	126.04	19675.65	566166.60
49	88821.81	34300.02	767021.70	133.42	19549.60	546491.00
50	88469.53	33506.98	732721.70	145.91	19416.18	526941.40
51	88076.71	32716.71	699214.70	147.52	19270.27	507525.20
52	87671.79	31940.02	666498.00	157.33	19122.75	488254.90
53	87231.47	31168.46	634558.00	171.99	18965.42	469132.20
54	86740.70	30397.08	603389.50	189.01	18793.44	450166.80
55	86190.78	29623.51	572992.40	188.21	18604.43	431373.30
56	85632.44	28865.62	543368.90	199.62	18416.22	412768.90
57	85028.64	28110.90	514503.30	210.57	18216.60	394352.70
58	84379.22	27359.73	486392.40	219.75	18006.03	376136.10
59	83688.22	26613.83	459032.70	222.74	17786.28	358130.00
60	82974.07	25879.29	432418.80	245.24	17563.54	340343.80
61	82172.38	25136.37	406539.50	258.16	17318.31	322780.20
62	81311.88	24394.82	381403.20	268.58	17060.14	305461.90
63	80399.12	23657.11	357008.30	281.03	16791.57	288401.80
64	79425.32	22921.14	333351.20	279.88	16510.54	271610.20
65	78436.48	22200.47	310430.10	284.94	16230.66	255099.70
66	77410.02	21488.60	288229.60	297.40	15945.72	238869.00
67	76317.65	20777.95	266741.00	310.32	15648.32	222923.30
68	75155.49	20068.06	245963.10	316.17	15338.00	207274.90
69	73948.22	19365.97	225895.00	334.98	15021.83	191937.00
70	72644.03	18658.56	206529.00	350.97	14686.85	176915.10
71	71250.81	17948.78	187870.50	377.94	14335.89	162228.30
72	69721.11	17225.68	169921.70	537.43	13957.95	147892.40
73	67503.22	16356.98	152696.00	551.92	13420.52	133934.40
74	65180.86	15490.51	136339.00	570.66	12868.60	120513.90
75	62732.57	14621.96	120848.50	579.45	12297.95	107645.30
76	60197.82	13761.32	106226.60	582.26	11718.50	95347.36
77	57600.82	12914.42	92465.26	596.64	11136.24	83628.86
78	54887.53	12069.43	79550.84	607.68	10539.60	72492.62
79	52069.82	11229.64	67481.42	613.37	9931.92	61953.02
80	49169.99	10400.32	56251.78	621.45	9318.56	52021.10
81	46174.33	9578.86	45851.45	630.98	8697.11	42702.54
82	43073.10	8763.68	36272.59	999.27	8066.13	34005.44
83	38065.42	7595.87	27508.91	1231.87	7066.86	25939.31
84	31771.05	6217.93	19913.04	1307.81	5834.98	18872.45
85	24957.64	4790.54	13695.11	1238.27	4527.17	13037.47

Actuarial table cohort 1945[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	18380.05	3460.15	8904.58	1061.00	3288.90	8510.30
87	12633.59	2332.61	5444.43	827.58	2227.91	5221.40
88	8063.47	1460.17	3111.82	588.36	1400.33	2993.49
89	4750.68	843.73	1651.65	380.60	811.97	1593.17
90	2565.67	446.90	807.92	223.11	431.37	781.20
91	1259.66	215.20	361.02	117.80	208.25	349.83
92	556.60	93.26	145.82	55.54	90.46	141.58
93	218.62	35.93	52.56	23.13	34.91	51.12
94	75.13	12.11	16.64	8.38	11.79	16.21
95	22.13	3.50	4.53	2.59	3.41	4.42
96	5.43	0.84	1.03	0.66	0.82	1.01
97	1.07	0.16	0.19	0.14	0.16	0.18
98	0.16	0.02	0.03	0.02	0.02	0.03
99	0.02	0.00	0.00	0.00	0.00	0.00
100	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1946.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3730654.00	5336.72	28256.65	1595622.00
1	94558.64	92740.21	3630654.00	394.44	22919.94	1567366.00
2	94148.58	90562.30	3537914.00	165.32	22525.50	1544446.00
3	93973.35	88655.40	3447351.00	117.56	22360.18	1521920.00
4	93846.29	86832.92	3358696.00	76.87	22242.61	1499560.00
5	93761.58	85086.19	3271863.00	71.24	22165.75	1477317.00
6	93681.54	83378.68	3186777.00	51.83	22094.51	1455152.00
7	93622.17	81723.42	3103398.00	45.67	22042.68	1433057.00
8	93568.82	80106.14	3021675.00	32.05	21997.01	1411014.00
9	93530.65	78533.59	2941569.00	28.34	21964.96	1389017.00
10	93496.24	76994.99	2863035.00	28.92	21936.62	1367052.00
11	93460.43	75485.40	2786040.00	24.95	21907.70	1345116.00
12	93428.94	74008.81	2710555.00	27.94	21882.76	1323208.00
13	93392.98	72557.62	2636546.00	27.39	21854.82	1301325.00
14	93357.03	71134.89	2563988.00	29.64	21827.43	1279471.00
15	93317.36	69737.27	2492853.00	35.83	21797.78	1257643.00
16	93268.48	68360.34	2423116.00	50.20	21761.95	1235845.00
17	93198.65	66995.52	2354756.00	59.57	21711.75	1214083.00
18	93114.15	65647.57	2287760.00	73.94	21652.18	1192372.00
19	93007.23	64311.18	2222113.00	70.16	21578.25	1170719.00
20	92903.77	63004.27	2157802.00	67.38	21508.08	1149141.00
21	92802.46	61725.27	2094797.00	63.05	21440.71	1127633.00
22	92705.81	60475.20	2033072.00	54.54	21377.66	1106192.00
23	92620.56	59257.67	1972597.00	50.13	21323.12	1084815.00
24	92540.67	58067.96	1913339.00	46.57	21272.98	1063492.00
25	92465.00	56904.71	1855271.00	47.25	21226.41	1042219.00
26	92386.71	55763.13	1798366.00	50.89	21179.16	1020992.00
27	92300.74	54639.87	1742603.00	50.94	21128.27	999813.10
28	92213.01	53538.17	1687963.00	44.72	21077.33	978684.80
29	92134.47	52463.87	1634425.00	44.59	21032.61	957607.50
30	92054.63	51410.35	1581961.00	47.63	20988.02	936574.90
31	91967.68	50374.07	1530551.00	46.96	20940.39	915586.90
32	91880.26	49358.37	1480177.00	49.59	20893.43	894646.50
33	91786.13	48359.58	1430819.00	54.99	20843.84	873753.10
34	91679.72	47374.60	1382459.00	51.69	20788.85	852909.20
35	91577.74	46411.87	1335084.00	49.13	20737.17	832120.40
36	91478.89	45470.20	1288673.00	49.16	20688.03	811383.20
37	91378.04	44546.61	1243202.00	54.36	20638.87	790695.20
38	91264.35	43635.58	1198656.00	59.32	20584.51	770056.30
39	91137.85	42737.12	1155020.00	69.10	20525.19	749471.80
40	90987.60	41846.15	1112283.00	75.00	20456.09	728946.60
41	90821.33	40966.42	1070437.00	75.87	20381.09	708490.50

Table A29: Actuarial table cohort 1946.

Actuarial table cohort 1946[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	90649.84	40102.74	1029471.00	79.45	20305.23	688109.40
43	90466.73	39252.08	989367.80	81.80	20225.78	667804.20
44	90274.51	38415.44	950115.70	88.85	20143.98	647578.40
45	90061.63	37587.83	911700.30	98.85	20055.13	627434.40
46	89820.14	36766.14	874112.40	104.35	19956.28	607379.30
47	89560.21	35954.75	837346.30	113.68	19851.93	587423.00
48	89271.49	35149.63	801391.60	110.90	19738.25	567571.10
49	88984.32	34362.78	766241.90	130.07	19627.36	547832.80
50	88640.88	33571.88	731879.10	134.59	19497.28	528205.50
51	88278.55	32791.68	698307.30	147.19	19362.69	508708.20
52	87874.55	32013.89	665515.60	160.35	19215.51	489345.50
53	87425.78	31237.89	633501.70	173.19	19055.16	470130.00
54	86931.56	30463.97	602263.80	167.18	18881.97	451074.80
55	86445.16	29710.94	571799.80	179.85	18714.79	432192.90
56	85911.63	28959.73	542088.90	191.44	18534.95	413478.10
57	85332.57	28211.38	513129.20	210.42	18343.51	394943.10
58	84683.62	27458.43	484917.80	204.46	18133.09	376599.60
59	84040.70	26725.93	457459.40	216.92	17928.63	358466.50
60	83345.21	25995.05	430733.40	233.38	17711.71	340537.90
61	82582.28	25261.76	404738.40	246.38	17478.33	322826.20
62	81761.07	24529.58	379476.60	250.86	17231.96	305347.80
63	80908.52	23807.00	354947.00	255.32	16981.10	288115.90
64	80023.81	23093.86	331140.00	269.28	16725.78	271134.80
65	79072.42	22380.47	308046.20	269.00	16456.50	254409.00
66	78103.37	21681.07	285665.70	283.04	16187.50	237952.50
67	77063.76	20981.09	263984.60	293.50	15904.46	221765.00
68	75964.58	20284.10	243003.60	309.78	15610.96	205860.50
69	74781.70	19584.24	222719.50	332.08	15301.18	190249.60
70	73488.81	18875.55	203135.20	347.51	14969.10	174948.40
71	72109.31	18165.04	184259.70	509.55	14621.59	159979.30
72	70046.89	17306.16	166094.60	524.79	14112.04	145357.70
73	67881.15	16448.56	148788.50	540.02	13587.25	131245.70
74	65608.86	15592.23	132339.90	559.55	13047.23	117658.40
75	63208.21	14732.82	116747.70	569.21	12487.67	104611.20
76	60718.26	13880.29	102014.90	572.41	11918.47	92123.55
77	58165.22	13040.96	88134.56	588.09	11346.06	80205.08
78	55490.79	12202.08	75093.60	600.37	10757.97	68859.02
79	52706.99	11367.05	62891.52	607.12	10157.60	58101.05
80	49836.68	10541.34	51524.47	616.30	9550.48	47943.44
81	46865.85	9722.32	40983.13	1040.36	8934.18	38392.96
82	41752.54	8495.00	31260.81	1321.39	7893.83	29458.78
83	35130.60	7010.24	22765.82	1430.73	6572.43	21564.95
84	27820.15	5444.69	15755.58	1375.52	5141.70	14992.52
85	20654.02	3964.47	10310.89	1194.01	3766.19	9850.82

Actuarial table cohort 1946[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	14311.54	2694.23	6346.42	942.22	2572.18	6084.64
87	9208.39	1700.19	3652.19	677.12	1629.96	3512.46
88	5469.13	990.38	1952.00	442.50	952.84	1882.50
89	2977.59	528.83	961.62	261.95	510.33	929.66
90	1473.73	256.70	432.80	139.62	248.38	419.33
91	656.47	112.15	176.09	66.44	108.76	170.95
92	259.93	43.55	63.94	27.92	42.32	62.18
93	90.06	14.80	20.39	10.20	14.41	19.86
94	26.74	4.31	5.59	3.18	4.20	5.45
95	6.62	1.05	1.28	0.82	1.02	1.25
96	1.31	0.20	0.24	0.17	0.20	0.23
97	0.19	0.03	0.03	0.03	0.03	0.03
98	0.02	0.00	0.00	0.00	0.00	0.00
99	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1947.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3746777.00	4918.07	27946.59	1608362.00
1	94985.49	93158.85	3646777.00	291.00	23028.52	1580415.00
2	94682.97	91076.34	3553618.00	141.20	22737.52	1557387.00
3	94533.30	89183.67	3462542.00	100.88	22596.32	1534649.00
4	94424.27	87367.72	3373358.00	80.42	22495.44	1512053.00
5	94335.65	85607.15	3285991.00	58.58	22415.02	1489557.00
6	94269.83	83902.27	3200384.00	52.24	22356.43	1467142.00
7	94209.99	82236.53	3116481.00	39.35	22304.19	1444786.00
8	94164.02	80615.71	3034245.00	35.26	22264.84	1422482.00
9	94122.04	79030.15	2953629.00	26.27	22229.59	1400217.00
10	94090.13	77484.07	2874599.00	32.90	22203.32	1377987.00
11	94049.40	75961.09	2797115.00	26.00	22170.42	1355784.00
12	94016.58	74474.30	2721154.00	30.53	22144.42	1333613.00
13	93977.29	73011.58	2646679.00	26.92	22113.90	1311469.00
14	93941.96	71580.59	2573668.00	31.58	22086.98	1289355.00
15	93899.70	70172.46	2502087.00	38.12	22055.39	1267268.00
16	93847.69	68784.87	2431915.00	52.33	22017.28	1245213.00
17	93774.90	67409.75	2363130.00	71.63	21964.95	1223195.00
18	93673.30	66041.78	2295720.00	77.16	21893.32	1201231.00
19	93561.71	64694.59	2229678.00	80.53	21816.16	1179337.00
20	93442.96	63369.93	2164984.00	67.89	21735.63	1157521.00
21	93340.88	62083.38	2101614.00	61.89	21667.73	1135785.00
22	93246.00	60827.58	2039531.00	57.24	21605.84	1114118.00
23	93156.53	59600.57	1978703.00	55.10	21548.59	1092512.00
24	93068.72	58399.31	1919102.00	47.00	21493.50	1070963.00
25	92992.35	57229.24	1860703.00	51.84	21446.49	1049470.00
26	92906.46	56076.84	1803474.00	52.22	21394.65	1028023.00
27	92818.24	54946.22	1747397.00	48.91	21342.43	1006629.00
28	92734.00	53840.65	1692451.00	44.18	21293.52	985286.20
29	92656.41	52761.07	1638610.00	44.95	21249.34	963992.70
30	92575.93	51701.49	1585849.00	42.68	21204.39	942743.30
31	92498.02	50664.55	1534148.00	49.07	21161.72	921538.90
32	92406.67	49641.17	1483483.00	48.47	21112.65	900377.20
33	92314.68	48638.06	1433842.00	49.01	21064.18	879264.60
34	92219.83	47653.70	1385204.00	49.05	21015.17	858200.40
35	92123.05	46688.23	1337550.00	54.78	20966.12	837185.20
36	92012.84	45735.60	1290862.00	54.15	20911.34	816219.10
37	91901.76	44801.92	1245126.00	59.24	20857.19	795307.80
38	91777.87	43881.11	1200324.00	56.77	20797.95	774450.60
39	91656.80	42980.47	1156443.00	62.47	20741.18	753652.60
40	91520.98	42091.46	1113463.00	66.62	20678.71	732911.50
41	91373.29	41215.39	1071371.00	78.55	20612.09	712232.70

Table A32: Actuarial table cohort 1947.

Actuarial table cohort 1947[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	91195.74	40344.24	1030156.00	80.48	20533.55	691620.70
43	91010.25	39487.91	989811.70	88.32	20453.07	671087.10
44	90802.71	38640.21	950323.80	91.71	20364.75	650634.00
45	90582.97	37805.41	911683.50	96.24	20273.04	630269.30
46	90347.85	36982.15	873878.10	103.33	20176.80	609996.20
47	90090.45	36167.62	836896.00	107.53	20073.46	589819.50
48	89817.35	35364.56	800728.40	123.67	19965.93	569746.00
49	89497.10	34560.80	765363.80	130.86	19842.26	549780.10
50	89151.59	33765.31	730803.00	133.22	19711.40	529937.80
51	88792.94	32982.75	697037.70	144.85	19578.18	510226.40
52	88395.33	32203.62	664055.00	153.85	19433.33	490648.20
53	87964.75	31430.46	631851.30	161.69	19279.48	471214.90
54	87503.36	30664.34	600420.90	173.60	19117.79	451935.40
55	86998.27	29901.05	569756.50	182.69	18944.19	432817.60
56	86456.31	29143.34	539855.50	192.65	18761.50	413873.40
57	85873.58	28390.24	510712.10	200.42	18568.85	395111.90
58	85255.46	27643.85	482321.90	205.11	18368.43	376543.10
59	84610.48	26907.12	454678.10	211.06	18163.32	358174.60
60	83933.78	26178.62	427770.90	220.47	17952.26	340011.30
61	83213.05	25454.71	401592.30	234.32	17731.78	322059.10
62	82432.04	24730.88	376137.60	237.21	17497.47	304327.30
63	81625.87	24018.08	351406.70	256.91	17260.26	286829.80
64	80735.65	23299.29	327388.60	262.16	17003.35	269569.60
65	79809.39	22589.06	304089.40	266.98	16741.19	252566.20
66	78847.62	21887.67	281500.30	287.24	16474.20	235825.00
67	77792.59	21179.51	259612.60	298.06	16186.96	219350.80
68	76676.34	20474.15	238433.10	316.30	15888.90	203163.90
69	75468.56	19764.12	217959.00	339.70	15572.60	187275.00
70	74146.00	19044.34	198194.80	472.49	15232.91	171702.40
71	72270.36	18205.61	179150.50	496.03	14760.41	156469.40
72	70262.67	17359.47	160944.90	511.63	14264.38	141709.00
73	68151.23	16514.01	143585.40	527.52	13752.75	127444.70
74	65931.54	15668.91	127071.40	547.76	13225.23	113691.90
75	63581.48	14819.82	111402.50	558.20	12677.47	100466.70
76	61139.65	13976.62	96582.66	561.76	12119.26	87789.21
77	58634.10	13146.08	82606.04	578.66	11557.50	75669.94
78	56002.56	12314.61	69459.96	592.10	10978.85	64112.44
79	53257.09	11485.69	57145.34	599.86	10386.74	53133.59
80	50421.10	10664.95	45659.65	1078.20	9786.88	42746.85
81	45223.70	9381.66	34994.70	1406.95	8708.68	32959.97
82	38308.60	7794.29	25613.04	1549.81	7301.73	24251.29
83	30542.01	6094.59	17818.75	1509.69	5751.92	16949.56
84	22828.12	4467.70	11724.16	1325.03	4242.24	11197.63
85	15925.00	3056.75	7256.46	1055.96	2917.20	6955.40

Actuarial table cohort 1947[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	10315.83	1942.01	4199.71	765.78	1861.25	4038.19
87	6168.32	1138.89	2257.70	504.75	1095.47	2176.95
88	3380.96	612.24	1118.81	301.26	590.72	1081.47
89	1684.70	299.21	506.57	161.85	289.46	490.75
90	755.52	131.60	207.36	77.62	127.61	201.29
91	301.17	51.45	75.76	32.86	49.99	73.67
92	105.05	17.60	24.31	12.10	17.13	23.68
93	31.41	5.16	6.71	3.80	5.03	6.54
94	7.82	1.26	1.55	0.99	1.23	1.51
95	1.56	0.25	0.29	0.21	0.24	0.28
96	0.23	0.04	0.04	0.03	0.04	0.04
97	0.02	0.00	0.00	0.00	0.00	0.00
98	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1948.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3781965.00	3842.64	27269.91	1630993.00
1	96082.02	94234.29	3681965.00	269.66	23427.27	1603723.00
2	95801.68	92152.43	3587730.00	139.71	23157.61	1580296.00
3	95653.59	90240.55	3495578.00	99.16	23017.90	1557138.00
4	95546.42	88406.00	3405337.00	71.42	22918.74	1534120.00
5	95467.72	86634.47	3316931.00	63.45	22847.33	1511202.00
6	95396.43	84904.97	3230297.00	44.95	22783.88	1488354.00
7	95344.93	83227.23	3145392.00	47.25	22738.92	1465570.00
8	95289.74	81579.46	3062165.00	32.08	22691.67	1442831.00
9	95251.54	79978.55	2980585.00	34.66	22659.60	1420140.00
10	95209.45	78405.83	2900607.00	29.29	22624.93	1397480.00
11	95173.18	76868.74	2822201.00	27.06	22595.64	1374855.00
12	95139.02	75363.43	2745332.00	29.41	22568.58	1352260.00
13	95101.16	73884.72	2669969.00	29.12	22539.17	1329691.00
14	95062.94	72434.74	2596084.00	36.86	22510.04	1307152.00
15	95013.61	71004.90	2523649.00	37.04	22473.18	1284642.00
16	94963.08	69602.38	2452644.00	60.25	22436.15	1262169.00
17	94879.26	68203.63	2383042.00	77.68	22375.89	1239732.00
18	94769.08	66814.33	2314838.00	78.85	22298.21	1217357.00
19	94655.05	65450.59	2248024.00	73.01	22219.36	1195058.00
20	94547.39	64118.92	2182573.00	61.85	22146.35	1172839.00
21	94454.40	62824.01	2118455.00	63.12	22084.50	1150693.00
22	94357.63	61552.74	2055631.00	59.80	22021.38	1128608.00
23	94264.17	60309.23	1994078.00	58.53	21961.58	1106587.00
24	94170.90	59090.91	1933769.00	51.44	21903.05	1084625.00
25	94087.31	57903.11	1874678.00	53.19	21851.61	1062722.00
26	93999.19	56736.40	1816775.00	48.95	21798.43	1040870.00
27	93916.51	55596.37	1760038.00	48.18	21749.48	1019072.00
28	93833.52	54479.03	1704442.00	48.49	21701.30	997322.60
29	93748.36	53382.86	1649963.00	50.92	21652.81	975621.30
30	93657.19	52305.35	1596580.00	48.35	21601.89	953968.50
31	93568.91	51251.12	1544275.00	47.28	21553.54	932366.60
32	93480.90	50218.25	1493023.00	48.59	21506.26	910813.10
33	93388.68	49203.92	1442805.00	49.34	21457.67	889306.80
34	93293.20	48208.35	1393601.00	51.13	21408.33	867849.10
35	93192.31	47230.14	1345393.00	56.48	21357.20	846440.80
36	93078.68	46265.39	1298163.00	52.51	21300.72	825083.60
37	92970.96	45323.15	1251897.00	59.88	21248.20	803782.90
38	92845.72	44391.67	1206574.00	64.56	21188.32	782534.70
39	92708.04	43473.43	1162183.00	64.89	21123.76	761346.40
40	92566.95	42572.51	1118709.00	71.92	21058.87	740222.60
41	92407.50	41681.89	1076137.00	73.68	20986.95	719163.70

Table A35: Actuarial table cohort 1948.

Actuarial table cohort 1948[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	92240.95	40806.63	1034455.00	85.28	20913.27	698176.80
43	92044.41	39936.61	993648.10	91.98	20827.99	677263.50
44	91828.27	39076.62	953711.50	94.13	20736.02	656435.50
45	91602.74	38231.02	914634.90	100.95	20641.89	635699.50
46	91356.11	37394.86	876403.80	102.55	20540.94	615057.60
47	91100.67	36573.18	839009.00	110.52	20438.39	594516.70
48	90819.97	35759.32	802435.80	117.47	20327.87	574078.30
49	90515.78	34954.18	766676.50	128.15	20210.40	553750.40
50	90177.43	34153.84	731722.30	132.65	20082.25	533540.00
51	89820.32	33364.38	697568.50	145.23	19949.60	513457.80
52	89421.68	32577.53	664204.10	151.73	19804.37	493508.20
53	88997.05	31799.31	631626.60	161.60	19652.65	473703.80
54	88535.91	31026.19	599827.20	170.41	19491.05	454051.10
55	88040.08	30259.12	568801.10	188.83	19320.63	434560.10
56	87479.91	29488.38	538541.90	186.06	19131.81	415239.50
57	86917.13	28735.24	509053.60	197.82	18945.75	396107.70
58	86307.04	27984.82	480318.30	209.82	18747.93	377161.90
59	85647.26	27236.83	452333.50	220.01	18538.11	358414.00
60	84941.87	26493.04	425096.70	229.43	18318.10	339875.90
61	84191.86	25754.13	398603.60	237.80	18088.67	321557.80
62	83399.24	25021.06	372849.50	248.06	17850.88	303469.10
63	82556.19	24291.82	347828.40	252.20	17602.81	285618.20
64	81682.29	23572.47	323536.60	259.27	17350.62	268015.40
65	80766.25	22859.88	299964.10	279.68	17091.34	250664.80
66	79758.73	22140.59	277104.30	282.72	16811.66	233573.40
67	78720.31	21432.09	254963.70	309.74	16528.94	216761.80
68	77560.34	20710.20	233531.60	325.94	16219.21	200232.80
69	76315.74	19985.98	212821.40	450.94	15893.27	184013.60
70	74560.06	19150.69	192835.40	460.53	15442.32	168120.40
71	72731.89	18321.88	173684.70	484.87	14981.79	152678.00
72	70769.40	17484.67	155362.80	500.85	14496.92	137696.30
73	68702.45	16647.57	137878.20	517.41	13996.07	123199.30
74	66525.30	15810.02	121230.60	538.39	13478.66	109203.30
75	64215.43	14967.59	105420.60	549.62	12940.27	95724.59
76	61811.15	14130.13	90452.97	553.52	12390.65	82784.33
77	59342.36	13304.88	76322.85	571.64	11837.13	70393.68
78	56742.74	12477.37	63017.97	586.25	11265.49	58556.55
79	54024.39	11651.17	50540.59	1120.05	10679.24	47291.05
80	48729.11	10307.07	38889.42	1497.39	9559.19	36611.82
81	41511.02	8611.46	28582.35	1674.28	8061.80	27052.62
82	33282.00	6771.57	19970.89	1649.36	6387.52	18990.82
83	25016.51	4991.99	13199.32	1461.25	4738.16	12603.30
84	17550.15	3434.75	8207.33	1174.22	3276.91	7865.15
85	11432.73	2194.48	4772.58	858.06	2102.70	4588.23

Actuarial table cohort 1948[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	6874.76	1294.21	2578.11	569.66	1244.63	2485.54
87	3789.45	699.67	1283.89	342.35	674.98	1240.91
88	1898.90	343.86	584.23	185.15	332.63	565.93
89	856.39	152.10	240.37	89.37	147.47	233.30
90	343.31	59.80	88.27	38.08	58.10	85.83
91	120.42	20.57	28.47	14.11	20.03	27.73
92	36.21	6.07	7.90	4.46	5.91	7.70
93	9.07	1.49	1.83	1.17	1.46	1.79
94	1.82	0.29	0.34	0.24	0.29	0.33
95	0.27	0.04	0.05	0.04	0.04	0.05
96	0.03	0.00	0.00	0.00	0.00	0.00
97	0.00	0.00	0.00	0.00	0.00	0.00

Actuarial table cohort 1949.

x	lx	Dx	Nx	Cx	Mx	Rx
0	100000.00	100000.00	3785896.00	3647.37	27194.31	1640025.00
1	96281.11	94429.55	3685896.00	227.00	23546.95	1612830.00
2	96045.13	92386.61	3591466.00	136.72	23319.95	1589283.00
3	95900.21	90473.22	3499080.00	87.80	23183.23	1565963.00
4	95805.32	88645.55	3408606.00	71.44	23095.43	1542780.00
5	95726.60	86869.39	3319961.00	52.30	23023.99	1519685.00
6	95667.84	85146.53	3233091.00	47.67	22971.70	1496661.00
7	95613.23	83461.43	3147945.00	39.53	22924.03	1473689.00
8	95567.06	81816.87	3064483.00	36.58	22884.50	1450765.00
9	95523.49	80206.89	2982667.00	30.91	22847.92	1427880.00
10	95485.96	78633.54	2902460.00	27.14	22817.01	1405033.00
11	95452.35	77094.21	2823826.00	26.84	22789.87	1382215.00
12	95418.47	75584.80	2746732.00	28.98	22763.03	1359426.00
13	95381.17	74102.26	2671147.00	33.71	22734.05	1336663.00
14	95336.92	72643.50	2597045.00	32.55	22700.34	1313929.00
15	95293.36	71213.96	2524401.00	37.29	22667.78	1291228.00
16	95242.49	69807.18	2453187.00	58.99	22630.50	1268560.00
17	95160.43	68405.74	2383380.00	75.43	22571.51	1245930.00
18	95053.43	67014.81	2314974.00	74.03	22496.07	1223358.00
19	94946.37	65652.03	2247960.00	62.88	22422.04	1200862.00
20	94853.65	64326.62	2182308.00	57.95	22359.16	1178440.00
21	94766.52	63031.61	2117981.00	65.06	22301.21	1156081.00
22	94666.78	61754.40	2054949.00	56.91	22236.15	1133780.00
23	94577.84	60509.91	1993195.00	58.07	22179.24	1111544.00
24	94485.29	59288.19	1932685.00	61.20	22121.17	1089365.00
25	94385.85	58086.83	1873397.00	50.62	22059.97	1067243.00
26	94301.98	56919.16	1815310.00	49.83	22009.35	1045183.00
27	94217.81	55774.73	1758391.00	53.36	21959.52	1023174.00
28	94125.89	54648.77	1702616.00	47.84	21906.16	1001215.00
29	94041.88	53549.99	1647967.00	46.56	21858.31	979308.40
30	93958.50	52473.62	1594417.00	50.72	21811.75	957450.10
31	93865.90	51413.80	1541944.00	46.22	21761.03	935638.30
32	93779.87	50378.85	1490530.00	47.86	21714.81	913877.30
33	93689.04	49362.17	1440151.00	47.52	21666.96	892162.50
34	93597.08	48365.38	1390789.00	49.97	21619.44	870495.50
35	93498.48	47385.31	1342424.00	53.69	21569.47	848876.10
36	93390.46	46420.36	1295038.00	59.97	21515.78	827306.60
37	93267.45	45467.69	1248618.00	63.32	21455.81	805790.80
38	93135.01	44529.99	1203150.00	60.71	21392.49	784335.00
39	93005.55	43612.94	1158620.00	66.93	21331.78	762942.60
40	92860.02	42707.30	1115007.00	74.16	21264.85	741610.80
41	92695.62	41811.84	1072300.00	82.02	21190.69	720345.90

Table A38: Actuarial table cohort 1949.

Actuarial table cohort 1949[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
42	92510.23	40925.76	1030488.00	87.41	21108.68	699155.20
43	92308.77	40051.31	989562.40	86.09	21021.27	678046.60
44	92106.47	39195.01	949511.00	92.23	20935.18	657025.30
45	91885.50	38349.03	910316.00	93.01	20842.96	636090.10
46	91658.27	37518.54	871967.00	109.42	20749.94	615247.20
47	91385.72	36687.61	834448.50	112.02	20640.53	594497.20
48	91101.22	35870.06	797760.90	116.71	20528.51	573856.70
49	90798.99	35063.54	761890.80	123.82	20411.80	553328.20
50	90472.07	34265.43	726827.20	141.35	20287.98	532916.40
51	90091.53	33465.12	692561.80	143.41	20146.63	512628.40
52	89697.88	32678.15	659096.70	152.93	20003.22	492481.80
53	89269.88	31896.80	626418.50	161.04	19850.29	472478.50
54	88810.35	31122.36	594521.70	169.82	19689.25	452628.30
55	88316.26	30354.04	563399.40	172.20	19519.43	432939.00
56	87805.42	29598.11	533045.30	181.41	19347.23	413419.60
57	87256.69	28847.50	503447.20	195.76	19165.82	394072.30
58	86652.96	28096.98	474599.70	210.74	18970.06	374906.50
59	85990.27	27345.91	446502.80	215.03	18759.32	355936.50
60	85300.83	26605.00	419156.80	226.13	18544.29	337177.10
61	84561.59	25867.23	392551.90	230.12	18318.16	318632.80
62	83794.57	25139.66	366684.60	239.25	18088.04	300314.70
63	82981.46	24416.96	341545.00	245.06	17848.78	282226.70
64	82132.29	23702.34	317128.00	263.05	17603.72	264377.90
65	81202.92	22983.48	293425.70	275.94	17340.68	246774.10
66	80208.88	22265.55	270442.20	288.32	17064.74	229433.50
67	79149.87	21549.04	248176.60	303.39	16776.42	212368.70
68	78013.68	20831.25	226627.60	417.14	16473.03	195592.30
69	76420.86	20013.52	205796.30	437.72	16055.89	179119.30
70	74716.66	19190.92	185782.80	447.31	15618.17	163063.40
71	72940.97	18374.55	166591.90	472.29	15170.86	147445.20
72	71029.37	17548.90	148217.40	488.57	14698.57	132274.40
73	69013.10	16722.85	130668.50	505.69	14210.00	117575.80
74	66885.26	15895.57	113945.60	527.29	13704.31	103365.80
75	64623.01	15062.59	98050.05	539.22	13177.01	89661.51
76	62264.23	14233.70	82987.46	543.42	12637.79	76484.50
77	59840.49	13416.56	68753.76	562.64	12094.37	63846.71
78	57281.77	12595.91	55337.19	1150.62	11531.73	51752.33
79	51946.55	11203.06	42741.29	1576.95	10381.11	40220.60
80	44491.12	9410.66	31538.23	1789.66	8804.15	29839.50
81	35864.20	7440.03	22127.58	1782.52	7014.50	21035.34
82	27103.17	5514.43	14687.55	1593.66	5231.97	14020.85
83	19116.83	3814.72	9173.12	1290.94	3638.32	8788.87
84	12520.65	2450.42	5358.40	950.33	2347.38	5150.56
85	7569.65	1452.97	2907.98	635.29	1397.05	2803.18

Actuarial table cohort 1949[cont'd].

x	lx	Dx	Nx	Cx	Mx	Rx
86	4195.04	789.74	1455.01	384.32	761.76	1406.13
87	2113.51	390.23	665.27	209.19	377.43	644.37
88	958.33	173.54	275.04	101.60	168.25	266.94
89	386.25	68.60	101.50	43.55	66.65	98.69
90	136.22	23.73	32.90	16.24	23.09	32.04
91	41.18	7.03	9.17	5.16	6.86	8.95
92	10.37	1.74	2.14	1.36	1.70	2.09
93	2.09	0.34	0.40	0.29	0.34	0.39
94	0.32	0.05	0.06	0.04	0.05	0.06
95	0.03	0.01	0.01	0.00	0.00	0.01
96	0.00	0.00	0.00	0.00	0.00	0.00

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