Appendix : Proof of Similarity properties

I. Proof of S1 properties

We prove that the advanced measure S1 satisfies all the similarity properties P1- P4.
for **P1** (reflexivity property): For 2 IT-2 FSs $\tilde{A}$ and $\tilde{B}$,
if $\tilde{A}$ = $\tilde{B}$ then

$$\overline{s}1 = 0 \tag{1}$$

because

$$\frac{1}{2n}\sum_{i=1}^{n}(\frac{|\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)|}{\overline{\mu}_{\tilde{A}}(x_i)+\overline{\mu}_{\tilde{A}}(x_i)}+\frac{|\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)|}{2-\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)} \tag{2}$$

and

$$\underline{s}1 = 0 \tag{3}$$

because

$$\frac{1}{2n}\sum_{i=1}^{n}(\frac{|\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)|}{\underline{\mu}_{\tilde{A}}(x_i)+\overline{\mu}_{\tilde{A}}(x_i)}+\frac{|\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)|}{2-\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)} \tag{4}$$

so $\underline{s}1 = \overline{s}1 = 0$ hence

$$S_1(\tilde{A},\tilde{B}) = 1-\frac{(\overline{s}1+\underline{s}1)}{2} = 1 \tag{5}$$

For **P2** property (symmetry), proof is trivial. We can prove that $S1(\tilde{A},\tilde{B}) = S1(\tilde{B},\tilde{A})$ due to absolute value properties.
For **P3** property (transivity), for 3 IT-2 FSs $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$,

$$if \tilde{A} < \tilde{B} < \tilde{C} \tag{6}$$

then

$$\overline{\mu}_{\tilde{A}}(x) < \overline{\mu}_{\tilde{B}}(x) < \overline{\mu}_{\tilde{C}}(x) \tag{7}$$

so

$$\overline{\mu}_{\tilde{A}}(x) - \overline{\mu}_{\tilde{B}}(x) < 0 \tag{8}$$

and

$$\overline{\mu}_{\tilde{C}}(x) - \overline{\mu}_{\tilde{B}}(x) > 0 \tag{9}$$

and

$$|\overline{\mu}_{\tilde{A}}(x) - \overline{\mu}_{\tilde{B}}(x)| \,/\, |(\overline{\mu}_{\tilde{A}}(x) + \overline{\mu}_{\tilde{B}}(x)| < 0 \tag{10}$$

and

$$|\overline{\mu}_{\tilde{C}}(x) - \overline{\mu}_{\tilde{B}}(x)| \,/\, |\overline{\mu}_{\tilde{C}}(x) + \overline{\mu}_{\tilde{B}}(x)| < 0 \tag{11}$$

...

$$\overline{s}1(\tilde{A},\tilde{B}) < 0 \tag{12}$$

$$\overline{s}1(\tilde{C},\tilde{B}) < 0 \tag{13}$$

Respectively for $\underline{s}1(\tilde{A},\tilde{B})$ and $\underline{s}1(\tilde{C},\tilde{B})$

$$S1(\tilde{A},\tilde{B}) = 1-\frac{(\overline{s}1+\underline{s}1)}{2} \tag{14}$$

So

$$S1(\tilde{A},\tilde{B}) > 0 \tag{15}$$

$$S1(\tilde{B},\tilde{C}) < 0 \tag{16}$$

so

$$S1(\tilde{A},\tilde{B}) > S1(\tilde{B},\tilde{C}) \tag{17}$$

For **P4 property** (overlapping) , if $\tilde{A}$ and $\tilde{B}$ are having common values, then the two sets are proportionally similar hence $S1(\tilde{A},\tilde{B})$ is > 0.

Proof of S2 properties

The measure S2 mentioned above fulfills the reflexivity, the transivity and the overlapping properties but it doesn't fulfill the symmetry property. This has been demonstrated through some numerical examples and mathematical definitions.
For **P1 property** (reflexivity): For 2 IT-2 FSs $\tilde{A}$ and $\tilde{B}$,
if $\tilde{A}$ = $\tilde{B}$ then

$$\overline{s}2 = 0 \tag{18}$$

because

$$\frac{1}{2n}\sum_{i=1}^{n}(\frac{|\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)|}{\overline{\mu}_{\tilde{A}}(x_i)+\overline{\mu}_{\tilde{A}}(x_i)}+\frac{|\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)|}{\overline{\mu}_{\tilde{A}}(x_i)-\overline{\mu}_{\tilde{A}}(x_i)} \tag{19}$$

and

$$\underline{s}2 = 0 \tag{20}$$

because

$$\frac{1}{2n}\sum_{i=1}^{n}(\frac{|\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)|}{\underline{\mu}_{\tilde{A}}(x_i)+\overline{\mu}_{\tilde{A}}(x_i)}+\frac{|\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)|}{\underline{\mu}_{\tilde{A}}(x_i)-\underline{\mu}_{\tilde{A}}(x_i)} \tag{21}$$

so $\underline{s}2 = \overline{s}2 = 0$

$$S2(\tilde{A},\tilde{B}) = 1-\frac{(\overline{s}2+\underline{s}2)}{2} = 1 \tag{22}$$

For **P2** property, because $\underline{\mu}_{\tilde{A}}(x_i) \neq \underline{\mu}_{\tilde{B}}(x_i)$ and $\overline{\mu}_{\tilde{A}}(x_i) \neq \overline{\mu}_{\tilde{B}}(x_i)$, the measure S2 doesn't satisfy the symmetry property $(S2(\tilde{A},\tilde{B}) \neq S2(\tilde{B},\tilde{A}))$
For **P3** property (transivity), for 3 IT-2 FSs $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$,

$$if \tilde{A} < \tilde{B} < \tilde{C} \tag{23}$$

then

$$\overline{\mu}_{\tilde{A}}(x) < \overline{\mu}_{\tilde{B}}(x) < \overline{\mu}_{\tilde{C}}(x) \tag{24}$$

so

$$\overline{\mu}_{\tilde{B}}(x) - \overline{\mu}_{\tilde{A}}(x) > 0 \tag{25}$$

and

$$\overline{\mu}_{\tilde{C}}(x) - \overline{\mu}_{\tilde{B}}(x) < 0 \tag{26}$$

and
...

$$\overline{s}2(\tilde{A},\tilde{B}) > 0 \tag{27}$$

$$\overline{s}2(\tilde{B},\tilde{C}) < 0 \tag{28}$$

Respectively for $\underline{s}2(\tilde{A},\tilde{B})$ and $\underline{s}2(\tilde{C},\tilde{B})$
because

$$S2(\tilde{A},\tilde{B}) = 1-\frac{(\overline{s}2+\underline{s}2)}{2} \tag{29}$$

so

$$S2(\tilde{A},\tilde{B}) > 0 \tag{30}$$

$$S2(\tilde{B},\tilde{C}) < 0 \tag{31}$$

So

$$S2(\tilde{A},\tilde{B}) > S2(\tilde{B},\tilde{C}) \tag{32}$$

As for S1 measure, the **P4** overlapping property is accomplished because $S2(\tilde{A},\tilde{B})$ is > 0.

PROOF OF S3 PROPERTIES

The **P1** reflexivity property is not satisfied. In fact, if $\tilde{A}$ and $\tilde{B}$ are equal then :

$$\overline{s}3 =$$

$$\frac{\sum_{i=1}^{n} min(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x_i)) * min(1-\overline{\mu}_{\tilde{A}}(x_i), 1-\overline{\mu}_{\tilde{A}}(x_i))}{\sum_{i=1}^{n} max(\overline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x_i)) * max(1-\overline{\mu}_{\tilde{A}}(x_i), 1-\overline{\mu}_{\tilde{A}}(x_i))} \tag{33}$$

so

$$\overline{s3} = \frac{\sum_{i=1}^{n} \overline{\mu}_{\tilde{A}}(x_i) * (1-\overline{\mu}_{\tilde{A}}(x_i))}{\sum_{i=1}^{n} \overline{\mu}_{\tilde{A}}(x_i) * (1-\overline{\mu}_{\tilde{A}}(x_i))} = 1 \tag{34}$$

Respectively for $\underline{s}3 = 1$.
so

$$S3(\tilde{A}, \tilde{B}) = 1 - \frac{1+1}{2} = 0 \neq 1 \tag{35}$$

So the similarity measure S3 doesn't satisfy the reflexivity because $S3(\tilde{A}, \tilde{B}) \neq 1$.
The proof of **P2** is trivial because min and max functions preserve the symmetry property. So, $\overline{s}3((\tilde{A}, \tilde{B})) = \overline{s}3((\tilde{B}, \tilde{A}))$ (respectively for $\underline{s}3((\tilde{A}, \tilde{B}))$). Hence $S3 = ((\tilde{A}, \tilde{B})) = S3 = ((\tilde{B}, \tilde{A}))$.

Through some experimental applications on Type-2 Fuzzy sets, we have noticed that the considered similarity measure fulfills the transivity and the overlapping properties.