A Direct and Generalized Construction of Polyphase Complementary Set With Low PMEPR

Palash Sarkar
Department of Mathematics
Indian Institute of Technology Patna
Bihar-801103, India
palash.pma15@iitp.ac.in

Sudhan Majhi
Department of EE
Indian Institute of Technology Patna
Bihar-801103, India
smajhi@iitp.ac.in

Zilong Liu
5G Innovation Centre
University of Surrey
UK-GU2 7XH
zilong.liu@surrey.ac.uk

***Abstract*—A salient disadvantage of orthogonal frequency division multiplexing (OFDM) systems is the high peak-to-mean envelope power ratio (PMEPR). The PMEPR problem can be solved by using complementary sequences with low PMEPR. In this paper, we present a new construction of complementary set (CS) by using generalized Boolean functions (GBFs), which generalizes the constructions given by Davis *et al.*, Paterson and Schmidt. The proposed CS provides lower PMEPR upper bound as compared to Schmidt's method for the sequences corresponding to higher order (≥ 3) GBFs. We obtain complementary sequences with maximum PMEPR of 2^{k+1} and $2^{k+2}-2M$ where k, M are non-negative integers that can be easily derived from the GBF associated with the CS.**

***Index Terms*—Complementary set (CS), Golay complementar pair (GCP), generalized Boolean function (GBF), orthogonal frequency-division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR), Reed-Muller (RM) code.**

I. Introduction

A major problem of orthogonal frequency-division multiplexing (OFDM) is its large peak-to-mean envelope power ratio (PMEPR) for the uncoded signals. PMEPR reduction through a coding technique can be achieved by designing a large codebook whose codewords, e.g., in the form of sequences, have low PEMPR values. This paper is focused on PMEPR reduction using codebooks from complementary sequences which to be introduced in the sequel.

The concept of Golay complementary pair (GCP) was introduced by M. J. E. Golay in [1], where either sequence from a GCP is known as Golay sequence. By definition, the aperiodic autocorrelation (AAC) sidelobes of two constituent sequences in a GCP sum to zero everywhere except for the in-phase time-shift position. GCP was extended to complementary sets (CSs) by Tseng and Liu in [2] where each CS consists of two or more constituent sequences, called complementary sequences, and with zero AAC sum property. A PMEPR reduction method was introduced by Davis and Jedwab in [3] to construct Golay sequences of length 2^m (m is a positive integer) by using second-order generalized Boolean function (GBF). Subsequently, Paterson employed complementary sequences to enlarge the code rate by relaxing the PMEPR of OFDM signal in [4]. Specifically, Paterson showed that each coset of generalized first-order Reed-Muller (RM) codes $RM_q(1,m)$ inside second-order generalized RM codes $RM_q(2,m)$ (q is an even number no less than 2) can be partitioned into CSs of size 2^{k+1} (where k is a non-negative integer depending only on $G(Q)$, a graph naturally associated with the quadratic form Q in m variables which defines the coset) and provided an upper bound on the PMEPR of arbitrary second-order cosets of $RM_q(1,m)$. The construction given in [4, Th. 12] was unable to provide a tight PMEPR bound for all the cases. By giving an improved version of [4, Th. 12] in [4, Th. 24], Paterson raised the following question :
"What is the strongest possible generalization of [4, Th. 12]?".

In [4, Th. 24], it is shown that after deleting k vertices in $G(Q)$, if the resulting graph contains a path and one isolated vertex, then $Q+RM_q(1,m)$ can be partitioned into CSs of size 2^{k+1} instead of 2^{k+2}, i.e., there is no need to delete the isolated vertex. Later, a generalization of [4, Th. 12] was made by Schmidt in [5] to establish a construction of complementary sequences that are contained in higher-order generalized RM codes. Schmidt shows that a GBF also can be used to generate a CS of a given size if it can be expressed by a path when it is restricted to certain variables. However, Schmidt's construction is not applicable when the graphs of some or all restricted Boolean functions are not path. As a result, a reasonable number of sequences with low PMEPR were excluded from the Schmidt's construction. Hence, an improved version of [5, Th. 5] or, a more generalized version of [4, Th. 12] is expected to extend the range of coding options with good PMEPR bound for practical applications of OFDM.

More constructions of CS with low PMEPR can be found in [6]–[9]. In [6], a framework is given to identify known Golay sequences and pairs of length 2^m ($m>4$) over $\mathbb{Z}_{2^h}$ (h is a positive integer) in explicit algebraic normal form. Nowadays, besides polyphase complementary sequences, the design of quadrature amplitude modulation (QAM) complementary sequences with low PMEPR is also an interesting research topic. In [8], QAM Golay seuences were introduced by Li, based on quadrature phase shift keying (PSK) GDJ-code. Later, Liu *et al.* constructed QAM Golay sequences by using properly selected Gaussian integer pairs [9]. Recently, some constructions on complementary or near-complementary sequences have been reported in [10]–[14]. These sequences may also be applicable in OFDM system to deal with PMEPR problem, in addition to their applications in scenarios such as asynchronous communications and channel estimation.

In this paper, we propose a novel construction to generate CS with low PMEPR. It uses higher order GBFs and graphical properties of restricted Boolean functions. By taking advantage of certain graphical properties related to both Path and isolated vertices associated to higher order GBFs, we show that the PMEPR upper bound can be further tightened. By moving to higher order RM code, we give a partial answer to the aforementioned open question raised by Paterson on strongest generalization of [4, Th. 12]. It will be shown that our proposed construction includes the previous ones by Davis and Jedwab [3], Paterson [4], and Schmidt [5] as special cases.

The remaining paper is organized as follows. In Section II, some useful notations and definitions are given. In Section III, a generalized construction of CS is presented. Section IV contains a discussion. Finally, concluding remarks are drawn in Section V.

II. Preliminary

A. Definitions of Correlations and Sequences

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{L-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{L-1})$ be two complex-valued sequences of equal length L and let τ be an integer. Define

$$C(\mathbf{a},\mathbf{b})(\tau) = \begin{cases} \sum_{i=0}^{L-1-\tau} a_{i+\tau} b_i^*, & 0 \leq \tau < L, \\ \sum_{i=0}^{L+\tau-1} a_i b_{i-\tau}^*, & -L < \tau < 0, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

and $A(\mathbf{a})(\tau) = C(\mathbf{a},\mathbf{a})(\tau)$. The above mentioned functions are called the aperiodic cross-correlation function between $\mathbf{a}$ and $\mathbf{b}$ and the AAC function of $\mathbf{a}$, respectively.

Definition 1: A set of n sequences $\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^{n-1}$, each of equal length L, is said to be a CS if

$$A(\mathbf{a}^0)(\tau) + A(\mathbf{a}^1)(\tau) + \cdots + A(\mathbf{a}^{n-1})(\tau) = \begin{cases} nL, & \tau = 0, \\ 0, & \text{otherwise}. \end{cases} \quad (2)$$

A CS of size two is called GCP. Sequences lying in a CS of size n (≥ 4) is called complementary sequences in this paper.

B. PMEPR of OFDM signal

For q-PSK modulation, the OFDM signal for the word $\mathbf{a} = (a_0, a_1, \ldots, a_{L-1})$ (where $a_i \in \mathbb{Z}_q$) can be modeled as the real part of

$$S(\mathbf{a})(t) = \sum_{j=0}^{L-1} \omega_q^{a_j} e^{2\pi i (f_0 + j f_s) t},$$

where $\omega_q = \exp(2\pi\sqrt{-1}/q)$ is a complex qth root of unity and $f_0 + j f_s$ ($0 \leq j < L$) is jth carrier frequency of the OFDM signal. We define the instantaneous envelope power of the OFDM signal as [4]

$$P(\mathbf{a})(t) = |S(\mathbf{a})(t)|^2.$$

From the above expression, it is easy to derive that

$$\begin{aligned} P(\mathbf{a})(t) &= \sum_{\tau=1-L}^{L-1} A(\mathbf{a})(\tau) \exp(2\pi\sqrt{-1}\tau f_s t) \\ &= A(\mathbf{a})(0) + 2 \cdot \mathrm{Re}\left\{ \sum_{\tau=1}^{L-1} A(\mathbf{a})(\tau) \exp(2\pi\sqrt{-1}\tau f_s t) \right\}, \end{aligned} \quad (3)$$

where $\mathrm{Re}\{x\}$ denotes the real part of a complex number x. We define the PMEPR of the signal $S(\mathbf{a})(t)$ as

$$\mathrm{PMEPR}(\mathbf{a}) = \frac{1}{n} \sup_{0 \leq f_s t < 1} P(\mathbf{a})(t).$$

The largest value that the PMEPR of an n-subcarrier OFDM signal is n. It is also noted that the PMEPR of a sequence lying in a CS of size K (where, K is a positive integer no less than 2) is upper bounded by K.

C. Generalized Boolean Functions

Let f be a function of m variables $x_0, x_1, \ldots, x_{m-1}$ over $\mathbb{Z}_q$. A monomial of degree r is defined as the product of any r distinct variables among $x_0, x_1, \ldots, x_{m-1}$. There are 2^m distinct monomials over m variables listed below:

$1, x_0, x_1, \ldots, x_{m-1}, x_0x_1, x_0x_2, \ldots, x_{m-2}x_{m-1}, \ldots, x_0x_1 \ldots x_{m-1}$.

A function f is said to be a GBF of order r if it can be uniquely expressed as a linear combination of monomials of degree atmost r, where the coefficient of each monomial is drawn from $\mathbb{Z}_q$. A GBF of order r can be expressed as

$$f = Q + \sum_{i=0}^{m-1} g_i x_i + g', \quad (4)$$

where

$$Q = \sum_{p=2}^{r} \sum_{0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{p-1} < m} a_{\alpha_0,\alpha_1,\ldots,\alpha_{p-1}} x_{\alpha_0} x_{\alpha_1} \cdots x_{\alpha_{p-1}}, \quad (5)$$

and $g_i, g', a_{\alpha_0,\alpha_1,\ldots,\alpha_{p-1}} \in \mathbb{Z}_q$.

D. Quadratic Forms and Graphs

Let f be a rth order GBF of m variables over $\mathbb{Z}_q$. Assume $\mathbf{x} = (x_{j_0}, x_{j_1}, \ldots, x_{j_{k-1}})$ and $\mathbf{c} = (c_0, c_1, \ldots, c_{k-1})$. Then $f\big|_{\mathbf{x}=\mathbf{c}}$ is obtained by substituting $x_{j_\alpha} = c_\alpha$ ($\alpha = 0, 1, \ldots, k-1$) in f. If $f\big|_{\mathbf{x}=\mathbf{c}}$ is a quadratic GBF, then $G(f\big|_{\mathbf{x}=\mathbf{c}})$ denotes a graph with $V = \{x_0, x_1, \ldots, x_{m-1}\} \setminus \{x_{j_0}, x_{j_1}, \ldots, x_{j_{k-1}}\}$ as the set of vertices. The $G(f\big|_{\mathbf{x}=\mathbf{c}})$ is obtained by joining the vertices x_{α_1} and x_{α_2} by an edge if there is a term $q_{\alpha_1\alpha_2} x_{\alpha_1} x_{\alpha_2}$ ($0 \leq \alpha_1 < \alpha_2 < m$, x_{α_1}, $x_{\alpha_2} \in V$) in the GBF $f\big|_{\mathbf{x}=\mathbf{c}}$ with $q_{\alpha_1\alpha_2} \neq 0$ ($q_{\alpha_1\alpha_2} \in \mathbb{Z}_q$). For $k = 0$, $G(f\big|_{\mathbf{x}=\mathbf{c}})$ is nothing but $G(f)$. Throughout this paper

E. Sequence Corresponding to a Boolean Function

Corresponding to a GBF f, we define a complex-valued vector (or, sequence) $\psi(f)$, as follows.

$$\psi(f) = (\omega_q^{f_0}, \omega_q^{f_1}, \ldots, \omega_q^{f_{2^m-1}}), \tag{6}$$

where $f_i = f(i_0, i_1, \ldots, i_{m-1})$ and $(i_0, i_1, \ldots, i_{m-1})$ is the binary vector representation of integer i ($i = \sum_{\alpha=0}^{m-1} i_\alpha 2^\alpha$).

Again, we define $\psi(f\big|_{\mathbf{x}=\mathbf{c}})$ as a complex-valued sequence with $\omega_q^{f(i_0,i_1,\ldots,i_{m-1})}$ as ith component if $i_{j_\alpha} = c_\alpha$ for each $0 \le \alpha < k$ and equal to zero otherwise.

Next, we present a lemma which will be used in our proposed construction.

Lemma 1 ([15]): Suppose that there are two GBFs f and f' of m-variables $x_0, x_1, \ldots, x_{m-1}$ over $\mathbb{Z}_q$, such that for some k $(\le m-3)$ restricting variables $\mathbf{x} = (x_{j_0}, x_{j_1}, \ldots, x_{j_{k-1}})$, $f\big|_{\mathbf{x}=\mathbf{c}}$ and $f'\big|_{\mathbf{x}=\mathbf{c}}$ is given by

$$\begin{aligned} f\big|_{\mathbf{x}=\mathbf{c}} &= P + L + g_l x_l + g, \\ f'\big|_{\mathbf{x}=\mathbf{c}} &= P + L + g_l x_l + \frac{q}{2}x_a + g, \end{aligned} \tag{7}$$

where $L = \sum_{\alpha=0}^{m-k-2} g_\alpha x_\alpha$, P is the quadratic form present in $f\big|_{\mathbf{x}=\mathbf{c}}$ or, $f\big|_{\mathbf{x}=\mathbf{c}}$, both $G(f\big|_{\mathbf{x}=\mathbf{c}})$ and $G(f'\big|_{\mathbf{x}=\mathbf{c}})$ consist of a path having the identical weight $q/2$ over $m-k-1$ vertices, given by $G(P)$, x_a be the either end vertex, x_l be an isolated vertex, and $g_l, g \in \mathbb{Z}_q$. Then for fixed $\mathbf{c}$ and $d_1 \neq d_2$ $(d_1, d_2 \in \{0,1\})$,

$$C(f\big|_{\mathbf{x}x_l=\mathbf{c}d_1}, f\big|_{\mathbf{x}x_l=\mathbf{c}d_2})(\tau) + C(f'\big|_{\mathbf{x}x_l=\mathbf{c}d_1}, f'\big|_{\mathbf{x}x_l=\mathbf{c}d_2})(\tau)$$
$$= \begin{cases} \omega_q^{(d_1-d_2)g_l} 2^{m-k}, & \tau = (d_2-d_1)2^l \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Throughout this paper, the number q will be taken as even and no less than 2.

III. Proposed Construction

In this section, we present a generalized construction of CS. For ease of presentation, whenever the context is clear, we use $C(f,g)(\tau)$ to denote $C(\psi(f), \psi(g))(\tau)$ for any two GBFs f and g. Similar changes are applied to restricted Boolean functions as well.

Theorem 1: Let f be a GBF of m variables over $\mathbb{Z}_q$ with the property that there exist M number of such $\mathbf{c}$ for which $G(f\big|_{\mathbf{x}=\mathbf{c}})$ is a path over $m-k$ vertices and there exist N_i number of such $\mathbf{c}$ for which $G(f\big|_{\mathbf{x}=\mathbf{c}})$ consists of a path over $m-k-1$ vertices and one isolated vertex x_{l_i} such that $M, N_i \ge 0, M + \sum_{i=1}^{p} N_i = 2^k$. Suppose further that all the relevant edges in $G(f\big|_{\mathbf{x}=\mathbf{c}})$ (for all $\mathbf{c}$) have identical weight of $q/2$. Then for any choice of $g_j, g' \in \mathbb{Z}_q$ (where, g_j is the coefficient of x_j and g' is a constant term present in f, $j = 0, 1, \ldots, m-1$), $\psi(f)$ lies in a set S of size 2^{k+1} with the following aperiodic auto-correlation property.

$$A(S)(\tau) = \begin{cases} 2^{m+1}\sum_{i=1}^{p} N_i + 2^{m+1}M, & \tau = 0, \\ \omega_q^{g_{l_i}} 2^m \sum_{\mathbf{c}\in S_{N_i}} \omega_q^{L_{\mathbf{c}}^{l_i}}, & \tau = 2^{l_i}, i=1,2,\ldots,p, \\ \omega_q^{-g_{l_i}} 2^m \sum_{\mathbf{c}\in S_{N_i}} \omega_q^{-L_{\mathbf{c}}^{l_i}}, & \tau = -2^{l_i}, i=1,2,\ldots,p, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

where, $g_{l_i} (\in \mathbb{Z}_q,\ i = 1, 2, \ldots, p)$ is the coefficient of x_{l_i} in f, S_{N_i} contains all those $\mathbf{c}$ for which $G(f\big|_{\mathbf{x}=\mathbf{c}})$ consists of a path over $m-k-1$ vertices and one isolated vertex labeled l_i ($l_i \in \{0, 1, \ldots, m-1\} \setminus \{j_0, j_1, \ldots, j_{k-1}\}$, and $l_1, l_2, \ldots, l_p$ are all distinct),

$$L_{\mathbf{c}}^{l_i} = \sum_{r=1}^{k} \sum_{0\le i_1<i_2<\cdots<i_r<k} \varrho^{l_i}_{i_1,i_2,\ldots,i_r} c_{i_1} c_{i_2} \cdots c_{i_r} \quad (\varrho^{l_i}_{i_1,i_2,\ldots,i_r}\text{'s} \in \mathbb{Z}_q),$$

and $\varrho^{l_i}_{i_1,i_2,\ldots,i_r}$ is the coefficient of $x_{j_{i_1}} x_{j_{i_2}} \cdots x_{j_{i_r}} x_{l_i}$ in f.

Proof: Due to page limitation, we cannot provide the proof here. The details of the proof can be found in [16]. ■

We have introduced M and N_i $(i = 1, 2, \ldots, p)$ in *Theorem* 1 with the condition $M + \sum_{i=1}^{p} N_i = 2^k$, $M, N_i \ge 0$. Therefore, M and N_i's range from 0 to 2^k.

Note 1 (Construction of GBFs as Defined in Theorem 1): The GBFs f, as defined in *Theorem* 1, can be expressed as

$$\begin{aligned} &\frac{q}{2}\sum_{\mathbf{c}\in S_M} \sum_{i=0}^{m-k-2} x_{\pi_{\mathbf{c}}(i)} x_{\pi_{\mathbf{c}}(i+1)} \prod_{\alpha=0}^{k-1} x_{j_\alpha}^{c_\alpha}(1-x_{j_\alpha})^{(1-c_\alpha)} \\ &\quad + \frac{q}{2}\sum_{\delta=1}^{p} \sum_{\mathbf{c}\in S_{N_\delta}} \sum_{i=0}^{m-k-3} x_{\pi^\delta_{\mathbf{c}}(i)} x_{\pi^\delta_{\mathbf{c}}(i+1)} \\ &\quad \times \prod_{\alpha=0}^{k-1} x_{j_\alpha}^{c_\alpha}(1-x_{j_\alpha})^{(1-c_\alpha)} \\ &+ \sum_{\delta=1}^{p}\sum_{r=1}^{k} \sum_{0\le i_1<i_2<\cdots<i_r<k} \varrho^{l_\delta}_{i_1,i_2,\ldots,i_r} x_{j_{i_1}} x_{j_{i_2}} \cdots x_{x_{i_r}} x_{l_\delta} \\ &\quad + \sum_{r=2}^{k} \sum_{0\le i_1<i_2<\cdots<i_r<k} \alpha_{i_1,i_2,\ldots,i_r} x_{j_{i_1}} x_{j_{i_2}} \cdots x_{x_{i_r}} \\ &\quad + \sum_{i=0}^{m-1} g_i x_i + g', \end{aligned} \tag{10}$$

where $\pi^\delta_{\mathbf{c}}$ are N_δ permutations of $\{0, 1, \ldots, m-1\} \setminus \{j_0, j_1, \ldots, j_{k-1}, l_\delta\}$ $(\delta = 1, 2, \ldots, p)$, $\pi_{\mathbf{c}}$ are M permutations of $\{0, 1, \ldots, m-1\} \setminus \{j_0, j_1, \ldots, j_{k-1}\}$, and $\alpha_{i_1,i_2,\ldots,i_r}$'s are belongs to $\mathbb{Z}_q$.

Remark 1: Let f be a quadratic GBF with the property that for all $\mathbf{c} \in \{0,1\}^k$, $G(f\big|_{\mathbf{x}=\mathbf{c}})$ is a path in $m-k$ vertices. Then from *Therorem* 1, we have $M=2^k$ and

$$A(S)(\tau)=\begin{cases}2^{m+k+1}, & \tau=0,\\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

Hence, S is a CS of size 2^{k+1} and therefore, Paterson's construction [4, Th. 12] turns to be a special case of our proposed one.

Remark 2: From *Remark* 1, for $k=0$, S is a CS of size 2, i.e., S is a GCP and thus the GDJ code in [3] is also a special case of *Theorem* 1.

Remark 3: Let f be a quadratic GBF with the property that for all $\mathbf{c} \in \{0,1\}^k$, $G(f\big|_{\mathbf{x}=\mathbf{c}})$ contains a path in $m-k-1$ vertices and one isolated vertex x_{l_1}. We also assume that all edges in the original graph between the isolated vertex and the k deleted vertices are weighted by $q/2$. Then, from *Theorem* 1, we have $N_1=2^k$, $S_{N_1}=\{0,1\}^k$, $L_{\mathbf{c}}^{l_1}=\frac{q}{2}\sum_{\alpha=0}^{k-1}c_\alpha$, and

$$A(S)(\tau)=\begin{cases}2^{m+k+1}, & \tau=0,\\ \omega_q^{g_{l_1}}2^{m+k}\displaystyle\sum_{\mathbf{c}\in S_{N_1}}\omega_q^{L_{\mathbf{c}}^{l_1}}, & \tau=2^{l_1},\\ \omega_q^{-g_{l_1}}2^{m+k}\displaystyle\sum_{\mathbf{c}\in S_{N_1}}\omega_q^{-L_{\mathbf{c}}^{l_1}}, & \tau=-2^{l_1},\\ 0, & \text{otherwise,}\end{cases} \tag{12}$$
$$=\begin{cases}2^{m+k+1}, & \tau=0,\\ 0, & \text{otherwise.},\end{cases}$$

Therefore, $\psi(f)$ lies in a CS of size 2^{k+1} and the result given by Paterson in [4, Th. 24] is a special case of *Theorem* 1.

Remark 4: Let f be a GBF with the property that for all $\mathbf{c} \in \{0,1\}^k$, $G(f\big|_{\mathbf{x}=\mathbf{c}})$ is a path in $m-k$ vertices. Then from *Therorem* 1, we have $M=2^k$ and

$$A(S)(\tau)=\begin{cases}2^{m+k+1}, & \tau=0,\\ 0, & \text{otherwise.}\end{cases} \tag{13}$$

From (13), it is clear that $\psi(f)$ lies in a CS of size 2^{k+1} and hence the PMEPR of $\psi(f)$ is atmost 2^{k+1}. Therefore, the result given by Schmidt in [5, Th. 5] is a special case of *Theorem* 1.

Note 2: Let f be GBF as defined in *Theroem* 1. If $\sum_{\mathbf{c}\in S_{N_i}}\omega_q^{L_c^{l_i}}\neq 0, \sum_{\mathbf{c}\in S_{N_i}}\omega_q^{-L_c^{l_i}}\neq 0$ $(i=1,2,\ldots,p)$, $\psi(f)$ lies in a CS, S', of size 2^{k+2} with at most PMEPR $2^{k+2}-2M$. The set S' can be expressed as

$$\left\{f+\frac{q}{2}\left(\mathbf{d}\cdot\mathbf{x}+d'\sum_{i=1}^{p}x_{l_i}+dx_{\mathbf{c}}\right):\mathbf{d}\in\{0,1\}^k,d,d'\in\{0,1\}\right\}$$
$$=S+\mathcal{S}_1, \tag{14}$$

where

$$S=\left\{f+\frac{q}{2}\left(\mathbf{d}\cdot\mathbf{x}+dx_{\mathbf{c}}\right):\mathbf{d}\in\{0,1\}^k,d,\in\{0,1\}\right\},$$
$$\mathcal{S}_1=\left\{f+\frac{q}{2}\left(\mathbf{d}\cdot\mathbf{x}+\sum_{i=1}^{p}x_{l_i}+dx_{\mathbf{c}}\right):\mathbf{d}\in\{0,1\}^k,d,\in\{0,1\}\right\}. \tag{15}$$

By using *Theorem* 1, we have

$$A(\mathcal{S}_1)(\tau)=\begin{cases}2^{m+1}\displaystyle\sum_{i=1}^{p}N_i+2^{m+1}M, & \tau=0,\\ -\omega_q^{g_{l_i}}2^m\displaystyle\sum_{\mathbf{c}\in S_{N_i}}\omega_q^{L_{\mathbf{c}}^{l_i}}, & \tau=2^{l_i}, i=1,2,\ldots,p,\\ -\omega_q^{-g_{l_i}}2^m\displaystyle\sum_{\mathbf{c}\in S_{N_i}}\omega_q^{-L_{\mathbf{c}}^{l_i}}, & \tau=-2^{l_i}, i=1,2,\ldots,p,\\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

Hence, from (9) and (16), we have

$$A(S')=\begin{cases}2^{m+k+2}, & \tau=0,\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

Also, since at most PMEPR of S and $\mathcal{S}_1$ is $2(M+\sum_{i=0}^{p}N_i)+2\sum_{i=0}^{p}N_i$ or, $2^{k+2}-2M$, the PMEPR of S' is upper bounded by $2^{k+2}-2M$. Therfore, $\psi(f)$ lies in a CS, S', of size 2^{k+2} with at most PMEPR $2^{k+2}-2M$.

Below we have illustrated our proposed constrcution by the following examples.

Example 1: Let f be a GBF of 4 variables over $\mathbb{Z}_2$, given by

$$f(x_0,x_1,x_2,x_3)=x_0x_1x_3+x_0x_3x_2+x_0x_2x_1+x_1x_2. \tag{18}$$

Here, Fig. 1 (a) and Fig. 1 (b) represents $G(f\big|_{x_0=1})$ and $G(f\big|_{x_0=0})$, respectively. It is clear that $G(f\big|_{x_0=1})$ is a path over the variables x_1, x_2, and x_3. $G(f\big|_{x_0=0})$ contains a path over the variables x_1, x_2 and one isolated vertex x_3. Therefore, $M=1$, $N_1=1$, $S_{N_1}=\{0\}$, $S_M=\{1\}$, $L_0^3=0$, $\pi_0:\{0,1\}\rightarrow\{1,2\}$, $\pi_1^1:\{0,1,2\}\rightarrow\{1,2,3\}$, $\varrho_0^3=0$, $g_i=0$ $(i=0,1,2,3)$ and $g'=0$ respectively. By using *Note* 1, we obtain the set S' given by

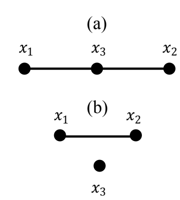

Fig. 1. The $G(f\big|_{x_0=0})$ and $G(f\big|_{x_0=1})$.

$$S'=\{f+(d_0x_0+d_1x_3+dx_2):d_0,d\in\{0,1\}\}, \tag{19}$$

which is a CS of size 8 with at most PMEPR 6 and the set S' cannot be obtained from Schmidt's construction by using the same f. Therefore, the PMEPR of $\psi(f)$ is at most 6 and from Schmidt's construction the PMEPR upper bound of $\psi(f)$ is at most 32.

Example 2: Let f be GBF of 5 variables over $\mathbb{Z}_4$, given by

$$f(x_0,x_1,x_2,x_3,x_4) = 2\left(x_0x_1x_2 + x_0x_1x_3 + x_1x_3 \right. \\ \left. + x_3x_2 + x_0x_4\right) + x_1 + 2x_2 + 2x_3 + 2x_4 + 3. \tag{20}$$

From (20), it is observed that $G(f\big|_{x_0=0})$ and $G(f\big|_{x_0=1})$ both contain a path over the vertices x_1, x_2, x_3 and one isoltaed vertex x_4. Thererfore, $p=1$, $N_1=2$, $S_{N_1}=\{0,1\}$, $\varrho_0^4=2$, $L_0^4=0$, $L_1^4=2$, $\pi_0^1,\pi_1^1 : \{0,1,2\} \rightarrow \{1,2,3\}$, $g_0=0$, $g_1=1$, $g_2=g_3=g_4=2$, and $g'=3$. Since, $\sum_{c\in S_{N_1}} \omega_4^{L_c^4} = \sum_{c\in S_{N_1}} \omega_4^{-L_c^4} = 0$, by using *Theorem* 1

$$S = \{2\left(x_0x_1x_2 + x_0x_1x_3 + x_1x_3 + x_3x_2 + x_0x_4\right) + x_1 \\ + 2x_2 + 2x_3 + 2x_4 + 3 + 2(d_0x_0 + dx_1) : d_0, d \in \{0,1\}\}, \tag{21}$$

is a CS of size 4. Therefore, the PMEPR of $\psi(f)$ is at most 4 and from Schmidt's construction, the PMEPR upper bound of $\psi(f)$ is 8.

IV. Discussion:

In this section, we discuss the efficiency of our constrcution as compare to Schmidt's construction.

The PMEPR upper bound of a sequence $\psi(f)$ corresponding to a GBF f having the graphical property as stated in *Theorem* 1 is $2^{k+2}-2M$ when $\sum_{\mathbf{c}\in S_{N_i}} \omega_q^{L_{\mathbf{c}}^{l_i}} \neq 0, \sum_{\mathbf{c}\in S_{N_i}} \omega_q^{-L_{\mathbf{c}}^{l_i}} \neq 0$. For the same $\psi(f)$, the PMEPR upper bound from Schmidt's construction is 2^{k+p+1} $(2^{k+2}-2M)$ only when $G(f\big|_{\mathbf{xx'}=\mathbf{cc'}})$ is a path for all $\mathbf{c} \in \{0,1\}^k, \mathbf{c'} \in \{0,1\}^p$, where $\mathbf{x'} = (x_{l_1}, x_{l_2}, \ldots, x_{l_p})$, $\mathbf{xx'}$ denotes the concatenation of $\mathbf{x}$ and $\mathbf{x'}$, and $\mathbf{cc'}$ denotes the concatenation of $\mathbf{c}$ and $\mathbf{c'}$. Otherwise, from Schmidt's construction, the PMEPR upper bound for $\psi(f)$ is strictly greater than 2^{k+p+1}. Specifically, as shown in *Example* 1, our proposed construction implies that the PMEPR upper bound for $\psi(f)$ is 6, whereas from Schmidt's construction the PMEPR upperbound for same $\psi(f)$ is 32. In (10), if $p=1$, $M=0$, $\varrho_{i_1,i_2,\ldots,i_r}^{l_1}=0$ $(r=2,3,\ldots,k)$, and $\varrho_{i_1}^{l_1} \in \{0,\frac{q}{2}\}$ (but $\varrho_0^{l_1}, \varrho_1^{l_1}, \ldots, \varrho_{k-1}^{l_1}$ all cannot be zero at the same time), then from *Theorem* 1, $\sum_{\mathbf{c}\in S_{N_1}} \omega_q^{L_{\mathbf{c}}^{l_1}} = 0, \sum_{\mathbf{c}\in S_{N_1}} \omega_q^{-L_{\mathbf{c}}^{l_1}} = 0$, and therefore the PMEPR upper bound for $\psi(f)$ is 2^{k+1} as in *Example* 2, we can see the PMEPR upper bound for $\psi(f)$ is at most 4 whereas Schmidt's construction provides a PMEPR bound of 8 for the same $\psi(f)$. As described above, our construction can provide tighter PMEPR upper bound as compare to Schmidt's construction.

V. Conclusions

In this paper, we have proposed a direct and generalized construction of polyphase CS by using higher GBFs whose restricted forms lead to graphs with isolated vertices. The proposed construction provides tighter PMEPR upper bound as compared to Schmidt's construction. By having a less restrictive propety than Paterson's and Schmidt's constrcution it is found that our proposed construction can provide a better code rate and the full results of this research which can be found in [16], will be submitted out as a journal paper. We have shown that our proposed construction generates sequences with maximum PMEPR upper bound 4 and 8. In addition to that we have obtained sequences with maximum PMEPR upper bound 6. The constructions given by Davis *et al.*, Paterson and Schmidt appear as special cases of our proposed construction.

Acknowledgment

This work of S. Majhi was supported by the Visvesvaraya Young Faculty Research Fellowship, Ministry of Electronics and Information Technology, Government of India, being implemented by the Digital India Corporation. The work of Z. Liu was supported in part by National Natural Science Foundation of China under Grant 61750110527, a Research Fund for International Young Scientists.

References

[1] M. J. E. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82–87, Apr. 1961.

[2] C.-C. Tseng and C. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. 18, no. 5, pp. 644–652, Sep. 1972.

[3] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.

[4] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, no. 1, pp. 104–120, Jan. 2000.

[5] K. U. Schmidt, "Complementary sets, generalized Reed-Muller codes, and power control for OFDM," *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 808–814, Feb. 2007.

[6] F. Fiedler, J. Jedwab, and M. G. Parker, "A framework for the construction of Golay sequences," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3114–3129, Jul. 2008.

[7] K. G. Paterson and V. Tarokh, "On the existence and construction of good codes with low peak-to-average power ratios," *IEEE Trans. Inf. Theory*, vol. 46, no. 6, pp. 1974–1987, Sept. 2000.

[8] Y. Li, "A construction of general QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 56, no. 11, pp. 5765–5771, Nov 2010.

[9] Z. Liu, Y. Li, and Y. L. Guan, "New constructions of general QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7684–7692, Nov. 2013.

[10] S. Das, S. Budišin, S. Majhi, Z. Liu, and Y. L. Guan, "A multiplier-free generator for polyphase complete complementary codes," *IEEE Trans. Signal Process.*, vol. 66, no. 5, pp. 1184–1196, Mar. 2018.

[11] S. Das, S. Majhi, and Z. Liu, "A novel class of complete complementary codes and their applications for APU matrices," *IEEE Signal Process. Lett.*, vol. 25, no. 9, pp. 1300–1304, Sept. 2018.

[12] P. Sarkar, S. Majhi, H. Vettikalladi, and A. S. Mahajumi, "A direct construction of inter-group complementary code set," *IEEE Access*, vol. 6, pp. 42 047–42 056, 2018.

[13] P. Sarkar, S. Majhi, and Z. Liu, "Optimal Z-complementary code set from generalized Reed-Muller codes," *IEEE Trans. Commun., To appear.*

[14] A. R. Adhikary, S. Majhi, Z. Liu, and Y. L. Guan, "New sets of even-length binary Z-complementary pairs with asymptotic ZCZ ratio of $3/4$," *IEEE Signal Process. Lett.*, vol. 25, no. 7, pp. 970–973, July 2018.

[15] T. E. Stinchcombe, "Aperiodic correlations of length 2^m sequences, complementarity, and power control for OFDM," Ph.D. dissertation, University of London, Apr. 2000.

[16] P. Sarkar, S. Majhi, and Z. Liu, "A direct and generalized construction of polyphase complementary set with low PMEPR and high code-rate for OFDM system." [Online]. Available: http://arxiv.org/abs/1901.05545