The Spectral Analysis of the Hodrick-Prescott Filter

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Abstract

The Hodrick-Prescott (HP) filter is a commonly used tool in macroeconomics to obtain the HP filter trend of a macroeconomic variable. In macroeconomics, the difference between the original series and this trend is called the "cyclical component". In this paper, we derive the autocovariance function and the spectrum of the cyclical component of a series that consists of a constant, a linear time trend, a unit root process, and a weakly stationary process. We show that the autocovariance function of the cyclical component of such a series depends on (1) the autocovariance of the innovations of the unit root process; (2) the autocovariance of the weakly stationary process and; (3) a component of the weights of the HP filter that is important in the middle of a large sample. The result for the spectrum of the cyclical component matches with earlier results in the literature that were obtained by using an approximate approach. Lastly, we derive the cross-covariance function and the cross-spectrum of the cyclical components of two cointegrated series.

Keywords: Hodrick-Prescott filter, autocovariance function, spectrum, cross-covariance function, cross-spectrum, cointegration.

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JEL codes: C01, C22, C49.

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1 Introduction

The HP filter is a commonly used tool in macroeconomics to separate the long-run movements and the short-run shocks of macroeconomic variables. The HP filter has a long history that goes back to Whittaker (1922) and Leser (1961) before it was introduced to Economics by Hodrick and Prescott (1997).

The HP filter smoothes the original series $\{y_t\}_{t=1}^T$ by minimizing

$$\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}(\tau_{t+1} - 2\tau_t + \tau_{t-1})^2, \tag{1}$$

over $\tau = (\tau_1, \tau_2, \ldots, \tau_T)' \in \mathbb{R}^T$. λ is a (nonnegative) smoothing parameter that is typically chosen to equal 1600 for quarterly data. There is a unique solution to this minimization problem which is denoted as $\hat{\tau} = (\hat{\tau}_{T1}, \hat{\tau}_{T2}, \ldots, \hat{\tau}_{TT})'$. The cyclical component is defined as the residual $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt}$ for $t = 1, 2, \ldots, T$. Note that throughout this paper we use "trend" and "cyclical component" of a series to refer to $\hat{\tau}_{Tt}$ and $\hat{c}_{Tt}$, respectively. Therefore, the term "cyclical component" is used to describe the residual obtained after applying the HP filter; this terminology is potentially confusing, but used thoughout in the literature on the HP filter. We refer to the periodic fluctuations of macroeconomic variables as "cycles"; these should be distinguished from the cyclical components. The HP filter is a special case of a general method for smoothing a time series that was suggested by Whittaker (1922). Leser (1961, p. 92, Equation (2)) also considered the HP filter.

The comovements of the cycles of various macroeconomic variables are studied extensively in the real business cycle literature. In this literature, the comovements of the cycles of variables are studied by analyzing the autocorrelation and cross-correlation functions of the cyclical components. Cogley and Nason (1995) compared the autocorrelation and cross-correlation functions of trend-stationary and difference-stationary processes and their cyclical components. Cogley and Nason (1995) contains no derivation of the autocorrelation or cross-correlation functions of the cyclical components of those processes, and instead they used simulations. In this paper, we derive the autocovariance function of the cyclical component of a data generating process that equals the sum of an intercept, a deterministic linear time trend, a unit root process, and a weakly stationary process. Also, we derive the cross-covariance function of the cyclical components of two cointegrated series.

Spectral analysis is a useful method for studying variables in terms of the duration of fluctuations, where long-run (short-run) fluctuations correspond to the low (high) frequency components. Hannan (1963), Howrey (1968), Phillips (1991), Corbae et al. (2002), Christiano and Fitzgerald (2003), Granger and Hatanaka (2015), and Müller and Watson (2018) are examples of papers that studied the spectral analysis of time series. Cogley and Nason (1995) analyzed the spectrum and the cross-spectrum of the cyclical component of trend-stationary and difference-stationary series to provide evidence that the HP filter has different impacts on trend-stationary and difference-stationary series. King and Rebelo (1993) also considered the HP filter in the frequency domain. Baxter and King (1999) compared the spectra that result from using the HP filter with the spectra that result from using the band pass filter.

The common approach of King and Rebelo (1993), Cogley and Nason (1995), Baxter and King (1999), and Christiano and Fitzgerald (2003) is to take the first-order condition of the minimization problem in Equation (1) as holds for $t = 3, .., T-2$ as the definition of the HP filter; that is, this equation is assumed to hold for all $t \in \mathbb{Z}$. This first order condition is

$$\begin{aligned} y_t &= (\lambda \bar{B}^2 - 4\lambda \bar{B} + (1 + 6\lambda) - 4\lambda B + \lambda B^2)\hat{\tau}_{Tt} \\ &= (1 + \lambda |1 - B|^4)\hat{\tau}_{Tt} \end{aligned} \tag{2}$$

for $t = 3, 4, \dots, T-2$, where B and $\bar{B}$ denote the backward and the forward operators, respectively. Equation (2) defines a two-sided weighted average representation that these authors use to analyze the behavior of the filter in the frequency domain. In Theorem 2, we rigorously derive the spectrum of the cyclical component of a process that equals the sum of an intercept, a deterministic linear time trend, a unit root process, and a weakly stationary process.

In this paper, we use the exact weights of the HP filter to derive our results. The weighted average representation of the trend of a series that is obtained by the HP filter is

$$\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts} y_s, \tag{3}$$

where w_{Tts} is the weight at the time point s with respect to the time point t for given sample size T. The weights also depend on λ, but this dependence is suppressed in the notation for brevity. The explicit formula of the weights is provided in Theorem 1 of de Jong and Sakarya (2016). In this paper, we first characterize the autocovariance function of the cyclical component of a series that includes an intercept, a linear time trend, a unit root process, and a stationary process by using the weighted average representation above. Then we derive the spectrum of the cyclical component of the same process. The cross-covariance function and cross-spectrum of the cyclical components of two cointegrated series are derived by using the same approach as in the first two results. Appendix A in the supplementary material provides a list of results that we use from de Jong and Sakarya (2016). Throughout the paper, we assume that λ is fixed with respect to T.[1]

Recently, the HP filter has received renewed attention; for example, Phillips and Jin (2015, 2020), de Jong and Sakarya (2016), Cornea-Madeira (2017), Hamilton (2018), Sakarya and de Jong (2020), and Phillips and Shi (2020). Phillips and Jin (2015, 2020) developed a limit theory for the HP filter when it is applied to stochastic trends, trend-stationary processes, and time series processes with trend breaks. They concluded that the HP filter is not capable of removing the stochastic trend of time series processes under certain conditions. de Jong and Sakarya (2016) and Cornea-Madeira (2017) derived the explicit formula for the weights of the HP filter. By using this formula for the weights of the HP filter, de Jong and Sakarya (2016) showed that the cyclical component of a unit root process is weakly dependent in the middle of a large sample under certain assumptions. Those assumptions are different from those in Phillips and Jin (2015, 2020). Also, de Jong and Sakarya (2016) considered adjusting the smoothing parameter for the data frequency. Sakarya and de Jong (2020) derived a property of the HP filter: the HP filter calculates the cyclical component of a series as the HP filter trend of the fourth difference of the series, except for the first and last two observations, for which different formulas are needed. By using this property, they characterized the effects of structural breaks on the cyclical components. They derived the cyclical components of deterministic polynomial trends, deterministic exponential trends and integrated processes that are integrated up to order 4. They used the latter result to establish the weak dependence properties of the cyclical component of a series that is integrated up to order 4. Both de Jong and Sakarya (2016) and Sakarya and de Jong (2020) derived the weak dependence properties of the cyclical component of integrated processes while the current paper derives the autocovariance function and spectrum of the cyclical component of a process that is described in Assumption 1.

Hamilton (2018) listed several reasons why the HP filter should not be used to detrend macroeconomic variables. Those reasons are, rephrasing Phillips and Shi (2020) p. 30, (1) the HP filter introduces spurious fluctuations; (2) the filter is not appropriate for unit root processes; (3) the cyclical component is a function of past, current and future observations; and (4) an AR(4) regression provides a better alternative. Recently, Phillips and Shi (2020) proposed an improved version of the HP filter which eliminates some of the drawbacks of the HP filter that Hamilton (2018) listed. Phillips and Shi (2020) showed that their "boosted" HP filter is capable of asymptotically removing unit roots and higher order polynomial trends with or without structural breaks. The simulation studies in Phillips and Shi (2020) illustrate that the cyclical component that is calculated by their "boosted" HP filter is robust to the initial choice of λ.

The organization of the paper is as follows. In Subsection 2.1, we derive the autocovariance function and the spectrum of the cyclical component of a series which is the sum of a constant, a linear time trend, a unit root process, and a weakly stationary process. In Subsection 2.2, we derive the cross-covariance and the cross-spectrum of two cointegrated series, and we conclude the paper in Section 3.

2 Main Results

2.1 The autocovariance function and the spectrum of the cyclical component

In this section, we derive the autocovariance function and the spectrum of the cyclical component of a series which is the sum of a linear trend, a unit root process, and a weakly stationary series. The autocovariance function of a series $\{z_{Tt}\}_{t=1}^{T}$ is defined as

$$\gamma_z(k) = \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(z_{Tt}, z_{T,t-k}), \tag{4}$$

where $k \in \mathbb{Z}$ and $cov(\cdot,\cdot)$ is the population covariance. The spectrum of $\{z_{Tt}\}_{t=1}^{T}$ is defined as

$$I_z(\omega) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma_z(k) e^{-i\omega k}, \tag{5}$$

where $\omega \in [-\pi, \pi]$. The spectrum of $\{z_{Tt}\}_{t=1}^{T}$ exists if $\sum_{k=-\infty}^{\infty} |\gamma_z(k)| < \infty$. The above definitions are in the triangular array context because they will be used for the HP trend and cyclical component, which are triangular arrays.

In macroeconomics, it is common practice to characterize the real economy by the autocorrelation and the cross-correlation of the cyclical component of macroeconomic variables (e.g., Kydland and Prescott, 1982; Cooley and Prescott, 1995; Müller and Watson, 2018). It has been a great concern how and to what extent the HP filter alters the autocovariance and the cross-covariance of the variables since the characterization of the real economy might be dramatically influenced by the filter. For example, King and Rebelo (1993) compared the autocorrelation and the cross-correlation of simulated series and the autocorrelation and the cross-correlation of their cyclical components to provide evidence that the HP filter alters the second moments of the original series (see Table 2 of King and Rebelo, 1993). In our first result, we derive the autocovariance function of the cyclical component of a process that is described in Assumption 1.

Assumption 1.

$$y_t = \alpha_1 + \alpha_2 t + \alpha_3 z_t + u_t, \tag{6}$$

where $z_t = \sum_{j=1}^{t} \varepsilon_j$, $\{\varepsilon_j\}_{j=-\infty}^{\infty}$ *and* $\{u_t\}_{t=-\infty}^{\infty}$ *are weakly stationary series with the autocovariance functions* $\gamma_\varepsilon(k)$ *and* $\gamma_u(k)$ *for* $k \in \mathbb{Z}$, *respectively. Additionally,* $\sum_{k=-\infty}^{\infty} |\gamma_\varepsilon(k)| < \infty$, $\sum_{k=-\infty}^{\infty} |\gamma_u(k)| < \infty$, *and* $cov(u_t, \varepsilon_j) = 0$ *for any* $t, j \in \mathbb{Z}$.

Theorem 1. *Define* $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt} = \sum_{s=1}^{T} (I(t=s) - w_{Tts}) y_s$. *Assume Assumption 1 holds. Then for* $k \in \mathbb{Z}$

$$\gamma_c(k) = \gamma_u(k) - 2 \sum_{j=-\infty}^{\infty} \gamma_u(j) f_\lambda(j+k) + \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \gamma_u(h) f_\lambda(h+j) f_\lambda(k+j) \tag{7}$$

$$
\begin{aligned}
&+\alpha_3^2 \sum_{m=0}^{\infty} \sum_{h=-\infty}^{m} \gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=m+1}^{\infty} f_\lambda(l-h)) \\
&-\alpha_3^2 \sum_{m=0}^{\infty} \sum_{h=m+1}^{\infty} \gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=-\infty}^{m} f_\lambda(l-h)) \\
&-\alpha_3^2 \sum_{m=-\infty}^{-1} \sum_{h=-\infty}^{m} \gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m} f_\lambda(s))(\sum_{l=m+1}^{\infty} f_\lambda(l-h)) \\
&+\alpha_3^2 \sum_{m=-\infty}^{-1} \sum_{h=m+1}^{\infty} \gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m} f_\lambda(s))(\sum_{l=-\infty}^{m} f_\lambda(l-h)),
\end{aligned}
$$

where for $m \in \mathbb{Z}$

$$f_\lambda(m) = \int_0^1 \frac{\cos(\pi r m)}{1 + 16\lambda \sin(\pi r/2)^4} dr.$$

The proof of this result and all other proofs for this paper can be found in Appendix B in the supplementary material.

The autocovariance function of $\hat{c}_{Tt}$ depends on the autocovariance functions of u_t and ε_t. The intercept and the deterministic linear time trend are fully absorbed into $\hat{\tau}_{Tt}$, as discussed in Section 6.1 of Sakarya and de Jong (2020). If we add a higher order deterministic polynomial trend to y_t ($y_t^* = y_t + d_t$ and $d_t = t^q$ for $t = 1, 2, \ldots, T$ and $q = 2, 3, \ldots$), then the cyclical component of y_t^* would be

$$\hat{c}_{Tt}^* = y_t + d_t - \sum_{s=1}^{T} w_{Tts}(y_s + d_s) = y_t - \sum_{s=1}^{T} w_{Tts} y_s + d_t - \sum_{s=1}^{T} w_{Tts} d_s$$

where $d_t - \sum_{s=1}^{T} w_{Tts} d_s$ is the cyclical component of d_t. Note that the cyclical component of d_t is not zero, as shown in Theorem 6 of Sakarya and de Jong (2020). Since the cyclical component of d_t is nonrandom, it will not appear in the autocovariance function of $\hat{c}_{Tt}^*$. Therefore, the autocovariance function of $\hat{c}_{Tt}^*$ will be the same as the expression in Equation (7).

The function $f_\lambda(\cdot)$ in Equation (7) arises as follows. The cyclical component is a weighted average of y_t, and the weights of the HP filter have eight components, as defined in Theorem 1 of de Jong and Sakarya (2016): $w_{Tts} = f_{T\lambda}(t-s) + \sum_{p=2}^{8} w_{Tts}^p$. A detailed description of the weights and their properties are provided in Appendix A in the supplemental material. The last seven components of the weights are important only in the beginning and at the end of a sample. The function $f_\lambda(m)$ that appears in the result of Theorem 1 is the pointwise limit of $f_{T\lambda}(m)$ as $T \to \infty$.

In Equation (7), the first three terms of the autocovariance function are the autocovariance function of $\hat{c}_{Tt}$ obtained from a series that is weakly stationary. Below, this special case is presented as a corollary.

Corollary 1. *Assume Assumption 1 holds and let $y_t = u_t$. Then for $k \in \mathbb{Z}$*

$$\gamma_c(k) = \gamma_y(k) - 2 \sum_{j=-\infty}^{\infty} \gamma_y(j) f_\lambda(j+k) + \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \gamma_y(h) f_\lambda(h+j) f_\lambda(k+j)$$

and

$$\gamma_{\tau}(k) = \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_y(h)f_\lambda(h+j)f_\lambda(k+j).$$

The result of Corollary 1 shows that the autocovariance function of $\hat{c}_{Tt}$ differs from the autocovariance function of y_t by some weighted average of the autocovariance function of y_t. Note that if $\lambda \to \infty$, $\gamma_c(k) \to \gamma_y(k)$ pointwise in k because $f_\lambda(m) \to 0$ for all m since $|\cos(\pi r m)| \leq 1$, $(1+16\lambda\sin(\pi r/2)^4)^{-1} \leq 1$ and $\lim_{\lambda\to\infty}(1+16\lambda\sin(\pi r/2)^4)^{-1} = 0$. King and Rebelo (1993) and Cogley and Nason (1995) raised the issue that the second moment of the original series and the second moment of the associated cyclical component are different, and provided simulation results that illustrate this. The fact that these second moments are different follows directly from Corollary 1 because $\gamma_c(0) \neq \gamma_y(0)$.

When we assume u_t is a white noise process, the results for $\gamma_c(k)$ and $\gamma_\tau(k)$ can be further simplified to

$$\gamma_c(k) = \sigma^2(I(k=0) - 2f_\lambda(k) + \sum_{j=-\infty}^{\infty} f_\lambda(j)f_\lambda(j+k)) \tag{8}$$

and

$$\gamma_\tau(k) = \sigma^2 \sum_{j=-\infty}^{\infty} f_\lambda(j)f_\lambda(k+j). \tag{9}$$

This implies that the cyclical component of a white noise process is correlated, i.e., $\gamma_c(k) \neq 0$ for some $k \geq 0$, whereas $\gamma_u(k) = 0$ for all $k \neq 0$. Figure 1 provides an illustration of autocovariance functions $\gamma_u(k)$, $\gamma_\tau(k)$, and $\gamma_c(k)$ for $k = 0, 1, \ldots, 20$ when u_t is a white noise process with $\sigma^2 = 1$ and $\lambda = 6.25$.

Next, we consider another special case of Theorem 1 in which the original series is a unit root process.

Corollary 2. *Assume Assumption 1 holds and let $y_t = z_t$. Then for $k \in \mathbb{Z}$*

$$\begin{aligned}
\gamma_c(k) = & \sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h)) \\
& - \sum_{m=0}^{\infty}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=-\infty}^{m}f_\lambda(l-h)) \\
& - \sum_{m=-\infty}^{-1}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h)) \\
& + \sum_{m=-\infty}^{-1}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_\lambda(s))(\sum_{l=-\infty}^{m}f_\lambda(l-h)).
\end{aligned}$$

Corollary 2 shows that when we calculate the correlogram of the cyclical component of a unit root process, we will find a correlogram that is reminiscent of a correlated weakly stationary process.

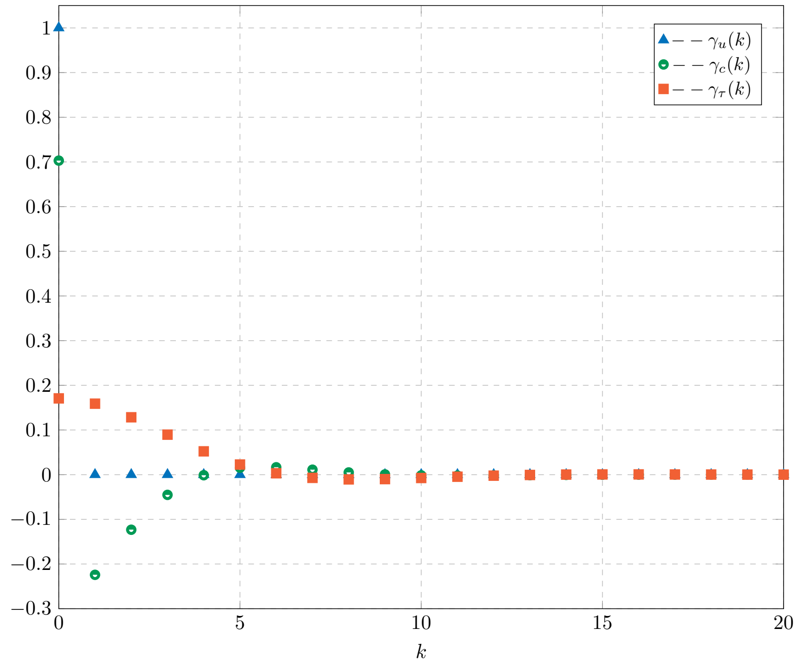

Figure 1: The plots of $\gamma_u(k) = \sigma^2 I(k=0)$ and the functions in Equations (8) and (9) for $\sigma^2 = 1$ and $\lambda = 6.25$.

It follows from Corollary 2 that even if the innovations of the unit root process are white noise, the correlogram of the cyclical component of the unit root process will resemble the correlogram of a correlated weakly stationary process and will not resemble the correlogram of a white noise process.

Next, we use the result of Theorem 1 to derive the spectrum of the cyclical component of y_t.

Theorem 2. *Assume Assumption 1 holds. Then* $\sum_{k=-\infty}^{\infty} |\gamma_c(k)| < \infty$ *and the spectrum of* $\hat{c}_{Tt}$ *is*

$$I_c(\omega) = (1 - h_\lambda(\omega))^2 I_u(\omega) + \alpha_3^2 (1 - h_\lambda(\omega))^2 (2\sin(\omega/2))^{-2} I_\varepsilon(\omega),$$

where $h_\lambda(\omega) = (1 + 16\lambda \sin(\omega/2)^4)^{-1}$ *for* $\omega \in [-\pi, \pi]$.

The above result shows that the HP filter suppresses the low-frequency component of the original series, because $1 - h_\lambda(\omega) \approx 0$ when ω is near zero. Therefore, $I_c(\omega) \approx 0$ when ω is near zero.

Figure 2 graphs $(1 - h_\lambda(\omega))^2$ for various values of the smoothing parameter that were suggested in the literature for different data frequencies.[2] Since $h_\lambda(\omega)$ is decreasing in λ, the HP filter allows more of the low frequencies into the cyclical component as λ takes higher values.

Below, we consider two special cases of the spectrum that is given in Theorem 2.

Corollary 3. *Assume Assumption 1 holds and let* $y_t = u_t$*. Then* $\sum_{k=-\infty}^{\infty} |\gamma_c(k)| < \infty$, $\sum_{k=-\infty}^{\infty} |\gamma_\tau(k)| < \infty$,

$$I_c(\omega) = (1 - h_\lambda(\omega))^2 I_y(\omega)$$

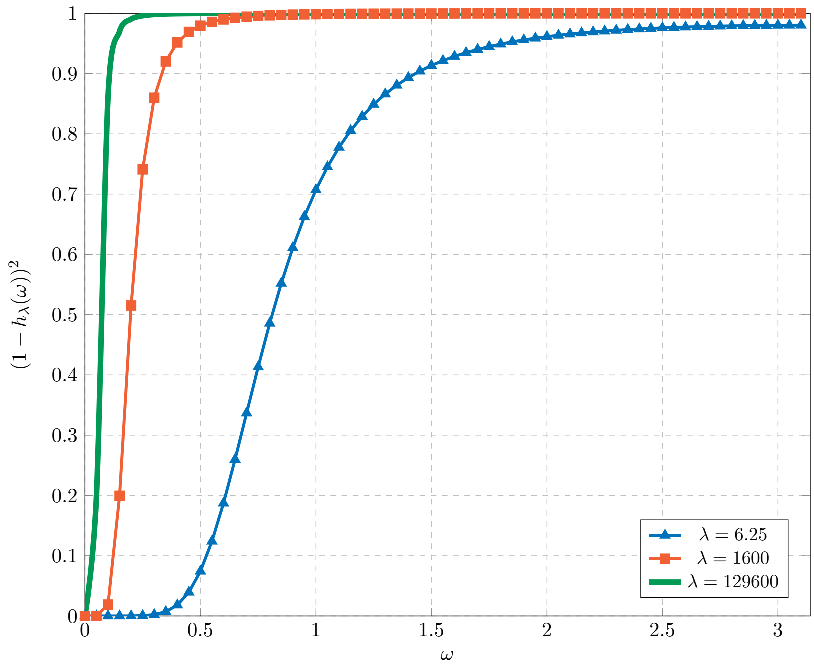

Figure 2: The function $(1 - h_\lambda(\omega))^2$ for different values of λ.

and

$$I_\tau(\omega) = h_\lambda(\omega)^2 I_y(\omega)$$

for $\omega \in [-\pi, \pi]$.

Unlike in Theorem 2, in Corollary 3 we can derive the spectrum of the trend because $y_t = u_t$ is a weakly stationary process. The results of Corollary 3 imply that as $\lambda \to \infty$, $I_c(\omega) \to I_y(\omega)$ and $I_\tau(\omega) \to 0$ pointwise in ω. This suggests that the spectrum of the HP filter trend approaches the spectrum of a constant when λ is sufficiently large because Equation (5) implies that if $I_\tau(\omega) \approx 0$, then $\gamma_\tau(k) \approx 0$ for all $k \in \mathbb{Z}$. This observation corresponds to the findings of de Jong and Sakarya (2016, p. 312) in the time domain, i.e., $\lim_{\lambda\to\infty} \hat{\tau}_{Tt} = T^{-1} \sum_{t=1}^{T} y_t$.

As a special case of Corollary 3, for the case where $y_t = u_t$ is white noise, we find

$$I_c(\omega) = \sigma^2 (1 - h_\lambda(\omega))^2 \tag{10}$$

and

$$I_\tau(\omega) = \sigma^2 h_\lambda(\omega)^2.$$

From the above equations, it follows that the spectrum of the cyclical component of a white noise process is not constant. Therefore, when confronted with the spectrum of Equation (10), a researcher might conclude that the cyclical component has cycles. The fact that filtering can suggest cycles is well-known since the work of Kuznets (1929) and Slutzky (1937). Cogley and Nason (1995) used simulations to investigate how the HP filter suggests cycles in the setting of trend-stationary and difference-stationary processes. Figure 2 graphs the spectrum of the cyclical component of a white noise process with $\sigma^2 = 1$ for different values of λ.

Next, we discuss the spectrum of the cyclical component of a unit root process.

Corollary 4. *Assume Assumption 1 holds and let $y_t = z_t$. Then $\sum_{k=-\infty}^{\infty} |\gamma_c(k)| < \infty$ and*

$$I_c(\omega) = (1 - h_\lambda(\omega))^2 (2\sin(\omega/2))^{-2} I_\varepsilon(\omega)$$

for $\omega \in [-\pi, \pi]$.

It has been noted in the literature by Cogley and Nason (1995) and Kaiser and Maravall (1999) that the HP filter is generally applied to macroeconomic variables that are nonstationary. Therefore, it is of great interest to derive the spectrum of a cyclical component obtained from a unit root process. Cogley and Nason (1995) and Kaiser and Maravall (1999) took the expression in Equation (2) as the definition of the HP filter and along these lines conjectured the above result. The result of Corollary 4 implies that as $\lambda \to \infty$, $I_c(\omega) \to (2\sin(\omega/2))^{-2} I_\varepsilon(\omega)$ pointwise in ω everywhere except $\omega = 0$.

2.2 The cross-covariance function and cross-spectrum of the cyclical components

In this section, we derive the cross-covariance function and the cross-spectrum of the cyclical components of two cointegrated series. As stated in Engle and Granger (1987), cointegrating relationships exist among macroeconomic variables such as consumption and income, short-term and long-term interest rates, and nominal GNP and M2. Cogley and Nason (1995) considered the cross-spectrum of the cyclical components of two cointegrated series.

Assumption 2. *Let $\{y_{1t}\}_{t=1}^{T}$ and $\{y_{2t}\}_{t=1}^{T}$ be cointegrated series such that*

$$y_{1t} = \phi y_{2t} + u_t,$$

where $y_{2t} = \sum_{j=1}^{t} \varepsilon_j$, $\{\varepsilon_j\}_{j=-\infty}^{\infty}$, $\{u_t\}_{t=-\infty}^{\infty}$ are weakly stationary processes with $cov(u_t, \varepsilon_j) = 0$ for any $t, j \in \mathbb{Z}$, and $\sum_{k=-\infty}^{\infty} |\gamma_\varepsilon(k)| < \infty$.

Theorem 3. *Define $\hat{c}_{Tt}^{i} = \sum_{s=1}^{T} (I(t = s) - w_{Tts}(\lambda_i)) y_{is}$ where $w_{Tts}(\lambda_i)$ denotes the weights of the HP filter that is calculated by smoothing parameter λ_i for $i = 1, 2$, and the cross-covariance function of $\hat{c}_{Tt}^{1}$ and $\hat{c}_{T,t-k}^{2}$ is*

$$\gamma_{c,\lambda_1,\lambda_2}^{1,2}(k) = \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(\hat{c}_{Tt}^{1}, \hat{c}_{T,t-k}^{2}),$$

for $k \in \mathbb{Z}$. Assume Assumption 2 holds. Then for $k \in \mathbb{Z}$

$$\begin{aligned}
\gamma^{1,2}_{c,\lambda_1,\lambda_2}(k) =& \phi \sum_{m=0}^{\infty} \sum_{h=-\infty}^{m} \gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_{\lambda_1}(s))(\sum_{l=m+1}^{\infty} f_{\lambda_2}(l-h)) \\
&- \phi \sum_{m=0}^{\infty} \sum_{h=m+1}^{\infty} \gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_{\lambda_1}(s))(\sum_{l=-\infty}^{m} f_{\lambda_2}(l-h)) \\
&- \phi \sum_{m=-\infty}^{-1} \sum_{h=-\infty}^{m} \gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m} f_{\lambda_1}(s))(\sum_{l=m+1}^{\infty} f_{\lambda_2}(l-h)) \\
&+ \phi \sum_{m=-\infty}^{-1} \sum_{h=m+1}^{\infty} \gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m} f_{\lambda_1}(s))(\sum_{l=-\infty}^{m} f_{\lambda_2}(l-h)).
\end{aligned}$$

Note that the HP filter is applied to y_{1t} and y_{2t} with different smoothing parameters (i.e. λ_1 and λ_2) to keep the result as general as possible. Next, we present the result for the cross-spectrum of two cointegrated series by using the cross-covariance result above.

Theorem 4. *The cross-spectrum of $\hat{c}^1_{Tt}$ and $\hat{c}^2_{T,t-k}$ is defined as*

$$I^{1,2}_{c,\lambda_1,\lambda_2}(\omega) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma^{1,2}_{c,\lambda_1,\lambda_2}(k) e^{-i\omega k}$$

for $\omega \in [-\pi, \pi]$. Assume Assumption 2 holds. Then

$$I^{1,2}_{c,\lambda_1,\lambda_2}(\omega) = \phi(1 - h_{\lambda_1}(\omega))(1 - h_{\lambda_2}(\omega))(2\sin(\omega/2))^{-2} I_\varepsilon(\omega). \tag{11}$$

In the setting of correlation properties of series, Hodrick and Prescott (1997) stated that "As the comovement results were not particularly sensitive to the value of the smoothing parameter λ selected... We do think it is important that all series be filtered using the same parameter λ." However, for the cross-spectrum, the suggestion that $I^{1,2}_{c,\lambda_1,\lambda_2}(\omega)$ resembles $I^{1,2}_{c,\lambda_1,\lambda_1}(\omega)$ for different values of λ_1 and λ_2 is not correct, as can be deduced from Theorem 4.

It is easy to observe from Equation (11) that

$$I^{1,2}_{c,\lambda_1,\lambda_2}(\omega) = ((1 - h_{\lambda_1}(\omega))/(1 - h_{\lambda_2}(\omega))) I^{1,2}_{c,\lambda_1,\lambda_1}(\omega).$$

Using elementary calculus, it is not hard to show that for $\lambda_1 < \lambda_2$ and $\omega \in [-\pi, \pi]$

$$\frac{1 - h_{\lambda_1}(\omega)}{1 - h_{\lambda_2}(\omega)} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1(1 + 16\lambda_2)}{\lambda_2(1 + 16\lambda_1)}\right],$$

and these bounds are attained on for $\omega = 0$ and $\omega = \pm\pi$, respectively. Therefore, taking λ_1/λ_2 as a scale factor, the two spectra will look similar as long as $(1 + 16\lambda_2)/(1 + 16\lambda_1)$ is close to 1. This quantity will equal 1.594 for $(\lambda_1, \lambda_2) = (6.25, 10)$, which is the best case we can ever encounter in practice, and equals 159.01 for $(\lambda_1, \lambda_2) = (10, 1600)$. Therefore, we should not extrapolate Hodrick and Prescott (1997)'s comment to suggest that the cross-spectrum will not be sensitive to the choice of smoothing parameters.

3 Conclusion

We derive the autocovariance function and the spectrum of the cyclical component of a process that is the sum of an intercept, a linear time trend, a unit root process and a weakly stationary process. We use the formula for the weights of the HP filter from de Jong and Sakarya (2016) to obtain our results. This is the first mathematically rigorous discussion of the autocovariance function and the spectrum of the cyclical component in the literature. We also derive the cross-covariance function and the cross-spectrum of the cyclical component of two cointegrated series. We show that the cross-spectrum of two cointegrated series depends on the cointegration coefficient, the smoothing parameter values and the spectrum of the innovations of the unit root process.

Notes

[1]The results of Phillips and Jin (2020) are based on the assumption that λ grows with T.

[2]$\lambda = 6.25, 1600, 129600$ are used for annual, quarterly, and monthly data, respectively. They are suggested by Hodrick and Prescott (1997), Ravn and Uhlig (2002), and de Jong and Sakarya (2016).

Data Availability Statement

The data that support the findings of this study, i.e., Figures 1 and 2, are openly available on the corresponding author's website at https://neslihansakarya.weebly.com/uploads/5/9/5/5/59554687/on_the_website.zip.

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