Meeting The Levenshtein Bound With Equality By Weighted-Correlation Complementary Set

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***Abstract*—Levenshtein improved the Welch bound on aperiodic correlation by weighting the cyclic shifts of the sequences over complex roots-of-unity. Although many works have been concerned on meeting the Welch bound with equality, no such effort has been reported for the Levenshtein bound. We show that the Levenshtein bound with equality is met if and only if the non-trivial aperiodic correlations have identical amplitude for all time-shifts, and the sequences form a novel class of complementary set whose aperiodic correlation is defined as the conventional aperiodic correlation modulated by a simplex weighting vector.**

I. Introduction

Welch obtained the correlation lower bounds for equi-energy sequence sets based on the inner product theorem of the vector set [1]. Massey *et al* obtained the necessary and sufficient condition of meeting the Welch bound (on total squared correlations) with equality [2],[3], and found that the sum capacity of synchronous code-division multiple-access (CDMA) system is maximized by the Welch bound equality (WBE) sequence sets [4]. Mow showed that for asynchronous CDMA system, the maximum worst-case signal-to-noise ratio (SNR) is met if and only if the sequences form a complementary set [5]-[7], which has zero non-trivial aperiodic correlation summations [8],[9]. Ever since then, a lot of works have been concerned on the WBE sequences [10]-[13].

In [14], Levenshtein proposed a lower bound on aperiodic correlation of binary sequence set by the inequality on the mean of the weighted squared correlations, and showed that it is tighter than the Welch bound for $K \geq 4$ and $N \geq 2$, where K denotes the sequence set size, and N the sequence length.

In this paper, we show how to meet the Levenshtein bound with equality by using a particular weighting vector, and when that equality is met, the sequences will form a *weighted-correlation complementary set* characterized by this weighting vector.

Notations and definitions are given in Section II.A, followed by a brief review of the Levenshtein bound in Section II.B. The problem of meeting the Levenshtein bound with equality is studied in Section III, and then concluded in Section IV.

II. Preliminaries

A. Notations and Definitions

A sequence set $\mathbf{A}$ contains K sequences, each of length N.

$$\begin{array}{ll}\mathbf{A} = \left\{\mathbf{a}^0, \mathbf{a}^1, \cdots, \mathbf{a}^u, \cdots, \mathbf{a}^{K-1}\right\}, & 0 \leq u \leq K-1 \\ \mathbf{a}^u = \left\{a_0^u, a_1^u, \cdots, a_t^u, \cdots, a_{N-1}^u\right\}, & 0 \leq t \leq N-1.\end{array}$$

Definition 1: Denote by $\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)$ the conventional aperiodic correlation function of $\mathbf{A}$ as

$$\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau) = \begin{cases} \sum\limits_{t=0}^{N-1-\tau} a_t^u (a_{t+\tau}^v)^*, & 0 \leq \tau \leq (N-1); \\ \sum\limits_{t=0}^{N-1+\tau} a_{t-\tau}^u (a_t^v)^*, & -(N-1) \leq \tau \leq -1; \\ 0, & |\tau| \geq N. \end{cases} \tag{1}$$

When $u = v$, $\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)$ is called the aperiodic auto-correlation function, and will be written as $\rho_{\mathbf{a}^u}(\tau)$ for ease of presentation; otherwise, aperiodic cross-correlation function.

Moreover, if

$$\sum_{u=0}^{K-1} \rho_{\mathbf{a}^u}(\tau) = 0 \tag{2}$$

for $\tau \neq 0$, we say that the sequences of $\mathbf{A}$ form a *conventional complementary set* with zero non-trivial aperiodic auto-correlation summations.

Definition 2: Denote by $\delta_{\max}$ the maximum *non-trivial aperiodic correlation magnitude* of $\mathbf{A}$ as

$$\delta_{\max} = \max\left\{|\rho_{\mathbf{a}^u,\mathbf{a}^v}(\tau)| : \ u \neq v, \text{ or } u = v, \tau \neq 0\right\}. \tag{3}$$

Definition 3: A simplex weighting vector $\mathbf{w} = (w_0, w_1, \cdots, w_{2N-2})^{\mathrm{T}}$ is defined as

$$w_i \geq 0, i = 0, 1, \cdots, 2N-2, \text{ and } \sum_{i=0}^{2N-2} w_i = 1. \tag{4}$$

Definition 4: For the simplex weighting vector $\mathbf{w}$ defined in (4), the *weighted aperiodic auto-correlation function* of $\mathbf{A}$

is defined as

$$\rho_{\mathbf{a}^u;\mathbf{w},\lambda}(\tau)=\begin{cases}(2N-1)\sum\limits_{t=0}^{N-1-\tau}a_t^u(a_{t+\tau}^u)^*w_{t+\lambda}, & 0\le\tau\le(N-1);\\ (2N-1)\sum\limits_{t=0}^{N-1+\tau}a_{t-\tau}^u(a_t^u)^*w_{t+\lambda}, & -(N-1)\le\tau\le-1;\\ 0, & |\tau|\ge N.\end{cases} \tag{5}$$

where $0\le\lambda\le 2N-2$, and the subscript addition $t+\lambda$ is performed *modulo* $2N-1$. Moreover, if

$$\sum_{u=0}^{K-1}\rho_{\mathbf{a}^u;\mathbf{w},\lambda}(\tau)=0 \tag{6}$$

for $\tau\neq 0$ and $0\le\lambda\le 2N-2$, we say that the sequences of $\mathbf{A}$ form a *weighted-correlation complementary set*.

***Remark** 1:* When the elements of $\mathbf{w}$ all are equal, i.e., $\mathbf{w}=\frac{1}{2N-1}(1,1,\cdots,1)$, $\rho_{\mathbf{a}^u;\mathbf{w},\lambda}(\tau)$ reduces to $\rho_{\mathbf{a}^u}(\tau)$.

The *right cyclic shift* of i (where $0\le i\le N-1$) positions for any vector $\mathbf{a}=(a_0,a_1,\cdots,a_{N-1})$ is denoted by

$$T^i\mathbf{a}=(a_{N-i},a_{N-i+1},\cdots,a_{N-1},a_0,a_1,\cdots,a_{N-i-1}).$$

For matrix $\mathbb{H}$ of size $p\times q$, we say that $\mathbb{H}(i,:)$, $\mathbb{H}(:,j)$, $\mathbb{H}(i,j)$ are the ith row, jth column, (i,j)th element of $\mathbb{H}$, respectively, where $0\le i\le p-1$ and $0\le j\le q-1$.

B. Review of the Levenshtein Bound

A brief review of the Levenshtein bound [14] will be given below.

For two binary sequence sets $\mathbf{A}$ and $\mathbf{B}$ in $\{1,-1\}^N$, define

$$F(\mathbf{A},\mathbf{B}):=\frac{1}{|\mathbf{A}||\mathbf{B}|}\sum_{\mathbf{a}\in\mathbf{A}}\sum_{\mathbf{b}\in\mathbf{B}}\sum_{s=0}^{2N-2}\sum_{t=0}^{2N-2}\left|\left\langle T^s(\mathbf{a},\mathbf{0}),T^t(\mathbf{b},\mathbf{0})\right\rangle\right|^2 w_sw_t, \tag{7}$$

where $|\mathbf{A}|$ and $|\mathbf{B}|$ respectively denote the cardinality of $\mathbf{A}$ and $\mathbf{B}$ (e.g., the set size of $\mathbf{A}$ and $\mathbf{B}$), $\mathbf{0}$ means a row vector of $N-1$ zeros, $(\mathbf{a},\mathbf{0})$ denotes the vector concatenation of row vector $\mathbf{a}$ and $\mathbf{0}$, $\langle\mathbf{x},\mathbf{y}\rangle$ denotes the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$, and the weighting vector $\mathbf{w}$ is defined in (4). Note that throughout this paper, $F(\mathbf{A},\mathbf{A})$ will be called the normalized *total squared weighted correlations* (TSWC) of $\mathbf{A}$.

Define the quadratic function

$$\begin{aligned}Q_{2N-1}(\mathbf{w},a)&:=\mathbf{w}^{\mathrm{T}}Q_{2N-1}(a)\mathbf{w}\\&=a\sum_{i=0}^{2N-2}w_i^2+\sum_{s,t=0}^{2N-2}\tau_{s,t,N}w_sw_t,\end{aligned} \tag{8}$$

where $Q_{2N-1}(a)$ is a quadratic matrix whose diagonal elements are all equal to a, and

$$\tau_{s,t,N}=\min\{|t-s|,2N-1-|t-s|\}. \tag{9}$$

***Lemma** 1:* For sequence set $\mathbf{A}\subseteq\{1,-1\}^N$, we have

1) (*Lemma 1* of [14])
$$F(\mathbf{A},\mathbf{A})\le\frac{1}{K}\left(\left(N^2-\delta_{\max}^2\right)\sum_{i=0}^{2N-2}w_i^2+K\delta_{\max}^2\right). \tag{10}$$

2) (*Lemma 3* of [14])
$$\begin{aligned}F(\mathbf{A},\mathbf{A})&\ge F\left(\{1,-1\}^N,\{1,-1\}^N\right)\\&=\sum_{s,t=0}^{2N-2}(n-\tau_{s,t,N})\,w_sw_t.\end{aligned} \tag{11}$$

3) By (10) and (11), the Levenshtain bound for binary sequence set is shown to be
$$\delta_{\max}^2\ge N-\frac{Q_{2N-1}\left(\mathbf{w},\frac{N(N-1)}{K}\right)}{1-\frac{1}{K}\sum\limits_{i=0}^{2N-2}w_i^2}. \tag{12}$$

III. Meeting the Levenshtein Bound with Equality

For any sequence $\mathbf{a}$ in $\mathbf{A}\subseteq\{1,-1\}^N$, construct a matrix $\mathbb{H}_{\mathbf{a}}$ of size $(2N-1)\times(2N-1)$, such that the sth row of $\mathbb{H}_{\mathbf{a}}$ is

$$\mathbb{H}_{\mathbf{a}}(s,:)=T^s(\mathbf{a},\mathbf{0}), \tag{13}$$

where $s\in\{0,1,\cdots,2N-2\}$. Let $f_{i,j,s}(\mathbf{a})=\mathbb{H}_{\mathbf{a}}(s,i)\mathbb{H}_{\mathbf{a}}(s,j)w_s$, then for the two binary sequence sets $\mathbf{A}$ and $\mathbf{B}$ defined in Section II.B, Levenshtein found that [14]:

$$F(\mathbf{A},\mathbf{B})=\frac{1}{|\mathbf{A}||\mathbf{B}|}\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2}\sum_{i,j=0}^{2N-2}f_{i,j,s}(\mathbf{a})f_{i,j,t}(\mathbf{b}) \tag{14}$$

and

$$(F(\mathbf{A},\mathbf{B}))^2\le F(\mathbf{A},\mathbf{A})F(\mathbf{B},\mathbf{B}). \tag{15}$$

Proof:

$$\begin{aligned}&\left|\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2}\sum_{i,j=0}^{2N-2}f_{i,j,s}(\mathbf{a})f_{i,j,t}(\mathbf{b})\right|^2\\
&=\left|\sum_{i,j=0}^{2N-2}\left(\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}f_{i,j,s}(\mathbf{a})\right)\cdot\left(\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2}f_{i,j,t}(\mathbf{b})\right)\right|^2\\
&\le\sum_{i,j=0}^{2N-2}\left|\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}f_{i,j,s}(\mathbf{a})\right|^2\cdot\sum_{i,j=0}^{2N-2}\left|\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2}f_{i,j,t}(\mathbf{b})\right|^2\\
&=\underbrace{\sum_{i,j=0}^{2N-2}\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}f_{i,j,s}(\mathbf{a})\sum_{\mathbf{b}\in\mathbf{A}}\sum_{t=0}^{2N-2}f_{i,j,t}(\mathbf{b})}_{|\mathbf{A}|^2F(\mathbf{A},\mathbf{A})}\\
&\quad\cdot\underbrace{\sum_{i,j=0}^{2N-2}\sum_{\mathbf{a}\in\mathbf{B}}\sum_{s=0}^{2N-2}f_{i,j,s}(\mathbf{a})\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2}f_{i,j,t}(\mathbf{b})}_{|\mathbf{B}|^2F(\mathbf{B},\mathbf{B})}\end{aligned} \tag{16}$$

■

Remark 2: According to [14],

$$F(\mathbf{A},\{1,-1\}^N)=F(\{1,-1\}^N,\{1,-1\}^N),$$

hence (11) can be obtained by setting $\mathbf{B}=\{1,-1\}^N$ in (15).

Remark 3: The equality of (16) is met if and only if

$$\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{a})=c\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2} f_{i,j,t}(\mathbf{b}), \tag{17}$$

for any constant c.

We are now ready to present the main result of this paper as follows.

Since

$$\begin{aligned}
&\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{a})\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2}\mathbb{H}_{\mathbf{a}}(s,i)\mathbb{H}_{\mathbf{a}}(s,j)w_s\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\big\langle\mathbb{H}_{\mathbf{a}}(:,i),\mathbf{w}\cdot\mathbb{H}_{\mathbf{a}}(:,j)\big\rangle,\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\sum_{t=0}^{N-\tau-1} a_t a_{t+\tau} w_{\lambda-t \bmod 2N-1},\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\sum_{t=0}^{N-\tau-1} a_t a_{t+\tau} \widetilde{w}_{t-\lambda \bmod 2N-1}\\
&=\frac{1}{2N-1}\sum_{\mathbf{a}\in\mathbf{A}}\rho_{\mathbf{a};\widetilde{\mathbf{w}},-\lambda}(\tau)
\end{aligned} \tag{18}$$

where $\mathbf{w}\cdot\mathbb{H}_{\mathbf{a}}(:,j)$ denotes the element-wise Hadamard product of $\mathbf{w}$ and $\mathbb{H}_{\mathbf{a}}(:,j)$, $\widetilde{\mathbf{w}}=\{\widetilde{w}_i=w_{2N-1-i},0\le i\le 2N-2\}$ denotes the reverse of $\mathbf{w}$,

$$\tau=\min\{|i-j|,2N-1-|i-j|\}, \tag{19}$$

and

$$\lambda=\begin{cases}\min\{i,j\} & \text{if } \tau=|i-j|,\\ \max\{i,j\} & \text{if } \tau=2N-1-|i-j|.\end{cases} \tag{20}$$

***Example** 1:* To verify (18)-(20), consider

$$\mathbf{A}=\{\mathbf{a}\}=\{a_1,a_2,a_3,a_4\},$$

where $\mathbf{A}\subseteq\{1,-1\}^4$, then $\mathbb{H}_{\mathbf{a}}$ can be shown below in the lower right corner.

$\widetilde{\mathbf{w}}$ $\downarrow$	i $\downarrow$		j_1 $\downarrow$				j_2 $\downarrow$
$\widetilde{w}_0$	a_0	a_1	a_2	a_3	0	0	0
$\widetilde{w}_6$	0	a_0	a_1	a_2	a_3	0	0
$\widetilde{w}_5$	0	0	a_0	a_1	a_2	a_3	0
$\widetilde{w}_4$	0	0	0	a_0	a_1	a_2	a_3
$\widetilde{w}_3$	a_3	0	0	0	a_0	a_1	a_2
$\widetilde{w}_2$	a_2	a_3	0	0	0	a_0	a_1
$\widetilde{w}_1$	a_1	a_2	a_3	0	0	0	a_0

For $i=0$ and $j=j_1=2$, we have $\tau=\min\{2,5\}=2$, therefore, $\lambda=\min\{0,2\}=0$, and

$$\begin{aligned}
&\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{a})\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\big\langle\mathbb{H}_{\mathbf{a}}(:,i),\mathbf{w}\cdot\mathbb{H}_{\mathbf{a}}(:,j)\big\rangle\\
&=\sum_{t=0}^{4-2-1} a_t a_{t+2}\widetilde{w}_{t-0 \bmod 7}\\
&=a_0a_2\widetilde{w}_0+a_1a_3\widetilde{w}_1,\\
&=\frac{\rho_{\mathbf{a};\widetilde{\mathbf{w}},0}(2)}{7}.
\end{aligned} \tag{21}$$

In a similar way, for $i=0$ and $j=j_2=6$, we have $\tau=\min\{1,6\}=1$, therefore, $\lambda=\max\{0,6\}=6=-1 \bmod 7$, and

$$\begin{aligned}
&\sum_{\mathbf{a}\in\mathbf{A}}\sum_{s=0}^{2N-2} f_{i,j,s}(\mathbf{a})\\
&=\sum_{\mathbf{a}\in\mathbf{A}}\big\langle\mathbb{H}_{\mathbf{a}}(:,i),\mathbf{w}\cdot\mathbb{H}_{\mathbf{a}}(:,j)\big\rangle\\
&=\sum_{t=0}^{4-1-1} a_t a_{t+1}\widetilde{w}_{t+1 \bmod 7}\\
&=a_0a_1\widetilde{w}_1+a_1a_2\widetilde{w}_2+a_2a_3\widetilde{w}_3,\\
&=\frac{\rho_{\mathbf{a};\widetilde{\mathbf{w}},1}(1)}{7}.
\end{aligned} \tag{22}$$

On the other hand, when $\mathbf{B}=\{1,-1\}^N$, we have

$$\begin{aligned}
&\sum_{\mathbf{b}\in\mathbf{B}}\sum_{t=0}^{2N-2} f_{i,j,t}(\mathbf{b})\\
&=\sum_{\mathbf{b}\in\{1,-1\}^N}\big\langle\mathbb{H}_{\mathbf{b}}(:,i),\mathbf{w}\cdot\mathbb{H}_{\mathbf{b}}(:,j)\big\rangle\\
&=\sum_{\mathbf{b}\in\{1,-1\}^N}\sum_{t=0}^{N-\tau-1} b_t b_{t+\tau} w_{\lambda-t \bmod 2N-1}\\
&=\sum_{t=0}^{N-\tau-1} w_{\lambda-t \bmod 2N-1}\sum_{\mathbf{b}\in\{1,-1\}^N} b_t b_{t+\tau}\\
&=2^{N-2}\sum_{t=0}^{N-\tau-1} w_{\lambda-t \bmod 2N-1}\left[\sum_{b_t=-1}^{1} b_t\right]\cdot\left[\sum_{b_{t+\tau}=-1}^{1} b_{t+\tau}\right]\\
&=0,
\end{aligned} \tag{23}$$

for any $\tau\neq 0$.

Now by (17), (18) and (23), one has

$$\sum_{\mathbf{a}\in\mathbf{A}}\rho_{\mathbf{a};\widetilde{\mathbf{w}},-\lambda}(\tau)=0, \quad \text{if } \tau\neq 0. \tag{24}$$

Running i and j over $\{0,1,\cdots,2N-2\}$, $-\lambda$ will run over $\{0,1,\cdots,2N-2\}$ as well, it thus follows by ***Definition** 4* that

***Theorem** 1:* The normalized *total squared weighted correlation* (TSWC) lower bound of any $\mathbf{A} \subseteq \{1, -1\}^N$ in (11), is met with equality if and only if $\mathbf{A}$ forms a *weighted-correlation complementary set*. Moreover, the additional condition for the Levenshtein bound in (12) to be met with equality is that all the *non-trivial aperiodic correlation magnitudes* of $\mathbf{A}$ should be identical.

IV. Conclusion

In this paper, we have shown that the Levenshtein bound (for aperiodic correlation) is met with equality if and only if the sequences of the binary sequence set form a *weighted-correlation complementary set* and their non-trivial aperiodic correlations take identical amplitude for all time-shifts. The same conclusion can also be easily shown to hold for sequence sets over complex roots-of-unity whose alphabet phases are more than 2 [15], yet the proof is omitted.

As expected, meeting the Levenshtein bound with equality is more stringent, and the search for the bound-achieving sequence set remains an open problem. For such bound-achieving sequence set, however, we note that:

1) If such a bound-achieving sequence set exists, the corresponding weighting vector, with which those bound-achieving sequences form a *weighted-correlation complementary set*, will be the one which maximizes the Levenshtein bound over all the weighting vectors.
2) For an arbitrary weighting vector $\mathbf{w}^*$ which does not maximize the Levenshtein bound, one cannot find a bound-achieving sequence set whose sequences form a *weighted-correlation complementary set* characterized by $\mathbf{w}^*$.
3) It is noted that the Welch bound cannot be improved for set size of 2[1], because the tightest Levenshtein bound in that case is achieved by the Welch bound when the weighting vector with all equal elements is used [14]. Hence, in the context of 1), such *weighted-correlation complementary set* will degenerate to a *Golay complementary pair*.
4) In the end, if a *Golay complementary pair* of length-N does not exist, the Levenshtein bound for set size of 2 will be loose because the equality cannot be met. For instance, it has been shown that the binary *Golay complementary pair* of $N < 100$ doesn't exist unless $N = 1, 2, 4, 8, 10, 16, 20, 26, 32, 40, 52, 64, 80$ [16].

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[1]In that case, a complementary set will be called a Golay complementary pair [5]. Note that a complementary set has a set size of 2 or more.

References

[1] L. R. Welch, “Lower bounds on the maximum cross-correlation of signals,”*IEEE Trans. Inform. Theory,* vol. IT-20, pp. 397-399, 1974.

[2] J. L. Massey, “On Welch's bound for correlation of a sequence set,” *Proceedings of the 1991 IEEE International Symposium on Information Theory (ISIT'91)*, pp. 385.

[3] J. L. Massey and T. Mittelholzer, “Welch's bound and sequence sets for code-division multiple-access systems,“ *in Sequences II: Methods in Communication, Security, and Computer Science*, R. Capocelli, et al., Eds. New York: Springer- Verlag, 1993.

[4] M. Rupf and J. L. Massey, “Optimum sequence multisets for synchronous code-division multiple-access channels,“ *IEEE Trans. Inform. Theory*, vol. 40, pp. 12611266, July 1994.

[5] M. J. E. Golay, “Complementary series,“ *IRE Trans. On Inform. Theory,* vol. IT-7, pp. 82-87, 1961.

[6] C. Tseng and C. Liu, “Complementary sets of sequences,”*IEEE Trans. Inform. Theory,* vol. IT-18, pp. 644-665, 1972.

[7] P. Z. Fan and M. Darnell, Sequence Design for Communications Applications. New York: Wiley, 1996.

[8] W. H. Mow, “Optimal Sequence Sets Meeting Welch's Lower Bound,“ *Proceedings of the 1995 IEEE International Symposium on Information Theory (ISIT'95)*, pp. 90, 17-22, September 1995, Whistler, British Columbia, Canada.

[9] W. H. Mow, *Sequence design for spread spectrum*, The Chinese University Press (Chinese University of Hong Kong, Shatin, Hong Kong), 1995.

[10] Dilip V. Sarwate, “Meeting the Welch bound with equality, Sequences and Their Applications: Proceedings of SETA '98“, ed. C. Ding, T. Helleseth, and H. Niederreiter, DMTCS Series, Springer-Verlag, 1999

[11] P. Viswanath and V. Anantharam, “Optimal sequences and sum capacity of synchronous CDMA systems,“ *IEEE Trans. Inform. Theory*, vol. 45, no. 6, pp. 1984C1993, Sep. 1999.

[12] S. Ulukus and R. D. Yates, “User Capacity of Asynchronous CDMA Systems With Matched Filter Receivers and Optimum Signature Sequences,“ *IEEE Trans. Inform. Theory*, vol. 50, no. 5, pp. 903-909, May 2004.

[13] P. Cotae, “On the Optimal Sequences and Total Weighted Square Correlation of Synchronous CDMA Systems in Multipath Channels,“ *IEEE Trans. Vehi. Techno.*, vol. 56, no. 4, pp. 2063-2072, Jul. 2007.

[14] V. I. Levenshtein, “New lower bounds on aperiodic crosscorrelation of binary codes,” *IEEE Trans. Inform. Theory,* vol. 45, no. 1, pp.284-288, 1999.

[15] S. Boztas, “New lower bounds on aperiodic cross-correlation of codes over nth roots of unity,” Research Report 13, Department of Mathematics, Royal Melbourne Institudte of Technology, Australia, 1998

[16] P. B. Borwein and R. A. Ferguson, “A complete description of Golay pairs for lengths up to 100,“ *Math. Comp.*, 73, pp. 967-985, Jul. 2003.