Improved Lower Bound for Quasi-Complementary Sequence Set

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***Abstract*—The Welch bound for aperiodic correlation for binary sequence set was improved by Levenshtein by weighting the cyclic shifts of the sequence vectors. Taking Levenshtein's idea, a new lower bound for quasi-complementary sequence set (QCSS) over the complex roots-of-unity is derived in this paper. It is shown to be tighter than the Welch bound for QCSS in one of the following cases:**

1) $K = 4M - 1$, $M \geq 2$ **and** $N > \frac{2}{1-\frac{1}{M}}$**;**
2) $K \geq 4M$, $M \geq 2$ **and** $N \geq 2$**.**

where K, M, N respectively denotes the set size, number of channels, elementary sequence length of QCSS.

I. Introduction

The Golay complementary sequence pair found by Golay in 1961 contains a pair of sequences whose non-trivial aperiodic auto-correlation summations for different time shifts are zero [1]. In [2], Tseng and Liu extended the Golay complementary pair to a set of 2 or more sequences whose non-trivial aperiodic auto-correlation and cross-correlation summations are zero. In [3], Suehiro proposed a construction for the complete complementary sequences, whose set size achieves the upper bound of the mutually orthogonal Golay complementary sequences. The readers are referred to [4] for a good survey of complementary sequences. Nowadays complementary sequences have found many applications, e.g., multi-carrier code-division multiple-access (MC-CDMA) communication ([5],[6]), peak-to-average power ratio (PAPR) control ([7]-[10]), inter-symbol interference (ISI) channel estimation ([11],[12]), and radar system [13]. However, a remarkable limitation of perfect complementary sequence set is their small set size. This limitation can be improved by allowing the aperiodic correlation summations to take low non-zero values. The resultant sequences are then called quasi-complementary sequences.

In [14], Welch obtained a collection of lower bounds for periodic and aperiodic correlation of sequence set by studying the minimax of inner products of vector set. The bounds include the lower bound of aperiodic correlation for quasi-complementary sequence set (QCSS):

$$\delta_{\max}^2 \geq M^2 N^2 \frac{\frac{K}{M} - 1}{K(2N-1) - 1}, \tag{1}$$

where $\delta_{\max}$ denotes for the maximum aperiodic correlation magnitude over all the non-trivial auto-correlations and cross-correlations, K denotes the set size of QCSS, M denotes the number of channels which is not less than 2, N denotes the elementary sequence length. Setting $M = 1$, (1) reduces to the lower bound of aperiodic correlation for conventional sequence set:

$$\delta_{\max}^2 \geq N^2 \frac{K - 1}{K(2N-1) - 1}. \tag{2}$$

In [15], Levenshtein showed that, for binary sequence set, the Welch bound in (2) can be strengthened for set size $K \geq 4$ and sequence length $N \geq 2$. The basic idea that the Levenshtein bound relied on is that, the mean of the weighted aperiodic correlation squares for any sequence subset over complex roots-of-unity should be equal to or greater than the mean over the whole set which includes all the complex roots-of-unity sequences. The strengthening of the Welch bound was extended to complex roots-of-unity sequences by Serdar in [16]. Later, Peng and Fan derived the lower bound on aperiodic correlation for low correlation zone sequence set by an approach similar to Levenshtein's [17]. However, there have been no similar effort on QCSS.

In this paper, a new lower bound for aperiodic correlation of QCSS is derived. The new bound will be shown to be tighter than the Welch bound for QCSS in (1) for certain range of K, M and N.

In section II, necessary notations and reviews of the Levenshtein bounds are given. In section III, we present the new lower bound for QCSS. We conclude this work in section IV.

II. Preliminaries

A. Notations

A sequence set $\mathbf{A}$ contains K sequences, each of length N.

$$\begin{aligned} \mathbf{A} &= \left\{\mathbf{a}^0, \mathbf{a}^1, \cdots, \mathbf{a}^u, \cdots, \mathbf{a}^{K-1}\right\}, \quad 0 \leq u \leq K-1 \\ \mathbf{a}^u &= \left\{a_0^u, a_1^u, \cdots, a_t^u, \cdots, a_{N-1}^u\right\}, \quad 0 \leq t \leq N-1. \end{aligned}$$

The aperiodic correlation function $\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)$ of $\mathbf{A}$ is defined as

$$\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)=\begin{cases}\sum\limits_{t=0}^{N-1-\tau} a_t^u (a_{t+\tau}^v)^*, & 0\leq\tau\leq(N-1);\\ \sum\limits_{t=0}^{N-1+\tau} a_{t-\tau}^u (a_t^v)^*, & -(N-1)\leq\tau\leq-1;\\ 0, & |\tau|\geq N.\end{cases} \tag{3}$$

When $u=v$, (3) is called the aperiodic auto-correlation function; otherwise, aperiodic cross-correlation function.

A complementary sequence set $\mathbf{C}$ contains K complementary sequences, each of $M\geq 2$ channels, and each channel has an elementary sequence of length N.

$$\begin{aligned}\mathbf{C}&=\left\{\mathbf{C}^0,\mathbf{C}^1,\cdots,\mathbf{C}^u,\cdots,\mathbf{C}^{K-1}\right\},\ 0\leq u\leq K-1\\ \mathbf{C}^u&=\left\{\mathbf{c}_0^u,\mathbf{c}_1^u,\cdots,\mathbf{c}_m^u,\cdots,\mathbf{c}_{M-1}^u\right\},\ 0\leq m\leq M-1\\ \mathbf{c}_m^u&=\left\{c_{m,0}^u,c_{m,1}^u,\cdots,c_{m,t}^u,\cdots,c_{m,N-1}^u\right\},\ 0\leq t\leq N-1.\end{aligned}$$

The aperiodic correlation function $\rho_{\mathbf{C}^u,\mathbf{C}^v}(\tau)$ of $\mathbf{C}$ is defined as

$$\rho_{\mathbf{C}^u,\mathbf{C}^v}(\tau)=\sum_{m=0}^{M-1}\rho_{\mathbf{c}_m^u,\mathbf{c}_m^v}(\tau), \tag{4}$$

where $0\leq u,v\leq K-1$.

Next, the aperiodic auto-correlation tolerance δ_{a} and the aperiodic cross-correlation tolerance δ_{c} are respectively defined below:

$$\begin{aligned}\delta_{\mathrm{a}}&=\max\{|\rho_{\mathbf{C}^u,\mathbf{C}^u}(\tau)|:\ 0\leq u\leq K-1,0<\tau\leq N-1\};\\ \delta_{\mathrm{c}}&=\max\{|\rho_{\mathbf{C}^u,\mathbf{C}^v}(\tau)|:\ 0\leq u,v\leq K-1,u\neq v,\\ &\qquad 0\leq\tau\leq N-1\}.\end{aligned}$$

Moreover, the aperiodic tolerance (or the maximum aperiodic correlation magnitude) $\delta_{\max}$ is defined to be $\delta_{\max}=\max\{\delta_{\mathrm{a}},\delta_{\mathrm{c}}\}$. When $\delta_{\max}>0$, we call $\mathbf{C}$ a quasi-complementary sequence set (QCSS); Otherwise, $\mathbf{C}$ is a set of mutually orthogonal Golay complementary sequences.

$\mathbf{A}$ (or $\mathbf{C}$) is over the complex roots-of-unity if every entry belongs to $E=\left\{1,\xi^1,\cdots,\xi^{H-1}\right\}$, where $\xi=\exp\sqrt{-1}\frac{2\pi}{H}$ and H is a positive integer greater than 1.

The cardinality of $\mathbf{A}$ (or $\mathbf{C}$), or the set size of $\mathbf{A}$ (or $\mathbf{C}$), is denoted by $|\mathbf{A}|$ (or $|\mathbf{C}|$).

The *right cyclic shift* of i (where $0\leq i\leq N-1$) positions for any vector $\mathbf{a}=(a_0,a_1,\cdots,a_{N-1})$ is denoted by

$$T^i\mathbf{a}=(a_{N-i},a_{N-i+1},\cdots,a_{N-1},a_0,a_1,\cdots,a_{N-i-1}).$$

B. Review of Levenshtein bound

For easy reference, most of the notations used in [15] are adopted here.

For two sequence subsets $\mathbf{A}$ and $\mathbf{B}$ in E^N, define

$$F(\mathbf{A},\mathbf{B}):=\frac{1}{|\mathbf{A}||\mathbf{B}|}\sum_{\mathbf{x}\in\mathbf{A}}\sum_{\mathbf{y}\in\mathbf{B}}\sum_{s=0}^{2N-2}\sum_{t=0}^{2N-2}\left|\left\langle T^s(\mathbf{x},\mathbf{0}),T^t(\mathbf{y},\mathbf{0})\right\rangle\right|^2 w_s w_t, \tag{5}$$

where $\mathbf{0}$ means a row vector of $N-1$ zeros, $(\mathbf{x},\mathbf{0})$ denotes the vector concatenation of row vector $\mathbf{x}$ and $\mathbf{0}$, $\langle\mathbf{a},\mathbf{b}\rangle$ denotes the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$, and the weighting vector $\mathbf{w}=(w_0,w_1,\cdots,w_{2N-2})^{\mathrm{T}}$ is constrained by

$$w_i\geq 0, i=0,1,\cdots,2N-2,\ \text{and}\ \sum_{i=0}^{2N-2}w_i=1. \tag{6}$$

Define the quadratic function

$$\begin{aligned}Q_{2N-1}(\mathbf{w},a)&:=\mathbf{w}^{\mathrm{T}}Q_{2N-1}(a)\mathbf{w}\\ &=a\sum_{i=0}^{2N-2}w_i^2+\sum_{s,t=0}^{2N-2}l_{s,t,N}w_s w_t,\end{aligned} \tag{7}$$

where $Q_{2N-1}(a)$ is a quadratic matrix whose diagonal elements are all equal to a given constant number a, and

$$l_{s,t,N}=\min\left\{|t-s|,2N-1-|t-s|\right\}. \tag{8}$$

Note that

$$\begin{aligned}&\left|\left\langle T^s(\mathbf{x},\mathbf{0}),T^t(\mathbf{y},\mathbf{0})\right\rangle\right|^2\\ &\quad+\left|\left\langle T^t(\mathbf{x},\mathbf{0}),T^s(\mathbf{y},\mathbf{0})\right\rangle\right|^2\\ &\quad=\left|\rho_{\mathbf{x},\mathbf{y}}(l_{s,t,N})\right|^2+\left|\rho_{\mathbf{y},\mathbf{x}}(l_{s,t,N})\right|^2.\end{aligned} \tag{9}$$

Lemma 1: For any sequence set $\mathbf{A}\subseteq E^N$, we have

1) (*Lemma 1* of [15])

$$F(\mathbf{A},\mathbf{A})\leq\frac{1}{K}\left(\left(N^2-\delta_{\max}^2\right)\sum_{i=0}^{2N-2}w_i^2+K\delta_{\max}^2\right). \tag{10}$$

2) (*Lemma 2* and *Lemma 3* of [15])

$$F(\mathbf{A},\mathbf{A})\geq F\left(E^N,E^N\right)=\sum_{s,t=0}^{2N-2}\left(n-l_{s,t,N}\right)w_s w_t. \tag{11}$$

3) (*Theorem 1* of [15])

$$\delta_{\max}^2\geq N-\frac{Q_{2N-1}\left(\mathbf{w},\frac{N(N-1)}{K}\right)}{1-\frac{1}{K}\sum\limits_{i=0}^{2N-2}w_i^2}. \tag{12}$$

III. Lower Bound For Quasi-Complementary Sequence Set

Let k be a positive integer. For two complementary sequence subsets $\mathbf{C}$ and $\mathbf{D}$ in $E^{M\times N}$, we define

$$\begin{aligned}&F(\mathbf{C},\mathbf{D}):=\\ &\quad\frac{1}{|\mathbf{C}||\mathbf{D}|}\sum_{\mathbf{C}^u\in\mathbf{C}}\sum_{\mathbf{D}^v\in\mathbf{D}}\sum_{s=0}^{2N-2}\sum_{t=0}^{2N-2}\\ &\quad\left|\left\langle T^s\left(\mathbf{C}^u,\mathbf{0}\right),T^t\left(\mathbf{D}^v,\mathbf{0}\right)\right\rangle\right|^{2k}w_s w_t,\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\left|\left\langle T^s\left(\mathbf{C}^u,\mathbf{0}\right),T^t\left(\mathbf{D}^v,\mathbf{0}\right)\right\rangle\right|^{2k}=\\ &\quad\left|\sum_{m=0}^{M-1}\left\langle T^s\left(\mathbf{c}_m^u,\mathbf{0}\right),T^t\left(\mathbf{d}_m^v,\mathbf{0}\right)\right\rangle\right|^{2k},\end{aligned} \tag{14}$$

$w_i (0 \leq i \leq 2N-2)$ is defined by (6). Note that the vector $(\mathbf{C}^u, \mathbf{0})$ is obtained by first doing the vector concatenation of each elementary sequence with $N-1$ zeros, then doing the vector concatenation of all the resultant vectors. With $l_{s,t,N}$ defined in (8), it is also noted that

$$\begin{aligned} &\left|\left\langle T^s\left(\mathbf{C}^u, \mathbf{0}\right), T^t\left(\mathbf{D}^v, \mathbf{0}\right)\right\rangle\right|^{2k} \\ &\quad + \left|\left\langle T^s\left(\mathbf{D}^v, \mathbf{0}\right), T^t\left(\mathbf{C}^u, \mathbf{0}\right)\right\rangle\right|^{2k} \\ &\qquad = \left|\rho_{\mathbf{C}^u, \mathbf{D}^v}(l_{s,t,N})\right|^{2k} + \left|\rho_{\mathbf{D}^v, \mathbf{C}^u}(l_{s,t,N})\right|^{2k}. \end{aligned} \tag{15}$$

In what follows, we will drop the subscripts of $l_{s,t,N}$ for ease of presentation, although l may vary with different values of s, t, and N. For complementary sequence sets over complex roots-of-unity, define $c^u_{m,e} := \xi^{\bar{c}^u_{m,e}}$ and $d^v_{m,e+l} := \xi^{\bar{d}^v_{m,e+l}}$, where $\bar{c}^u_{m,e}, \bar{d}^v_{m,e+l} \in \{0, 1, \cdots, H-1\}$. Let $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ and $\mathbf{e}$, $\mathbf{n}$ and $\mathbf{f}$ are similarly defined. Meanwhile, denote $\mathbf{M-1}$ (or $\mathbf{N-1-l}$) as an all $M-1$ (or $N-1-l$) vector of length k. Then

$$\begin{aligned} &\sum_{\mathbf{D}^v \in E^{M\times N}} \left|\rho_{\mathbf{C}^u, \mathbf{D}^v}(l)\right|^{2k} \\ &= \sum_{\mathbf{D}^v \in E^{M\times N}} \left|\sum_{m=0}^{M-1} \rho_{\mathbf{c}^u_m, \mathbf{d}^v_m}(l)\right|^{2k} \\ &= \sum_{\mathbf{D}^v \in E^{M\times N}} \left(\sum_{m,n=0}^{M-1} \sum_{e,f=0}^{N-1-l} \xi^{\bar{c}^u_{m,e} - \bar{d}^v_{m,e+l}} \xi^{-\bar{c}^u_{n,f} + \bar{d}^v_{n,f+l}}\right)^k \\ &= \sum_{\mathbf{D}^v \in E^{M\times N}} \sum_{\mathbf{m},\mathbf{n}=0^k}^{\mathbf{M-1}} \sum_{\mathbf{e},\mathbf{f}=0^k}^{\mathbf{N-1-l}} \\ &\qquad \prod_{t=1}^{k} \xi^{\bar{c}^u_{m_t,e_t} - \bar{c}^u_{n_t,f_t}} \xi^{-\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}} \\ &= \sum_{\mathbf{m},\mathbf{n}=0^k}^{\mathbf{M-1}} \sum_{\mathbf{e},\mathbf{f}=0^k}^{\mathbf{N-1-l}} \prod_{t=1}^{k} \xi^{\bar{c}^u_{m_t,e_t} - \bar{c}^u_{n_t,f_t}} \\ &\qquad \cdot \left(\sum_{\mathbf{D}^v \in E^{M\times N}} \prod_{t=1}^{k} \xi^{-\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}}\right) \\ &= S_1 + S_2, \end{aligned}$$

where S_1 and S_2 are the summation terms classified according to $a)$ $\mathbf{m} = \mathbf{n}$ and $\mathbf{e} = \mathbf{f}$; $b)$ $\mathbf{m} = \mathbf{n}$ and $\mathbf{e} \neq \mathbf{f}$, or $\mathbf{m} \neq \mathbf{n}$ and $\mathbf{e} = \mathbf{f}$, or $\mathbf{m} \neq \mathbf{n}$ and $\mathbf{e} \neq \mathbf{f}$, respectively. For case $a)$, we have

$$S_1 = \sum_{\mathbf{D}^v \in E^{M\times N}} \sum_{\mathbf{m}=0^k}^{\mathbf{M-1}} \sum_{\mathbf{e}=0^k}^{\mathbf{N-1-l}} 1 = H^{MN} M^k (N-l)^k. \tag{16}$$

For case $b)$, there must exist at least one position such that $\mathbf{m}$ is different from $\mathbf{n}$, or $\mathbf{e}$ is different from $\mathbf{f}$. Suppose that for the set $\mathbf{I} \subseteq \{0, 1, \cdots, k-1\}$ (where $0 < |\mathbf{I}| \leq k$), if and only if $t \in \mathbf{I}$, we have $m_t \neq n_t$ or $e_t \neq f_t$. Hence

$$\begin{aligned} &\sum_{\mathbf{D}^v \in E^{M\times N}} \prod_{t=1}^{k} \xi^{-\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}} \\ &= \sum_{\mathbf{D}^v \in E^{M\times N}} \xi^{\sum_{t\in\mathbf{I}} -\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}} \end{aligned}$$

One can see that, when $\mathbf{D}^v$ runs over $E^{M\times N}$, the chances that

$$\sum_{t\in\mathbf{I}} -\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}$$

takes any value in the set of $\{0, 1, \cdots, H-1\}$ are equal. Hence,

$$\sum_{\mathbf{D}^v \in E^{M\times N}} \xi^{\sum_{t\in\mathbf{I}} -\bar{d}^v_{m_t,e_t+l} + \bar{d}^v_{n_t,f_t+l}} = 0, \tag{17}$$

implying that $S_2 = 0$. Furthermore, by (16), we have

$$\sum_{\mathbf{D}^v \in E^{M\times N}} \left|\rho_{\mathbf{C}^u, \mathbf{D}^v}(l)\right|^{2k} = H^{MN} M^k (N-l)^k. \tag{18}$$

Using (18), and following the derivations of (11) in [15], we have

***Lemma** 2:* For any $\mathbf{C}^u \in E^{M\times N}$,

$$\begin{aligned} F(\mathbf{C}, \mathbf{C}) \quad &\geq \quad F\left(\{\mathbf{C}^u\}, E^{M\times N}\right) = F\left(E^{M\times N}, E^{M\times N}\right) \\ &= \quad \sum_{s,t=0}^{2N-2} M^k (N-l)^k w_s w_t. \end{aligned} \tag{19}$$

Expanding (13) accordingly, we obtain

***Lemma** 3:* For any $\mathbf{C} \subseteq E^{M\times N}$ of set size K,

$$F(\mathbf{C}, \mathbf{C}) \leq \frac{M^{2k} N^{2k}}{K} \sum_{i=0}^{2N-2} w_i^2 + \left(1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2\right) \delta^{2k}_{\max} \tag{20}$$

Together with ***Lemma** 2*, we have

***Theorem** 1:*

$$\begin{aligned} \left(1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2\right) \delta^{2k}_{\max} \geq \sum_{s,t=0}^{2N-2} M^k (N-l)^k w_s w_t \\ - \frac{M^{2k} N^{2k}}{K} \sum_{i=0}^{2N-2} w_i^2. \end{aligned} \tag{21}$$

For $k = 1$, ***Theorem** 1* reduces into

***Corollary** 1:*

$$\delta^2_{\max} \geq M \left(N - \frac{Q_{2N-1}\left(\mathbf{w}, \frac{N(MN-1)}{K}\right)}{1 - \frac{1}{K} \sum_{i=0}^{2N-2} w_i^2}\right). \tag{22}$$

***Remark** 1:* When $M = 1$, ***Corollary** 1* degenerates to the lower bound in (12) obtained by Levenshtein [15].

$$
\begin{aligned}
\Delta &= \frac{KM^2N^2(N-1)}{(KN-1)(K(2N-1)-1)}\left(\frac{K}{M}\left(1-\frac{(N+1)(2N-1)}{3N^2}\right)+\frac{N+1}{3MN^2}-1\right)\\
&= \frac{KM^2N^2(N-1)}{(KN-1)(K(2N-1)-1)}\left(\frac{K}{M}\left(\frac{1}{3}-\frac{N-1}{3N^2}\right)+\frac{N+1}{3MN^2}-1\right)\\
&= \frac{KM^2N^2(N-1)}{(KN-1)(K(2N-1)-1)}\left(\left(4-\frac{1}{M}\right)\left(\frac{1}{3}-\frac{N-1}{3N^2}\right)+\frac{N+1}{3MN^2}-1\right)\\
&= \frac{KM^2N^2(N-1)(N-2)}{3(KN-1)(K(2N-1)-1)N}\left(1-\frac{2}{N}-\frac{1}{M}\right).
\end{aligned} \tag{25}
$$

***Remark** 2:* When $\mathbf{w} = \frac{1}{2N-1}(1,1,\cdots,1)$ is used, ***Corollary** 1* reduces to the Welch bound for QCSS in (1).

By adopting the following weighting vector,

$$
w_i = \begin{cases} \frac{1}{N}, & i \in \{0,1,\cdots,N-1\}; \\ 0, & i \in \{N,N+1,\cdots,2N-2\} \end{cases} \tag{23}
$$

in ***Corollary** 1*, our improved QCSS lower bound is obtained as follows.

***Corollary** 2:* (Improved lower bound for QCSS)

$$
\delta_{\max}^2 \geq \frac{MN^2K - M^2N^2 - \frac{MK(N^2-1)}{3}}{NK-1}. \tag{24}
$$

The proposed bound is stronger than the Welch bound for QCSS in (1) if one of the following conditions is fulfilled:

1: $K = 4M-1$, $M \geq 2$ and $N > \frac{2}{1-\frac{1}{M}}$;
2: $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

Proof: We deal with the case of $K = 4M-1$ first. Let Δ denote the subtraction of the lower bound in (24) by the Welch bound for QCSS, then Δ can be simplified as shown in (25).

To improve the Welch bound for QCSS, i.e., to have $\Delta > 0$, it is required that

$$
N \geq 3 \text{ and } 1 - \frac{2}{N} - \frac{1}{M} > 0,
$$

which is equivalent to

$$
N > \frac{2}{1-\frac{1}{M}}. \tag{26}
$$

The condition in (26) can be explicitly stated as:

a: $K = 4M-1$, $M = 2$ and $N \geq 5$;
b: $K = 4M-1$, $M = 3$ and $N \geq 4$;
c: $K = 4M-1$, $M \geq 4$ and $N \geq 3$.

Similar proof for $K \geq 4M$ follows by repeating the derivation in (25) with $N \geq 2$ and $M \geq 2$. ■

The tightness of the proposed lower bound for case 1 of ***Corollary** 2* can be seen by the plot of Δ in Fig. 1. In general, the bound improvement increases with increasing M or N. Asymptotically, the bound improvement for both cases can be approximated as

$$
\Delta|_{N\to+\infty} = MN\left(\frac{1}{6} - \frac{M}{2K}\right). \tag{27}
$$

(27) can be easily obtained by setting N in the second row of (25) to be a very large number, and then discarding the relatively small terms.

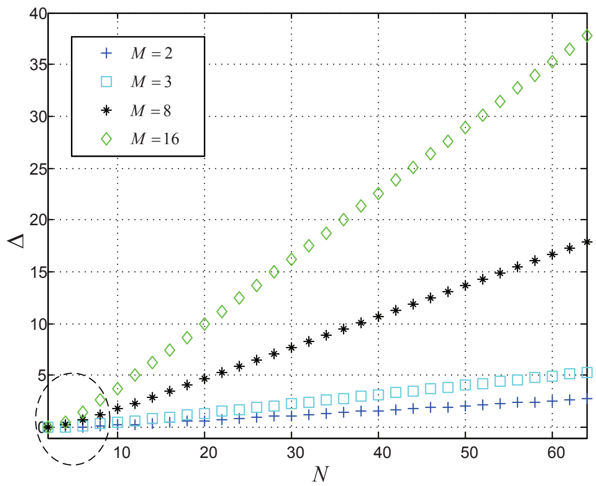

Fig. 1. The plot of Δ as a function of N for $K = 4M-1$

For small values of N, the tightness of the proposed bound for case 1 of ***Corollary** 2* may not be clearly reflected by Fig. 1, the readers are therefore directed to TABLE I. Row 1 of the table illustrates that the condition of $K = 4M-1$, $M = 2$, $N \geq 5$ (see case a in the proof of ***Corollary** 2*) leads to a tighter bound because $\Delta > 0$; Likewise, row 2 illustrates the condition in case b, and row 3 and 4 illustrate the condition in case c.

TABLE I
THE VALUES OF Δ FOR $2 \leq N \leq 7$ AND $K = 4M-1$

N	2	3	4	5	6	7
$M = 2$	0	-0.0137	0	0.0266	0.0599	0.0972
$M = 3$	0	0	0.0404	0.0998	0.1692	0.2446
$M = 8$	0	0.0583	0.2240	0.4403	0.6834	0.9425
$M = 16$	0	0.1480	0.5111	0.9755	1.4940	2.0445

IV. CONCLUSION

We have established a new lower bound for quasi-complementary sequence set (QCSS) over complex roots-of-unity, and shown that it is a generalization of the Levenshtein bound in [15]. Given the set size K, channel number M, and the elementary sequence length N, it is shown in ***Corollary** 2* that the proposed bound strengthens the Welch bound for QCSS in [14] under one of the following sets of conditions:

1) $K = 4M-1$, $M \geq 2$ and $N > \frac{2}{1-\frac{1}{M}}$.
2) $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

Interestingly, by setting $M = 1$ in the latter case, the proposed bound coincides with an earlier result of the Levenshtein bound that the Welch bound for binary sequence set can be improved for $K \geq 4$ and $N \geq 2$.

ACKNOWLEDGMENT

The work of Zi Long Liu and Yong Liang Guan was supported in full by the Advanced Communications Research Program DSOCL06271, a research grant from the Directorate of Research and Technology (DRTech), Ministry of Defence, Singapore. The work of Wai Ho Mow was supported by the National Science Foundation of China under the grant number 61028002/F0102.

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