New Optimal Binary Z-Complementary Pairs of Odd Lengths

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Abstract Existing systematic constructions of *optimal* odd-length binary Z-complementary pairs (OBZCPs) only generate pairs with lengths $2^\alpha+1$ (where α is a non-negative integer). In this paper, we propose a new construction of *optimal* OBZCPs having generic lengths of $2^\alpha 10^\beta 26^\gamma+1,\ \alpha \geq 1$ (where α, β and γ are non-negative integers). This is achieved with the aid of several interesting intrinsic structure properties of Golay complementary pairs obtained from Turyn's method.

key words: *Golay complementary pairs (GCPs), Zero correlation zone (ZCZ), Z-complementary pairs (ZCPs), Odd-length binary Z-complementary pairs (OBZCPs), Turyn's method.*

1. Introduction

The concept of "complementary pair" was first proposed by M. J. Golay in 1951 [1]. By definition, a pair of sequence whose out-of-phase aperiodic autocorrelations sum to zero, is called a complementary pair, commonly known as Golay complementary pair (GCP) [1]. Such sequences have found many applications in infra-red spectrometry, acoustic surface-wave devices [1], pulsed radar and navigation [2], synchronization, channel estimation [3], and peak power control in orthogonal frequency division multiplexing (OFDM) [4]–[6]. However, existing known binary GCPs only have even-lengths in the form of $2^\alpha 10^\beta 26^\gamma$ (where α, β, γ are non-negative integers) [7] [8].

In search of close alternative to GCPs, Fan *et. al.* introduced binary Z-complementary sequences (ZCPs) [9], which includes GCPs as special cases. In a ZCP, the out-of-phase aperiodic autocorrelations sum to zero within a region, called the zero correlation zone (ZCZ) [9]. Unlike GCPs, ZCPs exist for more lengths with various ZCZ widths. In 2011, Li *et. al.* [10] proved that the maximum ZCZ width that can be achieved by a binary ZCP of odd-length N is $(N+1)/2$. But the authors in [10] did not find a systematic way to construct odd length binary ZCPs, which achieve the maximum ZCZ width. A remarkable breakthrough was made by Liu *et. al.* in 2014 [11], [12] which gives a systematic construction of the odd-length binary ZCPs with maximum ZCZ widths. The constructed ZCPs also have the property of having minimum aperiodic autocorrelation sum magnitude (i.e., 2) outside the ZCZ. Therefore, it is called in [11] *optimal* odd-length binary ZCPs (OBZCPs). OBZCPs have potential applications in quasi-synchronous CDMA (QS-CDMA) systems to mitigate intersymbol interference (ISI) and multiple access interference (MAI) [13]–[15]. Existing systematic construction of *optimal* OBZCPs by Liu *et. al.* is based on generalized Boolean functions (GBFs) [11] by applying insertion method to Golay-Davis-Jedwab (GDJ) pairs [16] to obtain OBZCPs of length $2^\alpha+1$. Recently, [18] gives a direct construction of *optimal* OBZCPs of length $2^\alpha+1$, also based on GBFs. Similar to *optimal* OBZCPs, *optimal* binary periodic Z-complementary pairs (BPACPs) has been investigated in [17].

For more *optimal* OBZCPs, we apply insertion method to GCPs with lengths of $2^\alpha 10^\beta 26^\gamma,\ \alpha \geq 1$ (where α, β and γ are non-negative integers). These GCPs are constructed using Turyn's method and GCP kernels of lengths 2, 10 and 26. We exploit the polynomial form of Turyn's method to explore several interesting intrinsic structure properties of GCPs. We show that *optimal* OBZCPs of length $2^\alpha 10^\beta 26^\gamma+1$ can be obtained when insertion method is applied to certain binary GCPs of length $2^\alpha 10^\beta 26^\gamma,\ \alpha \geq 1$.

The paper is organised as follows. In Section 2, we introduce binary ZCPs, *optimal* OBZCPs and insertion method on sequences. In Section 3, we recall Turyn's method to construct binary GCPs of larger lengths and explore polynomial form of Turyn's method. We present a systematic construction of *optimal* OBZCPs of length $2^\alpha 10^\beta 26^\gamma+1,\ \alpha \geq 1$. We conclude the paper in Section 4.

2. Preliminaries

In this paper, 1 and -1 are represented by $+$ and $-$, respectively. "$\mathbf{a}||\mathbf{b}$" denotes the concatenation of sequences $\mathbf{a}$ and $\mathbf{b}$. $\underline{\mathbf{a}}$ is the reverse of sequence $\mathbf{a}$. Denote by $\mathbf{c}_L$ a length-L vector with identical entries of c. $\mathbf{a}\otimes\mathbf{b}$ denotes the Kronecker product of the sequences $\mathbf{a}$ and $\mathbf{b}$. For two length-N binary sequences $\mathbf{a}$ and $\mathbf{b}$ over $\{1,-1\}$, their aperiodic cross-correlation function (ACCF) is defined as

$$\rho_{\mathbf{a},\mathbf{b}}(\tau) := \begin{cases} \sum\limits_{k=0}^{N-1-\tau} a_k b_{k+\tau}, & 0 \leq \tau \leq N-1; \\ \sum\limits_{k=0}^{N-1-\tau} a_{k+\tau} b_k, & -(N-1) \leq \tau \leq -1; \\ 0, & \mid \tau \mid \geq N. \end{cases} \tag{1}$$

When $\mathbf{a}=\mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called aperiodic auto-correlation function (AACF) of $\mathbf{a}$ and will be denoted as $\rho_{\mathbf{a}}(\tau)$.

2.1 Introduction to OB-ZCPs

***Definition* 1** (Binary Z-complementary pair [11]): Let $(\mathbf{a};\mathbf{b})$ be a binary sequence pair of length N. It is said to be a ZCP with ZCZ width Z if and only if

$$\rho_{\mathbf{a}}(\tau)+\rho_{\mathbf{b}}(\tau)=0, \quad \text{for all } 1 \leq \tau \leq Z-1. \tag{2}$$

If $Z=N$, $(\mathbf{a};\mathbf{b})$ reduces to a GCP [1].

***Lemma* 1:** [11] Every OBZCP of length N has a maximum ZCZ width $(N+1)/2$, i.e.,

$$Z \leq (N+1)/2. \tag{3}$$

***Definition* 2:** [11] An OBZCP is said to be *Z-optimal* if $Z = (N+1)/2$.

***Lemma* 2:** [11] The magnitude of each out-of-zone aperiodic auto-correlation sum for a *Z-optimal* OBZCP $(\mathbf{a};\mathbf{b})$ of length N is lower bounded by 2, i.e.,

$$| \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) | \geq 2, \text{ for any } (N+1)/2 \leq \tau \leq (N-1). \tag{4}$$

***Definition* 3** (*Optimal* OBZCP [11]): An OBZCP is said to be *optimal* if it is *Z-optimal* and every out-of-zone aperiodic autocorrelation sums take the minimum magnitude value of 2.

***Example* 1:** Let

$$\begin{aligned} \mathbf{a} &= (+++-++-++), \\ \mathbf{b} &= (+++---+-+). \end{aligned} \tag{5}$$

$(\mathbf{a};\mathbf{b})$ is a length-9 *optimal* OBZCP with a ZCZ width of 5 because

$$\Big(| \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) | \Big)_{\tau=0}^{8} = (18,0,0,0,0,2,2,2,2). \tag{6}$$

2.2 Introduction to GCPs

***Lemma* 3:** [19] Let $\mathcal{G} \equiv (\mathbf{a};\mathbf{b})$ be a binary GCP of length N given by

$$\mathcal{G} = \begin{pmatrix} a_0, a_1, \ldots, a_{N-1} \\ b_0, b_1, \ldots, b_{N-1} \end{pmatrix} \tag{7}$$

Then,

$$\begin{aligned} & a_i a_{N-1-i} + b_i b_{N-1-i} = 0, \;\; 0 \leq i < N/2, \\ & \text{or, } a_i + a_{N-1-i} + b_i + b_{N-1-i} = \pm 2, \;\; 0 \leq i < N/2. \end{aligned} \tag{8}$$

Lemma 3 shows that if the i-column vector of $\mathcal{G}$ consists of identical elements, then the $(N-i)$-column vector consists of elements with opposite signs, and vice versa.

Let the kernels of binary GCPs of lengths $2,\ 10$ and 26 be denoted as in Table 1 [19].

***Definition* 4** (Insertion Function [11]): Let $\mathbf{a} = (a_0, a_1, \cdots, a_{N-1})$, be a sequence of length N. If an element x is inserted at r^{th} position of the sequence, then the insertion function $\mathcal{I}(\mathbf{a}, r, x)$ is defined as

$$\begin{aligned} &\mathcal{I}(\mathbf{a}, r, d) \\ &= \begin{cases} (x, a_0, a_1, \cdots, a_{N-1}), & \text{if } r = 0, \\ (a_0, a_1, \cdots, a_{N-1}, x), & \text{if } r = N, \\ (a_0, a_1, \cdots, a_{r-1}, x, a_r, \cdots, a_{N-1}), & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

3. CONSTRUCTIONS OF *optimal* OB-ZCPs

In this section, we present systematic constructions of new sets of optimal OB-ZCPs by employing insertion method to binary GCPs of length $2^{\alpha}10^{\beta}26^{\gamma},\ \alpha \geq 1$. In the next subsection, we recall Turyn's construction method for binary GCPs and uncover several intrinsic structure properties pertinent to these GCPs which are useful in the identifications of these newly constructed optimal OB-ZCPs.

3.1 Intrinsic Structure Properties of binary GCPs from Turyn's Construction:

For ease of presentation, we represent a GCP as a two-dimensional matrix with two rows corresponding to the two sequences of GCP. Our objective is to uncover several basic structure properties pertinent to the columns of binary GCPs which are generated from Turyn's construction.

***Lemma* 4** (Turyn's Method [20]): Let $\mathcal{A} \equiv (\mathbf{a};\mathbf{b})$ and $\mathcal{B} \equiv (\mathbf{c};\mathbf{d})$ be binary GCPs of lengths N and M respectively and denote $\mathcal{A}$ as the $1st$ pair and $\mathcal{B}$ as the $2nd$ pair. Then $(\mathbf{e};\mathbf{f}) = Turyn(\mathcal{A},\mathcal{B})$ is a GCP of length-MN where,

$$\begin{aligned} \mathbf{e} = \mathbf{c} \otimes (\mathbf{a}+\mathbf{b})/2 - \underline{\mathbf{d}} \otimes (\mathbf{b}-\mathbf{a})/2, \\ \mathbf{f} = \mathbf{d} \otimes (\mathbf{a}+\mathbf{b})/2 + \underline{\mathbf{c}} \otimes (\mathbf{b}-\mathbf{a})/2. \end{aligned} \tag{10}$$

Let $A(x)$ be a polynomial corresponding to sequence $\mathbf{a}$ of length N , defined as follows,

$$A(x) = \sum_{i=0}^{N-1} a_i x^i. \tag{11}$$

***Lemma* 5** (Turyn's method in polynomial form [21]): Let $\mathcal{A} \equiv (\mathbf{a};\mathbf{b})$ and $\mathcal{B} \equiv (\mathbf{c};\mathbf{d})$ be GCPs of lengths N and M respectively. Let $(\mathbf{e};\mathbf{f}) = Turyn(\mathcal{A},\mathcal{B})$. Also let $A(x)$, $B(x)$, $C(x)$, $D(x)$, $E(x)$, and $F(x)$ are polynomials corresponding to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$ respectively, and $F^*(x) = x^{\text{deg F(x)}} F(x^{-1})$. Then,

$$\begin{aligned} &\begin{pmatrix} E(x) & F(x) \\ -F^*(x) & E^*(x) \end{pmatrix} = \\ &\frac{1}{2}\begin{pmatrix} A(x) & B(x) \\ -B^*(x) & A^*(x) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C(x^N) & D(x^N) \\ -D^*(x^N) & C^*(x^N) \end{pmatrix}. \end{aligned} \tag{12}$$

Simplifying (12), we get

$$\begin{aligned} & E(x) \\ & = C(x^N)\left(\frac{A(x)+B(x)}{2}\right) - D^*(x^N)\left(\frac{B(x)-A(x)}{2}\right) \\ & F(x) \\ & = D(x^N)\left(\frac{A(x)+B(x)}{2}\right) + C^*(x^N)\left(\frac{B(x)-A(x)}{2}\right) \end{aligned} \tag{13}$$

Let $\mathcal{A} \equiv (\mathbf{a};\mathbf{b})$ be a binary GCP kernel of length $N \in \{2,10,26\}$ (i.e. K_2, K_{10} or K_{26} as shown in Table 1), and $\mathcal{B} \equiv (\mathbf{c};\mathbf{d})$ be any binary GCP of lenght M. Observe that, when $\mathcal{A} = K_2$, we have $a_0 = b_0$ and $a_1 = -b_1$. When $\mathcal{A} = K_{10}$, we have $a_i = b_i$ for $i \in \{0,1,2,3,5\}$, $a_i = -b_i$ for $i \in \{4,6,7,8,9\}$. When $\mathcal{A} = K_{26}$, we have $a_i = b_i$ for $i \in \{0,1,\cdots,11,13\}$, $a_i = -b_i$ for $i \in \{12,14,15,\cdots,25\}$. So,

Table 1 Kernels of GCPs of length 2, 10 and 26. [19]

N	$\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$	Denoted by
2	$\begin{pmatrix}++\\ +-\end{pmatrix}$	K_2
10	$\begin{pmatrix}++-+-+--++\\ ++-+++++--\end{pmatrix}$	K_{10}
26	$\begin{pmatrix}++++-++--+-+-+--+-+++--+++\\ ++++-++--+-+++++-+---++---\end{pmatrix}$	K_{26}

$$
\frac{A(x)+B(x)}{2}=\begin{cases} a_0 & N=2\\ \sum_{j=0}^{\frac{N}{2}-2} a_j x^j + a_{\frac{N}{2}} x^{\frac{N}{2}} & N=10,\ 26.\end{cases}
$$
$$
\frac{B(x)-A(x)}{2}=\begin{cases} b_1 x & N=2\\ -b_{\frac{N}{2}-1}x^{\frac{N}{2}-1} - \sum_{j=0}^{\frac{N}{2}-2} b_{\frac{N}{2}+1+j} x^{\frac{N}{2}+1+j} & \\ & N=10,\ 26.\end{cases} \tag{14}
$$

Next, we have,

$$
\begin{aligned}
C(x^N) &= \sum_{i=0}^{M-1} c_i (x^N)^i,\\
C^*(x) &= x^{M-1}\sum_{i=0}^{M-1} c_i x^{-i} = \sum_{i=0}^{M-1} c_{M-1-i} x^i,\\
C^*(x^N) &= \sum_{i=0}^{M-1} c_{M-1-i} x^{Ni},
\end{aligned} \tag{15}
$$

Similarly we calculate $D(x^N)$ and $D^*(x^N)$. By (13) we have,

$$
E(x)=\sum_{i=0}^{M-1} E_i(x) \text{ and } F(x)=\sum_{i=0}^{M-1} F_i(x), \tag{16}
$$

where,

$$
E_i(x) = \begin{cases} x^{Ni}[c_i a_0 - d_{M-1-i} b_1 x] & N=2\\ x^{Ni}\Big[\sum_{j=0}^{\frac{N}{2}-2} c_i a_j x^j - d_{M-1-i} b_{\frac{N}{2}-1} x^{\frac{N}{2}-1} + & \\ \quad c_i a_{\frac{N}{2}} x^{\frac{N}{2}} - \sum_{j=0}^{\frac{N}{2}-2} d_{M-1-i} b_{\frac{N}{2}+1+j} x^{\frac{N}{2}+1+j}\Big] & \\ & N=10,\ 26.\end{cases} \tag{17}
$$

And,

$$
F_i(x) = \begin{cases} x^{Ni}[d_i a_0 + c_{M-1-i} b_1 x] & N=2\\ x^{Ni}\Big[\sum_{j=0}^{\frac{N}{2}-2} d_i a_j x^j + c_{M-1-i} b_{\frac{N}{2}-1} x^{\frac{N}{2}-1} + & \\ \quad d_i a_{\frac{N}{2}} x^{\frac{N}{2}} + \sum_{j=0}^{\frac{N}{2}-2} c_{M-1-i} b_{\frac{N}{2}+1+j} x^{\frac{N}{2}+1+j}\Big] & \\ & N=10,\ 26.\end{cases} \tag{18}
$$

Observe that $\mathbf{e}$ and $\mathbf{f}$ are concatenation of M number of sequences $\mathbf{e}_i,\ \mathbf{f}_i$, each of length N, associated with $E_i(x)$ and $F_i(x)$ respectively. Since $(\mathbf{c},\mathbf{d})$ is a binary GCP, from *Lemma* 3, we assert that if c_i and d_i are of identical signs, c_{M-1-i} and d_{M-1-i} must be of different signs. Therefore, from (17) and (18), we see that the coefficients of the polynomials $E_i(x)$ and $F_i(x)$, are identical for each i, provided that c_i and d_i are identical. Specifically, provided that $c_i = d_i$, the length-N sequence pair corresponding to $E_i(x)$ and $F_i(x)$ are identical. On the other hand, if $c_i \neq d_i$, *Lemma* 3 shows that, c_{M-1-i} and d_{M-1-i} are of identical signs, and therefore every column of $(\mathbf{e}_i;\mathbf{f}_i)$ has different signs.

***Theorem* 1:** Let $(\mathbf{e};\mathbf{f})$ be a GCP of length $2^\alpha M$, constructed iteratively by employing Turyn's method on $K_2,\ K_{10}$ and K_{26} as follows:

$$
\begin{aligned}
&(\mathbf{e}_0;\mathbf{f}_0) = K_2,\\
&(\mathbf{e}_i;\mathbf{f}_i) = Turyn(\mathcal{A},(\mathbf{e}_{i-1};\mathbf{f}_{i-1})),\ \mathcal{A}=K_2,\ K_{10} \text{ or } K_{26},
\end{aligned} \tag{19}
$$

where $M = 10^\beta 26^\gamma$ and $\alpha,\ \beta,$ and γ are non-negative integers and $\alpha \geq 1$. Then the first $2^{\alpha-1}M$ columns of $(\mathbf{e};\mathbf{f})$ will have elements with identical sign in each column.

***Proof*:** Observe that the first column of K_2 has elements with same sign. And among the three kernels of length N ($N = 2,\ 10,$ or 26), only K_2 has first $N/2$ columns with identical elements in each column. From (16), (17) and (18), we see that, at the i-th iteration, the number of columns of the sequence pair $(\mathbf{e}_i;\mathbf{f}_i)$ having elements with identical signs and different signs completely depends upon the sequence pair $(\mathbf{e}_{i-1};\mathbf{f}_{i-1})$.

Let the p-th element of the sequence $\mathbf{e}_i$ and $\mathbf{f}_i$ be denoted by e_i^p and f_i^p, respectively. Now, if the j-th column of $(\mathbf{e}_{i-1};\mathbf{f}_{i-1})$ have elements with same sign, then we have $e_i^t = f_i^t$, for $Nj \leq t < N(j+1)$. In other words, each column of $(\mathbf{e}_i;\mathbf{f}_i)$ with column indices ranging from Nj to

$Nj+(N-1)$ will have elements with identical signs. If the j-th column of $(\mathbf{e}_{i-1};\mathbf{f}_{i-1})$ have elements with different signs, then we have $e_i^t=-f_i^t$, $Nj\le t<N(j+1)$. In other words, each column of $(\mathbf{e}_i;\mathbf{f}_i)$ with column indices ranging from Nj to $Nj+(N-1)$ will have elements with different signs.

Now, using the above argument in each step of the iteration, we see that at any point of the iteration $(\mathbf{e}_i,\mathbf{f}_i)$ has the first $N_i/2$ columns having elements with identical signs in each column, where N_i is the length of $(\mathbf{e}_i;\mathbf{f}_i)$. So, finally when $N=2^\alpha M$, then first $2^{\alpha-1}M$ columns of $(\mathbf{e};\mathbf{f})$ will have elements with identical sign in each column.

***Example* 2:** Let $(\mathbf{e};\mathbf{f})$ be a GCP of length 20, constructed via Turyn's method by setting $\mathcal{A}=K_{10}$ for $i=1$ in *Theorem* 1. The first $(2^0\times 10^1)=10$ columns of $\mathcal{C}$ has entries with identical signs in each column.

$$\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}=\begin{pmatrix}++-+-+--++--+-----++\\ ++-+-+--++++-+++++--\end{pmatrix} \tag{20}$$

For ease of presentation, from hereon, we assume $(\mathbf{a};\mathbf{b})$ is obtained from the construction in *Theorem* 1. The following lemma gives the AACSs at different time-shifts when $x,y\in\{1,-1\}$ are inserted at the front position of the sequences $\mathbf{a}$ and $\mathbf{b}$, respectively.

***Lemma* 6:** Let $(\mathbf{a};\mathbf{b})$ be GCP of length N. If $\mathbf{e}=\mathcal{I}(\mathbf{a},0,x)$ and $\mathbf{f}=\mathcal{I}(\mathbf{b},0,y)$, where $x,y\in\{1,-1\}$, then it satisfies the following:

$$\rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)=xa_{\tau-1}+yb_{\tau-1},\quad 1\le\tau<N. \tag{21}$$

***Proof*:** For $1\le\tau<N$. we get,

$$\begin{aligned}\rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)&=xa_{\tau-1}+yb_{\tau-1}+\rho_{\mathbf{a}}(\tau)+\rho_{\mathbf{b}}(\tau)\\ &=xa_{\tau-1}+yb_{\tau-1}\end{aligned} \tag{22}$$

Note that (22) can be further reduced to

$$\rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)=\begin{cases}0, & \text{if } xa_{\tau-1}\cdot yb_{\tau-1}=-1\\ \pm 2, & \text{if } xa_{\tau-1}\cdot yb_{\tau-1}=1\end{cases}. \tag{23}$$

By recalling *Theorem* 1, we obtain the following construction.

***Proposed Construction*:** Let $(\mathbf{a};\mathbf{b})$ be a GCP of length $N=2^\alpha M$, where $M=10^\beta 26^\gamma$, α, β, and γ are non-negative integers and $\alpha\ge 1$. If $x,y\in\{1,-1\}$ are of different signs and $\mathbf{e}=\mathcal{I}(\mathbf{a},0,x)$ and $\mathbf{f}=\mathcal{I}(\mathbf{b},0,y)$, then $(\mathbf{e};\mathbf{f})$ is a *optimal* OBZCP with ZCZ width of $2^{\alpha-1}M+1$ satisfying the following equation.

$$|\ \rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)\ |=\begin{cases}0, & \text{if } 0<\tau\le 2^{\alpha-1}M,\\ 2, & \text{if } 2^{\alpha-1}M<\tau<2^\alpha M.\end{cases} \tag{24}$$

***Proof*:** By *Theorem* 1, we have $a_i=b_i$ for $0<i\le 2^{\alpha-1}M$. Also, by *Lemma* 3, we have $a_i=-b_i$, where $2^{\alpha-1}M<i\le 2^\alpha M$. By (22), the proof of this corollary follows if x and y are of different signs $(x,y\in\{1,-1\})$ and inserted at the front position of $\mathbf{a}$ and $\mathbf{b}$, respectively.

***Remark* 1:** In *Corollary* 3.1, if $\beta=0$, $\gamma=0$ then it matches with the result of *Theorem* 4 of [11].

***Example* 3:** Let $(\mathbf{a};\mathbf{b})$ be the GCP of length 20 shown in (20). If $x=1$, $y=-1$, then $\mathbf{e}\equiv\mathcal{I}(\mathbf{a},0,x)$, $\mathbf{f}\equiv\mathcal{I}(\mathbf{b},0,y)$ is

$$\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}=\begin{pmatrix}+++-+-+--++--+-----++\\ -++-+-+--++++-+++++--\end{pmatrix} \tag{25}$$

One can readily show that $(\mathbf{e};\mathbf{f})$ is a length-21 *optimal* OBZCP with a ZCZ width of 11 because

$$\Big(|\ \rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)\ |\Big)_{\tau=0}^{20}=(42,\mathbf{0}_{10},\mathbf{2}_{10}). \tag{26}$$

***Example* 4:** Let $(\mathbf{a};\mathbf{b})$ be a GCP of length 104. The first $(2^{2-1}\times 26)=52$ columns of $(\mathbf{a};\mathbf{b})$ has entries with same sign in each column.

$$\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}=(A\|B\|C\|D), \tag{27}$$

where A,B,C and D are the following sequence pairs of length 26,

$$\begin{aligned}A&=\begin{pmatrix}++++-++--+-+-+--+-+++--+++\\ ++++-++--+-+-+--+-+++--+++\end{pmatrix},\\ B&=\begin{pmatrix}----+--++-+-----+-+++--+++\\ ----+--++-+-----+-+++--+++\end{pmatrix},\\ C&=\begin{pmatrix}----+--++-+-+-++-+---++---\\ ++++-++--+-+-+--+-+++--+++\end{pmatrix},\\ D&=\begin{pmatrix}----+--++-+-----+-+++--+++\\ ++++-++--+-++++++-+---++---\end{pmatrix}.\end{aligned} \tag{28}$$

Now, let $x=1$ and $y=-1$, Then, $\mathbf{e}=\mathcal{I}(\mathbf{a},0,x)$, $\mathbf{f}=\mathcal{I}(\mathbf{b},0,y)$ is

$$\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}=(X\|A\|B\|C\|D). \tag{29}$$

where $X=\binom{+}{-}$. Here, $(\mathbf{e};\mathbf{f})$ is a length-105 *optimal* OBZCP with a ZCZ width of 53 because

$$\Big(|\ \rho_{\mathbf{e}}(\tau)+\rho_{\mathbf{f}}(\tau)\ |\Big)_{\tau=0}^{104}=(210,\mathbf{0}_{52},\mathbf{2}_{52}). \tag{30}$$

4. Conclusion

In this paper, we have exploited several intrinsic structural properties of GCPs, which are constructed using Turyn's method. Based on these properties of GCPs, we have constructed *optimal* OBZCPs with lengths of $2^\alpha 10^\beta 26^\gamma+1$, $\alpha\ge 1$ (where α, β and γ are non-negative integers).

Acknowledgments

The work of Avik R. Adhikary is supported by Department of Science and Technology (DST) (INSPIRE Fellowship, Sanction no: 2014/292), Govt. of India. The work of Zilong Liu and Yong Liang Guan was supported by NSFC-NRF project under grant No. 61661146003/NRF2016NRF-NSFC001-089.

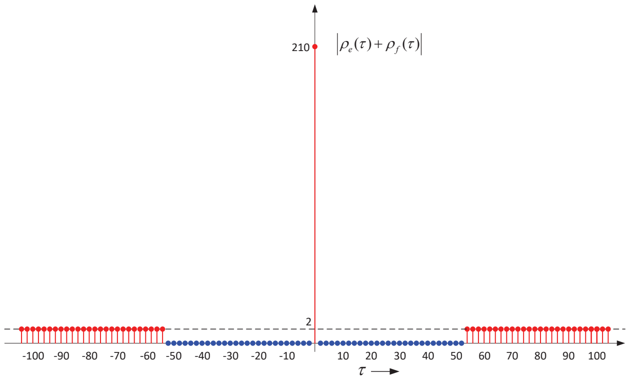

Fig. 1 Aperiodic autocorrelation sums of OBZCP in *Example 4*.

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