Sequence Design for Optimized Ambiguity Function and PAPR under Arbitrary Spectrum Hole Constraint

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Abstract In this paper, a novel sequence design algorithm under spectrum hole constraint is proposed which generates sequences resilient to time delays and Doppler shifts. In practical applications, it is desirable to have sequences with not only low ambiguity function (AF) in certain delay-Doppler region, but also low peak-to-average power ratio (PAPR). The proposed algorithm combines energy gradient method and Hu-Liu (HL) algorithm to jointly optimize AF and PAPR.

key words: *Ambiguity function (AF), peak-to-average power ratio (PAPR), local optimization, spectrum hole constraint*

1. Introduction

To solve the scarcity of available spectrum, cognitive radio (CR) is proposed to provide the capability of using and sharing spectrum in an opportunistic manner. The spectrum opportunity is defined as spectrum holes [1]–[4] which are not being used by the designated primary users at a particular time in a particular geographic area [5]. Traditional sequences with no spectrum hole constraint cannot be applied directly to CR systems. For high power transmission efficiency, it is also desirable to have sequences with low peak-to-average power ratio (PAPR).

Hu *et al.* [5] proposed a CR sequence design algorithm under arbitrary spectrum hole constraint by jointly optimizing the PAPR and aperiodic auto-correlation function (AACF) of sequences. However, the correlation of their proposed sequences in Doppler dimension was not considered. In 2015, bounds and constructions of optimal CR sequences with minimum total periodic autocorrelation sidelobe energy have been developed in [6].

In practice, Doppler shifts/spreads, caused by objects such as signal reflectors, moving transmitters and/or receivers, need to be carefully dealt with. As such, a two-dimensional correlation function, i.e., ambiguity function (AF) has been adopted in the literature to evaluate the goodness of sequences in Doppler environments [7]. AF can be used to evaluate both interference mitigation capabilities and the Doppler and range resolutions in radar systems [8]. Various modified definitions of AF were proposed based on different application scenarios [7], [9]–[11]. The ideal AF should exhibit thumbtack shape in whole delay-Doppler domain [7] which is in general a difficult design task [12]. Recently, local AF optimization over certain delay-Doppler bins of interest has received increasing attention [8]. Some efficient gradient methods for locally optimizing the AF and auto-correlation to design sequences have been proposed in [12], [13]. [14] reports a cyclic algorithm based on Singular Value Decompositions (SVD) to locally optimize AF. However, the spectrum hole constraint was not considered in these works [8], [12]–[14].

In this paper, the aperiodic discrete AF of a frequency-domain sequence is defined firstly. Then, the local AF and PAPR are jointly considered in the objective function to design sequences under arbitrary spectrum hole constraint. Based on these requirements, an optimization problem is formulated. Energy gradient method and Hu-Liu (HL) algorithm are used to solve the formulated optimization problem. Finally, a novel sequence design algorithm under arbitrary spectrum hole constraint is proposed. Through numerical experiments, it is shown that our constructed sequences possess better local AF than the CR sequences in [5] under the same spectrum hole constraint.

This paper is organized as follows. Necessary notation and the requirements of sequence design under spectrum hole constraint are presented in Section 2. The sequence design problem is formulated and a sequence design algorithm is proposed in section 3. Numerical experiments would be presented in Section 4. Finally, the conclusion is drawn in Section 5.

In this paper, $||\cdot||_2$ is the Euclidean norm, and the superscripts A^H and A^T denote the Hermitian and transposition of matrix A, respectively. $W_N = e^{-j\frac{2\pi}{N}}$,$j = sqrt(-1)$.

2. Notation and Requirements

Define a length-N frequency-domain complex sequence $X = [X_0, X_1, \cdots, X_{N-1}]^T$, with $X_k = A_k e^{j\phi_k}$. A_k is the predefined waveform magnitude. By using inverse discrete Fourier transform (IDFT), the time-domain form can be derived as $x = F_N^H X$, where $x = [x_0, x_1, \cdots, x_{N-1}]^T$. Denote the discrete Fourier transform (DFT) matrix of order N by $F_N = [f_{i,k}]_{i,k=0}^{N-1}$, i.e., $f_{i,k} = \frac{1}{\sqrt{N}} W_N^{ik}$.

The PAPR of a time-domain sequence x is defined as [5]

$$\text{PAPR}(x) = \frac{\max\limits_{0 \le i \le N-1} |x_i|^2}{\frac{1}{N}\sum\limits_{i=0}^{N-1} |x_i|^2}. \tag{1}$$

If x is a unimodular sequence, the ideal PAPR equals 1 (i.e., 0 dB).

In [12], a Doppler-shifted version of time-domain sequence is as $x_{f_d} = \{x_i e^{j2\pi f_d i T_b}\}_{i=0}^{N-1}$, where f_d and T_b represent the Doppler shift and bit time respectively. The relationship of the Doppler shift f_d and the normalized Doppler shift p can be denoted as $f_d = \frac{p}{NT_b}$. $x^{(p)} = \{x_i e^{j2\pi \frac{ip}{N}}\}_{i=0}^{N-1}$.

The correlation between $x^{(p)}$ and x can be written as [12]

$$r_{p,m} = \sum_{i=0}^{N-1} x_i x_{i+m}^* e^{j2\pi \frac{ip}{N}}, \quad (2)$$

where $m = -N+1, \cdots, N-1$; $p = -\frac{N}{2}, \cdots, \frac{N}{2}-1$. Assume $x_i = 0$ for $i < 0$ and $i > N-1$. Equation (2) can be regarded as the aperiodic discrete AF [10][14]. Similarly, the aperiodic discrete AF of frequency-domain sequence X can be defined as

$$r_{p,m} = \frac{1}{N} \sum_{i \in I} ((\sum_{k=0}^{N-1} X_k W_N^{-ik})(\sum_{k=0}^{N-1} X_k^* W_N^{(i+m)k}) W_N^{-ip}), \quad (3)$$

where $I = \{i | -m \le i \le N-1-m\} \cap \{i | 0 \le i \le N-1\}$.

The energy term which is used to calculate the gradient under a given normalized Doppler shift p can be defined as [12]

$$E_p = \sum_{m=-N+1}^{N-1} |r_{p,m}|^{2q} w_{p,m}, \quad (4)$$

where the exponent q allows control over the sidelobe level, and the coefficients $w_{p,m}$ control the shape of the correlation sidelobe.

In radar systems, joint resolution denotes the ratio of the squared magnitude of center peak to the ambiguity surface in the entire delay-Doppler domain [11]. The joint resolution equals from 0 to 1. The larger value of the joint resolution, the closer of AF to the ideal one. Similarly, in order to evaluate the AF performance of sequence in the area of interest in this paper, the joint resolution in the specific area of AF plane is defined as

$$\text{Res}(x) = \frac{|r_{0,0}|^2}{\sum_{p \in P} \sum_{m \in M} |r_{p,m}|^2}, \quad (5)$$

where the area of interest is determined by the set P and M jointly which contain a series of normalized Doppler shifts p and delays m. The ideal value of joint resolution is 1 which means the sidelobes equal 0 in the specific area.

For practical applications, we need sequences satisfying the following conditions:

1) The frequency-domain sequence subjects to a spectrum hole constraint;

2) Low PAPR value;

3) Good AF property in the area of interest.

Assume that there are N subcarriers in the entire spectrum. Let $S = [S_0, S_1, \cdots, S_{N-1}]$ be a subcarrier marking vector, in which $S_k = 1$ if the k-th subcarrier is available and 0 otherwise. $\Omega = \{k | S_k = 0\}$ is the set of all unavailable subcarrier positions which is also called as a spectrum hole constraint set. Let $B = [B_0, B_1, \cdots, B_{N-1}]^T$ be the frequency-domain sequence under spectrum hole constraint, where $B_k = 0$, if $k \in \Omega$.

3. Sequence Design for Optimized AF and PAPR

3.1 Problem Formulation

In [12], an energy gradient method is proposed to generate a time-domain sequence whose AF is optimized in the area of interest as shown in Fig.1. This method is used to optimize the local AF of a frequency-domain sequence in this paper.

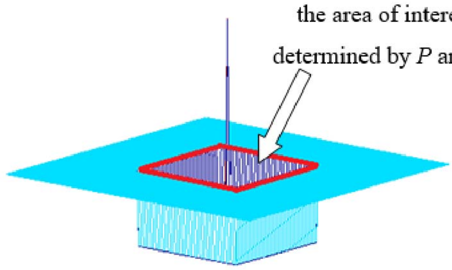

Fig. 1: Locally optimized ambiguity function in area of interest, in which P and M denote the maximum Doppler- and delay- shifts, respectively.

The idea is to calculate the gradient of E_p. As X is a complex frequency-domain sequence with a predefined magnitude, the gradient is written as the derivative of E_p with respect to its phase only. By using the chain rules, $\frac{\partial E_p}{\partial \phi_n}$ can be calculating by (6).

$$\frac{\partial E_p}{\partial \phi_n} = -Im(X_n)\frac{\partial E_p}{\partial Re(X_n)} + Re(X_n)\frac{\partial E_p}{\partial Im(X_n)}, \quad (6)$$

where $Re(X_n)$,$Im(X_n)$ stand for the real and imaginary parts of X_n. The final result is expressed in (7), where $\beta_{p,m} = w_{p,m}|r_{p,m}|^{2(q-1)}$.

It is obvious that the energy term in the area of interest can be expressed as

$$E = \sum_{p \in P} E_p, \quad (8)$$

and the gradient expression is

$$\frac{\partial E}{\partial \phi_n} = \sum_{p \in P} \frac{\partial E_p}{\partial \phi_n}. \quad (9)$$

Define the gradient vector as $\delta = [\frac{\partial E}{\partial \phi_0}, \cdots, \frac{\partial E}{\partial \phi_{N-1}}]$ and use backtracking line search to select the descent step α to update $X \leftarrow Xe^{-j\alpha\delta}$. The gradient method [12] can reduce the energy term E and achieve good local AF for X. The algorithm is summarized in ***Algorithm 1***.

$$\frac{\partial E_p}{\partial \phi_n} = \frac{2q}{N} \sum_{m=-N+1}^{N-1} (\beta_{p,m} \sum_{i \in I} Im(r_{p,m} \cdot X_n^* \cdot \sum_{\substack{k=0 \\ k \ne n}}^{N-1} (X_k W_N^{(n+p-k)i-mk}) + r_{p,m}^* \cdot X_n^* \cdot \sum_{\substack{k=0 \\ k \ne n}}^{N-1} (X_k W_N^{(n-p-k)i+mn}))). \quad (7)$$

Algorithm 1 :The gradient algorithm for optimizing the local AF of a frequency-domain sequence

Step 0: Input the initial frequency-domain sequence X. Define the area of interest: the set of Doppler P, the set of delay M and the weighting coefficients $w_{p,m}$.
Step 1: Gradient calculations for each normalized Doppler: Determination of $\delta_p = \left\{\frac{\partial E_p}{\partial \phi_n}\right\}_{n=0}^{N-1}$ for calculating $\delta = \left\{\frac{\partial E}{\partial \phi_n}\right\}_{n=0}^{N-1}$
Step 2: Determination of descent step α and update of X: $X \leftarrow Xe^{-j\alpha\delta}$
Step 3: Go to **Step 1** until an exit criterion is met.

Consider a known frequency-domain sequence X which has good local AF . In order to design frequency-domain sequence B with good local AF under spectrum hole constraint set Ω, we need to solve the following problem.

$$\begin{array}{l} \min\limits_{B,X} J_1(B,X) = \|B - X\|_2^2 \\ s.t. B_k, X_k = 0 \text{ if } k \in \Omega; \end{array} . \tag{10}$$

On the other hand, to optimize the PAPR of its time-domain sequence b, the following problem needs to be solved.

$$\begin{array}{l} \min\limits_{B,d} J_2(B,d) = \left\|F_N^H B - d\right\|_2^2 \\ s.t.(\text{i}): B_k = 0 \text{ if } k \in \Omega; \\ \quad (\text{ii}): |d_k| = 1, k = 0, 1, \cdots, N-1 \end{array} , \tag{11}$$

where $d = [d_0, d_1, \cdots, d_{N-1}]^T$ is a unimodular sequence.

In order to optimize AF and PAPR of sequence simultaneously, a parameter $\mu \in [0, 1]$ is used to control the relative weighting of J_1 and J_2. Therefore, the optimization problem can be formulated as

$$\begin{array}{l} \min\limits_{B,X,d} J(B,X,d) = \mu J_1(B,X) + (1-\mu) J_2(B,d) \\ s.t.(\text{iii}): B_k, X_k = 0 \text{ if } k \in \Omega; \\ \quad (\text{iv}): |d_k| = 1, k = 0, 1, \cdots, N-1 \end{array} . \tag{12}$$

3.2 Sequence Design for Optimized AF and PAPR

The HL algorithm which is proposed to generate a single CR sequence with low AACF and low PAPR [5]. According to the basic idea of HL algorithm, a new algorithm is proposed to solve the minimization problem in (12). The proposed algorithm is given as follows.

1) First, find the magnitudes of the frequency-domain sequence B. $\sum\limits_{k=0}^{N-1} |B_k|^2 = N$ and $B_k = 0$ if $k \in \Omega$.

2) Then, find the optimal phases of B, X, d, denoted by

$$\phi_B = [\phi_0^B, \phi_1^B, \cdots, \phi_{N-1}^B]^T,$$
$$\phi_X = [\phi_0^X, \phi_1^X, \cdots, \phi_{N-1}^X]^T,$$
$$\phi_d = [\phi_0^d, \phi_1^d, \cdots, \phi_{N-1}^d]^T$$

respectively.

a) Randomly initialize the values of ϕ_B. Set the value of the penalty factor μ. If the local AF has a higher priority, μ is set to be a larger value.

b) Fix B, and use B to be the initial sequence in Algorithm 1. Then, output the sequence X, $\phi_X = \arg\{X\}$.

c) Fix B, get the phase of d,

$$\phi_d = \arg\{F_N^H B\}. \tag{13}$$

d) Fix X and d, choose

$$\phi_B = \arg\{\mu X + (1-\mu) F_N d\} \tag{14}$$

Repeat steps b),c) and d) iteratively until an exit criterion is met.

The algorithm discussed above is introduced in Algorithm 2.

Algorithm 2 :Get a CR sequence b with good local AF and low PAPR under spectrum hole constraint set Ω

Step 0: Find the magnitude of B according to N and Ω. Randomly initialize $\phi_B^{(0)}$ and set the parameter μ.
Step 1: Fix B, calculate ϕ_X by using **Algorithm 1** and update X.
Step 2: Fix B, calculate ϕ_d by $\phi_d = \arg\{F_N^H B\}$ then update d.
Step 3: Fix X and d, calculate ϕ_B by $\phi_B = \arg\{\mu X + (1-\mu) F_N d\}$.
Step 4: Update B by ϕ_B, and go to **Step 1** until an exit criterion is met.
Step 5: Get the CR sequence b by using IDFT: $b = F_N^H B$.

The exit criterion in Algorithm 1 and Algorithm 2 is a lower threshold on the minimum improvement between two successive iterations or an upper limit on the number of iterations.

4. Numerical Experiments and Discussions

In this section, the performance of the new sequence generated by Algorithm 2 is compared with that of the CR sequences which are generated by HL algorithm in [5] under the same spectrum hole constraint. The PAPR and the joint resolution in the area of interest are two important aspects to evaluate sequence performance in our comparison.

Let $N = 64$, the set $P = \{-3, -2, \cdots, 3\}$ and $M = \{-10, -9, \cdots, -1, 0, 1, \cdots, 10\}$. The spectrum hole constraint set is $\Omega_1 = \{15, 16, 17, 18\} \cup \{45, 46, 47, 48\}$. The coefficients $w_{m,p}$ equal 1 if $m \in M$, $p \in P$, otherwise $w_{m,p} = 0$. The exponent q equals 4 and the parameter μ equals 0.5 in the experiments.

The AFs of CR sequences with different penalty factor and the proposed sequence are showed in Fig. 2, Fig. 3 and Fig. 4.

Fig. 2 presents the AF of the CR sequence with penalty factor 0.6. In the area of interest, the joint resolution is 0.32 and the maximum normalized sidelobe is 0.2993. The PAPR of the sequence is 2.9403dB.

Fig. 3 shows the AF of the CR sequence with penalty factor 0.2. The joint resolution is 0.2924 and the maximum normalized sidelobe is 0.3930 in the area of interest. The PAPR of the sequence is 1.0286dB.

Fig. 4 depicts the AF of the sequence that is designed by the proposed algorithm under Ω_1. The time-domain sequence will be presented in Appendix A. It is obvious that the AF in the area of interest is lower than other areas. The joint resolution is 0.7204 and the maximum normalized sidelobe is 0.0997 in the area. The PAPR is 2.9603dB.

We have the following observations for Fig. 2 to Fig. 4.

1. In order to focus on the evaluation of the local AF per-

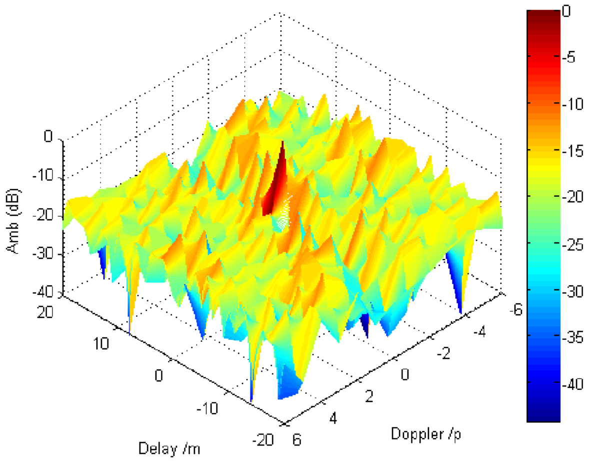

Fig. 2: AF of the CR sequence with penalty factor 0.6 [5].

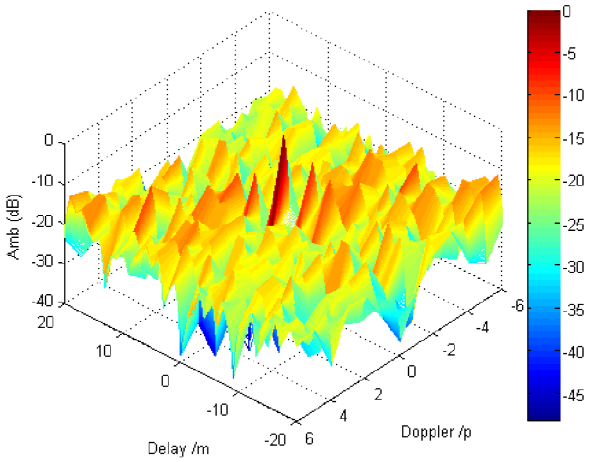

Fig. 3: AF of the CR sequence with penalty factor 0.2 [5].

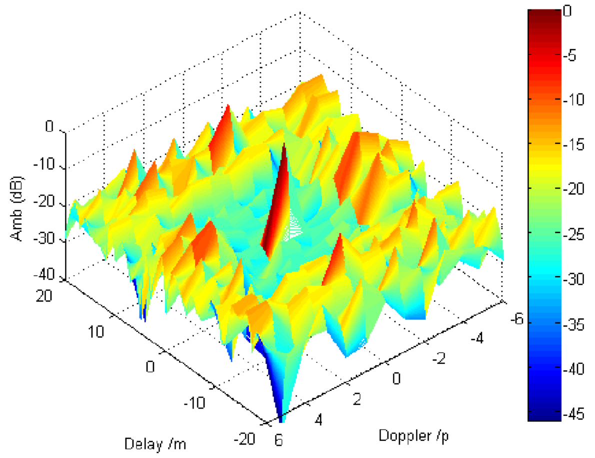

Fig. 4: AF of the sequence generated by the proposed algorithm under Ω_1.

formance only, a sequence is generated according to proposed scheme with the same PAPR as the CR sequence with 0.6. Meanwhile, the frequency-domain magnitudes of the two sequences are set same as well. Compared with the CR sequence with 0.6, the new sequence has higher joint resolution in the area of interest. The joint resolution is improved from 0.32 to 0.7204. The maximum normalized sidelobe is reduced from 0.2993 to 0.0997.

2. Compared with the CR sequence with 0.2, the PAPR of our new sequence is higher. But the joint resolution in the area of interest is improved further from 0.2924 to 0.7204. And the maximum sidelobe is lower than the CR sequence in the area decreased from 0.3930 to 0.0997.

Now, another spectrum hole constraint set is considered. The spectrum hole constraint set is $\Omega_2 = \{0\} \cup \{27, 28, \cdots, 37\}$ which is used in IEEE 802.11a to specify the nulled DC-subcarrier and guard bands. The set $P = \{-6, -5, \cdots, 6\}$ and $M = \{-5, -4, \cdots, -1, 1, \cdots, 5\}$. The coefficients $w_{m,p}$ equal 1 if $m \in M$, $p \in P$, otherwise $w_{m,p} = 0$. The exponent q equals 4 and the parameter μ equals 0.95 in the experiments.

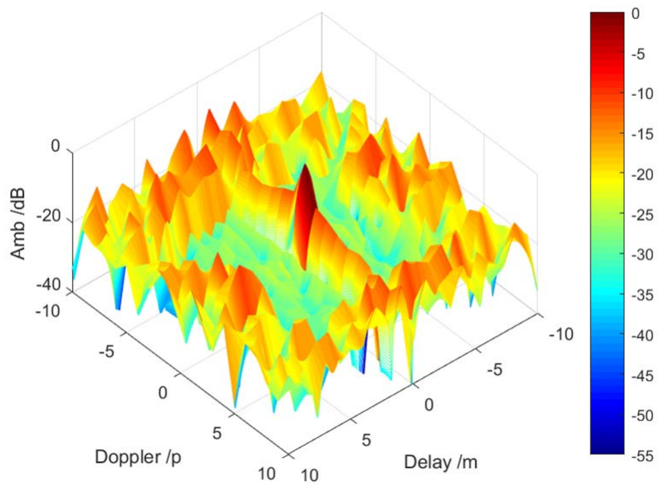

Fig. 5: AF of the sequence generated by the proposed algorithm under Ω_2.

Fig. 5 presents the AF of the sequence that is designed by the proposed algorithm under Ω_2. The PAPR is 6.4684dB. It is obvious that the sequence can be used to resist Doppler shift. In other word, the sequence is a local Doppler-tolerant sequence.

5. Conclusions

In this paper, an optimization problem is formulated by considering the local AF and PAPR of sequences under spectrum hole constraint. This problem is then solved by using energy gradient method and HL algorithm to generate sequences with good local AF and low PAPR. Compared with the CR sequences under the same spectrum hole constraint, the numerical results show that the proposed sequences possess larger joint resolution and lower maximum sidelobe in the area of interest, i.e., better the local AF performance. Our proposed algorithm can also generate local Doppler-tolerant sequences under arbitrary spectrum hole constraint by adjusting some parameters.

Acknowledgments

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Appendix A:

The sequence generated by the proposed algorithm 2 is represented. $b = [b_1, b_2, \cdots, b_{N-1}]$, with $b_i = |b_i|e^{j\phi_i}$. $|b|$ denotes the sequence magnitude vector, and $\phi\{b\}$ is the phase vector.

$$|\boldsymbol{b}| = \begin{bmatrix} 1.2456,0.9621,1.0319,1.2427,0.7367,0.6888, \\ 0.9731,0.9114,0.8885,1.2457,0.9864,1.1915, \\ 1.1921,0.6488,1.0677,1.1828,0.9013,0.8910, \\ 1.1431,0.7562,1.1069,0.8700,0.7879,0.9716, \\ 1.2494,1.3807,0.9614,0.9020,0.9719,0.8876, \\ 0.9921,0.5699,0.9228,0.8563,0.9122,1.1112, \\ 0.6149,0.9622,1.1832,1.1195,1.1100,0.8129, \\ 0.7579,1.0193,0.8943,0.8591,0.9430,0.9654, \\ 0.8001,1.1248,0.9238,0.8479,1.2897,0.7647, \\ 0.7496,1.2137,1.2633,0.6748,1.4061,1.2800, \\ 0.9091,1.0158,0.9367,1.0435 \end{bmatrix}$$

$$\boldsymbol{\phi}\{\boldsymbol{b}\} = \begin{bmatrix} 0.0743, 0.0412, 0.8821, 2.5199, 0.4484,-2.5843, \\ 0.4043,-2.9068, 2.1680, 1.1902, 1.6986,-0.2220, \\ -0.0481,-1.6217,-2.4788,-0.2232,-1.4501, 0.0661, \\ 1.3119, 1.7785, 2.1916,-1.9390, 2.3632, 1.4897, \\ 1.7583, 0.1366,-0.9647, 1.9896,-1.1340, 1.2072, \\ -2.9730, 0.3972, 2.8009,-2.8857,-2.0559,-1.3990, \\ 0.8753,-0.0081,-1.7302,-1.9939,-2.5063, 2.0628, \\ -1.1441, 1.9194,-2.0228,-0.6866, 2.3147,-1.1311, \\ -1.6952, 1.1147, 1.0458, 1.6849, 2.2452, 1.2186, \\ 0.4693,-0.0283, 2.4806,-1.1456, 2.3195, 0.3942, \\ -0.2457,-3.1303,-1.2001,-0.5120 \end{bmatrix}$$