

On the body of ample angles of asymptotically log Fano varieties

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Abstract

In dimension two, we reduce the classification problem for asymptotically log Fano pairs to the problem of determining generality conditions on certain blow-ups. In any dimension, we prove the rationality of the body of ample angles of an asymptotically log Fano pair, i.e., these convex bodies are always rational polytopes.

1 Introduction

Asymptotically log Fano pairs, introduced by Cheltsov–Rubinstein [5], generalizing work of [20], have received attention in the last decade within the theory of K-stability and the Calabi problem [6, 7, 10, 11]. We believe asymptotically log Fanos to be interesting objects in their own right. Indeed, while they belong to an infinite number of deformation families, one can expect a classification to be achievable. Alas, so far they have only been systematically studied under two special assumptions: the boundary having only one component or its generalization: the *strong* regime.

Our goal in this note is to expand the knowledge on the birational geometry of asymptotically log Fano pairs by considering two separate problems. First, we discuss some aspects of the classification of asymptotically log Fano pairs in dimension two, and reduce such a classification to the understanding of generality conditions on the location of points, possibly infinitely near, that are blown-up on a curve in a rational surface with Picard group of rank at most two. Second, we point out the rationality of certain convex bodies that arises in the study of asymptotically log Fano pairs in any dimension, making contact with recent advances on log-geography and the minimal model program.

1.1 The classification problem

A pair (X, D) consisting of a smooth complex projective variety X and a simple-normal-crossing (holomorphic) effective non-zero divisor $D = \sum_{i=1}^r D_i$ in X (where the D_i are distinct irreducible hypersurfaces) is called *asymptotically log Fano* in the sense of Cheltsov–Rubinstein [5, Definition 1.1] if there exists a sequence

$$\beta(j) = (\beta_1(j), \dots, \beta_r(j)) \in (0, 1]^r \cap \mathbb{Q}^r, \quad j \in \mathbb{N},$$

converging to the origin such that

$$-K_X - \sum_{i=1}^r (1 - \beta_i(j)) D_i \quad \text{is a } \mathbb{Q}\text{-ample divisor, for each } j \in \mathbb{N}. \quad (1.1)$$

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A pair (X, D) is called *strongly asymptotically log Fano* if (1.1) even holds in a semi-open sub-cube, not just along a sequence, i.e.,

$$-K_X - \sum_{i=1}^r (1 - \beta_i) D_i \quad \text{is a } \mathbb{Q}\text{-ample divisor, for all } \beta \in (0, \epsilon]^r \cap \mathbb{Q}^r \text{ for some } \epsilon \in (0, 1]. \quad (1.2)$$

Finally, (X, D) is called *log Fano* if (1.2) actually holds in a closed sub-cube, i.e., for $\beta \in [0, \epsilon]^r \cap \mathbb{Q}^r$ (to avoid confusion, we remark that some authors use this terminology to refer to rather more general objects [2, Definition 2.7] than Maeda’s Definition 2.1 below). In dimension two, we will denote a pair by (S, C) and refer to (1.1)–(1.2) as (*strongly*) *asymptotically log del Pezzo*.

By definition, a log Fano pair (X, D) is strongly asymptotically log Fano, and a strongly asymptotically log Fano pair (X, D) is asymptotically log Fano, but none of the reverse implications are true in general, already in dimension 2. Perhaps the simplest examples arise by considering anticanonical boundaries. If S is del Pezzo and $C \sim -K_S$ is smooth then (S, C) is strongly asymptotically log del Pezzo but not log del Pezzo. The notions of strongly asymptotically log del Pezzo and asymptotically log del Pezzo are actually equivalent when C is smooth, but as soon as C contains two components they are not: let $c \subset \mathbb{P}^2$ be a cubic curve smooth away from a double point p and let S be the blow-up of \mathbb{P}^2 at p . Let C be the total transform of c , i.e., $C = C_1 + C_2$ with C_1 the inverse image of p , and with C_2 the proper transform of c . In this case, (1.2) will not hold, in fact $-K_S - (1 - \beta_1)C_1 - (1 - \beta_2)C_2$ will be ample if and only if $0 < \beta_1 < 2\beta_2$ which defines a trapezoidal subset of the square $[0, 1]^2$ (see Figure 1).

The following problem, raised by Cheltsov and one of us [5] (see [22, §8–9] for detailed exposition), can be considered as a logarithmic generalization of the folklore classification [9, 16] of del Pezzo surfaces:

Problem 1.1. *Classify asymptotically log del Pezzo surfaces.*

The classification of the subclass of *strongly* asymptotically log del Pezzos has been achieved [5, Theorems 2.1, 3.1] (see also [25]), however, as we will see below, the non-strongly regime is considerably harder to classify. Part of the motivation in [5] is the theory of canonical Kähler metrics with edge singularities [18, 21] that is associated with such pairs. While Fano varieties exhibit remarkable finiteness properties in a given dimension, this is no longer the case for asymptotically log Fano varieties. In fact, already Maeda’s classification of log del Pezzo surfaces, that we review below, exhibits infinitely many pairs. However, it is precisely the case of nonzero β_i ’s, interpreted geometrically by the existence of Kähler metrics with cone angle $2\pi\beta_i(j)$ along C_i , that is of major interest in complex geometry, see [22] for a detailed survey and references. Moreover, as the recent works [4–8, 10–13, 22, 23] show, the notion of asymptotically log Fano varieties is interesting also purely from an algebraic geometry viewpoint. Finally, the family of strongly asymptotically log del Pezzos is already much more vast and rich geometrically than the class of log del Pezzos, and, as we try to explain in this note, the class of asymptotically log del Pezzos is, in a sense, yet an order of magnitude more vast. Our first goal in this note will be to illustrate with concrete examples some of the difficulties in solving Problem 1.1 and give some first steps in such a classification. More precisely, we will show that all asymptotically log del Pezzo pairs arise from pairs with Picard rank 2 in a very explicit way (Proposition 4.1) and classify the latter (Proposition 5.1). Classifying all asymptotically log del Pezzo pairs remains a difficult open problem.

1.2 The body of ample angles

One can collect all “admissible” coefficients in the sense of (1.1) (equivalently interpreted as angles for which there exists a Kähler edge metric with angles $2\pi\beta_i$ along C_i and with positive Ricci curvature on $S \setminus C$ [14, 18, 21, 22]) into a single body previously introduced by one of us [23, Definition 3.1]:

Definition 1.2. *The set*

$$\text{AA}(X, D) := \left\{ \beta = (\beta_1, \dots, \beta_r) \in (0, 1)^r : -K_X - \sum_{i=1}^r (1 - \beta_i) D_i \text{ is ample} \right\} \quad (1.3)$$

is called the body of ample angles of (X, D) .

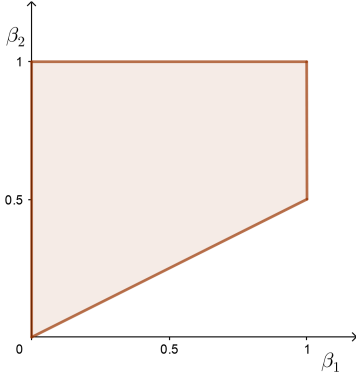


Figure 1: The body of ample angles for $(\mathbb{F}_1, C_1 + C_2)$ with $C_1 = Z_1$ and C_2 smooth in $|-K_{\mathbb{F}_1} - C_1|$ and intersecting C_1 transversally (see §2.1 for notation).

The problem of determining whether a given pair $(X, D = \sum_{i=1}^r D_i)$ is asymptotically log Fano amounts to determining whether

$$0 \in \overline{\text{AA}(X, D)}.$$

Thus, this set is a fundamental object in the study of asymptotically log Fano varieties.

When (X, D) is strongly asymptotically log Fano this body is simply a cube near the origin, in particular locally polyhedral near the origin.

It is easy to see that the body of ample angles of an asymptotically log Fano variety is a convex body [23, Lemma 3.3]. A natural question is therefore:

Problem 1.3. *Let (X, D) be asymptotically log Fano. Is $\overline{\text{AA}(X, D)}$ polyhedral?*

Of course, Problems 1.1 and 1.3 are related: if one had a classification as in Problem 1.1, then it would likely be possible to glean from it explicitly the bodies of ample angles, which would resolve Problem 1.3 in dimension 2. In fact, as shown by one of us, it turns out that Problem 1.3 can be solved affirmatively in dimension 2 by describing rather explicitly the linear inequalities cutting out the body of ample angles, however this approach specifically hinges on the Nakai–Moishezon criterion and other two-dimensional facts and does not generalize to higher dimensions [24]. Here, we resolve Problem 1.3 in any dimension. In fact, we prove a stronger statement that does not assume (X, D) is asymptotically log Fano (i.e., $0 \in \overline{\text{AA}(X, D)}$) but merely that $\text{AA}(X, D)$ is non-empty (note that by openness of ampleness the former implies the latter).

Theorem 1.4. *$\overline{\text{AA}(X, D)}$ is either empty or a rational polytope, i.e., cut out by finitely-many linear inequalities with rational coefficients in β_1, \dots, β_r .*

This result perhaps motivates renaming the body of ample angles the ‘angletope’ or ‘ample-angletope’.

Already in dimension two—at least from the point of view of the Nakai–Moishezon criterion—polyhedrality is not obvious: the aforementioned criterion involves infinitely-many linear inequalities (one for the intersection of (1.2) with each irreducible curve of S) as well as one quadratic inequality in the β_i ’s (corresponding to taking the self-intersection of (1.2)). The latter condition would not seem to affect the body being polyhedral *near the origin*. In fact, it only comes to play in the cases when a birational model (s, c) with Picard rank 2 of (S, C) , obtained by smooth contractions of curves intersecting C , satisfies $(K_s + c)^2 = 0$. Nevertheless, it is remarkable that even in these cases it can basically be replaced by a finite collection of linear inequalities, and that overall all the linear inequalities boil down to finitely-many. In fact, the proof of Theorem 1.4 uses the recent progress on the study of Shokurov’s log geography, due to one of us together with Birkar, Hacon and McKernan [1] (see also [3]) that was a crucial ingredient in the proof of the finite generation of the canonical ring.

1.3 Organization

The remainder of this note is organized as follows. In §2 we recall the classical result of Maeda on the classification of log del Pezzos and give a self-contained proof. In §3 we give a brief overview of the

classification of strongly asymptotically log del Pezzos following Cheltsov and one of us [5, 25]. In §4 we explain which aspects of the approach in op. cit. extend to the setting of asymptotically log del Pezzos and which ones do not. Our first main result is a description of all asymptotically log del Pezzos as a subset of a particular class of pairs that are obtained as certain blow-ups of asymptotically log del Pezzos with small Picard group (Proposition 4.1). In §5 we give an explicit classification of asymptotically log del Pezzos with small Picard group (Proposition 5.1). Propositions 4.1 and 5.1 combine to give a vast class of asymptotically log del Pezzos that are not strongly asymptotically log del Pezzo. We end in §6 with a proof of Theorem 1.4.

2 Log del Pezzo surfaces

While the classification of log del Pezzo surfaces is rather simple, it serves to illustrate the simplest setting and the origin of Problem 1.1.

The definition of log Fano manifolds goes back to work of Maeda [20].

Definition 2.1. *Let X be a smooth variety and let D be a simple normal crossing divisor in X . We say that the pair $(X, D = \sum D_i)$ is log Fano if $-K_X - D$ is ample.*

In dimension 2, these are also called log del Pezzo surfaces. The motivation for the adjective “logarithmic”, according to Maeda, is from the work of Iitaka on the classification of open algebraic varieties where logarithmic differential forms are used to define invariants of the pair. The open variety associated to (X, D) is the Zariski open set $X \setminus D$.

2.1 Some facts on Hirzebruch surfaces

The main difference between del Pezzo surfaces and log del Pezzo surfaces is the appearance of Hirzebruch surfaces, \mathbb{F}_n . Let us recall some basic facts and establish some notation for \mathbb{F}_n .

For each $n \geq 0$, denote by \mathbb{F}_n the unique rational ruled surface whose Picard group has rank two and contains a unique (if $n > 0$) smooth rational curve of self-intersection $-n$. We denote this curve by Z_n , and by F we denote the class of an irreducible smooth rational curve such that $F^2 = 0$ and $F \cdot Z_n = 1$. If $n = 0$ when we refer to Z_0 and F we intend that each is a fiber of a different projection to \mathbb{P}^1 . Hirzebruch surfaces are ruled toric surfaces and applying adjunction yields [15, Chapter 5, §2]

$$-K_S \sim 2Z_n + (n+2)F \tag{2.1}$$

Recall that every smooth irreducible curve in $|Z_n + nF|$ (a ‘zero section’) intersects each fiber transversally at a single point and does not intersect the ‘infinity section’ Z_n . Any curve C on \mathbb{F}_n satisfies

$$C \sim aZ_n + bF \tag{2.2}$$

with $a, b \in \mathbb{N} \cup \{0\}$. This, combined with the Nakai-Moishezon criterion implies

$$C \text{ is ample if and only if } a > 0 \text{ and } b > na, \tag{2.3}$$

and

$$C \text{ is nef if and only if } a \geq 0 \text{ and } b \geq na, \tag{2.4}$$

and furthermore,

$$C \text{ is an irreducible curve only if } C = Z_n \text{ or } b \geq na \geq 0, \tag{2.5}$$

and under such conditions the class (2.2) always contains an irreducible curve.

2.2 Maeda’s classification

Maeda classified log del Pezzo pairs [20, §3.4]. Let us review his classification as its rigidity serves as a contrast to the flexibility we demonstrate for asymptotically log del Pezzo pairs in Propositions 4.1 and 5.1 below.

Proposition 2.2. *Log del Pezzo surfaces $(S, C = \sum_{i=1}^r C_i)$ are classified as follows:*

- (i) $S = \mathbb{P}^2$, C_1 is a line,
- (ii) $S = \mathbb{P}^2$, C_1, C_2 , are lines,
- (iii) $S = \mathbb{P}^2$, and C_1 is a conic,
- (iv) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n$,
- (v) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n, C_2 \in |F|$,
- (vi) $S = \mathbb{F}_1$, $C_1 \in |Z_1 + F|$,
- (vii) $S = \mathbb{P}^1 \times \mathbb{P}^1$, C_1 is a $(1, 1)$ -curve.

Proof. By the Kodaira Vanishing Theorem and Kodaira–Serre duality, it follows that S is rational (cf. [5, §3]). When $S = \mathbb{P}^2$, and since $-K_S \sim 3H$ we see the possibilities are (i), (ii), or (iii). Assume now that $S = \mathbb{F}_n$. Denote

$$C_i \in |a_i Z_n + b_i F|.$$

Since $-K_S - C$ is ample and using (2.1) and (2.3), we see that

$$\sum_i a_i \in \{0, 1\}, \quad \sum_i b_i \in \{0, \dots, n+1\}.$$

and that at most one component of C is a fiber. Note that by (2.5) either $b_i \geq na_i > 0$ or else $(a_i, b_i) = (0, 1)$, i.e., C_i is a fiber. Thus, we may have at most one fiber, at most one $-n$ -curve, and at most one curve either in $|Z_n + nF|$ (if $n = 0, 1$) or in $|Z_n + (n+1)F|$ (if $n = 0$). Moreover, we cannot have all three. Thus, there are at most two components. If $r = 1$ then either $C_1 = Z_n$, so we are in case (iv), or else $C_1 \in |Z_1 + F|$ which is case (vi), or else $C_1 \in |Z_0 + F|$ which is case (vii). If $r = 2$ the only possibility is $C_1 = Z_n$ and $C_2 \in |F|$ (case (v)).

We proceed by reductio ad absurdum to discard other cases. If S had higher Picard rank, there would be a birational morphism $\pi: S \rightarrow s = \mathbb{F}_n$ for some $n \geq 0$ consisting on the consecutive contraction of -1 -curves. Since $-K_S - C$ is ample, it follows that any -1 -curve is either disjoint from C or in the support of C . Since the pushforward of ample divisors is ample, we may assume without loss of generality that π consists of one blow-up, let $c = \pi_*(C)$, E be the exceptional curve, $p = \pi(E)$ and \hat{F} be the proper transform in C of the fiber f through p of the projection $\mathbb{F}_n \rightarrow \mathbb{P}^1$. It follows that \hat{F} and E are both -1 -curves intersecting at one point and hence either they are both components of C or both disjoint of C . In the latter case, it turns out that $f \cap c = \emptyset$, so c consists only of fibers and by the above classification $(s, c) = (\mathbb{F}_0, f = Z_0)$, i.e. (s, c) is of type (iv). Let f' be the fiber passing through p of the other projection $\mathbb{F}_0 \rightarrow \mathbb{P}^1$. Then the proper transform \hat{F}' of F' via π satisfies $(-K_S - C) \cdot \hat{F}' = 0$, contradicting ampleness.

Hence $\hat{F}, E \subseteq \text{Supp}(C)$ and we will also obtain a contradiction. Write $C = \hat{F} + E + \Omega$ with $\hat{F}, E \not\subseteq \text{supp}(\Omega)$ and Ω some effective divisor. Let $\omega = \pi(\Omega)$. Notice that $f \not\subseteq \text{supp}(\omega)$ since C is an integral divisor with coefficients at most 1. Since $F, E \subset \text{supp}(C)$, we have that $-K_S - C = \pi^*(-K_s - c) + (\text{mult}_p \omega - 1)E$ and as $(-K_s - c)$ is ample, it follows that $\text{mult}_p \omega \leq 1$. But then it follows that $(-K_S - C) \cdot E = 1 - \text{mult}_p \omega \leq 0$, contradicting ampleness. \square

3 Strongly asymptotically log del Pezzo surfaces

The key ideas for the classification of strongly asymptotically log del Pezzos are quite different from the Maeda case. In fact, for Maeda the classification problem never leaves the realm of Picard rank at most 2. In other words, there is no need to consider blow-ups when carrying out the classification. However, for strongly asymptotically log del Pezzos there can be arbitrarily many blow-ups and there is no upper bound on the Picard rank, unlike the log del Pezzo or del Pezzo settings.

Thus, some sort of inductive reduction procedure is needed here. The key notion is that of minimality [5, Definition 2.8]:

Definition 3.1. *Let (S, C) be asymptotically log del Pezzo. We say it is minimal if it contains no smooth rational curves $E \not\subseteq C$ with $E^2 = -1$ and $E \cdot C = 1$.*

In an asymptotically log del Pezzo (S, C) , the dual graph of the boundary C can only be either a union of chains (in fact, at most two chains [5, Remark 3.7]) or one cycle (in which case $C \sim -K_S$) [5, Lemma 3.5]. In the former case we can talk of the ‘tail’ components of each chain and the ‘middle’ components. In the latter case all components are middle ones.

Moreover, in a strongly asymptotically log del Pezzo surface there can be no ‘middle’ curves with negative self-intersection in the boundary C [5, Lemma 3.6]. Thus, at the very worst there are a few such curves in the ‘tail’. In addition, there are the -1 -curves disjoint from the boundary or intersecting the boundary transversally at a single point. Contracting the latter kind, one ends up with a minimal strongly asymptotically log del Pezzo which in fact has Picard rank at most 2 (this is not obvious [5, Lemma 3.13], see also [25]) and those are readily classified. This yields a classification since, and this is the final step, one can verify generality conditions on the location of the blow-up points. Post factum one also finds that every pair that is obtained either as a tail blow-up or a blow-up away from the boundary can also be obtained by blowing-up points on the boundary on a possibly different base pair. Thus, tail blow-ups and ‘away’ blow-ups play no role in the original classification of [5]. In fact, we will use some variant of this argument in the proof of Proposition 4.1 below.

Remark 3.2. A systematic study of tail blow-ups has been initiated by one of us in the asymptotically log del Pezzo regime [23]. A systematic study of away blow-ups can be found in [24]. An alternative proof of the classification result of [5] has been given in [25].

4 Towards a classification of asymptotically log del Pezzo pairs

The key property from the classification of the strong regime that fails in the non-strong regime is precisely: no interior boundary curve with negative self-intersection. In particular, tail blow-ups occur at most once in the strong regime, and one can never blow-up the singular points of the boundary C let alone infinitely near such points.

But, if one carries out the inductive proof as in the strong regime allowing in addition to blow down (possibly many) boundary components—which is never needed in the strong regime—then, as we will see below, every asymptotically log del Pezzo will be obtained from a asymptotically log del Pezzo with Picard rank at most 2 via *both* proper boundary blow-ups of smooth boundary points, and total boundary blow-ups of singular boundary points. However, the order of the blow-ups matters: one may repeatedly blow-up singular points (i.e., blow-up infinitely near points) and then blow-up smooth points on the resulting new components of the boundary (exceptional curves). From the analysis of [23] it becomes clear that a daunting linear programming problem arises and we do not attempt to resolve it here.

Proposition 4.1. *Any asymptotically log del Pezzo (S, C) is obtained from a pair (s, c) with $\text{rk Pic}(s) \leq 2$ listed in Proposition 5.1 by a combination of the following operations: (i) blowing-up a collection of distinct points on the smooth locus of the boundary and replacing the boundary with its proper transform, (ii) blowing-up a collection of (possibly infinitely near) singular points of the boundary and replacing the boundary with its total transform.*

Remark 4.2. Again, we emphasize that Proposition 4.1 does *not* give a full classification by any means, and it does not imply that any such blow-ups will result in an asymptotically log del Pezzo pair; it merely says that any asymptotically log del Pezzo pair *can be described in such a way*, but we do not give that description here.

Proof. Reversing both operations (i) and (ii) preserves asymptotic log positivity. Indeed, contraction of a -1 -curve intersecting the boundary transversally at a single point preserves the property of being asymptotically log del Pezzo [5, Lemma 3.4]. And if $C = C_1 + E + \sum_{i=3}^r C_i$ with $E = C_2$ being a -1 -curve, and $E.C_i = 1$ for $i = 1, 3$ and zero otherwise with blow-down $\pi : S \rightarrow s$ contracting E to a point, then letting $c_1 := \pi(C_1)$, $c_i := \pi(C_{i+1})$, $i = 2, \dots, r-1$,

$$-K_S - \sum_{i=1}^r (1 - \beta_i) C_i \sim -\pi^* \left(K_s + \sum_{i=1}^{r-1} (1 - \beta_i) c_i \right) - (\beta_1 + \beta_3 - \beta_2) E,$$

so

$$-K_s - \sum_{i=1}^{r-1} (1 - \beta_i) c_i \sim \pi_* \left(-K_S - \sum_{i=1}^r (1 - \beta_i) C_i \right)$$

is therefore ample by the Nakai–Moishezon criterion (see, e.g., [5, (1.5)]), so $(s, \sum_{i=1}^{r-1} c_i)$ is asymptotically log del Pezzo.

Next, we claim that we can reverse the operations (i) and (ii) (in some order), until the Picard rank becomes at most 2 (while preserving the asymptotically log del Pezzo property as we just showed above). To prove the claim, first apply operations (i) to contract all -1 -curves in S that intersect the boundary transversally at a smooth point (note that these are in fact all -1 -curves not contained in C that intersect the boundary [5, Lemma 3.3]). Second, apply operations (ii) to contract any remaining -1 -curve in the resulting boundary that intersects two distinct other boundary components (when $r \geq 3$). We can also assume that $r \geq 2$ since if $r = 1$ the pair is strongly asymptotically log del Pezzo and so by the classification [5, Theorem 2.1] we can contract it down using only the operations (i). If after the above operations the the rank of the Picard group of the resulting surface is at most 2 we are done. Thus, we may assume that we are in one of the following mutually exclusive situations: (a) $\text{rk Pic}(S) \geq 3$, $r \geq 2$, $C \not\sim -K_S$, C contains some -1 -tail components and the only other -1 -curves in S are disjoint from C , or (b) $\text{rk Pic}(S) \geq 3$, $r \geq 2$, the only -1 -curves in S are disjoint from C , or (c) $\text{rk Pic}(S) \geq 3$, $r = 2$, $C_1 + C_2 \sim -K_S$ with $C_1^2 = -1$ and $C_1.C_2 = 2$. Here we used [5, Lemma 3.5] for the structure of C .

To treat (a), contract first all of the -1 -curves disjoint from C . The resulting pair, that we will still denote by (S, C) (without loss of generality) is asymptotically log del Pezzo [5, Lemma 3.4]. Furthermore, this contraction still preserves the fact that there are no -1 -curves not contained in the boundary that intersect the remaining tails: the contraction can only increase the self-intersection of curves in S but it is impossible to have a curve of negative self-intersection less than -1 not in C to begin with [5, Lemmas 2.5, 2.7]. This contraction also evidently preserves the property that there are still no -1 -curves in C that intersect two other components of C . Next, contract one of the -1 -tails in C . The new boundary will have the property that it contains no ‘middle’ -1 -curves while if the boundary component that intersected that tail upstairs was a -2 -curve it will now be itself a -1 -tail. We may therefore repeat this process (this corresponds to reverse operation to blowing-up infinitely near points) until the rank of the Picard group is exactly 3. By induction, in this process there is never a ‘middle’ -1 -component of the boundary. Denote by (S, C) (without loss of generality) the resulting asymptotically log del Pezzo pair [5, Lemma 3.12] with $\text{rk Pic}(S) = 3$. After this most recent contraction there is still at least one remaining -1 -tail, say $C_1 \subset C$, but no ‘middle’ -1 -curves in C (since every rational surface with Picard group of rank 3 has a -1 -curve and we know the only such curves in S must be tail components of C by our careful construction), and furthermore that C_1 still does not intersect any -1 -curves not contained in the boundary. Recall that rational surfaces of Picard rank 3 have precisely three curves of negative self-intersection namely $-k, -1, -1$, $k > 0$, with a -1 -curve being the ‘middle’ curve in such chain. The curve C must contain one of these -1 -curves, and by assumption it cannot contain a ‘middle’ -1 -curve and it cannot intersect a -1 -curve not in C , so all -1 -curves are in C . If $k = 1$ this means C has a ‘middle’ -1 -curve, giving a contradiction. If $k > 1$, the $-k$ -curve must be in C by [5, Lemma 2.5], giving a -1 -curve as a ‘middle’ curve. In conclusion then we obtain a contradiction which concludes our treatment of (a).

To treat (b), note that $C \not\sim -K_S$ as for any -1 -curve E have $-K_S.E = 1$ by adjunction which forces $E.C_i = 1$ for some i if $E \not\subset C$. Contract some of the -1 -curves on S until the Picard group has rank 3. As in case (a) this process still preserves the property that there are no -1 -curves not in the boundary that intersect the boundary. As in (a), there are 3-curves of negative self-intersection forming a chain. By [5, Lemma 2.5], any curve Z with $Z^2 < -1$ must be in C and thus intersect a -1 -curve, forcing S to be the del Pezzo surface of degree 7, with a chain of three -1 -curves e_1, l, e_2 , not intersecting C . We have that $-K_S \sim 2e_1 + 2e_2 + 3l > 0$ so one of these curves must intersect C by ampleness, giving a contradiction and concluding our treatment of (b).

To treat (c), observe that S contains no -1 -curves E disjoint from C since $-K_S.E = 1$ by adjunction, but on the other hand $-K_S \sim C_1 + C_2$ implies either $E.C_1 = 1$ or $E.C_2 = 1$. By our construction S also contains no -1 -curves intersecting C transversally so it follows that all -1 -curves on S are in C .

If C_1 is the only -1 -curve in S then $\text{rk Pic}(S) = 2$ and we are done. So assume that $\text{rk Pic}(S) = 3$ and that C_1 and C_2 are all the -1 -curves on S . Again, by the classification of rational surfaces the only rational surfaces with exactly two -1 -curves and Picard group of rank 3 are the blow-up of \mathbb{F}_n , $n \geq 2$ at a point or the blow-up of \mathbb{F}_1 on a point in Z_1 . However, in all of these cases the two -1 -curves intersect transversally at a single point and not in two points as we assumed in (c). Thus, (c) cannot happen and the proof of Proposition 4.1 is complete. \square

Remark 4.3. Proposition 4.1 shows that asymptotically log del Pezzo pairs are a subset of a rather explicit (infinite) family of pairs. Thus, it reduces Problem 1.1 to determining the generality conditions on the blow-ups of type (i) and (ii) of the pairs listed in Proposition 5.1. As an example, if (s, c) is (ALdP.4.n) and if $f \not\subset c$ is a fiber, we may not blow-up two smooth points on $f \cap c$ since then for any choice of β_i arbitrarily close to 0, $(-K_S - \sum_{i=1}^4 (1 - \beta_i)C_i) \cdot \tilde{f} = 0$. Hence, one generality condition would be “no two of the points blown-up may lie on a fiber not contained in the boundary.” Part of the difficulty in describing such generality conditions lies on the fact that there can infinitely-near points of any order in the operations (ii) and so one needs to describe these generality conditions on all the ‘intermediate surfaces’, so to speak.

5 Classification of asymptotically log del Pezzo pairs with small Picard group

In this section we classify asymptotically log del Pezzos (S, C) with $\text{rk}(\text{Pic}(S)) \leq 2$.

Proposition 5.1. *Let S be a smooth surface with $\text{rk}(\text{Pic}(S)) \leq 2$, and let C_1, \dots, C_r be distinct irreducible smooth curves on S such that $C = \sum_{i=1}^r C_i$ is a divisor with simple normal crossings. Then (S, C) is an asymptotically log del Pezzo pair if and only if it is one of the following pairs:*

- (I.1A) $S = \mathbb{P}^2$, C_1 is a cubic,
- (I.1B) $S = \mathbb{P}^2$, C_1 is a conic,
- (I.1C) $S = \mathbb{P}^2$, C_1 is a line,
- (I.2.n) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n$,
- (I.3A) $S = \mathbb{F}_1$, $C_1 \in |2(Z_1 + F)|$,
- (I.3B) $S = \mathbb{F}_1$, $C_1 \in |Z_1 + F|$,
- (I.4A) $S = \mathbb{P}^1 \times \mathbb{P}^1$, C_1 is a $(2, 2)$ -curve,
- (I.4B) $S = \mathbb{P}^1 \times \mathbb{P}^1$, C_1 is a $(2, 1)$ -curve,
- (I.4C) $S = \mathbb{P}^1 \times \mathbb{P}^1$, C_1 is a $(1, 1)$ -curve,
- (I.5.1) $S = \mathbb{F}_1$, $C_1 \in |2Z_1 + 3F|$,
- (I.6B.1) $S = \mathbb{F}_1$, $C_1 \in |Z_1 + 2F|$,
- (I.6C.1) $S = \mathbb{F}_1$, $C_1 \in |F|$,
- (II.1A) $S = \mathbb{P}^2$, C_1 is a conic, C_2 is a line,
- (II.1B) $S = \mathbb{P}^2$, C_1, C_2 are lines,
- (II.2A.n) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n, C_2 \in |Z_n + nF|$,
- (II.2B.n) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n, C_2 \in |Z_n + (n + 1)F|$,
- (II.2C.n) $S = \mathbb{F}_n$, $n \in \mathbb{N} \cup \{0\}$, $C_1 = Z_n, C_2 \in |F|$,

$$(II.3) \quad S = \mathbb{F}_1, C_1, C_2 \in |Z_1 + F|,$$

$$(II.4A) \quad S = \mathbb{P}^1 \times \mathbb{P}^1, C_1, C_2 \text{ are } (1, 1)\text{-curves},$$

$$(II.4B) \quad S = \mathbb{P}^1 \times \mathbb{P}^1, C_1 \text{ is a } (2, 1)\text{-curve}, C_2 \text{ is a } (0, 1)\text{-curve},$$

$$(II.5A.1) \quad S = \mathbb{F}_1, \text{ either } C_1 \in |2Z_1 + 2F|, C_2 \in |F|, \text{ and } |C_1 \cap C_2| = 2, \text{ or else } C_1 \in |Z_1 + 2F|, C_2 \in |Z_1 + F|,$$

$$(II.5B.1) \quad S = \mathbb{F}_1, C_1 \in |Z_1 + F|, C_2 \in |F|,$$

$$(ALdP.1.n) \quad S = \mathbb{F}_n, n \in \mathbb{N}, C_1 = Z_n, C_2 \in |Z_n + (n + 2)F|,$$

$$(III.1) \quad S = \mathbb{P}^2, C_1, C_2, C_3 \text{ are lines},$$

$$(III.2) \quad S = \mathbb{P}^1 \times \mathbb{P}^1, C_1, C_2, C_3 \text{ are } (1, 1)\text{-}, (0, 1)\text{-}, \text{ and } (1, 0)\text{-curves, respectively},$$

$$(III.3.n) \quad S = \mathbb{F}_n, n \in \mathbb{N} \cup \{0\}, C_1 = Z_n, C_2 \in |F|, C_3 \in |Z_n + nF|,$$

$$(III.4.1) \quad S = \mathbb{F}_1, C_1 \in |F|, C_2, C_3 \in |Z_1 + F|,$$

$$(ALdP.2.n) \quad S = \mathbb{F}_n, n \in \mathbb{N}, C_1 = Z_n, C_2 \in |Z_n + (n + 1)F|, C_3 \in |F|,$$

$$(ALdP.3.n) \quad S = \mathbb{F}_n, n \in \mathbb{N}, C_1 = Z_n, C_2, C_3 \in |F|,$$

$$(IV) \quad S = \mathbb{P}^1 \times \mathbb{P}^1, C_1, C_2 \text{ are } (1, 0)\text{-curves}, C_3, C_4 \text{ are } (0, 1)\text{-curves},$$

$$(ALdP.4.n) \quad S = \mathbb{F}_n, n \in \mathbb{N}, C_1 = Z_n, C_2, C_3 \in |F|, C_4 \in |Z_n + nF|.$$

Remark 5.2. Note that of course when we say, e.g., “ C_1 is a conic”, we mean it is a smooth conic since we assume, as in the statement that each C_i is smooth. Similarly, e.g., in (II.1A) the line and the conic *must* meet at two distinct point as we require $C_1 + C_2$ to have simple normal crossings.

Proof. We start with the sufficiency statement. All pairs in the statement, except (ALdP.1.n), ..., (ALdP.4.n), are strongly asymptotically log del Pezzo [5, Theorems 2.1,3.1]. The remaining four cases are asymptotically log del Pezzo but not strongly asymptotically log del Pezzo as can be seen from (2.3) which also computes their bodies of ample angles in terms of a single inequality, that we have recorded in Corollary 5.3 below and in Figures 1–2.

Let us turn to the necessity statement, i.e., suppose that (S, C) is asymptotically log del Pezzo with $\text{rk Pic}(S) \leq 2$. First, S is rational, hence it is either \mathbb{P}^2 or \mathbb{F}_n [5, §2–3]. When $S = \mathbb{P}^2$, $\text{rk}(\text{Pic}(S)) = 1$ so every asymptotically log del Pezzo is automatically strongly asymptotically log del Pezzo, and since $-K_S \sim 3H$ we see the possibilities are (I.1A), (I.1B), (I.1C), (II.1A), (II.1B), (III.1). Assume now that $S = \mathbb{F}_n$. Denote

$$C_i \in |a_i Z_n + b_i F|.$$

Since $-K_S - C$ is nef by [5, §2.1] and using (2.1) and (2.4), we see that

$$\sum_i a_i \in \{0, 1, 2\}, \quad \sum_i b_i \in \{0, \dots, n + 2\}.$$

Note that by (2.5) either $b_i \geq na_i > 0$, or $(a_i, b_i) = (1, 0)$, i.e. $C_i = Z_n$ in which case this can only happen for one i , or else $(a_i, b_i) = (0, 1)$, i.e., C_i is a fiber. Thus, at most two components of C_i are *not* fibers. Since every fiber intersects any other curve that is not a fiber by at least 1 and by [5, Lemma 3.5] the dual graph of C is either a union of chains or a cycle, if one C_i is not a fiber, then there are at most two fibers in C . In particular $1 \leq r \leq 4$ or C consists only of fibers. Also, if $a_i = 2$ for some i then $n \leq 2$ by $n + 2 \geq b_i \geq 2n$. So we get the following possibilities when $\max_i a_i = 2$:

$$\begin{aligned} [n, (a_1, b_1), \dots, (a_r, b_r)] \in \{ & [2, (2, 4)], [1, (2, 3)], [1, (2, 2), (0, 1)], \\ & [1, (2, 2)], [0, (2, 2)], [0, (2, 1), (0, 1)], [0, (2, 1)] \}. \end{aligned} \quad (5.1)$$

When $\max_i a_i = 1$, we split into two subcases: when there are at least two pairs (a_i, b_i) with all coefficients positive (in which case $n + 2 \geq \sum_i b_i \geq 2n$, as $b_i \geq na_i = n$ for at least two coefficients, so again $0 \leq n \leq 2$):

$$\begin{aligned} [n, (a_1, b_1), \dots, (a_r, b_r)] \in \{ & [2, (1, 2), (1, 2)], [1, (1, 2), (1, 1)], \\ & [1, (1, 1), (1, 1), (0, 1)], [1, (1, 1), (1, 1)], [0, (1, 1), (1, 1)] \}, \end{aligned} \quad (5.2)$$

and otherwise, still with $\max_i a_i = 1$, now for all n (so we omit the first index), and with $\max_i b_i = n$,

$$\begin{aligned} [(a_1, b_1), \dots, (a_r, b_r)] \in \{ & [(1, n), (1, 0), (0, 1), (0, 1)], [(1, n), (1, 0), (0, 1)], \\ & [(1, n), (1, 0)], [(1, n), (0, 1), (0, 1)], [(1, n), (0, 1)], [(1, n)] \}, \end{aligned} \quad (5.3)$$

with $\max_i b_i = n + 1$,

$$\begin{aligned} [(a_1, b_1), \dots, (a_r, b_r)] \in \{ & [(1, n + 1), (1, 0), (0, 1)], \\ & [(1, n + 1), (1, 0)], [(1, n + 1), (0, 1)], [(1, n + 1)] \}, \end{aligned} \quad (5.4)$$

with $\max_i b_i = n + 2$,

$$[(a_1, b_1), \dots, (a_r, b_r)] \in \{ [(1, n + 2), (1, 0)], [(1, n + 2)] \}, \quad (5.5)$$

when $\max_i b_i = 1$,

$$[(a_1, b_1), \dots, (a_r, b_r)] \in \{ [(1, 0)], [(1, 0), (0, 1)], [(1, 0), (0, 1), (0, 1)] \}, \quad (5.6)$$

since as we saw earlier if C has an irreducible curve in its support that it is not a fiber, then C can have at most 2 fibers in its support.

Finally when $\max_i a_i = 0$,

$$[(a_1, b_1), \dots, (a_r, b_r)] \in \{ [(0, 1)], \dots, \overbrace{[(0, 1), \dots, (0, 1)]}^{n+2} \}. \quad (5.7)$$

A few of these cases can be eliminated, though most of them actually occur. In (5.1), $[2, (2, 4)]$ corresponds to a smooth anticanonical curve in \mathbb{F}_2 , which is excluded as \mathbb{F}_2 is not del Pezzo. The remaining cases are: $[1, (2, 3)] = \text{I.5.1}$, $[1, (2, 2), (0, 1)] = \text{II.5A.1}$ (blow-up on the line in II.1A), $[1, (2, 2)] = \text{I.3A}$, $[0, (2, 1), (0, 1)] = \text{II.4B}$, $[0, (2, 1)] = \text{I.4B}$.

In (5.2) $[2, (1, 2), (1, 2)]$ is excluded as $Z_2 \cdot (Z_2 + 2F) = Z_2 \cdot (2Z_2 + 4F) = Z_2 \cdot (-K_{\mathbb{F}_2}) = 0$. The remaining cases are: $[1, (1, 2), (1, 1)] = \text{II.5A.1}$ (blow-up on the conic in II.1A);

$[1, (1, 1), (1, 1), (0, 1)] = \text{III.4.1}$, $[1, (1, 1), (1, 1)] = \text{II.3}$, $(0, (1, 1), (1, 1)) = \text{II.4A}$.

In (5.3), $[(1, n), (1, 0), (0, 1), (0, 1)]$ gives $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + nF) - (1 - \beta_2)Z_n - (1 - \beta_3)F - (1 - \beta_4)F \sim (\beta_1 + \beta_2)Z_n + (n\beta_1 + \beta_3 + \beta_4)F$, that is ample if and only if $n\beta_1 + \beta_3 + \beta_4 > n\beta_1 + n\beta_2$, i.e., $\beta_3 + \beta_4 > n\beta_2$, and this is ALdP.4.n if $n \geq 1$ or IV if $n = 0$; $[(1, n), (1, 0), (0, 1)] = \text{III.3.n}$, $[(1, n), (1, 0)] = \text{II.2A.n}$. For $[(1, n), (0, 1), (0, 1)]$ consider $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + nF) - (1 - \beta_2)F - (1 - \beta_3)F \sim (1 + \beta_1)Z_n + (n\beta_1 + \beta_2 + \beta_3)F$, that is ample if and only if $n\beta_1 + \beta_2 + \beta_3 > n + n\beta_1$, i.e., $n = 0$, and this is III.3.0. For $[(1, n), (0, 1)]$, consider $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + nF) - (1 - \beta_2)F \sim (1 + \beta_1)Z_n + (1 + n\beta_1 + \beta_2)F$, that is ample if and only if $1 + n\beta_1 + \beta_2 > n + n\beta_1$, i.e., $n = 0, 1$, and these are II.2C.0, II.5B.1. For $[(1, n)]$, $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + nF) \sim (1 + \beta_1)Z_n + (2 + n\beta_1)F$ implying $n = 0, 1, 2$ and these are I.2.0, I.3B, while the case $n = 2$ is excluded as in the first paragraph.

In (5.4): $[(1, n+1), (1, 0), (0, 1)] - K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + (n+1)F) - (1 - \beta_2)Z_n - (1 - \beta_3)F \sim (\beta_1 + \beta_2)Z_n + ((n+1)\beta_1 + \beta_3)F$ that is ample if and only if $(n+1)\beta_1 + \beta_3 > n(\beta_1 + \beta_2)$, i.e., $\beta_1 + \beta_3 > n\beta_2$, and this is ALdP.2.n if $n \geq 1$ and III.2 if $n = 0$; $[(1, n+1), (1, 0)] = \text{II.2B.n}$; $[(1, n+1), (0, 1)]$, $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + (n+1)F) - (1 - \beta_2)F \sim (1 + \beta_1)Z_n + ((n+1)\beta_1 + \beta_2)F$ that is ample if and only if $(n+1)\beta_1 + \beta_2 > n(1 + \beta_1)$, i.e., $n = 0$ and this is II.2B.0; $[(1, n+1)]$, $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + (n+1)F) \sim (1 + \beta_1)Z_n + (1 + (n+1)\beta_1)F$ that is ample if and only if $1 + (n+1)\beta_1 > n(1 + \beta_1)$, i.e., $n = 0, 1$, and these are I.4C, I.6B.1.

In (5.5), $[(1, n+2), (1, 0)]$, $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + (n+2)F) - (1 - \beta_2)Z_n \sim (\beta_1 + \beta_2)Z_n + (n+2)\beta_1 F$, i.e., $2\beta_1 > n\beta_2$, and this is ALdP.1.n; $[(1, n+2)]$, $-K_{\mathbb{F}_n} - (1 - \beta_1)(Z_n + (n+2)F) \sim (1 + \beta_1)Z_n + (n+2)\beta_1 F$, i.e., $(n+2)\beta_1 > n + n\beta_1$, i.e., $n = 0$ and this is I.4B.

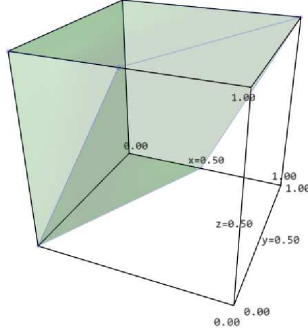


Figure 2: The body of ample angles for ALdP.k.2 for $k = 2, 3$ and the projection of ALdP.4.2 on $\beta_1 = m$ hyperplane for $m \in (0, 1)$.

In (5.6), there is one Z_n and $0 \leq k \leq 2$ fibers. When $k = 0$ this is I.2.n, when $k = 1$ this is II.2C.n, and when $k = 2$: $-K_{\mathbb{F}_n} - (1 - \beta_1)Z_n - (1 - \beta_2)F - (1 - \beta_3)F \sim (1 + \beta_1)Z_n + (n + \beta_2 + \beta_3)F$, that is ample if and only if $n(1 + \beta_1) < (n + \beta_2 + \beta_3)$, and is ALdP.3.n.

Finally, in (5.7), there are k fibers, so $-K_{\mathbb{F}_n} - (1 - \beta_1)F - \dots - (1 - \beta_k)F \sim 2Z_n + (n + 2 - k + \beta_1 + \dots + \beta_k)F$, that is ample if and only if $2n < (n + 2 - k + \beta_1 + \dots + \beta_k)$, i.e., $n = k = 1$ and I.6C.1, or $n = 0$ and $k = 1, 2$ and these are I.2.0, II.2A.0. \square

Corollary 5.3. *Let S be a smooth surface with $\text{rk}(\text{Pic}(S)) \leq 2$, and let C_1, \dots, C_r be irreducible smooth curves on S such that $C = \sum_{i=1}^r C_i$ is a divisor with simple normal crossings. Then (S, C) is an asymptotically log del Pezzo pair, but not strongly asymptotically log del Pezzo, if and only if it is one of the pairs ALdP.1.n, ALdP.2.n, ALdP.3.n, ALdP.4.n. Moreover,*

$$\text{AA}(S, C) = \begin{cases} \{(\beta_1, \beta_2) \in (0, 1]^2 : -n\beta_1 + 2\beta_2 > 0\} & \text{if } (S, C) \text{ is ALdP.1.n,} \\ \{(\beta_1, \beta_2, \beta_3) \in (0, 1]^3 : -n\beta_1 + \beta_2 + \beta_3 > 0\} & \text{if } (S, C) \text{ is ALdP.2.n or ALdP.3.n,} \\ \{(\beta_1, \beta_2, \beta_3, \beta_4) \in (0, 1]^4 : -n\beta_1 + \beta_2 + \beta_3 > 0\} & \text{if } (S, C) \text{ is ALdP.4.n.} \end{cases}$$

6 The body of ample angles

The goal of this section is to prove Theorem 1.4, as a consequence of the main result in the theory of Shokurov's log geography (cf. [1, 3, 19, 26]).

Before we proceed with the proof, we recall some of the notions and results which we will use later. As in Section 1, (X, D) is a pair consisting of a smooth complex projective variety X and a simple-normal crossing divisor $D = \sum_{i=1}^r D_i$. Let $\text{WDiv}_{\mathbb{R}}(X)$ be the real vector space spanned by all the divisors in X , and let $V \subset \text{WDiv}_{\mathbb{R}}(X)$ be the finite-dimensional vector subspace spanned by the basis D_1, \dots, D_r , and use that basis to identify $V \cong \mathbb{R}^r$. Fix an ample \mathbb{Q} -divisor A and define

$$\mathcal{L}(V) := \{B \in V : B = \sum_{i=1}^r a_i D_i \text{ where } a_1, \dots, a_r \in [0, 1]^r\},$$

and

$$\mathcal{A}_A(V) := \{B \in \mathcal{L}(V) : K_X + A + B \text{ is ample}\}.$$

We want to show that the set $\mathcal{A}_A(V)$ is a rational polytope. Indeed, let $\psi: X \rightarrow X$ be the identity morphism and let $\pi: X \rightarrow \text{Spec } \mathbb{C}$ be the structure morphism. By definition, $\text{Spec } \mathbb{C}$ is a point, so π is the trivial morphism to a point. Then, $\mathcal{A}_A(V)$ coincides with $A_{\psi, A, \pi}(V)$, as defined in [1, Definition 1.1.4]. Thus, by [1, Corollary 1.1.5], it follows that $\mathcal{A}_A(V)$ is a finite union of rational polytopes and, by convexity of $\mathcal{A}_A(V)$, the claim follows.

We now proceed with the proof of our Theorem.

Proof of Theorem 1.4. Suppose $\text{AA}(X, D)$ is non-empty. We claim that the closure of $\text{AA}(X, D)$ is a rational polyhedron. By openness of ampleness, there exists $\gamma = (\gamma_1, \dots, \gamma_r) \in (0, 1]^r \cap \mathbb{Q}$ such that $-K_X - \sum_{i=1}^r (1 - \gamma_i)D_i$ is ample. We fix such a γ for the rest of the proof. Let

$$\eta := \max \left\{ \max_i \frac{1 - \gamma_i}{\gamma_i}, \max_i \frac{\gamma_i}{1 - \gamma_i} \right\}.$$

For any $\beta \in (0, 1)^r$ we claim we can express

$$-K_X - \sum_{i=1}^r (1 - \beta_i)D_i = \eta(K_X + A + F(\beta)), \quad (6.1)$$

with two key properties:

- A ample and independent of β , and
- $F(\beta)$ effective, supported on D , and with the coefficient of each D_i in $(0, 1)$ (and, of course, depending on β), in such a way that the map between $\beta \in (0, 1)^r$ and the vector of coefficients of $F(\beta)$ in $(0, 1)^r$ is one-to-one.

More specifically, let

$$A := -\frac{1 + \eta}{\eta} \left(K_X + \sum (1 - \gamma_i)D_i \right),$$

and

$$F(\beta) := \sum_{i=1}^r \left(1 + \frac{\beta_i}{\eta} - \frac{1 + \eta}{\eta} \gamma_i \right) D_i.$$

Then, one readily checks that (6.1) holds for each $\beta \in (0, 1)^r$. It remains to check the two key properties. First, A is ample by definition since $\eta > 0$ and $\gamma \in \text{AA}(X, D)$; A is also clearly independent of β . Second, we need to verify that

$$0 \leq 1 + \frac{\beta_i}{\eta} - \frac{1 + \eta}{\eta} \gamma_i \leq 1, \quad (6.2)$$

for each $i = 1, \dots, r$ and each $\beta \in (0, 1)^r$. The upper bound amounts to

$$\frac{\beta_i}{\eta} \leq \frac{1 + \eta}{\eta} \gamma_i,$$

i.e.,

$$\frac{\beta_i}{\gamma_i} - 1 \leq \eta,$$

and this holds since

$$\frac{\beta_i}{\gamma_i} - 1 \leq \frac{1 - \gamma_i}{\gamma_i} \leq \max_i \frac{1 - \gamma_i}{\gamma_i} \leq \eta.$$

The lower bound amounts to

$$0 \leq 1 + \frac{\beta_i}{\eta} - \frac{1 + \eta}{\eta} \gamma_i,$$

i.e.,

$$\frac{\gamma_i - \beta_i}{1 - \gamma_i} \leq \eta,$$

and this holds since

$$\frac{\gamma_i - \beta_i}{1 - \gamma_i} \leq \frac{\gamma_i}{1 - \gamma_i} \leq \max_i \frac{\gamma_i}{1 - \gamma_i} \leq \eta.$$

In sum, we have proved the following. Let $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$ be the affine map

$$f(\beta) := (1, \dots, 1) + \frac{\beta}{\eta} - \frac{1 + \eta}{\eta} \gamma.$$

Then f maps the cube $(0, 1)^r$ into $(0, 1)^r$. We claim that these facts combined imply that

$$f^{-1}(\mathcal{A}_A(V)) \cap (0, 1)^r = \text{AA}(X, D), \quad (6.3)$$

where we identify $\mathcal{A}_A(V)$ with a subset of $(0, 1)^r$ via the basis D_1, \dots, D_r . Indeed, if $\beta \in \text{AA}(X, D)$ then by definition $\beta \in (0, 1)^r$, and, moreover, by (6.1) $f(\beta) \in \mathcal{A}_A(V)$, i.e., $\beta \in f^{-1}(\mathcal{A}_A(V)) \cap (0, 1)^r$. Conversely, if $\beta \in f^{-1}(\mathcal{A}_A(V)) \cap (0, 1)^r$, write $\beta = f^{-1}(\alpha)$ with $\alpha \in \mathcal{A}_A(V)$. One readily checks that f^{-1} can be written explicitly as

$$f^{-1}(\alpha) := \eta\alpha - (\eta, \dots, \eta) + (1 + \eta)\gamma. \quad (6.4)$$

By definition of α we know that $K_X + A + \sum \alpha_i D_i$, and hence also $\eta(K_X + A + \sum \alpha_i D_i)$, is ample. But the latter equals identically $-K_X - D + \eta \sum \alpha_i D_i + (1 + \eta) \sum \gamma_i D_i - \eta D$ which by (6.4) equals $-K_X - D + \sum (f^{-1}(\alpha))_i D_i = -K_X - \sum (1 - \beta_i) D_i$; recalling that by assumption $\beta \in (0, 1)^r$, we have shown that $\beta \in \text{AA}(X, D)$. Altogether, we have established (6.3), as desired. We remark in passing that it is not true in general that f^{-1} maps the cube $(0, 1)^r$ into itself, but that is not needed for our proof. Finally, using (6.3), that $\mathcal{A}_A(V)$ is a rational polyhedron by the discussion at the beginning of this section, and since f is an affine map, it follows that $\text{AA}(X, D)$ is a rational polyhedron. \square

Remark 6.1. We give a second proof of Theorem 1.4 that is a bit more highbrow. Let X be a smooth projective variety. We denote by $NS(X)$ the Néron-Severi group of X , i.e. the real vector space spanned by the divisors of X modulo numerical equivalence. We denote by $\text{Nef}(X) \subset NS(X)$ the nef cone of X , i.e., the cone spanned by the numerically effective (nef) divisors in $NS(X)$. Let L_1, \dots, L_k be divisors in X . The *Cox ring* of X associated to L_1, \dots, L_k is the ring

$$R(X, L_\bullet) := \bigoplus_{m_1, \dots, m_k \geq 0} H^0 \left(X, \mathcal{O}_X \left(\sum_{i=1}^k m_i L_i \right) \right).$$

The variety X is called a *Mori dream space* if $H^1(X, \mathcal{O}_X) = 0$ and there exist divisors L_1, \dots, L_k whose numerical classes generate $NS(X)$ and such that the ring $R(X, L_\bullet)$ is finitely generated. If X is a Mori dream space then, in particular, $\text{Nef}(X)$ is a rational polyhedral (cf. [17, Proposition 2.9 and Definition 1.10]). Assume now that $D = \sum_{i=1}^r D_i$ is a simple-normal-crossing divisor in X such that $\text{AA}(X, D)$ is not empty and let $\gamma = (\gamma_1, \dots, \gamma_r) \in \text{AA}(X, D)$. Let

$$\Delta = \sum_{i=1}^r (1 - \gamma_i) D_i.$$

Then (X, Δ) is log smooth and, in particular, it is divisorially log terminal (e.g. see [1, Definition 3.1.1]). Thus, [1, Corollary 1.3.2] implies that X is a Mori dream space. Consider the affine map

$$\Phi: \mathbb{R}^r \rightarrow NS(X)$$

given by, for any $(\beta_1, \dots, \beta_r) \in \mathbb{R}^r$,

$$\Phi(\beta_1, \dots, \beta_r) = [-K_X - \sum_{i=1}^r (1 - \beta_i) D_i].$$

Then, from the definitions,

$$\overline{\text{AA}(X, D)} = [0, 1]^r \cap \Phi^{-1}(\text{Nef}(X))$$

(here we denote by $\Phi^{-1}(\text{Nef}(X))$ the pre-image of $\text{Nef}(X)$). Since $NS(X)$ is a finite dimensional vector space, we may fix an isomorphism $NS(X) \simeq \mathbb{R}^d$ for some positive integer d and choose coordinates x_1, \dots, x_d on this vector space. Denote by Φ_1, \dots, Φ_d the map Φ expressed in these coordinates and note each $\Phi_j(\beta)$ is affine in β . Since we know that $\text{Nef}(X) = \bigcap_{k=1}^N \{ \sum_{j=1}^d a_{kj} x_j \geq 0 \}$ with $a_{kj} \in \mathbb{Q}$ and $N \in \mathbb{N}$, it follows that $\overline{\text{AA}(X, D)} = \{ \beta \in [0, 1]^r : \sum_{j=1}^d a_{kj} \Phi_j(\beta) \geq 0, k = 1, \dots, N \}$, a set cut out by finitely-many affine equations with rational coefficients in β , proving Theorem 1.4.

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