Optimal Spectrally-Constrained Sequences

Zilong Liu, Yong Liang Guan, Su Hu, Udaya Parampalli

***Abstract*—A sequence is said to be spectrally-constrained if it has to satisfy a spectral map consisting of several non-contiguous nulled frequency-slots. Such sequences play a key role in emerging spectrally-constrained systems such as cognitive radio and cognitive radar. In this paper, we study two types of spectrally-constrained sequences (SCSs), one with low periodic auto-correlation function (PACF) sidelobe, the other with zero auto-correlation zone (ZACZ). By deriving a correlation lower bound, we show that Type-I SCSs are optimal with *minimum total PACF sidelobe energy* provided that uniform power allocation is applied to all active (non-nulled) frequency-slots. We also propose optimal Type-II SCSs, each having *maximum ZACZ width*, for certain spectral map patterns.**

***Index Terms*—Spectrally-Constrained Sequences (SCS), Cognitive Radio, Cognitive Radar, Zero Auto-Correlation Zone (ZACZ), Periodic auto-correlation function (PACF).**

I. Introduction

The design of sequences with good correlation properties has been a classical research topic for several decades [1]. The study of traditional low correlation sequences has generally been motivated by design problems arising from application areas such as code-division multiple-access, channel estimation, security, etc. In recent years, this topic has been evolving in the research related to dynamic spectrum access systems, which introduce new *spectral constraints* in addition to the requirement of low correlation. Examples of such systems include cognitive radio [2], [3] and cognitive radar [4], which arise due to an increasingly congested spectrum.

The main objective of this paper is to design optimal spectrally constrained sequences (SCSs) for modern applications like those mentioned above. We note that SCSs are different from traditional sequences in that several frequency-slots of a (non-trivial) SCS should be nulled before transmission, whereas the use of traditional sequences assumes the availability of the entire spectral band. An application of SCSs is in overlay cognitive radio communication systems such as Transform Domain Communication Systems (TDCS) [5], where the entire spectral band is mostly occupied by primary users with a provision that secondary users can use certain idle spectral bands[1]. Such situations allow SCSs to play a key role in attaining higher spectral utilization.

Traditional sequences in general are not applicable in spectrally-constrained systems. As shown in [6], for instance, the correlation properties of a traditional zero-correlation zone (ZCZ) sequence set [7]−[10] will be damaged/lost if a spectrum hole constraint is imposed by spectral nulling of sequences. Hence analytical constructions, which have been widely used in traditional sequence design, seem ineffective.

[1]As such, SCSs may also be called cognitive radio sequences in some literature.

There have been several works concerning the design of SCSs from a numerical optimization point of view [11]−[14]. Very recently, a hybrid approach was developed in [6] to generate "quasi-ZCZ" SCSs with advantages of zero out-of-band power leakage, fast generation, and low computation load. As a follow-up of [6], we take a convex optimization approach and present two types of optimal SCSs in this paper. The first type of SCSs have *minimal PACF sidelobe energy* with uniform power allocation, while the second have *maximal width of zero auto-correlation zones (ZACZs)*. Furthermore, the peak-to-average power ratios (PAPRs) of these SCSs are optimized using the Gerchberg-Saxton (GS) algorithm [15].

II. Preliminaries

A. Background of SCS

Assume the entire frequency band is divided into N frequency-slots. Denote by $\mathcal{M} = [m_0, m_1, \cdots, m_{N-1}]^{\mathrm{T}}$ the "frequency-slot marking vector" which gives the status of all the N frequency-slots. Specifically, the value of $m_k (0 \leq k \leq N-1)$ is set to 1 if the kth frequency-slot is active, i.e., available for use; otherwise, $m_k = 0$, indicating that this frequency-slot is occupied/reserved by/for other application(s). Denote by Ω the positions of all nulled frequency-slots, i.e., $\Omega = \{k \mid m_k = 0\}$. In this paper, Ω is also referred to as a "spectrum hole constraint". A spectrum hole constraint is called non-trivial if Ω is a non-empty set, i.e., $|\Omega| > 0$.

Denote by $\mathcal{F}_N = [f_{i,j}]_{i,j=0}^{N-1}$ the (scaled) discrete Fourier transform (DFT) matrix of order N, i.e.,

$$f_{i,j} = \frac{1}{\sqrt{N}} \omega_N^{-ij}, \quad 0 \leq i, j \leq N-1, \tag{1}$$

where $\omega_N = \exp\left(\sqrt{-1}2\pi/N\right)$. Note that $\mathcal{F}_N$ is a unitary matrix, i.e., $\mathcal{F}_N \mathcal{F}_N^{\mathrm{H}} = \mathcal{I}_N$. Thus, the (scaled) inverse discrete Fourier transform (IDFT) matrix of order N is $\mathcal{F}_N^{\mathrm{H}}$.

Let

$$\Omega = \{i_0, i_1, \cdots, i_{k-1}\} \subset \{0, 1, 2, \cdots, N-1\} \tag{2}$$

where $0 \leq i_0 < i_1 < \cdots < i_{k-1} \leq N-1$. For ease of presentation, let

$$\begin{aligned} \overline{\Omega} &= \{0, 1, 2, \cdots, N-1\} \setminus \Omega \\ &= \{j_0, j_1, \cdots, j_{N-k-1}\}, \end{aligned} \tag{3}$$

with $0 \leq j_0 < j_1 < \cdots < j_{N-k-1} \leq N-1$. Consider

$$\mathbf{B} = [B_0, B_1, \cdots, B_\mu, \cdots, B_{N-1}]^{\mathrm{T}},$$

a length-N *frequency-domain* sequence with $B_\mu = 0$ if $\mu \in \Omega$. Based on $\mathbf{B}$, we have the corresponding *time-domain* sequence $\mathbf{b}$ as follows.

$$\mathbf{b} = [b_0, b_1, \cdots, b_i, \cdots, b_{N-1}]^{\mathrm{T}} = \mathcal{F}_N^{\mathrm{H}} \mathbf{B}.$$

Definition 1: $\mathbf{b}$ is called an SCS where the nulled frequency-slots [occupied/reserved by/for other application(s)] are characterized by Ω. In this sense, Ω is called the spectrum hole constraint of $\mathbf{b}$. Whenever the context is clear, we may also refer the frequency-domain sequence $\mathbf{B}$ as an SCS, for the ease of presentation.

Example 1: Let $N = 15$ and $\Omega = \{3, 4, 10, 11\}$. The corresponding spectral map is shown in Fig. 1, where the "lines with arrows" refer to active frequency-slots and the "diamond symbols" refer to nulled slots.

Remark 1: Without loss of generality, let $\|\mathbf{b}\|^2 = \sum_{i=0}^{N-1} |b_i|^2 = N$. It then follows from the Parseval's theorem that $\|\mathbf{B}\|^2 = \sum_{\mu\in\overline{\Omega}} |B_\mu|^2 = N$.

Define the PACF of $\mathbf{b}$ as

$$\theta_{\mathbf{b}}(\tau) := \sum_{i=0}^{N-1} b_i b^*_{i+\tau \bmod N},\ 0\leq\tau\leq N-1. \tag{4}$$

Also, let $\delta_{\mathbf{b}}$ be the maximum out-of-phase PACF magnitude, i.e.,

$$\delta_{\mathbf{b}} = \max\{|\theta_{\mathbf{b}}(\tau)| : 1 \leq |\tau| \leq N-1\}. \tag{5}$$

Definition 2: [PAPR of time-domain sequence]
The PAPR of *time-domain* sequence $\mathbf{b}$ is defined as

$$\mathrm{PAPR}(\mathbf{b}) := \frac{\max\limits_{0\leq n\leq N-1} |b_n|^2}{(1/N)\sum\limits_{n=0}^{N-1} |b_n|^2} = \max_{0\leq n\leq N-1} |b_n|^2. \tag{6}$$

For higher power transmission efficiency, ideally, we wish to find unimodular sequence which has PAPR of 1, meaning that all sequence elements have identical magnitude.

III. Optimal SCSs

In order to present the proposed optimal SCSs, we first express the PACF of $\mathbf{b}$ in the form of its frequency domain sequence $\mathbf{B}$.

$$\begin{aligned}
\theta_{\mathbf{b}}(\tau) &= \sum_{i=0}^{N-1} b_i b^*_{i+\tau} \\
&= \frac{1}{N}\cdot\sum_{i=0}^{N-1}\left(\sum_{\mu\in\overline{\Omega}} B_\mu \omega_N^{-i\mu}\right)\left(\sum_{\nu\in\overline{\Omega}} B^*_\nu \omega_N^{(i+\tau)\nu}\right) \\
&= \frac{1}{N}\sum_{\mu\in\overline{\Omega}}\sum_{\nu\in\overline{\Omega}} B_\mu B^*_\nu \omega_N^{\tau\nu}\cdot\sum_{i=0}^{N-1}\omega_N^{(\nu-\mu)i} \\
&= \sum_{\mu\in\overline{\Omega}}\sum_{\nu\in\overline{\Omega}} B_\mu B^*_\nu \omega_N^{\tau\nu}\delta(\mu-\nu) \\
&= \sum_{\mu\in\overline{\Omega}} |B_\mu|^2 \omega_N^{\tau\mu}.
\end{aligned} \tag{7}$$

Let

$$\begin{aligned}
\boldsymbol{\theta} &= [\theta_{\mathbf{b}}(1), \theta_{\mathbf{b}}(2), \cdots, \theta_{\mathbf{b}}(N-1)]^{\mathrm{T}}, \\
\mathbf{X} &= [X_0, X_1, \cdots, X_{N-k-1}]^{\mathrm{T}} \\
&= \left[|B_{j_0}|^2, |B_{j_1}|^2, \cdots, |B_{j_{N-k-1}}|^2\right]^{\mathrm{T}}
\end{aligned} \tag{8}$$

and

$$\mathbf{W} = \begin{bmatrix}
\omega_N^{j_0} & \omega_N^{j_1} & \cdots & \omega_N^{j_{N-k-1}} \\
\omega_N^{2j_0} & \omega_N^{2j_1} & \cdots & \omega_N^{2j_{N-k-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_N^{(N-1)j_0} & \omega_N^{(N-1)j_1} & \cdots & \omega_N^{(N-1)j_{N-k-1}}
\end{bmatrix}. \tag{9}$$

In addition, let $\mathcal{M} = \mathbf{W}^H\mathbf{W}$. It is easy to show that

$$\mathcal{M} = \begin{bmatrix}
N-1 & -1 & \cdots & -1 \\
-1 & N-1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & N-1
\end{bmatrix}_{(N-k)\times(N-k)}. \tag{10}$$

Based on (7), we have

$$\mathbf{W}\mathbf{X} = \boldsymbol{\theta}. \tag{11}$$

Now we consider the following optimization problem.

Problem 1: [minimization of "total PACF sidelobe energy"]

$$\begin{aligned}
&\min \mathcal{J}(\mathbf{X}), \\
&\text{where } \mathcal{J}(\mathbf{X}) := \boldsymbol{\theta}^H\boldsymbol{\theta} = \mathbf{X}^H\mathcal{M}\mathbf{X}, \\
&\text{subject to } \sum_{p=0}^{N-k-1} X_p = N,\ X_p \geq 0.
\end{aligned} \tag{12}$$

One can show that $\mathcal{J}(\mathbf{X})$ is convex where the minimum solution is given by

$$\mathbf{X}_{\min} = \frac{N}{N-k}\cdot[1, 1, \cdots, 1]^{\mathrm{T}}. \tag{13}$$

Substituting $\mathbf{X}_{\min}$ into $\mathcal{J}(\mathbf{X})$, we have

$$\mathcal{J}(\mathbf{X}) \geq \mathcal{J}(\mathbf{X}_{\min}) = N^2\cdot\gamma^2, \tag{14}$$

where

$$\gamma = \sqrt{\frac{N}{N-k} - 1} = \sqrt{\frac{k}{N-k}}. \tag{15}$$

Therefore,

$$(N-1)\delta^2_{\mathbf{b}} \geq N^2\cdot\gamma^2. \tag{16}$$

Consequently, We have the following PACF lower bound of $\mathbf{b}$.

Theorem 1:

$$\delta_{\mathbf{b}} \geq \frac{N}{\sqrt{N-1}}\cdot\gamma. \tag{17}$$

With respect to the minimal $\mathcal{J}(\mathbf{X})$ (i.e., the minimum *total PACF sidelobe energy*) in (14), we give the following definition.

Definition 3: [Optimal Type-I SCS] A sequence is said to optimal type-I SCS if (1): it has a non-trivial spectrum hole constraint (i.e., $k > 0$) and (2): uniform power allocation is assigned to all active frequency-slots [as indicated in (13)].

As opposed to optimal Type-I SCS which has low (but non-zero) autocorrelation over all non-zero time-shifts, we next

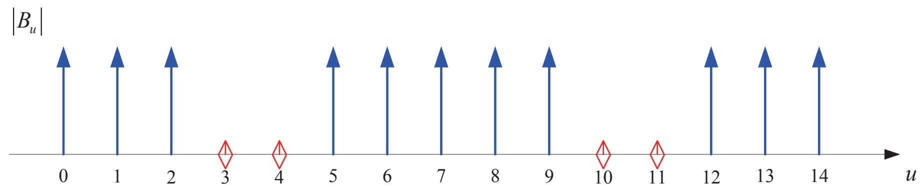


Fig. 1: SCS in frequency domain, where the "lines with arrows" refer to active frequency-slots and the "diamond symbols" refer to nulled slots.

introduce optimal Type-II SCS which has zero PACF over a window centered around the in-phase position. Before this, we first define ZACZ as follows.

Definition 4: [ZACZ] Sequence $\mathbf{b}$ is said to have ZACZ of Z if $\theta_{\mathbf{b}}(\tau)=0$ for any $\tau \in \{1,2,\cdots,Z-1\}$, where Z is an integer in the range of $[1,N]$. In particular, $\mathbf{b}$ is said to be a perfect sequence if it has ZACZ of $Z=N$.

We are now ready to define optimal Type-II SCS.

Definition 5: [Optimal Type-II SCS] A sequence is said to optimal type-II SCS if (1): it has a non-trivial spectrum hole constraint (i.e., $k>0$) and (2) its ZACZ is maximized.

Furthermore, we define

$$\mathbf{W}_Z =: \begin{bmatrix} \omega_N^{j_0} & \omega_N^{j_1} & \cdots & \omega_N^{j_{N-k-1}} \\ \omega_N^{2j_0} & \omega_N^{2j_1} & \cdots & \omega_N^{2j_{N-k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(Z-1)j_0} & \omega_N^{(Z-1)j_1} & \cdots & \omega_N^{(Z-1)j_{N-k-1}} \end{bmatrix} \tag{18}$$

and

$$\begin{aligned} \mathcal{M}_Z &:= \mathbf{W}_Z^H \mathbf{W}_Z, \\ \mathcal{J}_Z(\mathbf{X}) &:= \mathbf{X}^H \mathcal{M}_Z \mathbf{X}. \end{aligned} \tag{19}$$

Clearly, $\mathbf{W}_Z, \mathcal{M}_Z, \mathcal{J}_Z(\mathbf{X})$ reduce to the earlier defined $\mathbf{W}, \mathcal{M}, \mathcal{J}(\mathbf{X})$, respectively, if $Z=N$.

To design optimal Type-II SCS, we consider the optimization problem shown below.

Problem 2: [Maximization of ZACZ]

$$\begin{aligned} &\max Z, \\ \text{subject to } &(1): \mathcal{J}_Z(\mathbf{X})=0, \\ &(2): \sum_{p=0}^{N-k-1} X_p = N,\ X_p \geq 0. \end{aligned} \tag{20}$$

The optimization of *Problem 2* in general is challenging. However, we show below that *Problem 2* is solvable if all the active frequency-slots are uniformly distributed.

Theorem 2: Suppose $\frac{N}{N-k}$ is an integer, i.e., $(N-k)|N$. Suppose further that

$$\overline{\Omega} = \left\{ \frac{N}{N-k}x + y : \begin{array}{c} 0 \leq x \leq N-k-1, \\ 0 \leq y \leq N-1 \end{array} \right\}. \tag{21}$$

Then, $\mathbf{b}$ is an optimal Type-II SCS with maximal ZACZ of $Z=N-k$.

Proof: One can easily show $\mathcal{M}_{N-k}$ and $\mathcal{M}_{N-k+1}$ in (22). Similar to the proof for minimal $\mathcal{J}(\mathbf{X})$, we conclude that $\mathcal{J}_{N-k}(\mathbf{X})$ and $\mathcal{J}_{N-k+1}(\mathbf{X})$ are convex where the minimum solution is given by $\mathbf{X}_{\min}$ shown in (13). Therefore, we have

$$\mathcal{J}_{N-k}(\mathbf{X})=0 \text{ and } \mathcal{J}_{N-k+1}(\mathbf{X})=N^2,$$

showing that

$$|\theta_{\mathbf{b}}(\tau)| = \begin{cases} 0, & \tau \in \{0,1,\cdots,N-k-1\}; \\ N, & \tau = N-k. \end{cases}$$

Thus, we complete the proof. ■

To further optimize the PAPR of $\mathbf{b}$, we need to solve the following problem [6].

Problem 3: [Minimization of PAPR]

$$\begin{aligned} \min_{\mathbf{B},\mathbf{p}} \ & \mathcal{J}_2(\mathbf{B},\mathbf{p}) = \left\| \mathcal{F}_N^{\mathrm{H}} \mathbf{B} - \mathbf{p} \right\|_2^2, \\ \text{s.t.} \ & (1): B_k = 0 \text{ if } k \in \Omega; \\ & (2): |p_k| = 1, \quad k=0,1,\cdots,N-1, \end{aligned} \tag{23}$$

where $\mathbf{p}=[p_0,p_1,\cdots,p_{N-1}]^{\mathrm{T}}$.

As shown in [6], the above problem can be efficiently solved by the GS algorithm [15]. Consequently, designing an SCS $\mathbf{B}$ can now be summarized in the following two steps:

1) Solving *Problem 1* (or *Problem 2*) to get the magnitudes of $\mathbf{B}$;
2) Solving *Problem 3* by applying the GS algorithm to obtain the locally optimal phases of $\mathbf{B}$.

Example 2: Let $N=64$ and $\mathcal{M}=[\mathbf{1}_{16},\mathbf{0}_{16},\mathbf{1}_8,\mathbf{0}_{16},\mathbf{1}_8]^{\mathrm{T}}$ be the frequency-slot marking vector, where $\mathbf{1}_m$ and $\mathbf{0}_n$ denote a length-m all-1 row-vector and a length-n all-0 row-vector, respectively. Thus, we have $\gamma=\sqrt{\frac{k}{N-k}}=1$ and

$$\Omega = \{16,17,\cdots,31\} \cup \{40,41,\cdots,47\}. \tag{24}$$

Based on the solutions of *Problem 1* and by invoking the GS algorithm, we obtain and show in Table I the optimal SCS $\mathbf{B}_0$ which has uniform power allocation. The corresponding time-domain SCS $\mathbf{b}_0$ has PAPR of 1.7089. In particular, its corresponding normalized *total PACF sidelobe energy* is the minimal value of 1 (because $\gamma^2 = 1$). To visualize this

$$\begin{aligned} \mathcal{M}_{N-k} &= \begin{bmatrix} N-k-1 & -1 & \cdots & -1 \\ -1 & N-k-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-k-1 \end{bmatrix}_{(N-k-1)\times(N-k-1)}, \\ \mathcal{M}_{N-k+1} &= \begin{bmatrix} N-k & 0 & \cdots & 0 \\ 0 & N-k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N-k \end{bmatrix}_{(N-k)\times(N-k)}, \end{aligned} \tag{22}$$

TABLE I: Two optimal SCSs with *minimum total PACF sidelobe energy* and *maximum ZACZ*, respectively, where $i=\sqrt{-1}$.

$\mathbf{B}_0$	$-0.4324-1.3465i, -0.2255-1.3961i, -1.0922-0.8984i, -1.4135-0.0457i, -1.3148-0.5209i, -1.3106-0.5313i, 1.3801-0.3087i, -1.1687+0.7964i, -1.4019+0.1865i, -1.2240-0.7084i, 1.2477+0.6658i, 1.3569+0.3984i, -0.9180-1.0758i-1.1911-0.7624i, 0.9546+1.0434i, 0.6647-1.2483i, \mathbf{0}_{16}, -0.5364-1.3085i, -0.8043+1.1632i, 1.4095+0.1148i, -1.4073+0.1392i, 1.3537-0.4093i, -1.0283+0.9709i, -0.2843-1.3853i, 1.0680+0.9270i, \mathbf{0}_{16}, 0.7405-1.2048i, 1.1653-0.8013i, 1.3645-0.3718i, -1.2869+0.5864i, 0.1593+1.4052i, -0.9641+1.0347i, -1.3319+0.4755i, 0.7094-1.2234i.$
$\mathbf{B}_1$	$0.8354-0.4419i, 0.4507-0.7395i, 0.6977+0.8511i, 0.6704-0.8280i, -0.2354-0.8341i, 0.9046-0.1148i, 0.6189+1.1806i, 0, -0.7975+1.0454i, 0.7347+0.1366i, 0.9514-0.8254i, 0, -1.4286-0.2333i, -0.6012+0.2209i, -0.8961+0.2288i, 0.7564-0.9883i, 0.0527-0.6594i, 1.1152+0.6213i, 0, -0.2950+1.4713i, -0.1147-0.4745i, -0.2566-0.7490i, 0.2942-1.4409i, -0.2822+0.4484i, -0.5808+0.2074i, 1.4343-0.6161i, -0.2585+0.4827i, 0, -1.7014-0.1517i, 0.0811+0.1234i, -0.0545-0.1182i, -0.1640-1.7226i, 0, 0, -0.1058-1.7231i, 0, -0.3483-0.1322i, -0.6416+1.5154i, -0.4063+0.1242i, 0.4749+0.2436i, 0.4273+1.4531i, -0.6955+0.0092i, -0.2408+0.6294i, 0.9763-0.8451i, -0.9534+0.5341i, 0, 1.3749+0.0700i, 0.5330-0.7733i, 0.4667-0.6062i, 0.7960-0.7676i, -0.8782-0.7033i, -0.6209-0.5125i, 0.7531+0.6085i, -0.9090-0.7709i, 0.8259-0.2346i, 0.6788+0.5813i, 0.1526-1.1703i.$

sequence, we plot its magnitudes, their corresponding time-domain sequence magnitudes, and its normalized time-domain PACF in Fig. 2-a.

Although a general solution of *Problem 2* may be intractable, our computer search indicates that optimal Type-II SCSs exist for other spectral maps, in addition to that specified in *Theorem 2*. We show this by the following example.

Example 3: Let $N=57$ and

$$\Omega=\{7,11,18,27,32,33,35,45\}. \tag{25}$$

Based on the solutions of *Problem 2* and by provoking the GS algorithm, we obtain the optimal SCS $\mathbf{B}_1$ which has maximum ZACZ of $Z=18$ as shown in Table I. One can see that the corresponding time-domain SCS $\mathbf{b}_1$ has PAPR of 1.0576; One can also visualize this sequence in Fig. 2-b.

IV. Conclusions and Future Directions

Two types of optimal spectrally-constrained sequences (SCSs) with good periodic autocorrelation functions (PACFs) are proposed. Specifically, a Type-I optimal SCS has *minimum total PACF sidelobe energy*, while a Type-II has *maximum zero auto-correlation zone (ZACZ)*. We have made the following main contributions:

1) By the derived lower bound in (17), we show that every SCS with non-contiguous nulled frequency-slots should have low (but nonzero) PACF sidelobes over all out-of-phase time-shifts. Conversely, a perfect sequence with zero PACF sidelobe is possible if and only if no frequency is nulled.[2];
2) Our derivation of (17) shows that the *total PACF sidelobe energy* of a Type-I SCS is minimized (thus, *optimal*) if and only if uniform power allocation is applied to all active (non-nulled) frequency-slots;
3) We present in *Theorem 2* a class of optimal Type-II SCSs, each having uniformly distributed active frequency-slots.

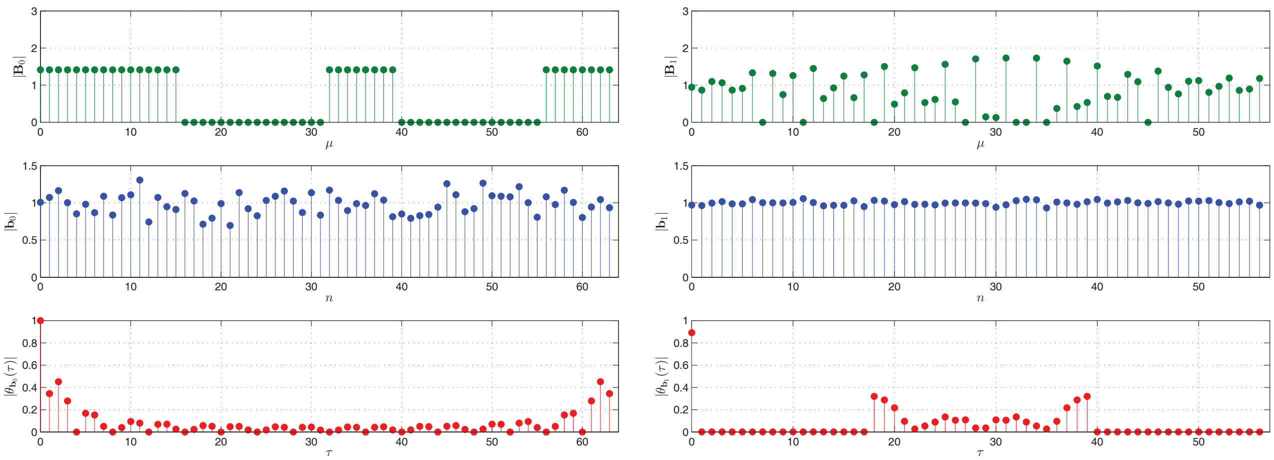
Finally, we note that the design of SCSs with good correlation properties is a largely open direction and there are many problems remaining unsolved. We summarize below several open problems as future lines of this research.

Q1: How to design strongly optimal Type-I SCSs, each having (1): maximum PACF magnitude meeting the correlation lower bound in (17) [other than the lower bound on *total PACF sidelobe energy* in (14)] with equality and (2): minimum PAPR of 1?

Q2: Can the derived correlation lower bound in (17) be generalized to the maximum periodic auto- and cross-correlation magnitudes of Type-I SCS set with two or more sequences? How to construct optimal Type-I SCSs approaching this generalized bound, numerically or analytically?

Q3: What are the necessary and sufficient conditions, especially on the patterns of spectral maps, for the existence of Type-II SCSs with ZACZ? Can we extend these conditions to SCSs with zero auto- and cross- correlation

[2]In this case, γ in (15) is equal to zero.



(a) Optimal Type-I SCS $\mathbf{B}_0$ with minimal total PACF sidelobe energy

(b) Optimal Type-II SCS $\mathbf{B}_1$ with maximal ZACZ

Fig. 2: Two optimal SCSs in *Example 1* and *Example 2*, respectively.

zone properties?

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