A New Class of Quadriphase Even-Shift Orthogonal Sequences

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Abstract In this paper, we systematically construct a new class of Quadriphase Even-Shift Orthogonal Sequences (or E-Sequences) from Golay Complementary codes using Sequence Interleaving. Besides having even-shift-orthogonal properties, the proposed E-Sequences have zero real aperiodic correlation functions for all shifts, so it is also a special class of Imaginary Sequences (IS). Due to its ideal real aperiodic correlation functions, the proposed E-Sequences perform much better than the traditional E-sequences which have zero aperiodic correlation functions only at even shifts. Through the properties of the proposed E-Sequences, we also establish the relationships between Golay-Complementary codes, Imaginary sequences, and E-sequences.

key words: *Even-Shift Orthogonal Sequences, Imaginary Sequences, Golay-Complementary codes, Sequence Interleaving.*

1. Introduction

Even-Shift Orthogonal Sequences, or E-Sequences, were found by Y. Taki *et al* in 1969. The aperiodic correlation functions of E-sequences (except for the zero shift for aperiodic auto-correlation functions), take constant zero values for all the even-shifts [1]. E-Sequences can be used to generate normal random signals [2], and they can also be employed as the orthogonal sequences for multiple-access in Pulse Spacing Modulation (PSM) communication system [3]. In 2004, N. Suehiro *et al* introduced the quadriphase E-sequences [4], but the sequence length was only limited to 8, and no systematic sequence construction method was given. Later, Matsufuji *et al* proposed the generating functions of E-sequences [5], and pointed out the intrinsic relationship between Golay Complementary codes and the E-sequences. However, his work was only limited to binary E-sequences. Furthermore, all the previous E-sequences don't have ideal real aperiodic correlation functions.

Imaginary Sequences (IS), on the other hand, have ideal real aperiodic correlation functions [6], so they can be used for timing synchronization and time domain channel estimation in communication system. Motivated by Matsufuji's methods, Lüke *et al* developed some efficient ways to obtain quadriphase and 8-phase sequences and sequences pairs with imaginary correlation sidelobes [7]. The IS found by Matsufuji *et al* were obtained from Golay complementary codes [8], so the sequences length is limited to $N = 2^\alpha 10^\beta 26^\gamma$ ($\alpha, \beta, \gamma = 0, 1, 2 \ldots$). However, with the aid of a kind of modified Kronecker product and the periodically repeated sequences product, Lüke *et al*. found that the sequences length of the IS can take far more options than the Golay length.

In this paper, we propose a new class of quadriphase E-Sequences, whose real aperiodic correlation functions are ideal, so it is also a new special class of Imaginary Sequences. We also present the generating functions of the new E-Sequences , as well as their other useful sequence properties.

This paper is organized as follows. In the next section, several necessary notations and definitions are presented for the description purpose. In Section 3, construction method for the new E-Sequences is introduced. In Section 4, we present the generating functions of the proposed E-Sequences. In Section 5, two sequence properties are discussed. Finally, some concluding remarks are addressed in Section 6.

2. PRELIMINARY

Notations: x^* denotes the conjugate of variable x, $||\mathbf{x}||_F$ is the Frobenius norm or the Hilbert-Schmidt norm of sequence $\mathbf{x}$. $[\mathbf{X};\mathbf{Y}]$ and $[\mathbf{X},\mathbf{Y}]$ are referred as the vertical concatenation and horizontal concatenation of matrix $\mathbf{X}$ and $\mathbf{Y}$ individually. $\delta(\tau)$ stands for the discrete Kronecker delta function. $Re\{x\}$ is the function of taking the real part of x. $\lfloor x \rfloor$ denotes the maximal integer not greater than x. $j = \sqrt[2]{-1}$. $\omega = e^{j\pi/2}$. $Z_H = \{1, e^{2\pi/H}, \cdots, e^{2\pi(H-1)/H}\}$ defines a unitary ring with H phase values, where H is an integer number which is greater than 1.

Definition 1: Let $\mathbf{a}$ (or $\{a_x\}$) and $\mathbf{b}$ (or $\{b_x\}$) be two arbitrary 1-D sequences (both are of length N), their apeiodic correlation function of timing shifts τ is defined as

$$\rho(\mathbf{a},\mathbf{b};\tau) = \begin{cases} \sum\limits_{x=0}^{N-1-\tau} a_x^* b_{x+\tau} & 0 \le \tau \le (N-1) \\ \sum\limits_{x=0}^{N-1+\tau} a_{x-\tau}^* b_x & 1-N \le \tau \le -1 \\ 0 & |\tau| \ge N \end{cases} \quad (1)$$

. In $\rho(\mathbf{a},\mathbf{b};\tau)$, when $\mathbf{a} = \mathbf{b}$, we regard $\rho(\mathbf{a},\mathbf{a};\tau)$ as the apeiodic auto-correlation function (AACF), otherwise, we regard it as the apeiodic cross-correlation function (ACCF).

Definition 2: Real aperiodic correlation function of $\mathbf{a}$ and $\mathbf{b}$ is defined as $Re\{\rho(\mathbf{a},\mathbf{b};\tau)\}$. $(\mathbf{a},\mathbf{b})$ is an IS pair if they have ideal real aperiodic correlation functions (or zero real aperiodic correlation functions), meaning that

$$Re\{\rho(\mathbf{a},\mathbf{b};\tau)\} = \begin{cases} ||\mathbf{a}||_F^2 & \text{if } \mathbf{a}=\mathbf{b} \text{ and } \tau=0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Definition 3: $(\mathbf{a},\mathbf{b})$ is an E-sequences pair if fulfills the following condition:

$$\rho(\mathbf{a},\mathbf{b};\tau) = \begin{cases} ||\mathbf{a}||_F^2 & \text{if } \mathbf{a}=\mathbf{b} \text{ and } \tau=0 \\ 0 & \text{if } \tau \text{ is an even number} \end{cases} \quad (3)$$

3. Sequence Construction

For description simplicity, we only consider the sequences which take the length of $2^n (n \geq 1)$. Those who are interested in other of sequence lengths may refer to [7].

Let $N = 2^n (n \geq 1)$ be the E-Sequences length. When $N = 2$, the corresponding E-Sequences pair can be easily shown to be $\mathbf{a} = [1, j]$, and $\mathbf{b} = [1, -j]$; When $N \geq 4$, we can find a binary Golay complementary code [8] as

$$\mathbf{g} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N/2-1} \\ d_0 & d_1 & \cdots & d_{N/2-1} \end{bmatrix} \qquad (4)$$

where $c_x, d_x \in \{1, -1\}$, $0 \leq x \leq (N/2-1)$, and

$$\sum_{x=0}^{N/2-1-\tau} c_x c_{x+\tau} + \sum_{x=0}^{N/2-1-\tau} d_x d_{x+\tau} = N\delta(\tau) \qquad (5)$$

Theorem 1: An E-Sequences pair $(\{a_x\}, \{b_x\})$ can be obtained by from Golay Complementary codes $(\{c_x\}, \{d_x\})$ by

$$\begin{aligned} a_x &= \left[\tfrac{1+(-1)^x}{2}\right] c_{\lfloor x/2 \rfloor} + j\left[\tfrac{1+(-1)^{x+1}}{2}\right] d_{\lfloor (x-1)/2 \rfloor} \\ b_x &= \pm j a_{N-1-x} \end{aligned}$$

Meanwhile, they have ideal real aperiodic correlation functions, so $(\{a_x\}, \{b_x\})$ is also an IS pair.

Proof. In fact, $(\{a_x\}, \{b_x\})$ can be expressed as

$$\begin{aligned} \mathbf{a} &= [c_0, jd_0, c_1, jd_1, \cdots, c_{N/2-1}, jd_{N/2-1}] \\ \mathbf{b} &= \pm[d_{N/2-1}, jc_{N/2-1}, \cdots, d_1, jc_0, d_0, jc_0] \end{aligned}$$

, hence essentially, this type of E-Sequences pair is built from the interleaving of Golay Complementary codes. Due to the symmetrical characteristics of the aperiodic correlation functions, we only present the proof procedure for $0 \leq \tau \leq N-1$.

A. aperiodic ACF

1. If τ is an even number, we have $\rho(\mathbf{a}, \mathbf{a}; \tau) = \rho(\mathbf{b}, \mathbf{b}; \tau)$, and

$$\begin{aligned} & \rho(\mathbf{a}, \mathbf{a}; \tau) \\ = & \sum_{x=0}^{N/2-1-\tau/2} \left[c_x c_{x+\tau/2} + d_x d_{x+\tau/2}\right] \\ = & N\delta(\tau) \end{aligned} \qquad (6)$$

2. If τ is an odd number, we have $\rho(\mathbf{a}, \mathbf{a}; \tau) = -\rho(\mathbf{b}, \mathbf{b}; \tau)$, and

$$\begin{aligned} & \rho(\mathbf{a}, \mathbf{a}; \tau) \\ = & j \times \left[\sum_{x=0}^{u} c_x d_{x+(\tau-1)/2} \right. \\ & \left. - \sum_{x=0}^{u-1} d_x c_{x+(\tau-1)/2+1}\right] \end{aligned} \qquad (7)$$

where $u = N/2 - 1 - (\tau - 1)/2$.

B. aperiodic CCF

1. If τ is an even number

$$\begin{aligned} & \rho(\mathbf{a}, \mathbf{b}; \tau) \\ = & \left[\sum_{x=0}^{u} c_x d_{u-x} - \sum_{x=0}^{u} c_x d_{u-x}\right] \\ = & 0 \end{aligned} \qquad (8)$$

where $u = N/2 - 1 - \tau/2$.

2. If τ is an odd number

$$\begin{aligned} & \rho(\mathbf{a}, \mathbf{b}; \tau) \\ = & \mp j \times \left[\sum_{x=0}^{u} d_x d_{u-x} + \sum_{x=0}^{u+1} c_x c_{u+1-x}\right] \end{aligned} \qquad (9)$$

where $u = N/2 - 2 - (\tau - 1)/2$.

When τ is an odd number, from equation (7) and (9), we know that there are odd number of multiplications within the summation operator, therefore, $|\rho(\mathbf{a}, \mathbf{a}; \tau)| \geq 1$ and $|\rho(\mathbf{a}, \mathbf{b}; \tau)| \geq 1$. So $(\{a_x\}, \{b_x\})$ is an E-sequences pair as well as an IS pair.

Q.E.D

4. Generating Functions

In [5] and [6], Matsufuji *et al* presented the generating functions for the conventional E-Sequences pair and IS pair, respectively. For easy reference, most of the notations in this paper are adopted from [6]. Let $(x_0, x_1, \cdots, x_{n-1})$ be the binary representation of x, meaning that $x = x_0 2^0 + x_1 2^1 + \cdots + x_{n-1} 2^{n-1}$, where $x_t \in \{0, 1\}$, $0 \leq t \leq n-1$, and $N = 2^n$. Let the generating functions of $\{a_x\}$ and $\{b_x\}$ as defined in **Theorem 1** are $f_n(x)$ and $g_n(x)$ respectively, or $a_x = \omega^{f_n(x)}$ and $b_x = \omega^{g_n(x)}$.

Theorem 2:

Case 1: $f_n(x) = 2(x_{i_1} x_{i_2} \oplus \cdots \oplus x_{i_{n-2}} x_{i_{n-1}} \oplus x_{i_{n-1}} x_{i_0} \oplus \beta_0 x_0 \oplus \cdots \oplus \beta_{n-2} x_{n-2} \oplus \beta_{n-1} x_{n-1} \oplus m) \oplus x_0$, and $g_n(x) = f_n(x) \oplus 2x_{i_1} \oplus 2\beta$,

Case 2: $f_n(x) = 2(x_{i_0} x_{i_1} \oplus \cdots \oplus x_{i_{n-2}} x_{i_{n-1}} \oplus \beta_0 x_0 \oplus \cdots \oplus \beta_{n-2} x_{n-2} \oplus \beta_{n-1} x_{n-1} \oplus m) \oplus x_0$, and $g_n(x) = f_n(x) \oplus 2x_{i_{n-1}} \oplus 2\beta$,

where $\oplus$ denotes for addition over Z_4, $i_0 = 0$ and $(i_1, \cdots, i_{n-1})$ is any permutation of $(1, \cdots, n-1)$, and $\beta_0, \beta_1, \cdots, \beta_{n-1}, \beta, m \in \{0, 1\}$.

Proof. Since

$$\begin{aligned} & a_x \\ = & \left[\tfrac{1+(-1)^x}{2}\right] c_{\lfloor x/2 \rfloor} + j\left[\tfrac{1+(-1)^{x+1}}{2}\right] d_{\lfloor (x-1)/2 \rfloor} \\ = & \left[\tfrac{1+(-1)^{x_0}}{2}\right] c_{x'} + j\left[\tfrac{1+(-1)^{x_0+1}}{2}\right] d_{x'} \end{aligned}$$

where $x' = (x_1 x_2 \cdots x_{n-1})$. Similarly, let $f_{n-1}(x')$ and $g_{n-1}(x')$ be the generating functions of Golay Complementary pair $(\{c_{x'}\}, \{d_{x'}\})$, or $c_{x'} = \omega^{f_{n-1}(x')}$ and $d_{x'} = \omega^{g_{n-1}(x')}$. Denote $s_{n-1}(x') = 2(x_{i_1} x_{i_2} \oplus \cdots \oplus x_{i_{n-2}} x_{i_{n-1}} \oplus \beta_1 x_1 \oplus \cdots \oplus \beta_{n-2} x_{n-2} \oplus \beta_{n-1} x_{n-1})$. Following *Corollary 5* in [9], $f_{n-1}(x')$ and $g_{n-1}(x')$ can be classified as follows,

$$
\begin{array}{ll}
\mathrm{I}: & \begin{array}{lcl} f_{n-1}(x') & = & s_{n-1}(x') \oplus 2m \\ g_{n-1}(x') & = & f_{n-1}(x') \oplus 2x_{i_{n-1}} \oplus 2m' \end{array} \\
\\
\mathrm{II}: & \begin{array}{lcl} f_{n-1}(x') & = & s_{n-1}(x') \oplus 2m \\ g_{n-1}(x') & = & f_{n-1}(x') \oplus 2x_{i_1} \oplus 2m' \end{array} \\
\\
\mathrm{III}: & \begin{array}{lcl} f_{n-1}(x') & = & s_{n-1}(x') \oplus 2x_{i_{n-1}} \oplus 2x_{i_1} \\ & & \oplus 2m \\ g_{n-1}(x') & = & f_{n-1}(x') \oplus 2x_{i_{n-1}} \oplus 2m' \end{array} \\
\\
\mathrm{IV}: & \begin{array}{lcl} f_{n-1}(x') & = & s_{n-1}(x') \oplus 2x_{i_{n-1}} \oplus 2x_{i_1} \\ & & \oplus 2m \\ g_{n-1}(x') & = & f_{n-1}(x') \oplus 2x_{i_1} \oplus 2m' \end{array}
\end{array}
$$

where $m, m' \in \{0, 1\}$. Essentially, generating functions in III and IV can be merged into I and II respectively because $2x_{i_k} \oplus 2c_{i_k}x_{i_k} = 2(1 \oplus c_{i_k})x_{i_k} = 2c'_{i_k}x_{i_k}$, where $k = 1$ or $n - 1$. So the proof only needs to be shown for the first two cases.

Case 1:

$$
\begin{array}{cl}
& a_x \\
= & \left[\frac{1+(-1)^{x_0}}{2}\right] c_{x'} + j \left[\frac{1+(-1)^{x_0+1}}{2}\right] d_{x'} \\
= & c_{x'}\omega^{x_0} \left[\frac{\omega^{-x_0}+\omega^{x_0}}{2} + \frac{\omega[\omega^{-x_0}-\omega^{x_0}]}{2}\omega^{2x_{i_{n-1}}+2m'}\right] \\
= & c_{x'}\omega^{x_0} \left[\cos(\pi x_0/2) + \sin(\pi x_0/2)\omega^{2x_{i_{n-1}}+2m'}\right] \\
= & c_{x'}\omega^{x_0} \left(\omega^{2x_{i_{n-1}}+2m'}\right)^{x_0} \\
= & c_{x'}\omega^{2x_{i_0}x_{i_{n-1}}+2m'x_0+x_0}
\end{array}
$$

Let $m' = \beta_0$. We have

$$
\begin{array}{cl}
& f_n(x) \\
= & 2(x_{i_1}x_{i_2} \oplus \cdots \oplus x_{i_{n-2}}x_{i_{n-1}} \oplus x_{i_{n-1}}x_{i_0} \\
& \oplus \beta_0 x_0 \oplus \cdots \oplus \beta_{n-2}x_{n-2} \oplus \beta_{n-1}x_{n-1}) \oplus x_0
\end{array}
$$

Since $N-1 = (x_0, x_1, \cdots, x_{n-1}) + (1-x_0, 1-x_1, \cdots, 1-x_{n-1})$, so we have

$$
\begin{array}{cl}
& g_n(x) \\
= & g_n(1-x_0, 1-x_1, \cdots, 1-x_{n-1}) \\
= & f_n(1-x_0, 1-x_1, \cdots, 1-x_{n-1}) \oplus 2m'' \oplus 1 \\
= & f_n(x) \oplus 2x_{i_1} \oplus 2\beta
\end{array}
$$

where $\beta = [n \oplus \beta_0 \oplus \cdots \oplus \beta_{n-1} \oplus m'' \oplus 1]$, and $m'' \in \{0, 1\}$.

Case 2:

Similar to *Case 1*.

Q.E.D

By recalling the generating functions in [6], we can see that *Case 2* is a special case of them when $i_0 = 0$ and $l' = 0$.

Interestingly, the multiplication operation within the generating functions in **Theorem 3** can be understood as multiplication performed along the unitary circle, as shown in Fig. 1 and Fig. 2. The gap of the outer circle shows clearly the missing multiplication terms within the generating functions, for instance, $x_{i_0}x_{i_1}$ and $x_{i_0}x_{i_{n-1}}$ are the missing terms for *Case 1* and *Case 2* respectively.

One example is given out to show the correctness of the generating functions, which is corresponding to *Case 1*. Assume $n = 3$, $(i_1, i_2) = (1, 2)$, and $(\beta_0, \beta_1, \beta_2, \beta, m) = (1, 0, 0, 1, 0)$. So,

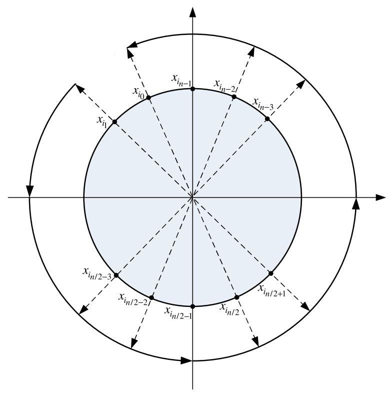

Fig. 1 Multiplication Order in Unitary Circle for *Case 1*

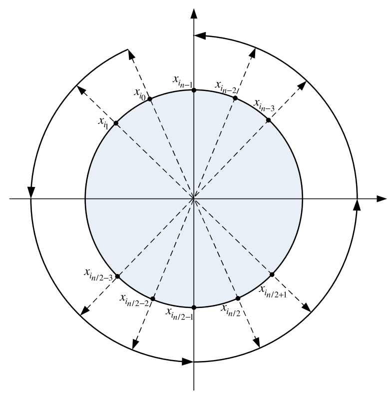

Fig. 2 Multiplication Order in Unitary Circle for *Case 2*

$$
\begin{array}{lcl}
f_3(x) & = & 2(x_1x_2 \oplus x_0x_2) \oplus 3x_0 \\
g_3(x) & = & 2(x_1x_2 \oplus x_0x_2) \oplus 3x_0 \oplus 2x_1 \oplus 2
\end{array}
$$

Therefore,

$$
\begin{array}{lcl}
\mathbf{a} & = & \{1, -j, 1, -j, 1, j, -1, -j\} \\
\mathbf{b} & = & \{-1, j, 1, -j, -1, -j, -1, -j\}
\end{array}
$$

Their aperiodic correlation functions can be expressed as,

$$
\begin{array}{cl}
& \rho(\mathbf{a}, \mathbf{a}; \tau) \\
= & \{j, 0, j, 0, j, 0, -3j, 8, 3j, 0, -j, 0, -j, 0, -j\} \\
& \rho(\mathbf{b}, \mathbf{b}; \tau) \\
= & \{-j, 0, -j, 0, -j, 0, 3j, 8, -3j, 0, j, 0, j, 0, j\} \\
& \rho(\mathbf{a}, \mathbf{b}; \tau) \\
= & \{-j, 0, j, 0, -j, 0, j, 0, j, 0, -5j, 0, -3j, 0, -j\}
\end{array}
$$

where $\tau = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$.

5. Other properties of the proposed E-Sequences pair

Let $\mathbf{a}_0$ and $\mathbf{a}_1$ be the first half and second half of $\mathbf{a}$, likewise, we define $\mathbf{b}_0$ and $\mathbf{b}_1$. We call $\mathbf{a}_0$ and $\mathbf{a}_1$, $\mathbf{b}_0$ and $\mathbf{b}_1$ the "half-partition-pair" of $\mathbf{a}$ and $\mathbf{b}$ respectively.

Theorem 3: $\{\mathbf{a}_0, \mathbf{a}_1\}$ and $\{\mathbf{b}_0, \mathbf{b}_1\}$ are E-Sequences pairs defined under **Theorem 1** if $i_1 = n - 1$ for *Case 1*, or if $i_{n-1} = n - 1$ for *Case 2*. This partition procedure can be repeated for each new resultant E-Sequences pair. In particular, if $i_k = n - k$ for *Case 1* or $i_k = k$ for *Case 2*, where $1 \leq k \leq n-1$, then all the "half-partition-pairs" can be classified as the new E-sequences pairs as defined in **Theorem 1**.

Proof. Denote $h^{\mathbf{a}_0}_{n-1}(\hat{x})$, $h^{\mathbf{a}_1}_{n-1}(\hat{x})$, $h^{\mathbf{b}_0}_{n-1}(\hat{x})$ and $h^{\mathbf{b}_1}_{n-1}(\hat{x})$ as the generating functions for $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$ and $\mathbf{b}_1$. We only give out the proof for *Case 1* because the proof for *Case 2* is very similar to *Case 1*. For *Case 1* when $i_1 = n-1$, $\hat{x} = (x_{i_0}, x_{i_2}, x_{i_3}, \cdots, x_{i_{n-2}}, x_{i_{n-1}})$. Let $\tilde{\beta} = \beta \oplus \beta_{n-1}$, and $\tilde{m} = \beta \oplus m$.

Therefore, we have

$$\begin{aligned}
& h^{\mathbf{a}_0}_{n-1}(\hat{x}) \\
= \; & f_n(x)|_{x_{i_1}=0} \\
= \; & 2(x_{i_2}x_{i_3} \oplus \cdots \oplus x_{i_{n-2}}x_{i_{n-1}} \oplus x_{i_{n-1}}x_{i_0} \\
& \oplus \beta_{i_0}x_{i_0} \oplus \beta_{i_2}x_{i_2} \oplus \cdots \oplus \beta_{i_{n-2}}x_{i_{n-2}} \\
& \oplus \beta_{i_{n-1}}x_{i_{n-1}} \oplus m) \oplus x_0
\end{aligned}$$

$$\begin{aligned}
& h^{\mathbf{a}_1}_{n-1}(\hat{x}) \\
= \; & f_n(x)|_{x_{i_1}=1} \\
= \; & h^{\mathbf{a}_0}_{n-1}(\hat{x}) \oplus 2x_{i_2} \oplus 2\beta_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
& h^{\mathbf{b}_0}_{n-1}(\hat{x}) \\
= \; & g_n(x)|_{x_{i_1}=0} \\
= \; & h^{\mathbf{a}_0}_{n-1}(\hat{x}) \oplus 2\beta \\
= \; & 2(x_{i_2}x_{i_3} \oplus \cdots \oplus x_{i_{n-2}}x_{i_{n-1}} \oplus x_{i_{n-1}}x_{i_0} \\
& \oplus \beta_{i_0}x_{i_0} \oplus \beta_{i_2}x_{i_2} \oplus \cdots \oplus \beta_{i_{n-2}}x_{i_{n-2}} \\
& \oplus \beta_{i_{n-1}}x_{i_{n-1}} \oplus \tilde{m}) \oplus x_0
\end{aligned}$$

$$\begin{aligned}
& h^{\mathbf{b}_1}_{n-1}(\hat{x}) \\
= \; & g_n(x)|_{x_{i_1}=1} \\
= \; & h^{\mathbf{b}_0}_{n-1}(\hat{x}) \oplus 2x_{i_2} \oplus 2\beta_{n-1}
\end{aligned}$$

We can see that $h^{\mathbf{a}_0}_{n-1}(\hat{x})$ and $h^{\mathbf{a}_1}_{n-1}(\hat{x})$, $h^{\mathbf{b}_0}_{n-1}(\hat{x})$ and $h^{\mathbf{b}_1}_{n-1}(\hat{x})$ have the same form as the generating functions in **Theorem 3**, thus $\{\mathbf{a}_0, \mathbf{a}_1\}$ and $\{\mathbf{b}_0, \mathbf{b}_1\}$ also belong to the new class of E-Sequences pairs.

Q.E.D

Corresponding to $\mathbf{a}$ and $\mathbf{b}$, we define two other sequences $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ as follows,

$$\begin{aligned}
\hat{\mathbf{a}} &= [c_0, -jd_0, c_1, -jd_1, \cdots, c_{N/2-1}, -jd_{N/2-1}] \\
\hat{\mathbf{b}} &= \pm[d_{N/2-1}, -jc_{N/2-1}, \cdots, d_1, -jc_0, d_0, -jc_0]
\end{aligned}$$

by changing the signs of $\mathbf{a}$ and $\mathbf{b}$ alternately.

Theorem 4: $\{\hat{\mathbf{a}}, \hat{\mathbf{b}}\}$ is also a pair of the new E-sequences defined in **Theorem 1**. In addition, when τ is an even-number, $\rho(\hat{\mathbf{a}}, \hat{\mathbf{a}}; \tau) = -\rho(\mathbf{a}, \mathbf{a}; \tau)$, $\rho(\hat{\mathbf{b}}, \hat{\mathbf{b}}; \tau) = -\rho(\mathbf{b}, \mathbf{b}; \tau)$, and $\rho(\hat{\mathbf{a}}, \hat{\mathbf{b}}; \tau) = -\rho(\mathbf{a}, \mathbf{b}; \tau)$. Therefore, $\mathbf{g}_0 = [\mathbf{a}; \hat{\mathbf{a}}]$ and $\mathbf{g}_1 = [\mathbf{b}; \hat{\mathbf{b}}]$ are a pair of Golay-Complementary codes which have ideal AACFs and ACCFs.

Proof. The proof of **Theorem 4** can be obtained following the same principle in Section III.

In fact, conventional E-sequences [1] have the same property as this new E-Sequences. Through the method in **Theorem 4**, we can build new Golay-Complementary codes or other perfect complementary codes whose element codes have ideal real correlation functions or even-shift orthogonal properties, which may be used to reduce the code acquisition complexity in the practical communication systems.

Before this work, E-Sequences pair and IS pair were studied individually. The new class of E-Sequences however, builds a link between E-Sequences pair and IS pair, and naturally they can be converted into each other by the proposed E-Sequences.

6. Conclusion

In this paper, we have proposed a new class of quadriphase E-Sequences and its generating functions. Its real aperiodic correlation functions are ideal, so the proposed E-Sequences is also a special class of Imaginary Sequences. Pertinent sequence properties are discussed. Through this work, E-sequences and Imaginary Sequences can be unified under the same framework.

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