Doppler Resilient Z-complementary Waveforms From ESP Sequences

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Abstract In this paper, we present a new method of constructing Z-complementary pulse trains from sequences with property of Equal Sums of (like) Powers (called ESP sequences) which are resilient to Doppler shifts. Numerical experiments indicate that the proposed Z-complementary pulse trains obtained from ESP sequences have better Doppler resilience than waveforms constructed from the well-known Prouhet-Thue-Morse (PTM) sequences.

key words: *Doppler resilient, Z-complementary sequences, ESP sequences.*

1. Introduction

In radar, we often need to localize the received signals by matched filter. A radar waveform is said to be desirable if it is resilient to both Doppler- and time- shifts, i.e., low ambiguity function. To generate this type of waveforms, a commonly used technique is phase coding [1], [2] in which a long pulse is phase coded with a unimodular sequence. The idea of phase coding is to control the autocorrelation function of the coded waveform through the autocorrelation function of the unimodular sequence.

Golay complementary pairs (GCP) and complementary sets of sequences play an important role in generating desirable radar waveforms [3]–[7]. A GCP has the property that the sum of aperiodic autocorrelations of two constituent sequences equals zero at all nonzero delays [8]. Unfortunately, the ambiguity function of GCP is only free of delay (range) sidelobes along the zero-Doppler axis, but may have large range sidelobes off the zero-Doppler axis [8].

Pezeshki et al. [8] proposed a systematic way of designing Doppler resilient waveforms by coding a train of GCPs with Prouhet-Thue-Morse (PTM) sequences [8]. It is noted that [8] mainly focused on the Doppler shifts around zero and achieved the Doppler resilient performance in the neighborhood of zero Doppler shift. Recent advances and extensions of the idea of [8] can be found in [9]–[13]. In particular, Chi et al. [9] considered the rational Doppler shift not limited in neighborhood of zero but in neighborhood of rational $\theta = \frac{2\pi l}{m}$ (away from zero Doppler) for which PTM sequences are still important. Doppler resilience (of a waveform) away from zero Doppler may be used to enhance the radar detection performance in scenarios when Doppler shifts are roughly known over certain region. Nguyen et al. [10] obtained new Doppler-resilient sequences by employing the Equal Sums of like Powers (ESP) sequences [10] to achieve similar Doppler tolerance in the neighborhood of zero Doppler shift.

However, no exploration of ESP sequences is known which can achieve Doppler resilience in the neighborhood of the $\theta = \frac{2\pi l}{m}$. Furthermore, all the existing works only considered GCP as "seed" sequences. It is noted that binary GCPs have very limited lengths, i.e., $L = 2^a 10^b 26^c$ for $a, b, c \geq 0$. As an alternative to GCP, Fan et al [14]–[20] proposed Z-complementary sequences which exist for essentially all lengths $L \geq Z$, where Z is the zero correlation zone. In fact, Golay complementary sequences may be viewed as a special case of Z-complementary sequences when $Z = L$.

In this paper, we use Z-complementary sequences associated with ESP sequences to analyze the Doppler resilient around the rational Doppler shift $\theta = \frac{2\pi l}{m}$. The numerical examples demonstrate that for Doppler resilience in rational Doppler intervals the ESP sequences are noticeably better than PTM sequences in a way.

2. Z-complementary Waveform

Let $x = (x[0], x[1], \cdots, x[L-1])$ and $y = (y[0], y[1], \cdots, y[L-1])$ be length-L binary unimodular sequences. We call them a Z-complementary pair if the sum of their autocorrelation functions satisfies

$$C_x(k)+C_y(k) = \begin{cases} 2L, & \text{for} \quad k = 0 \\ 0, & \text{for} \quad k = \pm 1, \pm 2, \cdots, \pm(Z-1), \end{cases}$$

where Z is called zero correlation zone (ZCZ), and $C_x(k) = \sum_{l=0}^{L-k-1} x[l]x^*[l+k]$ is the aperiodic autocorrelation of x. Of course, if $Z = L$, then the Z-complementary pair becomes the Golay complementary pair. [14]

In the following part, we use z-transform to analyze our problem, so we have to define Z-complementary sequence using z-transform like [8].

Let $X(z) = \mathcal{Z}\{x\}$ and $Y(z) = \mathcal{Z}\{y\}$ be the z-transforms of x and y so that

$$X(z) = x[0] + x[1]z^{-1} + \cdots + x[L-1]z^{-(L-1)},$$

$$Y(z) = y[0] + y[1]z^{-1} + \cdots + y[L-1]z^{-(L-1)}.$$

x and y (or alternatively $X(z)$ and $Y(z)$) are Z-complementary if $X(z)$ and $Y(z)$ satisfy

$$\begin{aligned} X(z)\widetilde{X}(z) + Y(z)\widetilde{Y}(z) &= |X(z)|^2 + |Y(z)|^2 \\ &= 2L + a(z) \end{aligned}$$

where $\widetilde{X}(z) = X^*(1/z^*)$ and $\widetilde{Y}(z) = Y^*(1/z^*)$ are the z-transforms of $\widetilde{x}[l] = x^*[-l]$ and $\widetilde{y}[l] = y^*[-l]$, and

$$a(z) = \sum_{i=Z}^{L-1} (C_x(i)z^{-i} + C_y(i)z^{-i} + C_x(i)z^{i} + C_y(i)z^{i}).$$

If we transmit the binary pulse train $s_0, s_1, \cdots, s_{N-1}$ whose lengths are L, then the z-transform of ambiguity function like [8] is given by

$$G(z,\theta) = \sum_{n=0}^{N-1} e^{jn\theta} |S_n(z)|^2. \tag{1}$$

We can analyze the Doppler resilient interval like [8] [9] to select Z-complementary train $s_0, s_1, \cdots, s_{N-1}$. So we consider the Taylor expansion of $G(z,\theta)$ around $\theta = \theta_0$, i.e.,

$$\begin{aligned}
G(z,\theta) &= \sum_{n=0}^{N-1} e^{jn\theta} |S_n(z)|^2 \\
&= \sum_{n=0}^{N-1} (e^{jn\theta_0} |S_n(z)|^2 + jne^{jn\theta_0} |S_n(z)|^2 (\theta-\theta_0) \\
&\quad + \cdots + \frac{1}{t!} (jn)^t e^{jn\theta_0} |S_n(z)|^2 (\theta-\theta_0)^t + \cdots) \\
&= \sum_{t=0}^{\infty} \frac{1}{t!} f^{(t)}(z,\theta_0) (j(\theta-\theta_0))^t
\end{aligned}$$

where

$$f^{(t)}(z,\theta_0) = \sum_{n=0}^{N-1} n^t |S_n(z)|^2 e^{jn\theta_0}. \tag{2}$$

It should be mentioned that $f^{(t)}(z,\theta_0)$ is still a polynomial of z as follows:

$$f^{(t)}(z,\theta_0) = \sum_{l=-(L-1)}^{L-1} \beta_{t,l} z^{-l}. \tag{3}$$

To achieve the goal of Doppler resilience in desired Doppler interval like the neighborhood of θ_0, the coefficients of all terms up to a certain order, say M, should vanish at all nonzero proper delays, i.e., $\pm 1, \pm 2 \cdots, \pm(Z-1)$. In other words, for any fixed $t \in \{1, 2 \cdots, M\}$, $f^{(t)}(z,\theta)$ as (3) should make $\beta_{t,-(Z-1)}, \cdots, \beta_{t,-1}, \beta_{t,1} \cdots, \beta_{t,Z-1}$ vanish.

Firstly, on the basis of [9], we also consider rational Doppler shifts $\theta_0 = 2\pi l/m$, where $l \neq 0$ and $m \neq 0$ are co-prime integers. We assume $N = mq$ for some integer q. Let $n = rm + i$, where $0 \le r \le q-1$ and $0 \le i \le m-1$. Then we use the binomial expansion for $n^t = (rm+i)^t$ to write $f^{(t)}(z,\theta_0)$ in (2) as follows:

$$\begin{aligned}
&f^{(t)}(z, 2\pi l/m) \\
&= \sum_{r=0}^{q-1} \sum_{i=0}^{m-1} (rm+i)^t |S_{rm+i}(z)|^2 e^{j\frac{2\pi l(rm+i)}{m}} \\
&= \sum_{r=0}^{q-1} \sum_{i=0}^{m-1} \sum_{u=0}^{t} \binom{t}{u} m^u i^{t-u} e^{j\frac{2\pi li}{m}} r^u |S_{rm+i}(z)|^2 \\
&= \sum_{u=0}^{t} \binom{t}{u} m^u \sum_{i=0}^{m-1} i^{t-u} e^{j\frac{2\pi li}{m}} [\sum_{r=0}^{q-1} r^u |S_{rm+i}(z)|^2].
\end{aligned}$$

To make $f^{(t)}(z,\theta_0)$ vanish at all nonzero proper delays $\pm 1, \pm 2, \cdots, \pm(Z-1)$, $\sum_{r=0}^{q-1} r^u |S_{rm+i}(z)|^2$ should vanish at all nonzero proper delays. This can be achieved by Z-complementary pulse train arranged by PTM sequences [9] or ESP sequences [10] so that

$$\begin{aligned}
&\sum_{r=0}^{q-1} r^u |S_{rm+i}(z)|^2 \\
&= \sum_{r\in A_0} r^u |X(z)|^2 + \sum_{r\in A_1} r^u |Y(z)|^2 \\
&= 2LP_u + \sum_{l=Z}^{L-1} (\bar{\beta}_l z^{-l} + \bar{\beta}_{-l} z^l)
\end{aligned}$$

where $A_0, A_1 \subset \{0, 1, \cdots, q-1\}$, x with y (or alternatively $X(z)$ with $Y(z)$) is a Z-complementary pair and

$$P_u := \sum_{r\in A_0} r^u = \sum_{r\in A_1} r^u. \tag{4}$$

It is worthy to mention that if $S_{rm+i}(z) = X(z) \oplus Y(z)$ where $X(z) \oplus Y(z)$ means x and y are simultaneously sent from two different antennas, then $|X(z) \oplus Y(z)|^2$ can be defined as $|X(z) \oplus Y(z)|^2 = |X(z)|^2 + |Y(z)|^2$.

In fact, the equation (4) is satisfied by PTM sequences as in [9]. The equation (4) is also satisfied by ESP sequences as will show in next section.

3. Z-complementary Pulse Train Constructed from ESP sequences

In this section, we will introduce the definition of ESP sequences which can be found in [10], [21].

Let A_0 and A_1 be two disjoint subsets of P whose elements are non-negative integers. Then A_0, A_1 are equal sums of (like) powers (ESP) sequences of degree t if

$$\sum_{n\in A_0} n^u = \sum_{n\in A_1} n^u,$$

for $u = 1, ..., t$ [10]. Based on the above definition, generalized ESP sequences can be defined if some numbers in $P \setminus (A_0 \cup A_1)$ are inserted into A_0 and A_1 simultaneously, then A_0 and A_1 can be called generalized ESP.

Example 1: Let $A_0 = (0, 4, 5)$ and $A_1 = (1, 2, 6)$ in $P = \{0, 1, \cdots, 6\}$, then A_0 and A_1 form an ESP pair of degree 2, since

$$0^u + 4^u + 5^u = 1^u + 2^u + 6^u, \; u = 1, 2$$

The generalized ESP sequences of degree 2 can be $A_0 = (0, 3, 4, 5)$ and $A_1 = (1, 2, 3, 6)$.

Similarly, $A_0 = (0, 5, 6, 16, 17, 22)$ and $A_1 = (1, 2, 10, 12, 20, 21)$ in $P = \{0, 1, \cdots, 22\}$ from a pair of ESP sequences of degree 5, since

$$0^u+5^u+6^u+16^u+17^u+22^u = 1^u+2^u+10^u+12^u+20^u+21^u,$$

for $u = 1, 2, 3, 4, 5$.

The generalized ESP sequences of degree 5 can be $A_0 = (0, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 22)$,

$A_1 = (1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21)$.

In example 1, the union of ESP sequences $A_0 = \{0, 4, 5\}$ and $A_1 = \{1, 2, 6\}$ compared with $\{0, 1, 2, 3, 4, 5, 6\}$ lacks number 3. This could result in pulse trains with gaps which is not permitted in radar [10]. Based on this fact, we can insert the number 3 into both ESP sequences so that the generalized ESP sequences $A_0 = (0, 3, 4, 5)$ and $A_1 = (1, 2, 3, 6)$. Note that MIMO transmission of both ESP sequences will be used since we transmit different signals at the same time [10]. It is clear that the generalized ESP sequences enable the satisfaction of (4).

Now we give an example to explain how the generalized ESP arrange the Z-complementary pulse train and to derive the concrete expression of composite ambiguity function [10].

Example 2: Let the generalized ESP sequences be $A_0 = \{0, 3, 4, 5\}$ and $A_1 = \{1, 2, 3, 6\}$, then the equation (4) is satisfied. For simplicity we denote $S_n(z)$ as S_n, $X(z)$ as X and $Y(z)$ as Y . If $m = 3$, then for S_{rm+i} in (4),we have

$$(S_{rm+i})_{r=0}^{6} = (X, Y, Y, X \oplus Y, X, X, Y), i = 0, 1, 2.$$

In other words,

$$\begin{aligned}(S_n)_{n=0}^{20} = (&X, X, X, Y, Y, Y, Y, Y, Y, X \oplus Y,\\ &X \oplus Y, X \oplus Y, X, X, X, X, X, X, Y, Y, Y)\end{aligned}$$

where $X \oplus Y$ means x and y are simultaneously sent by different antennas in some time slot. Indeed, $(X_n)_{n=0}^{20}$ can be obtained by two antennas:

$$\begin{aligned}T_0 &= (x, x, x, y, y, y, y, y, y, x, x, x),\\ T_1(9) &= (y, y, y, x, x, x, x, x, x, y, y, y).\end{aligned}$$

It's important to note that $T_1(9)$ transmits signals 9 PRIs (Pulse repetition intervals) later than T_0. $T_1(9) = (x_0, x_1, \cdots, x_{K-1})$ is a delayed pulse train if z-transform of its ambiguity function is as follows [10]

$$G_{T_1}(z, \theta, 9) = \sum_{n=0}^{K-1} |X_n(z)|^2 e^{j(n+9)\theta}.$$

Then z-transform of composite ambiguity function [10] of the two pulse trains is given by

$$\begin{aligned}&G(z, \theta)\\ &= G_{T_0}(z, \theta) + G_{T_1}(z, \theta, 9)\\ &= |X|^2 + |X|^2 e^{j\theta} + |X|^2 e^{j2\theta} + |Y|^3 e^{j3\theta} + |Y|^2 e^{j4\theta}\\ &+ |Y|^2 e^{j5\theta} + |Y|^2 e^{j6\theta} + |Y|^2 e^{j7\theta} + |Y|^2 e^{j8\theta}\\ &+ (|X|^2 + |Y|^2) e^{j9\theta} + (|X|^2 + |Y|^2) e^{j10\theta}\\ &+ (|X|^2 + |Y|^2) e^{j11\theta} + |X|^2 e^{j12\theta} + |X|^2 e^{j13\theta}\\ &+ |X|^2 e^{j14\theta} + |X|^2 e^{j15\theta} + |X|^2 e^{j16\theta} + |X|^2 e^{j17\theta}\\ &+ |Y|^2 e^{j18\theta} + |Y|^2 e^{j19\theta} + |Y|^2 e^{j20\theta}. \qquad (5)\end{aligned}$$

Obviously, the equation (5) is equivalent to (1), when $|X(z) \oplus Y(z)|^2 = |X(z)|^2 + |Y(z)|^2$ is used.

Fig. 1 shows the ambiguity function, i.e., the inverse transform of (5) in which ESP is degree 2. In fact, we can also use ESP with other degree up to degree 11 [21] . In next section, ESP with degree 5 will be used .

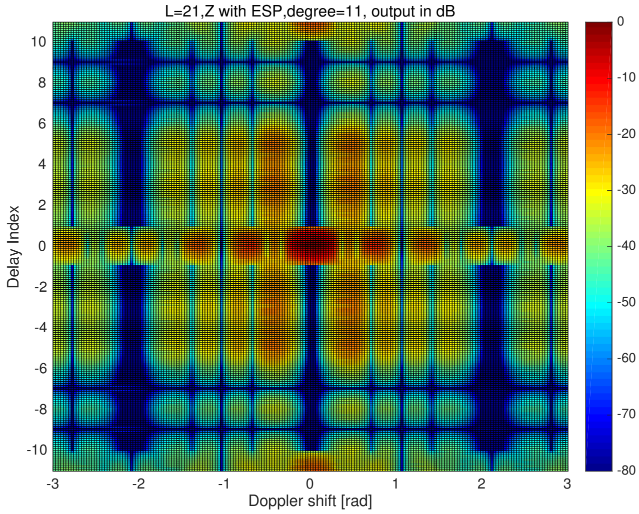

Fig. 1: Ambiguity function of the degree=2 ESP pulse train of Z-complementary waveform.

4. Numerical Experiments and Discussion

In this section we aim at comparing ESP with PTM to demonstrate the Doppler resilient performance. We choose a Z-complementary pair whose lengths are 21 and $Z_{max} = 11$ in [14]. We also choose ESP and PTM with degree 5 (or 5-order) and consider the rational Doppler shift $\theta = \frac{2\pi}{3}$, i.e., $m = 3, l = 1$ in $\frac{2\pi l}{m}$.

Some conclusions can be drawn based on Fig. 2 to Fig. 7.

- Fig. 2 and Fig. 3 show the ambiguity functions in the Doppler shift interval $[-3, 3]$, which is determined by ESP and PTM respectively. Compared the Fig. 2 with Fig. 3, ESP's performance is better than PTM's in some way. Since ESP's band around $\theta = 0$ is clearly broader than PTM's.
- Fig. 4 and Fig. 5 show the ambiguity functions in the Doppler shift interval $[\frac{2\pi}{3} - 0.1, \frac{2\pi}{3} + 0.1]$, which is determined by ESP and PTM respectively. Obviously, the dark color band of ESP is at least 0.04 broader than that of PTM .
- Fig. 5 and Fig. 6 show the ambiguity function determined by 11-order PTM pulse train is not better than that of 5-order PTM (see Fig. 5) or degree 5 ESP (see Fig. 4) in the Doppler shift interval $[\frac{2\pi}{3} - 0.1, \frac{2\pi}{3} + 0.1]$.
- Fig. 7 shows the ambiguity functions determined by ESP and that by PTM at Doppler shift $\theta = 2\pi/3 - 0.075$. In proper delay interval $[-9, 9]$, the peaks of sidelobe determined by ESP are at least 15dB smaller than those determined by PTM.

In short, ESP's Doppler resilient performance is significantly better than PTM's.

5. Conclusions

In this paper, we have constructed Doppler resilient pulse train of Z-complementary waveforms from Equal Sums of

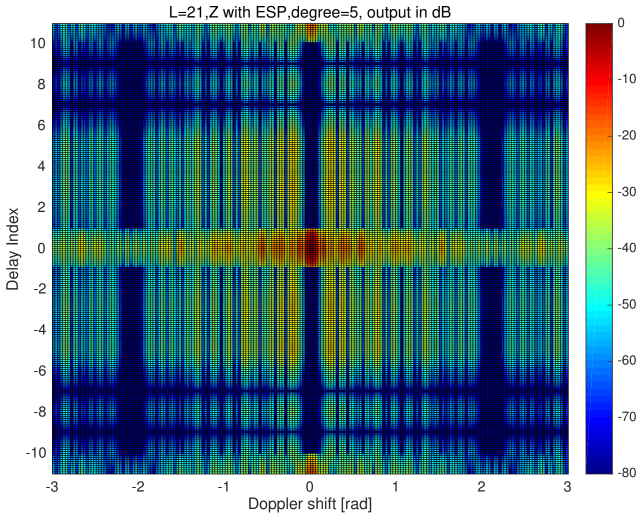

Fig. 2: Ambiguity function of the degree=5 ESP pulse train of Z-complementary waveform, the entire Doppler band.

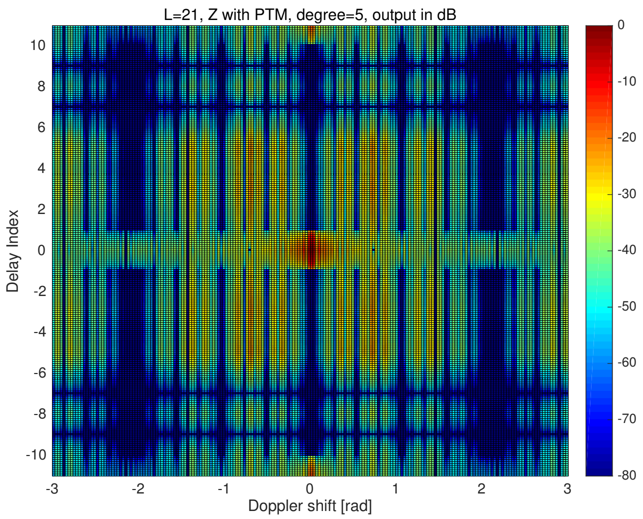

Fig. 3: Ambiguity function of the 5-order PTM pulse train of Z-complementary waveform, the entire Doppler band.

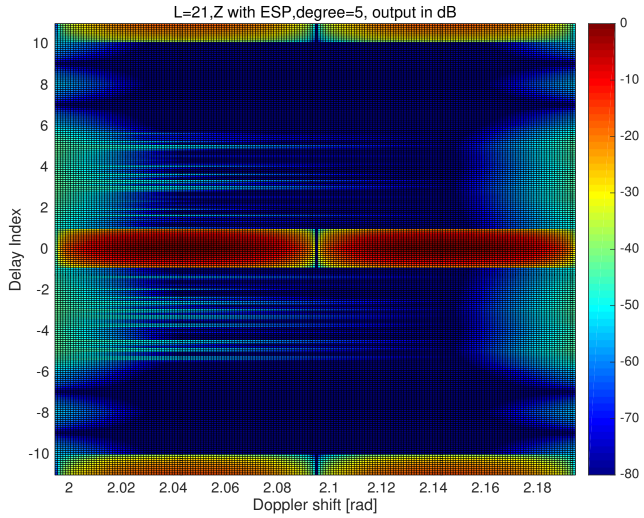

Fig. 4: Ambiguity function of the degree=5 ESP pulse train of Z-complementary waveform, in $[\frac{2\pi}{3}-0.1, \frac{2\pi}{3}+0.1]$.

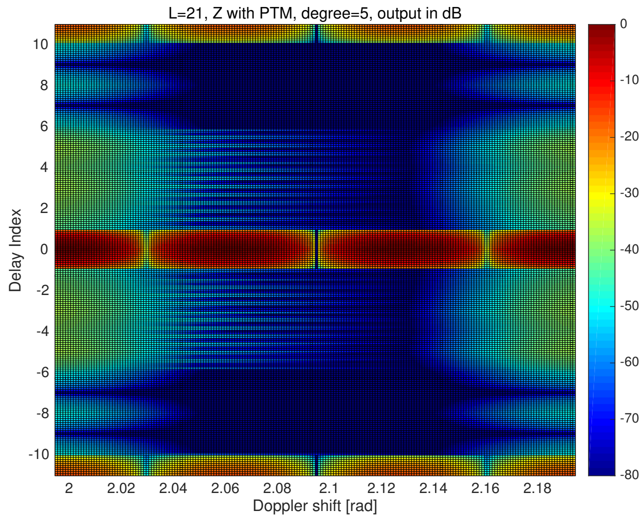

Fig. 5: Ambiguity function of the 5-order PTM pulse train of Z-complementary waveform, in $[\frac{2\pi}{3}-0.1, \frac{2\pi}{3}+0.1]$.

like Power (ESP) sequences for which the ambiguity function is not sensitive to nonzero proper delays in certain Doppler interval. The main contribution is a method for choosing Z-complementary sequences to force the low-order terms of the Taylor expansion of an ambiguity function to zero at all nonzero proper delays for which ESP sequences play a key role. ESP sequences are known to exist only for degree up to 11. However, in the desired Doppler resilient interval, the Doppler resilient performance is not strictly proportional to the degree. In the Numerical Experiments, degree 5 ESP sequences is compared with 5-order PTM sequences. It is shown that Doppler resilience of Z-complementary pulse trains determined by ESP sequences is significantly better than that by PTM sequences.

Acknowledgments

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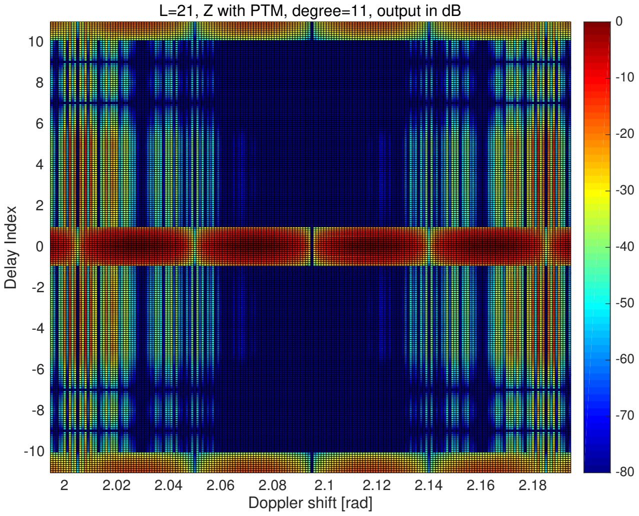

Fig. 6: Ambiguity function of the 11-order PTM pulse train of Z-complementary waveform, in $[\frac{2\pi}{3} - 0.1, \frac{2\pi}{3} + 0.1]$.

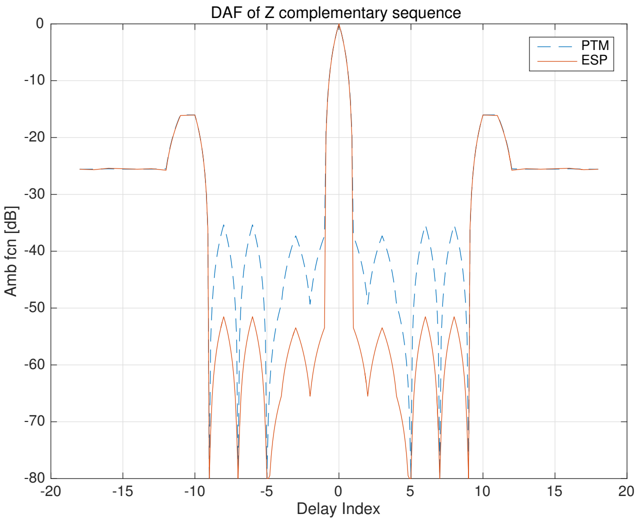

Fig. 7: A cut of ambiguity function of the 5-order PTM and degree 5 ESP at Doppler shift $\theta = 2\pi/3 - 0.075$ rad.

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