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The role of the wealth distribution on output volatility

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The role of the wealth distribution on output volatility*

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Abstract: *We explore the link between wealth inequality and business cycle fluctuations in a two-sector neoclassical growth model with endogenous labor and heterogeneous agents. Assuming that wealth inequality is described by the distribution of shares of capital, we show that in the most plausible situations wealth equality is a stabilizing factor. In particular, when wealth is Pareto distributed and preferences generate non linear absolute risk tolerance indices, a rise in the Gini index may only be associated to a rise in volatility. When individual preferences are such that the individual absolute risk tolerance indices are linear, as with HARA utility, even a low level of taste heterogeneity ensures that a rise in inequality may not reduce volatility, and this independently of the wealth distribution. Finally, we note that such a clear result is at odd with the existing related literature.*

Keywords: *Wealth Inequality, Pareto distribution, Gini index, Elastic Labor Supply, Macroeconomic Volatility, Endogenous Equilibrium Business Cycles.*

Journal of Economic Literature Classification Numbers: D30, D50, D90, O41.

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1 Introduction

Is there a relationship between income or wealth inequality and business cycle fluctuations? Recent data concerning the Latin American and OECD economies as well as the East Asian "tigers" suggests a positive answer. In 1990, the Gini coefficient of the distribution of income was on average 59.5% for Brazil, Chile, Mexico and Venezuela, and respectively 34% for the OECD countries and 35.5% for Hong Kong, Korea, Taiwan and Singapore. At the same time, the former were subject to much greater fluctuations in their respective growth rates than were the latter: during the 80's, the standard deviation of the rate of output growth was on average 5.9% for the above mentioned Latin American countries, and respectively 2.7% for the OECD and 2.8% for the East Asian countries. Building on these data, Breen and García-Peñalosa [11] show the existence of a significant positive correlation between a country's volatility and income inequality.

In the present paper we explore the role of the distribution of wealth on macroeconomic volatility from a theoretical perspective. The analysis is based on the fact that instability easily occurs in perfectly competitive multi-sector growth model. We then take the view that the link between wealth inequality and volatility should be first understood in the absence of any distortion. The simplest neoclassical economy allowing for dynamic instability is the Uzawa two-sector model with a consumption good and an investment good. We slightly enrich the standard model by endogeneising the labor supply. In this model the level of wealth inequality is characterized by the distribution of shares of capital. The message of the present paper is that within this standard framework in most of the realistic situations wealth equality is a stabilizing factor. More precisely, wealth inequality generates macroeconomic volatility when agents are heterogenous in their preferences and these belong to the HARA class, i.e. individual absolute risk tolerance is linear, a result holding independently of the wealth distribution. The conclusion also holds when agents have preferences characterized by non linear absolute risk tolerance and individual wealth follows a Pareto distribution.

The sharp results obtained in this paper are at odd with the previous related literature. Truly, the role of income and wealth inequality on macroeconomic volatility has been until recently largely ignored by the theoretical literature. Furthermore, the studies that do treat this issue consider models with externalities, increasing returns and rigidities in which dynamic instability of the long run equilibrium, either deterministic or stochastic, easily occurs. In these models, the various forms of market imperfections introduce many degrees of freedom and prevent from getting clear-cut results. Indeed, in some cases inequality is shown to be a stabilizing factor, as in Herrendorf *et al.* [26] or Ghiglino and Sorger [21], while in others the effect has opposite sign. A good example is here Aghion *et al.* [1] where inequality in the form of unequal access to investment opportunities across agents results in output and investment volatility. Closer to the present analysis, is the paper by Ghiglino and Venditti [22] that also investigates the link between inequality and instability in a neoclassical model. They conclude that wealth inequality may have an effect on instability only when the coefficient of absolute risk tolerance is not linear and that the direction of the effect depends on the fourth derivative of the utility function. The value of these results is highly reduced by these restrictions. Indeed, the exclusion of HARA preferences is not an advisable feature of the model while the sign of the fourth derivative is difficult to assess empirically. [1]

In the present paper we show that the lack of sharp results in Ghiglino and Venditti [22] is in part due to their assumption of inelastic labor supply. This is not too surprising as we know that the inclusion of leisure in the choices of the agents strongly affect the stability properties of the equilibrium in neoclassical growth models. In the present paper we also exploit preference heterogeneity further. Indeed, if agents with different wealth have different preferences, and thus different attitudes toward risk, the degree of inequality has a stronger effect than with homogeneous preferences. This allows us to obtain a relationship even when preferences belong to the HARA class. Finally, we exploit the shape of the wealth distribution, particularly by assuming a Pareto wealth distribution in which inequality is characterized by the Gini index. These three differences allow us to provide the clear-cut result that wealth equality has a positive effect on macroeconomic stability in realistic scenarios.

To conclude, we should note that there is a renewal of interest in the

[1] See also Ghiglino [18], Ghiglino and Olszack-Duquenne [19] and Bosi and Seegmuller [10] for similar analysis based on particular specifications for technologies and/or preferences.

possibility that macroeconomic volatility could affect the distribution of wealth, an issue we do not consider in the paper. For instance, Caroli and García-Peñalosa [13] show that higher volatility increases income inequality if agents with different endowments have different attitudes towards risk,[2] while García-Peñalosa and Turnovsky [16] provide a similar conclusion through the effect of greater production uncertainty.[3]

The rest of the paper is organized as follows: The methodology and the main results are briefly summarized in Section 2 while the model is introduced in Section 3. Section 4 is concerned with the definition of the equilibrium and the analysis of the effect of wealth inequality on the steady state. Section 5 discusses the existence of endogenous business cycle fluctuations. The occurrence of output volatility is related to wealth inequality in Section 6. Section 7 focusses on the particular case of a Pareto distribution with homogeneous preferences characterized by non linear individual absolute risk tolerance indices, while Section 8 concerns a general wealth distribution in the case of HARA preferences. Section 9 shows the robustness of our main conclusions when inelastic labor is considered and Section 10 concludes the paper. All the proofs are gathered into a final Appendix.

2 Summary of the paper

Starting from the decentralized model with many agents, our methodology consists in aggregating heterogeneous preferences within a central planner utility function which depends on a set of welfare weights. As the second welfare theorem ensures that any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments, we solve the weighted central planner problem. The competitive equilibrium is then obtained for a set of welfare weights associated with optimal allocations that saturate the budget constraints of all the consumers. Then we show that the welfare weights are continuous functions of the initial conditions so that the local dynamic properties of the general equilibrium model with heterogeneous agents and those of the planners' problem with welfare weights fixed at their steady state value are identical.

Building on Bosi *et al.* [9], we provide conditions on the technologies

[2] See also Cecchi and García-Peñalosa [14].

[3] See also García-Peñalosa and Turnovsky [15].

and the social utility function, aggregating individual preferences along the steady state, for the existence of endogenous business cycles fluctuations either damped in the long run or persistent through period-2 cycles.[4] As initially shown by Benhabib and Nishimura [5], we need a capital intensive consumption good sector in order to allow some oscillations of the capital stock "to get through" the Rybczinsky theorem. But the properties of preferences also matter: first, fluctuations in the consumption levels along the equilibrium path require a large enough social elasticity of intertemporal substitution in consumption, i.e. a large enough social absolute risk tolerance with respect to consumption. Second, a low enough social elasticity of labor supply, i.e. a low enough social absolute risk tolerance with respect to labor, is necessary to prevent the agents from smoothing the fluctuations of their wage and capital incomes associated to the fluctuations of the capital stock.[5]

The next step is to relate the conditions for the existence of endogenous fluctuations to the degree of wealth inequality. The measure of inequality in our framework is based on the distribution of capital shares across agents. Wealth inequality is thus rising when a bilateral transfer between two agents consists in increasing the income of the agent who is initially richer than the other. We show that when the social absolute risk tolerance indices with respect to consumption and labor are non linear, a modification of the degree of inequality affects the individual and the aggregate steady states and thus affects the local stability properties of the equilibrium. The non linearity of the social indices can be obtained in two cases: either when the individual absolute risk tolerance indices are non linear, or when they are linear (as with HARA preferences) but agents have heterogeneous preferences.

We then give clear-cut conditions on the slopes of the social absolute risk tolerance indices with respect to consumption and labor in order to

[4]It is worth noticing that following Benhabib and Nishimura [6] a more standard definition of macroeconomic volatility based on stochastic oscillations could be considered through the concept of cyclic sets. Indeed introducing small stochastic shocks into a deterministic model characterized by periodic cycles generates cyclic sets.

[5]When the labor supply is highly elastic, fluctuations of the wage rate and the rental rate of capital may be compensated by large modifications of the labor supply. The fluctuations of income are thus smoothed and the cycles can be eliminated. On the contrary, when the labor supply is weakly elastic, fluctuations of the capital stock generate fluctuations of incomes and cycles become persistent.

get a positive correlation between the degree of wealth inequality and the occurrence of macroeconomic volatility. The basic intuition for this result is the following: when a bilateral transfer increasing the degree of wealth inequality is considered, the consumption and labor choices of the two agents who are affected by the transfer are modified in such a way that the aggregate steady state levels of consumption and labor are changed. This in turn modifies the steady state values of the social absolute risk tolerance indices with respect to consumption and labor. When these modifications are such that the former increases while the latter decreases, or equivalently the social elasticity of intertemporal substitution in consumption increases while the social elasticity of labor decreases, endogenous business cycle fluctuations occur. In other words, increasing the level of wealth inequalities modifies the attitude towards risk of the planner and leads to macroeconomic volatility.

The conclusions appear quite robust as we are able to prove the results in three leading cases. First, the positive relationship between inequality and volatility is obtained when individual wealth is distributed according to a Pareto distribution and agents have homogeneous preferences characterized by non linear individual absolute risk tolerance indices. Second, a similar result holds for general wealth distributions when agents have heterogeneous HARA preferences characterized by linear individual absolute risk tolerance indices. Finally, we also show that all these results still hold even if the labor supply is inelastic. In particular, we prove that in this case the introduction of some heterogeneity of preferences across agents is a fundamental driving force leading to our results.

3 The model

3.1 Consumers

There are n agents and the total population is constant over time. In each period consumers provide elastically an amount of labor l_i, $i = 1, ..., n$, with $l_i \leq \bar{l}$ and $\sum_{i=1}^n l_i = \ell$, $\bar{l} > 0$ being the agent's endowment of labor. At the initial period $t = 0$, each agent i is also endowed with a share θ_i of the initial stock of capital k_0 with $\sum_{i=1}^n \theta_i = 1$. In order to simplify the formulation, we will assume that the n agents are ordered according to their initial capital endowment, i.e. $\theta_i > \theta_j$ for $i < j$. Let $(\theta_i)_{i=1}^n = \theta$ be the vector of initial shares. Consumer's preferences are characterized by a discounted additively separable utility function of the form

$$\mathcal{U}^i(x^i, \mathcal{L}^i) = \sum_{t=0}^{\infty} \delta^t \left[u_i(x_{it}) + v_i(\mathcal{L}_{it})\right] \tag{1}$$

where $\delta \in (0,1)$ is the discount factor, x_{it} the consumption of agent i at time t, $\mathcal{L}_{it} = \bar{l} - l_{it}$ its leisure at time t, and x^i, $\mathcal{L}^i$ are respectively its intertemporal streams of consumption and leisure. Agents are therefore different with respect to their preferences and their initial wealth. Each instantaneous utility function satisfies the following basic restrictions:

Assumption 1. *$u_i(x_i)$ and $v_i(\mathcal{L}_i)$ are C^2, such that $u_i'(x_i) > 0$, $v_i'(\mathcal{L}_i) > 0$, $u_i''(x_i) < 0$, $v_i''(\mathcal{L}_i) < 0$ for any $x_i > 0$, $\mathcal{L}_i > 0$, and satisfy the Inada conditions $\lim_{x_i \to 0} u_i'(x_i) = +\infty$, $\lim_{\mathcal{L}_i \to 0} v_i'(\mathcal{L}_i) = +\infty$.*

Denote by w_t the wage rate, r_t the gross rental rate of capital and p_t the price of investment good at time t, all in terms of the price of the consumption good. In a decentralized economy, an agent i maximizes his intertemporal utility function (1) subject to a single intertemporal budget constraint

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t w_t l_{it} + \theta_i r_0 k_0 \quad \text{with } i = 1, ..., n. \tag{2}$$

where the discount factors R_t are defined as:

$$R_t = \prod_{\tau=0}^{t} \frac{1}{1 + d_\tau}$$

with d_t the common interest rate which satisfies $d_0 = [r_0 - p_{-1}]/p_{-1}$ and $d_t = [r_t + (1-\mu)p_t - p_{t-1}]/p_{t-1}$ for any $t \geq 1$.[6]

3.2 Producers

We consider a two-sector economy with a consumption good y_0 and a capital good y. The consumption good is entirely consumed and the capital good

[6] This equation reflects the absence of arbitrage opportunities in a perfect foresight equilibrium. It is also called the *portfolio equilibrium condition* (see Becker and Boyd [2]). The difference between the equation evaluated at time $t = 0$ and $t \geq 1$ comes from the fact that at the initial date there is no residual capital coming from the previous period and in some sense we have $k_0 = y_{-1}$.

partially depreciates in each period at a constant rate $\mu \in [0,1]$. There are two inputs, capital and labor. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k^0, l^0), \quad y = f^1(k^1, l^1)$$

with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq \ell$, ℓ being the total amount of labor.

Assumption 2. *Each production function $f^j : \mathbb{R}^2_+ \to \mathbb{R}_+$, $j = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f^j_1(0, x) = f^j_2(x, 0) = +\infty$, $f^j_1(+\infty, x) = f^j_2(x, +\infty) = 0$.*

Notice that by definition, as $l_1 \leq \ell \leq n\bar{l}$, we have $y \leq f^1(k, \ell) \leq f^1(k, n\bar{l})$. The monotonicity properties and the Inada conditions in Assumption 2 then imply that there exists $\bar{k} > 0$ such that $f^1(k, n\bar{l}) > k$ when $k < \bar{k}$ while $f^1(k, n\bar{l}) < k$ when $k > \bar{k}$. The set of admissible 3-uples (k, y, ℓ) is thus defined as follows

$$\mathcal{D} = \left\{ (k, y, \ell) \in \mathbb{R}^3_+ | 0 \leq l \leq n\bar{l}, 0 \leq k \leq \bar{k},\ 0 \leq y \leq f^1(k, \ell) \right\} \tag{3}$$

It is easy to show that $\mathcal{D}$ is a compact, convex set.

There are two representative firms, one for each sector. For any given (k, y, ℓ), profit maximization in each representative firm is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} T(k, y, \ell) = \quad & \max_{(k^0, k^1, l^0, l^1)} \quad f^0(k^0, l^0) \\ & s.t. \quad y \leq f^1(k^1, l^1) \\ & \qquad k^0 + k^1 \leq k \\ & \qquad l^0 + l^1 \leq \ell \\ & \qquad k^0, k^1, l^0, l^1 \geq 0 \end{aligned} \tag{4}$$

The social production function $T(k, y, \ell)$ describes the frontier of the production possibility set associated with interior temporary equilibria such that $(k, y, \ell) \in \mathcal{D}$, and gives the maximal output of the consumption good. It also summarizes the trade-off between production of the final good and productive investment. Under Assumption 2, for any $(k, y, \ell) \in \mathcal{D}$, $T(k, y, \ell)$ is homogeneous of degree one, concave and we assume in the following that it is at least C^2.[7] We formulate the aggregate profit maximization as follows

$$\max_{(k,y,\ell)\in\mathcal{D}} T(k,y,\ell) + py - rk - w\ell \tag{5}$$

and we derive that for any $(k,y,\ell) \in int\mathcal{D}$, with $int\mathcal{D}$ denoting the interior of the set $\mathcal{D}$, the first order derivatives of the social production function give

$$T_1(k,y,\ell) = r,\ \ T_2(k,y,\ell) = -p,\ \ T_3(k,y,\ell) = w \tag{6}$$

4 Competitive equilibrium and Pareto optimum

From the first welfare theorem, we know that every competitive equilibrium obtained in the decentralized economy is a Pareto optimal allocation. Let

$$\Delta = \left\{ \eta_1, \ldots, \eta_n | \eta_i \geq 0 \text{ and } \sum_{i=1}^{n} \eta_i = 1 \right\}$$

be the unit simplex of $\mathbb{R}^n$. A Pareto optimal allocation is a solution to the following planner's problem for a given vector of nonnegative welfare weights $\eta = (\eta_1, \ldots, \eta_n) \in \Delta$:

$$\begin{aligned} \max_{\{x_{it}, l_{it}, y_t\}_{t\geq 0}} \quad & \sum_{i=1}^{n} \eta_i \sum_{t=0}^{\infty} \delta^t \left[u_i(x_{it}) + v_i(\bar{l} - l_{it}) \right] \\ s.t. \quad & \sum_{i=1}^{n} x_{it} = T(k_t, y_t, \ell_t) \\ & \sum_{i=1}^{n} l_{it} = \ell_t \\ & k_{t+1} = y_t + (1-\mu)k_t \\ & k_0 \text{ given,} \end{aligned} \tag{7}$$

The solution to the above program depends on the vector η and on k_0. The set of Pareto optima is obtained when η spans Δ. As markets are complete and under Assumptions 1 and 2, the second theorem of welfare economics also holds: any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments. We may then characterize an equilibrium with transfer by solving the weighted dynamic optimization

[7] Benhabib and Nishimura [4] show that $T(k,y,\ell)$ is C^1 under Assumption 2.

program (7).[8] A given competitive equilibrium is then obtained for a η such that the associated allocations saturate the budget constraint of all the consumers.

In order to simplify the analysis, we formulate the weighted dynamic optimization program (7) in reduced form. Let $U(x, \ell)$ be a social utility function such that for $\eta = (\eta_1, \ldots, \eta_n) \in \Delta$

$$\begin{aligned} U(x_t, \ell_t) \quad = \quad \max_{\{x_{it}, l_{it}\}_{t \geq 0}} \quad & \sum_{i=1}^{n} \eta_i \left[u_i(x_{it}) + v_i(\bar{l} - l_{it}) \right] \\ s.t \quad & \sum_{i=1}^{n} x_{it} = x_t \\ & \sum_{i=1}^{n} l_{it} = \ell_t \end{aligned} \tag{8}$$

The value function $U(x, \ell)$ can be characterized as follows:

Lemma 1. *Under Assumption 1, the value function of program (8) is additively separable, i.e.* $U(x, \ell) = u(x) - v(\ell)$ *with* $u(x)$ *and* $v(\ell)$ *some* C^2 *functions such that* $u'(x) > 0$, $v'(\ell) > 0$, $u''(x) < 0$, $v''(\ell) > 0$ *for any* $x > 0$, $\ell > 0$, *and* $\lim_{x \to 0} u'(x) = +\infty$, $\lim_{\ell \to n\bar{l}} v'(\ell) = +\infty$.

We may then define the indirect social utility function

$$V(k_t, k_{t+1}, \ell_t) = u(T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)) - v(\ell_t) \tag{9}$$

From (3), we also derive the set of admissible paths. As $\ell_t \leq n\bar{l}$ and $k_{t+1} = y_t + (1-\mu)k_t$, we have $k_{t+1} \leq f^1(k_t, \ell_t) + (1-\mu)k_t \leq f^1(k_t, n\bar{l}) + (1-\mu)k_t \equiv g(k_t)$. Assumption 2 implies that there exists $\tilde{k} > 0$ such that $g(k_t) > k_t$ when $k_t < \tilde{k}$ while $g(k_t) < k_t$ when $k_t > \tilde{k}$. It follows that it is not possible to maintain stocks beyond $\tilde{k}$. The set of admissible paths (k_t, k_{t+1}, ℓ_t) is thus defined as follows

$$\tilde{\mathcal{D}} = \Big\{ (k_t, k_{t+1}, \ell_t) \in \mathbb{R}^3_+ | 0 \leq \ell_t \leq n\bar{l}, \\ 0 \leq k_t \leq \tilde{k}, \ (1-\mu)k_t \leq y \leq f^1(k_t, \ell_t) + (1-\mu)k_t \Big\}$$

The planner's problem is then equivalent to

[8] This aproach has been pioneered by Negishi [31] and applied to dynamic models by Bewley [7] and Kehoe *et al.* [27] among others.

$$\max_{\{k_t,\ell_t\}_{t\geq 0}} \quad \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}, \ell_t)$$
$$s.t. \quad (k_t, k_{t+1}, \ell_t) \in \tilde{\mathcal{D}} \tag{10}$$
$$k_0 \text{ given}$$

Note that the solution depends on k_0.

In the present framework it is a standard result that the set of interior Pareto optima is the set of $\{k_t, \ell_t\}_{t\geq 0}$ that are solutions to the following system of Euler equations

$$V_2(k_t, k_{t+1}, \ell_t) + \delta V_1(k_{t+1}, k_{t+2}, \ell_{t+1}) = 0 \tag{11}$$
$$V_3(k_t, k_{t+1}, \ell_t) = 0 \tag{12}$$

and that satisfy the transversality condition

$$\lim_{t\to+\infty} \delta^t k_t V_1(k_t, k_{t+1}, \ell_t) = 0$$

Notice that using (6) and (9), the Euler equations become:

$$-u'(x_t)p_t + \delta u'(x_{t+1})\left[r_{t+1} - (1-\mu)p_{t+1}\right] = 0 \tag{13}$$
$$u'(x_t)w_t - v'(\ell_t) = 0 \tag{14}$$

Our methodology consists in providing a local stability analysis of the optimal path in a neighborhood of the steady state obtained as a stationary solution of the Euler equations. Within an optimal growth model with heterogeneous agents, the steady state has to be considered along two dimensions. At the aggregate level, an interior steady state is a sequence $(k_t, y_t, \ell_t) = (k^*, \mu k^*, \ell^*)$, $\forall t \geq 0$, with $x_t = x^* = T(k^*, \mu k^*, \ell^*)$, $p_t = p^* = -T_2(k^*, \mu k^*, \ell^*)$, $r_t = r^* = T_1(k^*, \mu k^*, \ell^*)$ and $w_t = w^* = T_3(k^*, \mu k^*, \ell^*)$, that solves the Euler equations (13)-(14). Since $T(k, y, \ell)$ is a linear homogeneous function, and denoting $\kappa = k/\ell$, an aggregate steady state may be also defined as a pair (κ^*, ℓ^*).

At the individual level, an interior steady state for agent i is a sequence of consumption and labor supply $(x_{it}, l_{it}) = (x_i^*, l_i^*)$ that solves the first order conditions corresponding to the individual maximization of the intertemporal utility function (1) subject to the intertemporal budget constraint (2). Of course, the whole set of individual steady states (x_i^*, l_i^*), $i = 1, \ldots, n$, satisfy $x^* = \sum_{i=1}^n x_i^*$ and $\ell^* = \sum_{i=1}^n l_i^*$. Moreover, the endogenous trade-off between consumption and leisure implies that these stationay values of individual consumption and labor supply depend on the initial distribution of capital $\theta = (\theta_i)_{i=1}^n$ with $\theta_j = 1 - \sum_{i=1, i\neq j}^n \theta_i$. We then provide a comparative statics analysis based on the consideration of an increase of some θ_i which necessarily implies a decrease of some other θ_j, everything set equal beside.

Theorem 1. *Let Assumptions 1 and 2 hold. Then:*

i) There exists a unique aggregate steady state (κ^*, ℓ^*) *which is the solution to the following pair of equations*

$$-\frac{T_1(\kappa, \mu\kappa, 1)}{T_2(\kappa, \mu\kappa, 1)} = f_1^1(k_1(\kappa, \mu\kappa, 1), l_1(\kappa, \mu\kappa, 1)) = (\delta\vartheta)^{-1}$$
$$u'(\ell T(\kappa, \mu\kappa, 1))T_3(\kappa, \mu\kappa, 1) - v'(\ell) = 0$$

with $\vartheta = [1 - \delta(1 - \mu)]^{-1}$.

ii) If $v_i'(\bar{l}) < u_i'((1 - \delta)\vartheta r^* \kappa^* \ell^* \theta_i)$ *for any* $i = 1, \ldots, n$, *there exist unique steady state values for the individual consumptions* $x_i^*(\theta) \in (0, x^*)$ *and labor supplies* $l_i^*(\theta) \in (0, \ell^*)$ *that satisfy the following system*

$$\begin{aligned} x_i^*(\theta) &= w^* l_i^*(\theta) + (1 - \delta)\vartheta r^* \kappa^* \ell^* \theta_i \\ u_i'(x_i^*(\theta))w^* &= v_i'(\bar{l} - l_i(\theta)) \end{aligned} \quad (15)$$

with $w^* = T_3(\kappa^*, \mu\kappa^*, 1)$, $r^* = T_1(\kappa^*, \mu\kappa^*, 1)$, $\ell^* = \sum_{i=1}^n l_i^*(\theta)$ *and* $x^* = \ell^* T(\kappa^*, \mu\kappa^*, 1) = \sum_{i=1}^n x_i^*(\theta)$.[9]

iii) $x_i^*(\theta)$ *and* $l_i^*(\theta)$ *are* C^1*-functions of* θ, *for any* $i = 1, \ldots, n$. *Moreover, a variation of the share* θ_i *implies for agent* i

$$\frac{\partial x_i^*(\theta)}{\partial \theta_i} > 0, \quad \frac{\partial \ell_i^*(\theta)}{\partial \theta_i} < 0 \quad (16)$$

and there is an agent $j \neq i$ *such that*

$$\frac{\partial x_j^*(\theta)}{\partial \theta_i} < 0, \quad \frac{\partial \ell_j^*(\theta)}{\partial \theta_i} > 0 \quad (17)$$

Remark 1. Using the linear homogeneity of $T(k, y, \ell)$, we get the following expression of the wage rate at the steady state $w^* = x^*/\ell^* - (1 - \delta)\vartheta r^* \kappa^*$. It follows from (15) that x_i^* and l_i^* can be also expressed as functions of the aggregate steady state values for consumption x^* and labor ℓ^*.

[9] In a similar but aggregate model, Sorger [36] shows that a continuum of stationary equilibria occurs. In our framework, as the steady state is obtained for a given set of welfare weights η, the same result is obtained and corresponds to the existence of a stationary equilibrium for each $\eta \in \Delta$.

The pair (κ^*, ℓ^*) is called the Modified Golden Rule. We may then provide a characterization of the aggregate consumption and labor supply (x^*, ℓ^*) which will be fundamental in the analysis of the link between wealth inequality and macroeconomic volatility.

We introduce two elasticities characterizing the agents' preferences: The elasticity of intertemporal substitution in individual consumption

$$\epsilon_x^i(x_i) = -u_i'(x_i)/u_i''(x_i)x_i > 0 \tag{18}$$

and the elasticity of the individual labor supply with respect to wage

$$\epsilon_l^i(l_i) = \frac{dl_i}{dw}\frac{w}{l_i} = -v_i'(\bar{l} - l_i)/v_i''(\bar{l} - l_i)l_i > 0 \tag{19}$$

which is derived from the first order condition (15). From these expressions we define the individual absolute risk tolerance indices for consumption and labor as follows

$$\rho_i(x_i) = \epsilon_x^i(x_i)x_i, \quad \gamma_i(l_i) = \epsilon_l^i(l_i)l_i \tag{20}$$

These are in fact the inverse of the corresponding absolute risk aversion.

Because leisure is an argument of the utility function, i.e. $\epsilon_l^i(l_i) > 0$ for any $i = 1, \ldots, n$, Theorem 1 implies that the individual decisions $(x_i^*(\theta), l_i^*(\theta))$ and the aggregate levels of consumption and labor supply depend on the initial distribution of capital. We have indeed:

Corollary 1. *Let Assumptions 1 and 2 hold. Then ℓ^* and x^* are C^1-functions of θ, and $\partial\ell^*/\partial\theta_i > 0$ if and only if*

$$\frac{\epsilon_l^j(l_j)}{\epsilon_x^j(x_j)} > \frac{\epsilon_l^i(l_i)}{\epsilon_x^i(x_i)} \tag{21}$$

Moreover, $\partial x^*/\partial\theta_i = T(\kappa^*, \mu\kappa^*, 1)\partial\ell^*/\partial\theta_i$.

Notice however that the aggregate consumption per worked hours, $\chi^* = x^*/\ell^*$, is invariant with respect to the initial distribution of capital, i.e. $\partial\chi^*/\partial\theta_j = 0$.

The main conclusions of Theorem 1 are quite intuitive. An increase of agent i's share of capital θ_i generates a higher wealth and allows him to enjoy higher consumption x_i and leisure $\mathcal{L}_i$. As a result his labor supply is decreased. But at the same time, when θ_i is increased, there must be some other agent j for which the share of capital is decreased as $\sum_{i=1}^n \theta_i = 1$ and thus $\theta_j = 1 - \sum_{i=1, i\neq j}^n \theta_i$. This explains why agent j has to lower his levels of consumption x_j and leisure $\mathcal{L}_j$. As a result his labor supply is increased. This explains the result of Theorem 1-iii).

Concerning the result of Corollary 1, the intuition is the following. From a global point of view, an increase of θ_i will generate larger aggregate amounts of consumption x^* and labor ℓ^* if agent i's reaction is relatively more important with respect to consumption and relatively less important with respect to leisure than agent j's reaction. This property is obtained under condition (21).

5 Endogenous competitive business cycles

Near the steady state the behavior of the non-linear dynamic system (13)-(14) is equivalent to the behavior of the linearized system. The dynamic properties of the steady state are then related to the eigenvalues of the Jacobian matrix associated with the linearized system. On the one hand, the characteristic roots depend on the first and second order derivatives of the social production function $T(k, y, \ell)$ through the relative capital intensity difference across sectors

$$b(k, y, \ell) = \frac{l^0}{T}\left(\frac{k^1}{l^1} - \frac{k^0}{l^0}\right) \qquad (22)$$

as well as on the elasticities of the consumption good's output and the rental rate with respect to the capital stock

$$\varepsilon_{ck}(k, y, \ell) = T_1 k/T > 0, \quad \varepsilon_{rk}(k, y, \ell) = -T_{11}k/T_1 > 0 \qquad (23)$$

Notice that $b(k, y, \ell) > (<)0$ if and only if the investment (consumption) good is capital intensive. As shown in Benhabib and Nishimura [5] and Bosi *et al.* [9], the existence of endogenous fluctuations requires $b(k, y, \ell) < 0$.

On the other hand, the characteristic roots depend on the first and second order derivatives of the social utility function through some standard curvature indices.

Definition 1. *Let $U(x, \ell) = u(x) - v(\ell)$ be the social utility function, as defined by (8) and Lemma 1 , and $\rho(x) = -u'(x)/u''(x) > 0$, $\gamma(\ell) = v'(\ell)/v''(\ell) > 0$ be the social absolute risk tolerance respectively for consumption and labor.*

As shown in Wilson [38], the social absolute risk tolerance indices are obtained from the individual ones, given by (20). The resulting expressions are

$$\rho(x) = \sum_{i=1}^{n} \rho_i(x_i(\theta)), \quad \gamma(\ell) = \sum_{i=1}^{n} \gamma_i(l_i(\theta)) \tag{24}$$

For given discount factor δ and technology parameters $(b, \varepsilon_{ck}, \varepsilon_{rk})$, the local stability properties of the steady state also depend on ℓ^*, $\rho(\ell^* T^*)$ and $\gamma(\ell^*)$. Notice that the model with inelastic labor can be obtained by assuming $\gamma(\ell^*) = 0$.

As explained in Section 4, our strategy of analysis consists in characterizing the competitive equilibrium through the analysis of the Pareto optimal solution of the central planner's program (10). The equilibrium path is then the solution to the planner's intertermporal maximation problem, where the planner's utility, or social utility, is the sum of the individual utilities weighted by the welfare weights. Consequently, the social utility function depends on the welfare weights, which themselves depend on the equilibrium allocations that in turn depend on the initial condition and on the distribution of individual capital endowments. This means that without regularity properties of the welfare weights, the local stability of the steady state cannot be obtained directly from the local stability of the planner's optimum with fixed welfare weights.

In fact, we have shown in Theorem 1 and Corollary 1 that the aggregate and individual steady states are continuous functions of the distribution of capital shares $\theta = (\theta_i)_{i=1}^n$. As shown in Kehoe *et al.* [27] and Santos [35], if the value function of the dynamic optimization program (10) is twice continuously differentiable, the welfare weights are continuous functions of θ, and the local dynamic properties of the competitive equilibrium can be analyzed from the planner's problem defined in terms of the social utility function with welfare weights fixed at their steady state value. Indeed, local stability means that with initial conditions slightly away from the steady state, the welfare weights will be close to their steady state values. The following Lemma gives a sufficient condition for the C^2 -differentiability of the value function of the dynamic optimization program (10) and thus for the continuity of the welfare weights.

Proposition 1. *Under Assumptions 1 and 2, the welfare weights* $(\eta_1, \dots, \eta_n) \in \Delta$ *are continuous functions of the initial individual shares of capital* $\theta = (\theta_i)_{i=1}$ *if for any* $(k_t, k_{t+1}, \ell_t) \in int\mathcal{D}$*:*

$$\begin{aligned} & T_2(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \\ + \quad & b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)T_1(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \neq 0 \end{aligned} \tag{25}$$

A sufficient condition for (25) to hold is $b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \leq 0$*, i.e. the consumption good is capital intensive.*

From (24) and Proposition 1 we derive that $\rho(x)$ and $\gamma(\ell)$ are continuous functions of the initial individual shares of capital $\theta = (\theta_i)_{i=1}^n$. As a direct consequence, we finally conclude that the dynamic properties of the competitive equilibrium can be analyzed from the planner's problem defined in terms of the social utility function with fixed welfare weights.

Proposition 2. *Under Assumptions 1 and 2, let condition (25) hold. Then the stability properties of the steady state of the general equilibrium model with equilibrium welfare weights and of the optimal growth model with appropriate fixed welfare weights are equivalent.*

Building on Proposition 2, we consider from now on capital intensity configurations under which condition (25) is satisfied and we pursue our analysis of the equilibrium path of the optimal growth model with welfare weights fixed at their steady state values. Our objective is to derive a relationship between the local dynamical properties of the equilibrium path in a neighborhood of the steady state and the degree of inequality in the economy referering to the distribution of capital across agents.

As we will measure wealth inequality through the vector θ of shares of capital and since the stability properties of the steady state depend on the absolute risk tolerence indices (as well as on technology) it is useful to define the bounds of the intervals of the admissible values for $\rho(x)$ and $\gamma(\ell)$ when the distribution of individual capital shares θ_i spans the feasible set. Let

$$\underline{\rho} = \min_\theta \rho(x^*(\theta)), \quad \bar{\rho} = \max_\theta \rho(x^*(\theta)), \quad \underline{\gamma} = \min_\theta \gamma(\ell^*(\theta)), \quad \bar{\gamma} = \max_\theta \gamma(\ell^*(\theta))$$

We then have by definition $\rho(x^*(\theta)) \in (\underline{\rho}, \bar{\rho})$, $\gamma(\ell^*(\theta)) \in (\underline{\gamma}, \bar{\gamma})$. As we will see there are many situations in which these bounds nay be computed.

We are now ready to relate the stability of the steady state with the absolute risk tolerance indices. However, first we analyse the role of technology. As shown in Bosi *et al.* [9], the existence of endogenous fluctuations in a two-sector optimal growth model with elastic labor requires a capital intensive consumption good with $b \in (-1/(1-\mu), 0)$. The intuition for this fact, initially provided by Benhabib and Nishimura (1985), is based on the Rybczinsky theorem: an instantaneous increase in the capital stock k_t implies a decrease of the output of the capital good y_t which lowers the investment and the capital stock in the next period k_{t+1}. But, from the same mechanism, this decrease of k_{t+1} implies an increase of the output of the capital good y_{t+1} which increases the investment and the capital stock in period $t+2$, k_{t+2}. Fluctuations of the capital stocks are obtained if this mechanism is strong enough with respect to the depreciation rate of capital.[10] It is worth noticing that the restriction on the capital intensity difference across sectors appears to be compatible with recent empirical evidences. Building on aggregate Input-Output tables, Takahashi *et al.* [37] have shown that over the last 30 years the OECD countries are characterized by a more capital intensive consumption good sector than the investment good sector.

The properties of preference also matter in the existence of fluctuations via the absolute risk tolerence indices. Indeed, the existence of persistent fluctuations requires two main ingredients. On the one hand, the agents have to accept fluctuations in their consumption levels and thus need a large enough elasticity of intertemporal substitution in consumption, i.e. a large enough absolute risk tolerance with respect to consumption. On the other hand, as labor is supplied endogenously a low enough elasticity of the labor supply, i.e. a low enough absolute risk tolerance with respect to labor, is necessary in order to prevent the agents from smoothing the fluctuations of their wage and capital incomes implied by the fluctuations of the capital stock.

The following two Propositions are adapted from Proposition 4 in Bosi *et al.* [9] and will be the basis for our main results stated in the next Section. The first one deals with the existence of damped fluctuations. Consider the following two critical values for $\rho(x^*)$ and $\gamma(\ell^*)$:

$$\begin{aligned}
\rho_c &= -\frac{\delta\vartheta^2\varepsilon_{ck}x^*}{b[1+(1-\mu)b]\varepsilon_{rk}} \equiv \Lambda_c x^* \\
\gamma_c &= -\frac{\frac{1-s}{s}\left(\frac{1}{1-\mu b}\right)^2\left[(1-\delta)\vartheta+\frac{1-s}{s}\right]b[1+(1-\mu)b]}{\left[\frac{1-s}{s}\frac{b}{1-\mu b}+\delta\vartheta\right]\left[\frac{1-s}{s}\frac{1+(1-\mu)b}{1-\mu b}+\vartheta\right]}\left[\rho(x^*)-\rho_c\right] \equiv \Gamma_c\left[\rho(x^*)-\rho_c\right]
\end{aligned} \quad (26)$$

[10]From the capital accumulation equation in (7), we easily get $dk_{t+1}/dk_t = b^{-1} + 1 - \mu$.

and assume $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$ so that $\rho_c > 0$, and $\gamma_c > 0$ for $\rho(x^*) > \rho_c$.

Proposition 3. *Under Assumptions 1 and 2, let the consumption good be capital intensive with $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$.*

1. If $\rho(x^*) < \rho_c$, then for any $\gamma(\ell^*) \in (\underline{\gamma}, \bar{\gamma})$, the steady state is saddle-point stable with monotone convergence.

2. If $\rho(x^*) > \rho_c$, then the steady state is saddle-point stable with monotone convergence when $\gamma(\ell^*) > \gamma_c$ while it is is saddle-point stable with oscillating convergence when $\gamma(\ell^*) < \gamma_c$.

Notice that the existence of damped fluctuations requires $\rho_c < \bar{\rho}$ and $\gamma_c \in (\underline{\gamma}, \bar{\gamma})$.

The second Proposition focusses on the existence of persistent oscillations. Let us introduce the following additional critical value:

$$\begin{aligned}
\rho_f &= -\frac{2\delta(1+\delta)\vartheta^2\varepsilon_{ck}x^*}{[1+(2-\mu)b]\left[\delta+[1+(1-\mu)\delta]b\right]\varepsilon_{rk}} \equiv \Lambda_f x^* \\
\gamma_f &= -\frac{\frac{1-s}{s}\left(\frac{1}{1-\mu b}\right)^2\left[(1-\delta)\vartheta+\frac{1-s}{s}\right][1+(2-\mu)b][\delta+b(1+\delta(1-\mu))]}{\left[\frac{1-s}{s}\frac{1+(2-\mu)b}{1-\mu b}+(1+\delta)\vartheta\right]\left[\frac{1-s}{s}\frac{\delta+b(1+\delta(1-\mu))}{1-\mu b}+2\delta\vartheta\right]}\left[\rho(x^*)-\rho_f\right] \\
&\equiv \Gamma_f\left[\rho(x^*)-\rho_f\right]
\end{aligned} \tag{27}$$

and assume $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$ so that $0 < \rho_c < \rho_f$, and $0 < \gamma_f < \gamma_c$ when $\rho(x^*) > \rho_f$.

Proposition 4. *Under Assumptions 1 and 2, let the consumption good be capital intensive with $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$.*

1. If $\rho(x^*) < \rho_c$, then for any $\gamma(\ell^*) \in (\underline{\gamma}, \bar{\gamma})$, the steady state is saddle-point stable with monotone convergence.

2. If $\rho(x^*) \in (\rho_c, \rho_f)$, then the steady state is saddle-point stable with monotone convergence when $\gamma(\ell^*) > \gamma_c$ while it is is saddle-point stable with oscillating convergence when $\gamma(\ell^*) < \gamma_c$.

3. If $\rho(x^*) > \rho_f$, then the steady state is saddle-point stable with monotone convergence when $\gamma(\ell^*) > \gamma_c$, saddle-point stable with oscillating convergence when $\gamma(\ell^*) \in (\gamma_f, \gamma_c)$ and becomes locally unstable with oscillating divergence when $\gamma(\ell^*) < \gamma_f$. Moreover, γ_f is a flip bifurcation value and there generically exist period-two cycles, in a left (or right) neighborhood of γ_f, which are saddle-point stable (or unstable).

Notice that $\rho_f < \bar{\rho}$ and $\gamma_f \in (\underline{\gamma}, \bar{\gamma})$ are necessary conditions for the existence of persistent oscillations.

6 The role of wealth inequality on output volatility

In this section we explore the link between wealth inequality and stability. Focussing on inequality rises two issues. The first concerns the economic variable under consideration. In the model, the primitives are the shares in initial capital, hour endowments and preferences. However, the relationship between these fundamentals and individual wealth at equilibrium is not straightforward because of the role of labor supply. Strictly speaking, consumers use their income to buy the consumption good and to consume leisure time. As time discount is identical across individuals, their consumption along the steady state is a good representation of individual wealth. Furthermore, in the present model commodity prices, interest rates and wages are independent of the distribution of capital shares, a property that greatly simplifies the computations. Formaly, the individual consumption is implicitly defined by equations (15). It follows that along the steady state the individual wealth $\Omega_i(\theta)$ of an agent i is given by $\Omega_i(\theta) = \sum_{t=0}^{\infty} \delta^t {x_i}^*(\theta) = \frac{1}{1-\delta}(w^* l_i^*(\theta) + (1-\delta)\vartheta r^* \kappa^* \ell^*(\theta)\theta_i) \equiv \phi_i(\theta)$, i.e. it depends on the individual labor supply and the steady state values of aggregate labor which themselves depend on the vector of shares θ. Consequently, the distribution of the initial shares of capital $\theta = (\theta_i)_{i=1}^n$ determines completely the distribution of individual wealth $\Omega = (\Omega_i(\theta))_{i=1}^n$. The notions of inequality in wealth or in shares of capital will then be used equivalently.

The second issue concerns the characterization of inequality. On the basis of Propositions 3 and 4, we need a measure of wealth inequality which can be linked with the steady state values of the absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$. As the steady state values x^* and ℓ^* are functions of the distribution $\theta = (\theta_i)_{i=1}^n$, we then consider a simple definition of increasing inequality based on the distribution of shares of capital and on the Pigou-Dalton transfer principle.

Definition 2. *Assume that the n agents are ordered according to their initial share of capital in decreasing order, i.e. $\theta_i \geq \theta_j$ for $i < j$. Con-*

sider a benchmark economy A characterized by a distribution $\theta^A = (\theta_k^A)_{k=1}^n$. *There is a larger inequality in some economy B if the associated distribution* $\theta^B = (\theta_k^B)_{k=1}^n$ *is such that* $\theta_i^A < \theta_i^B$ *and* $\theta_j^A > \theta_j^B$ *for some agents* i, j *such that* $j > i$*, while for the remaining agents* $k \neq i, j$, θ_k *is unaffected. Such a case in which economy A is less unequal than economy B is denoted* $\theta^A \prec_I \theta^B$.

We consider bi-lateral transfers between two agents i, j in which agent i, who owns intially a larger share of capital than agent j, ends up holding an even larger share while agent j, who initially owns a lower share of capital, ends up with an even lower share. Notice also that the transfers are mean-preserving so that the two distributions $\theta^A = (\theta_k^A)_{k=1}^n$ and $\theta^B = (\theta_k^B)_{k=1}^n$ may be easily compared.

The fact that wealth inequality may have an effect on macroeconomic volatility is a consequence of Propositions 3 and 4. Indeed, as shown by Theorem 1 and Corollary 1, a modification of the distribution of shares of capital θ implies modifications of $\rho(x^*)$ and $\gamma(\ell^*)$. Then the local stability properties of the steady state are affected by changes in the wealth distribution provided $\rho(x^*)$, ρ_c and ρ_f on the one hand, and $\gamma(\ell^*)$, γ_c and γ_f on the other hand, are changed at the same rate. In fact, the critical values $\rho_c, \rho_f, \gamma_c, \gamma_f$ are not constant but are linear functions of x^* and ℓ^* respectively. In the optimal growth literature (see Benhabib and Nishimura [5]) this fact generates the usual difficulty that the conditions for local stability or for the occurrence of cycles are implicit. In the present analysis, this fact allows for the existence of a correlation between wealth inequality and instability provided the social absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$ are not linear functions of x^* and ℓ^*. The difficulty is that the curvature of these indices involves the third and fourth order derivatives of the utility function, which are not limited by the standard assumptions on preferences.

However, an important point is that the non linearity of the *social* absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$ does not requires the non lineariry of the *individual* absolute risk tolerance indices. Building on Wilson [38], Hara *et al.* [25] have recently provided a characterization of the curvature properties of the social absolute risk tolerance index for consumption. Applied to our formulation and recalling from Remark 1 that the individual stationary values for consumption x_i^* are functions of the aggregate consumption level x^*, we get from formula (24):

Proposition 5. (Hara *et al.* [25]) *The curvature of the social absolute risk tolerance with respect to consumption is given by*

$$\rho''(x) \;=\; \sum_{i=1}^{n}\left(\frac{\partial x_i}{\partial x}\right)^2 \rho_i''(x_i) + \frac{1}{\rho(x)}\sum_{i=1}^{n}\frac{\partial x_i}{\partial x}\left[\rho_i'(x_i) - \rho'(x)\right]^2 \qquad (28)$$

with

$$\rho'(x) \;=\; \sum_{i=1}^{n}\frac{\partial x_i}{\partial x}\rho_i'(x_i) \qquad (29)$$

A similar conclusion is obviously obtained for the social absolute risk tolerance with respect to labor. This result shows that non linear social indices are obtained either when the individual indices are themselves non linear, or when there is some heterogeneity in agents' preferences. We thus introduce the following additional Assumption on individual preferences:

Assumption 3. *Either agents have homogeneous preferences characterized by non-linear $\rho_i(x_i)$ and $\gamma_i(l_i)$, or agents have heterogeneous preferences.*

Assumption 3 is not trully restrictive as we know that with homogeneous preferences leading to linear ρ_i and γ_i, inequality has no effect on stability.

Building on Theorem 1 and Corollary 1, we need also to introduce conditions that characterize the relationship between the aggregate steady state and the distribution of individual wealth. Consider the bound Λ_j and Γ_j, $j = c, f$, defined by (26) and (27), and let us denote $\Upsilon_j = \Gamma_j T^* \left[\rho'(x^*) - \Lambda_j\right]$.

Definition 3. *Let $(\Psi, \Phi) \in \{(\Lambda_c, \Upsilon_c), (\Lambda_f, \Upsilon_f)\}$ be given. Agents i and j are said to be elasticity-ordered if and only if one of the following sets of conditions hold:*

i) $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) < \epsilon_l^i(li)/\epsilon_x^i(x_i)$, $\rho'(x^*) < \Psi$ *and* $\gamma'(\ell^*) > \Phi$,

ii) $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) > \epsilon_l^i(li)/\epsilon_x^i(x_i)$, $\rho'(x^*) > \Psi$ *and* $\gamma'(\ell^*) < \Phi$.

Conditions i) and ii) are motivated by the result in Corrolary 1. They ensure that the aggregate labor supply is a decreasing or an increasing function of the share of capital θ_i owned by agent i and thus require different properties for the slopes of $\rho(x^*)$ and $\gamma(\ell^*)$. We will then consider a transfer between agents i and j with $i < j$, i.e. $\theta_i > \theta_j$. As $\theta = (\theta_i)_{i=1}^n$ with $\theta_j = 1 - \sum_{k=1,k\neq j}^n \theta_k$, this transfer will be such that the new distribution of capital shares is $\tilde{\theta} = (\tilde{\theta}_i)_{i=1}^n$ with $\tilde{\theta}_i = \theta_i + \epsilon$, $\tilde{\theta}_j = \theta_j - \epsilon$ and $\epsilon > 0$.[11]

[11] Notice that many transfers may be considered consecutively.

In order to simplify the formulation, we now introduce two different types of assumption concerning the capital intensity difference that are based on Propositions 3 and 4.

Assumption 4. *The consumption good is capital intensive with* $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$, $(\Psi, \Phi) = (\Lambda_c, \Upsilon_c)$, $\rho_c < \bar{\rho}$ *and* $\gamma_c \in (\underline{\gamma}, \bar{\gamma})$.

Assumption 5. *The consumption good is capital intensive with* $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$, $(\Psi, \Phi) = (\Lambda_f, \Upsilon_f)$, $\rho_f < \bar{\rho}$ *and* $\gamma_f \in (\underline{\gamma}, \bar{\gamma})$. *Moreover,* γ_f *is a flip bifurcation value giving rise to saddle-point stable period-two cycles in its left neighborhood.*

Assumption 4 is linked to the existence of damped fluctuations while Assumption 5 concerns the occurrence of persistent fluctuations. Although this last assumption concerns non trivial restrictions on the non-linear part of the Euler equation, a number of robust examples of saddle-point stable period-two cycles have been provided by Boldrin and Deneckere [8] and Mitra and Nishimura [28].

The following Theorem is the most general result of the paper. It shows that under the stated conditions a sufficiently high level of wealth inequality leads to endogenous business cycle fluctuations in a neighborhood of the steady state.

Theorem 2. *Let Assumptions 1, 2 and 3 hold, together with either Assumption 4 or Assumption 5. Let* $\prec_I^R$ *be the restriction of* $\prec_I$ *to the pairs of elasticity-ordered agents according to Definition 3. Then there exists a distribution* θ^0 *such that one of the following cases holds:*

1 - If Assumption 4 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy E *such that* $\theta^E \prec_I^R \theta^0$ *and is saddle-point stable with oscillations otherwise.*

2 - If Assumption 5 is satisfied, the steady state is saddle-point stable with oscillating convergence for any economy E *such that* $\theta^E \prec_I^R \theta^0$ *and is unstable with oscillating divergence otherwise. Moreover, there generically exist saddle-point stable period-two cycles for any economy* E *characterized by a distribution* θ^E *in a right neighborhood of* θ^0.

The results contained in the previous theorem are at the same time very general and specific. They can be viewed as general as no restriction is imposed on the distribution of wealth and both technology and preferences are not bound to belong to any given analytical class, as for example we do not impose them to be CES. They are also specific in the sense that the restriction to redistributions satisfying the property formalised in Definition 3 is necessary because we state the result without specifying the set of feasible wealth distributions. In fact, this condition is not necessary when individual wealth follows a Pareto distribution.

In the following two Sections we will sharpen our results by imposing reasonable restrictions on the fundamentals. The next Section restricts the attention to Pareto distributed individual wealth and preferences giving rise to individual absolute tolerance to risk which are not linear. The second next Section deals with the case in which the absolute tolerance to risk are linear. In this case, a preference heterogeneity is necessary, even in a very mild form, but the conclusions hold independently of the wealth distribution.

Finally, note that the results in Theorem 2 are obtained provided the stated conditions on absolute risk tolerance index for consumption $\rho(x^*)$ hold. This is important in respect with two recent contributions. First, as shown in Calvet *et al.* [12],[12] an increasing social absolute risk tolerance $\rho(x)$ is obtained when the individual absolute risk tolerance indices are increasing.[13] Second, Guiso and Paiella [24] have provided an empirical investigation of the absolute risk tolerance. They use household survey data to construct a direct measure of absolute risk tolerance based on the maximum price a consumer is willing to pay to buy a risky security. They relate this measure to consumers's endowment and attributes, and to measures of background risk and liquidity constraints. They find that risk tolerance is an increasing function of endowment. Theorem 2 then shows that with some appropriate restrictions on the slopes of $\rho(x^*)$ and $\gamma(\ell^*)$ which are compatible with standard positive individual elasticities of intertemporal substitution in consumption and of the labor supply, more wealth inequality generates more macroeconomic volatility.

[12] See also equation (29) in Proposition 5.

[13] Using (15) in Theorem 1 and Remark 1, and applying the same methodology as in Appendix 11.2 (proof of Theorem 1), it is easy to show that $\partial x_i/\partial x > 0$ and $\partial l_i/\partial \ell > 0$ for any $i = 1, \ldots, n$.

7 The role of inequality when wealth is Pareto distributed

The analysis presented so far holds for arbitrary distributions of shares in initial capital. However, there is strong empirical evidence that income and wealth distributions follow specific patterns and that these are persistent. In the case of the US economy, over the last century the data are characterized by skewed distributions of income and wealth with relatively large top shares,[14] and with heavy upper tails (power law behavior).[15] Building on these characteristics, in this Section we restrict the analysis to the family of Pareto wealth distributions and investigate the effect of increasing inequalities on the dynamics within this familly. Pareto wealth distributions are common choices as they are skewed to the right, display a heavy upper tail (slowly declining top wealth shares) and are very concentrated.

In the case of Pareto distributed wealth a useful result is that the Gini index G is connected to the exponent α of the Pareto distribution by

$$G = \frac{1}{2\alpha - 1}$$

Therefore, within this class of wealth distributions, large values of α correspond to more equal societies. Estimates of G vary greatly depending on the country and on the used data set. For the US in the 90's some studies indicate a Gini index as high as 0.78. Across countries the Gini index typically varies between 0.2 for very equal societies to 0.8 for unequal societies. These bounds on G imply a range for α, $\alpha = \frac{1}{2G} + \frac{1}{2} \in [1.125, 3]$, where the upper bound is for the most unequal societies. As the degree of inequality is directly given by the Gini index G throughout the parameter α, our aim is then to show that variations of α imply variations of the absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$.

It should be noted that while Pareto distributions appear to cover some stylized facts characterizing the data, there are however several issues related to this specific choice. First, Pareto distributions are not a good representation of the empirical distribution for low values of individual wealths. For the low wealth region, log-normal distributions provide a better fit. A second problem is that Pareto distributions have unbounded support, a fact

[14] See Piketty and Saez [33], Wolff [39, 40].

[15] See Nirei and Souma [32].

that generates a bias when compared to the data. Finally, a third set of issues concern whether the variable is continuous or discrete and whether it is assumed that the population is finite and countable or not. As the present model is specified for a discrete and finite set of individuals, we need to focus on the discrete version of the Pareto distribution, which is the Zeta distribution. In this case, the variable itself is also discretized.

7.1 The Zeta distribution and its implementation

The discrete version of the Pareto distribution is the Zeta distribution. Note that for a discrete random variable, the associated probability measure μ has a countable support $S = \{x_1, x_2, ...\}$ and μ is completely determined by $\mu(x_1), \mu(x_2), ...$ In the case of the Zeta distribution S is normalized in a way such that $S = \{1, 2, 3, ...\}$

Definition 4. *The Zeta distribution is characterized by the probability measure μ defined for natural values $s \in \mathbb{N}$ with $\mu(s) = \frac{s^{-\alpha}}{\sum_{j=1}^{\infty} j^{-\alpha}}$.*

We will assume that the true distribution of wealth is a Zeta distribution in which the support S is obtained by letting $s = \frac{w}{w_{\min}}$ where w is individual wealth and $w_{\min}$ the minimum of w, i.e. the subsistence level of wealth. This implies that wealth can only take values which are multiples of $w_{\min}$.

The model of the present paper involves a countable and finite population n. This implies that only integer values of $\mu(s)n$ can be modeled. However, in the sequel we show that for a sufficiently large population n, the Zeta distribution can be approximated by a distribution $\widehat{\mu}$ (of finite support) with integer values of $\widehat{\mu}(s)n$ for all s. We also show that this distribution can be decentralized, i.e. it is implementable. Consider the following definition.

Definition 5. *Let μ be the probability measure associated to the Zeta distribution. Let Γ_n be the distribution with probability measure η_n such that when $\mu(s)n \notin \mathbb{N}^*$, $\eta_n(s)$ is defined by the condition $|\mu(s) - \eta_n(s)| < 1/2n$ with $\eta_n(s)n \in \mathbb{N}^*$, and when $\mu(s)n \in \mathbb{N}^*$, $\eta_n(s) = \mu(s)$.*

For sufficiently large populations, Γ_n is a good approximation to the original Zeta distribution. Indeed, if we consider the sequence of economies indexed by their population n and characterized by a measure η_n we get:

Lemma 2. *The sequence of distributions* Γ_n *of measure* η_n *converges to the distribution of measure* μ*, i.e.* $\lim_{n\to\infty} |\mu(s) - \eta_n(s)| = 0$ *for all* s*.*

The previous results implies the following.

Corollary 2. *For any* $(x_{\min}, \alpha, \epsilon, \varepsilon)$ *with* $x_{\min} > 0, 0 < \alpha < 1, \epsilon > 0, \varepsilon > 0$*, there exists* n_0 *such that for any* $n > n_0$*, there exists a distribution* $\Gamma = \Gamma_n$ *of measure* $\eta = \eta_n$ *such that* $n\eta(s)$ *only takes integer values for all* $s \in \mathbb{N}$ *and which is* ϵ*-close to the original Zeta distribution, i.e.* $\|\eta - \mu\| < \epsilon$*, where* $\|.\|$ *is the weak (pointwise) norm for distributions. Furthermore, the associated Gini index is* ε*-close to* $1/(2\alpha - 1)$ *:*

$$\left| G(\Gamma) - \frac{1}{2\alpha-1} \right| < \varepsilon$$

The final issue is wether the distribution η can be implemented. We obtain the following conclusion

Lemma 3. *The distribution* Γ *of measure* η *obtained in Corollary 2 corresponds to a feasible finite sequence of individual endowments in initial capital* θ*, with* $\sum_{i=1}^{n} \theta_i$ *free but finite.*

We may then consider that the distribution of individual wealth follows with a sufficiently high level of approximation a Zeta distribution. As noted above, along the steady state the distribution of wealth and the distribution of individual consumption are identical.

Assumption 6. *Individual wealth is distributed according to a Zeta distribution of support* $S = \{w_{\min}, 2w_{\min}, 3w_{\min}, ...\}$

7.2 Inequality, Gini index and endogenous fluctuations

In order to obtain clear results we assume that agents are homogeneous with respect to their utility function and we focus on a specific class of preferences which corresponds to the slightest deviation with respect to the CRRA formulation.

Assumption 7. *Individual preferences are such that individual absolute risk tolerance indices regarding consumption and leisure are of the form*

$$\rho_i(x_i) = -\frac{u_i'(x_i)}{u_i''(x_i)} = \frac{x_i^{\varphi}}{\sigma} \text{ and } \gamma_i(l_i) = -\frac{v_i'(\mathcal{L}_i)}{v_i''(\mathcal{L}_i)} = \frac{(\bar{l}-\mathcal{L}_i)^{\nu}}{\gamma} = \frac{l_i^{\nu}}{\gamma}$$

with $\sigma, \varphi, \gamma, \nu > 0$ *and where* $\mathcal{L}_i = \bar{l} - l_i$ *is individual leisure.*

The class of preferences generating such individual risk tolerances is large and includes the CES (or CRRA) utility function. In fact a sufficient condition is that

$$u_i'(x_i) = \exp\left(-\sigma \frac{x_i^{1-\varphi}}{1-\varphi}\right) \text{ and } v_i'(\mathcal{L}_i) = \exp\left(\gamma \frac{(\bar{l}-\mathcal{L}_i)^{1-\nu}}{1-\nu}\right) \tag{30}$$

For consumption, when $\varphi < 1$ individual absolute risk tolerance $\rho_i(x_i)$ is a strictly concave function while for $\varphi > 1$, $\rho_i(x_i)$ is a strictly convex. When $\varphi \neq 1$ and according to Gollier [23], the most plausible value for σ is $\sigma = 2$. The CRRA specification is obtained when $\varphi = 1$. In this case $x_i^{1-\varphi}/(1-\varphi)$ becomes $\log x_i$ and $u_i'(x_i) = \exp(-\sigma \log x_i) = x_i^{-\sigma}$. The associated utility function is then $u_i(x_i) = x_i^{1-\sigma}/(1-\sigma)$. A similar expression holds for the utility of leisure. As a result, choosing values for φ and ν close enough to one allows to consider individual absolute risk tolerance indices arbitarily close to the linear formulation characterizing CRRA utility functions.

From Assumption 6 normalized incomes follow a Zeta distribution. As noted before, along the steady state we have $x_i = \omega_i$ so that normalized individual consumption $\tilde{x}_i = x_i/x_{\min}$ also follows a Zeta distribution. The associated value of $\rho(x)$ is

$$\rho(x) = \frac{x_{\min}^{\varphi}}{\sigma} \frac{\zeta(\alpha-\varphi)}{\zeta(\alpha)} = \bar{\rho}(\alpha) \tag{31}$$

where $\zeta(\alpha) = \sum_{s=1}^{\infty} s^{-\alpha}$ is the Riemann Zeta function. Notice that to ensure a finite value for $\zeta(\alpha-\varphi)$ we need to assume $\alpha > 1+\varphi$. From the distribution of consumption we compute the risk tolerance to leisure as follows

$$\gamma(\ell) = \frac{x_{\min}^{\frac{1-\varphi}{1-\nu}\nu}}{\gamma} \left[\frac{\nu-1}{\gamma(1-\varphi)}[\sigma - (1-\varphi)\hat{w}]\right]^{\frac{\nu}{1-\nu}} \frac{\zeta(\alpha + \frac{1-\varphi}{\nu-1}\nu)}{\zeta(\alpha)} \equiv \bar{\gamma}(\alpha) \tag{32}$$

with $\hat{w} = \log w$ and w as defined in (6). As soon as $(\nu-1)(1-\varphi)[\sigma - (1-\varphi)\hat{w}] > 0$, the functions $\bar{\rho}(\alpha)$ and $\bar{\gamma}(\alpha)$ are respectively decreasing and increasing in α. As inequalities increase when α decreases, we conclude that $\rho(x^*) = \bar{\rho}(\alpha)$ and $\gamma(\ell^*) = \bar{\gamma}(\alpha)$ are respectively increasing and decreasing functions of the level of inequality.

If the technologies are such that Assumption 4 or 5 holds, we then derive the following result:

Theorem 3. *Let Assumptions 2, 6 and 7 hold, together with either Assumption 4, or Assumption 5. Assume also that* $(1-\varphi)\hat{w} \leq 0$, $\alpha > 1+\varphi$ *and* $(\nu-1)(1-\varphi) > 0$. *Then there exists* $\bar{\sigma} > 0$, $\bar{\alpha} > 1+\varphi$ *and a level of wealth inequality characterized by a Gini index* $\bar{G} = 1/(2\bar{\alpha}-1)$ *such that* *when $\sigma \in (0, \bar{\sigma})$, one of the following cases holds:*

1 - If Assumption 4 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy E such that $G_E < \bar{G}$ and is saddle-point stable with oscillations otherwise.

2 - If Assumption 5 is satisfied, the steady state is saddle-point stable with oscillating convergence for any economy E such that $G_E < \bar{G}$ and is unstable otherwise. Moreover, there generically exist period-two cycles, in a right neighborhood of $\bar{G}$, which are saddle-point stable.

Theorem 3 improves the results of Theorem 2 when wealth is Pareto distributed. Importantly, we drop any restriction on the set of agents where the redistributions can take place. We provide in this particular case clear-cut conditions on the level of wealth inequality to get endogenous business cycle fluctuations. We show indeed that an increase of the Gini index implies the occurrence of macroeconomic volatility. Of course such a result is based on a large enough individual elasticity of intertemporal substitution $x_i^{\varphi-1}/\sigma$ as each consumer has to accept fluctuations of his consumption level.

Notice that if $\varphi = 1$, i.e. if the utility function is of the CES type with respect to consumption, the expression (56) no longer depends on α, while if $\nu = 1$, i.e. if the utility function is of the CES type with respect to leisure, the expression (56) is equal to zero as $\lim_{\nu \to 1} \zeta(\alpha + (1-\varphi)\nu/(\nu-1)) = 1$. Therefore, when $(\nu-1)(1-\varphi) = 0$ the occurrence of endogenous fluctuations no longer depends on the degree of inequality. As the nature of the results does not depend on whether $\varphi, \nu < 1$ or $\varphi, \nu > 1$, the problem occurring at $\varphi = 1$ or $\nu = 1$ can be considered as a discontinuity corresponding to the fact that Assumption 3 is not satisfied. It is worth pointing out however that Theorem 3 still holds for any values of φ or ν arbitrarily close to one.

8 The role of inequality with heterogeneous CES preferences

In Section 7, we have shown that large enough inequalities generate macroeconomic volatility when wealth is distributed in the economy according to a Pareto distribution and the common utility function of each agent is characterized by non linear individual risk tolerance to fluctuations in consumption and leisure. However, these results do not apply to CES utility functions because in this case risk tolerance is linear. In the current Section, we consider a general distribution of wealth and introduce a mild heterogeneity across agents concerning preferences. We then show that large enough inequalities may generate macroeconomic volatility even with CES utility functions.

We assume that consumer i's intertemporal utility function is given by

$$\mathcal{U}(x^i, \mathcal{L}^i) = \sum_{t=0}^{\infty} \delta^t \left[\frac{x_{it}^{1-\sigma_i}}{1-\sigma_i} - \frac{(\bar{l} - \mathcal{L}_{it})^{1+\gamma_i}}{1+\gamma_i} \right] \tag{33}$$

with $\delta \in (0,1)$, $\sigma_i \geq 0$ and $\gamma_i \geq 0$. Consequently, $\rho_i(x_i) = x_i/\sigma_i$ and $\gamma_i(l_i) = l_i/\gamma_i$, with $l_i = \bar{l} - \mathcal{L}_i$. Notice that the individual absolute risk tolerance indices are increasing functions of consumption and labor. As mentioned previously (see footnote 13), it is easy to show that $\partial x_i/\partial x > 0$ and $\partial l_i/\partial \ell > 0$ for any $i = 1, \ldots, n$, and Proposition 5 implies that the aggregate absolute risk tolerance indices are also increasing functions of consumption and labor.

We consider the simple case in which except agent 1, all the other $n-1$ agents have identical preferences. More precisely, agent 1 is characterized by the pair of parameters' values (σ_1, γ_1), while each agent $j = 2, \ldots, n$ is characterized by the common pair $(\sigma_j, \gamma_j) = (\sigma, \gamma)$. We also assume that all agents $j = 2, \ldots, n$ own the same share of capital θ_2 while agent 1 owns a share $\theta_1 > \theta_2$. For a given share $\theta_1 \in (1/n, 1)$, we thus derive $\theta_2 = (1-\theta_1)/(n-1)$.

The aggregate absolute risk tolerance indices are such that

$$\rho'(x) = \tfrac{\partial x_1}{\partial x}\tfrac{1}{\sigma_1} + \tfrac{\partial x_2}{\partial x}\tfrac{n-1}{\sigma} > 0 \quad \gamma'(\ell) = \tfrac{\partial l_1}{\partial \ell}\tfrac{1}{\gamma_1} + \tfrac{\partial l_2}{\partial \ell}\tfrac{n-1}{\gamma} > 0$$

It follows that for given values of (σ_1, γ_1) and a given set of technologies leading to the bounds (Ψ, Φ), appropriate choices of (σ, γ) allow to satisfy one of the two cases of the conditions reated to Definition 3 and thus show that an increase of wealth inequality may generate macroeconomic volatility. However, in order to prove that all the conditions of Theorem 2 may be satisfied, we need to specify a set of production functions.

Following Nishimura *et al.* [29] and Nishimura and Venditti [30], let us consider CES technologies such that

$$y_0 = \left(\alpha_0 l_0^{1-1/\varsigma} + \alpha_1 k_0^{1-1/\varsigma}\right)^{\varsigma/(\varsigma-1)}, \quad y_1 = \left(\beta_0 l_1^{1-1/\varsigma} + \beta_1 k_1^{1-1/\varsigma}\right)^{\varsigma/(\varsigma-1)}$$

with $\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = 1$. Both sectors are characterized by the same elasticity of capital-labor substitution $\varsigma > 0$. In order to simplify the formulation, we assume that the capital stock fully depreciates within one period, i.e. $\mu = 1$. We have also to impose the following restriction

$$\varsigma > \left(1 + \frac{ln\beta_1}{ln\delta}\right)^{-1} \equiv \hat{\varsigma} \in (0, 1)$$

in order to ensure the existence and uniqueness of the steady state values for the capital-labor ratio κ^* and the consumption per capita $T^* = T(\kappa^*, \kappa^*, 1)$.

In order to show that the local stability of the steady state is modified when the degree of wealth inequality is increased, i.e. when the value of θ_1 is increased, we provide illustrations for each of the cases 1- and 2- exhibited in Theorem 2. In the following results, we define an economy as a 8-uple of parameters $(\alpha_1, \beta_1, \varsigma, \delta, \sigma_1, \gamma_1, \sigma, \gamma)$. Following Propositions 3 and 4, we have to assume that the consumption good sector is capital intensive, namely $\alpha_1/\alpha_0 > \beta_1/\beta_0$.

The next Proposition deals with the correlation between wealth inequality and the existence of damped oscillations.

Proposition 6. *There exist a non-empty set of values of* n *and an open set of economies* $(\alpha_1, \beta_1, \varsigma, \delta, \sigma_1, \gamma_1, \sigma, \gamma)$ *with* $\varsigma > 1$, $1 > \sigma_1 > \sigma$ *and* $\gamma > 1 > \gamma_1$, *such that case 1- in Theorem 2 holds. Then, a sufficiently high level of wealth inequality leads to the existence of damped fluctuations in the neighborhood of the steady state.*

The following Proposition finally deals with the correlation between wealth inequality and the existence of persistent fluctuations.

Proposition 7. *There exist a non-empty set of values of* n *and an open set of economies* $(\alpha_1, \beta_1, \varsigma, \delta, \sigma_1, \gamma_1, \sigma, \gamma)$ *with* $\varsigma > 1$, $\sigma_1 > 1 > \sigma$ *and* $\gamma > \gamma_1 \geq 1$, *such that case 2- in Theorem 2 holds. Then, a sufficiently high level of wealth inequality leads to the existence of period-two cycles in the neighborhood of the steady state.*

Persistent fluctuations, i.e. the occurrence of saddle-point stable period-two cycles, is obtained if the flip bifurcation is super-critical, i.e. if the second part of Assumption 5 holds. In such a case, Proposition 7 shows that a low level of wealth inequality is associated to the existence of damped fluctuations in the neighborhood of the steady state, and an increase of wealth inequality will lead to the occurrence of persistent fluctuations. Although the second part of Assumption 5 corresponds to non-trivial conditions on the non-linear part of the Euler equation, a number of robust examples provided in the literature on optimal growth show that these conditions are usually satisfied.[16]

Propositions 6 and 7 then show that as soon as heterogeneous preferences are considered, a positive correlation between wealth inequality and business cycle fluctuations can be exhibited. Importantly, this can happpen even with increasing individual absolute risk tolerance in accordance with recent empirical findings by Guiso and Paiella [24].

9 The case with inelastic labor: comparisons and further results

In Ghiglino and Venditti [22], we have considered a similar two-sector optimal growth model but with inelastic labor. In such a framework, the aggregate amount of labor is normalized to 1 and each agent has a fixed endowment of labor l_i such that $\sum_{i=1}^n l_i = 1$. Consumer i's preferences are then characterized by a discounted utility function of the form

$$\mathcal{U}^i(x^i) = \sum_{t=0}^{\infty} \delta^t u_i(x_{it})$$

with $u_i(x_i)$ satisfying the corresponding part of Assumption 1 . If we consider the individual elasticities of the labor supply with respect to wage as defined by (19), assuming inelastic labor supplies yields $\epsilon_l^i(l_i) = 0$ for any $i = 1, \ldots, n$. It follows that the individual and social absolute risk tolerance indices for labor as defined by (20) and (24) are equal to zero, i.e. $\gamma_i(l_i) = \gamma(\ell) = 0$. A straightforward modification of Theorem 1 then gives the individual steady states as follows

$$x_i{}^*(x^*, \theta_i) = l_i\,[x^* - (1-\delta)\vartheta T_1^* k^*] + (1-\delta)\vartheta T_1^* k^* \theta_i \tag{34}$$

where $x^* = T(k^*, \mu k^*, 1) \equiv T^*$ is the aggregate consumption which is now independent from the distribution of wealth. Notice that the consumption level x_i^* of agent i only depends on the individual share of capital θ_i while in the case of endogenous labor, x_i^* depends on the whole distribution of shares $\theta = (\theta_i)_{i=1}^n$.

[16]See for instance Boldrin and Deneckere [8] and Mitra and Nishimura [28].

As the continuity of the welfare weights when labor is inelastic is obtained under the same condition given in Proposition 1, simple modifications of Propositions 3 and 4 when $\gamma(\ell)$ is set equal to 0 provide conditions for the existence of damped and persistent fluctuations. Consider indeed the critical bounds ρ_c and ρ_f respectively given by (26) and (27):

i) When $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$, the steady state is saddle-point stable with oscillations if $\rho(x^*) > \rho_c$.

ii) When $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$, the steady state is saddle-point stable with oscillating convergence if $\rho(x^*) \in (\rho_c, \rho_f)$ with ρ_f a flip bifurcation value, and there generically exist period-two cycles, in a right (or left) neighborhood of ρ_f, which are saddle-point stable (or unstable).

As in the case with endogenous labor, we now show that a positive correlation between wealth inequality an macroeconomic volatility can be easily obtained in two basic configurations: either when agents have heterogeneous preferences characterized by a linear absolute risk tolerance index under a general unspecified distribution of wealth, or when agents have homogeneous preferences characterized by a nonlinear absolute risk tolerance index under a Pareto distribution of wealth.

9.1 Preferences heterogeneity and output volatility

Although the aggregate consumption level is independent of the initial distribution of wealth when labor is inelastic, a modification of the degree of inequality based on bi-lateral transfers may still have an effect on the aggregate value of the absolute risk tolerance index $\rho(x^*)$. Indeed, building on (24) and the criteria introduced by Rothschild and Stiglitz [34], Ghiglino and Venditti [22] show that a rise of the degree of inequality characterizing the distribution of capital shares $\theta = (\theta_i)_{i=1}^n$ increases the value of $\rho(x^*)$ provided this function is strictly convex. As the local stability properties of the optimal path depend on the value of $\rho(x^*)$, this implies that sufficiently high levels of wealth inequality lead to endogenous fluctuations in a neighborhood of the steady state if the social absolute risk tolerance $\rho(x)$ is a strictly convex function.

These conclusions a priori suffer from a strong limitation. Indeed, restrictions on the curvature of the aggregate absolute risk tolerance index refer to the fourth order derivative of utility functions, and except weak indirect evidence given for instance by Gollier [23], the literature does not provide a clear characterization of the sign of this derivative. From this point of view, the conclusions obtained in Theorem 2 under elastic labor appear to be more powerful as they only refer to the third derivatives of utility functions which have been recently characterized on empirical ground.[17]

However, Proposition 5 implies that a convex social absolute risk tolerance index is obtained even when the individual indices are linear as soon as there is some heterogeneity of preferences across agents. As an illustration of this last case, consider the class of preferences given by a HARA utility function such that

$$u_i(x_i) = \frac{1-\sigma_i}{\sigma_i}\left(\frac{a_i x_i}{1-\sigma_i} + e_i\right)^{\sigma_i} \tag{35}$$

with $a_i > 0, e_i \geq 0$, $\sigma_i > 0$. The associated individual absolute risk tolerance is then

$$\rho_i(x_i) = \frac{x_i}{1-\sigma_i} + \frac{e_i}{a_i}$$

As shown in Ghiglino and Venditti [22], if all agents have homogeneous HARA preferences with $(a_i, e_i, \sigma_i) = (a, e, \sigma)$, the social absolute risk tolerance is linear, i.e. $\rho''(x) = 0$, and wealth inequality plays no role on the occurrence of macroeconomic volatility.[18] However, building on Proposition 5, we get from (34) that $\partial x_i/\partial x > 0$ and formula (28) implies $\rho''(x) > 0$ as soon as there are two agents i, j with different utility functions such that $(a_i, e_i, \sigma_i) \neq (a_j, e_j, \sigma_j)$.

Let us adapt Assumptions 4 and 5 to the case of inelastic labor.

Assumption 8. *$b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$ and $\rho_c \in (\underline{\rho}, \bar{\rho})$.*

Assumption 9. *$b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$, $\rho_f \in (\underline{\rho}, \bar{\rho})$ and the flip bifurcation generates saddle-point stable period-two cycles in a right neighborhood of ρ_f.*

We have then

Proposition 8. *Assume that individual preferences are represented by utility functions of the HARA class as defined by (35). If there exist at least two agents i, j with utility functions such that $(a_i, e_i, \sigma_i) \neq (a_j, e_j, \sigma_j)$, then*

[17] See Guiso and Paiella [24].

[18] Indeed, $\rho_i''(x_i) = 0$ and when $(a_i, e_i, \sigma_i) = (a, e, \sigma)$, then $\rho_i'(x_i) = \rho'(x) = 1/(1-\sigma)$.

the social absolute risk tolerance is convex and there exists a distribution θ^0 such that one of the following cases holds:

i) If Assumption 8 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy E such that $\theta^E \prec_I \theta^0$ and is saddle-point stable with oscillations otherwise.

ii) If Assumption 9 is satisfied, the steady state is saddle-point stable for any economy E such that $\theta^E \prec_I \theta^0$ and is unstable otherwise. Moreover, there generically exist period-two cycles for any economy E characterized by a distribution θ^E in a right neighborhood of θ^0, which are saddle-point stable.

This Proposition provides a strong and clear-cut result in the sense that individual HARA preferences always imply a positive relationship between wealth inequality and macroeconomic volatility provided a slight amount of preference heterogeneity across agents is considered.[19]

These results echo the conclusions obtained with endogenous labor. Indeed, in Section 8 we considered an economy with heterogeneous agents characterized by CES preferences leading to individual linear absolute risk tolerance indices. With Theorem 2 we showed a positive correlation between wealth inequality and macroeconomic volatility. Therefore, the present analysis show that this conclusion does not depend on whether the labor supply is elastic or not and that it is compatible with standard utility functions.

9.2 Pareto wealth distribution and output volatility

As in Section 7 we assume now that the distribution of individual wealth follows a Pareto distribution of parameter α (or more precisely its discrete analogue as given by the Zeta distribution, see Assumption 6). We also assume that agents are homogeneous with respect to their preferences and focus on a specific class of preferences. In particular, we assume that Assumption 7 holds so that their preferences give rise to $\rho_i(x_i) = x_i^{\varphi}/\sigma$, with $\sigma, \varphi > 0$. The associated absolute risk tolerance index $\rho(x)$ is obtained by equation (54). For any value of $\varphi \in (0,1) \cup (1, +\infty)$ such that $\alpha > 1 + \varphi$ we

[19] In Ghiglino and Venditti [22], we also provide illustrations of the correlation between wealth inequality and macroeconomic volatility using homogeneous preferences characterized by a non linear individual absolute risk tolerance index. We use the specification as given by Assumption 7 and (30).

find that $\rho(x^*) = \bar{\rho}(\alpha)$ is an increasing and convex function of the level of inequality, because inequalities increase when α decreases (see Figure 1 in Appendix 11.7 for an illustration). Supposing that the technology is such that Assumption 8 or 9 holds, Theorem 3 implies the following result:

Proposition 9. *Let Assumptions 2, 6 and 7 hold, together with either Assumption 8, or Assumption 9. Assume also that $\varphi \in (0,1) \cup (1,+\infty)$ and $\alpha > 1+\varphi$. Then there exists $\bar{\sigma} > 0$, $\bar{\alpha} > 1+\varphi$ and a level of wealth inequality characterized by a Gini index $\bar{G} = 1/(2\bar{\alpha}-1)$ such that when $\sigma \in (0,\bar{\sigma})$, one of the following cases holds:*

1 - If Assumption 8 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy E such that $G_E < \bar{G}$ and is saddle-point stable with oscillations otherwise.

2 - If Assumption 9 is satisfied, the steady state is saddle-point stable with oscillating convergence for any economy E such that $G_E < \bar{G}$ and is unstable otherwise. Moreover, there generically exist period-two cycles, in a right neighborhood of $\bar{G}$, which are saddle-point stable.

To conclude, we have shown that when individual wealth follows a Pareto distribution, an increase in inequality cannot reduce the level of macroeconomic volatility, a result that holds independently of whether labour is provided elastically or inelastically,

10 Conclusion

We have considered a two-sector optimal growth model with endogenous labor and heterogeneous agents with respect to preferences and their capital share. We have provided conditions on the slopes of the absolute risk tolerance indices for consumption and labor in order to get a positive correlation between wealth inequality and macroeconomic volatility. We have shown that such a conclusion is easily obtained either when the absolute risk tolerance indices are non linear, even with homogeneous preferences, or when the absolute risk tolerance indices are linear provided some degree of preference heterogeneity across agents is introduced.

11 Appendix

11.1 Proof of Lemma 1

Without loss of generality assume that there are three types of consumers. Let n_i be the number of agents of type $i = 1, 2, 3$ with $n_1 + n_2 + n_3 = n$. It is easy to show that all agents of the same type are given the same Pareto weight.[20] The social utility function is thus defined by

$$U(x,\ell) = \max_{(x_i,l_i)_{i=1}^2} \Big\{ \eta_1 n_1 u_1(x_1) + \eta_2 n_2 u_2(x_2) + \eta_3 n_3 u_3((x - n_1 x_1 - n_2 x_2)/n_3) \\ + \eta_1 n_1 v_1(\bar{l} - l_1) + \eta_2 n_2 v_2(\bar{l} - l_2) + \eta_3 n_3 v_3(\bar{l} - (\ell - n_1 l_1 - n_2 l_2)/n_3) \Big\}$$

with $\eta_i \geq 0$ and $\eta_1 + \eta_2 + \eta_3 = 1$. The first and second order derivatives of the social utility function can be related to the derivatives of the individual utility function of the agents. Indeed, the first order conditions associated with program (8) give

$$\Psi^1(x_1, x_2, x; \eta_1, \eta_2) = \eta_1 n_1 u_1'(x_1) - \eta_3 n_1 u_3'\left(\tfrac{x - n_1 x_1 - n_2 x_2}{n_3}\right) = 0 \quad (36)$$

$$\Psi^2(x_1, x_2, x; \eta_1, \eta_2) = \eta_2 n_2 u_2'(x_2) - \eta_3 n_2 u_3'\left(\tfrac{x - n_1 x_1 - n_2 x_2}{n_3}\right) = 0 \quad (37)$$

$$\Phi^1(l_1, l_2, \ell; \eta_1, \eta_2) = -\eta_1 n_1 v_1'(\bar{l} - l_1) + \eta_3 n_1 v_3'\left(\bar{l} - \tfrac{\ell - n_1 l_1 - n_2 l_2}{n_3}\right) = 0 \quad (38)$$

$$\Phi^2(l_1, l_2, \ell; \eta_1, \eta_2) = -\eta_2 n_2 v_2'(\bar{l} - l_2) + \eta_3 n_2 v_3'\left(\bar{l} - \tfrac{\ell - n_1 l_1 - n_2 l_2}{n_3}\right) = 0 \quad (39)$$

Notice that the first order conditions with respect to x_i are independent from the first order conditions with respect to l_i. It follows that the social utility function is additively separable, i.e. $U(x, \ell) = u(x) - v(\ell)$. The following expressions are easily obtained

$$\begin{aligned} u'(x) &= \eta_3 u_3'\left(\tfrac{x - n_1 x_1 - n_2 x_2}{n_3}\right) = \eta_1 u_1'(x_1) = \eta_2 u_2'(x_2) > 0 \\ u''(x) &= \eta_1 u_1''(x_1) \tfrac{\partial x_1}{\partial x} \\ v'(\ell) &= \eta_3 v_3'\left(\bar{l} - \tfrac{\ell - n_1 l_1 - n_2 l_2}{n_3}\right) = \eta_1 v_1'(\bar{l} - l_1) = \eta_2 v_2'(\bar{l} - l_2) > 0 \\ v''(\ell) &= -\eta_1 v_1''(\bar{l} - l_1) \tfrac{\partial l_1}{\partial \ell} \end{aligned} \quad (40)$$

[20] The first order conditions associated with the maximization program (8) that defines the social utility function give $\eta_i = \lambda / u_i'(x_i)$ with $\lambda \geq 0$ the Lagrange multiplier associated with the resources constraint. Consider then two agents j and k, $j \neq k$, of the same type, i.e. such that $u_j = u_k$, $\omega_j = \omega_k$ and $\theta_j = \theta_k$. It follows that x_j and x_k are solutions of two optimizations problems with the same utility function, the same initial resources and the same budget constraint. Therefore we obviously derive $x_j = x_k$ and thus $\eta_j = \eta_k$.

where x represents the aggregate consumption. The implicit function theorem applied to (36) and (37) allows us to express x_1 as a function of x. In matrix form we can write

$$\begin{pmatrix} \frac{\partial x_1}{\partial x} \\ \frac{\partial x_2}{\partial x} \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Psi^1}{\partial x_1} & \frac{\partial \Psi^1}{\partial x_2} \\ \frac{\partial \Psi^2}{\partial x_1} & \frac{\partial \Psi^2}{\partial x_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Psi^1}{\partial x} \\ \frac{\partial \Psi^2}{\partial x} \end{pmatrix}$$

We then get

$$\frac{\partial x_1}{\partial x} = \frac{\eta_2 \eta_3 u_2''(x_2) u_3''(x_3)}{\eta_1 \eta_2 n_3 u_1''(x_1) u_2''(x_2) + \eta_1 \eta_3 n_2 u_1''(x_1) u_3''(x_3) + \eta_2 \eta_3 n_1 u_2''(x_2) u_3''(x_3)} > 0 \qquad (41)$$

and thus $u''(x) < 0$. Similarly, the implicit function theorem applied to (38) and (39) allows us to express l_1 as a function of ℓ. In matrix form we have

$$\begin{pmatrix} \frac{\partial l_1}{\partial \ell} \\ \frac{\partial l_2}{\partial \ell} \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Phi^1}{\partial l_1} & \frac{\partial \Phi^1}{\partial l_2} \\ \frac{\partial \Phi^2}{\partial l_1} & \frac{\partial \Phi^2}{\partial l_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Phi^1}{\partial \ell} \\ \frac{\partial \Phi^2}{\partial \ell} \end{pmatrix}$$

We then get

$$\frac{\partial l_1}{\partial \ell} = \frac{\eta_2 \eta_3 v_2''(\bar{l}-l_2) v_3''(\bar{l}-l_3)}{\eta_1 \eta_2 n_3 v_1''(\bar{l}-l_1) v_2''(\bar{l}-l_2) + \eta_1 \eta_3 n_2 v_1''(\bar{l}-l_1) v_3''(\bar{l}-l_3) + \eta_2 \eta_3 n_1 v_2''(\bar{l}-l_2) v_3''(\bar{l}-l_3)} > 0 \qquad (42)$$

and $v''(x) > 0$. Under Assumption 1, we also derive from (40) that $\lim_{x \to 0} u'(x) = +\infty$ and $\lim_{\ell \to n\bar{l}} v'(\ell) = +\infty$. □

11.2 Proof of Theorem 1

Denoting $\kappa = k/\ell$, we derive from (13)-(14) and the first order conditions associated with program (4) that an aggregate steady state may be defined as a pair (κ^*, ℓ^*) solution of the following equations

$$-\frac{T_1(\kappa, \mu\kappa, 1)}{T_2(\kappa, \mu\kappa, 1)} = f_1^1(k_1(\kappa, \mu\kappa, 1), l_1(\kappa, \mu\kappa, 1)) = \delta^{-1} - (1-\mu) \qquad (43)$$

$$u'(\ell T(\kappa, \mu\kappa, 1)) T_3(\kappa, \mu\kappa, 1) - v'(\ell) = 0 \qquad (44)$$

Consider in a first step equation (43). Notice that the steady state value for κ only depends on the characteristics of the technologies and is independent from the utility function. Moreover, equation (43) is equivalent to the equation which defines the stationary capital stock of a two-sector optimal growth model with inelastic labor. The proof of Theorem 3.1 in Becker and Tsyganov [3] applies so that there exists one unique κ^* solution of (43).

Consider in a second step equation (44) evaluated at κ^*. We get:

$$T_3(\kappa^*, \mu\kappa^*, 1) = \frac{v'(\ell)}{u'(\ell T(\kappa^*,\mu\kappa^*,1))} \equiv \varphi(\ell)$$

The function $\varphi(\ell)$ is defined over $(0, \bar{\ell})$ and satisfies

$$\varphi'(\ell) = \frac{u'(x)v''(\ell) - u''(x)v'(\ell)T}{u'(x)^2} > 0$$

This monotonicity property together with the boundary conditions provided by Lemma 1 finally garantee the existence and uniqueness of a solution $\ell^* \in (0, \bar{\ell})$ of equation (44).

Let us now consider the first order conditions corresponding to the individual maximization of the intertemporal utility function (1) subject to the intertemporal budget constraint (2):

$$\delta^t u_i'(x_{it}) = \pi_i R_t \tag{45}$$

$$\delta^t v_i'(\bar{l} - l_{it}) = \pi_i R_t w_t \tag{46}$$

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t w_t l_{it} + \theta_i r_0 k_0 \tag{47}$$

$\forall t \geq 0$ and $i = 1, \dots, n$, where π_i is the Lagrange multiplier associated with the intertemporal budget constraint (2). From (6) we conclude that the interest rate satisfies

$$1 + d_t = \frac{r_t + (1-\mu)p_t}{p_{t-1}} = -\frac{T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)}{T_2(k_{t-1}, y_{t-1})}$$

for any $t \geq 1$ and $1 + d_0 = r_0/p_{-1}$ for $t = 0$. The Euler equation (13) evaluated at a steady state $x_{it} = x_i^*$ gives $1 + d^* = \delta^{-1}$ and thus $R_t = \delta^t$. Recall that from (6) we also get $T_1^* = r^*$, $T_2^* = -p^*$ and $w^* = T_3^*$. The intertemporal budget constraint (47) evaluated along the stationary path with $k_t = k^*$ for all $t \geq 0$ and $p_{-1} = p^*$ becomes $x_i^* = w^* l_i^* + (1-\delta)\theta_i p^* k^*/\delta$, with $p^* = \delta\vartheta r^*$. We then get

$$x_i^* = w^* l_i^* + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i \tag{48}$$

with $\kappa^* = k^*/\ell^*$. Using (45) and (46), we finally obtain

$$u_i'(w^* l_i^* + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i) w^* = v_i'(\bar{l} - l_i^*) \tag{49}$$

Assumption 1 implies $\lim_{l_i \to \bar{l}} v_i'(\bar{l} - l_i) = +\infty > u_i'(w^* \bar{l} + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i) w^*$. Therefore, if $v_i'(\bar{l}) < u_i'((1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i)$ for any $i = 1, \dots, n$, there exist unique steady state values for all individual consumptions x_i^* and labor supplies l_i^* solutions of equations (48) and (49). Notice that since equation (48) depends on θ_i and $\ell^* = \sum_{i=1}^n l_i^*$, we conclude that x_i^* and l_i^* are functions of the distribution of capital shares $\theta = (\theta_i)_{i=1}^n$, namely $x_i^*(\theta)$ and $l_i^*(\theta)$. For all $i = 1, \ldots, n$, consider finally equations (48) and (49) expressed as follows:

$$\begin{aligned} x_i^* - w^* l_i^* - (1-\delta)\vartheta r^* \kappa^* \left(\sum_{i=1}^n l_i^* \right) \theta_i &= 0 \\ u_i'(x_i^*) w^* - v_i'(\bar{l} - l_i^*) &= 0 \end{aligned} \tag{50}$$

with $\theta_j = 1 - \sum_{i=1, i \neq j}^n \theta_i$. Applying the implicit function theorem, we now show that $x_i^*(\theta)$ and $l_i^*(\theta)$ are C^1-functions. The $2n \times 2n$ Jacobian matrix of (50) with respect to $(x_i^*, l_i^*)_{i=1}^n$ is

$$J_{(x_i, l_i)} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

with $J_{11} = I_{n \times n}$,

$$J_{21} = w^* \begin{pmatrix} u_1''(x_1^*) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & u_n''(x_n^*) \end{pmatrix}, \quad J_{22} = \begin{pmatrix} v_1''(\bar{l} - l_1^*) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & v_n''(\bar{l} - l_n^*) \end{pmatrix}$$

and

$$J_{12} = -T^* I_{n \times n} + (1-\delta)\vartheta r^* \kappa^* \begin{pmatrix} 1-\theta_1 & \theta_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_1 \\ \theta_2 & 1-\theta_2 & \theta_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_2 \\ \vdots & & \ddots & \vdots & \vdots & & & & \vdots \\ \theta_{j-1} & \cdots & \theta_{j-1} & 1-\theta_{j-1} & \theta_{j-1} & \cdots & \cdots & \cdots & \theta_{j-1} \\ 1-\sum_{\substack{i=1 \\ i \neq j}}^n \theta_i & \cdots & \cdots & 1-\sum_{\substack{i=1 \\ i \neq j}}^n \theta_i & \sum_{\substack{i=1 \\ i \neq j}}^n \theta_i & 1-\sum_{\substack{i=1 \\ i \neq j}}^n \theta_i & \cdots & \cdots & 1-\sum_{\substack{i=1 \\ i \neq j}}^n \theta_i \\ \theta_{j+1} & \cdots & \cdots & \cdots & \theta_{j+1} & 1-\theta_{j+1} & \theta_{j+1} & \cdots & \theta_{j+1} \\ \vdots & & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & & & \ddots & \vdots \\ \theta_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_n & 1-\theta_n \end{pmatrix}$$

Consider the matrix $B = J_{22} - J_{21} J_{12}$. Tedious computations available upon request show that $|B| \neq 0$ with $sign|B| = (-1)^n$ and $B^{-1} = |B|^{-1} \left[b_{ij} \right]_{i,j=1}^n$ with $signb_{ii} = (-1)^{n-1}$ and $signb_{ij} = (-1)^n$ for $i \neq j$. The Jacobian matrix $J_{(x_i, l_i)}$ therefore admits an inverse such that

$$J_{(x_i, l_i)}^{-1} = \begin{pmatrix} I_{n \times n} + J_{12} B^{-1} J_{21} & -J_{12} B^{-1} \\ -B^{-1} J_{21} & B^{-1} \end{pmatrix}$$

Tedious computations available upon request also show that

$$I_{n\times n}+J_{12}B^{-1}J_{21}=|B|^{-1}\left[|B|I_{n\times n}+J_{12}\left[b_{ij}\right]J_{21}\right]\equiv|B|^{-1}\left[c_{ij}\right]_{i,j=1}^{n}$$

with $signc_{ii}=(-1)^n$ and $signc_{ij}=(-1)^{n-1}$ for $i\neq j$. The $2n\times(n-1)$ Jacobian matrix of (50) with respect to $\theta=(\theta_i)_{i=1}^n$ with $\theta_j=1-\sum_{i=1,i\neq j}^{n}\theta_i$

$$J_\theta=(1-\delta)\vartheta r^*\kappa^*\ell^*\begin{pmatrix}
-1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \ddots & & \vdots & \vdots & & & & \vdots\\
\vdots & & \ddots & 0 & \vdots & & & & \vdots\\
0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0\\
1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 1\\
0 & \cdots & \cdots & 0 & 0 & -1 & 0 & \cdots & 0\\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots\\
\vdots & & & \vdots & \vdots & & & \ddots & 0\\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
\vdots & & & & & & & & \vdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}\begin{array}{l}
\\ \left.\begin{array}{l}\\ \\ \end{array}\right\}\ j-1\\
\text{line } j\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\ n-j\\
\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\ n
\end{array}$$

We then conclude from the implicit function theorem that $x_i^*(\theta)$ and $l_i^*(\theta)$ are C^1-functions with

$$\left[\frac{\partial l_i}{\partial\theta_j}\right]_{n\times(n-1)}=\frac{(1-\delta)\vartheta r^*\kappa^*\ell^*}{|B|}\left[b_{ij}\right]J_{21}\begin{pmatrix}
-1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \ddots & & \vdots & \vdots & & & & \vdots\\
\vdots & & \ddots & 0 & \vdots & & & & \vdots\\
0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0\\
1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 1\\
0 & \cdots & \cdots & 0 & 0 & -1 & 0 & \cdots & 0\\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots\\
\vdots & & & \vdots & \vdots & & & \ddots & 0\\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1
\end{pmatrix}$$

$$=\frac{(1-\delta)\vartheta r^*\kappa^*w^*\ell^*}{|B|}\left[b_{ij}\right]\begin{pmatrix}
-u_1''(x_1^*) & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \ddots & & \vdots & \vdots & & & & \vdots\\
\vdots & & \ddots & 0 & \vdots & & & & \vdots\\
0 & \cdots & 0 & -u_{j-1}''(x_{j-1}^*) & 0 & \cdots & \cdots & \cdots & 0\\
u_j''(x_j^*) & \cdots & \cdots & u_j''(x_j^*) & u_j''(x_j^*) & u_j''(x_j^*) & \cdots & \cdots & u_j''(x_j^*)\\
0 & \cdots & \cdots & 0 & 0 & -u_{j+1}''(x_{j+1}^*) & 0 & \cdots & 0\\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots\\
\vdots & & & \vdots & \vdots & & & \ddots & 0\\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -u_n''(x_n^*)
\end{pmatrix}$$

and

$$\left[\frac{\partial x_i}{\partial \theta_j}\right]_{n\times(n-1)} = -\frac{(1-\delta)\vartheta r^*\kappa^*\ell^*}{|B|}[c_{ij}]\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & -1 \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$$

It follows that

$$\begin{aligned} \frac{\partial x_i^*(\theta)}{\partial \theta_i} &= -\frac{(1-\delta)\vartheta r^*\kappa^*\ell^*(c_{ij}-c_{ii})}{|B|} > 0 \\ \frac{\partial x_j^*(\theta)}{\partial \theta_i} &= -\frac{(1-\delta)\vartheta r^*\kappa^*\ell^*(c_{jj}-c_{ji})}{|B|} < 0 \\ \frac{\partial l_i^*(\theta)}{\partial \theta_i} &= \frac{(1-\delta)\vartheta r^*\kappa^* w^*\ell^*(b_{ij}u_j''(x_j^*)-b_{ii}u_i''(x_i^*))}{|B|} < 0 \\ \frac{\partial l_j^*(\theta)}{\partial \theta_i} &= \frac{(1-\delta)\vartheta r^*\kappa^* w^*\ell^*(b_{jj}u_j''(x_j^*)-b_{ji}u_i''(x_i^*))}{|B|} > 0 \end{aligned} \quad (51)$$

□

11.3 Proof of Corollary 1

As $\ell^*(\theta) = \sum_{i=1}^n l_i^*(\theta)$ and $x^*(\theta) = \sum_{i=1}^n x_i^*(\theta)$ with $l_i^*(\theta)$ and $x_i^*(\theta)$ some C^1-functions, we conclude that $\ell^*(\theta)$ and $x^*(\theta)$ are also C^1-functions. Moreover, we derive from (51)

$$\frac{\partial \ell^*(\theta)}{\partial \theta_i} = \sum_{k=1}^n \frac{\partial l_k^*(\theta)}{\partial \theta_i}$$

Recalling that $\theta_j = 1 - \sum_{i=1, i\neq j}^n \theta_i$, tedious computations available upon request then give

$$\begin{aligned} \frac{\partial \ell^*(\theta)}{\partial \theta_i} &= \frac{(1-\delta)\vartheta r^*\kappa^* w^*\ell^*}{|B|}\prod_{k\neq i,j}\left[v_k''(\bar{l}-l_k^*) + w^{*2}u_k''(x_k^*)\right] \\ &\times \; v_i''(\bar{l}-l_i^*)v_j''(\bar{l}-l_j^*)\left[\frac{u_j''(x_j^*)}{v_j''(\bar{l}-l_j^*)} - \frac{u_i''(x_i^*)}{v_i''(\bar{l}-l_i^*)}\right] \end{aligned}$$

Under Assumption 1, we know that $sign\prod_{k\neq i,j}[v_k''(\bar{l}-l_k^*) + w^{*2}u_k''(x_k^*)] = (-1)^{n-2}$ and thus $sign|B|\prod_{k\neq i,j}[v_k''(\bar{l}-l_k^*) + w^{*2}u_k''(x_k^*)] = (-1)^{2(n-1)} > 0$. It follows therefore that $\partial\ell^*(\theta)/\partial\theta_i > 0$ if and only if

$$\frac{u_j''(x_j^*)}{v_j''(\bar{l}-l_j^*)} > \frac{u_i''(x_i^*)}{v_i''(\bar{l}-l_i^*)}$$

Using (18) and (19), this condition becomes $(\epsilon_l^j(l_j)/\epsilon_x^j(x_j))(l_j/x_j) > (\epsilon_l^i(li)/\epsilon_x^i(x_i))(l_i/x_i)$. But, as the n agents are ordered according to their capital endowment, i.e. $\theta_i > \theta_j$ for $i < j$, starting from a configuration of pure equality with $\theta_k = 1/n$ for all k, we know from Theorem 1 that if we increase θ_i and thus decrease θ_j we get $x_i > x_j$ and $\ell_i < \ell_j$. It follows that $l_j/x_j > l_i/x_i$. The last result is finally obtained from the fact that $x^*(\theta) = T(\kappa^*, \mu\kappa^*, 1)\ell^*(\theta)$.

□

11.4 Proof of Proposition 1

In a one-sector economy with heterogeneous agents, Kehoe *et al.* [27] show that the welfare weights are continuous functions of the initial capital stock. This continuity property happens to be satisfied because the value function of the planner's problem (10) is C^2. However, in a multisector economy such a property is much more difficult to obtain. Santos [35] shows that the main sufficient condition to get this property is to assume strong concavity for the indirect utility function $\mathcal{V}(k_t, k_{t+1})$ (see Assumption B and Theorem 2.2 in Santos [35]). On a compact set, a C^2 function $\mathcal{V}(k_t, k_{t+1})$ is strongly concave if its Hessian matrix is always non-singular and negative-definite. In other words, the smallest eigenvalue in absolute value of the Hessian matrix needs to be strictly positive over the domain of definition of $\mathcal{V}(k_t, k_{t+1})$. In our two-sector model with endogenous labor, the indirect social utility function also depends on ℓ_t:

$$V(k_t, k_{t+1}, \ell_t) = u(T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)) - v(\ell_t)$$

with ℓ_t a solution of equation (14):

$$u'(T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t))w_t - v'(\ell_t) \equiv \phi(k_t, k_{t+1}, \ell_t) = 0$$

Since under Assumptions 1 and 2

$$\phi_3(k_t, k_{t+1}, \ell_t) = u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t) < 0$$

we derive from the implicit function theorem that $\ell_t = \psi(k_t, k_{t+1})$ with $\psi(.)$ a C^1-function such that

$$\begin{aligned} \psi_1(k_t, k_{t+1}) &= -\tfrac{u''(x_t)(r_t+(1-\mu)p_t)w_t + u'(x_t)(T_{31}-(1-\mu)T_{32})}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)} \\ \psi_2(k_t, k_{t+1}) &= -\tfrac{u'(x_t)T_{32} - u''(x_t)p_t w_t}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)} \end{aligned}$$

We then obtain

$$\mathcal{V}(k_t, k_{t+1}) = u(T(k_t, k_{t+1} - (1-\mu)k_t, \psi(k_t, k_{t+1}))) - v(\psi(k_t, k_{t+1}))$$

Consider now the social production function $T(k_t, y_t, \ell_t)$ with $y_t = k_{t+1} - (1-\mu)k_t$ and $\ell_t = \psi(k_t, k_{t+1})$. Proceeding in a similar way as for the set of admissible paths $\tilde{\mathcal{D}}$, we derive that $T(k_t, y_t, \ell_t) = 0$ if and only if $k_{t+1} = h(k_t)$. Assumption 2 then implies that there exists $\hat{k} > 0$ such that $h(k_t) > k_t$ when $k_t < \hat{k}$ while $h(k_t) < k_t$ when $k_t > \hat{k}$. As a result, $\mathcal{V}(k_t, k_{t+1})$ is defined over the compact, convex set

$$\hat{\mathcal{D}} = \left\{(k_t, k_{t+1}) \in \mathbb{R}^2_+ | 0 \leq k_t \leq \hat{k},\ (1-\mu)k_t \leq k_{t+1} \leq h(k_t)\right\}$$

We know that T is homogeneous of degree one so that its Hessian matrix $H_T(k_t, k_{t+1})$ is singular for any $(k_t, k_{t+1}) \in \hat{\mathcal{D}}$. As shown in Benhabib and Nishimura [5] and Bosi *et al.* [9], the second order derivatives of the social production function depend on the allocations of capital and labor across the two sectors. We get indeed

$$\begin{aligned}
T_{12}(k,y,\ell) &= -T_{11}(k,y,\ell)b(k,y,\ell) \\
T_{22}(k,y,\ell) &= T_{11}(k,y,\ell)b(k,y,\ell)^2 \\
T_{13}(k,y,\ell) &= -T_{11}(k,y,\ell)a(k,y,\ell) \\
T_{23}(k,y,\ell) &= T_{11}(k,y,\ell)a(k,y,\ell)b(k,y,\ell) \\
T_{33}(k,y,\ell) &= T_{11}(k,y,\ell)a(k,y,\ell)b(k,y,\ell)^2
\end{aligned} \tag{52}$$

where

$$b(k,y,\ell) = \frac{l^0}{T}\left(\frac{k^1}{l^1} - \frac{k^0}{l^0}\right) \tag{53}$$

is the relative capital intensity difference across sectors and $a(k,y,\ell) = k^0/l^0 > 0$ is the capital-labor ratio in the consumption good sector. We derive from (52):

$$H_T(k_t, k_{t+1}) = T_{11}\begin{pmatrix} 1 & -b_t & -a_t \\ -b_t & b_t^2 & a_t b_t \\ -a_t & a_t b_t & a_t^2 \end{pmatrix}$$

with $a_t = a(k_t, k_{t+1} - (1-\mu)k_t, \psi(k_t, k_{t+1}))$ and $b_t = b(k_t, k_{t+1} - (1-\mu)k_t, \psi(k_t, k_{t+1}))$. The Hessian matrix of $\mathcal{V}$ is then

$$\begin{aligned}
H_{\mathcal{V}}(k_t, k_{t+1}) = {} & u'T_{11}\begin{pmatrix} 1 & -(1-\mu) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b_t \\ -b_t & b_t^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -(1-\mu) & 1 \end{pmatrix} \\
& + u'T_{11}\begin{pmatrix} 1 & -(1-\mu) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -a_t & -a_t \\ a_t b_t & a_t b_t \end{pmatrix}\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \\
& + u''\begin{pmatrix} T_1 - (1-\mu)T_2 \\ T_2 \end{pmatrix}\begin{pmatrix} T_1 - (1-\mu)T_2 + T_3\psi_1 & T_2 + T_3\psi_2 \end{pmatrix}
\end{aligned}$$

Tedious but straightforward computations finally give the determinant of $H_{\mathcal{V}}(k_t, k_{t+1})$ as

$$|H_{\mathcal{V}}| = -\frac{u'(x_t)u''(x_t)v''(\ell_t)T_{11}(b_t r_t - p_t)^2}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)} \geq 0$$

Under Assumptions 1-2, we conclude from Lemma 1 that the Hessian matrix of $\mathcal{V}$ is non singular if over the interior of the set $\hat{\mathcal{D}}$ we have $b_t r_t - p_t \neq 0$ or equivalently, $T_2(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) + b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)T_1(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \neq 0$. This property also implies that the value function of the planner's problem (10) is C^2. □

11.5 Proof of Theorem 2

We begin by showing that if the social curvature indices $\rho(x^*)$ and $\gamma(\ell^*)$ are linear functions, then the degree of inequality does not have any influence on the existence of endogenous fluctuations. Linear expressions for $\rho(x^*)$ and $\gamma(\ell^*)$ are obtained in particular when agents are identical with respect to preferences and their utility function is CES (see (33)). In this case, $\rho_i(x_i) = x_i/\sigma$, $\gamma_i(l_i) = l_i/\gamma$, with $l_i = \bar{l} - \mathcal{L}_i$, $\sigma \geq 0$, $\gamma \geq 0$, and thus $\rho(x^*) = x^*/\sigma$ and $\gamma(\ell^*) = \ell^*/\gamma$. As a result, the inequalities entering the conditions in Propositions 3 and 4 become:

$$\begin{aligned} \rho(x^*) > \rho_i &\quad \Leftrightarrow \quad \tfrac{1}{\sigma} > \Lambda_i \\ \gamma(\ell^*) < \gamma_i &\quad \Leftrightarrow \quad \tfrac{1}{\gamma} < T^*\Gamma_i \left[\tfrac{1}{\sigma} - \Lambda_i\right] \end{aligned}$$

for $i = c, y$, since $x^* = \ell^* T^*$. As they are based only on parameters, it follows that inequality does not have any effect on the occurrence of cycles.

Assume therefore that Assumption 3 is satisfied. We focus on bilateral transfers between pairs of agents i and j that are elasticity-ordered according to Definition 3. Consider first the case in which Assumption 4 holds. Let us denote $\zeta_c(\theta) = \rho(T^*\ell^*(\theta)) - \rho_c$ and $\xi_c(\theta) = \gamma(\ell^*(\theta)) - \gamma_c$ with $\rho_c < \bar{\rho}$ and $\gamma_c \in (\underline{\gamma}, \bar{\gamma})$ as defined by (26). Proposition 3 shows that the steady state is saddle-point stable with oscillating convergence if $\rho(x^*) > \rho_c$ and $\gamma(\ell^*) < \gamma_c$. Therefore, an increase of wealth inequality implied by an increase of the share of capital θ_i owned by agent i, and thus a decrease of the share of capital θ_j owned by agents j, leads to damped fluctuations if the functions $\zeta_c(\theta)$ and $\xi_c(\theta)$ are respectively increasing and decreasing with respect to θ_i. We easily get

$$\frac{\partial \zeta_c(\theta)}{\partial \theta_i} = T^* \frac{\partial \ell^*(\theta)}{\partial \theta_i} \left[\rho'(x^*) - \Lambda_c\right]$$
$$\frac{\partial \xi_c(\theta)}{\partial \theta_i} = \frac{\partial \ell^*(\theta)}{\partial \theta_i} \Big[\gamma'(\ell^*) - \Gamma_c T^* \left[\rho'(x^*) - \Lambda_c\right]\Big]$$

These derivatives are respectively positive and negative in the following two configurations:

i) if $\partial\ell^*(\theta)/\partial\theta_i < 0$, $\rho'(x^*) < \Lambda_c$ and $\gamma'(\ell^*) > \Gamma_c T^* \left[\rho'(x^*) - \Lambda_c\right]$,

ii) if $\partial\ell^*(\theta)/\partial\theta_i > 0$, $\rho'(x^*) > \Lambda_c$ and $\gamma'(\ell^*) < \Gamma_c T^* \left[\rho'(x^*) - \Lambda_c\right]$.

The final result then follows from Corollary 1.

Consider now the case in which Assumption 5 holds. Let us denote $\zeta_f(\theta) = \rho(T^*\ell^*(\theta)) - \rho_f$ and $\xi_f(\theta) = \gamma(\ell^*(\theta)) - \gamma_f$ with $\rho_f < \bar{\rho}$ and $\gamma_f \in (\underline{\gamma}, \bar{\gamma})$ as defined by (27). Proposition 4 shows that the steady state is locally unstable with oscillating divergence if $\rho(x^*) > \rho_f$ and $\gamma(\ell^*) < \gamma_f$, and γ_f is a flip bifurcation value so that there generically exist period-two cycles, in a left neighborhood of γ_f, which are saddle-point stable. Therefore, an increase of wealth inequality implied by an increase of the share of capital θ_i owned by agent i, and thus a decrease of the share of capital θ_j owned by agents j, leads to persistent fluctuations if the functions $\zeta_f(\theta)$ and $\xi_c(\theta)$ are respectively increasing and decreasing with respect to θ_i. We easily get

$$\frac{\partial \zeta_f(\theta)}{\partial \theta_i} = T^* \frac{\partial \ell^*(\theta)}{\partial \theta_i} \left[\rho'(x^*) - \Lambda_f\right]$$
$$\frac{\partial \xi_f(\theta)}{\partial \theta_i} = \frac{\partial \ell^*(\theta)}{\partial \theta_i} \Big[\gamma'(\ell^*) - \Gamma_f T^* \left[\rho'(x^*) - \Lambda_f\right]\Big]$$

These derivatives are respectively positive and negative in the following two configurations:

i) if $\partial\ell^*(\theta)/\partial\theta_i < 0$, $\rho'(x^*) < \Lambda_f$ and $\gamma'(\ell^*) > \Gamma_f T^* \left[\rho'(x^*) - \Lambda_f\right]$,

ii) if $\partial\ell^*(\theta)/\partial\theta_i > 0$, $\rho'(x^*) > \Lambda_f$ and $\gamma'(\ell^*) < \Gamma_f T^* \left[\rho'(x^*) - \Lambda_f\right]$.

The final result then follows from Corollary 1. □

11.6 Proof of lemma 3

As there is a unique consumption good and agents have a common time discount factor, along the steady state the distribution of wealth is identical to the distribution of individual consumption. On the other hand we have seen that $x_i(\theta) = wl_i(\theta) + (1-\delta)\vartheta r\kappa l\theta_i = wl_i(\theta) + (1-\delta)\vartheta rk\theta_i$. However, the steady state quantities depend on the distribution of shares of capital, so that even if k^* is a steady state with a distribution θ, it is not a steady state with a distribution θ'. We therefore define a distribution χ such $\chi_i = k\theta_i$

so that $\sum_{i=1}^{n} \chi_i$ is not constrained to be 1. We then have $x_i(\chi) = wl_i(\chi) + (1-\delta)\vartheta r \chi_i$. For given values of all the x_i it is possible to find values of the χ_i such that the above n equations hold. Note that this is true even when several agents have the same consumption and therefore get associated with the same χ.

□

11.7 Proof of Theorem 3

As normalized incomes follow a Zeta distribution and along the steady state $x_i = \omega_i$, normalized individual consumption $\tilde{x}_i = x_i/x_{\min}$ also follows a Zeta distribution. The associated value of $\rho(x)$ is

$$\begin{aligned} \rho(x) &= \sum_{s=1}^{\infty} \frac{1}{\zeta(\alpha)} \frac{1}{s^{\alpha}} \rho(x(s)) = \frac{1}{\zeta(\alpha)} \sum_{s=1}^{\infty} s^{-\alpha} \frac{(x(s))^{\varphi}}{\sigma} \\ &= \frac{1}{\zeta(\alpha)} \sum_{s=1}^{\infty} s^{-\alpha} \frac{(s x_{\min})^{\varphi}}{\sigma} = \frac{x_{\min}^{\varphi}}{\sigma} \frac{\zeta(\alpha - \varphi)}{\zeta(\alpha)} = \bar{\rho}(\alpha) \end{aligned} \tag{54}$$

where $\zeta(\alpha) = \sum_{s=1}^{\infty} s^{-\alpha}$ is the Riemann Zeta function. As explained in Section 7.1, the sequence x is not implementable. However, it can be approximated by an implementable distribution of measure η that gives rise to a ρ which is arbitrarily close to $\rho(x)$. Indeed, consider the distribution Γ obtained in Lemma 2. Clearly, in this case

$$\lim_{n \to \infty} \rho(v(x)) = \rho(x)$$

We therefore can ignore this issue and assume that x is implementable. Notice that to ensure a finite value for $\zeta(\alpha - \varphi)$ we need to assume $\alpha > 1 + \varphi$.

As the stability properties depend also on the risk tolerance to leisure we need to know the distribution of leisure along the steady state. From (15) we derive that within the class of preferences considered here, leisure is related to consumption as follows

$$-\sigma \frac{x_i^{1-\varphi}}{1-\varphi} + \hat{w} = \gamma \frac{l_i^{1-\nu}}{1-\nu}$$

with $\hat{w} = \log w$ and w as defined in (6), giving

$$l_i = \left[\frac{\nu - 1}{\gamma(1-\varphi)} [\sigma - (1-\varphi)\hat{w}] \right]^{\frac{1}{1-\nu_i}} x_i^{\frac{1-\varphi}{1-\nu}} \tag{55}$$

Notice that this last equation requires $(\nu - 1)(1-\varphi)[\sigma - (1-\varphi)\hat{w}] > 0$. Combining (54) and (55) we then get

$$\gamma(\ell) = \frac{1}{\gamma}\left[\frac{\nu-1}{\gamma(1-\varphi)}[\sigma-(1-\varphi)\hat{w}]\right]^{\frac{\nu}{1-\nu}} \frac{1}{\zeta(\alpha)}\sum_{s=1}^{\infty} s^{-\alpha}(x(s))^{\frac{1-\varphi}{1-\nu}\nu}$$
$$= \frac{x_{\min}^{\frac{1-\varphi}{1-\nu}\nu}}{\gamma}\left[\frac{\nu-1}{\gamma(1-\varphi)}[\sigma-(1-\varphi)\hat{w}]\right]^{\frac{\nu}{1-\nu}} \frac{\zeta(\alpha+\frac{1-\varphi}{\nu-1}\nu)}{\zeta(\alpha)} \equiv \bar{\gamma}(\alpha) \quad (56)$$

Some calculations show that as soon as $(\nu-1)(1-\varphi)[\sigma-(1-\varphi)\hat{w}] > 0$, the functions $\bar{\rho}(\alpha)$ and $\bar{\gamma}(\alpha)$ are respectively decreasing and increasing in α. An illustration is given by the following Figure with $\varphi = 0.5$ and $\nu = 1.5$:

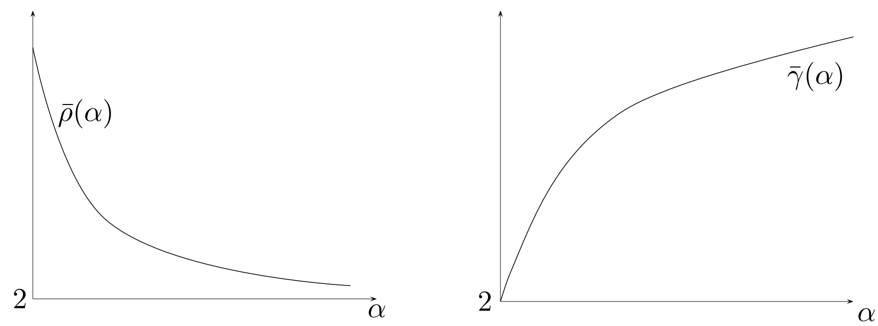

Figure 1: $\bar{\rho}(\alpha)$ and $\bar{\gamma}(\alpha)$.

Notice that these properties do not depend on the curvature of $\rho(x^*)$ and $\gamma(\ell^*)$, i.e. on whether φ or ν are larger or smaller than 1. For instance, a similar table is obtained if we set $\varphi = 1.1$ and $\nu = 0.5$

As inequalities increase when α decreases, we then conclude that $\rho(x^*) = \bar{\rho}(\alpha)$ and $\gamma(\ell^*) = \bar{\gamma}(\alpha)$ are respectively increasing and decreasing functions of the level of inequality. Building on Propositions 3 and 4, we may then drop the conditions on the elasticity in Definition 3 which simply becomes in the current framework: $\rho'(x^*) > \Psi$ with $\Psi \in \{\Lambda_c, \Lambda_f\}$. For a given set of technologies giving $\hat{w} = \log w$, and assuming that φ is such that $(1-\varphi)\hat{w} \leq 0$, it follows that this new version of Assumption 3 will be satisfied if the slope of $\bar{\rho}(\alpha)$ is large enough in absolute value, i.e. if σ is low enough. Notice that as $\bar{\gamma}(\alpha)$ is an increasing function of α , we do not need any particular restriction on its slope. We may also compute precisely the bounds $\underline{\rho}$, $\bar{\rho}$, $\underline{\gamma}$ and $\bar{\gamma}$. As $\lim_{\alpha\to 1+\varphi}\zeta(\alpha-\varphi) = +\infty$ and $\lim_{\alpha\to+\infty}\zeta(\alpha) = 1$, we get from (54) and (56):

$$\underline{\rho} = \lim_{\alpha\to+\infty}\bar{\rho}(\alpha) = \frac{1}{\sigma}, \quad \bar{\rho} = \lim_{\alpha\to 1+\varphi}\bar{\rho}(\alpha) + \infty \qquad (57)$$

and

$$\underline{\gamma} = \lim_{\alpha \to 1+\varphi} \bar{\gamma}(\alpha) \quad = \quad \frac{1}{\gamma}\left[\frac{\nu-1}{\gamma(1-\varphi)}[\sigma - (1-\varphi)\hat{w}]\right]^{\frac{\nu}{1-\nu}} \frac{\zeta(1+\varphi+\frac{1-\varphi}{\nu-1}\nu)}{\zeta(1+\varphi)} \tag{58}$$

$$\bar{\gamma} = \lim_{\alpha \to +\infty} \bar{\rho}(\alpha) \quad = \quad \frac{1}{\gamma}\left[\frac{\nu-1}{\gamma(1-\varphi)}[\sigma - (1-\varphi)\hat{w}]\right]^{\frac{\nu}{1-\nu}} \tag{59}$$

11.8 Computations for the CES example

As shown in Nishimura *et al.* [29] and Nishimura and Venditti [30], with the externality parameters set equal to zero and an identical elasticity of capital-labor substitution across sectors, the steady state values for the capital-labor ratio κ^* and the consumption per capita $T^* = T(\kappa^*, \kappa^*, 1)$ are given by:

$$\kappa^* = \frac{\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma}\left(\frac{(\delta\beta_1)^{1-\varsigma}-\beta_1}{\beta_0}\right)^{\frac{\varsigma}{1-\varsigma}}}{1-(\delta\beta_1)^{\varsigma}\left(1-\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma}\right)}$$

$$T^* = \frac{[1-(\delta\beta_1)^{\varsigma}]\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma}}{1-(\delta\beta_1)^{\varsigma}\left(1-\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma}\right)}\left[\alpha_0\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{1-\varsigma} + \alpha_1\frac{\beta_0}{(\delta\beta_1)^{1-\varsigma}-\beta_1}\right]^{\frac{\varsigma}{\varsigma-1}}$$

with $(\delta\beta_1)^{1-\varsigma} > \beta_1$. We also derive the prices

$$r^* = T_1^* = \alpha_1\left[\alpha_0\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{1-\varsigma}\frac{(\delta\beta_1)^{1-\varsigma}-\beta_1}{\beta_0} + \alpha_1\right]^{\frac{1}{\varsigma-1}}$$

and

$$p^* = -T_2^* = \delta r^*, \quad w^* = T_3^* = r^*\frac{\beta_0}{\beta_1}\left(\frac{(\delta\beta_1)^{1-\varsigma}-\beta_1}{\beta_0}\right)^{\frac{1}{1-\varsigma}}$$

At the steady state we get the capital intensity difference across sectors

$$b = (\delta\beta_1)^{\varsigma}\left[1 - \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma}\right]$$

the share of capital in total income

$$s = \left[1 + \frac{\beta_0}{\beta_1}\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{-\varsigma}\left(\frac{(\delta\beta_1)^{1-\varsigma}-\beta_1}{\beta_0}\right)(1-\delta b)\right]^{-1}$$

and the ratio of elasticities

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} = \frac{\alpha_1\beta_0}{\alpha_0}\frac{\varsigma\left((\delta\beta_1)^{1-\varsigma}-\beta_1 b\right)}{(\delta\beta_1)^{1-\varsigma}\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\varsigma-1}[1-(\delta\beta_1)^{\varsigma}][(\delta\beta_1)^{1-\varsigma}-\beta_1]} > 0$$

Consider now the consumption side of the model. Recall that except agent 1, all the other $n-1$ agents have identical preferences. More precisely, agent 1 is characterized by the pair (σ_1, γ_1), while each agent $j = 2, \ldots, n$ is characterized by the common pair $(\sigma_j, \gamma_j) = (\sigma, \gamma)$. We also assume that all agents $j = 2, \dots, n$ own the same share of capital θ_2 while agent 1 owns a share $\theta_1 > \theta_2$. It follows that for a given share $\theta_1 \in (0,1)$, we get $\theta_2 = (1-\theta_1)/(n-1)$. The first order conditions (48) give

$$\begin{aligned} x_1 &= w^* l_1 + (1-\delta) r^* \kappa^* \left(l_1 + (n-1) l_2 \right) \theta_1 \\ x_j = x_2 &= w^* l_2 + (1-\delta) r^* \kappa^* \left(l_1 + (n-1) l_2 \right) (1-\theta_1)/(n-1) \end{aligned}$$

with

$$l_1 = x_1^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1}, \quad l_2 = l_j = x_j^{-\sigma/\gamma} (w^*)^{1/\gamma} \tag{60}$$

Solving the first equation with respect to l_2 gives

$$l_2 = l_2(x_1) \quad = \quad \frac{x_1 - x_1^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1} [w^* + (1-\delta) r^* \kappa^* \theta_1]}{(1-\delta) r^* \kappa^* \theta_1 (n-1)}$$

Solving the second equation and using the previous one then yields:

$$\begin{aligned} & (l_2(x_1))^{-\gamma/\sigma} (w^*)^{1/\sigma} - l_2(x_1) \left[w^* + (1-\delta) r^* \kappa^* (1-\theta_1) \right] \\ - \quad & (1-\delta) r^* \kappa^* \tfrac{1-\theta_1}{n-1} x_1^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1} = 0 \end{aligned}$$

$x_1^*(\theta)$ is obtained as a solution of this equation and allows to compute $l_2^*(\theta)$, $l_1^*(\theta)$ and $x_2^*(\theta)$. We then get $x^*(\theta) = x_1^*(\theta) + (n-1)x_2^*(\theta)$, $\ell^*(\theta) = l_1^*(\theta) + (n-1)l_2^*(\theta)$, and thus $\rho(x^*(\theta)) = x_1^*(\theta)/\sigma_1 + (n-1)x_2^*(\theta)/\sigma$, $\gamma(\ell^*(\theta)) = l_1^*(\theta)/\gamma_1 + (n-1)l_2^*(\theta)/\gamma$.

We need also to define the intervals of admissible values for $\rho(x)$ and $\gamma(\ell)$. We start with the most equal distribution $\theta^e = (1/n, 1/n, \dots, 1/n)$. The first order conditions (48) give

$$\begin{aligned} x_1^e &= w^* l_1^e + (1-\delta) r^* \kappa^* \left(l_1^e + (n-1) l_2^e \right) / n \\ x_j^e = x_2^e &= w^* l_2^e + (1-\delta) r^* \kappa^* \left(l_1^e + (n-1) l_2^e \right) / n \end{aligned}$$

with (60). Solving the first equation with respect to l_2^e gives

$$l_2^e = l_2^e(x_1^e) \quad = \quad \frac{n\left[x_1^e - (x_1^e)^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1} [w^* + (1-\delta) r^* \kappa^*/n] \right]}{(1-\delta) r^* \kappa^* (n-1)}$$

Solving the second equation and using the previous one then yields:

$$\begin{aligned} & (l_2^e(x_1^e))^{-\gamma/\sigma} (w^*)^{1/\sigma} - l_2^e(x_1^e) \left[w^* + (1-\delta) r^* \kappa^* \tfrac{n-1}{n} \right] \\ - \quad & (1-\delta) \tfrac{r^* \kappa^*}{n} (x_1^e)^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1} = 0 \end{aligned}$$

$x_1(\theta^e)$ is obtained as a solution of this equation and allows to compute $l_2(\theta^e)$, $l_1(\theta^e)$ and $x_2(\theta^e)$. We then get $x(\theta^e) = x_1(\theta^e) + (n-1)x_2(\theta^e)$, $\ell(\theta^e) = l_1(\theta^e) + (n-1)l_2(\theta^e)$, and thus $\hat{\rho} = x_1(\theta^e)/\sigma_1 + (n-1)x_2(\theta^e)/\sigma$, $\hat{\gamma} = l_1(\theta^e)/\gamma_1 + (n-1)l_2(\theta^e)/\gamma$.

We deal now with with the most unequal distribution $\theta^u = (1, 0, \ldots, 0)$. From the first order conditions (48) we get

$$\begin{aligned} x_1^u &= w^* l_1^u + (1-\delta) r^* \kappa^* \left(l_1^u + (n-1) l_2^u\right) \\ x_j^u = x_2^u &= w^* l_2^u \end{aligned}$$

with (60). From the second equation we derive

$$x_2(\theta^u) = (w^*)^{\frac{1+\gamma}{\sigma+\gamma}}, \quad l_2(\theta^u) = (w^*)^{\frac{1-\sigma}{\sigma+\gamma}}$$

Solving the first equation with respect to x_1^u then yields:

$$x_1^u - (x_1^u)^{-\sigma_1/\gamma_1} (w^*)^{1/\gamma_1} T^* - (1-\delta) r^* \kappa^* (n-1) (w^*)^{\frac{1-\sigma}{\sigma+\gamma}} = 0$$

since $T^* = w^* + (1-\delta) r^* \kappa^*$. It follows that $x_1(\theta^u)$ is obtained as a solution of this equation and allows to compute $l_1(\theta^u)$. We then get $x(\theta^u) = x_1(\theta^u) + (n-1)(w^*)^{(1+\gamma)/(\sigma+\gamma)}$, $\ell(\theta^u) = l_1(\theta^u) + (n-1)(w^*)^{(1-\sigma)/(\sigma+\gamma)}$, and thus $\tilde{\rho} = x_1(\theta^e)/\sigma_1 + (n-1)(w^*)^{(1+\gamma)/(\sigma+\gamma)}/\sigma$, $\tilde{\gamma} = l_1(\theta^e)/\gamma_1 + (n-1)(w^*)^{(1-\sigma)/(\sigma+\gamma)}/\gamma$.

11.9 Proof of Proposition 6

We proceed numerically by finding parameters' values which satisfy $b \in (-\infty, -1] \cup [-\delta, 0)$, or equivalently

$$\alpha_1\beta_0/(\alpha_0\beta_1) \in (0, (1 + (\delta^{(\varsigma-1)/\varsigma}\beta_1)^{-\varsigma})^{1/\varsigma}] \cup [(1 + (\delta\beta_1)^{-\varsigma})^{1/\varsigma}, +\infty) \quad (61)$$

with $\rho_c < \bar{\rho}$ and $\gamma_c \in (\underline{\gamma}, \bar{\gamma})$.

Let $\alpha_1 = 0.45$, $\beta_1 = 0.2$, $\delta = 0.6$, $\varsigma = 1.187$, $\hat{\varsigma} = 0.241$, $n = 25$, $\sigma_1 = 0.65$, $\sigma = 0.582$, $\gamma_1 = 0.5$ and $\gamma = 7.9$. Then, using the expressions given in Appendix 11.8, we find that $b \approx -0.249$, i.e. (61) is satisfied, $x_1(\theta^e) \approx 0.246$, $x_2(\theta^e) \approx 0.349$, $l_1(\theta^e) \approx 0.6$, $l_2(\theta^e) \approx 0.93$, $x_1(\theta^u) \approx 1.458$, $x_2(\theta^u) \approx 0.294$, $l_1(\theta^u) \approx 0.059$ and $l_2(\theta^u) \approx 0.944$. It follows that $x(\theta^e) \approx 8.62$, $\ell(\theta^e) \approx 22.98$, $x(\theta^u) \approx 8.52$, $\ell(\theta^u) \approx 22.72$, $\underline{\rho} = \tilde{\rho} = \rho(x(\theta^u)) = 14.38$, $\bar{\rho} = \hat{\rho} = \rho(x(\theta^e)) = 14.768$, $\underline{\gamma} = \tilde{\gamma} = \gamma(\ell(\theta^u)) = 2.987$ and $\bar{\gamma} = \hat{\gamma} = \gamma(x(\theta^e)) = 4.036$.

Let us first consider an economy A with $\theta_1^A = 0.05$, $\theta_2^A = 0.038$ and thus $\theta^A = (0.05, 0.038, \cdots, 0.038)$. We get $x_1^*(\theta^A) \approx 0.253$, $x_2^*(\theta^A) \approx 0.348$, $l_1^*(\theta^A) \approx 0.579$, $l_2^*(\theta^A) \approx 0.932$ and thus $x^*(\theta^A) \approx 8.61$, $\ell^*(\theta^A) \approx 22.96$, $\rho(x^*(\theta^A)) \approx 14.754$, $\gamma(\ell^*(\theta^A)) \approx 3.99$, $\rho_c \approx 11.277$ and $\gamma_c \approx 3.79$.

Let us now consider an economy B with $\theta_1^B = 0.5$, $\theta_2^B = 0.0208$ and thus $\theta^B = (0.5, 0.0208, \cdots, 0.0208)$. We get $x_1^*(\theta^B) \approx 0.76$, $x_2^*(\theta^B) \approx 0.322$,

$l_1^*(\theta^B) \approx 0.138$, $l_2^*(\theta^B) \approx 0.937$ and thus $x^*(\theta^B) \approx 8.49$, $\ell^*(\theta^B) \approx 22.648$, $\rho(x^*(\theta^B)) \approx 14.46$, $\gamma(\ell^*(\theta^B)) \approx 3.12$, $\rho_c \approx 11.123$ and $\gamma_c \approx 3.64$.

For any $\theta_1 \in (0.05, 0.5)$, we then find that $\rho_c < \underline{\rho} < \bar{\rho}$, $\gamma_c \in (\underline{\gamma}, \bar{\gamma})$, $\rho(x^*(\theta)) \in (\underline{\rho}, \bar{\rho})$ and $\gamma(\ell^*(\theta)) \in (\underline{\gamma}, \bar{\gamma})$. For economy A, we have $\gamma(\ell^*(\theta^A)) \in (\gamma_c, \bar{\gamma})$ while for economy B we have $\gamma(\ell^*(\theta^B)) \in (\underline{\gamma}, \gamma_c)$. Therefore there exists $\theta_1^0 \approx 0.1058 \in (0.05, 0.5)$, $\theta_2^0 \approx 0.03726$ and thus $\theta^0 = (0.1058, 0.03726, \cdots, 0.03726)$ such that $\gamma(\ell^*(\theta^0)) = \gamma_c$. Case i) in Definition 3 and Assumption 4 hold. Therefore, case 1- in Theorem 2 implies that a sufficiently high level of wealth inequality leads to a value of $\gamma(\ell^*)$ lower than γ_c and thus to the existence of damped fluctuations. By continuity there exists an open set of parameters' values close to the previous values such that the same result holds. □

11.10 Proof of Proposition 7

We proceed numerically by finding parameters' values which satisfy $b \in (-1, -\delta)$, or equivalently

$$\alpha_1\beta_0/(\alpha_0\beta_1) \in ([1 + (\delta^{(\varsigma-1)/\gamma}\beta_1)^{-\gamma}]^{1/\gamma}, [1 + (\delta\beta_1)^{-\gamma}]^{1/\gamma}) \tag{62}$$

with $\rho_f < \bar{\rho}$ and $\gamma_f \in (\underline{\gamma}, \bar{\gamma})$.

Let $\alpha_1 = 0.85$, $\beta_1 = 0.2$, $\delta = 0.1$, $\varsigma = 1.49$, $\hat{\varsigma} = 0.589$, $n = 26$, $\sigma_1 = 1.666$, $\sigma = 0.375$, $\gamma_1 = 1$ and $\gamma = 24.2$. Then, using the expressions given in Appendix 11.8, we find that $b \approx -0.304$, i.e. (62) is satisfied, $x_1(\theta^e) \approx 0.175$, $x_2(\theta^e) \approx 0.168$, $l_1(\theta^e) \approx 1.044$, $l_2(\theta^e) \approx 0.913$, $x_1(\theta^u) \approx 2.927$, $x_2(\theta^u) \approx 0.053$, $l_1(\theta^u) \approx 0.0096$ and $l_2(\theta^u) \approx 0.93$. It follows that $x(\theta^e) \approx 4.38$, $\ell(\theta^e) \approx 23.88$, $x(\theta^u) \approx 4.267$, $\ell(\theta^u) \approx 23.26$, $\underline{\rho} = \tilde{\rho} = \rho(x(\theta^u)) = 5.33$, $\bar{\rho} = \hat{\rho} = \rho(x(\theta^e)) = 11.32$, $\underline{\gamma} = \tilde{\gamma} = \gamma(\ell(\theta^u)) = 0.97$ and $\bar{\gamma} = \hat{\gamma} = \gamma(x(\theta^e)) = 1.987$.

Let us first consider an economy A with $\theta_1 = 0.04$, $\theta_2 = 0.0384$ and thus $\theta^A = (0.04, 0.0384, \cdots, 0.0384)$. We get $x_1^*(\theta^A) \approx 0.178$, $x_2^*(\theta^A) \approx 0.168$, $l_1^*(\theta^A) \approx 1.016$, $l_2^*(\theta^A) \approx 0.914$ and thus $x^*(\theta^A) \approx 4.377$, $\ell^*(\theta^A) \approx 23.86$, $\rho(x^*(\theta^A)) \approx 11.3$, $\gamma(\ell^*(\theta^A)) \approx 1.96$, $\rho_f \approx 1.496$ and $\gamma_f \approx 1.72$.

Let us now consider an economy B with $\theta_1 = 0.2$, $\theta_2 = 0.032$ and thus $\theta^B = (0.2, 0.032, \cdots, 0.032)$. We get $x_1^*(\theta^B) \approx 0.587$, $x_2^*(\theta^B) \approx 0.145$, $l_1^*(\theta^B) \approx 0.14$, $l_2^*(\theta^B) \approx 0.915$ and thus $x^*(\theta^B) \approx 4.22$, $\ell^*(\theta^B) \approx 23.03$, $\rho(x^*(\theta^B)) \approx 10.05$, $\gamma(\ell^*(\theta^B)) \approx 1.086$, $\rho_f \approx 1.44$ and $\gamma_f \approx 1.51$.

For any $\theta_1 \in (0.04, 0.2)$, we then find that $\rho_f < \underline{\rho} < \bar{\rho}$, $\gamma_f \in (\underline{\gamma}, \bar{\gamma})$, $\rho(x^*(\theta)) \in (\underline{\rho}, \bar{\rho})$ and $\gamma(\ell^*(\theta)) \in (\underline{\gamma}, \bar{\gamma})$. For economy A, we have $\gamma(\ell^*(\theta^A)) \in (\gamma_f, \bar{\gamma})$ while for economy B we have $\gamma(\ell^*(\theta^B)) \in (\underline{\gamma}, \gamma_f)$. Therefore there exists $\theta_1^0 \approx 0.05792 \in (0.04, 0.2)$, $\theta_2^0 \approx 0.03768$ and thus $\theta^0 = (0.05792, 0.03768, \cdots, 0.03768)$ such that $\gamma(\ell^*(\theta^0)) = \gamma_f$. Case i) in Definition 3 and Assumptions 5 hold. Therefore, case 2- in Theorem 2 applies: increasing the level of wealth inequality leads to decreasing values of $\gamma(\ell^*)$ that will cross the flip bifurcation value γ_f, and thus imply the existence of period-two cycles. By continuity there exists an open set of parameters' values close to the previous values such that the same result holds. □

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