ON A THEOREM OF WOLFF REVISITED

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Dedicated to the memory of Thomas Wolff

Abstract. We study p-harmonic functions, $1 < p \neq 2 < \infty$, in $\mathbb{R}^2_+ = \{z = x + iy : y > 0, -\infty < x < \infty\}$ and $B(0,1) = \{z : |z| < 1\}$. We first show for fixed p, $1 < p \neq 2 < \infty$, and for all large integers $N \geq N_0$ that there exists a p-harmonic function on $B(0,1)$, $V = V(re^{i\theta})$, which is $2\pi/N$ periodic in the θ variable, and Lipschitz continuous on $\partial B(0,1)$ with Lipschitz norm $\leq cN$, satisfying $V(0) = 0$ and $c^{-1} \leq \int_{-\pi}^{\pi} V(e^{i\theta})d\theta \leq c$. In case $2 < p < \infty$ we give a more or less explicit example of V and our work is an extension of a result of Wolff in [Wol07, Lemma 1] on $\mathbb{R}^2_+$ to $B(0,1)$. Using our first result, we extend the work of Wolff in [Wol07] on the failure of Fatou type theorems for $\mathbb{R}^2_+$ to $B(0,1)$ for p-harmonic functions, $1 < p \neq 2 < \infty$. Finally, we also outline the modifications needed for extending the work of Llorente, Manfredi, and Wu in [LMW05] regarding the failure of subadditivity of p-harmonic measure on $\partial\mathbb{R}^2_+$ to $\partial B(0,1)$.

Contents

1. Introduction

Throughout this paper we mix complex and real notation, so $z = x + iy$ and $\bar{z} = x - iy$ whenever $x, y \in \mathbb{R}$ where $i = \sqrt{-1}$. Moreover, we let $\mathbb{R}^2_+ = \{z = x + iy : y > 0\}$ and $B(z_0, \rho) = \{z : |z - z_0| < \rho\}$ whenever $z_0 \in \mathbb{R}^2$ and $\rho > 0$. We consider for fixed p, $1 < p \neq 2 < \infty$, weak solutions u (called p-harmonic functions) to the p-Laplace equation

$$\mathcal{L}_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \tag{1.1}$$

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on $B(0,1)$ or $\mathbb{R}^2_+$ (see section 2 for the definition of a p-harmonic function). In (1.1), ∇u denotes the gradient of u and $\nabla\cdot$ denotes the divergence operator. In 1984 Wolff brilliantly used ideas from harmonic analysis and PDE to prove that the Fatou theorem fails for p-harmonic functions when $2 < p < \infty$.

Theorem 1.1 ([Wol07, Theorem 1]). *If $p > 2$, then there exist bounded weak solutions of $\mathcal{L}_p \hat{u} = 0$ in $\mathbb{R}^2_+$ such that $\{x \in \mathbb{R} : \lim\limits_{y\to 0} \hat{u}(x+iy)$ exists$\}$ has Lebesgue measure zero. Also there exist positive bounded weak solutions of $\mathcal{L}_p \hat{v} = 0$ such that $\{x \in \mathbb{R} : \limsup\limits_{y\to 0} \hat{v}(x+iy) > 0\}$ has Lebesgue measure 0.*

The key to his proof and the only obstacle in extending Theorem 1.1 to $1 < p < \infty, p \neq 2$, was the validity of the following theorem for $1 < p < 2, p \neq 2$, stated as Lemma 1 in [Wol07].

Theorem 1.2 ([Wol07, Lemma 1]). *If $p > 2$ there exists a bounded Lipschitz function Φ on the closure of $\mathbb{R}^2_+$ with $\Phi(z+1) = \Phi(z)$ for $z \in \mathbb{R}^2_+$, $\mathcal{L}_p\Phi = 0$ weakly on $\mathbb{R}^2_+$, $\int_{(0,1)\times(0,\infty)} |\nabla\Phi|^p \, dxdy < \infty$, and*

$$\lim_{y\to\infty} \Phi(x+iy) = 0 \text{ for } x \in \mathbb{R}, \quad \text{but} \quad \int_0^1 \Phi(x)dx \neq 0. \tag{1.2}$$

Theorem 1.2 was later proved for $1 < p < 2$, by the second author of this article in [Lew88] (so Theorem 1.1 is valid for $1 < p \neq 2 < \infty$). Wolff remarks above the statement of his Lemma 1, that Theorem 1.1 " should generalize to other domains but the arguments are easiest in a half space since $\mathcal{L}_p$ behaves nicely under Euclidean operations".

In fact Wolff makes extensive use of the fact that $\Phi(Nz+z_0)$, $z = x+iy \in \mathbb{R}^2_+$, N a positive integer, $z_0 \in \mathbb{R}^2_+$, is p-harmonic in $\mathbb{R}^2_+$, and $1/N$ periodic in x, with Lipschitz norm $\approx N$ on $\mathbb{R} = \partial\mathbb{R}^2_+$. Also he used functional analysis-PDE type arguments, involving the Fredholm alternative and perturbation of certain p-harmonic functions (when $2 < p < \infty$) to get Φ satisfying (1.2).

In this paper we first give, in Lemma 3.1, a hands on example of a Φ for which Theorem 1.2 is valid. We then use this example and basic properties of p-harmonic functions to give a more or less explicit construction of $V = V(\cdot, N, p)$ for $2 < p < \infty$ in the following theorem.

Theorem A. *Given p, $1 < p \neq 2 < \infty$, there exist N_0 and a constant $c_1 \geq 1$, all depending only on p, such that if $N \geq N_0$ is a positive integer, then there is a p-harmonic function V in $B(0,1)$ with continuous boundary values satisfying*

$$\begin{aligned}
&(a) \quad -c_1 \leq V(te^{i\theta}) = V(te^{i(\theta+2\pi/N)}) \leq c_1 \quad \text{for } 0 \leq t \leq 1 \text{ and } \theta \in \mathbb{R},\\
&(b) \quad \int_{B(0,1)} |\nabla V|^p dxdy \leq c_1 N^{p-1},\\
&(c) \quad V(0) = 0 \text{ and } c_1 \int_{-\pi}^{\pi} V(e^{i\theta})d\theta \geq 1,\\
&(d) \quad V|_{\partial B(0,1)} \text{ is Lipschitz with norm } \leq c_1 N.
\end{aligned} \tag{1.3}$$

We were not able to find a more or less explicit example for which Theorem A holds when $1 < p < 2$. Instead for $1 < p \neq 2 < \infty$, we also use a finesse type argument to eventually obtain Theorem A from the perturbation method used by Wolff in proving Theorem 1.2 and a limiting type argument. In this proof of Theorem A we also interpret rather loosely the phrase "c_1 depends only on p". However constants will always be independent of $N \geq N_0$. We shall make heavy use of Wolff's arguments in proving Theorem A, as well as arguments of Varpanen in [Var15], who adapted Wolff's perturbation argument for constructing solutions to a linearized p-harmonic periodic equation in $\mathbb{R}^2_+$ to certain periodic p-harmonic functions in the θ variable, defined on $B(0,1)$. In section 4 we use Theorem A and modest changes in Wolff's argument to obtain the following analogue of Theorem 1.1.

Theorem B. *If $1 < p \neq 2 < \infty$, then there exist bounded weak solutions of $\mathcal{L}_p \hat{u} = 0$ in $B(0,1)$ such that $\{\theta \in \mathbb{R} : \lim_{r\to 1} \hat{u}(re^{i\theta})$ exists$\}$ has Lebesgue measure zero. Also there exist bounded positive weak solutions of $\mathcal{L}_p \hat{v} = 0$ such that $\{\theta \in \mathbb{R} : \limsup_{r\to 1} \hat{v}(re^{i\theta}) > 0\}$ has Lebesgue measure 0.*

Next for fixed $p > 1$, and E a subset of $\partial B(0,1)$, let $\mathcal{C}(E)$, denote the class of all non-negative p-superharmonic functions ζ on $B(0,1)$ (i.e., $\mathcal{L}_p \zeta \leq 0$ weakly in $B(0,1)$) with

$$\liminf_{\substack{z\in B(0,1)\\ z\to e^{i\theta}}} \zeta(z) \geq 1 \quad \text{for all } e^{i\theta} \in E. \tag{1.4}$$

Put $\omega_p(z_0, E) = \inf\{\zeta(z_0) : \zeta \in \mathcal{C}(E)\}$ when $z_0 \in B(0,1)$. Then $\omega_p(z_0, E)$ is usually referred to as the p-harmonic measure of E relative to z_0 and $B(0,1)$. In section 5 we use Theorem A and follow closely Llorente, Manfredi, and Wu in generalizing their work, [LMW05], on p-harmonic measure in $\mathbb{R}^2_+$ to $B(0,1)$. We prove

Theorem C. *If $1 < p \neq 2 < \infty$ there exist finitely many sets $E_1, E_2, \ldots, E_\kappa \subset \partial B(0,1)$, such that*

$$\omega_p(0, E_k) = 0, \quad \omega_p(0, \partial B(0,1) \setminus E_k) = 1 \text{ for } 1 \leq k \leq \kappa, \quad \text{and} \quad \bigcup_{k=1}^{\kappa} E_k = \partial B(0,1). \tag{1.5}$$

Furthermore, $\partial B(0,1) \setminus E_k$ has one Lebesgue measure 0 for $1 \leq k \leq \kappa$.

As for the plan of this paper, in section 2 we give some definitions and state some basic properties of p-harmonic functions. As outlined above, in sections 3, 4, 5, we prove Theorems A, B, C, respectively. In section 6, we make closing remarks.

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2. Basic estimates and definitions for p-harmonic functions

In this section we first introduce some notation, then give some definitions, and finally state some fundamental estimates for p-harmonic functions when p is fixed,

$1 < p < \infty$. As in the introduction we set $B(z_0, \rho) = \{z : |z - z_0| < \rho\}$ and $\mathbb{R}^2_+ = \{z = x + iy : y > 0\}$. Concerning constants, unless otherwise stated, in this section, and throughout the paper, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on p. Sometimes we write $c = c(p)$ to indicate this dependence. Also $A \approx B$ means A/B is bounded above and below by positive constants depending only on p. Let $d(E_1, E_2)$ denote the distance between the sets E_1 and E_2. For short we write $d(z, E_2)$ for $d(\{z\}, E_2)$. Let $\operatorname{diam}(E)$, $\bar{E}$, and ∂E denote the diameter, closure, and boundary of E respectively. We also write $\max_E \hat{v}$, $\min_E \hat{v}$ to denote the essential supremum and infimum of $\hat{v}$ on E whenever $E \subset \mathbb{R}^2$ and $\hat{v}$ is defined on E.

If $O \subset \mathbb{R}^2$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions h with distributional gradient ∇h, both of which are q-th power integrable on O. Let

$$\|h\|_{1,q} = \|h\|_q + \| \, |\nabla h| \, \|_q$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ is the usual Lebesgue q norm of functions in the Lebesgue space $L^q(O)$. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^2$. Given an open set O and $1 < p < \infty$, we say that $\hat{v}$ is p-harmonic in O provided $\hat{v} \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int |\nabla \hat{v}|^{p-2} \langle \nabla \hat{v}, \nabla \theta \rangle \, dxdy = 0 \quad \text{whenever } \theta \in W_0^{1,p}(G). \tag{2.1}$$

We say that $\hat{v}$ is a p-subsolution (p-supersolution) in O provided $\hat{v} \in W^{1,p}(G)$ whenever G is as above and (2.1) holds with $=$ replaced by $\leq$ ($\geq$) whenever $\theta \in W_0^{1,p}(G)$ with $\theta \geq 0$. We begin our statement of lemmas with the following maximum principle.

Lemma 2.1. *Given $1 < p < \infty$, if $\hat{v}$ is a p-subsolution and $\hat{h}$ is a p-supersolution in O with $\max(\hat{v} - \hat{h}, 0) \in W_0^{1,p}(G)$, whenever G is an open set with $\bar{G} \subset O$, then $\max_O (\hat{v} - \hat{h}) \leq 0$.*

Proof. A proof of this lemma can be found in [HKM06, Lemma 3.18]. □

Lemma 2.2. *Given $p, 1 < p < \infty$, let $\hat v$ be p-harmonic in $B(z_0, 4\rho)$ for some $\rho > 0$ and $z_0 \in \mathbb{R}^2$. Then*

(2.2)

$$(a) \quad \max_{B(z_0,\rho/2)} \hat v - \min_{B(z_0,\rho/2)} \hat v \leq c \left(\rho^{p-2} \int_{B(z_0,\rho)} |\nabla \hat v|^p \, dxdy \right)^{1/p} \leq c^2 \, (\max_{B(z_0,2\rho)} \hat v - \min_{B(z_0,2\rho)} \hat v).$$

Furthermore, there exists $\tilde\alpha = \tilde\alpha(p) \in (0,1)$ such that if $s \leq \rho$ then

$$(b) \quad \max_{B(z_0,s)} \hat v - \min_{B(z_0,s)} \hat v \leq c \left(\frac{s}{\rho}\right)^{\tilde\alpha} \left(\max_{B(z_0,2\rho)} \hat v - \min_{B(z_0,2\rho)} \hat v \right).$$

(c) *If $\hat v \geq 0$ in $B(z_0, 4\rho)$, then $\max_{B(z_0,2\rho)} \hat v \leq c \min_{B(z_0,2\rho)} \hat v$.*

Proof. Lemma 2.2 is well known. A proof of this lemma, using Moser iteration of positive solutions to PDE of p-Laplace type, can be found in [Ser64]. (2.2) (c) is called Harnack's inequality. □

Lemma 2.3. *Let $\Omega = B(0,1)$ or $\mathbb{R}^2_+$ and $1 < p < \infty$. Let $z_0 \in \partial\Omega$ and suppose $\hat v$ is p-harmonic in $\Omega \cap B(z_0, 4\rho)$ for $0 < \rho < diam(\Omega)$ with $\hat h \in W^{1,p}(\Omega \cap B(z_0, 4\rho))$ and $\hat v - \hat h \in W^{1,p}_0(\Omega \cap B(z_0, 4\rho))$. If $\hat h$ is continuous on $\bar\Omega \cap B(z_0, 4\rho)$ then $\hat v$ has a continuous extension to $\bar\Omega \cap B(z_0, 4\rho)$, also denoted $\hat v$, with $\hat v \equiv \hat h$ on $\partial\Omega \cap B(z_0, 4\rho)$. If*

$$|\hat h(z) - \hat h(w)| \leq M' |z - w|^{\hat\sigma} \quad \text{whenever} \quad z, w \in \partial\Omega \cap B(z_0, 4\rho),$$

for some $\hat\sigma \in (0, 1]$, and $1 \leq M' < \infty$, then there exists $\hat\sigma_1 \in (0, 1]$, depending only on $\hat\sigma$ and p, such that

$$|\hat v(z) - \hat v(w)| \leq 2M' \rho^{\hat\sigma} + \ (|z - w|/\rho)^{\hat\sigma_1} \max_{\Omega \cap \bar B(z_0,2\rho)} |\hat v| \tag{2.3}$$

whenever $z, w \in \Omega \cap B(z_0, \rho)$.

If $\hat h \equiv 0$ on $\partial\Omega \cap B(z_0, 4\rho)$, $\hat v \geq 0$ in $B(z_0, 4\rho)$, $\hat c \geq 1$, and $z_1 \in \Omega \cap B(z_0, 4\rho)$ with $\hat c \, d(z_1, \partial\Omega) \geq \rho$, then there exists $\tilde c$, depending only on $\hat c$ and p, such that

$$(+) \quad \max_{B(z_0,2\rho)} \hat v \leq \tilde c \left(\rho^{p-2} \int_{B(z_0,3\rho)} |\nabla \hat v|^p \, dxdy \right)^{1/p} \leq (\tilde c)^2 \, \hat v(z_1). \tag{2.4}$$

Furthermore, using (2.3), it follows for $z, w \in \bar\Omega \cap B(z_0, 2\rho)$ that

$$(++) \quad |\hat v(z) - \hat v(w)| \leq c \, \hat v(z_1) \left(\frac{|z - w|}{\rho} \right)^{\hat\sigma_1}.$$

Proof. Continuity of $\hat v$ given continuity of $\hat h$ in $\bar\Omega \cap B(z_0, 4\rho)$ follows from Corollary 6.36 in [HKM06] with Ω in this lemma replaced by $\Omega \cap B(z_0, 4\rho)$. This Corollary and the Hölder continuity estimate on $\hat h$ above, are then used in Theorem 6.44 of [HKM06] to prove an inequality analogous to (2.3). Proofs involve Wiener type estimates for subsolutions of p-Laplace type that vanish on $\partial\Omega \cap B(z_0, 4\rho)$. (2.4) $(+)$ is sometimes referred to as Carleson's inequality, see [AS05]. □

Lemma 2.4. *Let $p, \hat{v}, z_0, \rho$, be as in Lemma 2.2. Then $\hat{v}$ has a representative locally in $W^{1,p}(B(z_0, 4\rho))$, with Hölder continuous partial derivatives in $B(z_0, 4\rho)$ (also denoted $\hat{v}$), and there exist $\hat{\gamma} \in (0, 1]$ and $c \geq 1$, depending only on p, such that if $z, w \in B(z_0, \rho/2)$, then*

(2.5)

$$(\hat{a}) \quad c^{-1}\,|\nabla\hat{v}(z) - \nabla\hat{v}(w)| \,\leq\, (|z-w|/\rho)^{\hat{\gamma}} \max_{B(z_0,\rho)} |\nabla\hat{v}| \leq\, c\,\rho^{-1}\,(|z-w|/\rho)^{\hat{\gamma}} \max_{B(z_0,2\rho)} |\hat{v}|.$$

Also $\hat{v}$ has distributional second partials with

$$(\hat{b}) \quad \int_{B(z_0,\rho)\cap\{\nabla\hat{v}\neq 0\}} |\nabla\hat{v}|^{p-2}\,\left(|\hat{v}_{xx}|^2(z) + |\hat{v}_{yy}|^2(z) + |\hat{v}_{xy}|^2(z)\right) dxdy \leq c\,\rho^{-p} \max_{B(z_0,2\rho)} |\hat{v}|.$$

$(\hat{c})$ *If $\nabla\hat{v}(z_0) \neq 0$, then $\hat{v}$ is infinitely differentiable in $B(z_0, s)$ for some $s > 0$.*

Proof. For a proof of the left-hand inequality in (2.5) $(\hat{a})$, see Theorem 1 in [Lew83]. (2.5) $(\hat{b})$ can be proved using the method of difference quotients or by using (2.5) $(\hat{a})$ and carefully taking limits as $\epsilon \to 0$ in (2.8) of [Lew83]. (2.5) $(\hat{c})$ follows from $(\hat{a})$, $(\hat{b})$, and Schauder type estimates (see [GT01]). □

Lemma 2.5. *Let $x_0 \in \mathbb{R}$, $\rho > 0$, $1 < p < \infty$, and suppose $\hat{u}$ and $\hat{v}$ are non-negative p-harmonic functions in $\mathbb{R}^2_+ \cap B(x_0, 4\rho)$ with continuous boundary values $\hat{v} \equiv \hat{u} \equiv 0$ on $\mathbb{R} \cap B(x_0, 4\rho)$. There exists $c = c(p)$ such that*

$$\frac{\hat{u}(z)}{\hat{v}(z)} \leq c\frac{\hat{u}(x_0+\rho i)}{\hat{v}(x_0+\rho i)} \quad \textit{whenever} \quad z \in \mathbb{R}^2_+ \cap B(x_0, 2\rho). \tag{2.6}$$

Also $\hat{v}$ has a p-harmonic extension to $B(x_0, 4\rho)$ obtained by requiring $\hat{v}(z) = -\hat{v}(\bar{z})$ for $z \in B(x_0, 4\rho) \setminus \mathbb{R}^2_+$.

Proof. Here (2.6) in Lemma 2.5 follows from essentially barrier estimates for non-divergence form PDE. See for example [AKSZ07]. The extension process for $\hat{v}$ is generally referred to as Schwarz reflection. □

Next given $\eta > 0$ and $x_0 \in \mathbb{R}$ let

$$S(x_0, \eta) := \{z = x + iy : \ |x - x_0| < \eta/2, \ 0 < y < \infty\}.$$

For short we write $S(\eta)$ when $x_0 = 0$. For fixed p, $1 < p < \infty$, let $R^{1,p}(S(\eta))$ denote the Riesz space of equivalence classes of functions f on $\mathbb{R}^2_+$ with $f(z+\eta) = f(z)$ when $z \in \mathbb{R}^2_+$ and norm

$$\|f\|_* = \|f\|_{*,p} = \left(\int_{S(\eta)} |\nabla f|^p \, dxdy\right)^{1/p} < \infty. \tag{2.7}$$

Also let $R^{1,p}_0(S(\eta))$ denote functions in $R^{1,p}(S(\eta))$ which can be approximated arbitrarily closely in the norm of $R^{1,p}(S(\eta))$ by functions in this space which are infinitely differentiable and vanish in an open neighbourhood of $\mathbb{R}$. It is well known, see [Wol07, section 1], that given $f \in R^{1,p}(S(\eta))$ there exists a unique p-harmonic function $\tilde{v}$ on $\mathbb{R}^2_+$ with $\tilde{v}(z + \eta) = \tilde{v}(z)$ for $z \in \mathbb{R}^2_+$ with $\tilde{v} - f \in R^{1,p}_0(S(\eta))$. In fact the usual minimization argument yields that $\tilde{v}$ has minimum norm among all functions h in

$R^{1,p}(S(\eta))$ with $h - f \in R_0^{1,p}(S(\eta))$. Uniqueness of $\tilde{v}$ is a consequence of the maximum principle in Lemma 2.1.

Next we state

Lemma 2.6. *Given $1 < p < \infty$, let $\hat{v}$ be p-harmonic in $\mathbb{R}^2_+$ and $\hat{v} \in R^{1,p}(S(\eta))$. Then there exists $c = c(p)$ and $\xi \in \mathbb{R}$ such that*

$$|\hat{v}(z) - \xi| \leq c \liminf_{t \to 0} \left(\max_{\mathbb{R}\times\{t\}} \hat{v} - \min_{\mathbb{R}\times\{t\}} \hat{v} \right) \exp\left(-\frac{y}{c\eta} \right) \tag{2.8}$$

whenever $z = x + iy \in \mathbb{R}^2_+$.

Proof. The lim inf in (2.8) need not be finite, but clearly $= \max_{\mathbb{R}\times\{0\}} \hat{v} - \min_{\mathbb{R}\times\{0\}} \hat{v}$ when $\hat{v}$ has a continuous extension to the closure of $\mathbb{R}^2_+$. Lemma 2.6 is proved in Lemma 1.3 of [Wol07] using η periodicity of $\hat{v}$ and facts about p-harmonic functions similar to Lemmas 2.1 and 2.2. □

Finally, we state an analogue of Lemma 2.6 for $B(0,1)$.

Lemma 2.7. *Given $1 < p < \infty$, let $\hat{v}$ be p-harmonic in $B(0,1)$, $\hat{v} \in W^{1,p}(B(0,1))$, and $\hat{v}(re^{i\theta}) = \hat{v}(re^{i(\theta+\eta)})$, when $z = re^{i\theta} \in B(0,1)$ and $2\pi/\eta$ is a positive integer. Then there exists $c = c(p) \geq 1$ such that*

$$|\hat{v}(re^{i\theta}) - \hat{v}(0)| \leq c \lim_{t \to 1} (\max_{B(0,t)} \hat{v} - \min_{B(0,t)} \hat{v})\, r^{\frac{1}{c\eta}}. \tag{2.9}$$

Proof. Fix $n = 0, 1, 2, \ldots$ Using η periodicity of $\hat{v}$ we deduce from Lemma 2.2 (b) applied $\approx (1000\eta)^{-1}$ times in balls of radius $\eta/10$ that

$$\max_{B(0,2^{-n-1})} \hat{v} - \min_{B(0,2^{-n-1})} \hat{v} \leq c^{-\tilde{\alpha}/\eta} \left(\max_{B(0,2^{-n})} \hat{v} - \min_{B(0,2^{-n})} \hat{v} \right),$$

where $c \geq 1$ depends only on p. Iterating this inequality we obtain (2.9). □

3. Proof of Theorem A

In this section we prove Theorem A and as stated in the introduction we give two proofs of Theorem A when $2 < p < \infty$. An important role in each proof is played by homogeneous p-harmonic functions of the form:

$$z = re^{i\theta} \to r^{\lambda}\, \phi(\theta) \quad \text{for } |\theta| < \alpha \text{ and } r > 0, \tag{3.1}$$

satisfying $\phi(0) = 1$, $\phi(\alpha) = 0$, $\phi(\theta) = \phi(-\theta)$, $\phi' < 0$ on $(0, \alpha]$, and $\phi \in C^\infty([-\alpha, \alpha])$ with $\lambda = \lambda(\alpha) \in (-\infty, \infty)$. Regarding (3.1), Krol' in [Kro73] (see also [Aro86]) used (3.1) and separation of variables to show for $1 < p < \infty$,

$$\begin{aligned} 0 = &\frac{d}{d\theta} \left\{ [\lambda^2 \phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2}\, \phi'(\theta) \right\} \\ &+ \lambda[\lambda(p-1) + (2-p)][\lambda^2\phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2} \phi(\theta). \end{aligned}$$

Letting $\psi = \phi'/\phi$ in the above equation and proceeding operationally he obtained, the first order equation

$$\begin{aligned} 0 =&((p-1)\psi^2+\lambda^2)\,\psi' \\ &+(\lambda^2+\psi^2)[(p-1)\,\psi^2+\lambda^2(p-1)+\lambda(2-p)]. \end{aligned} \tag{3.2}$$

Separating variables in (3.2) one gets

$$\frac{\lambda d\psi}{\lambda^2+\psi^2} - \frac{(\lambda-1)\,d\psi}{\lambda^2+\psi^2+\lambda(2-p)/(p-1)} + d\theta = 0. \tag{3.3}$$

Integrating (3.3) and using $\psi(0)=0$ we obtain for $0 \leq |\theta| < \alpha$ that

(3.4)

$$(\lambda/|\lambda|)\arctan(\psi/\lambda) - \frac{\lambda-1}{\sqrt{\lambda^2+\lambda(2-p)/(p-1)}}\arctan\left(\frac{\psi}{\sqrt{\lambda^2+\lambda(2-p)/(p-1)}}\right) = -\theta$$

Letting $\theta \to \alpha$ from the left and using $\psi(\pm\alpha) = -\infty$ we get

$$\pm\, 1 - \frac{\lambda-1}{\sqrt{\lambda^2+\lambda(2-p)/(p-1)}} = \frac{2\alpha}{\pi} \tag{3.5}$$

where $+1$ is taken if $\lambda > 0$ and -1 if $\lambda < 0$. Using the quadratic formula it is easily seen that for fixed $\alpha \in (0,\pi]$ each equation has exactly one λ satisfying it and $\lambda > 0$ if the $+$ sign is taken while $\lambda < 0$ if the - sign is taken in (3.5). Using these values of λ it follows that the operational argument can now be made rigorous by reversing the steps leading to (3.5). Then (3.2), $\psi(0)=0$, and calculus imply that ψ is decreasing and negative on $(0,\alpha)$. Integrating ψ over $[0,\theta)$, $\theta<\alpha$, and exponentiating it follows that $\phi > 0$ is decreasing on $(0,\alpha)$ with $\phi(\alpha)=0$. Symmetry and smoothness properties of ϕ listed above can be proved using ODE theory or Lemma (2.5) $(\hat{c})$ and Schwarz reflection.

To avoid confusion later on let $-\hat{\lambda}$ denote the value of λ in (3.5) with -1 taken, $\alpha = \pi/2$, and let $\hat{\phi}$ correspond to $-\hat{\lambda}$ as in (3.1) for given p, $1<p<\infty$. After some computation one obtains from (3.5) as in [LV13] that

$$\hat{\lambda} = \hat{\lambda}(p) = (1/3)\left(-p+3+2\sqrt{p^2-3p+3}\ \right)/(p-1). \tag{3.6}$$

3.1. **Proof of Theorem 1.2.** In this section we provide a hands on proof of Theorem 1.2 when $2<p<\infty$. To this end, given $0<t<10^{-10}$, let $a(\cdot)$ be a C^∞ smooth function on $\mathbb{R}$ with compact support in $(-t,t)$, $0\leq a\leq 1$ with $a\equiv 1$ on $(-t/2,t/2)$, and $|\nabla a| \leq 10^5/t$. Let $f(z) = a(x)a(y)$ when $z = x+iy \in \mathbb{R}^2$ and for fixed p, $1<p\neq 2<\infty$ let $\hat{u}$ be the unique p-harmonic function on $\mathbb{R}^2_+$ with $0\leq \hat{u} \leq 1$ satisfying

$$\int_{\mathbb{R}^2_+} |\nabla \hat{u}|^p dxdy \leq \int_{\mathbb{R}^2_+} |\nabla f|^p dxdy \leq c\, t^{2-p}, \tag{3.7}$$

and $\hat{u} - f \in W^{1,p}_0(\mathbb{R}^2_+ \cap B(0,\rho))$ whenever $0<\rho<\infty$. Existence and uniqueness of $\hat{u}$ follows with small changes from the usual calculus of variations argument for bounded domains (see [Eva10]). We assert that there exists $\beta_* \in (0,1]$ such that if $z, w \in B(0,\rho) \cap \bar{\mathbb{R}}^2_+$, then

$$|\hat{u}(z) - \hat{u}(w)| \leq c \left(\frac{|z-w|}{\rho} \right)^{\beta_*} \quad \text{and} \quad |\hat{u}(z)| \leq c \left(\frac{t}{|z|} \right)^{\beta_*} \text{ for } z \in \bar{\mathbb{R}}^2_+. \tag{3.8}$$

The left hand inequality in (3.8) follows from Lemma 2.3. To prove the right hand inequality in (3.8) observe from the boundary maximum principle in Lemma 2.1 and $0 \leq \hat{u} \leq 1$, that $\max_{\partial B(0,r)} \hat{u}$ is decreasing for $r \in (t, \infty)$. Using this fact and Harnack's inequality in Lemma 2.2 (c) applied to $\max_{\partial B(0,r)} \hat{u} - \hat{u}$, and (2.4) (++) we deduce the existence of $\theta \in (0,1)$ with

$$\max_{\partial B(0,2r)} \hat{u} \leq \theta \max_{\partial B(0,r)} \hat{u}.$$

Iterating this inequality we get the right hand inequality in (3.8).

Next we show as in [LV13] that

$$\hat{u}(i) \approx t^{\hat{\lambda}} \tag{3.9}$$

where $\hat{\lambda}$ is as in (3.6). To prove (3.9), let $z = re^{i\theta}$ for $r > 0$ and $0 \leq \theta \leq \pi$, and put

$$v(z) = v(re^{i\theta}) = (t/r)^{\hat{\lambda}}\, \hat{\phi}(\theta - \pi/2) \tag{3.10}$$

where $\hat{\lambda}$ and $\hat{\phi}$ as defined before (3.6). Then v is p-harmonic in $\mathbb{R}^2_+$ with $v \equiv 0$ on $\mathbb{R} \setminus \{0\}$ and $v(it) = 1$. Also from Harnack's inequality and (2.4) of Lemma 2.3 with $\hat{v} = 1 - \hat{u}$, $\hat{u}$, we find that $\hat{u}(it) \approx 1$. In view of the boundary values of $\hat{u}$, v and $\hat{u}(it) \approx v(it) = 1$, as well as Harnack's inequality in (2.2) (c), we see that Lemma 2.5 can be applied to get

$$\hat{u}/v \approx 1 \tag{3.11}$$

in $\mathbb{R}^2_+ \cap [B(0,4t) \setminus B(0,2t)]$. From (3.8) for $\hat{u}, v$, and $\hat{\lambda} > 0$ we find first that $\hat{u}(z), v(z) \to 0$ as $z \to \infty$ in $\mathbb{R}^2_+$ and thereupon from Lemma 2.1 that (3.11) holds in $\mathbb{R}^2_+ \setminus \bar{B}(0,2t)$. Since $v(i) = t^{\hat{\lambda}}$ we conclude from (3.11) that claim (3.9) is true.

Finally observe from (3.6) that for $1 < p < \infty$

$$\begin{aligned} (3/2)(p-1)^2(p^2-3p+3)^{1/2}\, d\hat{\lambda}/dp &= (p-1)(p-3/2) - (p^2-3p+3) - \sqrt{p^2-3p+3} \\ &= p/2 - 3/2 - \sqrt{p^2-3p+3} < 0. \end{aligned} \tag{3.12}$$

Indeed, the inequality in the second line in (3.12) is clearly true if $p \leq 3$ and for $p > 3$ is true because

$$(p-3)^2 < 4(p^2-3p+3) \text{ or } 0 < 3(p^2-2p+1) = 3(p-1)^2.$$

Since $\hat{\lambda}(2) = 1$ we see that

$$\hat{\lambda}(p) > 1 \text{ for } 1 < p < 2 \quad \text{and} \quad \hat{\lambda}(p) < 1 \text{ for } p > 2. \tag{3.13}$$

Let $\tilde{a}$ denote the one periodic extension of $a|_{[-1/2,1/2]}$ to $\mathbb{R}$. That is $\tilde{a}(x+1) = \tilde{a}(x)$ for $x \in \mathbb{R}$ and $\tilde{a} = a$ on $[-1/2, 1/2]$. Also let Ψ be the p-harmonic function on $\mathbb{R}^2_+$ with

$$\begin{aligned} &(a) \quad \Psi(z+1) = \Psi(z), \quad \text{whenever } z \in \mathbb{R}^2_+, \\ &(b) \quad \Psi - \tilde{a}(x)a(y) \in R_0^{1,p}(S(1)) \quad \text{and} \quad 0 \leq \Psi \leq 1 \text{ in } \mathbb{R}^2_+, \\ &(c) \quad \int_{S(1)} |\nabla \Psi|^p dxdy \leq c\, t^{2-p} < \infty, \\ &(d) \quad \lim_{y \to \infty} \Psi(x+iy) = \xi \quad \text{whenever } x \in \mathbb{R}. \end{aligned} \tag{3.14}$$

Existence of Ψ satisfying $(a) - (d)$ of (3.14) follows from the discussion after (2.7), and (2.8) of Lemma 2.6 (see also (3.7) for (c)). Comparing boundary values of $\hat{u}$ and Ψ we see that $\hat{u} \leq \Psi$ on $\mathbb{R}$. Using this fact and Lemma 2.1 we find in view of (3.8) that

$$\hat{u} \leq \Psi \quad \text{in} \quad \mathbb{R}^2_+. \tag{3.15}$$

From (3.15), (3.9), and Harnack's inequality for $\hat{u}$, we have

$$\int_0^1 \Psi(x+i)\, dx = \int_{-1/2}^{1/2} \Psi(x+i) dx \geq \int_{-1/2}^{1/2} \hat{u}(x+i)\, dx \approx t^{\hat{\lambda}}. \tag{3.16}$$

Also from (3.14) and (2.3) we obtain

$$\int_0^1 \Psi(x+si)\, dx = \int_{-1/2}^{1/2} \Psi(x+si)\, dx \leq c\, t \tag{3.17}$$

for some small $s > 0$. Thus

$$\int_0^1 \Psi(x+si)\, dx \leq c t^{1-\hat{\lambda}} \int_0^1 \Psi(x+i) dx \tag{3.18}$$

where c depends only on p. Recall from (3.13) that $\hat{\lambda} < 1$ if $p > 2$. So from (3.18) and (3.14) (d) we see for $t > 0$ sufficiently small that

Lemma 3.1. *Theorem 1.2 is valid for one of the four functions* $\Phi(z) = \pm(\Psi(z+i) - \xi)$ *or, for* $s > 0$ *small enough,* $\Phi(z) = \pm(\Psi(z+is) - \xi)$ *whenever* $z \in \mathbb{R}^2_+$.

This completes the hands on proof of Theorem 1.2 when $p > 2$.

Remark 3.2. *The above proof of Theorem 1.2 fails when* $1 < p < 2$ *as now* $\hat{\lambda} > 1$, *so* $t^{1-\hat{\lambda}} \to \infty$ *in* (3.18) *as* $t \to 0$. *In short, our hands on example could still be valid for* $1 < p < 2$, *but in this case one needs to make a better estimate than* (3.18).

3.2. **Hands on proof of Theorem A when** $2 < p < \infty$. To provide examples in $B(0,1)$, satisfying Theorem A, we need to make somewhat better estimates than in Lemma 3.1 since p-harmonic functions are not invariant under dilatation in polar coordinates.

Proof. For this purpose let $0 < b << t << 10^{-10}$. For the moment we allow both b and t to vary subject to these requirements but shall later fix $t = t_0$ and then essentially choose $b_0 << t_0$ so that if $0 < b \leq b_0$, then Theorem A is true for our examples. Let T be the triangular region whose boundary consists of the horizontal line segment from $-b - bt\,i$ to $b - bti$ and the line segments joining i to $\pm b - bti$ (see Figure 1). Let v_1 be the p-harmonic function in T with $v_1(z) - f(z/b + ti) \in W_0^{1,p}(T)$ where f is defined above (3.7). Then from Lemma 2.3 and translation, dilation invariance of p-harmonic functions, we see that v_1 has continuous boundary values with $v_1 \equiv 1$ on the open line segment from $-bt/2 - bti$ to $bt/2 - bti$, and $v_1 \equiv 0$ on $\partial T \setminus \bar{B}(-bti, bt)$. From the definition of $\hat{u}$ above (3.7) we find that

$$v_1(z) \leq \hat{u}(z/b + ti) \tag{3.19}$$

in the $W^{1,p}$ Sobolev sense, when $z = x + iy \in \partial T$. Thus by Lemma 2.1 this inequality holds in T. Also from (3.9), (3.19), and Harnack's inequality we get

$$v_1(bi) \leq c\,\hat{u}(i) \approx t^{\hat{\lambda}}. \tag{3.20}$$

On the other hand since both functions in (3.19) have the same boundary values on $\partial T \cap \{z = x - bti : -b \leq x \leq b\}$ it follows from (3.20), (2.3), Lemma 2.5, and Harnack's inequality that

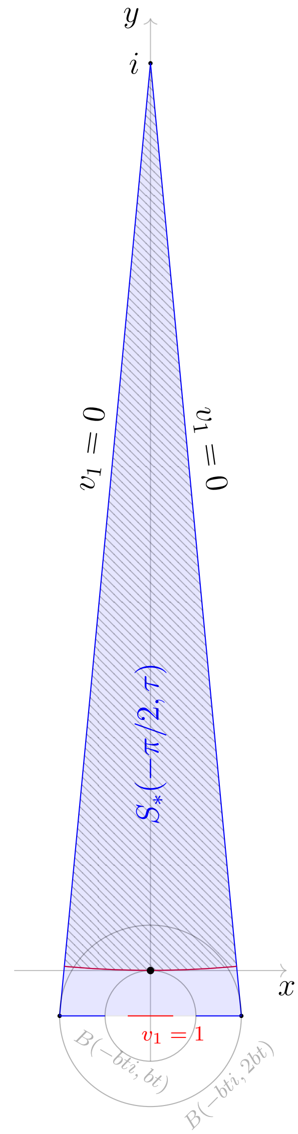

FIGURE 1. Domain T and $S_*(-\pi/2, \tau)$

$$\hat{u}(z/b + ti) \leq c_+(v_1(z) + t^{\hat{\lambda}}) \quad \text{for } z \in T \cap \bar{B}(-ibt, 2b). \tag{3.21}$$

Also from (3.11) and the definition of v we have

$$\hat{u}(i/\check{c}) \geq 2c_+ t^{\hat{\lambda}} \tag{3.22}$$

provided t is small enough, say $t \leq t_1$, and $\check{c}$ is large enough where $\check{c}, t_1$, depend on c_+ so only on $p > 2$. Using (3.22) in (3.21) with $z = \frac{-\check{c}t+1}{\check{c}}\, bi$ we obtain first that $v_1(\frac{-\check{c}t+1}{\check{c}}\, bi) \geq t^{\hat{\lambda}}$, and second from Harnack's inequality for v_1 that

$$v_1(bi) \approx t^{\hat{\lambda}}. \tag{3.23}$$

Next if $\theta_0 \in \mathbb{R}$ and $\eta > 0$, we let

$$S_*(\theta_0, \eta) := \{z : z = i + \rho e^{i\theta} :\ 0 \leq \rho < 1,\ |\theta - \theta_0| < \pi\eta\}.$$

From high school geometry we see that if $\pi\tau = \arctan(\frac{b}{1+bt})$, then the rays $\theta = -\pi/2 \pm \pi\tau$ drawn from i to $\pm b - bti$ make an angle $\pi\tau$ with the y axis and consequently (see Figure 1)

$$\bar{T} \cap \partial B(i,1) = \partial S_*(-\pi/2, \tau) \cap \partial B(i,1).$$

Given N a large positive integer choose b so that $\tau = N^{-1} \approx b$. We claim that

$$\int_{\bar{T}\cap\partial B(i,1)} v_1(z)|dz| \leq c\, b\, t. \tag{3.24}$$

To prove (3.24) we parametrize $\bar{T} \cap \partial B(i,1)$ by $z(x) = x + iy(x)$ for $-s \leq x \leq s$ where $s \approx b$ (so $y = 1 - \sqrt{1-x^2}$). Then from (3.19), (3.11), $b << t$, and the fact that in (3.10), $\hat{\phi}(\theta - \pi/2) \leq c \min(\theta, \pi - \theta)$ for $\theta \in [0,\pi]$, we see as in the proof of (3.23) that if $2bt \leq |x| \leq b$ then $|dz| \approx dx$ and

$$v_1(z(x)) \leq cv(z(x)/b + ti) \leq c^2 (bt)^{\hat{\lambda}}\, |x|^{-\hat{\lambda}} \left(\frac{|x|^2 + bt}{|x|}\right) \approx (bt)^{1+\hat{\lambda}} |x|^{-\hat{\lambda}-1}. \tag{3.25}$$

Thus

$$\int_{\bar{T}\cap\partial B(i,1)} v_1(z)|dz| \leq cbt + c\int_{bt}^{b} (bt)^{1+\hat{\lambda}} x^{-\hat{\lambda}-1} dx \leq c^2 bt,$$

so (3.24) is true. Let $\breve{h}(z) = v_1(z)$ when $z \in \bar{T} \cap \bar{B}(i,1)$ and extend $\breve{h}$ to $\bar{B}(i,1)$ by requiring that $\breve{h}(i + \rho e^{i\theta}) = \breve{h}(i + \rho e^{i(\theta + 2\pi/N)})$ for $0 < \rho \leq 1$, and $\theta \in \mathbb{R}$. Let $\breve{v}$ be the p-harmonic function in $B(i,1)$ with $\breve{v} \equiv \breve{h}$ on $\partial B(i,1)$ in the $W^{1,p}$ Sobolev sense. From the usual calculus of variations argument we see that

$$\begin{aligned} &(a') \quad 0 \leq \breve{v}(i + \rho e^{i\theta}) = \breve{v}(i + \rho e^{i(\theta + 2\pi/N)}) \leq 1 \quad \text{for } 0 \leq \rho \leq 1,\ \theta \in \mathbb{R}, \\ &(b') \quad \int_{S_*(-\pi/2,\tau)} |\nabla \breve{v}|^p dxdy \leq \int_T |\nabla v_1|^p dxdy \;\leq\; c\,(t/N)^{2-p}. \end{aligned} \tag{3.26}$$

We assert that

$$\begin{aligned} &(c') \quad \int_{-\pi}^{\pi} \breve{v}(i + e^{i\theta}) d\theta \leq ct, \quad \text{and} \quad \breve{v}(i) \geq c^{-1} t^{\hat{\lambda}}, \\ &(d') \quad |\breve{v}(z) - \breve{v}(w)| \leq c\,(N/t)\; |z - w| \quad \text{whenever } z, w \in \partial B(i,1). \end{aligned} \tag{3.27}$$

The left-hand inequality in (3.27) (c') follows from (3.24), (2.3) of Lemma 2.3, $v_1 = \breve{v}$ on $\bar{T} \cap \partial B(i,1)$, and (3.26) (a'). To prove the right-hand inequality in (3.27) (c'), we note that $\breve{v} \geq c^{-1} t^{\hat{\lambda}}$ on $\partial B(i, 1 - 1/N)$, as we see from Harnack's inequality for $\breve{v}$, (3.23), $v_1 \leq \breve{v}$ in $\bar{T} \cap \bar{B}(i,1)$, and (3.26) (a'). This inequality and the minimum principle for p-harmonic functions give the right-hand inequality in (3.27) (c'). To prove (3.27) (d'), let $z \in \bar{T} \cap \partial B(i,1), z_0 \in \partial T$, and suppose $|z_0 - z|$ is the distance from z to ∂T. If $|z - z_0| \geq bt/4$, then v_1 is p-harmonic in $B(z, bt/4)$. Otherwise from Schwarz reflection we see that v_1 has a p-harmonic extension to $B(z_0, bt/2)$ (also denoted v_1). Thus in either case v_1 is p-harmonic in $B(z, bt/4)$ so from (2.5) $(\hat{a})$ of Lemma 2.4, we have

$$|\nabla v_1|(z) \leq c\, (\max_{B(z,bt/4)} |v_1|)/bt \leq c/bt. \tag{3.28}$$

Now (3.28) and $\breve{v} = v_1$ on $\bar{T} \cap \partial B(i,1)$ give (3.27) (d'). We now choose $0 < t_0 < t_1 < 1 < N_0$, depending only on $p > 2$, so that if N is a positive integer with $N \geq N_0$, then (3.26), (3.27), are valid with $t = t_0$ and also,

$$0 \leq \int_{-\pi}^{\pi} \breve{v}(i + e^{i\theta}) d\theta \leq \breve{v}(i)/2. \tag{3.29}$$

With t_0 now fixed, put

$$V(z) = \breve{v}(i) - \breve{v}(z + i) \quad \text{whenever } z \in B(0,1).$$

Then from (3.26), (3.27), (3.29), we conclude that Theorem A is valid for fixed $p > 2$. □

3.3. **Finesse Proof of Theorem A for $1 < p \neq 2 < \infty$.** In this section we give a proof of Theorem A valid for $1 < p \neq 2 < \infty$, modelled on proofs of Wolff [Wol07] and Varpanen [Var15], which however does not produce explicit examples. To briefly outline our proof, we note that Wolff (see also [DS, Section 3]) constructed for fixed p, $1 < p \neq 2 < \infty$, a p-harmonic function, F, of the form

$$F(z) = F(x + iy) = e^{-\gamma y} f(x) \quad \text{for } z \in \mathbb{R}^2_+ \tag{3.30}$$

where $\gamma > 0$ and f satisfies

$$\begin{cases} f(x + 2\pi) = f(x) = f(-x), \\ f(\pi/2 - x) = -f(\pi/2 + x), \\ f(0) = 1 \quad \text{and} \quad f(\pm\pi/2) = 0, \\ f'(0) = 0 \quad \text{and} \quad f' < 0 \quad \text{on} \quad (0, \pi/2]. \end{cases}$$

He then perturbed off of F and used the Fredholm alternative to eventually construct Φ in Theorem 1.2. Varpanen generalized much of Wolff's argument to the $B(0,1)$ setting, but since the p-Laplacian is not invariant under dilations in polar coordinates he needed to make estimates on Lipschitz, L^∞ constants similar to those in Theorem A for each N. Unfortunately the functional analysis - Fredholm alternative part of Wolff's argument does not seem to allow for specific estimates of constants. Our idea for making estimates began from observing some similarities between (3.31), (3.32), and a $1/N$ scaling of ϕ in the θ variable (corresponding to $\alpha = \pi/(2N)$) in an ODE arising from (3.3), (3.4) , as $N \to \infty$. This eventually led us to Lemma 3.3 and after that in (3.45) to uniform convergence of $\tilde{u}_N$ and all its partial derivatives to F and the corresponding partial derivatives of F on $\mathbb{R}^2$. $\tilde{u}_N$ is defined in (3.39). Next we state in Lemma 3.4 and for $2 < p < \infty$, the solutions obtained by Wolff and Lewis (using the Fredholm alternative) to divergence form elliptic regularity Dirichlet problems. In view of Lemmas 3.3, 3.4, we study for each N (large) in (3.50)-(3.55) a similar elliptic regularity variational problem whose boundary functions are the same as the functions obtained by Wolff and Lewis. Moreover the coefficients of our operator and all its derivatives each converge uniformly on $\bar{\mathbb{R}}^2_+$ to corresponding derivatives of the coefficients used by Wolff and Lewis. From this fact, interior estimates for elliptic PDE with smooth coefficients (Moser, De-Giorgi iteration, the method of difference quotients, Sobolev theorems) and Schauder type boundary estimates (instead of the

Fredholm alternative) we are able to show our solutions and their derivatives of order $\leq l$ (l a fixed positive integer) are bounded in $\bar{\mathbb{R}}^2_+$ by constants that are independent of N provided $N \geq \tilde{N}(l)$. The information so obtained is transferred to $B(0,1)$ in (3.56) - (3.63) using a change of variables argument as in [Var15]. We note that (3.58) (see also (3.57)) answers in the affirmative an integrability question in section 1.4 of [Var15] but only for N sufficiently large. Finally we repeat arguments in Wolff - Lewis for $p > 2$, keeping careful track of the constants which now depend on N. In the last paragraph of our proof we use Lewis' "conjugate function argument" to get Theorem A for $1 < p < 2$.

To continue the proof of Theorem A for $1 < p < \infty, p \neq 2$, we note from p-harmonicity of F, and separation of variables, that it follows from (3.30), as in (3.3), that if $\sigma(x) = f'(x)/f(x)$ whenever $x \in [0, \pi/2]$ then

$$-\frac{d\sigma}{dx} = (p-1)\frac{(\gamma^2+\sigma^2)^2}{\gamma^2+(p-1)\sigma^2}, \quad \sigma(0) = 0 \text{ and } \sigma(\pi/2) = -\infty \tag{3.31}$$

where the last equality means as a limit from the left. Integrating (3.31) we get

$$\frac{p}{2(p-1)\gamma}\arctan(\sigma(x)/\gamma) - \frac{(p-2)\sigma(x)}{2(p-1)(\sigma^2(x)+\gamma^2)} = -x. \tag{3.32}$$

where we have used $\sigma(0) = 0$. Letting $x \to \pi/2$ it follows from (3.32) and $\sigma(\pi/2) = -\infty$ that

$$\gamma = \frac{p}{2(p-1)}. \tag{3.33}$$

Next we take the + sign and $\alpha = \pi/(2N)$ in (3.5). We obtain

$$1/N = 1 - (1-1/\lambda)(1 - \tfrac{(p-2)}{\lambda(p-1)})^{-1/2} \ . \tag{3.34}$$

Now since $\lambda > 0$ and $N > 1$, we see from (3.34) that $\lambda > 1$. Using this fact and taking logarithmic derivatives of the right-hand side of (3.34) with respect to $1/\lambda$, we find that it is decreasing as a function of $1/\lambda$. Thus $\lambda \to \infty$ as $N \to \infty$. Expanding (3.34) in powers of $1/\lambda$ we obtain

$$\begin{aligned} 1/N &= 1 - (1-1/\lambda)[1 + \tfrac{(p-2)}{2\lambda(p-1)} + O(1/\lambda^2)] \\ &= \frac{p}{2(p-1)\lambda} + O(1/\lambda^2) \ \text{ as } \ \lambda \to \infty. \end{aligned} \tag{3.35}$$

From (3.35) we conclude that

$$\frac{p}{2(p-1)}N = \gamma N = \lambda + O(1) \quad \text{as } N \to \infty \tag{3.36}$$

where γ is as in (3.33). Now suppose for the rest of the proof of Theorem A that $N \geq 10^{10}$ is a positive integer. Let

$$\lambda = \lambda(\pi/(2N), p) \quad \text{and} \quad \phi = \phi(\cdot, \pi/(2N), p)$$

be the value and function in (3.1) corresponding to $\alpha = \pi/(2N)$. Then $\phi(\pm\pi/(2N)) = 0$ so from Schwarz reflection with $\mathbb{R}$ replaced by $\theta = (2k-1)\pi/(2N)$ for $k = 1, \ldots, N$ (see Lemma 2.5) it follows that $z = re^{i\theta} \to r^\lambda \phi(\theta)$ extends to a p-harmonic function in $\mathbb{R}^2 \setminus \{0\}$, which is $2\pi/N$ periodic in the θ variable. Moreover since $\lambda > 1$ in (3.34), we claim that if $G(z) = G_N(z)$ denotes this extension and we define $G(0) = 0$, then G is p-harmonic in $\mathbb{R}^2$. To verify this claim given $q \in C_0^\infty(\mathbb{R}^2)$ and $\epsilon > 0$, we multiply q by a suitable cutoff so that $q = q_1 + q_2$ where q_1, q_2, are infinitely differentiable with q_1 having compact support in $B(0, \epsilon)$ while $q_2 \equiv 0$ in $B(0, \epsilon/2)$. Also $|\nabla q_1| \leq C\epsilon^{-1}$ where C is a positive constant that depends on q but is independent of ϵ. Then

(3.37)
$$\int_{\mathbb{R}^2} |\nabla G|^{p-2} \langle \nabla G, \nabla q \rangle dxdy = \int_{\mathbb{R}^2} |\nabla G|^{p-2} \langle \nabla G, \nabla q_1 \rangle dxdy \leq \hat{C} \epsilon^{(\lambda-1)(p-1)+1} \to 0$$

as $\epsilon \to 0$, where $\hat{C}$ has the same dependence as C. Our claim follows from (3.37), the definition of p-harmonicity, and the fact that functions in $C_0^\infty(\mathbb{R}^2)$ are dense in $W^{1,p}(\mathbb{R}^2)$.

Let $g(x) = g_N(x) = \phi(x/N)$ for $x \in \mathbb{R}$ where we now regard $\phi = \phi(\cdot, N)$, as defined on $\mathbb{R}$. Then

(3.38)

(α) $g = g_N(\cdot)$ is 2π periodic on $\mathbb{R}$, $g(x) = g(-x)$, $g(\pi/2 + x) = -g(\pi/2 - x)$, for $x \in \mathbb{R}$, and $g' \leq 0$ on $(0, \pi/2], g(\pm\pi/2) = 0$,

(β) $\max_{\mathbb{R}} |g| = 1 = g(0)$ and $c^{-1} \leq |g'(x)| + |g(x)| \leq c, x \in \mathbb{R}$, where $c = c(p)$.

Here (3.38) (α) and the left hand inequality in (3.38) (β), follow from the properties of ϕ listed after (3.1) and discussed after (3.5). To get the estimate from below in the right hand inequality of (3.38) (β) observe from Harnack's inequality and (3.38) (α) that we only need prove this inequality for x near $\pi/2$. Now comparing G to a linear function vanishing on the rays $\theta = \pm\pi/2$, using Lemma 2.5 with $\hat{u} = G, \hat{v}$ a linear function vanishing on the ray $\theta = \pi/2$, and taking limits as $z \to e^{i\pi/2}$, we deduce $c^{-1} \leq |g'(\pi/2)| \leq c$. The rest of (3.38) ($\beta$) follows from (3.36) and Lemma 2.4. We prove

Lemma 3.3. *For fixed p, $1 < p \neq 2 < \infty$, let f be as in (3.30) and $g = g_N$ as in (3.38). Then $g_N^{(k)}(x) \to f^{(k)}(x)$ as $N \to \infty$, uniformly on $\mathbb{R}$ for $k = 0, 1, 2, \ldots$.*

Proof of Lemma 3.3. Given $z = x + iy$, N a large positive integer, and G as defined below (3.36) let $\tilde{u}(z) = G(1 + iz/N)$ for $z \in \mathbb{R}^2$. From the definition of ϕ we see that if $|z| < N$,

$$\tilde{u}(z) = \tilde{u}_N(x + iy) = \left[(1 - y/N)^2 + (x/N)^2\right]^{\lambda/2} \phi\left(\arctan\left[\frac{x/N}{1 - y/N}\right]\right). \tag{3.39}$$

Let

$$H(z) := H_N(x + iy) = \left[(1 - y/N)^2 + (x/N)^2\right]^{\lambda/2},$$
$$K(z) := K_N(x + iy) = \phi\left(\arctan\left[\frac{x/N}{1 - y/N}\right]\right)$$

so $\tilde{u}(z) = H(z)\, K(z)$ when $|z| < N$. Fix $R > 100$. Then from L' Hospital's rule, (3.39), and (3.36), (3.38), we find uniformly for $z \in B(0, 2R)$, that

$$\text{(3.40)} \qquad \lim_{N\to\infty} H(z) = e^{-\gamma y}, \quad \lim_{N\to\infty} H_x(z) = 0, \quad \text{and} \quad \lim_{N\to\infty} H_y(z) = -\gamma\, e^{-\gamma y}.$$

From (3.36), (3.38), and the same argument as above we see that if N' is large enough then $|\tilde{u}_N|$ is uniformly bounded for $N \geq N'$, so from Lemmas 2.2 - 2.4, there exists $1 < M < \infty$ with

$$\text{(3.41)} \qquad \max_{B(0,4R)} (|\tilde{u}_N| + |\nabla \tilde{u}_N|) \leq M < \infty$$

for $N \geq N'$. From (3.41), (2.5) $(\hat{a})$, and Ascoli's theorem we see that a subsequence say $(\tilde{u}_{N_l}), (\nabla \tilde{u}_{N_l})$, converges uniformly in $B(0, 2R)$ to $u, \nabla u$, and u is p-harmonic in $B(0, 2R)$. Next we observe that $(|H_N|)$ is uniformly bounded below in $B(0, 4R)$ for $N \geq N'$ for N' large enough. Using this fact, (3.38), and (3.41) we see that

$$\text{(3.42)} \qquad |K_x|(z) = N^{-1} |\phi'| \left(\arctan \left[\frac{x/N}{1 - y/N} \right] \right) \frac{(1 - y/N)}{(1 - y/N)^2 + (x/N)^2} \leq M' < \infty$$

for $N \geq N', N \in \{N_l\}$. Choosing $y = 0$ in (3.42) and using (3.38), properties of arctan function we deduce

$$\text{(3.43)} \qquad |g_N'(\hat{x})| = N^{-1} |\phi'(\hat{x}/N)| \leq 2M' \quad \text{for } \hat{x} \in [-2R, 2R].$$

From (3.43) and the chain rule it follows easily that

$$\text{(3.44)} \qquad \lim_{l\to\infty} (K_{N_l})_y(z) = 0 \text{ uniformly in } B(0, 2R).$$

Thus in view of (3.44), (3.40), we get $u(z) = e^{-\gamma y} \nu(x)$ for $z \in B(0, 2R)$, so by uniqueness of f in (3.30) we have $\nu \equiv f$ in $B(0, 2R)$. Since every subsequence of $(\tilde{u}_N)$ converges uniformly to F and $R > 100$ is arbitrary we conclude Lemma 3.3 when $k = 0$.

Now from (3.38) (β) and uniform convergence of $(\nabla \tilde{u}_N)$ to $\nabla F \neq 0$ on compact subsets of $\mathbb{R}^2$, we deduce for $N \geq N'$ that $\nabla \tilde{u}_N \neq 0$ in $B(0, R)$. Then from (2.5) $(\hat{c})$ we see first that $\tilde{u}_N$ is infinitely differentiable in $B(0, R)$, for $N \geq N'$ and second from Schauder type arguments using (2.5) $(\hat{a}), (\hat{b})$, as in [GT01], that

$$\text{(3.45)} \qquad D^{(l)} \tilde{u}_N \to D^{(l)} F = D^{(l)} (e^{-\gamma y} f(x)), \quad \text{for } l = 0, 1, \ldots$$

uniformly on compact subsets of $\mathbb{R}^2$ where $D^{(l)}$ denotes an arbitrary l^{th} derivative in either x or y. To finish the proof of Lemma 3.3, we proceed by induction. Suppose by way of induction that Lemma 3.3 is valid for $k = l$, a non-negative integer. Using the product formula for derivatives and (3.39) we find that taking m partial derivatives in x on H gives an expression that is $O(N^{-m/2})$ when m is even and $O(N^{-(m+1)/2})$ when m is odd, for $z \in B(0, R)$ as $N \to \infty$. Also $n \leq l$ derivatives on K produces an expression that is $O(1)$ in $B(0, R)$ as $N \to \infty$, thanks to global p-harmonicity of F. Moreover in this $O(1)$ term the only way to get a non-zero term in the limit as $N \to \infty$ is to put all derivatives on ϕ, which then gives from the induction hypothesis a term converging to $f^{(n)}(x)$, as $N \to \infty$. From these observations and the product formula for derivatives we conclude that

$$\lim_{N\to\infty} \frac{\partial^{l+1}\tilde{u}_N(z)}{\partial x^{l+1}} = \lim_{N\to\infty} \left[(1+y/N)^2 + (x/N)^2\right]^{\lambda/2} g^{(l+1)}(x) = e^{-\gamma y} f^{(l+1)}(x). \tag{3.46}$$

From (3.46), L' Hospital's rule, and induction we see that Lemma 3.3 is true. □

In order to use Lemma 3.3 we briefly outline Wolff's proof of Theorem A for $p > 2$ and also the extension to $1 < p < 2$ of this theorem in [Lew88], tailored to 2π periodic rather than one periodic p-harmonic functions on $\mathbb{R}^2_+$. Let F, f, γ be as in (3.30), $2 < p < \infty$, and for $z \in \mathbb{R}^2$ set

$$\begin{aligned} A(z) &= A(x+iy) \\ &= ((f')^2 + \gamma^2 f^2)^{(p-4)/2} e^{-\gamma(p-2)y} \begin{pmatrix} \gamma^2 f^2 + (p-1)(f')^2 & -(p-2)\gamma f' f \\ -(p-2)\gamma f' f & (p-1)\gamma^2 f^2 + (f')^2 \end{pmatrix} (x). \end{aligned} \tag{3.47}$$

Note that A is 2π periodic in the x variable. Moreover, if $A(z) = (a_{ij}(z))$ for $z = x + iy \in \bar{\mathbb{R}}^2_+$ and $\xi = \xi_1 + i\xi_2 \in \mathbb{R}^2$, then

$$c^{-1}|\xi|^2 e^{-\gamma(p-2)y} \leq \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \leq c|\xi|^2 e^{-\gamma(p-2)y} \tag{3.48}$$

whenever $\xi \in \mathbb{R}^2$. Here (3.48) follows from (3.30), Harnack's inequality for F, as well as the analogue of (3.38) (β) for f. For the rest of this section we regard $\nabla\psi$ in rectangular coordinates, as a 2×1 column matrix whose top entry is ψ_x. Also, $\nabla\cdot$ is a 1×2 row matrix whose first or leftmost entry is $\frac{\partial}{\partial x}$. Finally if ξ is a 2×1 column matrix and ξ^t is the transpose of ξ, then $\langle A^*\nabla\psi, \xi\rangle = \xi^t A^*\nabla\psi$ whenever A^* is a 2×2 matrix with real entries.

Lemma 3.4. *Given p, $2 < p < \infty$, there exists $\zeta_i = \zeta_i(\cdot, p) \in C^\infty(\bar{\mathbb{R}}^2_+)$ for $i = 1, 2$, with $\nabla \cdot (A \nabla \zeta_i) = 0$ in $\mathbb{R}^2_+$ satisfying*

(3.49)

$(\bar{a})$ $\zeta_i(z + 2\pi) = \zeta_i(z)$, $z \in \mathbb{R}^2_+$, *and* $\max_{\mathbb{R}} |\zeta_i| = 1$.

$(\bar{b})$ $$\int_{S(2\pi)} \langle A \nabla \zeta_i, \nabla \zeta_i \rangle dxdy \approx \int_{S(2\pi)} e^{-\gamma(p-2)y} \, |\nabla \zeta_i|^2 dxdy < \infty \, .$$

$(\bar{c})$ *There exist $\delta = \delta(p) \in (0, 1]$ and $\mu_i \in \mathbb{R}$ with $\lim_{y \to \infty} \zeta_i(x + iy) = \mu_i$ for $x \in \mathbb{R}$ and $|\zeta_i(z) - \mu_i| = |\zeta_i(x + iy) - \mu_i| \leq 2 \, e^{-\delta y}$ for $y \geq 0$.*

$(\bar{d})$ $$\max_{\mathbb{R}^2_+} |\nabla \zeta_i| \leq M < \infty \quad \textit{and} \quad \int_{S(2\pi)} |\nabla \zeta_i|^q \, dxdy \leq M_q < \infty \textit{ for } q \in (0, \infty).$$

$(\bar{e})$ *There exist $y_0 \in (0, 1)$, c_+, and $c_{++} \geq 1$, with* $c_+^{-1} \leq \int_{-\pi}^{\pi} \frac{\partial \zeta_1}{\partial y}(x + iy) \, dx \leq c_+$ *and*
$$c_{++}^{-1} \leq \int_{-\pi}^{\pi} \langle A \nabla \zeta_2, e_1 \rangle (x + iy) \, dx \leq c_{++} \textit{ for } 0 \leq y \leq y_0, \textit{ where } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof of Lemma 3.4. The proof of Lemma 3.4 for ζ_1 and essentially also for $(\bar{a}) - (\bar{d})$ of ζ_2, is given in section 3 of [Wol07]. The proof of $(\bar{e})$ in Lemma 3.4 for ζ_2 is in [Lew88]. □

Next for for fixed p, $2 < p < \infty$ and $\lambda = \lambda(N, p)$, let $T^p(S(2\pi))$ be equivalence classes of functions h on $\mathbb{R}^2_+$ with $h(z + 2\pi) = h(z)$ for $z \in \mathbb{R}^2_+$, distributional partial derivatives ∇h, and norm,

$$\|h\|_{+,p} = \int_{S(2\pi)} e^{-(\lambda-1)(p-2)y/N} \, |\nabla h|^2 (x + iy) dxdy < \infty.$$

Also let $T^p_0(S(2\pi)) \subset T^p(S(2\pi))$ be functions in this space that can be approximated arbitrarily closely in the above norm by C^∞ functions in $T^p(S(2\pi))$ that vanish in an open neighbourhood of $\mathbb{R}$. For $g = g_N$ as in (3.38) and $z \in \bar{\mathbb{R}}^2_+$ set

$$\begin{aligned} \breve{A}(z) =& \breve{A}_N(x + iy) \\ =& ((g')^2 + (\lambda/N)^2 g^2)^{(p-4)/2} e^{-(\lambda-1)(p-2)y/N} \\ & \times \begin{pmatrix} (\lambda/N)^2 g^2 + (p-1)(g')^2 & -(p-2)(\lambda/N) g' g \\ -(p-2)(\lambda/N) g' g & (p-1)(\lambda/N)^2 g^2 + (g')^2 \end{pmatrix} (x). \end{aligned} \tag{3.50}$$

From (3.38) we observe that $\breve{A}(z + 2\pi) = \breve{A}(z)$ for $z \in \bar{\mathbb{R}}^2_+$ and from (3.36), (3.38), Lemma 3.3, that if $\breve{A}(z) = (\breve{a}_{ij}(z))$, then (3.48) holds with a_{ij} replaced by $\breve{a}_{ij}$. Moreover constants are independent of N, provided $N \geq N'$ and N' is large enough. Let $\breve{\zeta}_i = \breve{\zeta}_i(\cdot, N), i = 1, 2$, be the weak solution to $\nabla \cdot (\breve{A} \nabla \breve{\zeta}) = 0$ in $\mathbb{R}^2_+$ with $\breve{\zeta}_i - \zeta_i \in T^p_0(S(2\pi))$ and $\max_{\mathbb{R}^2_+} |\breve{\zeta}_i| = 1$. Existence and uniqueness of $\breve{\zeta}_1$, for example, follows from (3.48) for $\breve{A}$ and a slight modification of the usual calculus of variations minimization argument often given for bounded domains. To indicate this modification, let

$$I(h) = \int_{S(2\pi)} \sum_{i,j=1}^{2} \breve{a}_{ij}\, h_{x_i} h_{x_j} dxdy$$

where the functional $I(\cdot)$ is evaluated at functions in

$$\mathcal{F} := \{h : \ h \in T^p(S(2\pi)) \text{ with } h - \zeta_1 \in T_0^p(S(2\pi))\}.$$

For fixed $\rho >> 2\pi$, we can choose $h_j \in \mathcal{F}$ for $j = 1, 2, \dots$ so that

(3.51)
$$\begin{cases} \lim_{j\to\infty} I(h_j) = \inf\{I(h) : h \in \mathcal{F}\}, \\ h_j|_{S(2\pi)\cap B(0,\rho)} \to \tilde{h} = \tilde{h}(\cdot,\rho) \quad \text{strongly in } L^2(S(2\pi) \cap B(0,\rho)), \\ \text{each component of } \nabla h_j \text{ tends weakly to a function } k_i, i = 1, 2 \text{ with } k_i \text{ in } L^2(S(2\pi)). \end{cases}$$

Integrating by parts and using the definition of a distributional derivative, it follows from (3.51) that $\nabla\tilde{h}$ exists in the distributional sense and $\nabla h_j \rightharpoonup \nabla\tilde{h}$ weakly in $L^2(S(2\pi) \cap B(0,\rho))$. Using the Cantor diagonal argument we may suppose $\tilde{h}$ is independent of ρ so (3.51) holds for $0 < \rho < \infty$. From lower semicontinuity of the functional we conclude that $\tilde{h} \in \mathcal{F}$ and $I(\tilde{h}) = \min_{h\in\mathcal{F}} I(h)$. The rest of the proof is unchanged from the usual one for bounded domains.

Arguing as in section 3 of [Wol07] we deduce the existence of $\breve{\delta} \in (0, 1]$, depending only on p, and $\breve{\mu}_i \in [-1, 1]$ for $i = 1, 2$, satisfying

$$\max_{\mathbb{R}^2_+} |\breve{\zeta}_i| = 1, \quad \lim_{y\to\infty} \breve{\zeta}_i(x + iy) = \breve{\mu}_i, \quad \text{and} \quad |\breve{\zeta}_i\,(x + iy) - \breve{\mu}_i| \le 2e^{-\breve{\delta}y} \tag{3.52}$$

for $x \in \mathbb{R}$ and $y \ge 0$. From Lemma 3.3 and (3.36) we see that $D^{(l)}\breve{A} \to D^{(l)}A$ for $l = 0, 1, \dots,$ uniformly on $\bar{\mathbb{R}}^2_+$ as $N \to \infty$, where $D^{(l)}A$ denotes an arbitrary l-th partial derivative of A. From this observation, (3.52), and interior estimates for divergence form PDE with smooth coefficients (see [Eva10, Section 8.3]) we deduce that if l is a non-negative integer and $z = x + iy \in \mathbb{R}^2_+$, then

$$|D^{(l)}\breve{\zeta}_i(z)| \le \begin{cases} c_+(l)e^{-\breve{\delta}y} & \text{if } y \ge 1, \\ c_{++}(l)y^{-l} & \text{if } y < 1 \end{cases} \tag{3.53}$$

where constants depend only on l, and the ellipticity - smoothness constants for A provided $N \ge N'(l)$. We also note from smoothness of ζ_i and Theorem 8.30 in [GT01] that $\breve{\zeta}_i$ has a continuous extension to the closure of $\mathbb{R}^2_+$ with $\breve{\zeta}_i = \zeta_i$ on $\mathbb{R}, i = 1, 2$.

Next if $\breve{w}(z) = \theta - i \log r$ when $z = re^{i\theta}$, then $\breve{w}$ maps $\{z : |\theta| < \pi, e^{-1} < r < 1\}$, 1-1 and onto $S(2\pi) \cap \{\breve{w} = \breve{w}_1 + i\breve{w}_2 : 0 < \breve{w}_2 < 1\}$. Using periodicity of $\breve{\zeta}_i$, (3.53), and smoothness of ζ_i, we see that $\breve{\zeta}_i \circ \breve{w}$ extends to a function with l continuous derivatives in $B(0, 1) \setminus \bar{B}(0, e^{-1})$. Also this function has continuous boundary values equal to boundary functions with l continuous derivatives. Finally $\breve{\zeta}_i \circ \breve{w}$ is a solution to a uniformly elliptic divergence form PDE in this annulus with l continuous derivatives for which a boundary maximum principle holds. Smoothness and ellipticity constants can be chosen independent of N provided $N \geq \tilde{N}(l)$. Details to a more general case will be given after (3.55). We can now apply Theorems 6.14 and 6.6 in [GT01] to conclude for $l \geq 3$ that $\breve{\zeta}_i \circ \hat{w}$ has $l-1$ continuous derivatives in $\bar{B}(0,1) \setminus B(0, e^{-1})$ with L^∞ norms bounded independently of N, provided that $N \geq \tilde{N}(l)$. From this conclusion, (3.52), (3.53), elliptic regularity theory, and Ascoli's theorem it follows that

(3.54)
$$\lim_{N\to\infty} D^{(k)}\breve{\zeta}_i(\cdot, N) \to D^{(k)}\zeta_i \text{ uniformly in } \bar{\mathbb{R}}^2_+ \text{ as } N \to \infty, \text{ for } k = 0, 1, \dots, l-1.$$

In view of (3.54) and (3.49) $(\bar{d}), (\bar{e})$, we see for N' large enough and $N \geq N'(l)$, that

(3.55)

$$(\alpha) \quad \max_{\mathbb{R}^2_+} |\nabla \breve{\zeta}_i| \leq \breve{M} < \infty \text{ and } \int_{S(2\pi)} |\nabla \breve{\zeta}_i|^q dxdy \leq \breve{M}_q < \infty \text{ for } q \in (0, l-1).$$

$$(\beta) \quad c_*^{-1} \leq \int_{-\pi}^{\pi} \tfrac{\partial \breve{\zeta}_1}{\partial y}(x+iy)\, dx \leq c_* \text{ for } 0 \leq y \leq y_0,$$

$$(\gamma) \quad c_{**}^{-1} \leq \int_{-\pi}^{\pi} \langle \breve{A} \nabla \breve{\zeta}_2, e_1 \rangle (x+iy)\, dx \leq c_{**} \text{ for } 0 \leq y \leq y_0.$$

Constants in (3.55) are independent of $N \geq N'(l)$ and $\breve{M}, \breve{M}_q, c_*, c_{**}$, depend only on p, as well as the corresponding constants for ζ_1, ζ_2, in (3.47) $(\bar{d}), (\bar{e})$ of Lemma 3.4.

To continue the proof of Theorem A for $1 < p \neq 2 < \infty$, given $z = re^{i\theta} \in B(0,1)$, $N \geq N'$, we follow [Var15] and let $w = w(z) = N\theta - iN \log r \in \mathbb{R}^2_+$. Then w maps $\{z = re^{i\theta} : 0 < r < 1, |\theta| < \pi/N\}$ one-one and onto $S(2\pi)$. If $z = x + iy = re^{i\theta}$, put $\tilde{A}(z) = \breve{A}(w(z))$, when $z \in B(0,1) \setminus \{0\}$, and $\tilde{A}(0) = 0$. We note from (3.50) for $z \in B(0,1)$ that

$$\begin{aligned} \tilde{A}(z) &= \breve{A}_N(N\theta - iN\log r) \\ &= \tau(re^{i\theta}) \begin{pmatrix} \lambda^2\phi^2 + (p-1)(\phi')^2 & -(p-2)\lambda\phi'\phi \\ -(p-2)\lambda\phi'\phi & (p-1)\lambda^2\phi^2 + (\phi')^2 \end{pmatrix} (\theta) \end{aligned} \tag{3.56}$$

where

$$\tau(re^{i\theta}) \;=\; N^{(2-p)} r^{(\lambda-1)(p-2)} ((\phi')^2 + (\lambda)^2\phi^2)^{(p-4)/2}(\theta).$$

Here $\lambda = \lambda(2\pi/N)$ and $\phi = \phi(\cdot, N)$ is the extension of $\phi = \phi(\cdot, 2\pi/N)$ in (3.1) to $\mathbb{R}$. Let $\tilde{\zeta}_i(z) = \breve{\zeta}_i(w(z))$ for $z \in B(0,1) \setminus \{0\}$ and observe that $\tilde{\zeta}_i$ is $2\pi/N$ periodic in the θ variable. From the chain rule and (3.52) - (3.55) we see for $i = 1, 2$, that

$$\max_{\partial B(0,r)} [(r/N)\, |\nabla \tilde{\zeta}_i| + |\tilde{\zeta}_i - \breve{\mu}_i|] \leq \tilde{M}\, r^{N\breve{\delta}} \quad \text{for } 0 < r \leq 1, \tag{3.57}$$

where $\tilde{M}$ is independent of N for $N \geq N'$. Put $\tilde{\zeta}_i(0) = \breve{\mu}_i$. Then (3.57), (3.55) ($\alpha$), and the chain rule imply that

$$\int_{B(0,1)} |\nabla \tilde{\zeta}_i|^q \, r dr d\theta \leq N^{q-1} \tilde{M}_q < \infty \quad \text{for } q \in (0, N\delta/2), \tag{3.58}$$

where $\tilde{M}_q$ is independent of $\tilde{\zeta}_i$ for $N \geq N'$. Also from (3.55) (β), (γ), (3.56), we deduce for $N \geq N'$, that

$$\begin{aligned} &(+) \quad (2c_*)^{-1} N \leq \int_{-\pi}^{\pi} \tfrac{\partial \tilde{\zeta}_1}{\partial r}(re^{i\theta}) \, d\theta \leq 2c_* N \quad \text{for } 1 - \frac{y_0}{2N} \leq r \leq 1, \\ &(++) \quad (2c_{**})^{-1} N \leq \int_{-\pi}^{\pi} \langle \tilde{A} \nabla' \tilde{\zeta}_2, \, e_1 \rangle (re^{i\theta}) \, d\theta \leq 2c_{**} N \quad \text{for } 1 - \frac{y_0}{2N} \leq r \leq 1, \end{aligned} \tag{3.59}$$

where

$$\nabla' \tilde{\zeta}_2(re^{i\theta}) = \begin{pmatrix} r^{-1} \frac{\partial \tilde{\zeta}_2}{\partial \theta} \\ -\frac{\partial \tilde{\zeta}_2}{\partial r} \end{pmatrix}.$$

Next we observe from $\nabla \cdot (\breve{A} \nabla \breve{\zeta}_i) = 0$ for $i = 1, 2$, and the change of variables formula that if $\chi \in C_0^\infty(B(0,1) \setminus \{0\})$ then

$$I = \int_{B(0,1)} \langle \tilde{A} \nabla' \tilde{\zeta}_i, \nabla' \chi \rangle r dr d\theta = 0 \tag{3.60}$$

From (3.57) and the same limiting argument as in (3.37), we see that (3.60) still holds if $\chi \in C_0^\infty(B(0,1))$. Finally, if $\bar{v}(z) = \bar{v}(re^{i\theta}) = r^\lambda \phi(\theta, N)$, and

$$\bar{A}(z) = |\nabla \bar{v}|^{p-4} \begin{pmatrix} (p-1)\bar{v}_x^2 + \bar{v}_y^2 & (p-2)\bar{v}_x \bar{v}_y \\ (p-2)\bar{v}_x \bar{v}_y & (p-1)\bar{v}_y^2 + \bar{v}_x^2 \end{pmatrix} (z) \tag{3.61}$$

when $z = re^{i\theta} \in B(0,1)$, then (3.60) can be rewritten as

$$I = \int_{B(0,1)} \langle \bar{A} \nabla \tilde{\zeta}_i, \nabla \chi \rangle dx dy = 0 \tag{3.62}$$

so $\nabla \cdot (\bar{A} \nabla \tilde{\zeta}_i) = 0$ in $B(0,1)$. Here (3.62) can be verified by using the chain rule to switch (3.60) from polar to rectangular coordinates but also as in [Var15] by noticing that if $a(\cdot, \epsilon) = \bar{v} + \epsilon \tilde{l}$ for $\tilde{l} \in \{\tilde{\zeta}_i, i = 1, 2\}$, then

$$\frac{\partial}{\partial \epsilon} \left(\nabla \cdot (|\nabla a|^{p-2} \nabla a) \right)_{\epsilon = 0} = \nabla \cdot (\bar{A} \, \nabla \tilde{l}) = 0.$$

The left hand side of this equation can be evaluated independent of the coordinate system, so letting $\bar{v}_\xi$ and $\bar{v}_\eta$ denote directional derivatives of $\bar{v}$ at z, where $\xi = ie^{i\theta}$ and $\eta = -e^{i\theta}$, we obtain

$$\bar{v}_\xi = r^{-1} \bar{v}_\theta \quad \text{and} \quad \bar{v}_\eta = -\bar{v}_r.$$

Using this fact, replacing $\bar{v}_x$ and $\bar{v}_y$ in (3.61) and (3.62) by $\bar{v}_\xi$ and $\bar{v}_\eta$, and computing $\nabla \tilde{\zeta}_i$ and $\nabla \chi$, in the ξ and η coordinate system, we arrive at (3.60). Moreover, (3.59)

(++) can be rewritten as

$$(2c_{**})^{-1}\lambda^{p-2}N \leq \int_{-\pi}^{\pi} \langle \bar{A}\,\nabla\tilde{\zeta}_2,\, e_\theta\rangle(re^{i\theta})\,d\theta \leq 2c_{**}\lambda^{p-2}N \tag{3.63}$$

for $1-\frac{y_0}{2N}\leq r\leq 1$ where

$$e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Armed with (3.57)-(3.59) and (3.62), we can now essentially copy the proof of Lemmas 3.16-3.19 in [Wol07] for $2<p<\infty$ and the argument leading to (12)-(13) in [Lew88] for $1<p<2$. Thus the reader should have these papers at hand. Since constants now depend on N, we briefly indicate the slight changes in lemmas and displays. In the proof we let $C\geq 1$ be a constant, not necessarily the same at each occurrence, which may depend on other quantities besides p, such as c_*, c_{**}, but is independent of N and ϵ, for $N\geq N'$, $0<\epsilon\leq\epsilon'$. Given p, $1<p\neq 2<\infty$, and $\epsilon>0$ small, for $i=1,2$, let $k_i=k_i(\cdot,N)$ be the p-harmonic function in $B(0,1)$ with $k_i=\bar{v}+\epsilon\,\tilde{\zeta}_i$ on $\partial B(0,1)$ in the $W^{1,p}$ Sobolev sense. From Lemma 2.3 we see that k_i is Hölder continuous in $\bar{B}(0,1)$. Also from the boundary maximum principle for p-harmonic functions we deduce for $z=re^{i\theta}\in\bar{B}(0,1)$ that

$$k_j(re^{i\theta}) = k_j(re^{i(\theta+2\pi/N)}) \quad \text{for } j=1,2.$$

We note that f, v, and g in Wolff's notation in [Wol07] corresponds to our $\bar{v}$, $\tilde{\zeta}_i$, and k_i respectively. If $q\in W_0^{1,p}(B(0,1))$ and $2<p<\infty$ then the analogue of the display in Lemma 3.16 of [Wol07] in our notation relative to $B(0,1)$ is

$$\left|\int_{B(0,1)}\langle\nabla q, \nabla(\bar{v}+\epsilon\tilde{\zeta}_i)\rangle|\nabla(\bar{v}+\epsilon\tilde{\zeta}_i)|^{p-2}dxdy\right| \leq C\epsilon^{\sigma}\,N^{(p-1)/p'}\,\||\nabla q|\|_p \tag{3.64}$$

for $N\geq N'$, where $p'=p/(p-1)$, $\sigma=\min(2,p-1)$, and $\||\nabla q|\|_p$ is the Lebesgue p norm of $|\nabla q|$ on $B(0,1)$. To get this estimate we use Hölder's inequality, (3.57), and our knowledge of $\bar{v}$ to estimate the term in brackets in display (3.17) of [Wol07].

Lemma 3.18 of this paper follows easily from Lemma 3.16 with $q=\bar{v}+\epsilon\,\tilde{\zeta}_i-k_i$ for $i=1,2$, and now reads,

$$\||\nabla\bar{v}+\epsilon\nabla\tilde{\zeta}_i|\|_p^p \leq \||\nabla k_i|\|_p^p + C\epsilon^{\sigma}\,N^{(p-1)/p'}\,\||\nabla k_i-\nabla\bar{v}-\epsilon\nabla\tilde{\zeta}_i|\|_p \tag{3.65}$$

where all norms are relative to $B(0,1)$.

The new version of the conclusion in Lemma 3.19 of [Wol07] is: *There exists* $\epsilon'\in(0,1/2)$ *and* $C\geq 1$ *such that,*

$$\int_{B(0,1)\setminus B(0,1-\frac{y_0}{2N})}|\nabla(\bar{v}+\epsilon\tilde{\zeta}_i)-\nabla k_i|\,dxdy \leq C\epsilon^{\tilde{\tau}} \tag{3.66}$$

for $0<\epsilon\leq\epsilon'$ *where* $\tilde{\tau}=\sigma p'/2>1$. To get this new conclusion first replace ϵ^σ by $\epsilon^\sigma N^{(p-1)/p'}$ and S^λ by $B(0,1)$, in the last display on page 392 of [Wol07], as follows from the new version of Lemma 3.18. Second argue as in Wolff to get the top display on page 393 of his paper with $\epsilon^{\sigma/(p-1)}$ replaced by $\epsilon^{\sigma/(p-1)}N^{1-1/p}$. Using this display one gets the second display from the top on page 393 with $\epsilon^{\sigma p'}$ replaced by $N^{(p-1)}\epsilon^{\sigma p'}$

and f, v, g replaced by $\bar{v}, \tilde{\zeta}_i, k_i$, respectively. To get the next display choose $0 < \epsilon'$, in addition to the above requirements, so that

$$|\nabla \bar{v} + \epsilon \nabla \tilde{\zeta}_i|^{p-2} \geq \hat{C}^{-1} N^{p-2} \tag{3.67}$$

for $N \geq N'$, $0 < \epsilon \leq \epsilon'$, and $1 - \frac{y_0}{2N} \leq r \leq 1$. This choice is possible as we see from (3.36), (3.38) (β), and (3.57). We can now estimate the integral in (3.66), using Schwarz's inequality and (3.67) as in [Wol07]. We get the conclusion of Lemma 3.19 in Wolff's paper [Wol07], except the integral in this display is now taken over $B(0,1) \setminus B(0, 1 - \frac{y_0}{2N})$. Now (3.66), (3.59) $(+)$, and the fact that $\bar{v}$ has average 0 on circles with center at the origin, are easily seen to imply as in [Wol07] that

$$\int_{-\pi}^{\pi} k_1((1 - \tfrac{y_0}{2N})e^{i\theta}) d\theta - \int_{-\pi}^{\pi} k_1((1 - \tfrac{y_0}{4N})e^{i\theta}) d\theta \geq C^{-1}\epsilon \tag{3.68}$$

provided $0 < \epsilon \leq \epsilon'$ and ϵ' is small enough. From the triangle inequality we conclude that there is a $d \in \{1 - \frac{y_0}{2N}, 1 - \frac{y_0}{4N}\}$, for which if $\tilde{V}(z) = k_1(dz) - k_1(0)$ for $z \in B(0,1)$, then either $V = \tilde{V}$ or $V = -\tilde{V}$ satisfies (1.3) (c) in Theorem A. Also the usual calculus of variations argument giving k_1 and the maximum principle for p-harmonic functions, as well as either (3.57) or (3.58) and (3.38) (β), give (1.3) $(a), (b)$ in Theorem A with c replaced by C. Finally (1.3) (d) of Theorem A follows from these inequalities and Lemma 2.4. The proof of Theorem A is now complete for $2 < p < \infty$.

To avoid confusion we prove Theorem A, for $1 < p' < 2$, rather than $1 < p < 2$, where as usual $p' = p/(p-1)$ and $p > 2$. To do this we first replace the right-hand side in display (13) of [Lew88] by $C\epsilon^{\tilde{\tau}} N^{p-1}$, as we deduce in view of the new second display from the top on page 393 of [Wol07]. Second we use (13) and Schwarz's inequality in the second line of display (12) in [Lew88] (with Q replaced by $B(0,1) \setminus B(0, 1 - \frac{y_0}{2N})$, $q = p$), and either (3.57) or (3.58) to get

$$\left| \int_{B(0,1)\setminus B(0,1-\frac{y_0}{2N})} r^{-2} \left[|\nabla k_2|^{(p-2)} (k_2)_\theta - |\nabla \bar{v} + \epsilon \nabla \tilde{\zeta}_2|^{(p-2)} (\bar{v} + \epsilon \tilde{\zeta}_2)_\theta \right] dxdy \right|^2 \leq C N^{2(p-2)} \epsilon^{2\tilde{\tau}}, \tag{3.69}$$

where $\tilde{\tau}$ is as in (3.66). Taking square roots in (3.69), using (3.63), the fact that $|\nabla \bar{v}|^{p-2} \bar{v}_\theta$ has average 0 on circles with center at the origin, and arguing as in [Lew88] we get

$$\int_{B(0,1-\frac{y_0}{4N})\setminus B(0,1-\frac{y_0}{2N})} r^{-1} |\nabla k_2|^{p-2}(re^{i\theta}) (k_2)_\theta (re^{i\theta}) dr d\theta \geq C^{-1} N^{p-2} \epsilon \tag{3.70}$$

for $N \geq N'$ and $0 < \epsilon \leq \epsilon'$. Let k be the p'-harmonic function in $B(0,1)$ with $k(0) = 0$ satisfying

$$k_r = N^{2-p} r^{-1} |\nabla (k_2)|^{p-2} (k_2)_\theta \quad \text{and} \quad r^{-1} k_\theta = -N^{2-p} |\nabla k_2|^{p-2} (k_2)_r.$$

Existence of k follows from simple connectivity of $B(0,1)$ and the usual existence theorem for exact differentials. Then (3.70) implies

$$\int_{-\pi}^{\pi} k(1-\tfrac{y_0}{4N}e^{i\theta})d\theta - \int_{-\pi}^{\pi} k(1-\tfrac{y_0}{2N}e^{i\theta})d\theta \geq C^{-1}\epsilon \tag{3.71}$$

for $N \geq N'$ and $0 < \epsilon \leq \epsilon'$. Finally (3.71) and a similar argument to the one from (3.68) on in the first case considered, give Theorem A for $1 < p' < 2$. This completes the proof of Theorem A for $1 < p \neq 2 < \infty$.

4. Proof of Theorem B

In this section we first state Wolff's main lemma for applications (Lemma 1.6 in [Wol07]), in the unit disk setting and then use it to prove Theorem B. The proof of Theorem B is essentially unchanged from Wolff's proof of Theorem 1.1. However for the readers convenience we outline his proof, indicating how to resolve a few problems in converting this proof from a half space to $B(0,1)$. We also note that if V as in Theorem A is $2\pi/N$ periodic in the θ variable, where $N = kN_0$, $k = 1, 2, \dots,$ then V is $2\pi/k$ periodic in this variable. Also since N_0 depends only on p in the wider context discussed below the statement of Theorem A in section 1, we may as well assume $N_0 = 1$. Finally in the proof of Theorem B, we let $c \geq 1$, denote a positive constant depending only on p in this wider context.

4.1. **Main Lemma for applications of Theorem A.** Given $h \in W^{1,p}(B(0,1))$, let $\hat{h}$ be the p-harmonic function in $B(0,1)$ with boundary values $\hat{h} = h$ on $\partial B(0,1)$ in the $W^{1,p}(B(0,1))$ Sobolev sense. We also let $\|h\breve{\|}$ denote the Lipschitz norm of h restricted to $\partial B(0,1)$ and $\|h\|_\infty = \max_{\partial B(0,1)} |h|$. Next we state an analogue Lemma 1.6 in [Wol07].

Lemma 4.1. *Let $1 < p < \infty$. Define $\alpha = 1 - 2/p$ if $p \geq 2$, and $\alpha = 1 - p/2$, if $p < 2$. Let $\epsilon > 0$ and $0 < M < \infty$. Then there are $A = A(p,\epsilon,M) > 0$ and $\nu_0 = \nu_0(\epsilon,p,M) < \infty$, such that if $\nu > \nu_0 \geq 1$ is an integer, f, g, and q are periodic on $\partial B(0,1)$ in the θ variable with periods, 2π, 2π, and $2\pi\nu^{-1}$, respectively and if*

$$\max(\|f\|_\infty, \|g\|_\infty, \|q\|_\infty, \|f\breve{\|}, \|g\breve{\|}, \nu^{-1}\|q\breve{\|}) \leq M, \tag{4.1}$$

then for $z = re^{i\theta} \in B(0,1)$,

$$|\widehat{qf+g}(re^{i\theta}) - f(e^{i\theta})\hat{q}(re^{i\theta}) - g(e^{i\theta})| < \epsilon \quad \text{for } 1 - r < A\nu^{-\alpha}. \tag{4.2}$$

If, in addition, $\hat{q}(0) = 0$, then

$$|\widehat{qf+g}\,((1 - A\nu^{-\alpha})e^{i\theta}) - g(e^{i\theta})| < \epsilon \tag{4.3}$$

and

$$|\widehat{qf+g}(re^{i\theta}) - \hat{g}(re^{i\theta})| < \epsilon \quad \text{if } r < 1 - A\nu^{-\alpha}. \tag{4.4}$$

Proof. Lemma 4.1 is just a restatement for $B(0,1)$, of Lemma 1.6 in [Wol07]. To briefly outline the proof of Lemma 4.1, we note that Lemma 1.4 in [Wol07] is used to prove Lemma 1.6 in [Wol07]. This lemma relative to $B(0,1)$ states for fixed p, $1<p<\infty$, that if u and v are p-harmonic in $B(0,1)$, bounded, $u,v\in W^{1,p}(B(0,1))$, and if $u\leq v$ on $\{e^{i\theta}:|\theta-\theta_0|\leq 2\eta\}$ for $0<\eta<1/4$ in the $W^{1,p}(B(0,1))$ Sobolev sense, then for $0<t\leq 1/2$,

(4.5)
$$\int_{1-t}^{1}\int_{\theta_0-\eta}^{\theta_0+\eta}|\nabla(u-v)^+|\,rdrd\theta\leq c\eta^{-1}t^{1/p'}(\||\nabla u|\|_p+\||\nabla v|\|_p)^{\alpha}\,[\max_{\partial B(0,1)}(u-v)^+]^{1-\alpha}$$

where $a^+=\max(a,0)$. It follows from a Caccioppoli type inequality for $(u-v)^+$ that (4.5) holds.

To begin the proof of Lemma 4.1, if $z=re^{i\theta}\in B(0,1)$, let

$$J(re^{i\theta})=\widehat{qf+g}(re^{i\theta})-\hat{q}(re^{i\theta})f(e^{i\theta})-g(e^{i\theta}).$$

The first step in the proof of Lemma 4.1 is to show for given $\beta\in(0,10^{-5})$ that there is a $A=A(p,\epsilon,M,\beta)$ for which (4.4) holds (so $|J|(te^{i\theta})<\epsilon$) when $\beta\nu^{-1}<1-t<A\nu^{-\alpha}$, for $\nu\geq\nu_0=\nu_0(p,\epsilon,M,\beta)$. Indeed if $J(te^{i\theta_0})>\epsilon$, then (4.1), Lemmas 2.2, 2.3, and invariance of p-harmonic functions under a rotation, are used in [Wol07] to show that if $\eta=\frac{\epsilon}{10^5(M^2+M)}$, then there is a set $W\subset\{te^{i\theta}:|\theta-\theta_0|\leq\eta\}$ of Lebesgue measure $\delta\geq\rho\beta\eta/100$, where $\rho=\rho(p,M,\epsilon)$, with

$$\widehat{qf+g}(te^{i\theta})-\hat{q}(te^{i\theta})f(e^{i\theta_0})-g(e^{i\theta_0})>\epsilon/2. \tag{4.6}$$

Also (4.1) and the choice of η yield

$$|\widehat{qf+g}(e^{i\theta})-\hat{q}(e^{i\theta})f(e^{i\theta_0})-g(e^{i\theta_0})|<\epsilon/200\quad\text{when } |\theta-\theta_0|<\eta. \tag{4.7}$$

Using (4.7), our knowledge of W, and (4.5) it follows that if $u(re^{i\theta})=\widehat{qf+g}(re^{i\theta})$ and $v(re^{i\theta})=q(re^{i\theta})f(e^{i\theta_0})+g(e^{i\theta_0})$, then

$$\begin{aligned}\delta\epsilon/4&\leq\int_0^t\int_{\theta_0-\eta}^{\theta_0+\eta}|\nabla(u-v)^+|\,rdrd\theta\\&\leq c(M,\epsilon)t^{1/p'}(\||\nabla u|\|_p+\||\nabla v|\|_p)^{\alpha}\\&\leq c'(M,\epsilon)t^{1/p'}\nu^{\alpha/p'}.\end{aligned} \tag{4.8}$$

The estimate on $\||\nabla u|\|_p$ and $\||\nabla v|\|_p$, in the second line of (4.8) follows from (4.1) and the minimization property of p-harmonic functions using, for example,

$$\psi(re^{i\theta})=u(e^{i\theta})\chi(r)\quad\text{where}\quad\chi\in C_0^{\infty}(1-2/\nu,1+2/\nu)$$

with $\psi=1$ on $(1-1/\nu,1+1/\nu)$ and $|\nabla\psi|\leq c\nu$. Now (4.8) yields after some arithmetic that $t>\tilde{A}(\epsilon,M,\beta)\nu^{-\alpha}$. Thus (4.2) of Lemma 4.1 is true when $\beta\nu^{-1}<1-r<A\nu^{-\alpha}$, subject to fixing $\beta=\beta(\epsilon,M)$. To do this we apply (2.3) of Lemma 2.3 with $\hat{v}=\widehat{qf+g}$,

q, and with $\rho = \beta^{1/2}\nu^{-1}$, $\sigma = 1$, $M' = \nu$, to get for $1 - r < \beta\nu^{-1}$,

$$|J(re^{i\theta}) - J(e^{i\theta})| \leq c(M)\left(\nu\,(\beta^{1/2}\nu^{-1}) + (\frac{\beta\nu^{-1}}{\beta^{1/2}\nu^{-1}})^{\sigma_1}\right) \leq c'(M)\,\beta^{\sigma_1/2}. \tag{4.9}$$

Choosing $\beta = \beta(\epsilon, M) > 0$ small enough and then fixing β we obtain (4.2) for $1 - r < \beta\nu^{-1}$.

To prove (4.3) we note from (2.9) of Lemma 2.7 that

$$|q(re^{i\theta}) - q(0)| \leq cMr^{\nu/c} \tag{4.10}$$

where $c = c(p)$. Using (4.10) with $q(0) = 0$, $r = 1 - A\nu^{-\alpha}$, and choosing ν_0, still larger if necessary we get (4.3). Now (4.4) follows from (4.3) and (2.3) of Lemma 2.3 with $\hat{v} = \hat{g}$ and $\rho = A\nu^{-\alpha/2}$ in the same way as in the proof of (4.9) for ν_0 large enough. This finishes the sketch of proof of Lemma 4.1. $\square$

4.2. **Lemmas on Gap Series.** The examples in Theorem B will be constructed using Theorem A as the uniform limit on compact subsets of $B(0,1)$ of a sequence of p-harmonic functions in $B(0,1)$, whose boundary values are partial sums of Φ_j in Theorem B with periods $2\pi/N_j$ where $N_{j+1}/N_j >> 1$. Lemma 4.1 will be used to make estimates on this sequence. Throughout this subsection we let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}$. We begin with

Lemma 4.2. *For $j = 1, 2, \ldots$, let ψ_j be Lipschitz functions defined on $\partial B(0,1)$ with*

$$\int_{-\pi}^{\pi} \psi_j(e^{i\theta})d\theta = 0 \quad \textit{and} \quad \|\psi_j\|_\infty + \|\psi_j\check{\|} \leq C_1 < \infty. \tag{4.11}$$

For $j = 1, 2, \ldots$, let $(N_j)_1^\infty$ be a sequence of positive integers with $N_{j+1}/N_j \geq 2$. Also let $(a_j)_1^\infty$ be a sequence of real numbers with $\sum_{j=1}^{\infty} a_j^2 < \infty$.

If

$$s^*(e^{i\theta}) := \sup_k \left|\sum_{j=1}^{k} a_j\psi_j(e^{iN_j\theta})\right|$$

then

$$\int_{-\pi}^{\pi} (s^*)^2(e^{i\theta})d\theta \leq c\,C_1^2 \sum_{j=1}^{\infty} a_j^2 \tag{4.12}$$

where c is an absolute constant. Consequently,

$$\begin{aligned} &(a) \quad s(e^{i\theta}) := \lim_{k\to\infty} \sum_{j=1}^{k} a_j\psi_j(e^{iN_j\theta}) \quad \textit{exists for almost every } \theta \in [-\pi, \pi], \\ &(b) \quad |\{\theta \in [-\pi, \pi] : s^*(e^{i\theta}) > \lambda\}| \leq \frac{c\,C_1^2}{\lambda^2}. \end{aligned} \tag{4.13}$$

Proof. Using elementary properties of Fourier series (see [Zyg68]) and $\||\frac{d\psi_j}{d\theta}|\|_\infty \leq C_1$ we find that

$$\psi_j(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_{jn} e^{in\theta} \text{ where } b_{j0} = 0 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} n^2 b_{jn}^2 \leq c\, C_1^2. \tag{4.14}$$

Now

$$s^*(e^{i\theta}) \leq \sum_{n=-\infty}^{\infty} \sup_k \left| \sum_{j=1}^{k} a_j b_{jn} e^{inN_j\theta} \right| = \sum_{n=-\infty}^{\infty} l_n^*(e^{i\theta}) \tag{4.15}$$

where l_n^* is the maximal function of $\sum_{j=1}^{\infty} a_j b_{jn} e^{inN_j\theta}$. It is well known (see [Zyg68]) that

$$\int_{-\pi}^{\pi} (l_n^*)^2(e^{i\theta}) d\theta \leq c' \sum_{j=1}^{\infty} (a_j b_{jn})^2. \tag{4.16}$$

Using (4.15), (4.16), and Cauchy's inequality we get

$$\begin{aligned} \int_{-\pi}^{\pi} (s^*)^2(e^{i\theta}) d\theta &\leq \left(\sum_{n=-\infty}^{\infty} (\int_{-\pi}^{\pi} (l_n^*)^2(e^{i\theta}) d\theta)^{1/2} \right)^2 \leq c' \left(\sum_{n=-\infty}^{\infty} (\sum_{j=1}^{\infty} a_j^2 b_{jn}^2)^{1/2} \right)^2 \\ &\leq 2c' (\sum_{n=1}^{\infty} n^{-2}) \sum_{j=1}^{\infty} \sum_{n=-\infty}^{\infty} (a_j\, n b_{jn})^2 \leq c\, C_1^2 \sum_{j=0}^{\infty} a_j^2. \end{aligned} \tag{4.17}$$

Therefore, (4.12) is valid. Now (4.13) follows from standard arguments, using (4.12) (see [Zyg68]).

To prove Theorem B, let N_j be a sequence of positive integers with N_{j+1}/N_j a positive integer > 2. Let Φ_j be the p-harmonic function in Theorem A with period $2\pi/N_j$ and set $\tilde{\Phi}_j = \frac{\Phi_j}{\|\Phi_j\|_\infty}$. Also for $\theta \in \mathbb{R}$ and $j = 1, 2, \ldots$, we set

$$\begin{aligned} &\phi_j(e^{i\theta}) = \tilde{\Phi}_j(e^{i\theta/N_j}), \\ &d_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Phi}_j(e^{i\theta}) d\theta, \\ &\psi_j = \phi_j - d_j. \end{aligned} \tag{4.18}$$

Note from Theorem A that $c^{-1} \leq d_j \leq 1$ and that ψ_j satisfies (4.11) of Lemma 4.2 for $j = 1, 2, \ldots,$. For $j = 1, 2, \ldots$, set

$$G_j := \{[\pi k/N_j, \pi(k+2)/N_j],\ k \text{ an integer}\}$$

and let $\{L_j(e^{i\theta})\}$ be continuous functions on $\partial B(0,1)$ satisfying

(4.19)

$$\begin{cases} L_1 \equiv 1,\ 0 < L_{j+1} \leq L_j, \text{ and} \\ L_{j+1}/L_j \text{ considered as a function of } \theta \text{ on } \mathbb{R} \text{ is linear on the intervals in } G_j. \end{cases}$$

Let $\sigma(e^{i\theta})$ for $\theta \in \mathbb{R}$, be the formal series defined by

$$\sigma(e^{i\theta}) := R + \sum_{j=1}^{\infty} a_j\, L_j(e^{i\theta})\, \tilde{\Phi}_j(e^{i\theta}) \quad \text{and} \quad \tilde{s}(e^{i\theta}) := \sum_{j=1}^{\infty} a_j \tilde{\Phi}_j(e^{i\theta}) \tag{4.20}$$

with

$$0 \le |R| \le 1 \quad \text{and} \quad \sum_{j=1}^{\infty} a_j^2 < 1\,. \tag{4.21}$$

Finally let $\tilde{s}_n$ and σ_n denote corresponding n-th partial sums of $\tilde{s}$ and σ respectively.

Given $I \in G_j$, let $\tilde{I}$ denote the interval with the same center as I and three times its length. Using the gap assumption on (N_j), (4.19), and induction we find that

$$\|L_{j+1}\breve{\|} \le cN_j. \tag{4.22}$$

Using the gap assumption on (N_j), Theorem A, and (4.22), (4.21), we deduce for $n = 1, 2, \ldots$, that

$$\beta_n = N_n^{-1}(\|s_n\breve{\|} + \|\sigma_n\breve{\|}) \le c \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 0 \text{ as } n \to \infty. \tag{4.23}$$

Moreover, from (4.13)(b) of Lemma 4.2 and (4.18) we have

$$|\{\theta \in [-\pi, \pi] : \sup_n |\tilde{s}_n(e^{i\theta}) - \sum_{j=1}^{n} d_j a_j| > \lambda\}| \le c\,\lambda^{-2} \sum_{j=1}^{\infty} a_j^2. \tag{4.24}$$

First let $R = 0$ and choose (a_n) satisfying (4.21), so that $\sum_{j=1}^{\infty} d_j a_j$ is a divergent series whose partial sums are bounded. Then from (4.24) we deduce that

$$\sup_n |\tilde{s}_n(e^{i\theta})| < \infty \quad \text{and} \quad \tilde{s}(e^{i\theta}) \text{ does not exist for almost every } \theta \in [-\pi, \pi]. \tag{4.25}$$

Using (4.19)-(4.25), Wolff (see [Wol07, Lemma 2.12]) essentially proves

Lemma 4.3. *If $N_{j+1} > N_j(\log(2 + N_j))^3$ for $j = 1, 2, \ldots,$ then there is a choice of (L_j) satisfying (4.19) such that $\sup_j \|\sigma_j\|_\infty < \infty$ and σ diverges for almost every $\theta \in [-\pi, \pi]$.*

Proof. To outline the proof of this lemma, for $n = 1, 2, \ldots$, let Υ_n denote all intervals $I \subset \mathbb{R}$ that are maximal (in length) with the property that $I \in G_j$ for some j and $\max_I |s_j| > n$. From (4.22), (4.23), and (4.21) we see that if $I \in \Upsilon_n \cap G_j$, and $\tilde{c}$ is large enough (depending only on p), then $|s_j| > n - \tilde{c}$ on $\tilde{I}$ where $\tilde{c}$ depends only on p. Using (4.24) with $\lambda = n - \tilde{c}$ and boundedness of the partial sums of $\sum_{j=1}^{n} a_j d_j$ we get $c \ge 1$ depending only on p, and the choice of (a_j) such that

$$\sum_{\tilde{I} \in \Upsilon_n} |\tilde{I} \cap [-3\pi, 3\pi]| \le c\, n^{-2} \quad \text{for } n = 1, 2, \ldots. \tag{4.26}$$

Thus, from the usual measure theory argument,

$$\Big|\{\theta \in [-\pi, \pi] : \text{ for infinitely many } n,\ \theta \in \tilde{I} \text{ with } I \in \Upsilon_n\}\Big| = 0. \tag{4.27}$$

Finally, for $j = 2, \ldots,$ define L_j by induction as follows

(4.28)

(a) If L_k has been defined and $I \in G_k$ is also in $\cup\Upsilon_n$, put $L_{k+1} = \frac{1}{2}L_k$ on I.

(b) If none of the three intervals in G_k contained in $\tilde{I}$ are in $\cup\Upsilon_n$, set $L_{k+1} = L_k$ on I.

(c) If neither (a) nor (b) holds for $I \in G_k$ use (4.19) to define L_{k+1}.

From (4.27) and the definition of L_j we see for almost every $\theta \in [-\pi, \pi]$ that there exists a positive integer $m = m(\theta)$ such that $L_j(\theta) = L_m(\theta)$ for $j \geq m$. From (4.25) we conclude that (σ_k) diverges for almost every $\theta \in [-\pi, \pi]$. Also if $|\tilde{s}_k(e^{i\theta})| > n$, then since $|a_k|\|\tilde{\Phi}_k\|_\infty \leq 1$, we see from (4.28) that there exist n distinct integers, $j_1 < j_2 < \ldots j_n \leq k$ with $L_{j_i+1}(e^{i\theta}) = \frac{1}{2}L_{j_i}(e^{i\theta})$. Thus $L_{k+1}(e^{i\theta}) \leq 2^{-n}$. Using this fact and summing by parts Wolff gets, $\sup_k \|\sigma_k\|_\infty < \infty$. □

Next we state

Lemma 4.4. *If $N_{j+1} > N_j(\log(2 + N_j))^3$ for $j = 1, 2, \ldots,$ then there is a choice of (L_j) satisfying (4.19) such that $\sigma_j > 0$ for $j = 1, 2, \ldots$ and $\sup_j \|\sigma_j\|_\infty < \infty$ on $\mathbb{R}$. Also,*

$$\sigma(e^{i\theta}) = \lim_{j\to\infty} \sigma_j(e^{i\theta}) = 0 \quad \textit{for almost every } \theta \in [-\pi, \pi].$$

Proof. Lemma 4.4 is essentially Lemma 2.13 in [Wol07]. To outline his proof let

$$R = 1 \text{ and } a_j = -\frac{1}{4j} \text{ for } j = 1, 2, \ldots \text{ in (4.21)}. \tag{4.29}$$

We also set

$$\Upsilon_{kn} := \{I \in G_k : \max_I \tilde{s}_k > n \text{ and } I \not\subset J \in \Upsilon_{jn} \text{ for any } j < k \}.$$

Define $\mathcal{F}_{kn}$ and $\mathcal{H}_{kn}$, by induction as follows : Let $\sigma_1 = 1 + a_1\tilde{\Phi}_1$ be the first partial sum of σ in (4.20). By induction, suppose L_j and corresponding σ_j have been defined for $j \leq k$. Assume also that $\mathcal{F}_{jn}, \mathcal{H}_{jn} \subset G_j$ have been defined for $j < k$ and all positive integers n with $\mathcal{F}_{0n} = \emptyset = \mathcal{H}_{0n}$. If n is a positive integer and $I \in G_k$, we put $I \in \mathcal{F}_{kn}$ if $\min_I \sigma_k < 2^{-n}$ and this interval is not in $\mathcal{F}_{jn}$ for some $j < k$. Moreover we put $I \in \mathcal{H}_{kn}$ if $\min_I \tilde{s}_k < -\frac{2^n}{n+1}$ and $\max_I L_k > 2^{-n}$. Then

(4.30)

(a) $L_{k+1} = \dfrac{1}{2}L_k$ on $I \in G_k$ if $I \in \cup_n(\mathcal{F}_{kn} \cup \mathcal{H}_{kn} \cup \Upsilon_{kn})$

(b) $L_{k+1} = L_k$ on I if none of the three intervals in $\tilde{I}$ are in $\cup_n(\mathcal{F}_{kn} \cup \mathcal{H}_{kn} \cup \Upsilon_{kn})$.

(c) If neither (a) nor (b) hold for $I \in G_k$, use (4.19) to define L_{k+1}.

This definition together with (4.20) define L_{k+1} on G_k so by induction we get (L_m), (σ_m), and also $(\mathcal{F}_{mn})$, $(\mathcal{H}_{mn})$, (Υ_{mn}) whenever m, n are positive integers.

As in Lemma 4.3 we have $L_{k+1} < 2^{-n}$ on $I \in \Upsilon_{kn}$. Also if $2^{-(n+1)} \leq \min_I \sigma_j < 2^{-n}$ and $L_{j+1} \leq 2^{-n}$ on $I \in G_j$, then from (4.29) we see that

$$\sigma_{j+1} \geq \sigma_j - 2^{-(n+2)} \geq 2^{-(n+2)} \quad \text{on } I.$$

Using this observation and induction on n one can show for all positive integer k and n that

$$\text{if } I \in G_k \text{ and } \min_I \sigma_k < 2^{-n} \text{ then } L_{k+1} < 2^{-n} \text{ on } I \in G_k. \tag{4.31}$$

Now (4.31) implies that

$$\sigma_k > 0 \text{ for } k = 1, 2, \ldots, \text{ on } [-\pi, \pi] \tag{4.32}$$

since $\sigma_1 > 0$ and if $2^{-(n+1)} \leq \sigma_k < 2^{-n}$ on $I \in G_k$. Using this observation and (4.31) again we have

$$\sigma_{k+1} > \sigma_k - 2^{-(n+2)} > 0 \quad \text{on } I.$$

Thus to show that (σ_j) is bounded it suffices to show that $\max_k \sigma_k < c < \infty$. Using this fact and repeating the argument for boundedness of (σ_j) in Lemma 4.3 we obtain boundedness of (σ_j). It remains to prove that

$$\tilde{s}_k(e^{i\theta}) \to 0 \quad \text{for almost every } \theta \in [-\pi, \pi]. \tag{4.33}$$

We shall need

$$c^{-1} L_{k+1}(e^{i\theta_2}) \leq L_{k+1}(e^{i\theta_1}) \leq c\, L_{k+1}(e^{i\theta_2}) \tag{4.34}$$

whenever $\theta_1, \theta_2 \in \tilde{I}$ and $I \in G_k$ for $k = 1, 2 \ldots$. This follows easily from (4.30) and the gap assumption on (N_j). To prove (4.33) let E_n denote the set of all $\theta \in \mathbb{R}$ for which there exist k and l positive integers with $k < l$ satisfying

$$\tilde{s}_l > -\frac{2^n}{2(n+1)} \quad \text{while} \quad \tilde{s}_k < -\frac{2^n}{n+1}.$$

From $a_j < 0$ and $c^{-1} \leq d_j \leq 1$ for $j = 1, 2, \ldots,$ we obtain that

$$\max \left[|\tilde{s}_l(e^{i\theta}) - \sum_{j=1}^{l} a_j d_j|\,,\, |\tilde{s}_k(e^{i\theta}) - \sum_{j=1}^{k} a_j d_j| \right] \geq \frac{2^n}{8(n+1)} \tag{4.35}$$

for $n \geq 100$. If we let

$$\Lambda := \{\theta \in \mathbb{R} : \theta \in E_n \text{ for infinitely many } n\} \cup \{\theta \in \mathbb{R} : \limsup_{j\to\infty} \tilde{s}_j(e^{i\theta}) > -\infty\}$$

then using (4.35) and (4.24) we deduce

$$|\Lambda| = 0. \tag{4.36}$$

Next from induction on m and the definition of $\mathcal{H}_{km}$, it follows that if $\tilde{s}_k(e^{i\theta}) < -\frac{2^m}{m+1}$ on $I \in G_k$ then $L_{k+1}(e^{i\theta}) \leq 2^{-m}$. Therefore if $\theta_0 \notin \Lambda$ then

$$\lim_{k\to\infty} \tilde{s}_k\,(e^{i\theta_0}) = -\infty \quad \text{and} \quad \lim_{k\to\infty} (\tilde{s}_k\, L_{k+1})(e^{i\theta_0}) = 0.$$

These equalities and

$$0 \le \sigma_j(e^{i\theta_0}) = 1 + \sum_{l\le j}(L_l - L_{l+1})\tilde{s}_l(\theta_0) + s_j L_{j+1}(\theta_0) \tag{4.37}$$

imply that if $\theta_0 \notin \Lambda$ then it must be true that $\lim_{j\to\infty}\sigma_j(e^{i\theta_0})$ exists and is non-negative.

Suppose this limit is positive. Then from (4.23) we find that

$$\sup_j\{\max_{\tilde{I}} \tilde{s}_j : \theta_0 \in I \in G_j\} < \infty \quad \text{and} \quad \inf_j\{\min_{\tilde{I}} \sigma_j : \theta_0 \in I \in G_j\} > 0. \tag{4.38}$$

So θ_0 belongs to at most a finite number of $\tilde{I}$ with $I \in \Upsilon_{kn} \cup \mathcal{F}_{k,n}$ for $k, n = 1, 2, \dots$. Then since $L_k(e^{i\theta_0}) \to 0$ and $\tilde{s}_k(e^{i\theta_0}) \to -\infty$ as $k \to \infty$ we deduce that given m a sufficiently large positive integer, say $m \ge m_0$, there exists $m' < m$ with

$$m' = \max\{j : j < m \text{ and } L_j(\theta_0) \ne L_m(\theta_0)\}$$

such that $\theta_0 \in \tilde{I}$, $I \in \mathcal{H}_{m'n}$, for some positive integer n. This inequality and (4.34) yield that if $L_k(\theta_0) < 2^{-l}$, then $\tilde{s}_k(\theta_0) < -c\,\frac{2^l}{l+1}$ for $l \ge l_0$ where $c \ge 1$ is independent of k and l. Using this fact and choosing an increasing sequence (i_l) for $l \ge l_0$ so that $L_{i_l}(\theta_0) = 2^{-l}$ for $l \ge l_0$, it follows from (4.37) that $\sigma(e^{i\theta_0}) = -\infty$ which contradicts (4.32). This first shows that $\sigma(e^{i\theta_0}) = 0$ and this completes the proof of Lemma 4.4. □

4.3. **Construction of Examples.** To finish the proof of Theorem B we again follow Wolff in [Wol07] closely and use Lemmas 4.1, 4.3, and 4.4 to construct examples. Let $N_1 = 1$ and by induction suppose $N_2, \dots, N_k$ have been chosen, as in Lemmas 4.3 and 4.4, with σ as in (4.19)-(4.21). Let $g = \sigma_k$, $f = a_{k+1}L_{k+1}$, $q = \Phi_{k+1}$, and suppose

$$\max(\|f\|_\infty, \|g\|_\infty, \|q\|_\infty, \|f\breve{\|}, \|g\breve{\|}, N_{k+1}^{-1}\|q\breve{\|}) \le M \tag{4.39}$$

where $M = M(N_1, \dots, N_k)$ is a constant and Φ_{k+1} is p-harmonic in $B(0,1)$ with Lipschitz continuous boundary values and $\Phi_{k+1}(0) = 0$. Next apply Lemma 4.1 with M as in (4.39) and $\epsilon = 2^{-(k+1)}$ obtaining $A = A_k$ and ν_0 so that (4.1)-(4.4) are valid. We also choose $N_{k+1} > \nu_0$ and so that $A_k N_{k+1}^{-\alpha} < \frac{1}{2}A_{k-1}N_k^{-\alpha}$ where $\alpha = 1 - p/2$ if $p < 2$ and $\alpha = 1 - 2/p$ if $p > 2$. By induction we now get σ as in Lemma 4.3 or Lemma 4.4. Then

$$|\hat{\sigma}_{j+1}(re^{i\theta}) - \hat{\sigma}_j(re^{i\theta})| < 2^{-(j+1)} \quad \text{when } r < 1 - A_j N_{j+1}^{-\alpha}, \tag{4.40}$$

and

$$|\hat{\sigma}_{j+1}(re^{i\theta}) - \sigma_j(e^{i\theta})| < 2^{-j} + |a_{j+1}| \quad \text{when } r > 1 - A_j N_{j+1}^{-\alpha}. \tag{4.41}$$

From (4.40) we see that $(\hat{\sigma}_{j+1})$ converges uniformly on compact subsets of $B(0,1)$ to a p-harmonic function $\tilde{\sigma}$ satisfying

$$|\tilde{\sigma}(re^{i\theta}) - \hat{\sigma}_k(re^{i\theta})| < 2^{-k} \quad \text{when } r < 1 - A_k N_{k+1}^{-\alpha}. \tag{4.42}$$

Using (4.40), (4.42), and the triangle inequality we also have for $1 - A_k N_{k+1}^{-\alpha} < r < 1 - A_{k+1} N_{k+2}^{-\alpha}$ that

$$
\begin{aligned}
|\tilde{\sigma}(re^{i\theta}) - \sigma_k(e^{i\theta})| &\leq |\tilde{\sigma}(re^{i\theta}) - \hat{\sigma}_{k+1}(re^{i\theta})| + |\hat{\sigma}_{k+1}(re^{i\theta}) - \sigma_k(e^{i\theta})| \\
&< 2^{-(k+1)} + 2^{-k} + |a_{k+1}|.
\end{aligned} \tag{4.43}
$$

From (4.43) and our choice of (a_k) we see for (σ_k) as in Lemma 4.3 that $\lim_{r\to 0} \tilde{\sigma}(re^{i\theta})$ does not exist for almost every $\theta \in [-\pi, \pi]$ while if (σ_k) is as in Lemma 4.4, $\lim_{r\to 0} \tilde{\sigma}(re^{i\theta}) = 0$ almost everywhere. Moreover from boundedness of (σ_k) and the maximum principle for p-harmonic functions we deduce that $\tilde{\sigma}$ is bounded in Lemma 4.3 or 4.4, as well as non-negative in Lemma 4.4. To conclude the proof of Theorem B, put $\tilde{\sigma} = \hat{u}$ and $\tilde{\sigma} = \hat{v}$ if Lemma 4.3 and Lemma 4.4, respectively, was used to construct $\tilde{\sigma}$. □

5. Proof of Theorem C

In this section we use Theorem A to prove Theorem C. Let $(N_j)_1^\infty$ be a sequence of positive integers with $N_1 = 1$ and with $(N_j)_2^\infty$ to be chosen later in order to satisfy several conditions. For the moment we assume only that $N_{j+1}/N_j \geq 2$. Let Φ_j for $j = 1, 2, \dots$ be the p-harmonic function in $B(0,1)$ with period $2\pi/N_j$ constructed in Theorem B with $\Phi_j = V$. Apart from some details, which need to be worked out, arising from the fact that we have to work with (Φ_j) rather than just Φ, we can essentially copy the proof in [LMW05]. For the readers convenience we give details. Once again Lemma 4.1 plays an important role in the estimates.

We assume as we may that

$$
\|\Phi_j\|_\infty \leq 1/2 \quad \text{and} \quad \int_{-\pi}^{\pi} \log(1 + \Phi_j)(e^{i\theta}) d\theta \geq c_2^{-1} \tag{5.1}
$$

for $j = 1, 2, \dots$, where $c_2 \geq 1$ depends only on p. Indeed otherwise, we replace Φ_j by $\tilde{\Phi}_j = c^{-1}\Phi_j$ and observe from Theorem A, elementary facts about power series that for $c >> c_1$,

$$
\int_{-\pi}^{\pi} \log(1 + \tilde{\Phi}_j(e^{i\theta})) d\theta \geq \int_{-\pi}^{\pi} \tilde{\Phi}_j(e^{i\theta}) d\theta - 2\pi(c_1/c)^2 \geq (2c_1 c)^{-1}.
$$

Thus we assume (5.1) holds. We claim that there exists a positive integer $\kappa >> 1$ and a positive constant $C = C(\kappa) > 1$ such that for $j = 1, 2, \dots$,

$$
\sum_{l=1}^{\kappa} a_{lj} \geq C^{-1} \quad \text{and} \quad \prod_{l=1}^{\kappa} (1 + a_{lj}) > 1 + C^{-1} \tag{5.2}
$$

where for $l = 1, \dots, \kappa$,

$$
a_{lj} := \min\{\Phi_j(e^{i\theta/N_j}) : \ \theta \in [-\pi + \frac{(2l-2)\pi}{\kappa}, -\pi + \frac{2l\pi}{\kappa}]\}.
$$

To prove (5.2), let $\phi_j(e^{i\theta}) = \Phi_j(e^{i\theta/N_j})$ for $\theta \in \mathbb{R}$. Then from Theorem A we see that ϕ_j is continuous and 2π periodic on $\mathbb{R}$ with $\|\phi_j\check{\|} \leq c_1$, where c_1 depends only on p. Using these facts we get

$$2\pi\kappa^{-1}\sum_{l=1}^{\kappa} a_{lj} \geq \int_{-\pi}^{\pi} \phi_j(e^{i\theta})d\theta - \hat{c}\,\kappa^{-1} \geq \frac{1}{2c_1} > 0$$

for κ large enough thanks to (1.3) (c) . Likewise, from Theorem A and (5.1) it follows that

$$2\pi\sum_{l=1}^{\kappa} \log(1+a_{lj}) \geq \kappa\int_{-\pi}^{\pi} \log(1+\phi_j)(e^{i\theta})d\theta - c' > \frac{\kappa}{2c_2}$$

for κ large enough where c_2 depends only on p. Dividing this inequality by 2π and exponentiating we get the second inequality in (5.2). Hence (5.2) is valid. From (5.2) we deduce for $j = 1, 2, \ldots,$ the existence of Λ and $\tilde{N}_0$ so that

$$\begin{aligned} &(a) \quad 1 < \Lambda < (1+C^{-1})^{1/\kappa} < \prod_{l=1}^{\kappa}(1+a_{lj})^{1/\kappa}, \\ &(b) \quad 3^{-\tilde{N}_0} < \min_j \left[1 + \max_{1\leq l\leq\kappa} a_{lj} - \Lambda,\, (c_1\kappa)^{-1}\right]. \end{aligned} \tag{5.3}$$

Fix κ subject to the above requirements. For $\theta \in \mathbb{R}$ and $k = 1, \ldots, \kappa$, we let

$$q_1^k(e^{i\theta}) := \Phi_1(-e^{i(\theta+2k\pi/\kappa)}) \quad \text{and} \quad f_1^k(e^{i\theta}) := 1 + q_1^k(e^{i\theta}). \tag{5.4}$$

Moreover, for $\theta \in \mathbb{R}$, $j = 2, 3, \ldots,$ and $k = 1, \ldots, \kappa$ set

$$q_j^k(e^{i\theta}) := \Phi_j(-e^{i(\theta+2k\pi/\kappa)}) \quad \text{and} \quad f_j^k(e^{i\theta}) := (1 + q_j^k(e^{i\theta}))f_{j-1}^k(e^{i\theta}). \tag{5.5}$$

Observe from (5.1), (5.3), (5.5) that

$$\prod_{k=1}^{\kappa} f_j^k(e^{i\theta}) = \prod_{l=1}^{j}\prod_{k=1}^{\kappa}\left(1 + \Phi_l(-e^{i(\theta+2k\pi/\kappa)})\right) > \Lambda^{\kappa j}. \tag{5.6}$$

Let

$$E_k := \{e^{i\theta} \in \partial B(0,1) : f_j^k(e^{i\theta}) > \Lambda^j \text{ for infinitely many } j\}. \tag{5.7}$$

From (5.6) we see that

$$\bigcup_{k=1}^{\kappa} E_k = \partial B(0,1). \tag{5.8}$$

From (5.8) we conclude that to finish the proof of Theorem C it suffices to show $k = 1, \ldots, \kappa$ that

$$\omega_p(0, E_k) = 0, \quad \omega_p(0, \partial B(0,1)\setminus E_k) = 1, \quad \text{and} \quad |\partial B(0,1)\setminus E_k| = 0 \tag{5.9}$$

where ω_p is defined after (1.4). To do this we use Lemma 4.1 and an inductive type argument to choose $(N_j)_2^\infty$. First we require that $N_1 = 1$ and N_{j+1}/N_j is divisible by κ for $j = 1, 2, \ldots$. Second for fixed k and $j = 1, 2, \ldots$ we apply Lemma 4.1 with $f = g = f_j^k$ and $q = q_{j+1}^k$ From (5.1) and Theorem B we see that $\|q_j^k\|_\infty \leq 1/2$, and $\|q_j^k\breve{\|} \leq c_1 N_j$ for $j = 1, 2, \ldots$ Thus,

$$2^{-j} \leq \|f_j^k\|_\infty \leq (3/2)^j \quad \text{and} \quad \|f_j^k\breve{\|} \leq c_1\, 2^j N_j. \tag{5.10}$$

Let $M_j = c_1 4^j N_j$ and $\epsilon = \epsilon_j = 3^{-j-1}$. Then there exists small $A_j = A_j(p, \epsilon_j, M_j)$, and large $\nu_0(p, \epsilon_j, N_j)$ such that if $N_{j+1} > \nu_0$, then

$$|\hat{f}^k_{j+1}(re^{i\theta}) - f^k_j(e^{i\theta})(q_{j+1}(re^{i\theta}) + 1)| < 3^{-(j+1)} \quad \text{for } 1 - A_j N_{j+1}^{-\alpha} < r < 1 \tag{5.11}$$

and

$$|\hat{f}_{j+1}(re^{i\theta}) - \hat{f}_j(re^{i\theta}| < 3^{-(j+1)} \quad \text{for } r \leq 1 - A_j N_{j+1}^{-\alpha}. \tag{5.12}$$

Now using (2.3) as in the derivation of (4.4) from (4.3) we see that we may also assume

$$|\hat{f}_j(re^{i\theta}) - \hat{f}_j(e^{i\theta}| < 3^{-(j+1)} \quad \text{for } r \geq 1 - A_j N_{j+1}^{-\alpha}. \tag{5.13}$$

Finally, we may choose (A_j) and (N_j) so that

$$100\, N_{j+1}^{-1} < t_j = A_j N_{j+1}^{-\alpha} < (c_1 N_j 6^{j+1})^{-1} \quad \text{and} \quad t_{j+1} < \frac{t_j}{\kappa} \tag{5.14}$$

for $j = 1, 2, \dots,$. From (5.12) and (5.14), we deduce for $m > j$, a positive integer, and for $k = 1, 2, \dots, \kappa$, that

$$|\hat{f}^k_m(re^{i\theta}) - \hat{f}^k_j(re^{i\theta}| \leq 3^{-j} \quad \text{for } r \leq 1 - t_j. \tag{5.15}$$

From (5.15) and Lemmas 2.2 - 2.4 we obtain that $\hat{f}^k_j$ and $\nabla \hat{f}^k_j$ converge uniformly as $j \to \infty$ to a locally p-harmonic $\hat{f}^k$, $\nabla \hat{f}^k$, on compact subsets of $B(0,1)$ satisfying (5.15) with $\hat{f}^k_m$ replaced by $\hat{f}^k$. Also from (5.11), (5.1), and (5.15) with j replaced by $j+1$, it follows that

$$\hat{f}^k_m(re^{i\theta}) \geq \frac{1}{2} \hat{f}^k_j(re^{i\theta}) - 3^{-j} \quad \text{for } 1 - t_j \leq r \leq 1 - t_{j+1}. \tag{5.16}$$

Next for fixed k, $1 \leq k \leq \kappa$, let $G^k_j = \{e^{i\theta} : f^k_j(e^{i\theta}) > \Lambda^j\}$. Then

$$E_k = \bigcap_{n=1}^{\infty} \left(\bigcup_{j=n}^{\infty} G^k_j \right) \quad \text{where } E_k \text{ is as in (5.7)}. \tag{5.17}$$

By monotonicity of p-harmonic measure it suffices to show that

$$\omega_p \left(0, \bigcup_{j=n}^{\infty} G^k_j \right) \leq \tilde{C} \Lambda^{-n} \quad \text{for } n = 1, 2, \dots \tag{5.18}$$

where $\tilde{C} \geq 1$ does not depend on n. Moreover, from Theorems 11.3-11.4 and Corollary 11.5 in [HKM06] applied to $\omega_p \left(0, \partial B(0,1) \setminus \bigcup_{j=n}^{N} G^k_j \right)$ we see that

$$\lim_{N \to \infty} \omega_p \left(0, \bigcup_{j=n}^{N} G^k_j \right) = \omega_p \left(0, \bigcup_{j=n}^{\infty} G^k_j \right).$$

Therefore, instead of proving (5.18), we need only show that

$$\omega_p \left(0, \bigcup_{j=n}^{N} G^k_j \right) \leq \tilde{C} \Lambda^{-n} \quad \text{for } N > n \tag{5.19}$$

in order to conclude that $\omega_p(0, E_k) = 0$. This conclusion and Theorem 11.4 in [HKM06] then yield $\omega_p(0, \partial B(0,1) \setminus E_k) = 1$ for $k = 1, 2, \ldots, \kappa$.

To prove (5.19) we temporarily drop the k and write f_j, G_j for f_j^k, G_j^k. Let

$$H_j := \bigcup \{I \subset \mathbb{R} : I \text{ is a closed interval of length } t_j, \max_{\theta \in I} f_j(e^{i\theta}) \geq \Lambda^j - 3^{-j-1}\}$$

and let $\mathring{H}$ denote the interior of H relative to $\mathbb{R}$. Clearly,

$$f_j(e^{i\theta}) < \Lambda^j - 3^{-(j+1)} \quad \text{if } \theta \in H_j \setminus \mathring{H}_j. \tag{5.20}$$

Hence $\{\theta : e^{i\theta} \in \bar{G}_j\} \subset \mathring{H}_j$. From (5.10), (5.14) we see that

$$|f_j(e^{i\theta_1}) - f_j(e^{i\theta_2})| \leq c_1 2^j t_j \leq 3^{-j} 6^{-1} \text{ if } |\theta_1 - \theta_2| \leq t_j. \tag{5.21}$$

Thus

$$\min_{\theta \in H_j} f_j(e^{i\theta}) \geq \Lambda^{-j} - 3^{-j} 2^{-1}. \tag{5.22}$$

Let

$$T_j = \bigcup \{I \times [0, t_j] : I \in H_j\} \subset \bar{\mathbb{R}}^2_+ \text{ for } j = 1, 2, \ldots$$

Using (5.22), (5.13), (5.14), we conclude that

$$\hat{f}_j(re^{i\theta}) > \Lambda^j - 3^{-j} \quad \text{if } (\theta, 1-r) \in T_j. \tag{5.23}$$

At this point the authors in [LMW05] note that if it were true that

$$\hat{f}_N(re^{i\theta}) > \bar{C}^{-1} \Lambda^j \text{ for } (\theta, 1-r) \text{ in the closure of } \mathbb{R}^2_+ \cap \partial T_j \text{ for } N \geq j \geq n > \tilde{N}_0,$$

then it would follow from the boundary maximum principle for p-harmonic functions applied to $\bar{C} \Lambda^{-n} f_N$ in

$$B(0,1) \setminus \{re^{i\theta} : (\theta, 1-r) \in \bigcup_{j=n}^{N} \bar{T}_j\}$$

and convergence of $(\hat{f}_j)$ to $\hat{f}$ that (5.19) is valid. Unfortunately, this inequality need not hold so the authors modify the components of T_j as follows. Observe that T_j has a finite number of components having a non-empty intersection with $[-\pi, \pi]$. If $Q = [a, b] \times [0, t_j]$ is one of these components then

$$f_j(e^{ia}), f_j(e^{ib}) < \Lambda^j - 3^{-j-1} \text{ thanks to (5.20)}. \tag{5.24}$$

If $\max_{\theta \in [a,b]} f_j(e^{i\theta}) \leq \Lambda^j$, then from (5.24) and the definition of G_j we deduce that

$$\{e^{i\theta} : \theta \in [a, b]\} \cap \bar{G}_j = \emptyset$$

so in this case put $Q^* = \emptyset$. Otherwise $\max_{\theta \in [a,b]} f_j(e^{i\theta}) > \Lambda^j$, and from (5.10), (5.14), (5.24),(5.23), we see that if $I_j^Q = [a, a + t_j]$ and $J_j^Q = [b - t_j, b]$, then

$$\Lambda^j - 3^{-j} < \; f_j(e^{i\theta}) \; < \Lambda^j - 3^{-j-2} \quad \text{on } I_j^Q \cup J_j^Q. \tag{5.25}$$

We note from (5.14) that $N_{j+1}t_j > 100$ and $q_{j+1}(e^{i\theta})$ is $\frac{2\pi}{N_{j+1}}$ periodic in θ so from the definition of $(a_{l(j+1)})$ and (5.14) we can find intervals I^Q_{j+1} and J^Q_{j+1} with

$$I^Q_{j+1} \subset I^Q_j, J^Q_{j+1} \subset J^Q_j, \text{ and } \max_{I_{j+1}} q_{j+1}, \max_{J_{j+1}} q_{j+1} \geq \max_{1\leq l\leq \kappa} a_{l(j+1)}. \tag{5.26}$$

Moreover I_{j+1} and J_{j+1} each have length t_{j+1}. Then from (5.3) (b) and (5.26) we get for $L = I_{j+1}$ or J_{j+1} and $j > \tilde{N}_0$ that

$$\max_L f_{j+1}(e^{i\theta}) \geq (1 + \max_{1\leq l\leq\kappa} a_{l(j+1)})(\Lambda^j - 3^{-j}) > \Lambda^{j+1}. \tag{5.27}$$

From (5.27), (5.10), (5.14), with j replaced by $j+1$, we deduce that

$$\min_L f_{j+1}(e^{i\theta}) \geq \Lambda^{j+1} - 3^{-j-2}. \tag{5.28}$$

We can now argue by induction to get nested closed intervals $(I^Q_l)^\infty_j$ and $(J^Q_l)^\infty_j$, for which I^Q_l and J^Q_l have length t_l and (5.27), (5.28), are valid with $j+1$ replaced by l. Then

$$a < a^* := \bigcap_{l=j}^{\infty} I^Q_l \quad \text{and} \quad b^* := \bigcap_{l=j}^{\infty} J^Q_l < b. \tag{5.29}$$

Set $Q^* = [a^*, b^*] \times [0, t_j]$ and

$$T^*_j = \bigcup \{Q^* : Q \text{ is a component of } T_j\}.$$

Then by construction and (5.25)

$$\bar{G}_j \subset \mathring{H}^*_j \subset H^*_j = \partial T^*_j \cap \mathbb{R}.$$

Finally the authors show

$$\hat{f}_N(re^{i\theta}) > \frac{1}{3}\Lambda^j \text{ for } (\theta, 1-r) \text{ in } \partial T^*_j \setminus \mathring{H}^*_j \text{ and } N \geq j. \tag{5.30}$$

For $N = j$ this inequality is implied by (5.23) while if $t_{j+1} \leq 1-r \leq t_j$ we see from (5.16) and (5.23) that (5.30) is valid for $(\theta, 1-r)$ in $\partial T^*_j \cap [t_{j+1} \leq 1-r \leq t_j]$. The only remaining segments of $\partial T^*_j \cap \mathbb{R}^2_+$ are of the form $\{a^*\} \times [0, t_{j+1}]$, $\{b^*\} \times [0, t_{j+1}]$, where a^*, b^* are as in (5.29). If $(\theta, 1-r) \in \{a^*\} \times [t_{l+1}, t_l]$ or $\{b^*\} \times [t_{l+1}, t_l]$ for $j+1 \leq l < N$ we can use (5.16) with $m = N, j = l$, (5.13) with $j = l$, and (5.28) with $j+1 = l$ to get that (5.30) is valid on $\partial T^*_j \cap [t_{l+1} \leq 1-r \leq t_l]$. If $(\theta, 1-r) \in \{a^*\} \times [0, t_N]$ or $\{b^*\} \times [0, t_N]$ then from (5.13) with $j = N$ and (5.28) with $j+1 = N$, we obtain (5.30) on $\partial T^*_j \cap (0 < 1-r \leq t_N]$. Thus (5.30) is valid and from the discussions after (5.23), (5.19), we conclude $\omega_p(0, E_k) = 0$ for $1 \leq k \leq \kappa$.

It remains to prove $|\partial B(0,1) \setminus E_k| = 0$ for $1 \leq k \leq \kappa$. To do this, for $j = 1, 2, \dots$, let $\tau_j(e^{i\theta}) = \log(1 + a_{lj})$ when $\theta \in [-\pi + \frac{(2l-2)\pi}{\kappa}, -\pi + \frac{2l\pi}{\kappa})$ and $1 \leq l \leq \kappa$. We regard τ_j as a 2π periodic function on $\mathbb{R}$. For $\theta \in \mathbb{R}$, $j = 1, 2, \dots$, and $k = 1, \dots, \kappa$, let

$$h^k_j(e^{i\theta}) = \tau_j(-e^{i(N_j\theta + 2k\pi/\kappa)}) - m_j \quad \text{where} \quad m_j = \frac{2\pi}{\kappa}\sum_{l=1}^{\kappa} \log(1 + a_{lj}).$$

Then for fixed k, h_{j+1}^k is $2\pi/N_{j+1}$ periodic and has average 0 on intervals where h_j^k is constant since N_{j+1}/N_j is divisible by κ. Thus for fixed k, the functions $h_j^k(e^{i\theta})$ are orthogonal in $L_2(\partial B(0,1))$ and also uniformly bounded for $j = 1, 2, \dots$. Using this fact one can show (see page 182 in [KW85]) that

$$\sum_{l=1}^{j} h_l^k = O(j^{3/4}) \text{ for almost every } e^{i\theta} \in \partial B(0,1) \tag{5.31}$$

with respect to Lebesgue measure on $\partial B(0,1)$. Since

$$\log f_j^k \geq \sum_{l=1}^{j} (h_l^k + m_l)$$

it follows from (5.3) (a), (5.31) that for almost every $e^{i\theta} \in \partial B(0,1)$ there exists $j_0(\theta)$ such that for $j \geq j_0$, $f_j^k(e^{i\theta}) > \Lambda^j$. From the definition of E_k we arrive at $|\partial B(0,1) \setminus E_k| = 0$ for $1 \leq j \leq k$.

6. Closing Remarks

We note that in [LMW05, section 4], the authors discuss some interesting open questions for p-harmonic measure. Theorems A, B, C were inspired by these questions. One natural question is to what extent Theorem 1.1 or Theorem B has an analogue in other domains? For example, can one prove similar theorems in the unit ball, say $\breve{B}$, of $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}$, $n \geq 3$, when $1 < p < \infty, p \neq n$? For $p = n$, one can map $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ by way of a linear fractional transformation, conformally onto $\breve{B}$ and use invariance of the n-Laplacian under conformal mappings to conclude that the conclusion in Theorem 1.1 extends to $\breve{B}$. Theorems B and C generalize to $B(0,1) \times \mathbb{R}^{n-2}, n \geq 3$, by adding $n-2$ dummy variables. We note that for $p > 2$, the Martin function in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ relative to 0 is homogeneous of degree $-\lambda$ where $0 < \lambda < N - 1$ as follows from Theorem 1.1 in [LMTW19] (see also [DS18]). Using this fact one can construct examples in $\mathbb{R}^n_+$ similar to the hands on examples constructed in $\mathbb{R}^2_+$ for Theorem A.

Another interesting question is whether the set in Theorem 1.1 or Theorem B where radial limits exist can have Hausdorff dimension < 1? This set has dimension $\geq a = a(p) > 0$ thanks to work of [MW88] and [FGMMS88].

Also an interesting question to us is whether Theorem 1.1 or Theorem B have analogues for solutions to more general PDE of p-Laplace type. To give an example, given p, $1 < p < \infty$, suppose $f : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ has continuous third partials on $\mathbb{R}^n \setminus \{0\}$ with

(6.1)

(a) $f(t\eta) = t^p f(\eta)$ when $t > 0$ and $\eta \in \mathbb{R}^n$,

(b) There exists $\tilde{a}_1 = \tilde{a}_1(p) \geq 1$ such that if $\eta, \xi \in \mathbb{R}^n \setminus \{0\}$, then

$$\tilde{a}_1^{-1} \, |\xi|^2 |\eta|^{p-2} \leq \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \eta_i \partial \eta_j}(\eta) \, \xi_i \, \xi_j \leq \tilde{a}_1 \, |\xi|^2 |\eta|^{p-2}.$$

Put $\mathcal{A} = \nabla f$ for fixed $p > 1$. Given an open set O we say that v is $\mathcal{A}$-harmonic in O provided $v \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int \langle \mathcal{A}(\nabla v(y)), \nabla \theta(y) \rangle \, dy = 0 \quad \text{whenever } \theta \in W_0^{1,p}(G). \tag{6.2}$$

Note that if $f(\eta) = |\eta|^p$ in (6.1) then v as in (6.2) is p-harmonic in O. Also observe that if v is $\mathcal{A} = \nabla f$-harmonic in $\mathbb{R}^2_+$ then $\tilde{v}(z) = v(Nz + z_0)$ is also $\mathcal{A} = \nabla f$ harmonic in $\mathbb{R}^2_+$ for $z, z_0 \in \mathbb{R}^2_+$ and $N \in \mathbb{R}$. As mentioned earlier Wolff made important use of similar translation, dilation invariance for p-harmonic functions. Thus we believe Theorems 1.1 stands a good chance of generalizing to the $\mathcal{A}$-harmonic setting. On the other hand we made important use of rotational invariance of p-harmonic functions in our proof of Theorem B. Since this invariance is not true in general for $\mathcal{A} = \nabla f$-harmonic functions on $B(0, 1)$, an extension of Theorem B to the $\mathcal{A}$-harmonic setting would require new techniques.

Finally, we note that in a bounded domain $D \subset \mathbb{R}^n$ with $0 \in D$ and for $p = 2$ one can show $\omega_2(\cdot)$ (known as harmonic measure) is a positive Borel measure on ∂D, associated with the Green's function of D having a pole at 0. This notion of harmonic measure led the authors with various co-authors in [BL05, LNPC11, Lew15, LNV13, ALV15] to study the Hausdorff dimension of a positive Borel measure with support in ∂D, associated with a positive p-harmonic function defined in $D \cap N$ and with continuous boundary value 0 on ∂D. Here N is an open neighbourhood of ∂D. Moreover, many of the dimension results we obtained for these "p harmonic measures" in the above papers were also shown in [Akm14, ALV17] to hold for the positive Borel measures associated with positive $\mathcal{A} = \nabla f$-harmonic functions in $D \cap N$, vanishing on ∂D.

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