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RESEARCH ARTICLE

An algebraic approach to revealed preference

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Abstract

We propose and develop an algebraic approach to revealed preference. Our approach dispenses with non-algebraic structure, such as topological assumptions. We provide algebraic axioms of revealed preference that subsume previous classical revealed preference axioms and show that a data set is rationalizable if and only if it is consistent with an algebraic axiom.

Keywords Revealed preferences · Unified approach · Algebraic axiom · Revealed preference axiom · Preference extension

JEL Classification C02 · D01 · D11

1 Introduction

The revealed preference approach to consumer choice, pioneered by Samuelson (1938), builds on the fact that, although we cannot observe the complete preference relation profiles of economic agents, we can observe their choices over some budget sets. Starting with the work Richter (1966) and Afriat (1967), this approach has been used to construct tests of rational decision making (see Chambers and Echenique 2016, for a recent comprehensive overview).

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Contribution

We propose an algebraic version of revealed preference approach. That is, we consider theories about preferences in their logical structure together with the underlying algebraic structure. By a theory about preferences, we mean a statement about preferences such as "If x is better than y then $f(x)$ is better than $f(y)$, where $f \in \mathcal{F}$." In this statement, f is some function over alternatives, and $\mathcal{F}$ is a family of functions that actually defines the theory. The algebraic structure we impose considers the algebra of $(\mathcal{F}, \circ)$, where $\circ$ is the composition operator. In particular, we propose *algebraic axioms of revealed preferences*, and show that if $(\mathcal{F}, \circ)$ is a group,[1] then the observed set of data could be generated by a binary relation consistent with the theory $\mathcal{F}$ if and only if it is consistent with the algebraic axiom of revealed preferences.

Since the definition of the theory does not include transitivity, we consider two versions of the rationalization: excluding and including the transitivity of the underlying binary relation. Hence, we define the weak and strong algebraic axioms similarly to the weak and strong axioms of revealed preferences. We show that the weak (algebraic) axiom is the equivalent to the possibility that the data could be generated by a binary relation consistent with the theory $\mathcal{F}$; while the strong (algebraic) axiom is equivalent to the possibility that the data could be generated by a transitive binary relation consistent with the theory $\mathcal{F}$. Moreover, we show that if we do not require the preference relation to be transitive, we obtain completeness for free. That is, a data set could be generated by a binary relation consistent with theory $\mathcal{F}$ if and only if it could be generated by a complete binary relation consistent with the theory $\mathcal{F}$. If we also require the preference relation to be transitive, in addition to being consistent, we need an additional assumption in order to get the complete rationalization.

We also show that our result provides a unified framework for the classical revealed preference axioms (see Afriat 1967; Varian 1983; Forges and Minelli 2009; Heufer 2013; Nishimura et al. 2017; Castillo and Freer 2021). In particular, we provide applications to the existence of transitive, homothetic, and quasilinear preferences, as well as preferences that satisfy independence. Finally, let us note that our approach does not guarantee the existence of a utility function that represents the underlying preference relation. This is a consequence of considering only the algebraic structure of preferences.

Related literature

Our work is linked to the literature on generalized revealed preferences. Several authors in this literature provide a generalization of the revealed preference approach, but keeping some topological assumptions in place. Topological assumptions are necessary to guarantee existence of a convenient utility function representing the underlying preference relation. Seminal examples are Chavas and Cox (1993), Forges and Minelli (2009) and Nishimura et al. (2017), who generalize (Afriat 1967) theorem for general shapes of budgets and topological spaces. Recently, Polisson et al. (2020) have pro-

[1] A tuple $(\mathcal{F}, \circ)$ is said to be group if the set of functions $\mathcal{F}$ contains an identify function, $\mathcal{F}$ is closed, every function in $\mathcal{F}$ has an inverse that also belongs to $\mathcal{F}$, and the composition operator $\circ$ is associative.

posed a lattice approach, and provided conditions that guarantee the rationalization of an observed set of data with theories such as expected utility, ranked dependent expected utility, and cumulative prospect theory. The papers mentioned above construct tests which can be easily applied to the data.

Other authors generalize (Szpilrajn 1930) result, concentrating on the completion of the revealed preference relation. Seminal papers by Suzumura (1976), Duggan (1999) and Demuynck (2009) provide revealed preference tests (in the shape of *Suzumura consistency*) for transitive, acyclic, homothetic, and convex preferences. However, the Suzumura consistency condition may be complicated for practical implementation, which is an important difference with the papers mentioned in the previous paragraph. Let us also note that Suzumura consistency is not a revealed preference axiom in the orthodox sense, as it is not stated in terms of choices and budgets. In this sense, we provide a link between these two strands of revealed preference research. That is, we adopt a scope of theories comparable to one presented in Demuynck (2009), while providing a tractable and simple revealed preference axiom.

Unlike most of the previous literature, we do not require the revealed preference relation to be generated by a complete preference relation. As we know from Chambers et al. (2014), completeness does not bring additional empirical content if transitivity is required. Our results in terms of adding completeness to other properties can be interpreted in that spirit. Instead of taking completeness as a desiderata, we focus on whether we can get it and at what costs. Unless we require the resulting binary relation to be transitive, the completeness is obtained for free, in the sense that adding the assumption of completeness to other assumptions about the binary relation does not bring any additional empirical content.

Organization of the paper

The remainder of the paper is organized as follows. We present the necessary definitions for algebraic revealed preferences in Sect. 2. We show our rationalizability result in Sect. 3. We present applications in Sect. 4. We provide some concluding remarks in Sect. 5. All proofs omitted in the text are collected in an “Appendix”.

2 Preliminaries

Let X be the space of alternatives. Let $R \subseteq X \times X$ be a *binary relation*. A binary relation is said to be *reflexive* if $(x, x) \in R$ for every $x \in X$. Let $R \subseteq X \times X$ be a *preference relation* if it is a reflexive binary relation. A binary relation is said to be *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for every $x, y, z \in X$. A binary relation is said to be *complete* if all pairs are comparable, i.e. $(x, y) \in R$ or $(y, x) \in R$ for every $x, y \in X$. Let $P(R)$ be the (asymmetric) strict part of the relation; that is, $P(R) = \{(x, y) \in R : \ (y, x) \notin R\}$.

2.1 Theories about preferences

Next we present our notion of a theory about preferences. Our notion is meant to capture the idea that a theory is a collection of desirable properties of preference relations. We first discuss some usual theories to motivate our notion.

Example Let a convex cone $X \subseteq \mathbb{R}^N_+$ be a set of alternatives. Consider a theory that imposes *homothetic preferences*, that is $(x, y) \in R$ implies $(\alpha x, \alpha y) \in R$ for every $\alpha \in \mathbb{R}_{++}$.

Example Let $X \subseteq \mathbb{R} \times \mathbb{R}^{N-1}_+$ be a set of alternatives. Consider a theory that imposes *quasilinear preferences*, that is $(x, y) \in R$ implies $(x + \alpha e, y + \alpha e) \in R$ for every $\alpha \in \mathbb{R}$, where $e = (1, 0, \ldots, 0, \ldots, 0)$ is the vector of zeros with unique 1 element.[2]

We can generalize the structure observed in both of these theories as follows. Both αx and $x + \alpha e$ can be in general presented as functions $f : X \to X$. Then, the theory itself can be described as a collection of functions $\mathcal{F}$ preserving the preference relation. For homothetic preferences the allowed functions would be all linear functions with positive slope and zero intercept. For quasilinear preferences the allowed functions would be all linear functions with slope of one and intercept of αe for every real α. We generalize this idea allowing for other collections of functions, so that every allowed collection of functions $\mathcal{F}$ defines a theory. In particular, we require the collection of functions $\mathcal{F}$ endowed with the composition operator to be a group.

Definition 1 A tuple $(\mathcal{F}, \circ)$ is a **group** if it

contains identity:
$I \in \mathcal{F}$, where $I(x) = x$ for every $x \in X$;
is closed:
$f, f' \in \mathcal{F}$ implies $f \circ f' = f'' \in \mathcal{F}$;
has inverse element:
$\forall f \in \mathcal{F}\ \exists f^{-1} \in \mathcal{F}$ such that $f \circ f^{-1} = f^{-1} \circ f = I$;
is associative:
$(f \circ f') \circ f'' = f \circ (f' \circ f'')$ for all $f, f', f'' \in \mathcal{F}$.

As is well-known, the collection of all bijective functions from a set to itself constitutes a group. Thus, the assumption of group structure is not very restrictive once we can characterize a theory as a collection of transformations. The requirement of identity guarantees that applying a theory we are not losing information about preferences. Closedness implies a theory is self-contained; without this requirement a theory may generate some implications which are not directly prescribed by this theory. Associativity is a technical property of the composition operator.

The requirement of inverse element is more substantial; intuitively, it means that if it is desirable that $(x, y) \in R$ implies $(f(x), f(y)) \in R$, then the reverse implication is desirable as well. The inverse implication is actually guaranteed by existence of the inverse element for every function $f \in \mathcal{F}$. As we want to consider not only complete

[2] We assume that the first good is a numeraire without a loss of generality. Otherwise, one can simply renumerate the dimensions in order to ensure the first good to be a numeraire.

relations being consistent with the theory, the reverse implication is crucial. For an example of its usefulness, one can see Aumann (1962) where it is illustrated how the independence axiom needs to be modified for the representation of incomplete preference relation.

Definition 2 Let $(\mathcal{F}, \circ)$ be a group. A preference relation R is **consistent with theory** $\mathcal{F}$ if

$$(x, y) \in R \text{ implies } (f(x), f(y)) \in R \text{ for every } f \in \mathcal{F}.$$

Note that our notion of theory does not involve completeness or transitivity. Neither of these assumptions can be expressed in the simple terms of the theory as we define it. Completeness and/or transitivity will be considered as additional assumptions when introducing the notions of rationalization of the data set with a preference relation. This also allows us to consider the interaction between these assumptions and our notion of the theory.

2.2 Data and rationalization

The essence of the revealed preference problem is to extract the (unobserved) preference relations which generated (observed) choices over budgets. If there is such a preference relation, then the corresponding data set (collection of choices from budgets) is rationalizable. Next, we formally define the rationalizable data sets.

Let $B \subseteq X$ be a budget set, where B is any nonempty subset of X. Let $\mathcal{B}$ be a collection of budgets. Let $C : \mathcal{B} \rightrightarrows X$ be a choice correspondence. Denote by $(\mathcal{B}, C)$ a data set. We say that $\max(B, R) \subseteq B$ is the *set of maximal points* of budget set B according to preference relation R if $(x, y) \in R$ for every $x \in \max(B, R)$ and $y \in B$; and $(x, y) \in P(R)$ for every $x \in \max(B, R)$ and $y \in B \setminus \max(B, R)$.[3] While the first part of the definition of the set of maximal points is standard, the second one may seem unnecessary. For instance, it is trivially satisfied for any transitive and complete R. However, we do need this part as we consider possibly intransitive or incomplete preference relations for which it is non-trivial.

Rationalizability requires that the set of chosen points coincides with the set of maximal points at every budget for a preference relation that is consistent with the theory $\mathcal{F}$.

We present four different versions of rationalizability, according to whether they include or not transitivity and completeness as desiderata. Rationalizability without transitivity is denoted ‘weak’ and rationalizability with completeness is denoted ‘complete.’

Definition 3 A data set $(\mathcal{B}, C)$ is

weakly rationalizable if there is a preference relation R^* consistent with theory $\mathcal{F}$,

[3] Our notion of maximal points (even being standard for revealed preferences) already implicitly makes an assumption about comparability. That is, the chosen point is at least as good as any other point in the budget, and not just an undominated point. However, relaxing it would result in the absence of empirical content of any theory that does not explicitly assumes completeness.

completely weakly rationalizable if there is a complete preference relation R^* consistent with theory $\mathcal{F}$,
rationalizable if there is a transitive preference relation R^* consistent with theory $\mathcal{F}$,
completely rationalizable if there is a transitive and complete preference relation R^* consistent with theory $\mathcal{F}$,

such that

$$C(B) = \max(B, R^*) \text{ for every } B \in \mathcal{B}.$$

The definition of rationalizability specifies that there is a preference relation consistent with a given theory, such that the observed choices can be generated by maximization of this preference relation. Notably, we abstain from any restrictions on the cardinality of the data set and any topological structure of the space of alternatives, since we are pursuing a purely algebraic approach. A disadvantage of this level of generality is that we cannot talk about a utility function representing the preference relation.

3 Results

In what follows, we establish the equivalence between the axioms of revealed preference and the four notions of rationalization according to $\mathcal{F}$. In presenting some of the results, we denote the composition of $f_1, \ldots, f_n \in \mathcal{F}$ by

$$\bigcirc_{j=1}^{n} f_j = f_1 \circ f_2 \circ \ldots \circ f_n.$$

3.1 Non-complete rationalization

Following the revealed preference tradition, we define two axioms: weak and strong. Parallel to standard results in revealed preference theory, the weak axiom does not account for transitivity, while the strong one does.

Definition 4 A data set satisfies the **weak algebraic axiom of revealed preference (WAARP)** if for every $x_i \in C(B_i)$ and $x_j \in C(B_j)$ and every $f \in \mathcal{F}$,

$$f(x_i) \in B_j \text{ implies } f^{-1}(x_j) \notin B_i \backslash C(B_i).$$

Recall that the standard weak axiom implies that if $x_i \in C(B_i)$ is available at B_j, then $x_j \in C(B_j)$ should not be available at $B_i \backslash C(B_i)$. Otherwise, there would be a contradiction between x_j being better than x_i and x_i being better than x_j. Since now we incorporate the theory $\mathcal{F}$, we also need to make sure that there is no $f \in \mathcal{F}$ such that x_j is better than $f(x_i)$, and $f(x_i)$ is better than x_j. Taking the inverse f of x_j is crucial because if $(x_i, f^{-1}(x_j)) \in R^*$ and R^* is consistent with the theory, then $(f(x_i), [f \circ f^{-1}](x_j)) = (f(x_i), x_j) \in R^*$. This reasoning illustrates the necessity of the weak axiom.

Proposition 1 *A data set is weakly rationalizable if and only if it satisfies WAARP.*

In parallel to the standard logic behind the strong axiom, we need to account for the indirect preference relation. This is done via replacing a single point $x_j \in B_i$ (which would correspond to direct revealed preference) by the sequence of $x_n \in B_{n-1}, x_{n-1} \in B_{n-2}, \dots, x_2 \in B_1$, which would imply that $x_n \notin B_1$. The existence of the sequence described above would ensure that x_1 is better than x_n if the preference relation is transitive, while the conclusion guarantees that x_n is not better than x_1.

Definition 5 A data set $(\mathcal{B}, C)$ satisfies the **strong algebraic axiom of revealed preference (SAARP)** if for every sequence $x_1, \dots, x_n$ such that $x_j \in C(B_j)$ for every $j \in \{1, \dots, n\}$ for some $B_1, \dots, B_n \in \mathcal{B}$, and every sequence $f_1, \dots, f_{n-1} \in \mathcal{F}$ such that

$$f_j(x_{j+1}) \in B_j \text{ for every } j \in \{1, \dots, n-1\},$$

we have

$$\left[\bigcirc_{j=1}^{n-1} f_j\right]^{-1} (x_1) \notin B_n \backslash C(B_n).$$

SAARP incorporates not only the structure of the standard strong axiom but also the transformations allowed by the theory. Assuming that $f_j(x_{j+1}) \in B_j$ implies that x_j is better than $f_j(x_{j+1})$. Since we require the preference relation to be transitive and consistent with $\mathcal{F}$, then x_j being better than $f_j(x_{j+1})$, implies that x_{j-1} is better than $[f_{j-1} \circ f_j](x_{j+1})$. Applying this logic sequentially we obtain that

$$x_1 \text{ is better than } \left[\bigcirc_{j=1}^{n-1} f_j\right](x_n).$$

Hence, we need to ensure that

$$\left[\bigcirc_{j=1}^{n-1} f_j\right](x_n) \text{ is not strictly better than } x_1.$$

Given that the preference relation is consistent with $\mathcal{F}$, the claim above is equivalent to taking the inverse of the function from both sides, i.e.

$$x_n \text{ is not strictly better than } \left[\bigcirc_{j=1}^{n-1} f_j\right]^{-1} (x_1).$$

While this argument illustrates the necessity of SAARP, the axiom is also sufficient for rationalization.

Proposition 2 *A data set is rationalizable if and only if it satisfies SAARP.*

3.2 Complete rationalization

The results above consider only consistency with the theory and transitivity as desiderata for rationalization. As it happens, we can obtain complete weak rationalization without any additional assumptions. That is, the very same weak axiom is necessary and sufficient for complete weak rationalization.

Proposition 3 *A data set is completely weakly rationalizable if and only if it satisfies WAARP.*

The idea of the proof is to provide a (rather arbitrary) algorithm which completes the observed revealed preference relation in a way that it is consistent with the theory. While in the case of weak rationalization such algorithm clearly converges, once we add the requirement of transitivity convergence cannot be guaranteed without additional structure. In particular, we employ the notion of order. This additional structure is illustrated by the examples below.

Example Recall a theory that imposes homothetic preferences, that is $(x, y) \in R$ implies $(\alpha x, \alpha y) \in R$ for every $\alpha \in \mathbb{R}_{++}$. Note that the space of functions is completely ordered, that is for every $\alpha, \alpha' \in \mathbb{R}_{++}$ either $\alpha \geq \alpha'$ or $\alpha' \geq \alpha$, where $\geq$ is the standard greater or equal order on $\mathbb{R}_{++}$. Hence, either $\alpha x \geq \alpha' x$ or $\alpha' x \geq \alpha x$ for every $x \in X \subseteq \mathbb{R}_{++}$. Moreover, e.g. in dealing with baskets of goods, it makes sense to assume that homotheticity would also imply that $(\alpha x, \alpha' x) \in R$ if $\alpha \geq \alpha'$ for every $x \in X \subseteq \mathbb{R}_{++}$.

Example Recall a theory that imposes quasilinear preferences, that is $(x, y) \in R$ implies $(x + \alpha e, y + \alpha e) \in R$ for every $\alpha \in \mathbb{R}$, where $e = (1, 0, \ldots, 0, \ldots, 0)$ is the vector of zeros with unique 1 element. Note that the functions we consider are completely ordered, that is for every $\alpha, \alpha' \in \mathbb{R}$ either $\alpha \geq \alpha'$ or $\alpha' \geq \alpha$, where $\geq$ is the standard greater or equal order on $\mathbb{R}$. Hence, either $x + \alpha e \geq x + \alpha' e$ or $x + \alpha' e \geq x + \alpha e$ for every $x \in X$. Similarly to homotheticity, it makes sense to assume monotonicity with respect to α. That is $(x + \alpha e, x + \alpha' e) \in R$ if $\alpha \geq \alpha'$.

Given a group $(\mathcal{F}, \circ)$, a triple $(\mathcal{F}, \circ, \geq)$ is an *ordered group* if $\geq$ is a complete order such that

$$f \geq f' \text{ implies } f'' \circ f \geq f'' \circ f',$$
$$f \geq f' \text{ implies } f \circ f'' \geq f' \circ f''.$$

for every $f, f', f'' \in \mathcal{F}$. Note that we have to use both left- and right-ordered assumptions as we did not assume that the $\circ$ operator is commutative. A theory $\mathcal{F}$ is *ordered* if $(\mathcal{F}, \circ, \geq)$ is an *ordered group*.

We need to modify the notion of consistency in order to take into account that a theory implements the underlying order of the group $(\mathcal{F}, \circ, \geq)$. A preference relation R is *consistent with ordered theory* $\mathcal{F}$ if

$$(x, y) \in R \text{ implies } (f'(x), f(y)) \in R \text{ for every } f' \geq f \in \mathcal{F}.$$

The order imposed on the theory is passed to the preferences. Moreover, the completeness assumption on the order is in some sense passed as well. Introducing a partial order rather than a complete order would not be enough for the complete rationalization provided below.

Finally, given the refinements of the theory, we need to introduce some additional constraints on the data set. A data set $(\mathcal{B}, C)$ is *regular* if for every $x \in X$ and $B \in \mathcal{B}$

$$f(x) \in B \text{ implies } f'(x) \in B \text{ for every } f' \in \mathcal{F} \text{ such that } f' \leq f,$$
$$f(x) \in C(B) \text{ implies } f'(x) \notin B \text{ for every } f' \in \mathcal{F} \text{ such that } f' > f.$$

Even though we introduce regularity as assumption, it can be incorporated into the testable conditions. The condition on budgets corresponds to the notion of downward closure of the budget frequently employed in revealed preference literature. The condition on the chosen point refers to the fact that chosen point should be on the border of the budget set.

Our result for completely rationalizable theory differs from previous ones in three ways. First and foremost, we assume that $\mathcal{F}$ is a completely order theory, which is the key assumption. Second, we modify the notion of consistency and use *consistency with ordered theory* when talking about *complete rationalizability*. And third, we impose the corresponding technical assumption of regularity on the budget set. Both regularity and consistency with ordered theory are technical assumptions introduced to incorporate the change in the notion of theory into the notion of rationalization.

Proposition 4 *Let $\mathcal{F}$ be a completely ordered theory and $(\mathcal{B}, C)$ be a regular data set. A data set is completely rationalizable (with ordered theory $\mathcal{F}$) if and only if it satisfies SAARP.*

4 Applications

In this section we show that classical theories of preferences fit in the algebraic framework. We focus on rationalization and complete rationalization, as for classical applications, transitivity is commonly considered as a requirement. One can simplify these axioms down to their weak counterparts, by considering sequences of the length no more than two.

For brevity of exposition, we consider applications in the real hyperplane ($\mathbb{R}^N$) or subsets of it, even though the results hold for more general structures. In particular, $\mathbb{R}^N$ is a vector space over a fully ordered field which is the more general algebraic requirement we need for applications.

Transitive preferences

Let $X \subseteq \mathbb{R}^N$ be a universal set of alternatives. Note that transitivity is already embedded in the notion of rationalization. Hence, a statement of the theory in terms of functions can look simply as

$$(x, y) \in R \text{ implies } (x, y) \in R.$$

Hence, we can define

$$\mathcal{T} = \{I\}.$$

Trivially, $(\mathcal{T}, \circ)$ is a group, since it contains a unique element that is the identity function. Moreover, it can be easily seen that $\mathcal{T}$ is the correct theory describing transitive preferences given the definition above. Moreover, AARP in this case is equivalent to SARP.

Definition 6 A data set $(\mathcal{B}, C)$ satisfies the **Strong Axiom of Revealed Preference (SARP)** if for every sequence $x_1, \ldots, x_n$ such that $x_j \in C(B_j)$ for every $j \in \{1, \ldots, n\}$ for some $B_1, \ldots, B_n \in \mathcal{B}$, if

$$x_{j+1} \in B_j \text{ for every } j \in \{1, \ldots, n-1\}$$

then

$$x_1 \notin B_n \setminus C(B_n).$$

$(\mathcal{T}, \circ, \geq)$ is a trivial (as it contains single element) and therefore ordered group. Therefore, SARP is equivalent to the (complete) rationalization by means of the main results.

Corollary 1 *A regular data set is completely rationalizable if and only if it satisfies SARP.*

Homothetic preferences

Let the convex cone $X \subseteq \mathbb{R}_+^N$ be a space of alternatives.[4]

Definition 7 A preference relation is said to be **homothetic** if

$$(x, y) \in R \text{ implies } (\alpha x, \alpha y) \in R$$

for every $x, y \in X$ and $\alpha \in \mathbb{R}_{++}$.

We can define the

$$\mathcal{H} = \{f(x) = \alpha x : \alpha \in \mathbb{R}_{++}\}.$$

It is easy to see that $(\mathcal{H}, \circ)$ is a group. Moreover $(\mathcal{H}, \circ, \geq)$ is an ordered group as $(\mathbb{R}_{++}, *, \geq)$ is an ordered group. In this case AARP is equivalent to a homothetic axiom of revealed preferences that generalizes the one proposed by Varian (1983), Heufer (2013), and Heufer and Hjertstrand (2019).

[4] A more general algebraic requirement would be for X to be a vector space over a fully ordered field $(A, *, +, \geq)$.

Definition 8 A data set $(\mathcal{B}, C)$ satisfies the **Homothetic Axiom of Revealed Preference (HARP)** if for every sequence $x_1, \ldots, x_n$ such that $x_j \in C(B_j)$ for every $j \in \{1, \ldots, n\}$ for some $B_1, \ldots, B_n \in \mathcal{B}$, and every sequence $\alpha_1 \ldots, \alpha_{n-1} \in \mathbb{R}_{++}$ such that

$$\alpha_j x_{j+1} \in B_j \text{ for every } j \in \{1, \ldots, n-1\},$$

we have

$$\frac{x_1}{\Pi_{j=1}^{n-1}\alpha_j} \notin B_n \backslash C(B_n).$$

Note that HARP uses division as the operator being equivalent to the multiplication by α^{-1}. Since $(\mathcal{H}, \circ, \geq)$ is an ordered group, HARP is equivalent to the (complete) rationalization by the means of the main results.

Corollary 2 *A regular data set is completely rationalizable with homothetic preferences if and only if it satisfies HARP*

Quasilinear preferences

Let $X \subseteq \mathbb{R} \times \mathbb{R}_+^{N-1}$ be a space of alternatives.[5] Denote by $e = (1, 0, 0, \ldots)$ the vector with unique 1 element in the first place and zeros elsewhere; that is, the first good is the numeraire. We make this assumption to simplify notation; as long as the vector space is of finite or countable dimension, this assumption is without loss of generality.

Definition 9 A preference relation is said to be **quasilinear** if

$$(x, y) \in R \text{ implies } (x + \alpha e, y + \alpha e) \in R$$

for every $x, y \in X$ and every $\alpha \in \mathbb{R}$.

We can define

$$\mathcal{Q} = \{f(x) = x + \alpha e : \alpha \in \mathbb{R}\}.$$

It is easy to see that $(\mathcal{Q}, \circ)$ is a group. Moreover, $(\mathcal{Q}, \circ, \geq)$ is an ordered group. In this case, AARP is equivalent to a quasilinear axiom of revealed preferences that generalizes the one proposed by Rochet (1987), Brown and Calsamiglia (2007), and Castillo and Freer (2021).

Definition 10 A data set $(\mathcal{B}, C)$ satisfies the **Quasilinear Axiom of Revealed Preference (QARP)** if for every sequence $x_1, \ldots, x_n$ such that $x_j \in C(B_j)$ for every

[5] The more general algebraic restriction would require X to be a vector space over a fully ordered field $(A, +, *, \geq)$.

$j \in \{1, \dots, n\}$ for some $B_1, \dots, B_n \in \mathcal{B}$, and every sequence $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$ such that

$$x_{j+1} + \alpha_j e \in B_j \text{ for every } j \in \{1, \dots, n-1\},$$

we have

$$x_1 - \sum_{j=2}^{n} \alpha_j \notin B_n \backslash C(B_n).$$

Since $(\mathcal{Q}, \circ, \geq)$ is an ordered group, QARP is equivalent to the (complete) rationalization by the means of the main results.

Corollary 3 *A regular data set is completely rationalizable with quasilinear preferences if and only if it satisfies QARP*

Preferences satisfying independence

Consider a set Ω of N potential outcomes, possibly infinite. Let $X = \mathbb{R}^N$ be a space of lotteries.[6] Note that the dimension (N) does not have to be finite; the same reasoning also holds for infinite number of outcomes. That is, X represents "probability distributions" over the prizes in Ω. Technically, we define the space of lotteries to be bigger than usual; properly, we should define it as a simplex over the space of outcomes. A problem which would appear in that case is that applying independence one can easily jump out of the simplex.[7]

Definition 11 A preference relation is said to satisfy **independence** if

$$(x, y) \in R \text{ implies } (\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) \in R$$

for every $x, y, z \in X$ and $\alpha \in \mathbb{R}_{++}$.

We use Aumann (1962) version of the independence axiom exactly because there is no completeness assumed, and we allow α to go above 1. This statement of independence guarantees that corresponding family of functions given the composition operator form a group.

We define

$$\mathcal{I} = \{f(x) = \alpha x + (1-\alpha)z : \alpha \in \mathbb{R}_{++}; z \in X\}.$$

Let us verify that $(\mathcal{I}, \circ)$ is a group, given that this example is less straightforward than previous examples. Denote $f(x) = \alpha x + (1-\alpha)z$ by $f_{\alpha,z}(x)$. Hence, the inverse for

[6] A more general algebraic version would require A to be a field that contains both 0 and 1 and X to be a vector space over A.

[7] A way out is to consider a theory not as being a group of functions, but being "generated" by a group of functions. By being generated we mean that we can truncate both domain and image of the function to fit the set of alternatives.

this function is $f_{\frac{1}{\alpha},z}(x)$, which also an eligible function. Composition of the functions is given by

$$[f_{\alpha,z}(x) \circ f_{\alpha',z'}](x) = f_{\tilde{\alpha},\tilde{z}},$$

where

$$\tilde{\alpha} = \alpha\alpha' \text{ and } \tilde{z} = \frac{\alpha'(1-\alpha)}{1-\alpha'\alpha} z + \frac{1-\alpha'}{1-\alpha'\alpha} z'$$

if $\alpha\alpha' \neq 1$.[8] In this case, $\tilde{\alpha} \in \mathbb{R}_{++}$, and $z \in X$ is the convex combination of two elegible lotteries. Note that when stating the notion of theory we did not require the group $(\mathcal{F}, \circ)$ to be commutative, i.e. it may be the case that $f \circ f' \neq f' \circ f$. Independence illustrates why we did not impose this condition, which would have simplified the exposition. In the case of independence, it is easy to see that commutativeness does not have to be satisfied. Next, we provide the specification of the AARP for the case of independence.

Definition 12 A data set $(\mathcal{B}, C)$ satisfies the **Independence Axiom of Revealed Preference (IARP)** if for every sequence $x_1, \ldots, x_n$ such that $x_j \in C(B_j)$ for every $j \in \{1, \ldots, n\}$ for some $B_1, \ldots, B_n \in \mathcal{B}$, and every sequences $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}_{++}$ and $z_1, \ldots, z_n \in X$ such that

$$f_{\alpha_j,z_j}(x_{j+1}) \in B_j \text{ for every } j \in \{1, \ldots, n-1\},$$

we have

$$\left[\bigcirc_{j=n-1}^{1} f_{\frac{1}{\alpha_j},z_j}\right](x_1) \notin B_n \backslash C(B_n).$$

Note that IARP is only equivalent to rationalization. We cannot represent $\mathcal{I}$ as an ordered group, even though this group can be partially ordered. Therefore, we cannot obtain complete rationalization in this case and focus on rationalization with preferences satisfying independence.

Corollary 4 *A data set is rationalizable with preferences satisfying independence if and only if it satisfies IARP.*

5 Concluding remarks

We develop an algebraic approach to study revealed preference. In our algebraic framework, a theory about preferences can be thought of as a collection of functions which impose the desired properties. Endowing this collection of functions with a composition operator the theory can be considered as an algebra. In particular, we consider a

[8] If $\alpha = 1$, then the function is the identify mapping, and it is inverse to itself.

set of theories which satisfy the group structure. For this set of theories, we provide a (algebraic) revealed preference axiom, and show that this axiom provides a criterion for an observed data set to be generated by a preference relation consistent with the theory. We show that our algebraic axiom subsumes the classical axioms of revealed preferences including those for transitive (see Afriat 1967; Diewert 1973; Forges and Minelli 2009; Nishimura et al. 2017), homothetic (see Varian 1983; Heufer 2013; Heufer and Hjertstrand 2019), and quasilinear preferences (see Brown and Calsamiglia 2007; Castillo and Freer 2021), as well as preferences satisfying independence (Demuynck and Lauwers 2009).

In addition we investigate the role of requiring the underlying preference relation to be transitive and/or complete. Studying transitivity requires us to consider the two versions of algebraic axiom: weak and strong. These axioms work similarly to the well-known weak and strong axioms of revealed preferences. The weak algebraic axiom corresponds to the case in which the underlying preference relation does not have to be transitive, as it does not account for indirect revealed preference relation. The strong algebraic axiom does take into account indirect preference revelation, and therefore corresponds to the case of transitive preferences. As far as completeness is concerned, we found that unless we require the underlying preference relation to be transitive, then a data set can be generated by a preference relation consistent with an algebraic theory if and only if it can be generated by a complete preference relation consistent with an algebraic theory. That is, we get the completeness for free. If we require the underlying preference relation to be transitive, we need to make extra assumptions about the structure of the theory to obtain completeness.

As directions for further research, note that our algebraic axioms are compatible with De Clippel and Rozen (2021) acyclic satisfiability and Hu et al. (2021) revealed indirect preference approaches. That is, using algebraic axioms may allow for an immediate generalization of those approaches to a wider set of desiderata. Moreover, the acyclic satisfiability procedure itself has an interesting algebraic structure. Therefore, developing and studying it further may provide a tractable and unified approach to study behavioral theories.

Another fruitful set of applications comes from the revealed preference analysis of group behavior. There has been recently some interest in applying revealed preference to social choice theory, e.g. Duggan (2016, 2019); voting theory, e.g. Kalandrakis (2010) and Gomberg (2018), and game theoretic models, e.g. Brown and Matzkin (1996), Echenique and Saito (2013), Cherchye et al. (2013, 2017), and Castillo et al. (2019). The unified algebraic approach may help to make progress along these lines, allowing to evaluate the set of questions which can be tackled with revealed preference theory.

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A Proofs for Sect. 3

Before we proceed with the proofs, we need to introduce some auxiliary notation. Denote by $\mathcal{R}$ the space of preference relations (reflexive binary relations). Let $F : \mathcal{R} \to \mathcal{R}$ be a *theory closure* corresponding to theory $\mathcal{F}$, that is $(x, y) \in F(R)$ if there is $f \in \mathcal{F}$ such that $(f(x), f(y)) \in R$. We would omit an explicit reference to the theory when it can be done without confusing the reader. Let $T : \mathcal{R} \to \mathcal{R}$ be a *transitive closure* that is $(x, y) \in T(R)$ if there is a finite sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R$ for every $j \in \{1, \ldots, n-1\}$. The idea behind the closure is that it is a constructive counterpart of the consistency of the preference relation with the theory (for theory closure) and of the transitivity of the preference relation (for transitive closure). Hence, the important construct is the fixed point of the closure, i.e. $R = F(R)$ and $R = T(R)$. Note that the transitive and theory closures are of distinct (mathematical) nature, even though being closures they share some useful properties.

Next we show that F and T are indeed closures, that is that they satisfy the following properties. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be *increasing* if $R \subseteq G(R)$. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be *monotone* if $R \subseteq R'$ implies $G(R) \subseteq G(R')$. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be *idempotent* if $G(G(R)) = G(R)$. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be a *closure* if it is increasing, monotone and idempotent.

Lemma A.1 (Demuynck 2009) *$T : \mathcal{R} \to \mathcal{R}$ is a closure.*

Lemma A.2 *$F : \mathcal{R} \to \mathcal{R}$ is a closure.*

Proof **F is increasing.**
Recall that $I \in \mathcal{F}$. Hence, for every $(x, y) \in R$ then letting $f = I = I^{-1}$, we know that $I(x, y) = (x, y) \in R$, and therefore $(x, y) \in F(R)$.
F is monotone.
Take $R \subseteq R'$, $(x, y) \in F(R)$ if there is $f \in \mathcal{F}$ such that $(f(x), f(y)) \in R \subseteq R'$. Hence, $(x, y) \in F(R')$ and therefore, $F(R) \subseteq F(R')$.
F is idempotent.
Since F is increasing we know that $F(R) \subseteq F(F(R))$. Hence, we are left to show that $F(F(R)) \subseteq F(R)$. Consider $(x, y) \in F(F(R))$, then there is $f \in \mathcal{F}$ such that $(f(x), f(y)) \in F(R)$. Then, in its order there is $f' \in \mathcal{F}$ such that $(f'(f(x)), f'(f(y))) \in R$. Given that $(\mathcal{F}, \circ)$ is a group, then there is $\hat{f} = f' \circ f \in \mathcal{F}$ and $(\hat{f}(x), \hat{f}(y)) \in R$, and therefore, $(x, y) \in F(R)$. □

Next, we show that the fixed point of the corresponding closure (theory or transitive) exhibits the properties reflected in the assumption. That is, a fixed point of a theory closure is consistent with the theory, and a fixed point of the transitive closure is transitive. The result for the transitive closure is straightforward and based on previous literature.

Lemma A.3 (Demuynck 2009; Freer and Martinelli 2021) *$T(R)$ is transitive, i.e. $(x, y), (y, z) \in T(R)$ implies $(x, z) \in T(R)$ for every $x, y, z \in X$.*

The result for theory closure is stronger than one would think we need. Recall that consistency with the theory requires that $(x, y) \in R$ implies $(f(x), f(y)) \in R$ for every $x, y \in X$ and $f \in \mathcal{F}$. Equivalence instead of implication is needed in the further proofs.

Lemma A.4 *$(x, y) \in F(R)$ if and only if $(f(x), f(y)) \in F(R)$ for every $x, y \in X$ and $f \in \mathcal{F}$.*

Proof ($\Rightarrow$) Since $(x, y) \in F(R)$, then there are $(z, w) \in R$ such that $z = \bar{f}(x)$ and $w = \bar{f}(y)$ for some $\bar{f} \in \mathcal{F}$. Since $(\mathcal{F}, \circ)$ is a group, then there are $f^{-1}, \bar{f}^{-1} \in \mathcal{F}$ and $\hat{f} = [\bar{f}^{-1} \circ f^{-1}] \in \mathcal{F}$. Hence, we can express $x = \bar{f}^{-1}(z)$ and $y = \bar{f}^{-1}(w)$. Then, $\hat{f}(f(x)) = \bar{f}^{-1}(x) = z$ and $\hat{f}(f(y)) = \bar{f}^{-1}(x) = w$. Hence, by construction of F we can conclude that $(f(x), f(y)) \in F(R)$.

($\Leftarrow$) Consider $(f(x), f(y)) \in F(R)$, then there is $(\bar{f}(f(x)), \bar{f}(f(y))) \in R$ by construction of F. Since $(\mathcal{F}, \circ)$ is a group, then there is $\hat{f} = \bar{f} \circ f \in \mathcal{F}$. Hence, $(\hat{f}(x), \hat{f}(y)) \in R$ and therefore $(x, y) \in F(R)$ by construction of F. □

Next, show the connection between the theory and the transitive closures. In particular, we show that the transitive closure of a fixed point of F is also a fixed point of F. Recall that a fixed point of F is a preference relation consistent with the theory $\mathcal{F}$. Hence, the transitive closure of a preference relation consistent with the algebraic theory is another (now transitive) preference relation consistent with the theory $\mathcal{F}$.

Lemma A.5 *Let $R = F(R)$, then $F(T(R)) = T(R)$.*

Proof Since F is increasing, then $T(R) \subseteq F(T(R))$. Hence, we are left to show $F(T(R)) \subseteq T(R)$. Take $(x, y) \in F(T(R))$, then there is $f \in \mathcal{F}$, such that $(f(x), f(y)) \in T(R)$. That is there is a sequence $f(x) = s_1, \ldots, s_n = f(y)$ such that

$$(s_j, s_{j+1}) \in R \text{ for every } j \in \{1, \ldots, n-1\}.$$

Given that $R = F(R)$ and $(\mathcal{F}, \circ)$ is a group, then there is $f^{-1} \in \mathcal{F}$ such that

$$(f^{-1}(s_j), f^{-1}(s_{j+1})) \in R \text{ for every } j \in \{1, \ldots, n-1\}.$$

Hence, $(f^{-1}(s_1), f^{-1}(s_n)) = (x, y) \in T(R)$. □

Next we define a *revealed preference relation* denoted by R_E. That is, $(x, y) \in R_E$ if $x \in C(B)$, $y \in B$ for some $B \in \mathcal{B}$ or $x = y$. The first part defines the revealed preference part of the relation, while the second part guarantees reflexivity. Let $R \preceq R'$ (R' is an **extension** of R) if $R \subseteq R'$ and $P(R) \subseteq P(R')$. We start with introducing an equivalent definition of extension, that can be linked back to Suzumura's consistency.

Lemma A.6 (Demuynck 2009; Freer and Martinelli 2021) *Let $R \subseteq R'$. $R \preceq R'$ if and only if $P^{-1}(R) \cap R' = \emptyset$.*

Given the revealed preference relation and extension notation introduced, we can restate the conditions for (weak) rationalization of the data. In order to find a preference relation which generates the observed choices, we need to find R^* such that $R_E \preceq R^*$. This condition guarantees that for every budget set the choice correspondence is *equal to* the set of maximal points. To guarantee that the preference relation R^* is consistent with the theory we need to ensure that $R^* = F(R^*)$. To guarantee that the binary relation R^* is transitive we need to ensure that $R^* = T(R^*)$. Hence, the rationalizability conditions can be restated as follows.

A data set is

- *weakly rationalizable* if there is a preference relation $R^* = F(R^*)$ such that $R_E \preceq R^*$;
- *rationalizable* if there is a preference relation $R^* = T(F(R^*))$ such that $R_E \preceq R^*$;
- *completely weakly rationalizable* if there is a complete preference relation $R^* = F(R^*)$ such that $R_E \preceq R^*$;
- *completely rationalizable* if there is a complete preference relation $R^* = T(F(R^*))$ such that $R_E \preceq R^*$.

A.1 Proof of Proposition 1

Proof (⇒) We prove this statement by contradiction. That is a data set is weakly rationalizable, though fails WAARP, That is, there are $x_i \in C(B_i)$ and $x_j \in C(B_j)$ such that

$$f(x_i) \in B_j \text{ and } f^{-1}(x_j) \in B_i \backslash C(B_i).$$

Since the data set is rationalizable and $f(x_i) \in B_j$, then $(x_j, f(x_i)) \in R^*$. Since R^* is consistent with the theory and $(\mathcal{F}, \circ)$ is a group, then $(f^{-1}(x_j), [f^{-1} \circ f](x_i)) = (f^{-1}(x_j), x_i) \in R^*$. At the same time $f^{-1}(x_j) \in B_i \backslash C(B_i)$ implies that $(x_i, f^{-1}(x_j)) \in P(R^*)$. That is a contradiction to the fact that $C(B_j) = \max(B_j, R^*)$.

(⇐) Assume that WAARP is satisfied. Let us start with showing that $R_E \preceq F(R_E)$. On the contrary assume that $(z, w) \in P^{-1}(R_E) \cap F(R_E)$. By construction there is $f \in \mathcal{F}$ such that $(f(z), f(w)) \in R_E$. That is, $f(z) = x_j \in C(B_j)$ and $f(w) \in B_j$. At the same time $(w, z) \in P(R_E)$ implies that $w = x_i \in C(B_i)$ and $f^{-1}(x_j) \in B_i \backslash C(B_i)$. That is an immediate contradiction to WAARP. Hence, $R_E \preceq F(R_E)$.

Lemma A.4 implies that $F(R_E)$ is consistent with the theory and $R_E \preceq F(R_E)$ implies that $C(B) = \max(B, F(R_E))$. Let us conclude by providing a formal argument for the latter point. Consider $x \in C(B)$, then $(x, y) \in R_E$ for every $y \in B$ and $(y, x) \in R_E$ only if $y \in C(B)$. Since $R_E \preceq F(R_E)$, then $(x, y) \in R_E$ for every $y \in B$ since $R_E \subseteq F(R_E)$; and $(y, x) \in R_E$ only if $y \in C(B)$ since $P(R_E) \subseteq P(F(R_E))$ and $(x, y) \in P(R_E)$ for every $y \in B \backslash C(B)$ by construction of R_E. □

A.2 Proof of Proposition 2

Proof (⇒) We assume on the contrary that the data is rationalizable and there is a violation of SAARP. That is there are sequences $x_1, \ldots, x_n$ such that $x_j \in C(B_j)$ for every $j \in \{1, \dots, n\}$ for some $B_1, \dots, B_n \in \mathcal{B}$, and $f_1, \dots, f_{n-1} \in \mathcal{F}$ such that

$$f_j(x_{j+1}) \in B_j \text{ for every } j \in \{1, \dots, n-1\},$$

and

$$\left[\bigcirc_{j=1}^{n-1} f_j\right]^{-1} (x_1) \in B_n \backslash C(B_n).$$

Since the data set is rationalizable, there is a preference relation R^* transitive and consistent with the theory $\mathcal{F}$ such that

$$(x, y) \in R^* \text{ for every } x \in C(B);\ \ y \in B \text{ for some } B \in \mathcal{B}.$$

Hence, $f_j(x_{j+1}) \in B_j$ implies that

$$(x_j, f_j(x_{j+1})) \in R^* \text{ for every } j \in 1, \dots, n-1.$$

However, to show the relation between x_1 and x_n we need to make sure that the the less preferred element in the pair j is the more preferred element in the pair $j+1$. Consider the first two pairs in this sequence, namely $(x_1, f_1(x_2))$ and $(x_2, f_2(x_3))$.

Given that R^* is consistent with the theory, then $(x_2, f_2(x_3)) \in R^*$ implies that $(f_1(x_2), [f_1 \circ f_2](x_3)) \in R^*$. Extending the same logic we can conclude that

$$(x_j, f_j(x_{j+1})) \in R^* \text{ implies } \left(\left[\bigcirc_{i=1}^{j-1} f_i\right](x_j), \left[\bigcirc_{i=1}^{j} f_i\right](x_{j+1})\right) \in R^*$$

for every $j \in \{2, \dots, n-1\}$.

Then, by transitivity we can conclude that

$$\left(x_1, \left[\bigcirc_{i=1}^{n-1} f_i\right](x_n)\right) \in R^*.$$

Given that $(\mathcal{F}, \circ)$ is a group, there is an inverse $\left[\bigcirc_{i=1}^{n-1} f_i\right]^{-1} \in \mathcal{F}$ and given that R^* is consistent with $\mathcal{F}$ we can conclude that

$$\left(\left[\bigcirc_{i=1}^{n-1} f_i\right]^{-1} (x_1), x_n\right) \in R^*.$$

On the other hand the violation of AARP implies that

$$\left[\bigcirc_{j=1}^{n-1} f_j\right]^{-1} (x_1) \in B_n \backslash C(B_n),$$

then the fact that R^* is a preference relation that rationalizes the data set implies

$$\left(x_n, \left[\bigcirc_{i=1}^{n-1} f_i\right]^{-1}(x_1)\right) \in P(R^*).$$

That is a direct contradiction given that by construction $(x, y) \in P(R^*)$only if $(x, y) \in R^*$ and $(y, x) \notin R^*$, i.e. $P(R^*)$ denotes the strict (asymmetric) part of the preference relation R^*.

($\Leftarrow$) Since data set satisfies SAARP, then it satisfies WAARP. Therefore, as we have proven above $R_E \preceq F(R_E)$. Hence, to complete the proof (given Lemma A.5) we need to show that $R_E \preceq T(F(R_E))$ if the data satisfies SAARP. We proceed with the proof by contradiction. That is, assume that there is $(x, y) \in P^{-1}(R_E) \cap T(F(R_E))$. Then, there is a sequence of $x = s_1, \dots, s_n = y$ such that $(s_j, s_{j+1}) \in F(R_E)$ for every $j \in \{1, \dots, n-1\}$. Since $(s_j, s_{j+1}) \in F(R_E)$, then there is $f_j \in \mathcal{F}$ such that $(f_j(s_j), f_j(s_{j+1})) \in R_E$. Hence, we can restate the sequence that adds (x, y) to $T(F(R_E))$ as follows. There are sequences $x = s_1, \dots, s_n = y$ and $f_1, \dots, f_{n-1} \in \mathcal{F}$ such that

$$(f_j(s_j), f_j(s_{j+1})) \in R_E \text{ for every } j \in \{1, \dots, n-1\}.$$

Given the construction of the revealed preference relation we know that $f_j(s_j) = x_j \in B_j$ for some $B_j \in \mathcal{B}$. Moreover, $f_j(s_{j+1}) = [f_j \circ f_{j+1}^{-1}](x_{j+1}) \in B_j$, where $f_{j+1}^{-1}, [f_j \circ f_{j+1}^{-1}] \in \mathcal{F}$ since $(\mathcal{F}, \circ)$ is a group. Denote by $\hat{f}_j = [f_j \circ f_{j+1}^{-1}]$ for every $j \in \{1, \dots, n-2\}$ and $\hat{f}_{n-1} = f_{n-1}$. Then, we can rewrite the sequence as

$$\hat{f}_j(x_{j+1}) \in B_j \text{ for every } j \in \{1, \dots, n-1\}.$$

At the same time we know that $(y, x) \in P(R_E)$, that is

$$y = s_n = x_n \in C(B_n) \text{ and } x = f_1^{-1}(x_1) = [f_1^{-1} \circ f_1](s_1) \in B_n \backslash C(B_n).$$

Let us compute the

$$\left[\bigcirc_{j=1}^{n-1} \hat{f}_j\right]^{-1} = \left[[f_1 \circ f_2^{-1}] \circ [f_2 \circ f_3^{-1}] \circ \dots \circ [f_{n-2} \circ f_{n-1}^{-1}] \circ [f_{n-1}]\right]^{-1} = f_1^{-1}.$$

Therefore, the following holds

$$\left[\bigcirc_{j=1}^{n-1} \hat{f}_j\right]^{-1}(x_1) = f_1^{-1}(x_1) \in B_n \backslash C(B_n)$$

that is a direct contradiction of SAARP. □

A.3 Proof of Proposition 3

Necessity of WAARP for complete weak rationalization obviously follows from the necessity of WAARP for the weak rationalization. Hence, we are left to provide a proof for the sufficiency part. For the sufficiency proof, we need a couple of auxiliary properties of closures and an auxiliary result. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be *algebraic* if for any $R \in \mathcal{R}$ and all $(x, y) \in G(R)$ there is a finite relation $R' \subseteq R$ such that $(x, y) \in G(R')$. That is, to for any comparison in $G(R)$ there is a finite sub-relation to add this comparison. Denote by $N(R) = X \times X \backslash (R \cup R^{-1})$, that is a set of non-comparable pairs. A function $G : \mathcal{R} \to \mathcal{R}$ is said to be *weakly expansive* if for any $R = G(R)$ such that $N(R) \neq \emptyset$, then there is a non-empty $S \subseteq N(R)$ such that $R \cup S \preceq F(R \cup S)$. These two properties allow us to restate the result from Demuynck (2009).

Lemma A.7 (Demuynck 2009 Extension Theorem) *Let $G : \mathcal{R} \to \mathcal{R}$ be a weakly expansive, algebraic closure and let R_E be a revealed preference relation. There is a complete preference relation $R^* = G(R^*)$ such that $R_E \preceq R^*$ if and only if $R_E \preceq F(R_E)$.*

Lemma A.7 already guarantees us the sufficiency proof once we show that F is weakly expansive and algebraic. Recall that the condition of $R_E \preceq F(R_E)$ is equivalent to WAARP, and existence of complete preference relation $R^* = F(R^*)$ is equivalent to the existence of complete and consistent with the theory and guarantees that the observed choices are the best in the given budget set. Hence, to complete the proof we show that F is algebraic and weakly expansive.

Lemma A.8 *A theory closure F is algebraic and weakly expansive.*

Proof **F is algebraic**.
Consider a relation R and an element $(x, y) \in F(R)$, then there is $f \in \mathcal{F}$ such that $(f(x), f(y)) \in R$. Let $D = \{f(x), f(y)\}$ and let $R' = R \cap (D \times D)$. Then, $(x, y) \in F(R')$ and R' is finite by construction.
F is weakly expansive.
Consider $R = F(R)$ and $(x, y) \in N(R)$. Let $R' = R \cup \{(x, y)\}$ and let us show that $R' \preceq F(R')$. On the contrary assume that $(z, w) \in F(R') \cup P^{-1}(R')$. Let us start from showing that $(z, w) = (f(x), f(y))$ for some $f \in \mathcal{F}$. On the contrary assume that there are $(x, y) \neq (x', y') \in R'$, such that $z = f(x')$ and $w = f(y')$. Then, $(z, w) \in R$ and therefore, $(w, z) \notin P(R')$ since $P(R')$ is the asymmetric (strict) part of R'.

Hence, further we consider the case $(z, w) = (f(x), f(y))$ for some $f \in \mathcal{F}$. Since $(w, z) \in P(R') \subseteq R \cup \{(x, y)\}$, then either (i) $f = I$ and $(y, x) \in R$ that contradicts the fact that $(x, y) \in N(R)$, or $(f(y), f(x)) \in R$ for some $I \neq f \in \mathcal{F}$. The latter statement (given that $(\mathcal{F}, \circ)$ is a group) implies that there is $f^{-1} \in \mathcal{F}$, and therefore $([f^{-1} \circ f(y)], [f^{-1} \circ f](x)) = (y, x) \in R$ that is a contradiction to $(x, y) \in R$. □

A.4 Proof of Proposition 4

Recall that for Proposition 4 we need to consider the notion of ordered theory. We first introduce a modified theory closure.

We say that $\bar{F} : \mathcal{R} \to \mathcal{R}$ is an *ordered theory closure* if for every $(x, y) \in \bar{F}(R)$, there are $\bar{f} \le f \in \mathcal{F}$ such that $(\bar{f}(x), f(y)) \in R$. Note that in the definition of the ordered theory closure we have to follow the inverse (in terms of ordering the functions) logic. That is, while the theory states that $(x, y) \in R$ implies $(\bar{f}(x), f(y)) \in R$ for $\bar{f} \ge f \in \mathcal{F}$, the ordered theory closure uses the inverse functions. To wit, letting $z = \bar{f}(x)$ and $w = f(y)$, we can denote $x = \bar{f}^{-1}(z)$ and $y = f^{-1}(w)$; and as the functions are inverse we have $\bar{f}^{-1} \le f^{-1}$. It is useful to keep in mind this observation to avoid confusion while reading the proofs.

To proceed we need to follow a scheme similar to the proofs provided before.

1. $T(\bar{F}) : \mathcal{R} \to \mathcal{R}$ is an algebraic closure.
2. $T(\bar{F}(R))$ is transitive and consistent with the ordered theory.
3. $T(\bar{F}) : \mathcal{R} \to \mathcal{R}$ is weakly expansive.
4. SAARP (on the regular data set) is equivalent to $R_E \preceq T(\bar{F}(R_E))$.

(1) $T(\bar{F}) : \mathcal{R} \to \mathcal{R}$ **is a closure.**
Let us note that a composition of closures is a closure as well (see Lemma B.1). Hence, given that we already shown that T is a closure (see Lemma A.1), we are left to show that $\bar{F}$ is a closure to complete this part.

Lemma A.9 *$\bar{F} : \mathcal{R} \to \mathcal{R}$ is an algebraic closure.*

Proof $\bar{F}$ **is increasing.**
Since $I \in \mathcal{F}$, then $(x, y) \in R$ implies that $(x, y) \in \bar{F}(R)$.

$\bar{F}$ **is monotone.**
Let $R \subseteq R'$ and assume on the contrary that there is $(x, y) \in \bar{F}(R)$ and $(x, y) \notin \bar{F}(R')$. Since $(x, y) \in \bar{F}(R)$, then there are $\bar{f} \le f \in \mathcal{F}$ such that $(\bar{f}(x), f(y)) \in R \subseteq R'$. Latter implies that $(\bar{f}(x), f(y)) \in R'$ and therefore $(x, y) \in \bar{F}(R')$.
$\bar{F}$ **is idempotent.**
Since $\bar{F}$ is increasing and monotone we already know that $\bar{F}(R) \subseteq \bar{F}(\bar{F}(R))$. Hence, we are left to show that $\bar{F}(\bar{F}(R)) \subseteq \bar{F}(R)$. Consider $(x, y) \in \bar{F}(\bar{F}(R))$, then there are $\bar{f} \le f \in \mathcal{F}$ such that $(\bar{f}(x), f(y)) \in \bar{F}(R)$. Then, there are $\bar{f}' \le f' \in \mathcal{F}$ such that $([\bar{f}' \circ \bar{f}](x), [f' \circ f](y)) \in R$. Recall that $\mathcal{F}$ is a group, therefore, $[\bar{f}' \circ \bar{f}], [f' \circ f] \in \mathcal{F}$. Moreover, $\mathcal{F}$ is an ordered group, and therefore, $[\bar{f}' \circ \bar{f}] \le [f' \circ f]$. Then, there are $\bar{f}^* = [\bar{f}' \circ \bar{f}] \le [f' \circ f] = f^*$ such that $(\bar{f}^*(x), f^*(y)) \in R$ and therefore, $(x, y) \in \bar{F}(R)$.
$\bar{F}$ **is algebraic.**
Consider $(x, y) \in \bar{F}(R)$, then there are $\bar{f} \le f \in \mathcal{F}$ such that $(\bar{f}(x), f(y)) \in R$. Hence, the finite relation can be $D = \{\bar{f}(x), f(y)\}$ and let $R' = D \times D \cap R$. Then, we have $(x, y) \in \bar{F}(R')$ and R' is finite by construction. □

A finite composition of algebraic closures is also an algebraic closure (see Lemma B.1 for the formal proof). Hence, knowing that T and $\bar{F}$ are both algebraic closures, we conclude that $T \circ \bar{F}$ is an algebraic closure as well. Hence, we can conclude this part of the proof, and proceed to show that the composition of closures delivers a transitive preference relation consistent with the ordered theory.

(2) $T(\bar{F}(R))$ is transitive and consistent with the ordered theory.

Lemma A.10 *$R = \bar{F}(R)$ is consistent with the ordered theory.*

Proof Suppose the contrary, i.e. $R = \bar{F}(R)$, but R is not consistent with the theory. Hence, there are $\bar{f} \leq f \in \mathcal{F}$ such that $(f(x), \bar{f}(y)) \notin R$. Let $z = f(x)$ and $w = \bar{f}(y)$, then (given that $(\mathcal{F}, \circ)$ is a group), there are $\bar{f}^{-1}, f^{-1} \in \mathcal{F}$ such that $\bar{f}^{-1} \geq f^{-1}$ (given that $(\mathcal{F}, \circ, \geq)$ is an ordered group). Hence, $(f^{-1}(z), \bar{f}^{-1}(w)) \in R = \bar{F}(R)$ and $f^{-1} \leq \bar{f}^{-1}$ implies that $(z, w) \in \bar{F}(R) = R$. The latter observation is equivalent to $(f(x), \bar{f}(y)) \in R$, that is a contradiction. □

Lemma A.11 *If $R = \bar{F}(R)$, then $T(R) = \bar{F}(T(R))$.*

Proof Since $\bar{F}$ is increasing then $T(R) \subseteq \bar{F}(T(R))$. We are left to show that $\bar{F}(T(R)) \subseteq T(R)$. Assume $(x, y) \in \bar{F}(T(R))$, then there are $\bar{f} \leq f \in \mathcal{F}$ such that $(\bar{f}(x), f(y)) \in T(R)$. Then, there is a sequence $\bar{f}(x) = s_1, \ldots, s_n = f(y)$ such that

$$(s_j, s_{j+1}) \in R \text{ for every } j \in \{1, \ldots, n-1\}.$$

Since $(\mathcal{F}, \circ, \geq)$ is an ordered group, then there are $\bar{f}^{-1}, f^{-1} \in \mathcal{F}$ and $\bar{f}^{-1} \geq f^{-1}$. Since $R = \bar{F}(R)$ is consistent with ordered theory $\mathcal{F}$ (see Lemma A.10) and $(\bar{f}(x), s_2) \in R$, then

$$([\bar{f}^{-1} \circ \bar{f}](x), f^{-1}(s_1)) = (x, f^{-1}(s_1)) \in R.$$

Since R is consistent with the theory, then $(f^{-1}(s_j), f^{-1}(s_{j+1})) \in R$ for $2 \leq j \leq n-1$. Finally, $f^{-1}(s_n) = [f^{-1} \circ f](y)$, therefore, $(x, y) \in T(R)$. □

(3) $T(\bar{F}) : \mathcal{R} \to \mathcal{R}$ is weakly expansive.

Lemma A.12 *$T \circ \bar{F} : \mathcal{R} \to \mathcal{R}$ is weakly expansive.*

Proof We consider $R = T(\bar{F}(R))$ such that $N(R) \neq \emptyset$. Let $(x, y) \in N(R)$ and $\bar{R} = R \cup \{(x, y)\}$. Suppose on the contrary to the weak expansiveness that there is $(z, w) \in T(\bar{F}(\bar{R}))$ and $(w, z) \in P(\bar{R})$. Since we are interested in the composition of the closures, we need to proceed proving the weak expansiveness sequentially. We start from making the claim that inner (ordered theory) closure is weakly expansive, with the proceeding similarly to one of Lemma A.8.

Claim 1 $\bar{F} : \mathcal{R} \to \mathcal{R}$ is weakly expansive.

***Proof of Claim* 1** Suppose on the contrary that there is $(z, w) \in \bar{F}(\bar{R})$ and $(w, z) \in P(\bar{R})$. We start from showing that $(z, w) = (\bar{f}(x), f(y))$ for some $\bar{f} \leq f \in \mathcal{F}$. Suppose the contrary, there is $(\bar{x}, \bar{y}) \neq (z, w) \in \bar{R}$ such that $(\bar{f}(\bar{x}), f(\bar{y})) \in \bar{R}$.

Since, $\bar{R} = R \cup \{(x, y\}$, then $(\bar{f}(\bar{x}), f(\bar{y})) \in R$ then $(z, w) \in \bar{F}(R) = R$, that is a contradiction to the fact that $(w, z) \in P(R)$ as $P(\cdot)$ being the asymmetric part of the relation.

Hence, $(z, w) = (\bar{f}(x), f(y))$ for some $\bar{f} \leq f \in \mathcal{F}$. Let us note that $f \neq I$, otherwise we would obtain an immediate contradiction. Recall that $(\mathcal{F}, \circ, \geq)$ is an ordered group, then there are $\bar{f}^{-1} \geq f^{-1} \in \mathcal{F}$. Since $(w, z) \in P(\bar{R})$, then $(\bar{f}^{-1}(w), f^{-1}) = (y, x) \in R$ since R is consistent with the theory. That is contradiction to the fact that $(x, y) \in N(R)$. □

Given Claim 1, we know that $(z, w) \notin \bar{F}(\bar{R})$. Hence, $(z, w) \in T(\bar{F}(\bar{R}))$ implies there is a non-trivial sequence of $z = s_1, \ldots, s_n = y$ such that

$$(\bar{f}_j(s_j), f_j(s_{j+1})) \in \bar{R} \text{ for some } \bar{f}_j \leq f_j \text{ for every } j \in \{1, \ldots, n-1\}.$$

Let us consider one of the *shortest sequences* that add (z, w), i.e. such that there is no shorter sequence. To complete the proof we need to show that there is exactly one entry of (x, y) to this sequence. This claim would significantly simplify our proof as the rest of the comparisons would belong to R.

Claim 2 There is a unique $k \leq n-1$ such that $(\bar{f}_k(s_k), f_k(s_{k+1})) = (x, y) \in \bar{R}$.

Proof of Claim 2 We start from illustrating why there should be at least one entry of $(\bar{f}_k(s_k), f_k(s_{k+1}) = (x, y)$. If there is none, then $(\bar{f}_j(s_j), f_j(s_{j+1}) \in R$ for every j and therefore, $(z, w) \in R = T(\bar{F}(R))$. That is a contradiction to the fact that $(w, z) \in P(R)$. Given that there is at least one entry of (x, y), we next proceed with proving that this entry should be unique. Consider on the contrary that there are at least two entries and assume without loss of generality that $k < l$ and correspond to the first and second entry of the (x, y) to the sequence. Since the $(\mathcal{F}, \circ, \geq)$ is ordered group, we can proceed by considering two cases.

Case 1: $f_k \leq \bar{f}_l$. Given that $f_k \leq \bar{f}_l$ and $\bar{f}_l \leq f_l$, we can conclude that $f_k \leq f_l$. Recall that by assumption $f_k(s_{k+1}) = y = f_l(s_{l+1})$. Given that $(\mathcal{F}, \circ, \geq)$ is an ordered group, there are $f_k^{-1} \geq f_l^{-1} \in \mathcal{F}$. Since R is reflexive and consistent with the theory, then $(y, y) \in R$ implies that $(f_k^{-1}(y), f_l^{-1}(y)) = (s_{k+1}, s_{l+1}) \in R \subseteq \bar{R}$. That is the very same sequence can be expressed without using the second entry of the (x, y). That is a contradiction.

Case 2: $f_k > \bar{f}_l$. Since k and l correspond to the first and second entry of (x, y), then (1) $(s_{k+1}, s_l) \in R$ since $R = T(\bar{F}(R))$, and (2) $(f_k(s_{k+1}), \bar{f}_l(s_l)) \in R$ since R is consistent with the theory ($R = \bar{F}(R)$) and $f_k > \bar{f}_l \in \mathcal{F}$. Moreover, $f_k(s_{k+1}) = y$ and $\bar{f}_l(s_l) = x$, therefore, $(y, x) \in R$ that is a contradiction to $(x, y) \in N(R)$. □

To complete the proof let us appeal to the Claim 2, we can ensure that there is a unique entry of (x, y) to the sequence. Given that $(w, z) \in R$, we can conclude that $(s_{k+1}, s_k) \in R$. Given that $f_k \geq \bar{f}_k$ and R is consistent with the theory, then $(y, x) = (f_k(s_{k+1}), \bar{f}_k(s_k)) \in R$. That is a contradiction to the fact that $(x, y) \in N(R)$. Hence, $T \circ \bar{F}$ is weakly expansive. □

(3) SAARP is equivalent to $R_E \preceq T(\bar{F}(R_E))$.

Lemma A.13 *If a data set satisfies SAARP, then $R_E \preceq T(\bar{F}(R_E))$.*

Proof Suppose on the contrary that there is $(z, w) \in T(\bar{F}(R_E))$ and $(w, z) \in P(R_E)$. Since $(z, w) \in T(\bar{F}(R_E))$, then there is a sequence $z = s_1, \ldots, s_n$ such that for every $j \in \{1, \ldots, n-1\}$ there are $\bar{f}_j \leq f_j \in \mathcal{F}$ such that

$$(\bar{f}_j(s_j), f_j(s_{j+1})) \in R_E.$$

Since $(\bar{f}_j(s_j), f_j(s_{j+1})) \in R_E$, the construction of preference relation implies that $\bar{f}_j(s_j) = x_j \in C(B_j)$ for some $B_j \in \mathcal{B}$. Recall that $(\mathcal{F}, \circ, \geq)$ is ordered group, then there are $f_j^{-1} \leq \bar{f}_j^{-1} \in \mathcal{F}$. Hence, $f_j(s_{j+1}) = [f_j \circ \bar{f}_{j+1}^{-1}](x_{j+1}) \in B_j$, then $[f_j \circ f_{j+1}^{-1}](x_{j+1}) \in B_j$ given that data set is regular and $[f_j \circ \bar{f}_{j+1}^{-1}](x_{j+1}) \leq [f_j \circ f_{j+1}^{-1}](x_{j+1})$. Hence, denoting by $\hat{f}_j = [f_j \circ f_{j+1}^{-1}]$ for every $j \leq n-1$ and $\hat{f}_{n-1} = f_{n-1}$ we obtain

$$\hat{f}_j(x_{j+1}) \in B_j \text{ for every } j \in \{1, \ldots, n-1\}.$$

Then,

$$\left[\bigcirc_{j=1}^{n-1} \hat{f}_j\right]^{-1} = \Big[[f_1 \circ f_2^{-1}] \circ [f_2 \circ f_3^{-1}] \circ \ldots \circ [f_{n-2} \circ f_{n-1}^{-1}] \circ [f_{n-1}]\Big]^{-1} = f_1^{-1}.$$

Recall that $f^{-1} \leq \bar{f}^{-1}$ and $\bar{f}^{-1}(x_1) \in B_n \backslash C(B_n)$, then $\bar{f}^{-1}(x_1) \in B_n \backslash C(B_n)$ since data set is regular. Therefore, the following holds

$$\left[\bigcirc_{j=1}^{n-1} \hat{f}_j\right]^{-1} (x_1) = f_1^{-1}(x_1) \in B_n \backslash C(B_n)$$

that is a contradiction of SAARP. □

***Proof of Proposition* 4** (⇒) Necessity of SAARP follows from an argument similar to the one in the proof of Proposition 2. Therefore, we leave it as an exercise.

(⇐) Lemma A.13 guarantees that if the data set satisfies SAARP, then $R_E \preceq T(\bar{F}(R_E))$. Given that we have shown that $T(\bar{F}(R_E))$ is weakly expansive algebraic closer, we can apply Demuynck Extension Theorem (see Lemma A.7) to guarantee that there is a complete R^* such that $R_E \preceq R^* = T(\bar{F}(R^*))$. Since, $T(\bar{F}(R^*)) = R^*$ it is consistent with ordered theory and transitive (see Lemmas A.10 and A.11). Given that $R_E \preceq R^*$ then, R^* rationalizes the data set. □

B Auxiliary results

We provide two groups of auxiliary results. The first group are results about the compositions of closures; in particular, we show that some properties are transferred to the composition of the closures. The second group are the basic properties of ordered theories used in the proofs above.

Lemma B.1 *Let $F_1, F_2 : \mathcal{R} \to \mathcal{R}$ be algebraic closures, then $F = F_2 \circ F_1 : \mathcal{R} \to \mathcal{R}$ is increasing, monotone, and algebraic.*

Proof **F is increasing.**
If $(x, y) \in R$, then $(x, y) \in F_1(R)$ since F_1 is increasing. If $(x, y) \in F_1(R)$, then $(x, y) \in F_2(F_1(R))$, since F_2 is increasing. Hence, $(x, y) \in R$, then $(x, y) \in F(R) = F_2(F_1(R))$.
F is monotone.
If $R \subseteq R'$, then $F_1(R) \subseteq F_1(R')$ since F_1 is monotone. Given $F_1(R) \subseteq F_1(R')$, then $F_2(F_1(R)) \subseteq F_2(F_1(R'))$ since F_2 is monotone. Hence, $R \subseteq R'$ implies that $F_2(F_1(R)) = F(R) \subseteq F(R') = F_2(F_1(R'))$ since F_2.
F is algebraic.
Consider $(x, y) \in F(R) = F_2(F_1(R))$. Given that F_2 is algebraic then there is a finite $D \subseteq F_1(R)$ such that $(x, y) \in F_2(F_1(D))$. Given that D is finite, then for every $(x_i, y_i) \in D \subseteq F_1(R)$ there is D_i such that $(x_i, y_i) \in F_1(D_i)$, given that F_1 is algebraic as well. Hence, let

$$\bar{D} = \bigcup_{(x_i, y_i) \in D} D_i$$

then by construction $(x, y) \in F_2(F_1(\bar{D}))$. Therefore, $F = F_2 \circ F_1$ is algebraic. □

Lemma B.2 *Let $(\mathcal{F}, \circ, \geq)$ be ordered group, then*

(i) *$f \geq f'$ and $\bar{f} \geq \bar{f}'$ implies $f \circ \bar{f} \geq f' \circ \bar{f}'$,*
(ii) *$f \geq \bar{f}$ implies $f^{-1} \leq \bar{f}^{-1}$.*

Proof **Part (i).**
If $f \geq f'$, then $f \circ \bar{f} \geq f' \circ \bar{f}$, since $(\mathcal{F}, \circ, \geq)$ is an ordered group. If $\bar{f} \geq \bar{f}'$ then $f' \geq \bar{f} \geq f' \circ \bar{f}'$, since $(\mathcal{F}, \circ, \geq)$ is an ordered group. Then,

$$f \circ \bar{f} \geq f' \circ \bar{f} \geq f' \circ \bar{f}' \Rightarrow f \circ \bar{f} \geq f' \circ \bar{f}'.$$

Part (ii).
Since $(\mathcal{F}, \circ, \geq)$ is an ordered group and $f \geq \bar{f}$ we can conduct the following reasoning

$$f \geq \bar{f} \Rightarrow f^{-1} \circ f \geq f^{-1} \circ \bar{f} \Rightarrow I \geq f^{-1} \circ \bar{f} \Rightarrow$$
$$\Rightarrow I \circ \bar{f}^{-1} \geq f^{-1} \circ \bar{f} \circ \bar{f}^{-1} \Rightarrow \bar{f}^{-1} \geq f^{-1}.$$

□

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