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Games of Social Influence

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Abstract

This paper considers a game played among players who seek to extract payoffs from a group of individuals subject to local interaction effects. We are interested in the relation between the network of social interaction and equilibrium actions and payoffs.

We start with an analysis of two economic examples – strategic advertising in the presence of word of mouth advertising and social non-competitive marketing – to bring out the simple point that changing network connections can increase as well as decrease equilibrium actions and payoffs. This leads to an investigation of general conditions on payoffs under which equilibrium actions and payoffs increase/decrease with an increase in density of connections. We also develop conditions under which a greater dispersion in network connections leads unambiguously to positive and negative effects on actions as well as payoffs.

JEL classification:

Keywords: Social interaction, seeding the network, word of mouth communication, diffusion strategy.

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1 Introduction

There are two groups of agents, $\mathcal{M}$ and $\mathcal{N}$. Members in group $\mathcal{M}$ each choose a strategy – which may be advertising, sending out free product samples or spreading health messages in a community– with a view to influencing individuals $\mathcal{N}$ to choose certain actions – such as the purchase of their product or the adoption of a particular safety measure. The behavior of individuals in group $\mathcal{N}$ is subject to local influence: for example a member of group $\mathcal{N}$ receives some information she shares it with her friends and colleagues. Group $\mathcal{N}$ members make decisions based on the strategies of group $\mathcal{M}$ members and the structure of social interaction. We study the relation between the structure of social interaction and equilibrium actions and payoffs of players in $\mathcal{M}$. In particular, we are interested in the following questions.

1. How do changes in the network of relations – adding links to some players or redistributing links among members of $\mathcal{N}$ – affect equilibrium strategies of players in group $\mathcal{M}$.

2. What is the relation between network structures and profits of players in $\mathcal{M}$?

3. How doe this relation depend on the nature of the game that players of group $\mathcal{M}$ are engaged in?

Social interaction among members of $\mathcal{N}$ is modeled as a network. Clearly, different dimensions of networks can potentially influence individual behavior. Here, we focus on the degree distribution of the network. There are many advantages to using a degree distribution. First, the degree distribution summarizes a large amount of information about the social interaction in a very simple and natural way. Second, it allows us to formalize ideas about adding links or redistributing links in the network. The notion of adding links to a network is formalized in terms of first order stochastic relations between the degree distributions, while the idea of redistributing links is formalized in terms of second order dominance relations. Third, with this approach we can study the effects of additional information about social structures in a natural way. At one extreme $\mathcal{M}$ members know nothing more than the degree distribution of the network, while at the other extreme, the complete network architecture is common knowledge. In between there is a whole range of knowledge levels which reflect how much group $\mathcal{M}$ members know about the local network of each member of group $\mathcal{N}$. For example members of group $\mathcal{M}$ may know the degree distribution and the degree of each member of group $\mathcal{N}$. This possibility of varying information structures allows us to study the value of network information.

The other important dimension of the problem is the nature of relation between the strategic variables. In some contexts an increase in the action of a member of $\mathcal{M}$ may be beneficial for the other players, i.e. the actions generate positive externalities, while in other contexts the variables may be negatively related. Similarly, the strategic variables have effects on the marginal returns of different players and in some contexts the marginal returns are increasing in the choices of others, i.e., games of strategic complements, while in others are decreasing, i.e. games of strategic substitutes.

Our analysis develops sufficient conditions on payoff functions under which equilibrium strategies and payoffs increase or decrease with different types of changes in the network. We illustrate these findings via a detailed discussion of two economic examples: strategic advertising by firms and social marketing by non-governmental agencies.

Our model of strategic advertising draws on the pioneering work of Butters (1977) and Grossman and Shapiro (1984). Group $\mathcal{M}$ represents a set of firms which are choosing advertising strategies, while group $\mathcal{N}$ may be viewed as consumers who want to buy from the cheapest firm. Buyers are not aware about the existence of the product and firms advertise to inform consumers of their product. Consumers are connected in a network and they share information received from firms, via word of mouth communication. In this example of advertising actions generate negative externalities and the game is of strategic substitutes. Proposition 4.1 tells us that there exists a symmetric equilibrium in pure strategies in this game. Proposition 3.1 tells us that the intensity of advertising increases (decreases) with an increase in word of mouth communication if costs of advertising are high (low). Proposition 3.1 also says that equilibrium payoffs are increasing if costs of advertising are low. We then provide an example to show that equilibrium profits may fall as word of mouth communication increases if costs of advertising are high. Finally, we show that a redistribution of links in the mean preserving spread sense leads to an increase in actions and a fall in profits if costs of advertising are small.

The second example is a model of social marketing where non-governmental organizations wish to spread a health message in a community. This model draws on the theory of thresholds based behavior as in Granovetter (1978) and Lopez-Pintado (2004). An individual is more likely to be persuaded about a safety measure if her information is reinforced by the information of her friends and neighbors. Also, an individual is more likely to be persuaded about a particular safety measure if she receives messages from a number of different organizations. In contrast with the advertising example, in this application actions generate positive externalities and the game is of strategic complements. Propositions 4.2 and 4.3 tell us that the equilibrium actions as well as payoffs increase with an increase in the density of links. Similarly, Propositions 3.4 shows that equilibrium actions and profits are both increasing with a redistribution in the degree distribution.

A comparison of the results in the two examples highlights the main point we wish to bring out in this paper: that equilibrium behavior and profits of players are jointly shaped by the structure of interaction and the strategic nature of the variables. Moreover, our results develop conditions under which we can identify how changes in the network alter equilibrium actions and payoffs.

This paper is a contribution to the study of the role of social structure in economic activity.[1] This theory studies how individual behavior is shaped by the network of social connections, such as neighbors, friends, colleagues and research collaborators. Influential papers in this line of work include Ballester, Calvo-Armengol and Zenou (2006), Bala and Goyal (1998), Bramoulle and Kranton (2995), Ellison and Fudenberg (1993,1995), Galeotti (2005), Galeotti et al. (2006), Goyal and Moraga (2001), and Morris (2000). The existence of substantial social influence leads us to study ways in which external players such as firms and governments should incorporate social structure in the design of optimal strategies. In our earlier paper, Galeotti and Goyal (2007), we studied the case of a single outside player; in the present paper we extend the model to examine how social structure shapes strategic interaction among external players. This extension shows that the ideas we introduced in the earlier paper – such as ordering networks in terms of degree distributions and the central role of the relation of marginal returns and degrees – are also important in understanding play and payoffs in games of social influence. Our analysis of such games yields new insights with regard to the effects of networks: for instance, in our earlier paper, Galeotti and Goyal (2007) we showed that with a monopoly firm profits always increase with an increase in word of mouth communication; by contrast, in the present paper we find that firm profits may increase or decrease, and that this depends on the costs of advertising.

The questions we address bear some similarity to the issues discussed in Ballester, Calvo-Armengol and Zenou (2005) and Calvo-Armengol and Barreda (2006). They study criminal activity with local spillovers. In their framework individuals located in a network choose actions which affect the payoffs of other individuals within the network. They examine the question which individuals should be omitted from the network if the objective is to minimize crime. Our paper differs in two ways, we consider a different type of strategy – investments in influence – and we allow for many external players.

Section 2 presents the general model. Section 3 studies two specific models. Section 4 provides the general analysis, while section 5 concludes. The appendix contains technical details of some of the proofs.

[1] The important role of friends, neighbors and colleagues in shaping individual choices has been brought out in a number of empirical studies over the years; influential works include Coleman (1966), Conley and Udry (2004), Foster and Rosenzweig (1995), Feick and Price (1987).

2 Model

We study a class of games in which there is a finite group of players $\mathcal{M} = \{1, 2, ..., m\}$, with $m > 1$, who exert costly effort with a view to getting a group of individuals to choose an action.[2] There is a unit measure of individuals $\mathcal{N} = [0, 1]$ and their behavior is influenced by *social interaction*. In particular, individuals are located in a social network and in principle the structure of the network can be complex and take on a variety of forms. Players in $\mathcal{M}$ have limited knowledge of this network: they only knows the proportion of individuals having different levels of social interaction.

For an individual $i \in \mathcal{N}$, the level of social interaction is parameterized by a number k, where k is termed the degree. We will suppose that each individual draws k others with probability $P(k) \geq 0$, $k \in \{1, 2, ..., \bar{k}\} = O$ and $\sum_{k \in O} P(k) = 1$. She uses an (atomless) uniform distribution on the unit interval to pick her sample. So, if she has a k sized sample, she makes k draws, and each draw is independent. Now suppose that the draw of the sample size is independent across individuals. We can then say that there is fraction $P(k)$ of individuals who choose a k sized sample. We will refer to P as the degree distribution. Define $\hat{k} = \sum_{k \in O} P(k)k$ as the average degree of social interaction.

Given that the only information that players in $\mathcal{M}$ posses on the network is the degree distribution, each player $i \in \mathcal{M}$ takes an action $x_i \in \mathcal{X} = [0, 1]$. A strategy profile is then $\mathbf{x} = \{x_1, x_2, ..., x_M\}$. As is usual, $\mathbf{x_{-i}} = \{x_1, ...x_{i-1}, x_{i+1}, x_M\}$ denotes the strategies of all players other than i.

Given a strategy profile $\mathbf{x}$, the payoffs to any player $i \in \mathcal{M}$ can be expressed as the sum of profits that she gets from each individual in the network. Let profits from an individual of degree $k \in O$ be given by $\phi_k(x_i, \mathbf{x_{-i}})$, where $\phi_k(.) : \mathcal{X}^M \to R$. We assume that $\phi_k(., .)$ is a twice continuously differentiable function of all strategies, and that it is increasing and concave in own action. i.e. $\frac{\partial \phi_k(x_i, \mathbf{x}_{-i})}{\partial x_i} \geq 0$ and $\frac{\partial^2 \phi_k(x_i, \mathbf{x}_{-i})}{\partial x_i^2} \leq 0$.

The expected net payoffs to player i given a degree distribution P and a strategy profile $(x_i, \mathbf{x}_{-i})$ are:

$$\Pi_i(x_i, \mathbf{x}_{-i}|P) = \sum_{k \in O} P(k)\phi_k(x_i, \mathbf{x}_{-i}) - C(\alpha, x_i), \tag{1}$$

where $C(.) : \mathcal{X} \to R$ is the cost of effort and we assume that it is increasing and convex in x_i. The parameter $\alpha \geq 0$ indicates the efficiency in generating efforts and we will assume that the costs of effort and the marginal costs are strictly increasing in α.

[2] The analysis for the case in which $m = 1$ can be found in our companion paper Galeotti and Goyal (2007).

The payoffs that player $i \in \mathcal{M}$ earns from choosing a strategy x_i depend on the actions of other players in $\mathcal{M}$ and on the way these actions are mediated via the relationships between individuals in the network. Different assumptions pertaining to such effects define different games and it is useful to state the conditions briefly here.

Definition 2.1 *The game exhibits strategic complements if for all $k \in O$, $x_i > x_i^{'}$ and $\mathbf{x}_{-i} \geq \mathbf{x}_{-i}^{'}$: $\phi_k(x_i, \mathbf{x}_{-i}) - \phi_k(x_i^{'}, \mathbf{x}_{-i}) \geq \phi_k(x_i, \mathbf{x}_{-i}^{'}) - \phi_k(x_i^{'}, \mathbf{x}_{-i}^{'})$. Analogously, the game exhibits strategic substitutes if for all $k \in O$, $x_i > x_i^{'}$ and $\mathbf{x}_{-i} \geq \mathbf{x}_{-i}^{'}$: $\phi_k(x_i, \mathbf{x}_{-i}) - \phi_k(x_i^{'}, \mathbf{x}_{-i}) \leq \phi_k(x_i, \mathbf{x}_{-i}^{'}) - \phi_k(x_i^{'}, \mathbf{x}_{-i}^{'})$.*

Definition 2.2 *The game exhibits positive externalities if for all $k \in O$ and $(x_i, \mathbf{x}_{-i}) \geq (x_i, \mathbf{x'}_{-i})$: $\phi_k(x_i, \mathbf{x}_{-i}) \geq \phi_k(x_i, \mathbf{x'}_{-i})$. Analogously, the game exhibits negative externalities if for all $k \in O$ and $(x_i, \mathbf{x}_{-i}) \geq (x_i, \mathbf{x'}_{-i})$: $\phi_k(x_i, \mathbf{x}_{-i}) \leq \phi_k(x_i, \mathbf{x'}_{-i})$.*

Definition 2.3 *A game exhibits increasing marginal returns in degree if for all $x_i > x_i^{'}$, $\mathbf{x}_{-i}$ and $k < \bar{k} - 1$: $\phi_{k+1}(x_i, \mathbf{x}_{-i}) - \phi_{k+1}(x_i^{'}, \mathbf{x}_{-i}) \geq \phi_k(x_i, \mathbf{x}_{-i}) - \phi_k(x_i^{'}, \mathbf{x}_{-i})$. Analogously, a game exhibits decreasing marginal returns in degree if for all $x_i > x_i^{'}$, $\mathbf{x}_{-i}$ and $k < \bar{k} - 1$: $\phi_{k+1}(x_i, \mathbf{x}_{-i}) - \phi_{k+1}(x_i^{'}, \mathbf{x}_{-i}) \leq \phi_k(x_i, \mathbf{x}_{-i}) - \phi_k(x_i^{'}, \mathbf{x}_{-i})$.*

A (pure-strategy) Nash equilibrium is a strategy profile $\mathbf{x}^* = (x_1^*, ..., x_M^*)$ such that for every $i \in \mathcal{M}$,

$$\Pi_i(x_i^*, \mathbf{x}_{-i}^*|P) \geq \Pi_i(x_i, \mathbf{x}_{-i}^*|P), \forall x_i \in \mathcal{X} \tag{2}$$

The different conditions on payoffs are said to hold strictly if the inequalities are strict.

Finally, with regard the structure of the network, we note that there are two issues we will specially be concerned with, the addition of links and greater dispersion in the links. In our context, where networks are measured in terms of degree distributions, it is natural to view the addition of links in terms of first order stochastic dominance (FOSD) shifts in the degree distribution. Similarly, we can naturally view redistributions of links – say from regular graphs to unequal networks – in terms of second order stochastic dominance shifts and mean preserving spreads (MPS) of the degree distribution.

3 Economic Examples

In this section we develop some of the main ideas of our paper via a detailed discussion of equilibrium strategies in two examples. The first is an example of strategic advertising in the presence of word-of-mouth communication, while the second is a model of social marketing.

3.1 Strategic Advertising with word-of-mouth communication

Consider two firms advertising to a group of consumers, who share product information among themeselves.[3] The set of firms is $\mathcal{M} = \{1, 2\}$ and they produce a homogeneous good at zero marginal cost. Each firm sells the object at price 1. The set of buyers is $\mathcal{N} = [0, 1]$; each buyer has inelastic demand and her reservation value for the object is $v = 1$. Suppose that potential buyers are not aware of the existence of the product and that firms undertake costly informative advertising.

A firm $i \in \mathcal{M}$ chooses $x_i \in [0, 1]$, which specifies the fraction of buyers in $\mathcal{N}$ who receive advertisement directly from firm i. This comes at a cost $\alpha x^2/2$, $\alpha > 0$. A buyer may also receive information via word of mouth communication. In this example, a buyer with degree k contacts k other consumers, from whom she obtains the information about available products in the market, if any. If a consumer is only informed about firm i's product, then he buys the object from that firm. If he is informed about the products of both firms then he picks one firm at random.

It then follows that for a given strategy profile $\mathbf{x}$, the expected gross profits to firm i from a k degree buyer are:

$$\begin{aligned}\phi_k(x_i, x_j) &= \left[1 - (1 - x_i)^{k+1}\right] \left[(1 - x_j)^{k+1} + \frac{1}{2}(1 - (1 - x_j)^{k+1})\right] \\ &= \frac{1}{2}\left[1 - (1 - x_i)^{k+1}\right]\left[1 + (1 - x_j)^{k+1}\right].\end{aligned}$$

This is simply the probability that a consumer with k friends becomes either aware only of firm i or of both firms. It is then easy to verify that this game satisfies strict strategic substitutes and strict negative externalities. Also note that if $x = x_1 = x_2$ then $\phi_k(x, x)$ is increasing in k.

For a given distribution of connections P, we can then write the expected profits of firm i under strategy $\mathbf{x}$ as:

$$\Pi_i(x_i, x_j|P) = \sum_{k \in O} \frac{P(k)}{2}[1 - (1 - x_i)^{k+1}][1 + (1 - x_j)^{k+1}] - \frac{\alpha x_i^2}{2},$$

An interior symmetric equilibrium x_P^* solves:

$$\frac{\partial \Pi_i(x_P^*, x_P^*|P)}{\partial x_i} = \sum_{k \in O} \frac{P(k)}{2}(k + 1)(1 - x_P^*)^k[1 + (1 - x_P^*)^{k+1}] - \alpha x_P^* = 0, \quad (3)$$

[3]This example builds on the model of advertising introduced in Galeotti and Goyal (2007). That paper looked at the case of a single firm, whereas the interest in this paper is in the case of 2 or more firms.

and it is easy to see that an interior symmetric equilibrium always exists.

We now examine the ways in which equilibrium advertising strategies and equilibrium profits vary with the level and dispersion of word of mouth communication. We start with the effects of an increase in the level of word of mouth communication.

Proposition 3.1 *Suppose P' FOSD P. There exists $\underline{\alpha}$ and $\bar{\alpha}$ such that if $\alpha > \bar{\alpha}$ then $x^*_{P'} \geq x^*_P$, while if $\alpha < \underline{\alpha}$ then $x^*_{P'} \leq x^*_P$. Furthermore, if $\alpha < \underline{\alpha}$ then $\Pi(x^*_{P'}, x^*_{P'}|P') \geq \Pi(x^*_P, x^*_P|P)$*

Proof: The derivative of the marginal returns with respect to degree in a symmetric equilibrium x^*_P is:

$$\frac{\partial^2 \phi_k(x^*_P, x^*_P)}{\partial x_i \partial k} = \frac{(1-x^*_P)^k}{2}\left[1 + (1-x^*_P)^{k+1} + (k+1)\left[1 + 2(1-x^*_P)^{k+1}\right]\ln(1-x^*_P)\right] \tag{4}$$

It is easy to see that for sufficiently low x^*_P the game exhibits increasing marginal returns in degree, while for sufficiently high x^*_P it exhibits decreasing marginal returns in degree. Also note that, by investigation of the equilibrium condition 3, x^*_P is decreasing in α and that x^*_P goes to zero when $\alpha \to \infty$, while x^*_P goes to 1 when $\alpha \to 0$. The first part of the proposition then follows from Proposition 4.2 and Remark 4.2.

Note that this game satisfies the property of negative externalities, $\phi_k(.,.)$ is increasing in k and for all $\alpha < \underline{\alpha}$ the game exhibits decreasing marginal returns at x^*_P. The result on payoffs then follows from Proposition 4.3.

■

When the costs of advertising are low, firms advertise with high intensity. In this case an increase in the level of word of mouth communication decreases the marginal returns of advertising because buyers are very likely to become aware of products from their cohorts. This observation and the fact that firms' advertisements are strategic substitutes imply that in the new equilibrium firms advertise less. The reverse holds for high costs of advertising.

First note that for a given advertising level an increase in network density increases the demand of each firm, so that firms' profits increase. This is true regardless of the level of the costs of advertising. However, an addition of links changes the equilibrium advertising level which in turn alters the intensity of firms' competition. Proposition 3.1 tells us that when advertising is sufficiently cheap, FOSD shifts lower equilibrium efforts; the negative externality property implies that a firm can increase profits by retaining old equilibrium advertising levels. So this must also be true in equilibrium. Matters are more complicated when costs of advertising are high. Proposition 3.1 tells us that for large costs of advertising a FOSD shift in connections raises equilibrium advertising. This negative externality property pushes toward lower profits, while the FOSD shift in degrees pushes toward higher profits. The following example clarifies this issue.

Example: Suppose that $P(2) = 0.2$, $P(3) \in [0, 0.8]$ and $P(1) = 1 - P(2) - P(3)$ and that $C(\alpha, x) = \alpha x$. In Figure 1 we plot the expected equilibrium profits of a firm as function of $P(3)$ for a high level of α and a low level of α, respectively. Note that an increase in $P(3)$ corresponds to a FOSD shifts in the degree distribution. Figure 1 shows that for low costs of advertising profits increase with FOSD shifts. However, for high costs of advertising equilibrium firms' profits fall with an increase in the level of work of mouth communication.

We now turn to the effects of greater dispersion in word of mouth communication. It turns out that the equilibrium response depends on whether the marginal returns to advertisements are concave or convex in the degrees. In the former case, greater dispersion implies a fall in expected marginal returns and under the concavity of returns with respect to own advertisement as well as the strategic substitutability between firms' advertising this means that firms advertise less. The converse holds if marginal returns are convex in degrees. Furthermore, for low costs of advertising also the returns from an individual are concave in the degree, so that greater dispersion decreases equilibrium profits. These ideas are summarized in the next proposition.

Proposition 3.2 *Suppose P' is a MPS of P. There exists $\underline{\underline{\alpha}}$ and $\bar{\bar{\alpha}}$ such that if $\alpha > \bar{\bar{\alpha}}$ then $x^*_{P'} \leq x^*_P$, while if $\alpha < \underline{\underline{\alpha}}$ then $x^*_{P'} > x^*_P$. Furthermore, if $\alpha < \underline{\underline{\alpha}}$ then $\Pi(x^*_{P'}, \mathbf{x}^*_{-i,P'}|P') \leq \Pi(x^*_P, \mathbf{x}^*_{-i,P}|P)$.*

Proof: Note that:

$$\frac{\partial^3 \phi_k(x^*_P, x^*_P)}{\partial x_i \partial k^2} = (1-x^*_P)^k \ln(1-x^*_P)[2+4(1-x^*_P)^{k+1}+(k+1)(1+4(1-x^*_P)^{k+1}) \ln(1-x^*_P)]. \tag{5}$$

For sufficiently low x^*_P the marginal returns are concave in k, while they are convex for sufficiently large x^*_P. The change in x_P then follows by noting that x^*_P is decreasing in α and invoking Proposition 4.4.
Note that at equilibrium the returns from a consumer are concave in degree. Further, the game satisfies negative externalities and when $\alpha < \underline{\underline{\alpha}}$ the marginal returns are convex in k at equilibrium. The prove then follows from Proposition 4.5.

■

3.2 Social marketing

Consider $M = 2$ non-governmental organizations (NGOs) trying to spread a health message concerning AIDS in a community. On the one hand, an individual is more likely to be persuaded about a particular safety measure if her information is reinforced by the information of her friends and neighbors. On the other hand, messages sent by different organizations are complements so that an individual is more likely to be persuaded about a particular safety measure if she receives different messages from different NGOs. NGOs take this reinforcement mechanism into account when they choose their social marketing activity. We formalize this example using a threshold model, which we now introduce.

Each organization chooses a level of social marketing $x \in [0, 1]$ which specifies the fraction of individuals in $\mathcal{N}$ directly exposed to the social message. An individual observes messages directly received from NGOs and messages received by his neighbors. If an individual observes s messages from one NGO and l messages from the other organization he adopts a particular safety measure with some probability $\psi(s, l)$.

Given our assumptions, the expected returns to organization i from a k degree individual under a strategy profile (x_1, x_2) are

$$\phi_k(x_1, x_2) = \sum_{s \in O} \binom{k+1}{s} x_1^s (1 - x_1^s)^{k+1-s} \left[\sum_{l \in O} \binom{k+1}{l} x_2^l (1 - x_2)^{k+1-l} \psi(s, l) \right] \quad (6)$$

The properties of these payoffs ultimately depend on the properties of the function $\psi(k, s)$. For simplicity, here we focus on the case in which $\psi(s, l) = sl/(\bar{k} + 1)^2$. That is, the probability that an individual is persuaded to adopt a certain behavior increases with the number of social messages he observes, but it is independent of his neighbor's size. In this case, we can rewrite expression (6) as:

$$\phi_k(x_1, x_2) = \frac{x_1 x_2 (k+1)^2}{(\bar{k}+1)^2}. \quad (7)$$

It is easy to verify that this payoff satisfies the properties of strict strategic complements, strict positive externalities and increasing marginal returns in degree. Using expression (7), the expected payoffs to agency i given a network distribution P, and strategy profile (x_i, x_j), are:

$$\Pi_i(x_1, x_2 | P) = \sum_{k \in O} P(k) \frac{x_1 x_2 (k+1)^2}{(\bar{k}+1)^2} - \alpha x_i.$$

An interior symmetric equilibrium $x_P^* \in (0, 1)$ solves:

$$\frac{\partial \Pi_i(x_P^*, x_P^* | P)}{\partial x_i} = \sum_{k \in O} P(k) \frac{x_P^* (k+1)^2}{(\bar{k}+1)^2} - \alpha = 0, \quad (8)$$

and, denoting by σ^2 the variance of P, we can rewrite this equilibrium condition as:

$$\frac{x_P^*(\sigma^2 + (\hat{k}+1)^2}{(\bar{k}+1)^2} - \alpha = 0. \tag{9}$$

It is now easy to see that an interior symmetric equilibrium exists if and only if $\alpha \in (0, \frac{\sigma^2+(\hat{k}+1)^2}{(\bar{k}+1)^2})$, which is assumed hereafter.

The following proposition considers the effects of adding links in the network on the equilibrium level of social marketing and NGOs' equilibrium profits.

Proposition 3.3 *Suppose that P' FOSD P. Then $x_{P'}^* \geq x_P^*$ and $\Pi(x_{P'}^*, x_{P'}^*|P') \geq \Pi(x_P^*, x_P^*|P)$.*

Proof: First observe that marginal returns are increasing in degree. The result on the effects of FOSD degree shifts on equilibrium actions then follows from Proposition 4.2. Note that the returns are increasing in degree and this is a game of positive externalities. Then Proposition 4.3 implies that profits go up with a first order shift in the degree distribution. ■

To get some intuition behind these results, note that the marginal returns are increasing in degree and so a FOSD shift in degrees leads to an increase in expected marginal returns and consequently to an increase in optimal actions. An increase in optimal actions of a player lead to higher best responses of others, due to strategic complements property of payoffs. The increase in equilibrium actions now follows naturally. This positive effect on actions taken along with the positive externalites property of payoffs immediately yields an increase in equilibrium profits.

We conclude by considering the effects of MPS shifts in the degree distribution.

Proposition 3.4 *Suppose P' is a MPS of P. Then $x_{P'}^* \geq x_P^*$ and $\Pi(x_{P'}^*, x_{P'}^*|P') \geq \Pi(x_P^*, x_P^*|P)$*

This result follows as a corollary of Proposition 4.5. Note that the marginal returns to send health messages is convex in degree. Thus, a MPS shift in the degree distribution increases the marginal returns from individual marketing activity. Similarly, the returns from sending health messages are convex in degree, so that in more dispersed networks profits are higher.

4 General Results

We now turn to the general model introduced in Section 2. We start by showing existence of equilibrium.

Proposition 4.1 *Suppose expected payoffs are given by (1). Assume that the expected payoffs are jointly continuous in players' strategies and concave in own strategy. Then there exists a symmetric equilibrium in pure strategies. Furthermore, if the game satisfies strict strategic substitutes then this equilibrium is unique.*

Proof: The existence of a pure strategy equilibrium follows from the Debreu-Fan-Glicksberg Theorem. The existence of a symmetric equilibrium follows from the symmetry in payoffs and actions sets; for completeness we provide the details in the appendix.

Finally, suppose that x and x' are two symmetric strategy profiles and that for each player $i \in \mathcal{M}$, $x_i' > x_i$. Suppose without loss of generality that $x_i > 0$. Equilibrium implies that expected marginal returns are exactly equal to marginal costs. Concavity in own strategy and strict strategic substitutes then implies that at x' the expected marginal returns are smaller than the marginal costs, which contradicts the hypothesis that x' is an equilibrium.

■

Remark 4.1 *We observe that concavity of expected payoffs follows from the concavity of ϕ and the convexity of $C(\alpha, x)$.*

In what follows we focus on interior symmetric equilibria.[4] An interior symmetric equilibrium is a strategy profile $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, ..., x_n^*)$ which solves:

$$\frac{\partial \Pi_i(x^*, \mathbf{x}_{-i}^*)}{\partial x_i} = \sum_{k \in O}^{\bar{k}} P(k) \frac{\partial \phi_k(x^*, \mathbf{x}_{-i}^*)}{\partial x_i} - \frac{\partial C(\alpha, x^*)}{\partial x_i} = 0, \forall i \in \mathcal{M}. \quad (10)$$

We now examine the effects of an increase in the level of social interaction on the equilibrium strategy. A FOSD shift in the degree distribution implies that the individuals located in the network are (on average) more connected. Intuitively, if marginal returns from an individual are increasing in the degree, then players would complement an increase of social interaction with higher effort. In contrast, if the marginal returns are decreasing in degree, players exert less effort when the level of social interaction increases.

The next proposition summarizes our analysis of FOSD shifts in degree distributions.

Proposition 4.2 *Assume that the game exhibits either strategic complements or strategic substitutes. Consider two degree distributions P and P' and suppose that P' FOSD P. For every symmetric equilibrium x_P^* under P there exists a symmetric equilibrium $x_{P'}^*$ under P' such that $x_{P'}^* \geq x_P^*$ ($x_{P'}^* \leq x_P^*$) if the game exhibits increasing (decreasing) marginal returns in degree.*

[4] Sufficient conditions for the existence of an interior symmetric equilibrium are $\frac{\partial \phi_k(0,0,...,0)}{\partial x_i} > \frac{\partial C(\alpha,0)}{x_i}$ and $\frac{\partial \phi_k(1,1,...,1)}{\partial x_i} < \frac{\partial C(\alpha,1)}{\partial x_i}$, for all $k \in O$.

Proof: Suppose the game exhibits increasing marginal returns in degree. We first prove the proposition for games of strategic complements. Start with x^*_P, since P' FOSD P and the game exhibits increasing marginal returns in degree it follows that for each $i \in \mathcal{M}$:

$$\sum_{k \in O} P'(k) \frac{\partial \phi_k(x^*_P, \mathbf{x}^*_{-i,P})}{\partial x_i} - \frac{\partial C(\alpha, x^*_P)}{\partial x_i} \geq 0.$$

If this expression equals 0, then x^*_P is an equilibrium and the proof follows. Suppose it is strictly positive and let each player $i \in \mathcal{M}$ play her highest best response, say x^1; by concavity of gross payoffs and strict convexity of costs, $x^1 \geq x^*$. If $\mathbf{x}^1$ is an equilibrium the proof follows. If it is not an equilibrium, iterate on the best response process. Due to strategic complements and concavity of gross payoffs in own action, the best response for each player is $x^2 \geq x^1$. Since $\mathcal{X}$ is compact, this process converges and the limit is a symmetric equilibrium with the desired property.

The proof for games with strategic substitutes is as follows. Suppose that $x^*_{P'} < x^*_P$. Since x^*_P is an interior equilibrium it must be the case that

$$\frac{\partial \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P)}{\partial x_i} = \sum_{k \in O} P(k) \frac{\partial \phi_k(x^*_P, \mathbf{x}^*_{-i,P})}{\partial x_i} - \frac{\partial C(\alpha, x^*_P)}{\partial x_i} = 0. \tag{11}$$

Note that:

$$\frac{\partial \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P)}{\partial x_i} \leq \frac{\partial \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P')}{\partial x_i} \tag{12}$$

$$< \frac{\partial \Pi_i(x^*_{P'}, \mathbf{x}^*_{-i,P'}|P')}{\partial x_i}, \tag{13}$$

where the first inequality is due to increasing marginal returns in degree and P' FOSD P, while the second inequality is due to strategic substitutes, concavity of $\phi_k(.,.)$ in own action, and strict convexity of the cost function. Thus, $x^*_{P'}$ cannot be an equilibrium.

The proof for games with decreasing marginal returns in degree is analogous and is omitted.

■

We note that the proof of Proposition 4.2 only exploits the property of increasing marginal returns in degree locally (at equilibrium).

Remark 4.2 *If the game satisfies increasing (decreasing) marginal returns in degree at equilibrium x^*_P then Proposition 4.2 holds.*

What are the effects of additional links on expected profits? An increase in the level of social interaction has two effects that we need to keep track of. *First*, we need to consider how FOSD shifts change profits, ceteris paribus. This depends on whether the returns from an individual are increasing or decreasing in the degree. In the former case FOSD shifts in the distribution of connections increase profits, while in the latter case profits go down. *Secondly*, we need to consider how the equilibrium response to FOSD shifts affects profits. Proposition 4.2 illustrates the impact of FOSD shifts on the equilibrium strategy. It tells us that players increase (decrease) effort if the marginal returns are increasing (decreasing) in degree. The impact of such changes on expected profits depend on whether the game satisfies positive or negative externalities.

Proposition 4.3 *Assume that the game is either of strategic complements or strategic substitutes. Consider two degree distributions P and P' and suppose that P' FOSD P.*

I. Suppose $\phi_{k+1}(x_i, \mathbf{x}_{-i}) \geq \phi_k(x_i, \mathbf{x}_{-i})$ for all $\mathbf{x}$ and k. If the game exhibits positive externalities and increasing marginal returns in degree then there exists x^*_P and $x^*_{P'}$ such that $\Pi(x^*_{P'}, \mathbf{x}^*_{-i,P'}|P') \geq \Pi(x^*_P, \mathbf{x}^*_{-i,P}|P)$. The same holds if the game exhibits negative externalities and decreasing marginal returns in degree.

II. Suppose $\phi_{k+1}(x_i, \mathbf{x}_{-i}) \leq \phi_k(x_i, \mathbf{x}_{-i})$ for all $\mathbf{x}$ and k. If the game exhibits negative externalities and increasing marginal returns in degree then there exists x^*_P and $x^*_{P'}$ such that $\Pi(x^*_{P'}, \mathbf{x}^*_{-i,P'}|P') \leq \Pi(x^*_P, \mathbf{x}^*_{-i,P}|P)$. The same holds if the game exhibits positive externalities and decreasing marginal returns in degree.

Proof: We prove Part I; the proof for Part II is analogous and omitted. First, assume increasing marginal returns in degree, positive externalities and $\phi_{k+1}(x, \mathbf{x}_{-i}) \geq \phi_k(x, \mathbf{x}_{-i})$. Since the game exhibits increasing marginal returns in degree, Proposition 4.2 implies that there exists $x^*_{P'} \geq x^*_P$. Then,

$$\begin{aligned}
\Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P) &\leq \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P') \\
&\leq \Pi_i(x^*_P, \mathbf{x}^*_{-i,P'}|P') \\
&\leq \Pi_i(x^*_{P'}, \mathbf{x}^*_{-i,P'}|P'),
\end{aligned}$$

where the first inequality follows because $\phi_{k+1}(x^*, \mathbf{x}^*_{-i}) \geq \phi_k(x^*, \mathbf{x}^*_{-i})$ for all k and because P' FOSD P. The second inequality follows because $x^*_{P'} \geq x^*_P$ and positive externalities, while the third inequality follows from optimality of equilibrium strategies.

Second, note that if we assume decreasing marginal returns in degree, negative externalities and $\phi_{k+1}(x, \mathbf{x}_{-i}) \geq \phi_k(x, \mathbf{x}_{-i})$, the inequalities above still hold and that completes the proof. ■

P' *first order stochastic dominate* P	**Positive Externalities**	**Negative Exteralities**
Decreasing Marginal Returns in Degree	Ambiguous	$\Pi^*_{P'} \geq \Pi^*_P$
Increasing Marginal Returns in Degree	$\Pi^*_{P'} \geq \Pi^*_P$	Ambiguous

Table 1: ϕ_k non-decreasing in k

Table 1 summarizes the first part of Proposition 4.3. By way of illustration, let us consider games of strategic substitutes and negative externalities (the strategic advertising game is an example). If returns from an individual are increasing in degree, greater social interaction makes a given strategy profile more effective in influencing network members. That is, FOSD shifts in the degree distribution increase expected payoffs, ceteris paribus. However, FOSD shifts also alter the equilibrium strategy. Suppose the game exhibits decreasing marginal returns in degree. Then Proposition 4.2 implies that the equilibrium strategy will entail lower efforts. Negative externalities then implies that the new equilibrium profits are higher. In this class of games profits will be higher in denser networks. In contrast, if the game exhibits increasing marginal returns in degree, a FOSD shift increases the equilibrium effort level, which, due to negative externalities, decreases profits. In this case the two effects go in opposite directions and whether profits increase or decrease will depend on the specific game. We note that in the strategic advertising example, the marginal returns are decreasing (increasing) in degree for low (high) values of α. It is for this reason that we are able to obtain clear cut results in the case of low costs of advertising, while the results in the case of high costs of advertising are ambiguous.

We now study the effects of redistributing links within a network by considering MPS shifts in the degree distribution. By inspection of the equilibrium condition (10) we note that the effect of such shifts depend on the curvature of the marginal returns from network members.

Proposition 4.4 *Assume that the game is either of strategic complements or strategic substitutes. Consider two degree distributions P and $P^{'}$ and suppose that $P^{'}$ is a MPS of P. If the marginal returns are concave (convex) in k, then for every symmetric equilibrium x^*_P under P there exists a symmetric equilibrium $x^*_{P'}$ under P' such that $x^*_{P'} \leq x^*_P$ ($x^*_P \leq x^*_{P'}$).*

Proof. Suppose that the marginal returns are concave in the degree. The proof for the case where marginal returns are convex in the degree is analogous and omitted. First, consider strategic complements. Start with x^*_P. Since P' is a MPS of P and since the marginal returns are concave in the degree, it follows that for all $i \in \mathcal{M}$:

$$\frac{\partial \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P')}{\partial x_i} \leq \frac{\partial \Pi_i(x^*_P, \mathbf{x}^*_{-i,P}|P)}{\partial x_i} = 0.$$

In the case of equality the proof follows. Suppose the inequality holds strictly. Let each player play her lowest best response, say x^1; by concavity of gross payoffs and strict convexity of costs, $x^1 < x_P^*$. If x^1 is an equilibrium the proof follows. Otherwise, note that the best response of each player, given x^1, is lower (due to strategic complements). We can then iterate the process and, since $\mathcal{X}$ is compact, this process converges and the limit is a symmetric equilibrium with the desired properties.

Second, consider games of strategic substitutes. Suppose that $x_{P'}^* > x_P^*$. Since x_P^* in an interior equilibrium it must be the case that:

$$\frac{\partial \Pi_i(x_P^*, \mathbf{x}_{-i,P}^*|P)}{\partial x_i} = 0.$$

Next, note that

$$\begin{aligned}\frac{\partial \Pi_i(x_P^*, \mathbf{x}_{-i,P}^*|P)}{\partial x_i} &\geq \frac{\partial \Pi_i(x_P^*, \mathbf{x}_{-i,P}^*|P')}{\partial x_i} \\ &> \frac{\partial \Pi_i(x_{P'}^*, \mathbf{x}_{-i,P'}^*|P')}{\partial x_i},\end{aligned}$$

where the first inequality follows because the marginal returns are concave in degree and P' is a MPS of P, while the second inequality follows from the hypothesis that $x_{P'}^* > x_P^*$, strategic substitutes, $\phi_k(.,.)$ is concave in own actions and $C(.,.)$ is strictly convex. This contradicts the hypothesis that $x_{P'}^* > x_P^*$ is an equilibrium. ■

The intuition underlying Proposition 4.4 is as follows: suppose that the game exhibits strategic complements. If the marginal returns from an individual are concave in degree, a MPS shift in the degree distribution decreases the expected marginal returns and therefore players exert less effort. In the case of strategic complements a fall in actions of others reinforces this pressure and compactness of strategy set guarantees the existence of a lower equilibrium. In the case of games with strategic substitutes, suppose that the equilibrium effort increases; concavity of gross payoffs in own action as well as falling marginal returns in degree and strategic substitutes imply that marginal returns are strictly lower. However, since the cost function is strictly convex, the marginal costs are strictly larger, which is inconsistent with equilibrium.

The next proposition concludes the analysis by showing the effects of greater dispersion on equilibrium profits

Proposition 4.5 *Assume that the game is either of strategic complements or strategic substitutes. Consider two degree distributions P and $P^{'}$ and suppose that P' is a MPS of P.*

- *I Suppose $\phi_k(x_i, \mathbf{x}_{-i}.)$ is concave in k for all $\mathbf{x}$. If marginal returns are concave in k and the game exhibits positive externalities then there exists x_P^* and $x_{P'}^*$ such that $\Pi(x_P^*, \mathbf{x}_{-i,P}^*|P) \geq \Pi(x_{P'}^*, \mathbf{x}_{-i,P'}^*|P')$. The same holds if marginal returns are convex in k and there are negative externalities.*

II Suppose $\phi_k(x_i, \mathbf{x}_i)$ is convex in k for all $\mathbf{x}$. If marginal returns are concave in k and the game exhibits negative externalities then there exists x_P^* and $x_{P'}^*$ such that $\Pi(x_P^*, \mathbf{x}_{-i,P}^*|P) \leq \Pi(x_{P'}^*, \mathbf{x}_{-i,P'}^*|P')$. The same holds if marginal returns are convex in k and there are positive externalities.

Proof: We prove Part I; the proof for Part II is analogous and omitted. First, suppose that marginal returns are concave in k and the game exhibits positive externalities. Since marginal returns are concave in k, Proposition 4.4 implies that there exists $x_{P'}^* \leq x_P^*$. Then,

$$\begin{aligned}\Pi_i(x_{P'}^*, \mathbf{x}_{-i,P'}^*|P') &\leq \Pi_i(x_{P'}^*, \mathbf{x}_{-i,P'}^*|P) \\ &\leq \Pi_i(x_{P'}^*, \mathbf{x}_{-i,P}^*|P) \\ &\leq \Pi_i(x_P^*, \mathbf{x}_{-i,P}^*|P),\end{aligned}$$

where the first inequality follows because $\phi_k(.,.)$ is concave in k and P' is a MPS of P, the second inequality follows because $x_{P'}^* \leq x_P^*$ and positive externalities. The third inequality follows from optimality of equilibrium strategy.

Second note that the same inequalities holds if marginal returns are convex in k and the game exhibits negative externalities.

■

5 Conclusion

This paper considers a game played among players who seek to extract payoffs from a group of individuals subject to local interaction effects. A well known example of such a game is strategic advertising by firms in the presence of word of mouth advertising among consumers. We are interested in issues such as the existence of equilibrium for given networks of interaction as well as the effects of changing networks on equilibrium actions and payoffs.

Networks are complicated objects and the first step is to develop a simple measure of ordering and classifying them; following the example of the recent literature in statistical physics, in this paper we focus on the distribution of connections, viz. the degree distribution. The analysis proceeds by looking at how equilibrium varies with changes in the degree distribution.

An analysis of economic examples suggests that changing network connections can increase as well as decrease equilibrium actions and payoffs. This led to an investigation of general conditions on payoffs under which equilibrium actions and payoffs increase/decrease with an increase in density of connections. We also developed conditions under which a greater dispersion in network connections lead to unambiguously positive and negative effects on actions as well as payoffs.

In this paper we have studied the simplest case where members of group $\mathcal{M}$ only know the degree distribution of the network. In future work, it will be interesting to examine the case where players know more about the network, and indeed to examine how valuable it is to learn about the network in a strategically competitive situation.

Appendix

Proof of existence of symmetric equilibrium: We now prove existence of symmetric equilibria. Consider player $i \in \mathcal{M}$ with strategy $x_i \in \mathcal{X}$ who faces a symmetric profile in which all $M-1$ players choose $x' \in \mathcal{M}$. Let $\mathbf{x'}_{-i}$ a $M-1$ vector in which each element is x'. Thus, we may define a payoff $\pi_i(x_i, \mathbf{x'}_{-i})$ of player i as follows

$$\pi(x_i, \mathbf{x'}_{-i}) = \sum_{k \in O} P(k)\phi_k(x_i, \mathbf{x'}_{-i}) - C(\alpha, x_i).$$

Since $\phi_k(.,.)$ is concave in own action for every k and $C(.,.)$ is strictly convex in own action, this function is strictly concave in x_i. Now define a best response correspondence $\rho_i : \mathcal{X} \to \mathcal{X}$ as follows

$$\rho_i(x') = \arg\max_{x_i \in \mathcal{X}} \pi_i(x_i, \mathbf{x'}_{-i}).$$

The compactness and non-emptiness of $\mathcal{X}$ and the joint continuity of $\pi_i(x_i, \mathbf{x'}_{-i})$ with respect to x_i and $\mathbf{x'}_{-i}$ implies that $\rho_i(.)$ is upper semi-continuous. On the other hand, the convexity and non-emptiness of $\mathcal{X}$ and the concavity of $\phi_i(x_i, \mathbf{x'}_{-i})$ with respect to x_i for every $\mathbf{x'}_{-i}$ imply that ρ_i is convex and non-empty for every $\mathbf{x'}_{-i}$, where $\mathbf{x'}_{-i}$ is defined as above. We can then invoke Kakutani's Fixed Point Theorem to conclude that there exists a fixed point x_i^* such that $x^* \in \rho_i(x^*)$.

■

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Figure 1. First order stochastic Dominance and Equilibrium Profits

P(1)=1-P(2)=P(3), P(2)=0.2 and P(3)=0…0.8