A Frequency-Domain Approach to Tightening the Generalized Levenshtein Bound

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***Abstract*—Generalized Levenshtein bound (GLB) is a lower bound on the maximum aperiodic correlation sum of quasi-complementary sequence set (QCSS) which refers to a set of two-dimensional matrices with low non-trivial aperiodic auto- and cross- correlation sums. GLB is an *indefinite* fractional quadratic function of a "simplex" weight vector $\mathbf{w}$ and three additional parameters associated with QCSS. We present a novel approach to *analytically* conduct fractional quadratic optimization for the tightening of the GLB. Our key idea is to apply the frequency-domain decomposition of the relevant circulant matrix (i.e., the numerator term of GLB) to convert the *non-convex* problem into a convex one. We derive a new weight vector which asymptotically leads to a tighter GLB (over the Welch bound) for all possible (K, M) cases, where K, M denote the set size, the number of channels, of QCSS, respectively.**

***Index Terms*—Fractional quadratic function, Welch Bound, generalized Levenshtein bound (GLB), perfect complementary sequence set (PCSS), quasi-complementary sequence set (QCSS)**

I. Introduction

A set of two-dimensional matrices is called a quasi-complementary sequence set (QCSS) if it has low non-trivial *aperiodic* auto- and cross- correlation sums. QCSSs have potential applications in enabling multicarrier code-division multiple-access (MC-CDMA) communications with low multiuser interference [1]. Compared with its counterpart perfect-complementary sequence set (PCSS) [2], QCSS is capable of supporting more MC-CDMA users for given number of subcarriers [3], [4]. In this paper, a complementary sequence is also called a complementary matrix, and vice versa.

A correlation lower bound of QCSS was first derived in Welch's 1974 paper [5] which is restated as

$$\delta_{\max}^2 \geq M^2N^2\frac{\frac{K}{M}-1}{K(2N-1)-1}, \tag{1}$$

where δ stands for the maximal magnitude of correlation sums, K, M, N denotes the set size K, the number of channels $M \geq 2$, and the row sequence length N, of QCSS, respectively.

In 2014, Liu, Guan and Mow derived generalized Levenshtein bound (GLB), a correlation lower bound for QCSS [6, *Theorem 1*]. For more information on Levenshtein bound, the reader is referred to [7], [8], [9]. It was shown that GLB includes Welch bound as a special case and is in general tighter except for an unknown zone of K/M. Specifically, in its bounding equation, GLB is a fractional quadratic function of the "simplex" weight vector $\mathbf{w}$. A necessary condition (shown in [6, *Theorem 2*]) for the GLB to be tighter than the Welch bound is that $K \geq \overline{K}+1$, where

$$\overline{K} \triangleq \left\lfloor 4(MN-1)N\sin^2\frac{\pi}{2(2N-1)}\right\rfloor. \tag{2}$$

Although a "step-function" weight vector was adopted in [6, (34)], it only leads to a tighter GLB for $K \geq 3M+1$. As a matter of fact, the tightness of GLB remains unknown for

$$\frac{\overline{K}}{M} < \frac{K}{M} < 3+\frac{1}{M},$$

when N is sufficiently large[1].

The principal objective of this paper is to *analytically* conduct fractional quadratic optimization to tighten the GLB for *all* $K \geq \overline{K}+1$ (instead of *some*). For this, we are to find a (locally) optimal weight vector which is used in the bounding equation. A similar research problem was raised in [7] for traditional binary sequences (i.e., non-QCSS with $M=1$). See [10] for more details. The optimization of GLB on QCSS (with $M \geq 2$), however, is more challenging because an analytical solution to a non-convex GLB (in terms of weight vector $\mathbf{w}$) for *all* possible cases of (K, M) is in general intractable.

We adopt a frequency-domain optimization approach in Section III to minimize the (non-convex) fractional quadratic function of GLB. This is achieved by properly exploiting the specific structure of the circulant quadratic matrix in the numerator of the fractional quadratic term of GLB. Following this optimization approach, we find a new weight vector which leads to a tighter GLB for *all* (K, M) cases satisfying $K \geq \overline{K}+1$ and $M \geq 2$, asymptotically (in N). Our finding shows that the condition of $K \geq \overline{K}+1$, shown in [6, Theorem 2], is not only necessary but also sufficient, as N tends to infinity.

II. Preliminaries

In this section, we first present some necessary notations and define QCSS. Then, we give a brief review of GLB.

[1] See Subsection III-A for details of the K/M zone in which the tightness of GLB remains unknown.

A. Introduction to QCSS

For two complex-valued sequences $\mathbf{a} = [a_0, a_1, \cdots, a_{N-1}]$ and $\mathbf{b} = [b_0, b_1, \cdots, b_{N-1}]$, their aperiodic correlation function at time-shift τ is defined as

$$\rho_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum\limits_{t=0}^{N-1-\tau} a_t b_{t+\tau}^*, & 0 \leq \tau \leq (N-1); \\ \sum\limits_{t=0}^{N-1+\tau} a_{t-\tau} b_t^*, & -(N-1) \leq \tau \leq -1; \\ 0, & |\tau| \geq N. \end{cases} \tag{3}$$

When $\mathbf{a} \neq \mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called the aperiodic cross-correlation function (ACCF); otherwise, it is called the aperiodic auto-correlation function (AACF). For simplicity, the AACF of $\mathbf{a}$ is denoted by $\rho_{\mathbf{a}}(\tau)$.

Let $\mathcal{C} = \{\mathbf{C}^0, \mathbf{C}^1, \cdots, \mathbf{C}^{K-1}\}$ be a set of K matrices, each of order $M \times N$ (where $M \geq 2$), i.e.,

$$\mathbf{C}^\nu = \begin{bmatrix} \mathbf{c}_0^\nu \\ \mathbf{c}_1^\nu \\ \vdots \\ \mathbf{c}_{M-1}^\nu \end{bmatrix}_{M \times N} \tag{4}$$

where $\mathbf{c}_m^\nu = [c_{m,0}^\nu, c_{m,1}^\nu, \cdots, c_{m,N-1}^\nu] (0 \leq m \leq M-1)$ and $0 \leq \nu \leq K-1$. Define the "aperiodic correlation sum" of matrices $\mathbf{C}^\mu$ and $\mathbf{C}^\nu$ as follows,

$$\rho_{\mathbf{C}^\mu,\mathbf{C}^\nu}(\tau) = \sum_{m=0}^{M-1} \rho_{\mathbf{c}_m^\mu,\mathbf{c}_m^\nu}(\tau), \quad 0 \leq \mu, \nu \leq K-1. \tag{5}$$

Also, define the aperiodic auto-correlation tolerance δ_a and the aperiodic cross-correlation tolerance δ_c of $\mathcal{C}$ as

$$\delta_{\mathrm{a}} \triangleq \max \left\{ \left| \rho_{\mathbf{C}^\mu,\mathbf{C}^\mu}(\tau) \right| : \begin{array}{c} 0 < \tau \leq N-1, \\ 0 \leq \mu \leq K-1. \end{array} \right\},$$
$$\delta_{\mathrm{c}} \triangleq \max \left\{ \left| \rho_{\mathbf{C}^\mu,\mathbf{C}^\nu}(\tau) \right| : \begin{array}{c} 0 \leq \tau \leq N-1, \\ \mu \neq \nu, 0 \leq \mu, \nu \leq K-1. \end{array} \right\}$$

respectively. Moreover, define the aperiodic tolerance (also called the "maximum aperiodic correlation magnitude") of $\mathcal{C}$ as $\delta_{\max} \triangleq \max\{\delta_{\mathrm{a}}, \delta_{\mathrm{c}}\}$. When $\delta_{\max} = 0$, $\mathcal{C}$ is called a *perfect complementary sequence set* (PCSS); otherwise, it is called a *quasi-complementary sequence set* (QCSS)[2].

Note that the transmission of a PCSS or a QCSS requires a multi-channel system. Specifically, every matrix in a PCSS (or a QCSS) needs $M \geq 2$ non-interfering channels for the separate transmission of M row sequences. This is different from the traditional single-channel sequences with $M = 1$ only.

B. Review of GLB

Let $\mathbf{w} = [w_0, w_1, \cdots, w_{2N-2}]^{\mathrm{T}}$ be a "simplex" weight vector which is constrained by

$$w_i \geq 0, \quad i = 0, 1, \cdots, 2N-2, \quad \text{and} \quad \sum_{i=0}^{2N-2} w_i = 1. \tag{6}$$

[2]QCSS can also be defined with respect to the "periodic correlation sums". The interested reader may refer to [3].

Define a quadratic function

$$\begin{aligned} Q(\mathbf{w}, a) &\triangleq \mathbf{w}^{\mathrm{T}} \mathbf{Q}_a \mathbf{w} \\ &= a \sum_{i=0}^{2N-2} w_i^2 + \sum_{s,t=0}^{2N-2} \tau_{s,t,N} w_s w_t, \end{aligned} \tag{7}$$

where $\mathbf{Q}_a$ is a $(2N-1) \times (2N-1)$ circulant matrix with all of its diagonal entries equal to a, and its off-diagonal entries $\mathbf{Q}_a(s,t) = \tau_{s,t,N}$, where $s \neq t$ and

$$0 \leq \tau_{s,t,N} \triangleq \min\{|t-s|, 2N-1-|t-s|\} \leq N-1. \tag{8}$$

The GLB for QCSS over complex roots of unity in [6] is shown below.

Lemma 1:

$$\delta_{\max}^2 \geq M \left[N - \frac{Q\left(\mathbf{w}, \frac{N(MN-1)}{K}\right)}{1 - \frac{1}{K} \sum\limits_{i=0}^{2N-2} w_i^2} \right]. \tag{9}$$

A weaker simplified version of (9) is given below.

$$\delta_{\max}^2 \geq M \left[N - Q\left(\mathbf{w}, \frac{MN^2}{K}\right) \right]. \tag{10}$$

Remark 1: Setting $\mathbf{w} = \frac{1}{2N-1}(1, 1, \cdots, 1)$, the GLB reduces to the Welch bound for QCSS in (1).

Remark 2: [6, *Theorem 2*] For the GLB to be tighter than the corresponding Welch bound, it is *necessary* that $K \geq \overline{K} + 1$, where $\overline{K}$ is defined in (2).

Remark 3: [6, *Corollary 1*] Applying the weight vector $\mathbf{w}$ with

$$w_i = \begin{cases} \frac{1}{m}, & i \in \{0, 1, \cdots, m-1\}; \\ 0, & i \in \{m, m+1, \cdots, 2N-2\}; \end{cases} \tag{11}$$

where $1 \leq m \leq N$, to (9), we have

$$\delta_{\max}^2 \geq \max_{1 \leq m \leq N} \frac{3MNKm - 3M^2N^2 - MK(m^2-1)}{3(mK-1)}. \tag{12}$$

The lower bound in (12) is tighter than the Welch bound for QCSS in (1) if one of the two following conditions is fulfilled:

(1): $3M + 1 \leq K \leq 4M - 1$, $M \geq 2$ and

$$N \geq \left\lfloor \frac{K - 1 + \sqrt{-3K^2 + (12M-6)K + 12M + 1}}{2(K - 3M)} \right\rfloor + 1; \tag{13}$$

(2): $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

III. Fractional Quadratic Programming of GLB

A. Motivation

The necessary condition in *Remark 2* implies that for a given M, N, the Welch bound for QCSS cannot be improved if $K \leq \overline{K}$, where $\overline{K}$ is defined in (2). On the other hand, the weight vector in (11) can only lead to a tighter GLB for $K \geq 3M + 1$. Because of this, the tightness of GLB is unknown in the following ambiguous zone.

$$\frac{\overline{K}}{M} < \frac{K}{M} < 3 + \frac{1}{M}. \tag{14}$$

We are therefore interested in finding a weight vector which is capable of optimizing and tightening the GLB for *all* (rather than *some*) $K \geq \overline{K}+1$. However, the optimization of GLB in (9) is challenging because its fractional quadratic term (in terms of $\mathbf{w}$) is indefinite. More specifically, the quadratic term $Q\left(\mathbf{w}, \frac{N(MN-1)}{K}\right)$ in the numerator is indefinite as some eigenvalues of the corresponding circulant matrix are negative when $K \geq \overline{K}+1$ [6, Appendix B]. According to [11], GLB may be classified as a standard FQP (StFQP) as the feasible set is the standard simplex. To the best of the authors' knowledge, Preisig pioneered an iterative algorithm for which convergence to a KKT point (not necessarily a local minimizer) of the StFQP can be proved [12]. Two algorithms for StFQP based on semidefinite programming (SDP) relaxations are presented in [11], yet the optimalities of the resultant solutions are unknown. As a matter of fact, the algorithms developed in [11], [12] may only be feasible for medium-scaled StFQP with $N \leq 200$. In contrast, we target at an *analytical* solution (as opposed to a numerical solution) which is applicable to GLB with an arbitrarily large sequence length (e.g., $N > 1000$). Thus, the techniques used in [11], [12] may not be useful for the specific StFQP problem considered in this paper.

In the sequel, we introduce a frequency-domain optimization approach which finds a local minimizer (i.e., a weight vector) of the GLB. We show that the obtained weight vector leads to a tighter GLB for *all* $K \geq \overline{K}+1$ and $M \geq 2$, asymptotically.

B. Proposed Weight Vector

To tighten the GLB in (9), we adopt a novel optimization approach in this subsection, motivated by the observation that any circulant matrix [e.g., $\mathbf{Q}_a$ in (7) which forms a part of the GLB quadratic function in (9)] can be decomposed in the frequency domain.

Define $\xi_L = \exp\left(-\sqrt{-1}2\pi/L\right)$ and the L-point discrete Fourier transform (DFT) matrix as

$$\mathbf{F}_L = [f_{m,n}]_{m,n=0}^{L-1}, \text{ where } f_{m,n} = \xi_L^{mn}. \tag{15}$$

Denote by $\mathbf{q}$ the first column vector of $\mathbf{Q}_a$ in (7), i.e.,

$$\mathbf{q} = [a, 1, 2, \cdots, N-1, N-1, \cdots, 2, 1]^{\mathrm{T}}. \tag{16}$$

Let

$$\mathbf{v} = \mathbf{F}_{2N-1}\mathbf{w} = [v_0, v_1, \cdots, v_{2N-2}]^{\mathrm{T}}. \tag{17}$$

It is noted that $v_0 = \sum_{i=0}^{2N-2} w_i = 1$. By [13], the circulant matrix $\mathbf{Q}_a$ defined in (7) can be expressed as

$$\mathbf{Q}_a = \frac{1}{2N-1}\mathbf{F}_{2N-1}^{\mathrm{H}}\mathrm{diag}(\boldsymbol{\lambda})\mathbf{F}_{2N-1}, \tag{18}$$

where

$$\boldsymbol{\lambda} = \mathbf{F}_{2N-1}\mathbf{q} = [\lambda_0, \lambda_1, \cdots, \lambda_{2N-2}]^{\mathrm{T}}, \tag{19}$$

and $\mathrm{diag}(\boldsymbol{\lambda})$ is the matrix with $\boldsymbol{\lambda}$ being the diagonal vector and zero for all the non-diagonal matrix entries. Consequently [14, Theorem 3.1],

$$Q(\mathbf{w}, a) = \frac{1}{2N-1}\sum_{l=0}^{2N-2} \lambda_l \left|v_l\right|^2. \tag{20}$$

Similarly,

$$\sum_{i=0}^{2N-2} w_i^2 = \frac{1}{2N-1}\sum_{l=0}^{2N-2} \left|v_l\right|^2. \tag{21}$$

By [6, Appendix B], we have

$$\lambda_0 = a + (N-1)N, \tag{22}$$

and

$$\lambda_l = a - \frac{1-(-1)^l\cos\frac{\pi l}{2N-1}}{2\sin^2\frac{\pi l}{2N-1}}, \tag{23}$$

for $l = 1, \cdots, 2N-2$. Note that $\lambda_l = \lambda_{2N-1-l}$ for $l \in \{1, 2, \cdots, N-1\}$. Moreover, we remark that

$$\lambda_l > \lambda_1, \quad \text{for } 2 \leq l \leq N-1. \tag{24}$$

This is because

1) if l is odd:

$$\lambda_l - \lambda_1 = \frac{\sin^2\frac{\pi l}{2(2N-1)} - \sin^2\frac{\pi}{2(2N-1)}}{4\sin^2\frac{\pi}{2(2N-1)}\sin^2\frac{\pi l}{2(2N-1)}} > 0. \tag{25}$$

2) if l is even:

$$\lambda_l - \lambda_1 = \frac{\cos\frac{\pi l}{2N-1} + \cos\frac{\pi}{2N-1}}{8\sin^2\frac{\pi}{2(2N-1)}\cos^2\frac{\pi l}{2(2N-1)}} > 0. \tag{26}$$

To maximize the GLB in (9), it is equivalent to consider the following optimization problem.

Problem 1:

$$\begin{aligned}&\min_{\mathbf{v}} \frac{\lambda_0 + \sum\limits_{l=1}^{2N-2}\lambda_l|v_l|^2}{2N-1-\frac{1}{K}-\frac{1}{K}\sum\limits_{l=1}^{2N-2}|v_l|^2},\\ &\text{subject to } \mathbf{w} = \frac{1}{2N-1}\mathbf{F}_{2N-1}^{\mathrm{H}}\mathbf{v} \geq 0.\end{aligned} \tag{27}$$

Since $\mathbf{w}$ is real-valued, $\mathbf{v}$ is conjugate symmetric, i.e., $v_l = v_{2N-1-l}^*$ for $l = 1, 2, \cdots, 2N-2$. Having this in mind, we define

$$r^2 = \sum_{l=1}^{N-1}|v_l|^2 = \sum_{l=N}^{2N-2}|v_l|^2. \tag{28}$$

Taking advantage of the fact that $\lambda_1 = \lambda_{2N-2}$ are strictly smaller than other λ_l's with nonzero l as shown in (24), we have

$$\sum_{l=1}^{2N-2}\lambda_l|v_l|^2 = 2\lambda_1 r^2 + \sum_{l=2}^{2N-3}(\lambda_l - \lambda_1)|v_l|^2 \geq 2\lambda_1 r^2, \tag{29}$$

where the equality is achieved if and only if $v_l = 0$ for $l = 2, 3, \cdots, 2N-3$. Inspired by this observation, we relax the non-negativity constraint on $\mathbf{w}$, i.e., some negative w_i's may be allowed (but the sum of all elements of $\mathbf{w}$ must still be equal to 1). With this, the optimization problem in (27) can be translated to

$$\begin{aligned}&\min_{r} \min_{\sum_{l=1}^{2N-2}|v_l|^2=2r^2} \frac{\lambda_0+\sum_{l=1}^{2N-2}\lambda_l|v_l|^2}{2N-1-\frac{1+2r^2}{K}},\\ &=\min_{r} \frac{\lambda_0+2\lambda_1 r^2}{2N-1-\frac{1}{K}-\frac{2r^2}{K}},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\lambda_0&=\frac{N(MN-1)}{K}+N(N-1),\\ \lambda_1&=\frac{N(MN-1)}{K}-\frac{1}{4\sin^2\frac{\pi}{2(2N-1)}}.\end{aligned} \tag{31}$$

From now on, we adopt the setting of $v_1 = v_{2N-2}^* = r\exp\left(\sqrt{-1}\theta\right)$ and $v_l = 0$, for $l = 2, 3, \cdots, 2N-3$, where r, θ denote the magnitude and phase of v_1, respectively. Since $\mathbf{w} = \frac{1}{2N-1}\mathbf{F}_{2N-1}^{\mathrm{H}}\mathbf{v}$, we obtain $\mathbf{w}$ with

$$w_i=\frac{1}{2N-1}\left[1+2r\cos\left(\theta+\frac{2\pi i}{2N-1}\right)\right],\ 0\le i\le 2N-2. \tag{32}$$

To optimize the fractional function in (30), we have the following lemma.

Lemma 2: The fractional function $\frac{\lambda_0+2\lambda_1 r^2}{2N-1-\frac{1}{K}-\frac{2r^2}{K}}$ in terms of r^2 in (30) is

Case 1: monotonically decreasing in r^2 if $K \geq \overline{K}+1$ and $\frac{\lambda_0}{|\lambda_1|} < (2N-1)K-1$;

Case 2: monotonically increasing in r^2 if $K \leq \overline{K}$, or $K \geq \overline{K}+1$ and $\frac{\lambda_0}{|\lambda_1|} \geq (2N-1)K-1$.

Proof: To prove Case 1, we first show that $\lambda_1 < 0$ if and only if

$$K\ge\overline{K}+1=\left\lfloor 4(MN-1)N\sin^2\frac{\pi}{2(2N-1)}\right\rfloor+1,$$

where $\overline{K}$ is defined in (2). For ease of analysis, we write

$$4(MN-1)N\sin^2\frac{\pi}{2(2N-1)}=n+\epsilon, \tag{33}$$

where n is a positive integer and $0 \leq \epsilon < 1$. Thus, $\overline{K}+1 = n+1$. Consequently, we have

$$\begin{aligned}\lambda_1&=a-\frac{1}{4\sin^2\frac{\pi}{2(2N-1)}}\\ &=\frac{N(MN-1)}{K}\left[1-\frac{K}{4N(MN-1)\sin^2\frac{\pi}{2(2N-1)}}\right]\\ &\le\frac{N(MN-1)}{K}\left(1-\frac{n+1}{n+\epsilon}\right)\\ &<0,\end{aligned} \tag{34}$$

with which the proof of Case 1 follows. The proof of Case 2 can be easily obtained by following a similar argument. ■

For Case 2 of *Lemma 2*, it can be readily shown that the minimum of the fractional function $\frac{\lambda_0+2\lambda_1 r^2}{2N-1-\frac{1}{K}-\frac{2r^2}{K}}$ in (30) is achieved at $r = 0$. Thus, the weight vector in (32) reduces to

$$\mathbf{w}=\frac{1}{2N-1}\cdot[1,1,\cdots,1]^{\mathrm{T}}, \tag{35}$$

where the corresponding GLB reduces to the Welch bound in (1).

Next, let us focus on the application of Case 1 for GLB tightening. In this case, we wish to know the upper bound of r^2 in order to minimize the fractional function of r^2 in (30). Coming back to the constraint of $\mathbf{w}$ given in (6), r and θ should satisfy

$$1+2r\min_{i=0,1,\cdots,2N-2}\cos\left(\theta+\frac{2\pi i}{2N-1}\right)\ge 0. \tag{36}$$

Thus,

$$0\le r\le \underbrace{\max_{\theta}\frac{-1}{2\cdot\min\limits_{i=0,1,\cdots,2N-2}\cos\left(\theta+\frac{2\pi i}{2N-1}\right)}}_{=\frac{1}{2\cos\left(\frac{\pi}{2N-1}\right)}}, \tag{37}$$

where the upper bound is achieved with equality when $\theta = \frac{2\pi j}{2N-1}$ for any integer j. By substituting $r = \frac{1}{2\cos\left(\frac{\pi}{2N-1}\right)}$ into (32), we obtain the following weight vector.

$$w_i=\frac{1}{2N-1}\left(1+\frac{\cos\frac{2\pi(i+j)}{2N-1}}{\cos\frac{\pi}{2N-1}}\right), \tag{38}$$

where $i = 0, 1, \cdots, 2N-2$ and j is any integer. The resultant GLB from this weight vector is shown in the following lemma.

Lemma 3: For $K \geq \overline{K}+1$ and $\frac{\lambda_0}{|\lambda_1|} < (2N-1)K-1$, we have

$$\delta_{\max}^2\ge M\left[N-\frac{K\left(\lambda_0-\frac{|\lambda_1|}{2\cos^2\frac{\pi}{2N-1}}\right)}{(2N-1)K-1-\frac{1}{2\cos^2\frac{\pi}{2N-1}}}\right], \tag{39}$$

where λ_0, λ_1 are given in (31).

To analyze the asymptotic tightness of the lower bound in (39), we note that when N is sufficiently large, the second condition in *Lemma 3*, i.e.,

$$\frac{\lambda_0}{|\lambda_1|}<(2N-1)K-1, \tag{40}$$

is true for $K \geq \overline{K}+1$. To show this, we substitute λ_0, λ_1 into (40). After some manipulations, one can see that the inequality in (40) holds if and only if

$$\begin{aligned}K>&4(MN-1)N\sin^2\frac{\pi}{2(2N-1)}\\ &+\frac{4N(N-1)\sin^2\frac{\pi}{2(2N-1)}+1}{2N-1}.\end{aligned} \tag{41}$$

Carrying on the expression in (33), we require

$$K-n\ge 1>\epsilon+\frac{4N(N-1)\sin^2\frac{\pi}{2(2N-1)}+1}{2N-1}, \tag{42}$$

which is guaranteed to hold for sufficiently large N because ϵ is strictly smaller than 1 by assumption. Furthermore, we note that

$$\begin{aligned} \lim_{N\to\infty} \frac{K\lambda_0}{N} &= (K+M)N, \\ \lim_{N\to\infty} \frac{K|\lambda_1|}{2N\cos^2\frac{\pi}{2N-1}} &= \left(\frac{2K}{\pi^2}-\frac{M}{2}\right)N. \end{aligned} \tag{43}$$

Therefore,

$$\lim_{N\to\infty} \frac{K\left(\lambda_0 - \frac{|\lambda_1|}{2\cos^2\frac{\pi}{2N-1}}\right)}{N\cdot\left[(2N-1)K-1-\frac{1}{2\cos^2\frac{\pi}{2N-1}}\right]} = \frac{3M}{4K}+\frac{1}{2}-\frac{1}{\pi^2}. \tag{44}$$

On the other hand, let us rewrite the Welch bound expression (1) as

$$M^2N^2\frac{\frac{K}{M}-1}{K(2N-1)-1} = M\left(N-\mathcal{R}_1\right) \tag{45}$$

with

$$\mathcal{R}_1 \triangleq \frac{N(MN-1)+N(N-1)K}{(2N-1)K-1}. \tag{46}$$

Then,

$$\lim_{N\to\infty}\frac{\mathcal{R}_1}{N} = \frac{1}{2}+\frac{M}{2K}. \tag{47}$$

With (46) and (47), one can show that the lower bound in *Lemma 3* is asymptotically tighter than the Welch bound in (1) if and only if the following equation is satisfied.

$$\frac{1}{2}+\frac{M}{2K} > \frac{3M}{4K}+\frac{1}{2}-\frac{1}{\pi^2}. \tag{48}$$

Equivalently, we need to prove that for $K=\left\lfloor\frac{\pi^2M}{4}\right\rfloor+1$ (as $N\to\infty$), the following inequality holds.

$$d_1(M) \triangleq \frac{\left\lfloor\frac{\pi^2M}{4}\right\rfloor+1}{M}-\frac{\pi^2}{4} > 0. \tag{49}$$

One can readily show that the condition $d_1(M)>0$ given in (49) is true for *all* $M\geq 2$. Therefore, we have the following theorem.

Theorem 1: The GLB in (39) which arises from the weight vector in (38) reduces to

$$\delta_{\max}^2 \gtrsim MN\left[\left(\frac{1}{2}+\frac{1}{\pi^2}\right)-\frac{3M}{4K}\right], \tag{50}$$

for sufficiently large N. Such an asymptotic lower bound is tighter than the Welch bound for *all* $K\geq\overline{K}+1$ and for *all* $M\geq 2$.

IV. Conclusions

This paper is motivated by the ambiguous zone [specified in (14)] in which the tightness of the generalized Levenshtein bound (GLB) over the Welch bound was unknown. We have taken a frequency-domain optimization approach to conduct fractional quadratic optimization of GLB. The most significant finding of this paper is the derived weight vector in (38) resulting in a tighter GLB in (39) (over the Welch bound) for *all* $K\geq\overline{K}+1$ and for *all* $M\geq 2$, asymptotically. This finding not only eliminates the aforementioned ambiguous zone of GLB, but also explicitly shows that the GLB tighter condition given in [6, *Theorem 2*] is necessary and sufficient, asymptotically. Interestingly, we have proved in our full paper [15] that the derived weight vector in (38) is a local minimizer of the GLB under certain asymptotic conditions. More numerical comparisons with GLBs from other weight vectors can also be found in [15].

V. Acknowledgement

The work of Z. Liu and Y. L. Guan was supported by the Advanced Communications Research Program under grant contract DSOCL14095, and the NRF-NSFC joint grant NRF2016NRF-NSFC001-089. The work of W. H. Mow was supported by the General Research Fund from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. 16233816).

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