Research Article

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Square function and non-tangential maximal function estimates for elliptic operators in 1-sided NTA domains satisfying the capacity density condition

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Abstract: Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided non-tangentially accessible domain (also known as uniform domain), that is, Ω satisfies the interior Corkscrew and Harnack chain conditions, which are respectively scale-invariant/quantitative versions of openness and path-connectedness. Let us assume also that Ω satisfies the so-called capacity density condition, a quantitative version of the fact that all boundary points are Wiener regular. Consider two real-valued (non-necessarily symmetric) uniformly elliptic operators

$$L_0 u = -\operatorname{div}(A_0 \nabla u) \quad \text{and} \quad Lu = -\operatorname{div}(A \nabla u)$$

in Ω, and write ω_{L_0} and ω_L for the respective associated elliptic measures. The goal of this article and its companion [M. Akman, S. Hofmann, J. M. Martell and T. Toro, Perturbation of elliptic operators in 1-sided NTA domains satisfying the capacity density condition, preprint (2021), https://arxiv.org/abs/1901.08261v3] is to find sufficient conditions guaranteeing that ω_L satisfies an A_∞-condition or a RH_q-condition with respect to ω_{L_0}. In this paper, we are interested in obtaining a square function and non-tangential estimates for solutions of operators as before. We establish that bounded weak null-solutions satisfy Carleson measure estimates, with respect to the associated elliptic measure. We also show that for every weak null-solution, the associated square function can be controlled by the non-tangential maximal function in any Lebesgue space with respect to the associated elliptic measure. These results extend previous work of Dahlberg, Jerison and Kenig and are fundamental for the proof of the perturbation results in the paper cited above.

Keywords: Uniformly elliptic operators, elliptic measure, Green function, 1-sided non-tangentially accessible domains, 1-sided chord-arc domains, capacity density condition, Ahlfors-regularity, A_∞ Muckenhoupt weights, reverse Hölder, Carleson measures, square function estimates, non-tangential maximal function estimates, dyadic analysis, sawtooth domains, perturbation

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1 Introduction and main results

The purpose of this article and its companion [2] is to study some perturbation problems for second order divergence form real-valued elliptic operators with bounded measurable coefficients in domains with rough boundaries. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and let $Lu = -\operatorname{div}(A\nabla u)$ be a second order divergence form real-valued elliptic operator defined in Ω. Here the coefficient matrix $A = (a_{i,j}(\cdot))_{i,j=1}^{n+1}$ is real-valued (not necessarily symmetric) and uniformly elliptic, with $a_{i,j} \in L^\infty(\Omega)$, that is, there exists a constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \quad |A(X)\xi \cdot \eta| \leq \Lambda|\xi|\,|\eta|$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$. Associated with L, one can construct a family of positive Borel measures $\{\omega_L^X\}_{X\in\Omega}$, defined on $\partial\Omega$ with $\omega^X(\partial\Omega) \leq 1$ for every $X \in \Omega$, so that for each $f \in C_c(\partial\Omega)$ one can define its associated weak-solution

$$u(X) = \int_{\partial\Omega} f(z)\,d\omega_L^X(z), \quad \text{whenever } X \in \Omega, \tag{1.1}$$

which satisfies $Lu = 0$ in Ω in the weak sense. These solutions are sometimes called Perron solutions (see, for example, the classical books [11, 16, 17]). In principle, unless we assume some further condition (e.g., existence of a barrier function or Wiener regularity), u need not be continuous all the way to the boundary, but still we think of u as the solution to the continuous Dirichlet problem with boundary data f. We call ω_L^X the elliptic measure of Ω associated with the operator L with pole at $X \in \Omega$. For convenience, we will sometimes write ω_L and call it simply the elliptic measure, dropping the dependence on the pole.

Given two such operators $L_0u = -\operatorname{div}(A_0\nabla u)$ and $Lu = -\operatorname{div}(A\nabla u)$, one may wonder whether one can find conditions on the matrices A_0 and A so that some "good estimates" for the Dirichlet problem or for the elliptic measure for L_0 might be transferred to the operator L. Similarly, one may try to see whether A being "close" to A_0 in some sense gives some relationship between ω_L and ω_{L_0}. In this direction, a celebrated result of Littman, Stampacchia, and Weinberger in [29] states that the continuous Dirichlet problem for the Laplace operator $L_0 = \Delta$, (i.e., A_0 is the identity) is solvable if and only if it is solvable for any real-valued elliptic operator L. By solvability here we mean that the elliptic measure solutions as in (1.1) are indeed continuous in $\overline{\Omega}$. It is well known that solvability in this sense is in fact equivalent to the fact that all boundary points are regular in the sense of Wiener (or, equivalently, existence of barrier functions), a condition which entails some capacitary thickness of the complement of Ω. Note that, for this result, one does not need to know that L is "close" to the Laplacian in any sense (other than the fact that both operators are uniformly elliptic).

On the other hand, if $\Omega = \mathbb{R}^2_+$ is the upper-half plane and $L_0 = \Delta$, then the harmonic measure associated with Δ is mutually absolutely continuous with respect to the surface measure on the boundary, and its Radon–Nikodym derivative is the classical Poisson kernel. However, Caffarelli, Fabes, and Kenig in [3] constructed a uniformly real-valued elliptic operator L in the plane (the pullback of the Laplacian via a quasiconformal mapping of the upper half plane to itself) for which the associated elliptic measure ω_L is not even absolutely continuous with respect to the surface measure (see also [31] for another example). This means that, although the continuous Dirichlet problem is solvable for L, one can not solve in general the associated problem for integrable data with respect to surface measure. Hence, in principle the "good behavior" of harmonic measure does not always transfer to any elliptic measure even in a nice domain such as the upper-half plane. Consequently, it is natural to see if those good properties can be transferred by assuming some conditions reflecting the fact that L is "close" to L_0 or, in other words, by imposing some conditions on the disagreement of A and A_0.

The goal of this article and its companion [2] is to solve some perturbation problems that go beyond [5, 6, 9, 12, 13, 30]. Our setting is that of 1-sided NTA domains satisfying the so-called capacity density condition (CDC for short); see Section 2 for the precise definitions. The latter is a quantitative version of the well-known Wiener criterion and it is weaker than the Ahlfors regularity of the boundary or the existence of exterior Corkscrews (see Definition 2.1); see the discussion after Definition 2.7. This setting guarantees among other things that any elliptic measure is doubling in some appropriate sense, and hence one can see that a suitable portion of the boundary of the domain endowed with the Euclidean distance and with a given elliptic measure ω_{L_0} is a space of homogeneous type. In particular, classes like $A_\infty(\omega_{L_0})$ or $\mathrm{RH}_p(\omega_{L_0})$ have the same good features of the corresponding ones in the Euclidean setting. However, our assumptions do not guarantee that the surface measure has any good behavior, and it could even be locally infinite. In one of our main results, we consider the case in which a certain disagreement condition, originating in [13], holds either with small or large constant. The small constant case can be seen as an extension of [13, 30] to a setting in which surface measure is not a good object. The large constant case is new even in nice domains such as balls, upper-half spaces, Lipschitz domains or chord-arc domains. To the best of our knowledge, our work is the first to establish perturbation results on sets with bad surface measures, and our large perturbation results are the first of their type. Finally, we do not require the operators to be symmetric. The precise results, along with its context in the historical developments, will be stated in the sequel to the present paper [2].

In the present article, we develop some of the needed tools, and present some other results which are of independent interest. Key to our argument is the construction of certain sawtooth domains adapted to a dyadic grid on the boundary and to the Whitney decomposition of the domain. These domains are shown to inherit the main geometrical/topological features of the original domain (see Proposition 2.12). With this in hand, we obtain a discrete sawtooth lemma for projections improving [10, Main Lemma]; see Lemma 3.2 and Lemma 3.3. These ingredients are crucial for the main results of this paper, which we state next. First, we establish that bounded weak-solutions satisfy Carleson measure estimates adapted to the elliptic measure.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain (cf. Definition 2.3) satisfying the capacity density condition (cf. Definition 2.7). Let $Lu = -\operatorname{div}(A\nabla u)$ be a real-valued (not necessarily symmetric) elliptic operator and write ω_L and G_L to denote, respectively, the associated elliptic measure and the Green function. There exists C depending only on the dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constant of L, such that for every $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ with $Lu = 0$ in the weak-sense in Ω, there holds*

$$\sup_B \sup_{B'} \frac{1}{\omega_L^{X_\Delta}(\Delta')} \iint_{B'\cap\Omega} |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX \leq C\|u\|^2_{L^\infty(\Omega)}, \tag{1.2}$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, X_Δ is a corkscrew point relative to Δ (cf. Definition 2.1), and the suprema are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$, and c_0 is the Corkscrew constant (cf. Definition 2.1).

This result is in turn the main ingredient to obtain that the conical square function can be locally controlled by the non-tangential maximal function in norm with respect to the elliptic measure, allowing us to extend some estimates from [10] to our general setting.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain (cf. Definition 2.3) satisfying the capacity density condition (cf. Definition 2.7). Let $Lu = -\operatorname{div}(A\nabla u)$ be a real-valued (non-necessarily symmetric) elliptic operator and write ω_L to denote the associated elliptic measure and the Green function. For every $0 < q < \infty$, there exists C_q depending only on the dimension n, the 1-sided NTA constants, the CDC constant, the ellipticity constant of L, and q, such that for every $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with $Lu = 0$ in the weak-sense in Ω, for every $Q_0 \in \mathbb{D}(\partial\Omega)$ (cf. Lemma 2.8), there holds*

$$\|\mathcal{S}_{Q_0} u\|_{L^q(Q_0, \omega_L^{X_{Q_0}})} \leq C_q \|\mathcal{N}_{Q_0} u\|_{L^q(Q_0, \omega_L^{X_{Q_0}})}, \tag{1.3}$$

where $\mathcal{S}_{Q_0}$ and $\mathcal{N}_{Q_0}$ are the localized dyadic conical square function and non-tangential maximal function, respectively (cf. (2.12) and (2.11)), and X_{Q_0} is a corkscrew point relative to Q_0 (see Section 2.4).

We note that the estimate (1.3) is written for the localized dyadic conical square function and non-tangential maximal function. It is not difficult to see that, as a consequence, one can obtain a similar estimate for the regular localized (or truncated) conical square function and non-tangential maximal function with arbitrary apertures (see [4, Lemma 4.8]), precise statements are left to the interested reader.

The plan of this paper is as follows. Section 2 presents some of the preliminaries, definitions, and tools which will be used throughout this paper. Section 3 contains a dyadic version of the main lemma of [10]. In Section 4, we prove our main results, Theorem 1.1 and Theorem 1.2.

We would like to mention that after an initial version of this work was posted on arXiv [1], Feneuil and Poggi in [14] obtained results related to ours; compare, for instance, Theorem 1.1 with [14, Theorem 1.27]. Also, the recent work [4] complements this paper and its companion [2]; see, for instance, [4, Corollary 1.4].

2 Preliminaries

2.1 Notation and conventions

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on the dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq \frac{a}{b} \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. Unless otherwise specified, upper case constants are greater than 1 and lower case constants are smaller than 1. In some occasions, it is important to keep track of the dependence on a given parameter γ; in that case, we write $a \lesssim_\gamma b$ or $a \approx_\gamma b$ to emphasize that the implicit constants in the inequalities depend on γ.
- Our ambient space is $\mathbb{R}^{n+1}$, $n \geq 2$.
- Given $E \subset \mathbb{R}^{n+1}$, we write $\operatorname{diam}(E) = \sup_{x,y \in E}|x - y|$ to denote its diameter.
- Given a domain $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- The open $(n + 1)$-dimensional Euclidean ball of radius r will be denoted by $B(x, r)$ when the center x lies on $\partial\Omega$, or by $B(X, r)$ when the center X lies in $\mathbb{R}^{n+1} \setminus \partial\Omega$. A *surface ball* is denoted by $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and unless otherwise specified it is implicitly assumed that $x \in \partial\Omega$.
- If $\partial\Omega$ is bounded, it is always understood (unless otherwise specified) that all surface balls have radii controlled by the diameter of $\partial\Omega$, that is, if $\Delta = \Delta(x, r)$, then $r \lesssim \operatorname{diam}(\partial\Omega)$. Note that in this way $\Delta = \partial\Omega$ if $\operatorname{diam}(\partial\Omega) < r \lesssim \operatorname{diam}(\partial\Omega)$.
- For $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.
- We let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\mathbf{1}_A$ denote the usual indicator function of A, i.e., $\mathbf{1}_A(X) = 1$ if $X \in A$, and $\mathbf{1}_A(X) = 0$ if $X \notin A$.
- We shall use the letter I (and sometimes J) to denote a closed $(n + 1)$-dimensional Euclidean cube with sides parallel to the coordinate axes, and we let $\ell(I)$ denote the side length of I. We use Q to denote dyadic "cubes" on E or $\partial\Omega$. The latter exist as a consequence of Lemma 2.8 below.

2.2 Some definitions

Definition 2.1 (Corkscrew condition). Following [27], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* if for some uniform constant $0 < c_0 < 1$ and for every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, if we write $\Delta := \Delta(x, r)$, there is a ball $B(X_\Delta, c_0 r) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *Corkscrew point relative to* Δ (or relative to B) and the constant c_0 is called the *Corkscrew constant*. We note that we may allow $r < C \operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c_0. We say that Ω satisfies the *exterior Corkscrew condition* if $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the Corkscrew condition.

Definition 2.2 (Harnack chain condition). Again following [27], we say that Ω satisfies the *Harnack chain condition* if there are uniform constants $C_1, C_2 > 1$ such that for every pair of points $X, X' \in \Omega$ there is a chain of balls $B_1, B_2, \dots, B_N \subset \Omega$ with $N \leq C_1(2 + \max\{\log_2 \Pi, 0\})$, where

$$\Pi := \frac{|X - X'|}{\min\{\delta(X), \delta(X')\}},$$

such that $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and for every $1 \leq k \leq N$,

$$C_2^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C_2 \operatorname{diam}(B_k). \tag{2.1}$$

The chain of balls is called a *Harnack chain*.

We note that in the context of the previous definition, if $\Pi \leq 1$, we can trivially form the Harnack chain $B_1 = B(X, 3\delta(X)/5)$ and $B_2 = B(X', 3\delta(X')/5)$, where (2.1) holds with $C_2 = 3$. Hence, the Harnack chain condition is non-trivial only when $\Pi > 1$.

Definition 2.3 (1-sided NTA and NTA). We say that a domain Ω is a 1-*sided non-tangentially accessible domain* (1-sided NTA) if it satisfies both the Corkscrew and Harnack chain conditions. Furthermore, we say that Ω is a *non-tangentially accessible domain* (NTA domain) if it is a 1-sided NTA domain and if, in addition, $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ also satisfies the Corkscrew condition.

Remark 2.4. In the literature, 1-sided NTA domains are also called *uniform domains*. We remark that the 1-sided NTA condition is a quantitative form of path connectedness.

Definition 2.5 (Ahlfors regular). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is *n-dimensional Ahlfors regular* (AR for short) if there is some uniform constant $C_1 > 1$ such that

$$C_1^{-1} r^n \leq \mathcal{H}^n(E \cap B(x, r)) \leq C_1 r^n, \quad x \in E,\ 0 < r < \operatorname{diam}(E).$$

Definition 2.6 (1-sided CAD and CAD). A 1-*sided chord-arc domain* (1-sided CAD) is a 1-sided NTA domain with AR boundary. A *chord-arc domain* (CAD) is an NTA domain with AR boundary.

We next recall the definition of the capacity of a set. Given an open set $D \subset \mathbb{R}^{n+1}$ (where we recall that we always assume that $n \geq 2$) and a compact set $K \subset D$, we define the capacity of K relative to D by

$$\operatorname{Cap}_2(K, D) = \inf\Big\{ \iint_D |\nabla v(X)|^2 \, dX : v \in C_0^\infty(D),\ v(x) \geq 1 \text{ in } K \Big\}.$$

Definition 2.7 (Capacity density condition). An open set Ω is said to satisfy the *capacity density condition* (CDC for short) if there exists a uniform constant $c_1 > 0$ such that

$$\frac{\operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega, B(x, 2r))}{\operatorname{Cap}_2(\overline{B(x,r)}, B(x, 2r))} \geq c_1$$

for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$.

The CDC is also known as the uniform 2-fatness as studied by Lewis in [28]. Using [16, Example 2.12], one has that

$$\operatorname{Cap}_2(\overline{B(x,r)}, B(x, 2r)) \approx r^{n-1} \quad \text{for all } x \in \mathbb{R}^{n+1} \text{ and } r > 0, \tag{2.2}$$

and hence the CDC is a quantitative version of the Wiener regularity; in particular, it implies that every $x \in \partial\Omega$ is Wiener regular. It is easy to see that the exterior Corkscrew condition implies CDC. Also, it was proved in [32, Section 3] and [18, Lemma 3.27] that a set with Ahlfors regular boundary satisfies the capacity density condition with constant c_1 depending only on n and the Ahlfors regular constant.

2.3 Existence of a dyadic grid

In this section, we introduce a dyadic grid along the lines of that obtained in [7]. More precisely, we will use the dyadic structure from [25, 26], with a modification from [24, Proof of Proposition 2.12].

Lemma 2.8 (Existence and properties of the "dyadic grid"). *Let $E \subset \mathbb{R}^{n+1}$ be a closed set. Then there exists a constant $C \geq 1$ depending just on n such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets (called "dyadic cubes" or "cubes" for simplicity)*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{J}_k\},$$

where $\mathfrak{J}_k$ denotes some countable (possibly finite) index set depending on k satisfying the following conditions:
(i) $E = \bigcup_{j\in\mathfrak{J}_k} Q_j^k$ *for each* $k \in \mathbb{Z}$.
(ii) *If* $m \le k$, *then either* $Q_j^k \subset Q_i^m$ *or* $Q_i^m \cap Q_j^k = \emptyset$.
(iii) *For each* $k \in \mathbb{Z}$, $j \in \mathfrak{J}_k$, *and* $m < k$, *there is a unique* $i \in \mathfrak{J}_m$ *such that* $Q_j^k \subset Q_i^m$.
(iv) *For each* $k \in \mathbb{Z}$ *and* $j \in \mathfrak{J}_k$, *there is* $x_j^k \in E$ *such that*

$$B(x_j^k, C^{-1}2^{-k}) \cap E \subset Q_j^k \subset B(x_j^k, C2^{-k}) \cap E.$$

Proof. We first note that E is geometric doubling. That is, there exists N depending just on n such that for every $x \in E$ and $r > 0$ one can cover the surface ball $B(x, r) \cap E$ with at most N surface balls of the form $B(x_i, r/2) \cap E$ with $x_i \in E$; observe that geometric doubling for E is inherited from the corresponding property on $\mathbb{R}^{n+1}$ and that is why N depends only on n and it is independent of E. Besides, letting $\eta = \frac{1}{16}$, for every $k \in \mathbb{Z}$ it is easy to find a countable collection $\{x_j^k\}_{j\in\mathfrak{J}_k} \subset E$ such that, for all $x \in E$,

$$|x_j^k - x_{j'}^k| \ge \eta^k, \quad j, j' \in \mathfrak{J}_k,\ j \ne j', \qquad \min_{j\in\mathfrak{J}_k} |x - x_j| < \eta^k.$$

Invoking then [25, 26] on E with the Euclidean distance and $c_0 = C_0 = 1$, one can construct a family of dyadic cubes associated with these families of points, say $\mathfrak{D}_k$ for $k \in \mathbb{Z}$. These satisfy (i)–(iv) in the statement with the only difference that we have to replace 2^{-k} by η^k in (iv).

At this point, we follow the argument in [24, Proof of Proposition 2.12] with $\eta = \frac{1}{16}$. For any $k \in \mathbb{Z}$, we set $\mathbb{D}_j = \mathfrak{D}_k$ for every $4k \le j < 4(k + 1)$. It is straightforward to show that properties (i)–(iii) for the families $\mathbb{D}_k$ follow at once from those for the families $\mathfrak{D}_k$. Regarding (iv), let $Q^i \in \mathbb{D}_j$ and let $k \in \mathbb{Z}$ such that $4k \le j < 4(k + 1)$, so that $Q^i \in \mathbb{D}_j = \mathfrak{D}_k$. Writing $x^i \in E$ for the corresponding point associated with $Q^i \in \mathfrak{D}_k$ and invoking (iv) for $\mathfrak{D}_k$, we conclude

$$B(x^i, C^{-1}2^{-j}) \cap E \subset B(x^i, C^{-1}\eta^k) \cap E \subset Q^i \subset B(x^i, C\eta^k) \cap E \subset B(x^i, 16C2^{-j}) \cap E,$$

and hence (iv) holds. □

A few remarks are in order concerning this lemma. Note that by construction, within the same generation (that is, within each $\mathbb{D}_k$), the cubes are pairwise disjoint (hence, there are no repetitions). On the other hand, repetitions are allowed in the different generations, that is, one could have that $Q \in \mathbb{D}_k$ and $Q' \in \mathbb{D}_{k-1}$ agree. Then, although Q and Q' are the same set, as cubes we understand that they are different. In short, it is then understood that $\mathbb{D}(E) := \{Q_j^k : j \in \mathfrak{J}_k,\ k \in \mathbb{Z}\}$ is an indexed collection of sets where repetitions of sets are allowed in the different generations but not within the same generation. With this in mind, we can give a proper definition of the "length" of a cube (this concept has no geometric meaning for the moment). For every $Q \in \mathbb{D}_k$, we set $\ell(Q) = 2^{-k}$, which is called the "length" of Q. Note that the "length" is well defined when considered on $\mathbb{D}$, but it is not well-defined on the family of sets induced by $\mathbb{D}$. It is important to observe that the "length" refers to the way the cubes are organized in the dyadic grid and in general may not have a geometrical meaning. It is clear from (iv) that $\operatorname{diam}(Q) \lesssim \ell(Q)$ (we will see below that in our setting the converse holds, see Remark 2.17). We warn the reader that we are abusing the notation as we use the symbol $\ell(\cdot)$ to denote two different things: $\ell(I)$ denotes the side length of I, a closed $(n + 1)$-dimensional Euclidean cube with sides parallel to the coordinate axes; and $\ell(Q)$ denotes the "length" of Q, a dyadic cube on E. This conflict of notation will cause no trouble as the meaning will always be clear from the context. Similarly, we write $k(Q) := k$ if $Q \in \mathbb{D}_k$, and if $Q = Q_j^k \in \mathbb{D}_k$ with $j \in \mathfrak{J}_k$, $k \in \mathbb{Z}$, we set $x_Q := x_j^k$ and $r_Q := (2C)^{-1}2^{-k}$, with C being the constant in Lemma 2.8, which depends only on the dimension n. We shall refer to the point x_Q as the "center" of Q. Observe that these are well-defined quantities when interpreted as functions on $\mathbb{D}$. Note however that having $Q_j^k = Q_{j'}^{k'}$ does not mean that x_j^k and $x_{j'}^{k'}$ agree; the same occurs with $r_{Q_j^k}$ and $r_{Q_{j'}^{k'}}$, or with $\ell(Q_j^k)$ and $\ell(Q_{j'}^{k'})$.

Let us observe that the generations run for all $k \in \mathbb{Z}$. However, as we are about to see, sometimes it is natural to truncate the generations. If E is bounded and $k \in \mathbb{Z}$ is such that $\operatorname{diam}(E) < C^{-1}2^{-k}$, then there cannot be two distinct cubes in $\mathbb{D}_k$, and thus $\mathbb{D}_k = \{E\}$. Therefore, we are going to ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$. Hence, we shall denote by $\mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e., $\mathbb{D}(E) := \bigcup_k \mathbb{D}_k$, where, if diam$(E)$ is finite, the union runs over those $k \in \mathbb{Z}$ such that $2^{-k} \lesssim \operatorname{diam}(E)$. The precise implicit constant, which is assumed to be large enough, is not relevant and may depend on the other relevant constants (n, CDC, Corkscrew, Harnack chain etc.), and it may slightly change in the different arguments of this paper.

In what follows, given $B = B(x, r)$ with $x \in E$, we will set $\Delta = \Delta(x, r) = B \cap E$. We write $\Xi = 2C^2$, with C being the constant in Lemma 2.8, which depends only on the dimension n. For each $Q \in \mathbb{D}$, we take its center $x_Q \in E$ and let $r_Q = (2C)^{-1}\ell(Q)$, and hence $\Xi^{-1}\ell(Q) \leq r_Q \leq \ell(Q)$. Lemma 2.8 (iv) yields

$$\Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, \Xi r_Q). \tag{2.3}$$

We shall denote these balls and surface balls by

$$B_Q := B(x_Q, r_Q), \qquad \Delta_Q := \Delta(x_Q, r_Q), \tag{2.4}$$

$$\widetilde{B}_Q := B(x_Q, \Xi r_Q), \quad \widetilde{\Delta}_Q := \Delta(x_Q, \Xi r_Q). \tag{2.5}$$

Much as before, these sets are well-defined quantities when interpreted as functions on $\mathbb{D}$. Observe that having two cubes which agree as sets does not imply that the associated balls or surface balls are the same.

Let $Q \in \mathbb{D}_k$ and consider the family of its dyadic children $\{Q' \in \mathbb{D}_{k+1} : Q' \subset Q\}$. Note that for any two distinct children Q', Q'', one has $|x_{Q'} - x_{Q''}| \geq r_{Q'} = r_{Q''} = r_Q/2$; otherwise, $x_{Q''} \in Q'' \cap \Delta_{Q'} \subset Q'' \cap Q'$, contradicting the fact that Q' and Q'' are disjoint. Also $x_{Q'}, x_{Q''} \in Q \subset \Delta(x_Q, r_Q)$, and hence, by the geometric doubling property, we have a purely dimensional bound for the number of such $x_{Q'}$. Therefore, the number of dyadic children of a given dyadic cube is uniformly bounded.

Lemma 2.9. *Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $\mathbb{D}(E)$ be the dyadic grid as in Lemma 2.8. Assume that there is a Borel measure μ which is doubling, that is, there exists $C_\mu \geq 1$ such that $\mu(\Delta(x, 2r)) \leq C_\mu \mu(\Delta(x, r)) < \infty$ for every $x \in E$ and $r > 0$. Then $\mu(\partial Q) = 0$ for every $Q \in \mathbb{D}(E)$. Moreover, there exist $0 < \tau_0 < 1$, C_1, and $\eta > 0$ depending only on the dimension and C_μ such that for every $\tau \in (0, \tau_0)$ and $Q \in \mathbb{D}(E)$,*

$$\mu(\{x \in Q : \operatorname{dist}(x, E \setminus Q) \leq \tau \ell(Q)\}) \leq C_1 \tau^\eta \mu(Q). \tag{2.6}$$

Proof. The argument is a refinement of that in [19, Proposition 6.3] (see also [15, p. 403], where the Euclidean case was treated). Fix an integer k, a cube $Q \in \mathbb{D}_k$, and a positive integer m to be chosen. Fix $\tau > 0$ small enough to be chosen and write

$$\Sigma_\tau = \{x \in \overline{Q} : \operatorname{dist}(x, E \setminus Q) < \tau \ell(Q)\}.$$

For fixed $Q \in \mathbb{D}(E)$, let

$$\mathbb{D}_Q := \{Q' \in \mathbb{D}(E) : Q' \subset Q\} \quad \text{and} \quad \{Q_i^1\} := \mathbb{D}^1 := \mathbb{D}_Q \cap \mathbb{D}_{k+m},$$

and make the disjoint decomposition $Q = \bigcup Q_i^1$. We then split $\mathbb{D}^1 = \mathbb{D}^{1,1} \cup \mathbb{D}^{1,2}$, where $Q_i^1 \in \mathbb{D}^{1,1}$ if $\overline{Q_i^1}$ meets Σ_τ, and $Q_i^1 \in \mathbb{D}^{1,2}$ otherwise. We then write $\overline{Q} = R^{1,1} \cup R^{1,2}$, where

$$R^{1,1} := \bigcup_{\mathbb{D}^{1,1}} \widehat{Q}_i^1, \quad R^{1,2} := \bigcup_{\mathbb{D}^{1,2}} Q_i^1,$$

and for each cube $Q_i^1 \in \mathbb{D}^{1,1}$ we construct $\widehat{Q}_i^1$ as follows. We enumerate the elements in $\mathbb{D}^{1,1}$ as

$$Q_{i_1}^1, Q_{i_2}^1, \dots, Q_{i_N}^1,$$

and then set $(Q_i^1)^* = Q_i^1 \cup (\partial Q_i^1 \cap \partial Q)$ and

$$\widehat{Q}_{i_1}^1 := (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_2}^1 := (Q_{i_2}^1)^* \setminus (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_3}^1 := (Q_{i_3}^1)^* \setminus ((Q_{i_1}^1)^* \cup (Q_{i_2}^1)^*), \dots$$

so that $R^{1,1}$ covers Σ_τ and the modified cubes $\widehat{Q}_i^1$ are pairwise disjoint.

We also note from (2.3) that if we fix m so that $2^{-m} < \Xi^{-2}/4$, then

$$\operatorname{dist}(\Delta_Q, E \setminus Q) \geq r_Q \geq \Xi^{-1}\ell(Q), \quad \operatorname{diam}(Q_i^1) \leq 2\Xi r_{Q_i^1} \leq 2\Xi\ell(Q_i^1) < \frac{\Xi^{-1}}{2}\ell(Q).$$

Then $R^{1,1}$ misses Δ_Q provided $\tau < \Xi^{-1}/2$. Otherwise, we can find $x \in \overline{Q_i^1} \cap \Delta_Q$ with $Q_i^1 \in \mathbb{D}^{1,1}$. The latter implies that there is $y \in \overline{Q_i^1} \cap \Sigma_\tau$. All these yield a contradiction:

$$\Xi^{-1}\ell(Q) \le \operatorname{dist}(\Delta_Q, E \setminus Q) \le |x-y| + \operatorname{dist}(y, E \setminus Q) \le \operatorname{diam}(\overline{Q_i^1}) + \tau\ell(Q) < \Xi^{-1}\ell(Q).$$

Consequently, by the doubling property,

$$\mu(\overline{Q}) \le \mu(2\widetilde{\Delta}_Q) \le C'_\mu \mu(\Delta_Q) \le C'_\mu \mu(R^{1,2}).$$

Since $R^{1,1}$ and $R^{1,2}$ are disjoint, the latter estimate yields

$$\mu(R^{1,1}) \le \Big(1 - \frac{1}{C'_\mu}\Big)\mu(\overline{Q}) =: \theta\mu(\overline{Q}),$$

where we note that $0 < \theta < 1$.

Let us now repeat this procedure, decomposing $\widehat{Q}_i^1$ for each $Q_i^1 \in \mathbb{D}^{1,1}$. We set $\mathbb{D}^2(Q_i^1) = \mathbb{D}_{Q_i^1} \cap \mathbb{D}_{k+2m}$ and split it into $\mathbb{D}^{2,1}(Q_i^1)$ and $\mathbb{D}^{2,2}(Q_i^1)$, where $Q' \in \mathbb{D}^{2,1}(Q_i^1)$ if $\overline{Q'}$ meets Σ_τ. Associated to any $Q' \in \mathbb{D}^{2,1}(Q_i^1)$, we set

$$(Q')^* = (Q' \cap \widehat{Q}_i^1) \cup (\partial Q' \cap (\partial Q \cap \widehat{Q}_i^1)).$$

Then we make these sets disjoint as before and we have that $R^{2,1}(Q_i^1)$ is defined as the disjoint union of the corresponding $\widehat{Q'}$. Note that

$$\widehat{Q}_i^1 = R^{2,1}(Q_i^1) \cup R^{2,2}(Q_i^1)$$

and this is a disjoint union. As before, $R^{2,1}(Q_i^1)$ misses $\Delta_{Q_i^1}$ provided $\tau < 2^{-m}\Xi^{-1}/2$, so that by the doubling property,

$$\mu(\widehat{Q}_i^1) \le \mu(2\widetilde{\Delta}_{Q_i^1}) \le C'_\mu \mu(\Delta_{Q_i^1}) \le C'_\mu \mu(R^{2,2}(Q_i^1)),$$

and thus

$$\mu(R^{2,1}(Q_i^1)) \le \theta\mu(\widehat{Q}_i^1).$$

Next, we set $R^{2,1}$ and $R^{2,2}$ as the union of the corresponding $R^{2,1}(Q_i^1)$ and $R^{2,2}(Q_i^1)$ with $Q_i^1 \in \mathbb{D}^{1,1}$. Then

$$\mu(R^{2,1}) := \mu\Big(\bigcup_{Q_i^1 \in \mathbb{D}^{1,1}} R^{2,1}(Q_i^1)\Big) = \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu(R^{2,1}(Q_i^1)) \le \theta \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu(\widehat{Q}_i^1) = \theta\mu(R^{1,1}) \le \theta^2\mu(\overline{Q}).$$

Iterating this procedure, we obtain that for every $k = 0, 1, \dots,$ if $\tau < 2^{-km}\Xi^{-1}/2$, then

$$\mu(R^{k+1,1}) \le \theta^{k+1}\mu(\overline{Q}).$$

Let us see that this leads to the desired estimates. Fix $\tau < \Xi^{-1}/2$ and find $k \ge 0$ such that

$$2^{-(k+1)m}\Xi^{-1}/2 \le \tau < 2^{-km}\Xi^{-1}/2.$$

By construction $\Sigma_\tau \subset R^{k+1,1}$, and thus

$$\mu(\Sigma_\tau) \le \mu(R^{k+1,1}) \le \theta^{k+1}\mu(\overline{Q}) \le (2\Xi)^{\frac{\log_2\theta^{-1}}{m}} \tau^{\frac{\log_2\theta^{-1}}{m}} \mu(\overline{Q}),$$

which easily gives (2.6) with

$$C_1 = (2\Xi)^{\frac{\log_2\theta^{-1}}{m}} \quad \text{and} \quad \eta = \frac{\log_2\theta^{-1}}{m}.$$

On the other hand, note that

$$\partial Q \subset \bigcap_{j:2^{-j}<\Xi^{-1}/2} \Sigma_{2^{-j}},$$

and also $\Sigma_{2^{-(j+1)}} \subset \Sigma_{2^{-j}}$. Thus clearly,

$$0 \le \mu(\partial Q) \le \lim_{j\to\infty} \mu(\Sigma_{2^{-j}}) \le \lim_{j\to\infty} C_1 2^{-j\eta}\mu(Q) = 0,$$

yielding that $\mu(\partial Q) = 0$. □

Remark 2.10. Note that the previous argument is local in the sense that if we just want to obtain the desired estimates for a fixed Q_0, we would only need to assume that μ is doubling in $2\widetilde{\Delta}_{Q_0}$. Indeed, we would just need to know that $\mu(\Delta(x, 2r)) \leq C\mu(\Delta(x, r)) < \infty$ for every $x \in Q_0$ and $0 < r < \Xi\ell(Q_0)$, and the involved constants in the resulting estimates will depend only on the dimension and C_μ. Further details are left to the interested reader.

We next introduce the "discretized Carleson region" relative to Q as $\mathbb{D}_Q = \{Q' \in \mathbb{D} : Q' \subset Q\}$. Let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}$ be a family of pairwise disjoint cubes. The "global discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes $Q \in \mathbb{D}$ that are not contained in any $Q_i \in \mathcal{F}$, that is,

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i}.$$

For a given $Q \in \mathbb{D}$, the "local discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes in $\mathbb{D}_Q$ that are not contained in any $Q_i \in \mathcal{F}$ or, equivalently,

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q.$$

We also allow $\mathcal{F}$ to be the null set, in which case $\mathbb{D}_\emptyset = \mathbb{D}$ and $\mathbb{D}_{\emptyset,Q} = \mathbb{D}_Q$.

With a slight abuse of notation, let Q^0 be either E, and in that case $\mathbb{D}_{Q^0} := \mathbb{D}$, or a fixed cube in $\mathbb{D}$, and hence $\mathbb{D}_{Q^0}$ is the family of dyadic subcubes of Q^0. Let μ be a non-negative Borel measure on Q^0 and assume that $0 < \mu(Q) < \infty$ for every $Q \in \mathbb{D}_{Q^0}$. For the rest of this section, we will be working with μ that is dyadically doubling in Q^0. This means that there exists C_μ such that $\mu(Q) \leq C_\mu \mu(Q')$ for every $Q, Q' \in \mathbb{D}_{Q^0}$ with $\ell(Q) = 2\ell(Q')$.

Definition 2.11 (A_∞^{dyadic}). Given Q^0 and a non-negative dyadically doubling measure μ in Q^0, a non-negative Borel measure ν defined on Q^0 is said to belong to $A_\infty^{\text{dyadic}}(Q^0, \mu)$ if there exist constants $0 < \alpha, \beta < 1$ such that for every $Q \in \mathbb{D}_{Q^0}$ and for every Borel set $F \subset Q$, we have that

$$\frac{\mu(F)}{\mu(Q)} > \alpha \quad \text{implies} \quad \frac{\nu(F)}{\nu(Q)} > \beta.$$

It is well known (see [8, 15]) that, since μ is a dyadically doubling measure in Q^0, $\nu \in A_\infty^{\text{dyadic}}(Q^0, \mu)$ if and only if $\nu \ll \mu$ in Q^0 (here $\ll$ means absolutely continuous, that is, if $\mu(F) = 0$, then $\nu(F) = 0$ whenever F is a Borel subset of Q^0) and there exists $1 < p < \infty$ such that $\nu \in \mathrm{RH}_p^{\text{dyadic}}(Q^0, \mu)$, that is, there is a constant $C \geq 1$ such that

$$\Big(\fint_Q k(x)^p \, d\mu(x)\Big)^{\frac{1}{p}} \leq C \fint_Q k(x) \, d\mu(x) = C\frac{\nu(Q)}{\mu(Q)}$$

for every $Q \in \mathbb{D}_{Q^0}$, and where $k = d\nu/d\mu$ is the Radon–Nikodym derivative.

For each family $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q^0}$ of pairwise disjoint dyadic cubes, and each $f \in L^1_{\text{loc}}(\mu)$, we define the projection operator

$$\mathcal{P}_{\mathcal{F}}^\mu f(x) = f(x)\mathbf{1}_{E \setminus (\bigcup_{Q_i \in \mathcal{F}} Q_i)}(x) + \sum_{Q_i \in \mathcal{F}} \Big(\fint_{Q_i} f(y) \, d\mu(y)\Big)\mathbf{1}_{Q_i}(x).$$

If ν is a non-negative Borel measure on Q^0, we may naturally define the measure $\mathcal{P}_{\mathcal{F}}^\mu \nu$ by

$$\mathcal{P}_{\mathcal{F}}^\mu \nu(F) = \int_E \mathcal{P}_{\mathcal{F}}^\mu \mathbf{1}_F \, d\nu,$$

that is,

$$\mathcal{P}_{\mathcal{F}}^\mu \nu(F) = \nu\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \nu(Q_i) \tag{2.7}$$

for each Borel set $F \subset Q^0$.

2.4 Sawtooth domains

In the sequel, $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, will be a 1-sided NTA domain satisfying the CDC. Write $\mathbb{D} = \mathbb{D}(\partial\Omega)$ for the dyadic grid obtained from Lemma 2.8 with $E = \partial\Omega$. In Remark 2.17 below, we shall show that under the present assumptions one has that $\operatorname{diam}(\Delta) \approx r_\Delta$ for every surface ball Δ. In particular, $\operatorname{diam}(Q) \approx \ell(Q)$ for every $Q \in \mathbb{D}$ in view of (2.3). Given $Q \in \mathbb{D}$, we pick (and fix) X_{Δ_Q}, a Corkscrew point relative to Δ_Q, and define $X_Q := X_{\Delta_Q}$, called the "Corkscrew point relative to Q". We note that there might be many choices for X_Q, but we just choose one which is fixed from now on. In any case, one has

$$\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q).$$

As done above, given $Q \in \mathbb{D}$ and $\mathcal{F}$ a possibly empty family of pairwise disjoint dyadic cubes, we can define the following: $\mathbb{D}_Q$, the "discretized Carleson region"; $\mathbb{D}_\mathcal{F}$, the "global discretized sawtooth" relative to $\mathcal{F}$; and $\mathbb{D}_{\mathcal{F},Q}$, the "local discretized sawtooth" relative to $\mathcal{F}$. Note that if $\mathcal{F}$ is the null set, then $\mathbb{D}_\emptyset = \mathbb{D}$ and $\mathbb{D}_{\emptyset,Q} = \mathbb{D}_Q$. We would like to introduce the "geometric" Carleson regions and sawtooths.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a fixed collection of (closed) dyadic Whitney cubes of $\Omega \subset \mathbb{R}^{n+1}$, so that the cubes in $\mathcal{W}$ form a covering of Ω with non-overlapping interiors and satisfy

$$4 \operatorname{diam}(I) \leq \operatorname{dist}(4I, \partial\Omega) \leq \operatorname{dist}(I, \partial\Omega) \leq 40 \operatorname{diam}(I) \quad \text{for all } I \in \mathcal{W}, \tag{2.8}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2), \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.}$$

Let $X(I)$ denote the center of I, let $\ell(I)$ denote the side length of I, and write $k = k_I$ if $\ell(I) = 2^{-k}$.

Given $0 < \lambda < 1$ and $I \in \mathcal{W}$, we write $I^* = (1 + \lambda)I$ for the "fattening" of I. By taking λ small enough, we can arrange matters, so that, first, $\operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J)$ for every $I, J \in \mathcal{W}$. Secondly, I^* meets J^* if and only if ∂I meets ∂J (the fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathcal{W}$ whose boundaries touch, so that the Harnack chain property then holds locally in $I^* \cup J^*$, with constants depending on λ). By picking λ sufficiently small, say $0 < \lambda < \lambda_0$, we may also suppose that there is $\tau \in (\frac{1}{2}, 1)$ such that for distinct $I, J \in \mathcal{W}$ we have that $\tau J \cap I^* = \emptyset$. In what follows, we will need to work with dilations $I^{**} = (1 + 2\lambda)I$ or $I^{***} = (1 + 4\lambda)I$, and in order to ensure that the same properties discussed above for I^* hold for I^{**} and I^{***}, we further assume that $0 < \lambda < \lambda_0/4$. We note that $I^{**} \neq (I^*)^*$.

For every $Q \in \mathbb{D}$, we can construct and fix a family $\mathcal{W}_Q^* \subset \mathcal{W}(\Omega)$ and define

$$U_Q := \bigcup_{I \in \mathcal{W}_Q^*} I^*,$$

satisfying the following properties: $X_Q \in U_Q$ and there are uniform constants k^* and K_0 such that

$$\begin{cases} k(Q) - k^* \leq k_I \leq k(Q) + k^* & \text{for all } I \in \mathcal{W}_Q^*, \\ X(I) \to_{U_Q} X_Q & \text{for all } I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I, Q) \leq K_0 2^{-k(Q)} & \text{for all } I \in \mathcal{W}_Q^*. \end{cases} \tag{2.9}$$

Here, $X(I) \to_{U_Q} X_Q$ means that the interior of U_Q contains all balls in a Harnack chain (in Ω) connecting $X(I)$ to X_Q, and moreover, for any point Z contained in any ball in the Harnack chain, we have

$$\operatorname{dist}(Z, \partial\Omega) \approx \operatorname{dist}(Z, \Omega \setminus U_Q)$$

with uniform control of the implicit constants. The constants k^*, K_0, and the implicit constants in the condition $X(I) \to_{U_Q} X_Q$ depend on the allowable parameters and on λ. Moreover, given $I \in \mathcal{W}(\Omega)$, we have that $I \in \mathcal{W}_{Q_I}^*$, where $Q_I \in \mathbb{D}$ satisfies $\ell(Q_I) = \ell(I)$, and contains any fixed $\hat{y} \in \partial\Omega$ such that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, \hat{y})$. The reader is referred to [19, Section 3] and [23] for full details.

For a given $Q \in \mathbb{D}$, the "Carleson box" relative to Q is defined by

$$T_Q := \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \Big).$$

For a given family $\mathcal{F} = \{Q_i\} \subset \mathbb{D}$ of pairwise disjoint cubes and a given $Q \in \mathbb{D}$, we define the "local sawtooth region" relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} = \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'} \Big) = \operatorname{int}\Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^* \Big), \tag{2.10}$$

where

$$\mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \mathcal{W}^*_{Q}.$$

Note that in the previous definition we may allow $\mathcal{F}$ to be empty, in which case clearly $\Omega_{\emptyset,Q} = T_Q$. Similarly, the "global sawtooth region" relative to $\mathcal{F}$ is defined by

$$\Omega_{\mathcal{F}} = \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} U_{Q'} \Big) = \operatorname{int}\Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F}}} I^* \Big),$$

where

$$\mathcal{W}_{\mathcal{F}} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} \mathcal{W}^*_{Q}.$$

If $\mathcal{F}$ is the empty set, clearly $\Omega_\emptyset = \Omega$. For a given $Q \in \mathbb{D}$ and $x \in \partial\Omega$, let us introduce the "truncated dyadic cone"

$$\Gamma_Q(x) := \bigcup_{x \in Q' \in \mathbb{D}_Q} U_{Q'},$$

where it is understood that $\Gamma_Q(x) = \emptyset$ if $x \notin Q$. Analogously, we can slightly fatten the Whitney boxes and use I^{**} to define new fattened Whitney regions and sawtooth domains. More precisely, for every $Q \in \mathbb{D}$,

$$T^*_Q := \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_Q} U^*_{Q'} \Big), \quad \Omega^*_{\mathcal{F},Q} := \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U^*_{Q'} \Big), \quad \Gamma^*_Q(x) := \bigcup_{x \in Q' \in \mathbb{D}_{Q_0}} U^*_{Q'},$$

where

$$U^*_Q := \bigcup_{I \in \mathcal{W}^*_Q} I^{**}.$$

Similarly, we can define T^{**}_Q, $\Omega^{**}_{\mathcal{F},Q}$, $\Gamma^{**}_Q(x)$, and U^{**}_Q by using I^{***} in place of I^{**}.

Given Q we next define the "localized dyadic non-tangential maximal function"

$$\mathcal{N}_Q u(x) := \sup_{Y \in \Gamma^*_Q(x)} |u(Y)|, \quad x \in \partial\Omega, \tag{2.11}$$

for every $u \in C(T^*_Q)$, where it is understood that $\mathcal{N}_Q u(x) = 0$ for every $x \in \partial\Omega \setminus Q$ (since $\Gamma^*_Q(x) = \emptyset$ in such a case). Finally, let us introduce the "localized dyadic conical square function"

$$\mathcal{S}_Q u(x) := \Big(\iint_{\Gamma_Q(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY \Big)^{\frac{1}{2}}, \quad x \in \partial\Omega, \tag{2.12}$$

for every $u \in W^{1,2}_{\mathrm{loc}}(T_{Q_0})$. Note that again $\mathcal{S}_Q u(x) = 0$ for every $x \in \partial\Omega \setminus Q$.

To define the "Carleson box" T_Δ associated with a surface ball $\Delta = \Delta(x, r)$, let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200r \le 2^{-k}$, and set

$$\mathbb{D}^\Delta := \{Q \in \mathbb{D}_{k(\Delta)} : \ Q \cap 2\Delta \neq \emptyset\}.$$

We then define

$$T_\Delta := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q} \Big).$$

We can also consider fattened versions of T_Δ given by

$$T^*_\Delta := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T^*_Q} \Big), \quad T^{**}_\Delta := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T^{**}_Q} \Big).$$

Following [19, Section 3] or [23], one can easily see that there exist constants $0 < \kappa_1 < 1$ and $\kappa_0 \geq 16\Xi$ (with Ξ being the constant in (2.3)), depending only on the allowable parameters, so that

$$\kappa_1 B_Q \cap \Omega \subset T_Q \subset T_Q^* \subset T_Q^{**} \subset \overline{T_Q^{**}} \subset \kappa_0 B_Q \cap \overline{\Omega} =: \frac{1}{2} B_Q^* \cap \overline{\Omega}, \tag{2.13}$$

$$\frac{5}{4} B_\Delta \cap \Omega \subset T_\Delta \subset T_\Delta^* \subset T_\Delta^{**} \subset \overline{T_\Delta^{**}} \subset \kappa_0 B_\Delta \cap \overline{\Omega} =: \frac{1}{2} B_\Delta^* \cap \overline{\Omega}, \tag{2.14}$$

and also

$$Q \subset \kappa_0 B_\Delta \cap \partial\Omega = \frac{1}{2} B_\Delta^* \cap \partial\Omega =: \frac{1}{2}\Delta^* \quad \text{for all } Q \in \mathbb{D}^\Delta,$$

where B_Q is defined as in (2.4), $\Delta = \Delta(x, r)$ with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and $B_\Delta = B(x, r)$ is so that $\Delta = B_\Delta \cap \partial\Omega$. From our choice of the parameters, one also has that $B_Q^* \subset B_{Q'}^*$ whenever $Q \subset Q'$.

In the remainder of this section, we show that if Ω is a 1-sided NTA domain satisfying the CDC, then Carleson boxes and local and global sawtooth domains are also 1-sided NTA domains satisfying the CDC. We next present some of the properties of the capacity which will be used in our proofs. From the definition of capacity, one can easily see that, given a ball B and compact sets $F_1 \subset F_2 \subset \overline{B}$, then

$$\operatorname{Cap}_2(F_1, 2B) \leq \operatorname{Cap}_2(F_2, 2B). \tag{2.15}$$

Also, given two balls $B_1 \subset B_2$ and a compact set $F \subset \overline{B_1}$, then

$$\operatorname{Cap}_2(F, 2B_2) \leq \operatorname{Cap}_2(F, 2B_1). \tag{2.16}$$

On the other hand, [16, Lemma 2.16] gives that if F is a compact set with $F \subset \overline{B}$, then there is a dimensional constant C_n such that

$$C_n^{-1} \operatorname{Cap}_2(F, 2B) \leq \operatorname{Cap}_2(F, 4B) \leq \operatorname{Cap}_2(F, 2B). \tag{2.17}$$

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain satisfying the CDC. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q}$ are 1-sided NTA domains and satisfy the CDC with uniform implicit constants depending only on the dimension and on the corresponding constants for Ω.*

Proof. A careful examination of the proofs in [19, Appendices A.1 and A.2] reveals that if Ω is a 1-sided NTA domain, then all Carleson boxes T_Q and T_Δ, and local and global sawtooth domains $\Omega_{\mathcal{F},Q}$ and $\Omega_{\mathcal{F}}$ inherit the interior Corkscrew and Harnack chain conditions, and hence they are also 1-sided NTA domains. Therefore, we only need to prove the CDC. We are going to consider only the case $\Omega_{\mathcal{F},Q}$ (which in particular gives the desired property for T_Q by allowing $\mathcal{F}$ to be the null set). The other proofs require minimal changes, which are left to the interested reader. To this end, fix $Q \in \mathbb{D}$ and $\mathcal{F} \subset \mathbb{D}_Q$ a (possibly empty) family of pairwise disjoint dyadic cubes. Let $x \in \partial\Omega_{\mathcal{F},Q}$ and $0 < r < \operatorname{diam}(\Omega_{\mathcal{F},Q}) \approx \ell(Q)$.

Case 1: $\delta(x) = 0$. In that case, we have that $x \in \partial\Omega$ and we can use that Ω satisfies the CDC with constant c_1, inequality (2.15) and the fact that $\Omega_{\mathcal{F},Q} \subset \Omega$ (see Figure 1) to obtain the desired estimate

$$c_1 r^{n-1} \lesssim \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega, B(x, 2r)) \leq \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x, 2r)).$$

Case 2: $0 < \delta(x) < r/M$ with M large enough to be chosen. In this case, $x \in \Omega \cap \partial\Omega_{\mathcal{F},Q}$, and hence there exist $Q' \in \mathbb{D}_{\mathcal{F},Q}$ and $I \in \mathcal{W}_{Q'}^*$ such that $x \in \partial I^*$. Note that, by (2.9),

$$|x - x_{Q'}| \leq \operatorname{diam}(I^*) + \operatorname{dist}(I, Q') + \operatorname{diam}(Q') \lesssim \ell(Q') \approx \ell(I) \approx \delta(x) \lesssim \frac{r}{M}.$$

Let $Q'' \in \mathbb{D}_Q$ be such that

$$x_{Q'} \in Q'' \quad \text{and} \quad \frac{r}{2M} \leq \ell(Q'') < \frac{r}{M} < \ell(Q).$$

If $Z \in B_{Q''}$, then

$$|Z - x| \leq |Z - x_{Q''}| + |x_{Q''} - x_{Q'}| + |x_{Q'} - x| \lesssim \ell(Q'') + \frac{r}{M} \lesssim \frac{r}{M} < r,$$

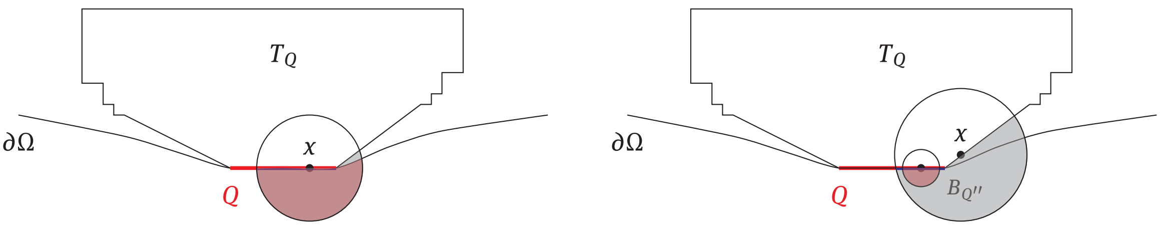

Figure 1: Case 1 and case 2 for T_Q.

provided M is taken large enough, and consequently $B_{Q''} \subset B(x,r)$ (see Figure 1). Finally, if $Z \in B(x,2r)$, we can obtain, by taking M larger if needed,

$$|Z - x_{Q''}| \le |Z - x| + |x - x_{Q'}| + |x_{Q'} - x_{Q''}| < 2r + C\frac{r}{M} + \Xi r_{Q''} < 6M\Xi r_{Q''},$$

so that $B(x,2r) \subset 6M\Xi B_{Q''}$. Once M has been fixed so that the previous estimates hold, we use them in conjunction with the fact that Ω satisfies the CDC with constant c_1, inequalities (2.15)–(2.17), and the fact that $\Omega_{\mathcal{F},Q} \subset \Omega$ to obtain

$$\begin{aligned}\frac{c_1}{(2M\Xi)^{n-1}} r^{n-1} &\le c_1 r_{Q''}^{n-1}\\ &\lesssim \mathrm{Cap}_2(\overline{B_{Q''}} \setminus \Omega, 2B_{Q''})\\ &\lesssim \mathrm{Cap}_2(\overline{B_{Q''}} \setminus \Omega, 6M\Xi B_{Q''})\\ &\le \mathrm{Cap}_2(\overline{B_{Q''}} \setminus \Omega, B(x,2r))\\ &\le \mathrm{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x,2r)),\end{aligned}$$

which gives us the desired lower bound in the present case.

Case 3: $\delta(x) > r/M$. In this case, $x \in \Omega \cap \partial\Omega_{\mathcal{F},Q}$, and hence there exist $Q' \in \mathbb{D}_{\mathcal{F},Q}$ and $I \in \mathcal{W}^*_{Q'}$ such that $x \in \partial I^*$ and $\mathrm{int}(I^*) \subset \Omega_{\mathcal{F},Q}$. Also there exists $J \in \mathcal{W}$, with $J \ni x$ such that $J \notin \mathcal{W}^*_{Q''}$ for any $Q'' \in \mathbb{D}_{\mathcal{F},Q}$, which implies that $\tau J \subset \Omega \setminus \Omega_{\mathcal{F},Q}$ for some $\tau \in (\frac{1}{2}, 1)$ (see Section 2.4). Note that $\ell(I) \approx \ell(J) \approx \delta(x) \gtrsim r$, and more precisely $r/M < \delta(x) < 41\,\mathrm{diam}(J)$ by (2.8).

Let $B' = B(x', s)$ with $s = r/(300M)$ and x' being the point in the segment joining x and the center of J at distance $2s$ from x (see Figure 2). It is easy to see that $B' \subset B(x,r) \subset B(x,2r) \subset 1000MB'$ and also $\overline{B'} \subset \mathrm{int}(J) \setminus \Omega_{\mathcal{F},Q}$. We can then use (2.2) and (2.15)–(2.17) to obtain the desired estimate:

$$\begin{aligned}\frac{1}{(300M)^{n-1}} r^{n-1} &= s^{n-1}\\ &\approx \mathrm{Cap}_2(\overline{B'}, 2B')\\ &\lesssim \mathrm{Cap}_2(\overline{B'}, 1000MB')\\ &\le \mathrm{Cap}_2(\overline{B'}, B(x,2r))\\ &\le \mathrm{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x,2r)).\end{aligned}$$

Collecting the three cases and using (2.2), we have been able to show that

$$\frac{\mathrm{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x,2r))}{\mathrm{Cap}_2(\overline{B(x,r)}, B(x,2r))} \gtrsim 1 \quad \text{for all } x \in \partial\Omega_{\mathcal{F},Q},\ 0 < r < \mathrm{diam}(\Omega_{\mathcal{F},Q}),$$

which eventually gives that $\Omega_{\mathcal{F},Q}$ satisfies the CDC. This completes the proof. □

Our next auxiliary result adapts [22, Lemma 4.44] to our current setting and constructs cut-off functions adapted to the sawtooth domains. These will be used later in the proof of Theorem 1.1.

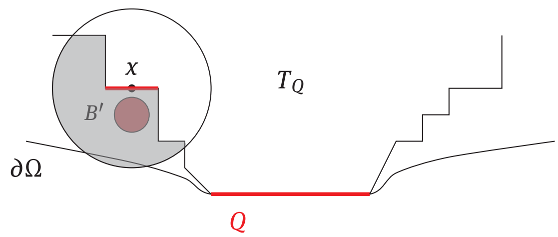

Figure 2: Case 3 for T_Q.

Lemma 2.13. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Given $Q_0 \in \mathbb{D}$ and $N \geq 4$, consider the family of pairwise disjoint cubes*

$$\mathcal{F}_N = \{Q \in \mathbb{D}_{Q_0} : \ell(Q) = 2^{-N}\ell(Q_0)\}$$

and let $\Omega_N := \Omega_{\mathcal{F}_N,Q_0}$ and $\Omega_N^* := \Omega^*_{\mathcal{F}_N,Q_0}$. There exist $\Psi_N \in C_c^\infty(\mathbb{R}^{n+1})$ and a constant $C \geq 1$ depending only on the dimension n, the 1-sided NTA constants, the CDC constant, and independent of N and Q_0 such that the following assertions hold:

(i) $C^{-1}\mathbf{1}_{\Omega_N} \leq \Psi_N \leq \mathbf{1}_{\Omega_N^*}$.

(ii) $\sup_{X\in\Omega}|\nabla\Psi_N(X)|\delta(X) \leq C$.

(iii) *Setting*

$$\mathcal{W}_N := \bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}} \mathcal{W}_Q^*, \quad \mathcal{W}_N^\Sigma := \{I \in \mathcal{W}_N : \textit{there exists } J \in \mathcal{W}\setminus\mathcal{W}_N \textit{ with } \partial I \cap \partial J \neq \emptyset\},$$

one has

$$\nabla\Psi_N \equiv 0 \quad \textit{in} \bigcup_{I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma} I^{**},$$

and there exists a family $\{\widehat{Q}_I\}_{I\in\mathcal{W}_N^\Sigma} \subset \mathbb{D}$ so that

$$C^{-1}\ell(I) \leq \ell(\widehat{Q}_I) \leq C\ell(I), \quad \operatorname{dist}(I,\widehat{Q}_I) \leq C\ell(I), \quad \sum_{I\in\mathcal{W}_N^\Sigma}\mathbf{1}_{\widehat{Q}_I} \leq C. \tag{2.18}$$

Proof. We proceed as in [22, Lemma 4.44]. Recall that, given any closed dyadic cube I in $\mathbb{R}^{n+1}$, we set $I^* = (1+\lambda)I$ and $I^{**} = (1+2\lambda)I$. Let us introduce $\widetilde{I^*} = (1+\frac{3}{2}\lambda)I$ so that

$$I^* \subsetneq \operatorname{int}(\widetilde{I^*}) \subsetneq \widetilde{I^*} \subset \operatorname{int}(I^{**}). \tag{2.19}$$

Given $I_0 := [-\frac{1}{2},\frac{1}{2}]^{n+1} \subset \mathbb{R}^{n+1}$, fix $\phi_0 \in C_c^\infty(\mathbb{R}^{n+1})$ such that $1_{I_0^*} \leq \phi_0 \leq 1_{\widetilde{I_0^*}}$ and $|\nabla\phi_0| \lesssim 1$ (the implicit constant depends on the parameter λ). For every $I \in \mathcal{W} = \mathcal{W}(\Omega)$, we set

$$\phi_I(\,\cdot\,) = \phi_0\Big(\frac{\cdot - X(I)}{\ell(I)}\Big),$$

so that

$$\phi_I \in C^\infty(\mathbb{R}^{n+1}), \quad 1_{I^*} \leq \phi_I \leq 1_{\widetilde{I^*}}, \quad |\nabla\phi_I| \lesssim \ell(I)^{-1},$$

with implicit constant depending only on n and λ.

For every $X \in \Omega$, we let $\Phi(X) := \sum_{I\in\mathcal{W}}\phi_I(X)$. It then follows that $\Phi \in C^\infty_{\text{loc}}(\Omega)$, since for every compact subset of Ω the previous sum has finitely many non-vanishing terms. Also, $1 \leq \Phi(X) \leq C_\lambda$ for every $X \in \Omega$ since the family $\{\widetilde{I^*}\}_{I\in\mathcal{W}}$ has bounded overlap by our choice of λ. Hence, we can set $\Phi_I = \phi_I/\Phi$ and one can easily see that $\Phi_I \in C_c^\infty(\mathbb{R}^{n+1})$, $C_\lambda^{-1}1_{I^*} \leq \Phi_I \leq 1_{\widetilde{I^*}}$ and $|\nabla\Phi_I| \lesssim \ell(I)^{-1}$. With this in hand, set

$$\Psi_N(X) := \sum_{I\in\mathcal{W}_N}\Phi_I(X) = \frac{\sum_{I\in\mathcal{W}_N}\phi_I(X)}{\sum_{I\in\mathcal{W}}\phi_I(X)}, \quad X \in \Omega.$$

We first note that the number of terms in the sum defining Ψ_N is bounded depending on N. Indeed, if $Q \in \mathbb{D}_{\mathcal{F}_N,Q_0}$, then $Q \in \mathbb{D}_{Q_0}$ and $2^{-N}\ell(Q_0) < \ell(Q) \leq \ell(Q_0)$, which implies that $\mathbb{D}_{\mathcal{F}_N,Q_0}$ has finite cardinality with bounds depending only on the dimension and N (here we recall that the number of dyadic children of a given cube is uniformly controlled). Also, by construction, $\mathcal{W}_Q^*$ has cardinality depending only on the allowable parameters. Hence, $\#\mathcal{W}_N \lesssim C_N < \infty$. This and the fact that each Φ_I is in $C_c^\infty(\mathbb{R}^{n+1})$ yield that $\Psi_N \in C_c^\infty(\mathbb{R}^{n+1})$. Note also that (2.19) and the definition of $\mathcal{W}_N$ give

$$\operatorname{supp}\Psi_I \subset \bigcup_{I\in\mathcal{W}_N} \widetilde{I^*} = \bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}} \bigcup_{I\in\mathcal{W}_Q^*} \widetilde{I^*} \subset \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}} \bigcup_{I\in\mathcal{W}_Q^*} I^{**}\Big) = \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}} U_Q^*\Big) = \Omega_N^*.$$

This, the fact that $\mathcal{W}_N \subset \mathcal{W}$, and the definition of Ψ_N immediately give that $\Psi_N \leq \mathbf{1}_{\Omega_N^*}$. On the other hand, if $X \in \Omega_N = \Omega_{\mathcal{F}_N,Q_0}$, there exists $I \in \mathcal{W}_N$ such that $X \in I^*$, in which case $\Psi_N(X) \geq \Phi_I(X) \geq C_\lambda^{-1}$. All these imply (i). Note that (ii) follows by observing that for every $X \in \Omega$,

$$|\nabla\Psi_N(X)| \leq \sum_{I\in\mathcal{W}_N} |\nabla\Phi_I(X)| \lesssim \sum_{I\in\mathcal{W}} \ell(I)^{-1} 1_{\widetilde{I^*}}(X) \lesssim \delta(X)^{-1},$$

where we have used that if $X \in \widetilde{I^*}$, then $\delta(X) \approx \ell(I)$, and also that the family $\{\widetilde{I^*}\}_{I\in\mathcal{W}}$ has bounded overlap.

To see (iii), fix $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$ and $X \in I^{**}$, and set $\mathcal{W}_X := \{J \in \mathcal{W} : \phi_J(X) \neq 0\}$ so that $I \in \mathcal{W}_X$. We first note that $\mathcal{W}_X \subset \mathcal{W}_N$. Indeed, if $\phi_J(X) \neq 0$, then $X \in \widetilde{J^*}$. Hence $X \in I^{**} \cap J^{**}$, and our choice of λ gives that ∂I meets ∂J, which in turn implies that $J \in \mathcal{W}_N$ since $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. All these yield

$$\Psi_N(X) = \frac{\sum_{J\in\mathcal{W}_N} \phi_J(X)}{\sum_{J\in\mathcal{W}} \phi_J(X)} = \frac{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X} \phi_J(X)}{\sum_{J\in\mathcal{W}_X} \phi_J(X)} = \frac{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X} \phi_J(X)}{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X} \phi_J(X)} = 1.$$

Hence, $\Psi_N|_{I^{**}} \equiv 1$ for every $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. This and the fact that $\Psi_N \in C_c^\infty(\mathbb{R}^{n+1})$ immediately give that $\nabla\Psi_N \equiv 0$ in $\bigcup_{I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma} I^{**}$.

We are left with showing the last part of (iv) and for that we borrow some ideas from [20, Appendix A.2]. Fix $I \in \mathcal{W}_N^\Sigma$ and let J be so that $J \in \mathcal{W} \setminus \mathcal{W}_N$ with $\partial I \cap \partial J \neq \emptyset$; in particular, $\ell(I) \approx \ell(J)$. Since $I \in \mathcal{W}_N^\Sigma$, there exists $Q_I \in \mathbb{D}_{\mathcal{F}_N,Q_0}$ (that is, $Q_I \subset Q_0$ with $2^{-N}\ell(Q_0) < \ell(Q_I) \leq \ell(Q_0)$ so that $I \in \mathcal{W}_{Q_I}^*$). Pick $Q_J \in \mathbb{D}$ so that $\ell(Q_J) = \ell(J)$ and it contains any fixed $\hat{y} \in \partial\Omega$ such that $\operatorname{dist}(J, \partial\Omega) = \operatorname{dist}(J, \hat{y})$. Then, as observed in Section 2.4, one has $J \in \mathcal{W}_{Q_J}^*$. But, since $J \in \mathcal{W} \setminus \mathcal{W}_N$, we necessarily have

$$Q_J \notin \mathbb{D}_{\mathcal{F}_N,Q_0} = \mathbb{D}_{\mathcal{F}_N} \cap \mathbb{D}_{Q_0}.$$

Hence,

$$\mathcal{W}_N^\Sigma = \mathcal{W}_N^{\Sigma,1} \cup \mathcal{W}_N^{\Sigma,2} \cup \mathcal{W}_N^{\Sigma,3},$$

where

$$\begin{aligned}
\mathcal{W}_N^{\Sigma,1} &:= \{I \in \mathcal{W}_N^\Sigma : Q_0 \subset Q_J\},\\
\mathcal{W}_N^{\Sigma,2} &:= \{I \in \mathcal{W}_N^\Sigma : Q_J \subset Q_0,\ \ell(Q_J) \leq 2^{-N}\ell(Q_0)\},\\
\mathcal{W}_N^{\Sigma,3} &:= \{I \in \mathcal{W}_N^\Sigma : Q_J \cap Q_0 = \emptyset\}.
\end{aligned}$$

For later use, it is convenient to observe that

$$\operatorname{dist}(Q_J, I) \leq \operatorname{dist}(Q_J, J) + \operatorname{diam}(J) + \operatorname{diam}(I) \approx \ell(J) + \ell(I) \approx \ell(I). \tag{2.20}$$

Let us first consider $\mathcal{W}_N^{\Sigma,1}$. If $I \in \mathcal{W}_N^{\Sigma,1}$, we clearly have

$$\ell(Q_0) \leq \ell(Q_J) = \ell(J) \approx \ell(I) \approx \ell(Q_I) \leq \ell(Q_0)$$

and, since $Q_I \in \mathbb{D}_{Q_0}$,

$$\operatorname{dist}(I, x_{Q_0}) \leq \operatorname{dist}(I, Q_I) + \operatorname{diam}(Q_I) \approx \ell(I).$$

In particular, $\#\mathcal{W}_N^{\Sigma,1} \lesssim 1$. Thus if we set $\widehat{Q}_I := Q_J$, it follows from (2.20) that the first two conditions in (2.18) hold, and also

$$\sum_{I\in\mathcal{W}_N^{\Sigma,1}} \mathbf{1}_{\widehat{Q}_I} \le \#\mathcal{W}_N^{\Sigma,1} \lesssim 1.$$

Consider next $\mathcal{W}_N^{\Sigma,2}$. For any $I \in \mathcal{W}_N^{\Sigma,2}$ we also set $\widehat{Q}_I := Q_J$, so that from (2.20) we clearly see that the first two conditions in (2.18) hold. It then remains to estimate the overlap. With this goal in mind, we first note that if $I \in \mathcal{W}_N^{\Sigma,2}$, the fact that $Q_I \in \mathbb{D}_{\mathcal{F}_N,Q_0}$ yields

$$2^{-N}\ell(Q_0) < \ell(Q_I) \approx \ell(I) \approx \ell(J) \approx \ell(Q_J) \le 2^{-N}\ell(Q_0),$$

and hence $\ell(I) \approx 2^{-N}\ell(Q_0)$. Suppose next that $Q_J \cap Q_J' = \widehat{Q}_I \cap \widehat{Q}_I \neq \emptyset$ for $I, I' \in \mathcal{W}_N^{\Sigma,2}$. Then, since I touches J and I' touches J',

$$\operatorname{dist}(I, I') \le \operatorname{diam}(J) + \operatorname{dist}(J, Q_J) + \operatorname{diam}(Q_J) + \operatorname{diam}(Q_J') + \operatorname{diam}(J') \approx \ell(J) + \ell(J') \approx 2^{-N}\ell(Q_0).$$

Hence, fixed $I \in \mathcal{W}_N^{\Sigma,2}$, there is a uniformly bounded number of $I' \in \mathcal{W}_N^{\Sigma,2}$ with $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$, and, in particular,

$$\sum_{I\in\mathcal{W}_N^{\Sigma,2}} \mathbf{1}_{\widehat{Q}_I} \lesssim 1.$$

We finally take into consideration the most delicate collection $\mathcal{W}_N^{\Sigma,3}$. In this case, for every $I \in \mathcal{W}_N^{\Sigma,3}$ we pick $\widehat{Q}_I \in \mathbb{D}$ so that $\widehat{Q}_I \ni x_{Q_J}$ and $\ell(\widehat{Q}_I) = 2^{-M'}\ell(Q_J)$ with $M' \ge 3$ large enough so that $2^{M'} \ge 2\Xi^2$ (cf. (2.3)). Note that, since $M' \ge 3$, we have that $\widehat{Q}_I \subset Q_J$, which, together with (2.20), implies

$$\operatorname{dist}(I, \widehat{Q}_I) \le \operatorname{dist}(I, Q_J) + \operatorname{diam}(Q_J) \lesssim \ell(I).$$

Hence, the first two conditions in (2.18) hold in the current situation.

On the other hand, the choice of M' and (2.3) guarantee that

$$\operatorname{diam}(\widehat{Q}_I) \le 2\Xi r_{\widehat{Q}_I} \le 2\Xi\ell(\widehat{Q}_I) = 2^{-M'+1}\,\Xi\,\ell(Q_J) \le \Xi^{-1}\ell(Q_J). \tag{2.21}$$

Also, since $2\Delta_{Q_J} \subset Q_J$, it follows that $Q_0 \cap 2\Delta_{Q_J} = \emptyset$, and therefore $2\Xi^{-1}\ell(Q_J) \le \operatorname{dist}(x_{Q_J}, Q_0)$. Besides, since $Q_I \subset Q_0$,

$$\operatorname{dist}(x_{Q_J}, Q_0) \le \operatorname{diam}(Q_J) + \operatorname{dist}(Q_J, J) + \operatorname{diam}(J) + \operatorname{diam}(I) + \operatorname{dist}(I, Q_I) + \operatorname{diam}(Q_I) \approx \ell(J) \approx \ell(I).$$

Thus,

$$2\Xi^{-1}\ell(Q_J) \le \operatorname{dist}(x_{Q_J}, Q_0) \le C\ell(J).$$

Suppose next that $I, I' \in \mathcal{W}_N^{\Sigma,3}$ are so that $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$ and assume without loss of generality that $\widehat{Q}_{I'} \subset \widehat{Q}_I$, and hence $\ell(J') \le \ell(J)$. Then, since $x_{Q_J} \in \widehat{Q}_I$ and $x_{Q_{J'}} \in \widehat{Q}_{I'} \subset \widehat{Q}_I$, from (2.21) we get

$$2\Xi^{-1}\ell(Q_J) \le \operatorname{dist}(x_{Q_J}, Q_0) \le |x_{Q_J} - x_{Q_{J'}}| + \operatorname{dist}(x_{Q_{J'}}, Q_0) \le \operatorname{diam}(\widehat{Q}_I) + C\ell(J') \le \Xi^{-1}\ell(Q_J) + C\ell(J')$$

Therefore, $\Xi^{-1}\ell(Q_J) \le C\ell(J)$, which in turn gives $\ell(I) \approx \ell(J) \approx \ell(J') \approx \ell(I')$. Note also that, since I touches J, I' touches J', and $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$, we obtain

$$\operatorname{dist}(I, I') \le \operatorname{diam}(J) + \operatorname{dist}(J, Q_J) + \operatorname{diam}(Q_J) + \operatorname{diam}(Q_{J'}) + \operatorname{dist}(Q_{J'}, J') + \operatorname{diam}(J') \approx \ell(J) + \ell(J') \approx \ell(I).$$

Consequently, fixed $I \in \mathcal{W}_N^{\Sigma,3}$, there is a uniformly bounded number of $I' \in \mathcal{W}_N^{\Sigma,3}$ with $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$. As a result,

$$\sum_{I\in\mathcal{W}_N^{\Sigma,3}} \mathbf{1}_{\widehat{Q}_I} \lesssim 1.$$

This clearly completes the proof of (iii), and hence that of Lemma 2.13. □

2.5 Uniformly elliptic operators, elliptic measure and the Green function

Next, we recall several facts concerning elliptic measure and the Green functions. To set the stage, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Throughout, we consider elliptic operators L of the form $Lu = -\operatorname{div}(A\nabla u)$ with $A(X) = (a_{i,j}(X))_{i,j=1}^{n+1}$ being a real-valued (non-necessarily symmetric) matrix such that $a_{i,j} \in L^\infty(\Omega)$ and there exists $\Lambda \geq 1$ such that the following uniform ellipticity condition holds:

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \quad |A(X)\xi \cdot \eta| \leq \Lambda|\xi|\,|\eta| \tag{2.22}$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$. We write $L^\top$ to denote the transpose of L, or, in other words, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A.

We say that u is a weak solution to $Lu = 0$ in Ω provided that $u \in W^{1,2}_{\text{loc}}(\Omega)$ satisfies

$$\iint A(X)\nabla u(X) \cdot \nabla\phi(X)dX = 0, \quad \text{whenever } \phi \in C_0^\infty(\Omega).$$

Associated with L, one can construct an elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ and a Green function G_L (see [23] for full details). Sometimes, in order to emphasize the dependence on Ω, we will write $\omega_{L,\Omega}$ and $G_{L,\Omega}$. If Ω satisfies the CDC, then it follows that all boundary points are Wiener regular, and hence for a given $f \in C_c(\partial\Omega)$ we can define

$$u(X) = \int_{\partial\Omega} f(z)d\omega_L^X(z), \quad \text{whenever } X \in \Omega,$$

so that $u \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfies $u = f$ on $\partial\Omega$ and $Lu = 0$ in the weak sense in Ω. Moreover, if Ω is bounded and $f \in \operatorname{Lip}(\partial\Omega)$, then $u \in W^{1,2}(\Omega)$. In the same context, the Green function satisfies the following properties which will be used throughout the paper:

$$0 \leq G_L(X,Y) \leq C|X-Y|^{1-n} \quad \text{for all } X, Y \in \Omega,\ X \neq Y, \tag{2.23}$$

$$G_L(\cdot, Y) \in W^{1,2}_{\text{loc}}(\Omega \setminus \{Y\}) \cap C(\overline{\Omega} \setminus \{Y\}) \quad \text{and} \quad G_L(\cdot, Y)|_{\partial\Omega} \equiv 0 \quad \text{for all } Y \in \Omega,$$

$$G_L(X,Y) = G_{L^\top}(Y,X) \quad \text{for all } X, Y \in \Omega,\ X \neq Y, \tag{2.24}$$

$$\iint_\Omega A(X)\nabla_X G_L(X,Y) \cdot \nabla\varphi(X)\,dX = \varphi(Y) \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.25}$$

We first define the reverse Hölder class and the A_∞ classes with respect to a fixed elliptic measure in Ω. We recall Definition 2.11 where A_∞^{dyadic} is introduced for either a fixed dyadic cube or the whole underlying space. Here we work with surface balls and with two elliptic measures whose poles are adapted to the surface ball in question. In turn, this allows us to introduce a global A_∞ condition. One reason we take A_∞ classes with respect to a fixed elliptic measure is that we do not know whether $\mathcal{H}^n|_{\partial\Omega}$ is well-defined since we do not assume any Ahlfors regularity. Hence, we have to develop these notions in terms of elliptic measures. To this end, let Ω satisfy the CDC and let L_0 and L be two real-valued (non-necessarily symmetric) elliptic operators associated with $L_0 u = -\operatorname{div}(A_0\nabla u)$ and $Lu = -\operatorname{div}(A\nabla u)$, where A and A_0 satisfy (2.22). Let $\omega_{L_0}^X$ and ω_L^X be the elliptic measures of Ω associated with the operators L_0 and L, respectively, with pole at $X \in \Omega$. Note that if we further assume that Ω is connected, then $\omega_L^X \ll \omega_L^Y$ on $\partial\Omega$ for every $X, Y \in \Omega$. Hence if $\omega_L^{X_0} \ll \omega_{L_0}^{Y_0}$ on $\partial\Omega$ for some $X_0, Y_0 \in \Omega$, then $\omega_L^X \ll \omega_{L_0}^Y$ on $\partial\Omega$ for every $X, Y \in \Omega$, and thus we can simply write $\omega_L \ll \omega_{L_0}$ on $\partial\Omega$. In the latter case, we will use the notation

$$h(\cdot\,; L, L_0, X) = \frac{d\omega_L^X}{d\omega_{L_0}^X}$$

to denote the Radon–Nikodym derivative of ω_L^X with respect to $\omega_{L_0}^X$, which is a well-defined function $\omega_{L_0}^X$-almost everywhere on $\partial\Omega$.

Definition 2.14 (Reverse Hölder and A_∞ classes). Fix $\Delta_0 = B_0 \cap \partial\Omega$, where $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$. Given p, $1 < p < \infty$, we say that $\omega_L \in \mathrm{RH}_p(\Delta_0, \omega_{L_0})$, provided that $\omega_L \ll \omega_{L_0}$ on Δ_0, and there exists $C \geq 1$ such that

$$\Big(\fint_\Delta h(y; L, L_0, X_{\Delta_0})^p \, d\omega_{L_0}^{X_{\Delta_0}}(y)\Big)^{\frac{1}{p}} \leq C \fint_\Delta h(y; L, L_0, X_{\Delta_0}) \, d\omega_{L_0}^{X_{\Delta_0}}(y) = C \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)}$$

for every $\Delta = B \cap \partial\Omega$, where $B \subset B(x_0, r_0)$, $B = B(x, r)$ with $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$. The infimum of the constants C as above is denoted by $[\omega_L]_{\mathrm{RH}_p(\Delta_0, \omega_{L_0})}$.

Similarly, we say that $\omega_L \in \mathrm{RH}_p(\partial\Omega, \omega_{L_0})$ provided that for every $\Delta_0 = \Delta(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ one has $\omega_L \in \mathrm{RH}_p(\Delta_0, \omega_{L_0})$ uniformly on Δ_0, that is,

$$[\omega_L]_{\mathrm{RH}_p(\partial\Omega, \omega_{L_0})} := \sup_{\Delta_0} [\omega_L]_{\mathrm{RH}_p(\Delta_0, \omega_{L_0})} < \infty.$$

Finally,

$$A_\infty(\Delta_0, \omega_{L_0}) = \bigcup_{p>1} \mathrm{RH}_p(\Delta_0, \omega_{L_0}) \quad \text{and} \quad A_\infty(\partial\Omega, \omega_{L_0}) = \bigcup_{p>1} \mathrm{RH}_p(\partial\Omega, \omega_{L_0}).$$

The following result lists a number of properties which will be used throughout this paper; the proofs may be found in [23].

Lemma 2.15. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a 1-sided NTA domain satisfying the CDC. Let $L_0 = -\operatorname{div}(A_0\nabla)$ and $L = -\operatorname{div}(A\nabla)$ be two real-valued (non-necessarily symmetric) elliptic operators. There exist $C_1 \geq 1$, $\rho \in (0, 1)$ (depending only on the dimension, the 1-sided NTA constants, the CDC constant, and the ellipticity of L), and $C_2 \geq 1$ (depending on the same parameters and on the ellipticity of L_0) such that for every $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$, $0 < r_0 < \operatorname{diam}(\partial\Omega)$, and $\Delta_0 = B_0 \cap \partial\Omega$, we have the following properties:*

(i) *$\omega_L^Y(\Delta_0) \geq C_1^{-1}$ for every $Y \in C_1^{-1}B_0 \cap \Omega$ and $\omega_L^{X_{\Delta_0}}(\Delta_0) \geq C_1^{-1}$.*

(ii) *If $B = B(x, r)$ with $x \in \partial\Omega$ and $\Delta = B \cap \partial\Omega$ is such that $2B \subset B_0$, then for all $X \in \Omega \setminus B_0$ we have that*

$$C_1^{-1}\omega_L^X(\Delta) \leq r^{n-1} G_L(X, X_\Delta) \leq C_1 \omega_L^X(\Delta).$$

(iii) *If $X \in \Omega \setminus 4B_0$, then $\omega_L^X(2\Delta_0) \leq C_1 \omega_L^X(\Delta_0)$.*

(iv) *For every $X \in \Omega \setminus 2\kappa_0 B_0$ with κ_0 as in (2.14), we have that*

$$\frac{1}{C_1} \frac{1}{\omega_L^X(\Delta_0)} \leq \frac{d\omega_L^{X_{\Delta_0}}}{d\omega_L^X}(y) \leq C_1 \frac{1}{\omega_L^X(\Delta_0)} \quad \text{for } \omega_L^X\text{-a.e. } y \in \Delta_0.$$

(v) *For every $X \in B_0 \cap \Omega$ and for any $j \geq 1$,*

$$\frac{d\omega_L^X}{d\omega_L^{X_{2^j\Delta_0}}}(y) \leq C_1 \Big(\frac{\delta(X)}{2^j r_0}\Big)^\rho \quad \text{for } \omega_L^X\text{-a.e. } y \in \partial\Omega \setminus 2^j\Delta_0.$$

Remark 2.16. We note that from (iv) in the previous result, Harnack's inequality, and (2.3) one can easily see that

$$\frac{d\omega_L^{X_{Q'}}}{d\omega_L^{X_{Q''}}}(y) \approx \frac{1}{\omega_L^{X_{Q''}}(Q')} \quad \text{for } \omega_L^{X_{Q''}}\text{-a.e. } y \in Q', \text{ whenever } Q' \subset Q'' \in \mathbb{D}. \tag{2.26}$$

Observe that, since $\omega_L^{X_{Q''}} \ll \omega_L^{X_{Q'}}$, an analogous inequality for the reciprocal of the Radon–Nikodym derivative follows immediately.

Remark 2.17. Given a 1-sided NTA domain Ω satisfying the CDC, we claim that if $\Delta = \Delta(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, then $\operatorname{diam}(\Delta) \approx r$. To see this, we first observe that $\operatorname{diam}(\Delta) \leq 2r$. If $\operatorname{diam}(\Delta) \geq c_0 r/4$, where c_0 is the Corkscrew constant, then clearly $\operatorname{diam}(\Delta) \approx r$. Hence, we may assume that $\operatorname{diam}(\Delta) < c_0 r/4$. Let $s = 2\operatorname{diam}(\Delta)$ so that $\operatorname{diam}(\Delta) < s < r$ and note that one can easily see that $\Delta = \Delta' := \Delta(x, s)$. Associated with Δ and Δ', we can consider the corresponding Corkscrew points X_Δ and $X_{\Delta'}$. These are different, despite the fact that $\Delta = \Delta(x, r)$. Indeed,

$$c_0 r \le \delta(X_\Delta) \le |X_\Delta - X_{\Delta'}| + |X_{\Delta'} - x| \le |X_\Delta - X_{\Delta'}| + s < |X_\Delta - X_{\Delta'}| + \frac{c_0}{2} r,$$

which yields that $|X_\Delta - X_{\Delta'}| \ge \frac{c_0}{2} r$. Note that $X_\Delta \notin 2B' := B(x, 2s)$, since otherwise we would get a contradiction: $c_0 r \le \delta(X_\Delta) \le |X_\Delta - x| < 2s < c_0 r$. Hence, we can invoke Lemma 2.15 (i) and (ii) and (2.23) to see that

$$1 \approx \omega_L^{X_\Delta}(\Delta) = \omega_L^{X_\Delta}(\Delta') \approx s^{n-1} G_L(X_\Delta, X_{\Delta'}) \lesssim s^{n-1} |X_\Delta - X_{\Delta'}|^{1-n} \lesssim \Big(\frac{s}{r}\Big)^{n-1}.$$

This and the fact that $n \ge 2$ easily yield that $r \lesssim s$, as desired.

We close this section by establishing an estimate for the non-tangential maximal function for elliptic-measure solutions.

Proposition 2.18. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Given $Q_0 \in \mathbb{D}$ and $f \in C_c(\partial\Omega)$ with $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$, let*

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \quad X \in \partial\Omega.$$

Then for every $x \in Q_0$,

$$\mathcal{N}_{Q_0} u(x) \lesssim \sup_{\substack{\Delta \ni x \\ 0 < r_\Delta < 4\Xi r_{Q_0}}} \fint_\Delta |f(y)|\, d\omega_L^{X_{Q_0}}(y), \tag{2.27}$$

and, as a consequence, for every $1 < q \le \infty$,

$$\|\mathcal{N}_{Q_0} u\|_{L^q(Q_0, \omega_L^{X_{Q_0}})} \lesssim \|f\|_{L^q(2\widetilde{\Delta}_{Q_0}, \omega_L^{X_{Q_0}})}. \tag{2.28}$$

Moreover, the implicit constants depend just on the dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constant of L and on q in (2.28).

Proof. By decomposing f into its positive and negative parts, we may assume that f is non-negative with $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$, and construct the associated u as in the statement, which is non-negative. Fix $x \in Q_0$ and let $X \in \Gamma_{Q_0}^*(x)$. Then, by definition, there are $Q \in \mathbb{D}_{Q_0}$ and $I \in \mathcal{W}_Q^*$ such that $x \in Q$ and $X \in I^{**}$. Hence, using Harnack's inequality and the notation introduced in (2.3)–(2.5), we obtain

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y) \approx \int_{\partial\Omega} f(y)\, d\omega_L^{X_Q}(y) \le \int_{4\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_Q}(y) + \sum_{j=3}^\infty \int_{2^j\widetilde{\Delta}_Q \setminus 2^{j-1}\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_Q}(y) =: \sum_{j=2}^\infty \mathcal{I}_j.$$

Let $k_0 \ge 0$ be such that $\ell(Q) = 2^{-k_0}\ell(Q_0)$. Observe that for every $j \ge k_0 + 3$ one has that $2\widetilde{\Delta}_{Q_0} \setminus 2^{j-1}\widetilde{\Delta}_Q = \emptyset$. Otherwise, there is $z \in 2\widetilde{\Delta}_{Q_0} \setminus 2^{j-1}\widetilde{\Delta}_Q$, and hence we get a contradiction:

$$4\Xi r_{Q_0} \le 2^{j-1-k_0}\Xi r_{Q_0} = 2^{j-1}\Xi r_Q \le |z - x_Q| \le |z - x_{Q_0}| + |x_Q - x_{Q_0}| \le 3\Xi r_{Q_0}.$$

With this in hand, and since $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$, we clearly see that $\mathcal{I}_j = 0$ for $j \ge k_0 + 3$.

In order to estimate the $\mathcal{I}_j$'s, we need some preparations. Note that for every $2 \le j \le k_0 + 2$ one has $2^j\widetilde{B}_Q \subset 5\widetilde{B}_{Q_0}$. We claim that

$$\frac{d\omega_L^{X_{2^j\widetilde{\Delta}_Q}}}{d\omega_L^{X_{Q_0}}}(y) \lesssim \frac{1}{\omega_L^{X_{Q_0}}(2^j\widetilde{\Delta}_Q)} \quad \text{for } \omega_L^{X_{Q_0}}\text{-a.e. } y \in 2^j\widetilde{\Delta}_Q,\ 2 \le j \le k_0 + 2. \tag{2.29}$$

Indeed, this estimate follows from Harnack's inequality and Lemma 2.15 (i) when $j \approx k_0$ since $2^j\ell(Q) \approx \ell(Q_0)$, and from Lemma 2.15 (iv) whenever $j \ll k_0$. We also observe that Lemma 2.15 (i) and Harnack's inequality readily give that

$$\omega_L^{X_{2^j\widetilde{\Delta}_Q}}(2^j\widetilde{\Delta}_Q) \approx 1 \quad \text{for every } 2 \le j \le k_0 + 2. \tag{2.30}$$

Finally, by Lemma 2.15 (v) and Harnack's inequality, it follows that

$$\frac{d\omega_L^{X_Q}}{d\omega_L^{X_{2^{j-1}\widetilde{\Delta}_Q}}}(y) \lesssim 2^{-j\rho} \quad \text{for } \omega_L^{X_Q}\text{-a.e. } y \in \partial\Omega \setminus 2^{j-1}\,\widetilde{\Delta}_Q,\ j \geq 3. \tag{2.31}$$

Let us start estimating $\mathcal{I}_2$. Use Harnack's inequality and (2.29) and (2.30) with $j = 2$ to conclude that

$$\mathcal{I}_2 \approx \fint_{4\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{\widetilde{\Delta}_Q}}(y) \approx \fint_{4\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y).$$

On the other hand, for $3 \leq j \leq k_0 + 2$, we employ (2.31), Harnack's inequality, (2.29) and (2.30) to obtain

$$\mathcal{I}_j \lesssim 2^{-j\rho} \int_{2^j\,\widetilde{\Delta}_Q \setminus 2^{j-1}\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{2^{j-1}\,\widetilde{\Delta}_Q}}(y) \lesssim 2^{-j\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{2^j\,\widetilde{\Delta}_Q}}(y) \approx 2^{-j\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y).$$

If we now collect all obtained estimates, we conclude (2.27) as desired:

$$\begin{aligned} u(X) &\lesssim \sum_{j=2}^{k_0+2} \mathcal{I}_j \\ &\lesssim \sum_{j=2}^{k_0+2} 2^{-j\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y) \\ &\leq \sup_{\substack{\Delta \ni x \\ 0 < r_\Delta < 8\Xi r_{Q_0}}} \fint_{\Delta} |f(y)|\, d\omega_L^{X_{Q_0}}(y) \sum_{j=2}^{\infty} 2^{-j\rho} \\ &\lesssim \sup_{\substack{\Delta \ni x \\ 0 < r_\Delta < 4\Xi r_{Q_0}}} \fint_{\Delta} |f(y)|\, d\omega_L^{X_{Q_0}}(y). \end{aligned}$$

To complete the proof we just need to obtain (2.28), but this follows at once upon using (2.27) and observing that the local Hardy–Littlewood maximal function on its right-hand side is bounded on $L^q(20\,\widetilde{\Delta}_{Q_0}, \omega_L^{X_{Q_0}})$, since $\omega_L^{X_{Q_0}}$ is a doubling measure in $20\,\widetilde{\Delta}_{Q_0}$ by Lemma 2.15 (i) and (iii). □

3 Dyadic sawtooth lemma for projections

In this section, we shall prove two dyadic versions of the main lemma in [10]. To set the stage we sate a result which was partially proved in [19, Proposition 6.7] under the further assumption that $\partial\Omega$ is Ahlfors regular.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain satisfying the CDC. Fix $Q_0 \in \mathbb{D}$ and let $\mathcal{F} = \{Q_k\}_k \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes. There exists*

$$Y_{Q_0} \in \Omega \cap \Omega_{\mathcal{F},Q_0} \cap \Omega^*_{\mathcal{F},Q_0}$$

so that

$$\operatorname{dist}(Y_{Q_0}, \partial\Omega) \approx \operatorname{dist}(Y_{Q_0}, \partial\Omega_{\mathcal{F},Q_0}) \approx \operatorname{dist}(Y_{Q_0}, \partial\Omega^*_{\mathcal{F},Q_0}) \approx \ell(Q_0),$$

where the implicit constants depend only on the dimension, the 1-sided NTA constants, the CDC constant, and is independent of Q_0 and $\mathcal{F}$. Additionally, for each $Q_j \in \mathcal{F}$, there is an n-dimensional cube $P_j \subset \partial\Omega_{\mathcal{F},Q_0}$, which is contained in a face of I^* for some $I \in \mathcal{W}$, and which satisfies

$$\ell(P_j) \approx \operatorname{dist}(P_j, Q_j) \approx \operatorname{dist}(P_j, \partial\Omega) \approx \ell(I) \approx \ell(Q_j) \tag{3.1}$$

and

$$\sum_j 1_{P_j} \lesssim 1, \tag{3.2}$$

where the implicit constants depend on allowable parameters.

Proof. Note first that $\Omega_{\mathcal{F},Q_0}$ is a 1-sided NTA domain satisfying the CDC (see Proposition 2.12). Pick an arbitrary $x_0 \in \partial\Omega_{\mathcal{F},Q_0}$ and let Y_0 be a Corkscrew point relative to the surface ball

$$B(x_0, \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0})/2) \cap \partial\Omega_{\mathcal{F},Q_0}$$

for the bounded domain $\Omega_{\mathcal{F},Q_0}$ (recall that one has $\operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0}) \approx \ell(Q_0) < \infty$ by (2.13)). Note that

$$Y_0 \in \Omega_{\mathcal{F},Q_0} \subset \Omega,$$

which is comprised of fattened Whitney boxes. Then $Y_0 \in I^{**}$ for some $I \in \mathcal{W}$ with $\operatorname{int}(I^{**}) \subset \Omega_{\mathcal{F},Q_0}$. Let $Y_{Q_0} = X(I)$ be the center of I so that $\delta(Y_0) \approx \ell(I) \approx \delta(Y_{Q_0})$. Then

$$\begin{aligned}\ell(Q_0) &\approx \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0})\\ &\approx \operatorname{dist}(Y_0, \partial\Omega_{\mathcal{F},Q_0})\\ &\leq \operatorname{dist}(Y_0, \partial\Omega_{\mathcal{F},Q_0^*})\\ &\leq \delta(Y_0)\\ &\approx \delta(Y_{Q_0})\\ &\approx \ell(I)\\ &\leq \operatorname{diam}(\Omega_{\mathcal{F},Q_0})\\ &= \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0})\\ &\approx \ell(Q_0).\end{aligned}$$

To continue, we note that the existence of the family $\{P_j\}_j$ so that (3.1) holds has been proved in [19, Proposition 6.7] under the further assumption that $\partial\Omega$ is Ahlfors regular. However, a careful examination of the proof shows that the same argument applies in our scenario. We are left with showing (3.2). To see this, observe that, as in [19, Remark 6.9], if $P_j \cap P_k \neq \emptyset$, then $\ell(Q_j) \approx \ell(Q_k)$. Indeed, from the previous result, $P_j \subset I_j^*$ and $P_k \subset I_k^*$ for some $I_j, I_k \in \mathcal{W}$. Thus I_j^* meets I_k^*, and by construction I_j and I_k meet. Using (3.1) and the nature of the Whitney cubes, we see that $\ell(Q_j) \approx \ell(I_j) \approx \ell(I_k) \approx \ell(Q_k)$. Using this and (3.1), one can also see that $\operatorname{dist}(Q_j, Q_k) \lesssim \ell(Q_j) \approx \ell(Q_k)$. Hence, fixing P_{j_0} and $x \in P_{j_0}$, we have some constant $k_0 \geq 1$ (depending on the allowable parameters) such that

$$\begin{aligned}\sum_j 1_{P_j}(x) &\leq \#\{P_j : P_j \cap P_{j_0} \neq \emptyset\}\\ &\leq \#\Big\{Q_j : 2^{-k_0} \leq \frac{\ell(Q_j)}{\ell(Q_{j_0})} \leq 2^{k_0},\ \operatorname{dist}(Q_j, Q_{j_0}) \leq 2^{k_0}\ell(Q_{j_0})\Big\}\\ &= \sum_{k=-k_0}^{k_0} \#\{Q_j : \ell(Q_j) = 2^k\ell(Q_{j_0}),\ \operatorname{dist}(Q_j, Q_{j_0}) \leq 2^{k_0}\ell(Q_{j_0})\}\\ &=: \sum_{k=-k_0}^{k_0} N_k.\end{aligned}$$

We next estimate each N_k. Fixed k, note that the Q_j's belong to the same generation, and hence they are pairwise disjoint and the same occurs for their corresponding Δ_{Q_j}'s, each of which has radius $(2C)^{-1}2^k\ell(Q_{j_0})$. In particular, $|x_{Q_j} - x_{Q_j'}| \gtrsim 2^k\ell(Q_{j_0}) \geq 2^{-k_0}\ell(Q_{j_0})$ for any such cubes Q_j and $Q_{j'}$ with $j \neq j'$. Moreover,

$$|x_{Q_j} - x_{Q_{j_0}}| \leq \operatorname{diam}(Q_j) + \operatorname{dist}(Q_j, Q_{j_0}) + \operatorname{diam}(Q_{j_0}) \lesssim 2^{k_0}\ell(Q_{j_0}).$$

Thus, it is easy to see (since $\mathbb{R}^{n+1}$ is geometric doubling) that $N_k \lesssim 2^{2k_0(n+1)}$. All these together gives us (3.2); we note in passing that the argument in [19, Remark 6.9] used the fact that there $\partial\Omega$ is AR to estimate each N_k, while here we are invoking the geometric doubling property of the ambient space $\mathbb{R}^{n+1}$. □

We are now ready to state the first main result of this section, which is a version of [19, Lemma 6.15] (see also [10]) valid in our setting.

Lemma 3.2 (Discrete sawtooth lemma for projections). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a* bounded 1*-sided NTA domain satisfying the CDC. Let $Q_0 \in \mathbb{D}$, let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes, and let μ be a dyadically doubling measure in Q_0. Given two real-valued (non-necessarily symmetric) elliptic L_0 and L, we write*

$$\omega_0^{Y_{Q_0}} = \omega_{L_0,\Omega}^{Y_{Q_0}} \quad \text{and} \quad \omega_L^{Y_{Q_0}} = \omega_{L,\Omega}^{Y_{Q_0}}$$

for the elliptic measures associated with L_0 and L for the domain Ω with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$ (cf. Lemma 3.1). Let

$$\omega_{L,*}^{Y_{Q_0}} = \omega_{L,\Omega_{\mathcal{F},Q_0}}^{Y_{Q_0}}$$

be the elliptic measure associated with L for the domain $\Omega_{\mathcal{F},Q_0}$ with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$. Consider the measure $\nu_L^{Y_{Q_0}}$ defined by

$$\nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\omega_L^{Y_{Q_0}}(F \cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i), \quad F \subset Q_0, \tag{3.3}$$

where P_i is the cube produced in Proposition 3.1. Then $\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}$ (see (2.7)) depends only on $\omega_0^{Y_{Q_0}}$ and $\omega_{L,*}^{Y_{Q_0}}$, but not on $\omega_L^{Y_{Q_0}}$. More precisely,

$$\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i), \quad F \subset Q_0. \tag{3.4}$$

Moreover, there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and all $F \subset Q$, we have

$$\left(\frac{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(Q)}\right)^{\theta} \lesssim \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q)} \lesssim \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(Q)}. \tag{3.5}$$

Proof. Our argument follows the ideas from [19, Lemma 6.15] and we use several auxiliary technical results from [19, Section 6] which were proved under the additional assumption that $\partial\Omega$ is AR. However, as we will indicate along the proof, most of them can be adapted to our setting. Those arguments that require new ideas will be explained in detail.

We first observe that (3.4) readily follows from the definitions of $\mathcal{P}_{\mathcal{F}}^{\mu}$ and $\nu_L^{Y_{Q_0}}$. We first establish the second estimate in (3.5). With this goal in mind, let us fix $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q_0$.

Case 1: There exists $Q_i \in \mathcal{F}$ such that $Q \subset Q_i$. By (3.4), we have

$$\frac{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q)} = \frac{\frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i)}{\frac{\mu(Q \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i)} = \frac{\mu(F)}{\mu(Q)} = \frac{\frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_L^{Y_{Q_0}}(Q_i)}{\frac{\mu(Q \cap Q_i)}{\mu(Q_i)} \omega_L^{Y_{Q_0}}(Q_i)} = \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(Q)}.$$

Case 2: $Q \not\subset Q_i$ for any $Q_i \in \mathcal{F}$, that is, $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. In particular, if $Q \cap Q_i \neq \emptyset$ with $Q_i \in \mathcal{F}$, then necessarily $Q_i \subsetneq Q$. Let $x_i^\star$ denote the center of P_i and pick $r_i \approx \ell(Q_i) \approx \ell(P_i)$ so that

$$P_i \subset \Delta_\star(x_i^\star, r_i) := B(x_i^\star, r_i) \cap \partial\Omega_{\mathcal{F},Q_0}.$$

Note that, by Proposition 2.12, Harnack's inequality and Lemma 2.15 (i) and (iii), we have that

$$\omega_{L,*}^{Y_{Q_0}}(P_i) \approx \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i)).$$

On the other hand, as in [19, Proposition 6.12], one can see that

$$\Delta_\star^Q := B(x_Q^\star, t_Q) \cap \partial\Omega_{\mathcal{F},Q_0} \subset \Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big) \tag{3.6}$$

with $t_Q \approx \ell(Q)$, $x_Q^\star \in \partial\Omega_{\mathcal{F},Q_0}$ and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$ with implicit constants depending on the allowable parameters. We note that the last expression is slightly different to that in [19, Proposition 6.2]; nonetheless, the one stated here follows from the proof in account of [19, (6.14) and Proposition 6.1] as ∂Q_i is contained in $\overline{T_{Q_i}}$. Besides, Proposition 3.1 easily yields

$$\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} P_i\Big) \subset \Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big) \subset C\Delta_\star^Q,$$

and hence

$$\omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big)\Big) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q).$$

Writing (see [19, Proposition 6.1])

$$E_0 = Q_0 \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i \subset \partial\Omega \cap \partial\Omega_{\mathcal{F},Q},$$

we have

$$\begin{aligned} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q) &\le \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i)) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &= \mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q) \end{aligned} \tag{3.7}$$

and, by (3.2),

$$\begin{aligned} \mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q) &= \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(Q \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &= \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}\Big((Q \cap E_0) \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} P_i\Big)\Big) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q). \end{aligned} \tag{3.8}$$

Since $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, we can invoke [19, Proposition 6.4] (which also holds in the current setting) to find $Y_Q \in \Omega_{\mathcal{F},Q_0}$, which serves as a Corkscrew point simultaneously for $\Omega_{\mathcal{F},Q_0}$ with respect to the surface ball $\Delta_\star(y_Q, s_Q)$ for some $y_Q \in \Omega_{\mathcal{F},Q}$ and some $s_Q \approx \ell(Q)$, and for Ω with respect to each surface ball $\Delta(x, s_Q)$, for every $x \in Q$. Applying (2.26) and Harnack's inequality to join Y_Q with X_Q and Y_{Q_0} with Y_Q, we have

$$\frac{d\omega_L^{Y_Q}}{d\omega_L^{Y_{Q_0}}} \approx \frac{1}{\omega_L^{Y_{Q_0}}(Q)}, \quad \omega_L^{Y_{Q_0}}\text{-a.e. in } Q. \tag{3.9}$$

On the other hand, one can see that

$$\widetilde{B}_Q \cup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} B(x_i^\star, r_i)\Big) \subset B(y_Q, \widehat{s}_Q) \tag{3.10}$$

for some $\widehat{s}_Q \approx s_Q$. Invoking then Proposition 2.12, and Lemma 2.15 (iii) and (iv) in the domain $\Omega_{\mathcal{F},Q_0}$, we can analogously see

$$\frac{d\omega_{L,*}^{Y_Q}}{d\omega_{L,*}^{Y_{Q_0}}} \approx \frac{1}{\omega_{L,*}^{Y_{Q_0}}(\Delta(y_Q, \widehat{s}_Q))} \approx \frac{1}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)}, \quad \omega_{L,*}^{Y_{Q_0}}\text{-a.e. in } \Delta(y_Q, \widehat{s}_Q).$$

Next, we invoke (3.7), (3.9), and (3.10) to obtain

$$\begin{aligned} \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q)} &\approx \frac{\omega_{L,*}^{Y_{Q_0}}(F \cap E_0)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \frac{\omega_{L,*}^{Y_{Q_0}}(P_i)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} \\ &\approx \omega_{L,*}^{Y_Q}(F \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_Q}(P_i). \end{aligned} \tag{3.11}$$

We claim that the following estimates hold:

$$\omega_{L,*}^{Y_Q}(F \cap E_0) \lesssim \omega_L^{Y_Q}(F \cap E_0), \quad \omega_{L,*}^{Y_Q}(P_i) \lesssim \omega_L^{Y_Q}(Q_i).$$

The first estimate follows easily from the maximum principle since $\Omega_{\mathcal{F},Q_0} \subset \Omega$ and $F \cap E_0 \subset \partial\Omega \cap \partial\Omega_{\mathcal{F},Q_0}$. For the second one, by the maximum principle, we just need to see that $\omega_L^X(Q_i) \gtrsim 1$ for $X \in P_i$, but this follows from Lemma 2.15 (i), (2.3), Harnack's inequality, and (3.1).

With the previous estimates at our disposal, we can the continue with our estimate (3.11):

$$\begin{aligned}\frac{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q)} &\lesssim \omega_L^{Y_Q}(F \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_L^{Y_Q}(Q_i) \\ &\approx \frac{\omega_L^{Y_{Q_0}}(F \cap E_0)}{\omega_L^{Y_{Q_0}}(Q)} + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \frac{\omega_L^{Y_{Q_0}}(Q_i)}{\omega_L^{Y_{Q_0}}(Q)} \\ &= \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)} \\ &= \frac{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(Q)},\end{aligned}$$

where we have used (3.10) and the fact that

$$\mathcal{P}_{\mathcal{F}}^{\mu} \omega_L^{Y_{Q_0}}(Q) = \omega_L^{Y_{Q_0}}(Q).$$

This proves the second estimate in (3.5) in the current case.

Once we have shown the second estimate in (3.5), we can invoke [19, Lemma B.7] (which is a purely dyadic result, and hence applies in our setting) along with Lemma 3.4 below to eventually obtain the first estimate in (3.5). □

As a consequence of the previous result, we can easily obtain a dyadic analog of the main lemma in [10].

Lemma 3.3 (Discrete sawtooth lemma). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a* bounded 1*-sided NTA domain satisfying the CDC. Let $Q_0 \in \mathbb{D}$ and let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes. Given two real-valued (non-necessarily symmetric) elliptic L_0 and L, we write*

$$\omega_0^{Y_{Q_0}} = \omega_{L_0,\Omega}^{Y_{Q_0}} \quad \textit{and} \quad \omega_L^{Y_{Q_0}} = \omega_{L,\Omega}^{Y_{Q_0}}$$

for the elliptic measures associated with L_0 and L for the domain Ω with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$ (cf. Lemma 3.1). Let

$$\omega_{L,*}^{Y_{Q_0}} = \omega_{L,\Omega_{\mathcal{F},Q_0}}^{Y_{Q_0}}$$

be the elliptic measure associated with L for the domain $\Omega_{\mathcal{F},Q_0}$ with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$. Consider the measure $\nu_L^{Y_{Q_0}}$ defined by (3.3). Then there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and all $F \subset Q$, we have

$$\left(\frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}\right)^{\theta} \lesssim \frac{\nu_L^{Y_{Q_0}}(F)}{\nu_L^{Y_{Q_0}}(Q)} \lesssim \frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}. \tag{3.12}$$

In particular, if $F \subset Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i$, we have

$$\left(\frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}\right)^{\theta} \lesssim \frac{\omega_{L,*}^{Y_{Q_0}}(F)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} \lesssim \frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}, \tag{3.13}$$

where $\Delta_\star^Q := B(x_Q^\star, t_Q) \cap \partial\Omega_{\mathcal{F},Q_0}$ with $t_Q \approx \ell(Q)$, $x_Q^\star \in \partial\Omega_{\mathcal{F},Q_0}$, and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$ *with implicit constants depending on the allowable parameters (cf.* [19, Proposition 6.12]*).*

Proof. Letting $\mu = \omega_L^{Y_{Q_0}}$, which is dyadically doubling in Q_0, one easily has

$$\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}} = \omega_L^{Y_{Q_0}} \quad \text{and} \quad \mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}} = \nu_L^{Y_{Q_0}}.$$

Thus, (3.5) in Lemma 3.2 readily yields (3.12). Next, to obtain (3.13) we may assume that F is non-empty. Observe that if $F \subset Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i$, then

$$\nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}(F).$$

On the other hand, if $F \subset Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i$, we must be in case 2 of the proof of Lemma 3.2, and hence (3.7) and (3.8) hold. With all these, we readily obtain (3.13). □

Our last result in this section establishes that both $\nu_L^{Y_{Q_0}}$ and $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$ are dyadically doubling on Q_0.

Lemma 3.4. *Under the assumptions of Lemma 3.2, $\nu_L^{Y_{Q_0}}$ and $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$ are dyadically doubling on Q_0.*

Proof. We follow the ideas in [19, Lemma B.2]. We shall first see that $\nu_L^{Y_{Q_0}}$ is dyadically doubling. To this end, let $Q \in \mathbb{D}_{Q_0}$ be fixed and let Q' be one of its dyadic children. We consider three cases.

Case 1: There exists $Q_i \in \mathcal{F}$ such that $Q \subset Q_i$. In this case, we have

$$\nu_L^{Y_{Q_0}}(Q) = \frac{\omega_L^{Y_{Q_0}}(Q)}{\omega_L^{Y_{Q_0}}(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i) \lesssim \frac{\omega_L^{Y_{Q_0}}(Q')}{\omega_L^{Y_{Q_0}}(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i) = \nu_L^{Y_{Q_0}}(Q'),$$

where we have used Harnack's inequality and Lemma 2.15 (i) and (iii).

Case 2: $Q' \in \mathcal{F}$. For simplicity say $Q' = Q_1 \in \mathcal{F}$, and in this case

$$\nu_L^{Y_{Q_0}}(Q') = \omega_{L,*}^{Y_{Q_0}}(P_1).$$

Note that then $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and we let $\mathcal{F}_1$ be the family of cubes $Q_i \in \mathcal{F}$ with $Q_i \cap Q \neq \emptyset$ and observe that if $Q_i \in \mathcal{F}_1$, then $Q_i \subsetneq Q$. Then, by (3.2),

$$\begin{aligned}\nu_L^{Y_{Q_0}}(Q) &= \omega_{L,*}^{Y_{Q_0}}\Big(Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) + \sum_{Q_i\in\mathcal{F}_1} \frac{\omega_L^{Y_{Q_0}}(Q\cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i)\\ &= \omega_{L,*}^{Y_{Q_0}}\Big(Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) + \sum_{Q_i\in\mathcal{F}_1} \omega_{L,*}^{Y_{Q_0}}(P_i)\\ &\lesssim \omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i\in\mathcal{F}_1} P_i\Big)\Big). \end{aligned} \tag{3.14}$$

Recall that in case 2 in the proof of Lemma 3.2 we mentioned that $P_1 \subset \Delta_\star(x_1^\star, r_1)$ with $x_1^\star$ being the center of P_1 and $r_1 \approx \ell(P_1) \approx \ell(Q_1) \approx \ell(Q)$ since Q is the dyadic parent of Q_1. Note that, since $Q_i \in \mathcal{F}_1$, by (3.1),

$$\ell(P_i) \approx \operatorname{dist}(P_i, Q) \approx \ell(Q_i) \lesssim \ell(Q) = 2\ell(Q_1) \approx \ell(P_1) \approx \operatorname{dist}(Q_1, P_1) \approx r_1.$$

Thus,

$$\Big(Q \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i\in\mathcal{F}_1} P_i\Big) \subset \Delta_\star(x_1^\star, Cr_1),$$

where we here and below we use the notation $\Delta_\star$ for the surface balls with respect to $\partial\Omega_{\mathcal{F},Q_0}$. Using this, (3.14), Lemma 2.15 (i) and (iii) and Harnack's inequality, we derive

$$\nu_L^{Y_{Q_0}}(Q) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_1^\star, Cr_1)) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_1^\star, r_1)) \lesssim \omega_{L,*}^{Y_{Q_0}}(P_1) = \nu_L^{Y_{Q_0}}(Q').$$

Case 3: None of the conditions in the previous cases happen, and necessarily $Q, Q' \in \mathbb{D}_{\mathcal{F},Q_0}$. We take the same set $\mathcal{F}_1$ as in the previous case and again, if $Q_i \in \mathcal{F}_1$, then $Q_i \subsetneq Q$ (otherwise, we are driven to case 1). Introduce $\mathcal{F}_2$ as the family of cubes $Q_i \in \mathcal{F}$ with $Q_i \cap Q' \neq \emptyset$. Again, if $Q_i \in \mathcal{F}_2$, we have $Q_i \subsetneq Q'$; otherwise either $Q' = Q_i$, which is case 2, or $Q' \subsetneq Q_i$, which implies $Q \subset Q_i$ and we are back to case 1.

Note that, since Q is the dyadic parent of Q', using the same notation as in (3.6) applied to $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$, we have that

$$\operatorname{dist}(x^\star_{Q'}, Q) \le \operatorname{dist}(x^\star_{Q'}, Q') \lesssim \ell(Q') \approx \ell(Q) \approx t_{Q'}.$$

Also by (3.1),

$$\operatorname{dist}(x^\star_{Q'}, P_i) \lesssim \operatorname{dist}(x^\star_{Q'}, Q) + \ell(Q) + \operatorname{dist}(Q, P_i) \lesssim \ell(Q) + \operatorname{dist}(Q_i, P_i) \lesssim \ell(Q) \approx t_{Q'}.$$

These readily give

$$\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big) \subset \Delta_\star(x^\star_{Q'}, C t_{Q'}).$$

We can then proceed as in the previous case (see (3.14)) to obtain

$$\nu_L^{Y_{Q_0}}(Q) \lesssim \omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \cup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big)\Big) \le \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x^\star_{Q'}, C t_{Q'})) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^{Q'}),$$

where $\Delta_\star^{Q'} = B(x^\star_{Q'}, t_{Q'}) \cap \partial\Omega_{\mathcal{F},Q_0}$ (see (3.6)) and we have used Lemma 2.15 (i) and (iii) and Harnack's inequality. On the other hand, proceeding as in (3.7) with Q' in place of Q since $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$, we obtain

$$\begin{aligned}
\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^{Q'}) &\le \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i)) \\
&\lesssim \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \omega_{L,*}^{Y_{Q_0}}(P_i) \\
&= \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \frac{\omega_L^{Y_{Q_0}}(Q' \cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \\
&= \nu_L^{Y_{Q_0}}(Q').
\end{aligned}$$

Eventually, we obtain that

$$\nu_L^{Y_{Q_0}}(Q) \lesssim \nu_L^{Y_{Q_0}}(Q'),$$

completing the proof of the dyadic doubling property of $\nu_L^{Y_{Q_0}}$.

We next deal with $\mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}$. We can simply follow the previous argument replacing $\omega_L^{Y_{Q_0}}$ by $\mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}$ to see that in cases 2 and 3 we have that

$$\mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}(Q) = \nu_L^{Y_{Q_0}}(Q) \quad \text{and} \quad \mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}(Q') = \nu_L^{Y_{Q_0}}(Q'),$$

and hence the doubling condition follows from the previous calculations and the constant depends on that of $\omega_{L,*}^{Y_{Q_0}}$. With regard to case 1, in which $Q \subset Q_i$ for some $Q_i \in \mathcal{F}$, one can easily see that

$$\mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}(Q) = \frac{\mu(Q)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \lesssim \frac{\mu(Q')}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) = \mathcal{P}^\mu_{\mathcal{F}} \nu_L^{Y_{Q_0}}(Q'),$$

which uses that μ is dyadically doubling in Q_0. Eventually, we have seen that the doubling constant depends on the ones of $\omega_{L,*}^{Y_{Q_0}}$ and μ as desired. This completes the proof. □

4 Proof of the main results

4.1 Proof of Theorem 1.1

By renormalization, we may assume without loss of generality that $\|u\|_{L^\infty(\Omega)} = 1$. We will first prove a dyadic version of (1.2). Let $\mathbb{D} = \mathbb{D}(\partial\Omega)$ be the dyadic grid from Lemma 2.8 with $E = \partial\Omega$. Our goal is to show that

$$M_0 := \sup_{Q^0 \in \mathbb{D}} \sup_{\substack{Q_0 \in \mathbb{D}_{Q^0} \\ \ell(Q_0) \le \frac{\ell(Q^0)}{M}}} \frac{1}{\omega_L^{X_{Q^0}}(Q_0)} \iint_{T_{Q_0}} |\nabla u(X)|^2 G_L(X_{Q^0}, X)\, dX \lesssim 1, \tag{4.1}$$

with $M \geq 4$ large enough. Assuming this momentarily, let us see how to derive (1.2). Fix B and B' as in the suprema in (1.2). Let $k, k' \in \mathbb{Z}$ be so that $2^{k-1} < r \leq 2^k$ and $2^{k'-1} < r' \leq 2^{k'}$, and define $k'' := \min\{k', k - 10k_M\}$ where $k_M \geq 1$ is large enough to be chosen depending on M and the allowable parameters. Set

$$\mathcal{W}' := \{I \in \mathcal{W} : I \cap B' \neq \emptyset,\ \ell(I) < 2^{k''}\} \cup \{I \in \mathcal{W} : I \cap B' \neq \emptyset,\ \ell(I) \geq 2^{k''}\} =: \mathcal{W}_1' \cup \mathcal{W}_2'.$$

Note that for every $I \in \mathcal{W}$ with $I \cap B' \neq \emptyset$ we have

$$\ell(I) < \operatorname{diam}(I) \leq \frac{\operatorname{dist}(I, \partial\Omega)}{4} < \frac{r'}{4} \leq 2^{k'-2}.$$

As a consequence, if $\mathcal{W}_2' \neq \emptyset$, then $k'' = k - 10k_M$, and, picking $I \in \mathcal{W}_2' \neq \emptyset$, one has

$$r \approx 2^k \approx_M 2^{k''} \leq \ell(I) \leq 2^{k'-2} \approx r' \lesssim r.$$

This gives $r' \approx_M r$ and $\#\mathcal{W}_2' \lesssim_M 1$.

To proceed, let us write

$$\iint_{B'\cap\Omega} |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX \leq \iint_{\bigcup_{I\in\mathcal{W}_1'} I} |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX + \sum_{I\in\mathcal{W}_2'} \iint_I |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX =: \mathcal{I} + \mathcal{II},$$

and we estimate each term in turn.

To estimate $\mathcal{II}$, we may assume that $\mathcal{W}_2' \neq \emptyset$, and hence $k'' = k - 10k_M$, $r' \approx r$ and $\#\mathcal{W}_2' \lesssim 1$. Then Lemma 2.15, the fact that $\omega(\partial\Omega) \leq 1$, Caccioppoli's inequality, the normalization $\|u\|_{L^\infty(\Omega)} = 1$, and Harnack's inequality give

$$\mathcal{II} = \sum_{I\in\mathcal{W}_2'} \iint_I |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX \lesssim \sum_{I\in\mathcal{W}_2'} \ell(I)^{1-n} \iint_I |\nabla u(X)|^2\, dX \lesssim \#\mathcal{W}_2' \lesssim 1 \approx \omega_L^{X_\Delta}(\Delta').$$

Next, we deal with $\mathcal{I}$. Introduce the disjoint family

$$\mathcal{F}' = \{Q \in \mathbb{D} : \ell(Q) = 2^{k''-1},\ Q \cap 3B' \neq \emptyset\}.$$

Given $I \in \mathcal{W}_1'$, let $X_I \in B' \cap I$, and let $Q_I \in \mathbb{D}$ be so that $\ell(Q_I) = \ell(I)$ and it contains some fixed $y_I \in \partial\Omega$ such that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, y_I)$. Then, as observed in Section 2.4, one has $I \in \mathcal{W}_{Q_I}^*$. Note that

$$|y_I - x'| \leq \operatorname{dist}(y_I, I) + \operatorname{diam}(I) + |X_I - x'| \leq \frac{5}{4}\operatorname{dist}(I, \partial\Omega) + |X_I - x'| \leq \frac{9}{4}|X_I - x'| < 3r',$$

and hence $y_I \in Q_I \cap 3\Delta'$. This and the fact that, as observed before, $\ell(Q_I) = \ell(I) < 2^{k''}$ imply that $Q_I \subset Q$ for some $Q \in \mathcal{F}'$. Hence, $I \subset (1+\lambda)I \subset U_{Q_I} \subset \overline{T_Q}$ for some $Q \in \mathcal{F}'$. This eventually shows that

$$\bigcup_{I\in\mathcal{W}_1'} I \subset \bigcup_{Q\in\mathcal{F}'} T_Q,$$

and therefore

$$\mathcal{I} \leq \sum_{Q\in\mathcal{F}'} \iint_{T_Q} |\nabla u(X)|^2 G_L(X_\Delta, X)\, dX.$$

For any $Q \in \mathcal{F}'$, pick the unique (ancestor) $\widehat{Q} \in \mathbb{D}$ with $\ell(\widehat{Q}) = 2^{k-1}$ and $Q \subset \widehat{Q}$. Note that

$$\delta(X_\Delta) \approx r, \quad \delta(X_{\widehat{Q}}) \approx \ell(\widehat{Q}) = 2^{k-1} \approx r.$$

Also,

$$\begin{aligned} |X_\Delta - X_{\widehat{Q}}| &\leq |X_\Delta - x| + |x - x'| + |x' - x_Q| + |x_Q - x_{\widehat{Q}}| + |x_{\widehat{Q}} - X_{\widehat{Q}}| \\ &< 3r + 3r' + \operatorname{diam}(Q) + \operatorname{diam}(\widehat{Q}) + \ell(\widehat{Q}) \\ &\lesssim r + 2^{k''} + 2^k \\ &\lesssim r. \end{aligned}$$

Hence, by the Harnack chain condition, one obtains that $G_L(X_\Delta, X) \approx G_L(X_{\widehat{Q}}, X)$ for every $X \in T_Q$ (in doing that, we need to make sure that k_M is large enough so that the Harnack chain joining X_Δ and $X_{\widehat{Q}}$, which is cr-away from $\partial\Omega$, does not get near T_Q, which is $\kappa_0 \ell(Q)$-close to $\partial\Omega$). Note also that

$$\frac{\ell(Q)}{\ell(\widehat{Q})} = 2^{k''-k} \leq 2^{-k_M} < M^{-1},$$

provided k_M is large enough depending on M. All these and (4.1) yield

$$\begin{aligned} \mathcal{I} &\lesssim \sum_{Q \in \mathcal{F}'} \iint_{T_Q} |\nabla u(X)|^2 G_L(X_{\widehat{Q}}, X)\, dX \\ &\lesssim M_0 \sum_{Q \in \mathcal{F}'} \omega_L^{X_{\widehat{Q}}}(Q) \\ &\lesssim M_0 \sum_{Q \in \mathcal{F}'} \omega_L^{X_\Delta}(Q) \\ &\leq M_0 \omega_L^{X_\Delta}\Big(\bigcup_{Q \in \mathcal{F}'} Q\Big) \\ &\leq M_0 \omega_L^{X_\Delta}(C\Delta') \\ &\lesssim M_0 \omega_L^{X_\Delta}(\Delta'), \end{aligned}$$

where we have used Lemma 2.15. This completes the prof of the fact that (4.1) implies (1.2).

We next focus on showing (4.1). With this goal in mind, we fix $Q^0 \in \mathbb{D} = \mathbb{D}(\partial\Omega)$ and let $Q_0 \in \mathbb{D}_{Q^0}$ with $\ell(Q_0) \leq \ell(Q^0)/M$ with M large enough so that $X_{Q^0} \notin 4B^*_{Q_0}$ (cf. (2.13)). Write $\omega_L = \omega_L^{X_{Q^0}}$ and $\mathcal{G}_L = G_L(X_{Q^0}, \cdot)$ and note that our choice of M, (2.24), and (2.25) guarantee that $L^\top \mathcal{G}_L = L^\top G_{L^\top}(\cdot, X_{Q^0}) = 0$ in the weak sense in $4B^*_Q$.

Fix $N \gg 1$, consider the family of pairwise disjoint cubes $\mathcal{F}_N = \{Q \in \mathbb{D}_{Q_0} : \ell(Q) = 2^{-N}\ell(Q_0)\}$ and let $\Omega_N := \Omega_{\mathcal{F}_N, Q_0}$ and $\Omega_N^* = \Omega_{\mathcal{F}_N, Q_0}$ (cf. (2.10)). Note that, by construction, $\Omega_N \subset \Omega_N^* \subset T_{Q_0}$ and Ω_N is an increasing sequence of sets converging to T_{Q_0}. Additionally, by construction, $\delta(X) \gtrsim 2^{-N}\ell(Q_0)$ for every $X \in \Omega_N^*$, and hence $\overline{\Omega_N^*}$ is a compact subset of Ω. Our goal is to show that for every $N \gg 1$ there holds

$$\iint_{\Omega_N} |\nabla u(X)|^2 \mathcal{G}_L(X)\, dX \leq M_0 \omega_L(Q_0), \tag{4.2}$$

with M_0 independent of Q^0, Q_0, and N. Hence, the monotone convergence theorem yields

$$\iint_{T_{Q_0}} |\nabla u(X)|^2 \mathcal{G}_L(X)\, dX = \lim_{N \to \infty} \iint_{\Omega_N} |\nabla u(X)|^2 \mathcal{G}_L(X)\, dX \leq M_0 \omega_L(Q_0),$$

which is (4.1).

Let us next start estimating (4.2). Using Ψ_N from Lemma 2.13 and the ellipticity of the matrix A, we have

$$\begin{aligned} &\iint_{\Omega_N} |\nabla u(X)|^2 \mathcal{G}_L(X)\, dX \\ &\quad \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla u(X)|^2 \mathcal{G}_L(X) \Psi_N(X)\, dX \\ &\quad \lesssim \iint_{\mathbb{R}^{n+1}} A(X)\nabla u(X) \cdot \nabla u(X) \mathcal{G}_L(X) \Psi_N(X)\, dX \\ &\quad = \iint_{\mathbb{R}^{n+1}} A(X)\nabla u(X) \cdot \nabla(u \mathcal{G}_L \Psi_N)(X)\, dX - \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla(u^2 \Psi_N)(X) \cdot \nabla \mathcal{G}_L(X)\, dX \\ &\qquad - \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla(u^2)(X) \cdot \nabla \Psi_N(X) \mathcal{G}_L(X)\, dX + \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla \Psi_N(X) \cdot \nabla \mathcal{G}_L(X) u(X)^2\, dX \\ &\quad =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \end{aligned}$$

We recall that M was taken large enough so that $X_{Q^0} \notin 4B^*_{Q_0}$ (cf. (2.13)); in particular, X_{Q^0} is away from $\Omega^*_N \subset B^*_{Q_0}$. As a result, $u\mathcal{G}_L\Psi_N$ and $u^2\Psi_N$ belong to $W^{1,2}(\Omega)$, since $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$, $\operatorname{supp}\Psi_N \subset \Omega^*_N$, $\delta(X) \gtrsim 2^{-N}\ell(Q_0)$ for every $X \in \Omega^*_N$ (hence $\overline{\Omega^*_N}$ is a compact subset of Ω), and by the properties of G_L. Using all these, one can easily see via a limiting argument (convolving with a compactly supported smooth approximation to the identity) that the fact that $Lu = 0$ in the weak sense in Ω implies that $\mathcal{J}_1 = 0$. Likewise, one can easily show that $\mathcal{J}_2 = 0$ by recalling that $\operatorname{supp}\Psi_N \subset \Omega^*_N \subset \frac{1}{2}B_Q{}^* \cap \Omega$ (see (2.13)) and that, as mentioned above, $L^\top\mathcal{G}_L = 0$ in the weak sense in $4B^*_Q$. Thus, we are left with estimating the terms $\mathcal{J}_3$ and $\mathcal{J}_4$. One can then show that

$$\begin{aligned}
|\mathcal{J}_3| + |\mathcal{J}_4| &\lesssim \iint_{\bigcup_{I\in\mathcal{W}^\Sigma_N} I^{**}} (|\nabla u|\mathcal{G}_L + |\nabla\mathcal{G}_L|)\delta(\cdot)^{-1}\,dX \\
&\lesssim \sum_{I\in\mathcal{W}^\Sigma_N} \ell(I)^{\frac{n-1}{2}}\Big(\Big(\iint_{I^{**}}|\nabla u|^2\,dX\Big)^{\frac12}\mathcal{G}_L(X(I)) + \Big(\iint_{I^{**}}|\nabla\mathcal{G}_L|^2\,dX\Big)^{\frac12}\Big) \\
&\lesssim \sum_{I\in\mathcal{W}^\Sigma_N} \ell(I)^{\frac{n-3}{2}}\Big(\Big(\int_{I^{***}}|u|^2\,dX\Big)^{\frac12} + \ell(I)^{\frac{n+1}{2}}\Big)\mathcal{G}_L(X(I)) \\
&\lesssim \sum_{I\in\mathcal{W}^\Sigma_N} \ell(I)^{n-1}\mathcal{G}_L(X(I)),
\end{aligned}$$

where in the first inequality we have used Lemma 2.13 (ii) and (iii), and the normalization $\|u\|_{L^\infty(\Omega)} = 1$. The second follows from Cauchy–Schwarz's and Harnack's inequalities, and the fact that $\delta(X) \approx \ell(I)$ for every $X \in I \in \mathcal{W}$. The third estimate follows from Caccioppoli's inequality since $L^\top\mathcal{G}_L = 0$ and $Lu = 0$ in the weak sense in $I^{***} \subset \frac{1}{2}B^*_Q \cap \Omega$ (see (2.13)), and Harnack's inequality. And the last one uses again that $\|u\|_{L^\infty(\Omega)} = 1$. Invoking Lemmas 2.15 and Lemma 2.13, one can see that $\ell(I)^{n-1}\mathcal{G}_L(X(I)) \lesssim \omega_L(\widehat{Q}_I)$ for every $I \in \mathcal{W}^\Sigma_N$. This together with Lemma 2.13 allows us to conclude

$$|\mathcal{J}_3| + |\mathcal{J}_4| \lesssim \sum_{I\in\mathcal{W}^\Sigma_N}\omega_L(\widehat{Q}_I) \lesssim \omega_L\Big(\bigcup_{I\in\mathcal{W}^\Sigma_N}\widehat{Q}_I\Big).$$

Note that, if $y \in \widehat{Q}_I$ with $I \in \mathcal{W}^\Sigma_N$, one has

$$|y - x_{Q_0}| \le \operatorname{diam}(\widehat{Q}_I) + \operatorname{dist}(\widehat{Q}_I, I) + \operatorname{diam}(I) + \operatorname{dist}(I, x_{Q_0}) \lesssim \ell(I) + \ell(Q_0) \lesssim \ell(Q_0),$$

where we have used (2.13) and (2.18). Thus, Lemma 2.15, (2.3), and (2.4) give

$$|\mathcal{J}_3| + |\mathcal{J}_4| \lesssim \omega_L(C\Delta_{Q_0}) \lesssim \omega_L(\Delta_{Q_0}) \le \omega_L(Q_0).$$

This allows us to complete the proof of Theorem 1.1. □

4.2 Proof of Theorem 1.2

We borrow some ideas from [21]. Given $k \in \mathbb{N}$, introduce the truncated localized conical square function: for every $Q \in \mathbb{D}_{Q_0}$ and $x \in Q$, let

$$\mathcal{S}^k_Q u(x) := \Big(\iint_{\Gamma^k_Q(x)} |\nabla u(Y)|^2\delta(Y)^{1-n}\,dY\Big)^{\frac12}, \quad \text{where } \Gamma^k_Q(x) := \bigcup_{\substack{x\in Q'\in\mathbb{D}_Q\\ \ell(Q')\ge 2^{-k}\ell(Q_0)}} U_{Q'},$$

where if $\ell(Q) < 2^{-k}\ell(Q_0)$ it is understood that $\Gamma^k_Q(x) = \emptyset$ and $\mathcal{S}^k_Q u(x) = 0$. Note that by the monotone convergence theorem, $\mathcal{S}^k_Q u(x) \nearrow \mathcal{S}_Q u(x)$ as $k \to \infty$ for every $x \in Q$.

Fixed k_0 large enough (eventually, $k_0 \to \infty$), our goal is to show that we can find $\vartheta > 0$ (independent of k_0) such that for every $\beta, \gamma, \lambda > 0$ we have

$$\omega_L^{X_{Q_0}}(\{x \in Q_0 : \mathcal{S}^{k_0}_{Q_0}u(x) > (1+\beta)\lambda,\ \mathcal{N}_{Q_0}u(x) \le \gamma\lambda\}) \lesssim \Big(\frac{\gamma}{\beta}\Big)^\vartheta \omega_L^{X_{Q_0}}(\{x \in Q_0 : \mathcal{S}^{k_0}_{Q_0}u(x) > \beta\lambda\}), \tag{4.3}$$

where the implicit constant depends on the allowable parameters and it is independent of k_0. The passage from this good-λ inequality to (1.3) is included at the end of the proof. To prove (4.3), we fix $\beta, \gamma, \lambda > 0$ and set

$$E_\lambda := \{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > \lambda\}.$$

Consider first the case $E_\lambda \subsetneq Q_0$. Note that if $x \in E_\lambda$, by definition, $\mathcal{S}_{Q_0}^{k_0} u(x) > \lambda$. Let $Q_x \in \mathbb{D}_{Q_0}$ be the unique dyadic cube such that $Q_x \ni x$ and $\ell(Q_x) = 2^{-k_0}\ell(Q_0)$. Then it is clear from the construction that for every $y \in Q_x$ one has

$$\Gamma_{Q_0}^{k_0}(x) = \bigcup_{Q_x \subset Q \subset Q_0} U_Q = \Gamma_{Q_0}^{k_0}(y) \quad \text{and} \quad \lambda < \mathcal{S}_{Q_0}^{k_0} u(x) = \mathcal{S}_{Q_0}^{k_0} u(y).$$

Hence, $Q_x \subset E_\lambda$ and we have shown that for every $x \in E_\lambda$ there exists $Q_x \in \mathbb{D}_{Q_0}$ such that $Q_x \ni x$ and $Q_x \subset E_\lambda$. We then take the ancestors of Q_x, and look for the one with maximal side length $Q_x^{\max} \supset Q_x$ which is contained in E_λ. That is,

$$Q \subset E_\lambda \quad \text{for every } Q_x \subset Q \subset Q_x^{\max}$$

and

$$\widehat{Q}_x^{\max} \cap Q_0 \setminus E_\lambda \neq \emptyset,$$

where $\widehat{Q}_x^{\max}$ is the dyadic parent of $Q_x^{\max}$ (during this proof we will use $\widehat{Q}$ to denote the dyadic parent of Q, that is, the only dyadic cube containing it with double side length). Note that the assumption $E_\lambda \subsetneq Q_0$ guarantees that $Q_x^{\max} \in \mathbb{D}_{Q_0} \setminus \{Q_0\}$. Let $\mathcal{F}_0 = \{Q_j\}_j$ be the collection of such maximal cubes as x runs in E_λ, and we clearly have that the family is pairwise disjoint and also

$$E_\lambda = \bigcup_{Q_j \in \mathcal{F}_0} Q_j.$$

Also, by construction $\ell(Q_j) \geq 2^{-k_0}\ell(Q_0)$, and by the maximality of each Q_j we can select $x_j \in \widehat{Q}_j \setminus E_\lambda$.

On the other hand, for any $x \in Q_j$, using that $x_j \in \widehat{Q}_j \setminus E_\lambda$, we have

$$\Gamma_{Q_0}^{k_0}(x) = \bigcup_{\substack{x \in Q \in \mathbb{D}_{Q_0} \\ \ell(Q) \geq 2^{-k_0}\ell(Q_0)}} U_Q = \Gamma_{Q_j}^{k_0}(x) \cup \Big(\bigcup_{Q_j \subsetneq Q \subset Q_0} U_Q \Big) \subset \Gamma_{Q_j}^{k_0}(x) \cup \Gamma_{Q_0}^{k_0}(x_j),$$

and therefore

$$\mathcal{S}_{Q_0}^{k_0} u(x) \leq \mathcal{S}_{Q_j}^{k_0} u(x) + \mathcal{S}_{Q_0}^{k_0} u(x_j) \leq \mathcal{S}_{Q_j}^{k_0} u(x) + \lambda.$$

As a consequence,

$$\{x \in Q_j : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda\} \subset \{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda\}$$

and

$$\begin{aligned} \{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda\} &= \{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda\} \cap E_\lambda \\ &= \bigcup_{Q_j \in \mathcal{F}_0} \{x \in Q_j : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda\} \\ &\subset \bigcup_{Q_j \in \mathcal{F}_0} \{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda\}. \end{aligned}$$

This has been done under the assumption that $E_\lambda \subsetneq Q_0$. In the case $E_\lambda = Q_0$, we set $\mathcal{F}_0 = \{Q_0\}$. Then in both cases we obtain

$$\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda\} \subset \bigcup_{Q_j \in \mathcal{F}_0} \{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda\}. \tag{4.4}$$

Thus, to obtain (4.3) it suffices to see that for every $Q_j \in \mathcal{F}_0$,

$$\omega_L^{X_{Q_0}}(\{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda,\ \mathcal{N}_{Q_0} u(x) \leq \gamma\lambda\}) \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \omega_L^{X_{Q_0}}(Q_j). \tag{4.5}$$

From this, we just need to sum in $Q_j \in \mathcal{F}_0$ to see that (4.4) together with the previous facts yields the desired estimate (4.3):

$$
\begin{aligned}
\omega_L^{X_{Q_0}}(\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\lambda,\ \mathcal{N}_{Q_0} u(x) \le \gamma\lambda\})
&\le \sum_{Q_j \in \mathcal{F}_0} \omega_L^{X_{Q_0}}(\{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda,\ \mathcal{N}_{Q_0} u(x) \le \gamma\lambda\}) \\
&\lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \sum_{Q_j \in \mathcal{F}_0} \omega_L^{X_{Q_0}}(Q_j) \\
&= \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \omega_L^{X_{Q_0}}\Big(\bigcup_{Q_j \in \mathcal{F}_0} Q_j\Big) \\
&= \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \omega_L^{X_{Q_0}}(E_\lambda).
\end{aligned}
$$

Let us then obtain (4.5). Fix $Q_j \in \mathcal{F}_0$ and to ease the notation write $P_0 = Q_j$. Set

$$
\widetilde{E}_\lambda = \{x \in P_0 : \mathcal{S}_{P_0}^{k_0} u(x) > \beta\lambda\}, \quad F_\lambda = \{x \in P_0 : \mathcal{N}_{Q_0} u(x) \le \gamma\lambda\}. \tag{4.6}
$$

If $\omega_L^{X_{Q_0}}(F_\lambda) = 0$, then (4.5) is trivial, and hence we may assume that $\omega_L^{X_{Q_0}}(F_\lambda) > 0$ so that $P_0 \cap F_\lambda = F_\lambda \neq \emptyset$. We subdivide P_0 dyadically and stop the first time that $Q \cap F_\lambda = \emptyset$. If one never stops, we write $\mathcal{F}_{P_0}^* = \{\emptyset\}$; otherwise, $\mathcal{F}_{P_0}^* = \{P_j\}_j \subset \mathbb{D}_{P_0} \setminus \{P_0\}$ is the family of stopping cubes which is maximal (and hence pairwise disjoint) with respect to the property $F_\lambda \cap Q = \emptyset$. In particular,

$$
F_\lambda \subset P_0 \setminus \Big(\bigcup_{\mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}} P_j\Big).
$$

Next, we claim that

$$
\bigcup_{x \in F_\lambda} \Gamma_{P_0}^{k_0}(x) \subset \bigcup_{\substack{Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0} \\ \ell(Q) \ge 2^{-k_0}\ell(Q_0)}} U_Q \subset \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}} U_Q^*\Big) = \Omega_{\mathcal{F}_{P_0}^*, P_0}^* =: \Omega_*. \tag{4.7}
$$

To verify the first inclusion, we fix $Y \in \Gamma_{P_0}^{k_0}(x)$ with $x \in F_\lambda$. Then $Y \in U_Q$, where $x \in Q \in \mathbb{D}_{P_0}$. Since $x \in F_\lambda$, we must have $Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*}$ (otherwise $Q \subset P_j$ for some $P_j \in \mathcal{F}_{P_0}^*$, and this would imply that $x \in P_j \cap F_\lambda = \emptyset$), and therefore $Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}$, which gives the first inclusion. The second inclusion in (4.7) is trivial (since $U_Q \subset \operatorname{int}(U_Q^*)$).

To continue, we see that

$$
|u(Y)| \le \gamma\lambda \quad \text{for all } Y \in \Omega_*. \tag{4.8}
$$

Fix such a Y so that $Y \in U_Q^*$ for some $Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}$. If $Q \cap F_\lambda = \emptyset$, by maximality of the cubes in $\mathcal{F}_{P_0}^*$, it follows that $Q \subset P_j$ for some $P_j \in \mathcal{F}_{P_0}^*$, which contradicts the fact $Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}$. Thus, $Q \cap F_\lambda \neq \emptyset$ and we can select $x \in Q \cap F_\lambda$ so that by definition $|u(Y)| \le \mathcal{N}_{Q_0} u(x) \le \gamma\lambda$, since $Y \in U_Q^* \subset \Gamma_{Q_0}^*(x)$.

Apply Lemma 3.1 to find $X_* := Y_{P_0} \in \Omega_* \cap \Omega$ so that

$$
\ell(P_0) \approx \operatorname{dist}(X_*, \partial\Omega_*) \approx \delta(X_*). \tag{4.9}
$$

Let $\omega_L^* := \omega_{L,\Omega_*}^{X_*}$ be the elliptic measure associated with L relative to Ω_* with pole at X_* and write

$$
\delta_* = \operatorname{dist}(\cdot, \partial\Omega_*).
$$

Given $Y \in \Omega_*$, we choose $y_Y \in \partial\Omega_*$ such that $|Y - y_Y| = \delta_*(Y)$. By definition, for $x \in F_\lambda$ and $Y \in \Gamma_{P_0}(x)$, there is a $Q \in \mathbb{D}_{P_0}$ such that $Y \in U_Q$ and $x \in Q$. Thus, by the triangle inequality and the definition of U_Q, we have that for $Y \in \Gamma_{P_0}(x)$,

$$
|x - y_Y| \le |x - Y| + \delta_*(Y) \approx \delta(Y) + \delta_*(Y) \approx \delta_*(Y)
$$

where in the last step we have used that

$$\delta(Y) \approx \delta_*(Y) \quad \text{for } Y \in \bigcup_{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}} U_Q. \tag{4.10}$$

On the other hand, as observed above (see [19, Proposition 6.1]),

$$F_\lambda \subset P_0 \setminus \Big(\bigcup_{\mathcal{F}} Q_j\Big) \subset \partial\Omega \cap \partial\Omega_*.$$

Using this and the fact that $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$ if $Q \cap F_\lambda \neq \emptyset$, we have

$$\begin{aligned}
\int_{F_\lambda} \mathcal{S}^{k_0}_{P_0} u(x)^2\, d\omega^*_L(x) &= \int_{F_\lambda} \iint_{\Gamma^{k_0}_{P_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY\, d\omega^*_L(x) \\
&\leq \int_{F_\lambda} \sum_{\substack{x\in Q\in\mathbb{D}_{P_0}\\ \ell(Q)\geq 2^{-k_0}\ell(Q_0)}} \iint_{U_Q} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY\, d\omega^*_L(x) \\
&\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}} \Big(\iint_{U_Q} |\nabla u(Y)|^2\, dY\Big) \ell(Q)^{1-n} \omega^*_L(Q\cap F_\lambda) \\
&\lesssim \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}\\ \ell(Q)\geq M^{-1}\ell(P_0)}} \cdots + \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}\\ \ell(Q)< M^{-1}\ell(P_0)}} \cdots \\
&=: \Sigma_1 + \Sigma_2,
\end{aligned} \tag{4.11}$$

where M is a large constant to be chosen.

We start by estimating Σ_1. Note first that

$$\#\{Q :\in \mathbb{D}_{P_0} : \ell(Q) \geq M^{-1}\ell(P_0)\} \leq C_M,$$

and thus

$$\begin{aligned}
\Sigma_1 &\lesssim \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}\\ \ell(Q)\geq M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I\in\mathcal{W}^*_Q} \iint_{I^*} |\nabla u(Y)|^2\, dY \\
&\lesssim \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F}^*_{P_0},P_0}\\ \ell(Q)\geq M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I\in\mathcal{W}^*_Q} \ell(I)^{-2} \iint_{I^{**}} |u(Y)|^2\, dY \\
&\lesssim (\gamma\lambda)^2 \sum_{\substack{Q\in\mathbb{D}_{P_0}\\ \ell(Q)\geq M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I\in\mathcal{W}^*_Q} \ell(I)^{n-1} \\
&\lesssim_M (\gamma\lambda)^2,
\end{aligned}$$

where we have used (4.8), along with the fact that $\operatorname{int}(I^{**}) \subset \operatorname{int}(U^*_Q) \subset \Omega_*$ for any $I \in \mathcal{W}^*_Q$ with $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$, and the fact that $\mathcal{W}^*_Q$ has uniformly bounded cardinality. To estimate Σ_2, picking $y_Q \in Q \cap F_\lambda$, we have that

$$Q \cap F_\lambda \subset B(y_Q, 2\operatorname{diam}(Q)) \cap \partial\Omega_* =: \Delta^*_Q.$$

Write X^*_Q for the Corkscrew relative to Δ^*_Q with respect to Ω_* so that $\delta_*(X^*_Q) \approx \operatorname{diam}(Q) \lesssim M^{-1}\ell(P_0)$. Note that, by (4.9), we clearly have $X_* \in \Omega \setminus B(y_Q, 4\operatorname{diam}(Q))$ provided M is sufficiently large. Hence, by Lemma 2.15 (ii) applied in Ω_*, which is a 1-sided NTA domain satisfying the CDC by Proposition 2.12, for every $Y \in U_Q$ we obtain

$$\ell(Q)^{1-n}\omega^*_L(Q\cap F_\lambda) \lesssim \operatorname{diam}(Q)^{1-n}\omega^*_L(\Delta^*_Q) \lesssim G_{L,*}(X_*, X^*_Q) \approx G_{L,*}(X_*, Y), \tag{4.12}$$

where $G_{L,*}$ is the Green function for the operator L relative to the domain Ω_*. Above, the last estimate uses Harnack's inequality (we may need to take M slightly larger) and the fact that, by (4.10), one has $\delta_*(Y) \approx \ell(Q) \approx \operatorname{diam}(Q) \approx \delta_*(X^*_Q)$ (see Remark 2.17) and that if $I \ni Y$ with $I \in \mathcal{W}^*_Q$, then

$$|Y - X_*| \leq \operatorname{diam}(I) + \operatorname{dist}(I, Q) + \operatorname{diam}(Q) + |y_Q - X^*| \lesssim \operatorname{diam}(Q).$$

Write $\{P_0^i\}_i \subset \mathbb{D}_{P_0}$ for the collection of dyadic cubes with $M\ell(P_0) \le \ell(P_0^i) < 2M\ell(P_0)$, which has uniformly bounded cardinality depending on M. Note that

$$\{Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0} : \ell(Q) < M^{-1}\ell(P_0)\} \subset \bigcup_i \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0^i}.$$

For each i, if

$$\mathbb{D}_{\mathcal{F}_{P_0}^*, P_0^i} \neq \emptyset,$$

then

$$P_0^i \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0},$$

and hence $P_0^i \cap F_\lambda \neq \emptyset$. Pick then $y_i \in P_0^i \cap F_\lambda$ and note that for every $Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0^i}$, by (2.13), it follows that

$$U_Q \subset T_{P_0^i} \cap \Omega_* \subset B_{P_0^i}^* \cap \Omega_* \subset B(y_i, C\kappa_0 \ell(P_0^i)) \cap \Omega_* =: B_i \cap \Omega_*.$$

Then, using (4.12), we have

$$\begin{aligned}
\Sigma_2 &\lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0} \\ \ell(Q) < M^{-1}\ell(P_0)}} \iint_{U_Q} |\nabla u(Y)|^2 G_{L,*}(X_*, Y)\, dY \\
&\lesssim \sum_i \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}_{P_0}^*, P_0^i} \\ \ell(Q) < M^{-1}\ell(P_0)}} \iint_{U_Q} |\nabla u(Y)|^2 G_{L,*}(X_*, Y)\, dY \\
&\lesssim \sum_i \iint_{B_i \cap \Omega_*} |\nabla u(Y)|^2 G_{L,*}(X_*, Y)\, dY \\
&\lesssim \|u\|_{L^\infty(\Omega_*)}^2 \sum_i \omega_L^*(B_i \cap \partial\Omega_*) \\
&\lesssim (\gamma\lambda)^2,
\end{aligned}$$

where we have invoked Theorem 1.1 applied in Ω_*, which is a 1-sided NTA domain satisfying the CDC by Proposition 2.12, and we may need to take M slightly larger and use Harnack's inequality, (4.8), and the fact that $\{P_0^i\}_i \subset \mathbb{D}_{P_0}$ has uniformly bounded cardinality.

Using Chebyshev's inequality, (4.11), and collecting the estimates for Σ_1 and Σ_2, we conclude that

$$\omega_L^*(\widetilde{E}_\lambda \cap F_\lambda) \le \frac{1}{(\beta\lambda)^2} \int_{\widetilde{E}_\lambda \cap F_\lambda} (\mathcal{S}_{P_0}^{k_0} u)^2\, d\omega_L^* \le \frac{1}{(\beta\lambda)^2} \int_{F_\lambda} \mathcal{S}_{P_0}^{k_0} u(x)^2\, d\omega_L^*(x) \lesssim \Big(\frac{\gamma}{\beta}\Big)^2.$$

At this point, we invoke Lemma 3.3 in P_0 with $\mathcal{F}_{P_0}^*$; we warn the reader that P_0 and $\mathcal{F}_{P_0}^* = \{P_j\}_j$ play the role of Q_0 and $\{Q_j\}_j$ and that associated to each P_j one finds $\widetilde{P}_j$ as in Proposition 3.1, which now plays the role of P_j in that result. Furthermore, $\mu = \omega_L^{X_*}$ (recall that $X_* = Y_{P_0}$) and observe that

$$F_\lambda \subset P_0 \setminus \Big(\bigcup_{\mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}} P_j\Big)$$

implies on account of (3.13) that for some $\vartheta > 0$ we have

$$\frac{\omega_L^{X_*}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_*}(P_0)} \lesssim \Big(\frac{\omega_L^*(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^*(\Delta_\star^{P_0})}\Big)^{\frac{\vartheta}{2}} \lesssim \Big(\frac{\gamma}{\beta}\Big)^\vartheta,$$

where we have used that $\omega_L^*(\Delta_\star^{P_0}) \approx 1$ since

$$\Delta_\star^{P_0} := B(x_{P_0}^\star, t_{P_0}) \cap \partial\Omega_* \quad \text{with } t_{P_0} \approx \ell(P_0) \approx \operatorname{diam}(\partial\Omega_*),$$

and we have used $x^\star_{P_0} \in \partial\Omega_*$, relation (4.9), Harnack's inequality, and Lemma 2.15 (i). We can then use Remark 2.16, Harnack's inequality, and (4.9) to conclude that

$$\frac{\omega_L^{X_{Q_0}}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_{Q_0}}(P_0)} \approx \frac{\omega_L^{X_{P_0}}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_{P_0}}(P_0)} \approx \frac{\omega_L^{X_*}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_*}(P_0)} \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta}.$$

By recalling that $P_0 = Q_j \in \mathcal{F}_0$, and the definitions of $\widetilde{E}_\lambda$ and F_λ in (4.6), the previous estimates readily lead to (4.5).

To conclude, we need to see how (4.3) yields (1.3). With this goal in mind, we first observe that for every $x \in Q_0$ and $Y \in \Gamma^{k_0}_{Q_0}(x)$ one has that $Y \in \overline{B^*_{Q_0}} \cap \Omega$ (see (2.13)) and also $\delta(Y) \gtrsim 2^{-k_0}\ell(Q_0)$. Hence, since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, one has

$$\begin{aligned}\sup_{x\in Q_0} \mathcal{S}^{k_0}_{Q_0} u(x) &= \sup_{x\in Q_0}\Big(\iint_{\Gamma^{k_0}_{Q_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY\Big)^{\frac12}\\ &\lesssim (2^{-k_0}\ell(Q_0))^{\frac{1-n}{2}}\Big(\iint_{B^*_{Q_0}\cap\{Y\in\Omega:\delta(Y)\gtrsim 2^{-k_0}\ell(Q_0)\}} |\nabla u(Y)|^2\, dY\Big)^{\frac12}\\ &< \infty. \end{aligned} \tag{4.13}$$

On the other hand, given $0 < q < \infty$, we can use (4.3) to obtain

$$\begin{aligned}(1+\beta)^{-q}\|\mathcal{S}^{k_0}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})} &= \int_0^\infty q\lambda^q \omega_L^{X_{Q_0}}(\{x\in Q_0 : \mathcal{S}^{k_0}_{Q_0}u(x) > (1+\beta)\lambda\})\frac{d\lambda}{\lambda}\\ &\le \int_0^\infty q\lambda^q \omega_L^{X_{Q_0}}(\{x\in Q_0 : \mathcal{S}^{k_0}_{Q_0}u(x) > (1+\beta)\lambda,\ \mathcal{N}_{Q_0}u(x)\le\gamma\lambda\})\frac{d\lambda}{\lambda}\\ &\quad + \int_0^\infty q\lambda^q \omega_L^{X_{Q_0}}(\{x\in Q_0 : \mathcal{N}_{Q_0}u(x) > \gamma\lambda\})\frac{d\lambda}{\lambda}\\ &\lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\int_0^\infty q\lambda^q \omega_L^{X_{Q_0}}(\{x\in Q_0 : \mathcal{S}^{k_0}_{Q_0}u(x) > \beta\lambda\})\frac{d\lambda}{\lambda} + \gamma^{-q}\|\mathcal{N}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})}\\ &\lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\beta^{-q}\|\mathcal{S}^{k_0}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})} + \gamma^{-q}\|\mathcal{N}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})}.\end{aligned}$$

We can then choose γ small enough so that we can hide the first term in the right-hand side of the last quantity (which is finite by (4.13)), and eventually conclude that

$$\|\mathcal{S}^{k_0}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})} \lesssim \|\mathcal{N}_{Q_0}u\|^q_{L^q(Q_0,\omega_L^{X_{Q_0}})}.$$

Since the implicit constant does not depend on k_0, and $\mathcal{S}^k_{Q_0}u(x) \nearrow \mathcal{S}_{Q_0}u(x)$ as $k \to \infty$ for every $x \in Q$, the monotone convergence theorem yields at once (1.3), and the proof Theorem 1.2 is complete.

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