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On Cyclically Presented Groups of Positive Words Length Four Relators.

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March 15, 2022
Colchester

Contents

List of Tables

CHAPTER 1

Summary.

Substantial progress has been made into the Tits alternative for cyclically presented groups of positive word length four relators. A classification theorem of the finiteness of the abelianisation of said groups is established. A few theorems are proven on the structure of cyclically presented groups of positive word length four for low numbers of generators.

CHAPTER 2

Introduction.

Cyclically presented groups of positive word length four, defined in the preliminaries below, are the focus of this thesis. We set out to investigate whether the Tits alternative holds for this class of groups. Some explorations are made into the structure of these groups for small values of n (the number of generators in the defining presentation).

Jacques Tits first proved the Tits alternative in 1972 for finitely generated linear groups [31]. Since then, the Tits alternative has been proven for Hyperbolic groups [16, Theorem 5.3.E][1], Mapping class groups [26], $\mathrm{Out}(F_n)$ [4] for natural n, and certain groups of birational transformations of algebraic structures [8].

Neither the Grigorchuk group [29] nor Thompson's group F [7] satisfy the Tits alternative, in the sense that, for every subgroup H of these groups, H is neither virtually solvable nor contains a nonabelian free subgroup.

The class is partitioned into eight subclasses according to whether each group's defining numbers, $n, j, k,$ and l, where n is the number of generators in the defining presentation and $j, k, l \in \{0, \ldots, n-1\}$ are the subscripts of generators the defining word w is written in, satisfy three congruence conditions, (A), (B), and (C), defined below and in [5]. The positive word length three case has similar, in fact more, congruence conditions (see [14]).[2]

The preliminaries define what the Tits alternative is, alongside cyclically presented groups

[1] We are grateful to an anonymous Mathematics Stack Exchange user, user1729, for this reference.

[2] More specifically, instead of their being three congruence conditions, there are four; (A′), (B′), (C′), and (D), where I have used the ′ symbol to distinguish them from the congruence conditions for the positive length four case.

in general. Largeness and SQ-universality are delineated. A note is made about when $(n, j, k, l) > 1$ for $G_n(x_0x_jx_kx_l) = G_n(w)$; for details on this notation, see Section 3.1. The congruence conditions (A), (B), (C) of [5] are introduced. Results about deficiency are cited. The shift extension, alluded to below, is defined. A result by Collins in [11] about small cancellation conditions is stated.

The first, non-preliminary section, discusses a handy number γ from [5] defined as the greatest common divisor of n together with linear combinations in j, k, l. A shift extension $E_n(w)$ of an arbitrary cyclically presented group $G = G_n(w)$ of positive word length four by $\mathbb{Z}_n$ along an automorphism θ of G is considered in relation to γ. It is concluded, through certain Tietze transformations and the divisibility conditions they define on n and linear combinations of j, k, l, that we may assume $\gamma = 1$ for the investigation of the Tits alternative in the positive word length four case. For details on how the shift extension is used to show that we may assume $\gamma = 1$, see Section 4.1

The next three sections of this dissertation explore what happens when each of (A), (B), and (C), respectively, are true. This is because some useful results can be gathered from them in isolation, regardless as to what the other two congruence conditions are.

The cases when both (B) and (C) are false are considered together in one section, after the section on when (C) is true. The main tool used is small cancellation theory.

Section 4.6 is on the case when (A), (B), and (C) are true. The Tits alternative in this scenario is deduced almost immediately from the congruence conditions.

I use the notation XYZ to indicate the case when the condition (A) has truth value X, (B) has truth value Y, and (C) has truth value Z; I use T for "true" and F for "false"; so, for example, TTF means (A) and (B) are true while (C) is false.

Next, we consider TTF. Since Bogley & Parker prove in [5] that, in this case, when $\gamma = 1$, the cyclically presented groups of positive word length four are finite, so the Tits alternative holds there.

The Tits alternative for TFT is dealt with in Section 4.8. Here $G_n(w)$ is shown to be large for $n \geq 3$ using, primarily, Williams' [34]. There is no $n = 1$ case here. The remaining $n = 2$ cases are isomorphic to the Baumslag-Solitar group $\mathrm{BS}(1, -1)$, which, in turn, is isomorphic to one of Collins' groups from [11], so cannot have a free, non-abelian subgroup, yet, because of some abelian subgroup of index 2, is solvable.

The FTT case that is Section 4.9 is handled almost entirely by applying a theorem of Bogley

ABC	Tits alternative	Reference	Group ($\gamma = 1$)
TTT	Y	Th4.6.1	$n = 1, \mathbb{Z}_4$ $n = 4, F_3$
TTF	Y	Cor4.7.1	finite
TFT	Y	Th4.8.2	$n \neq 1$, $n = 2$, infinite, solvable $n > 2$, large
TFF FFF	Y Y	Th4.5.1	free subgroup of rank two
FTT	Y	Th4.9.1, Th4.9.2	$\mathbb{Z}_4$
FTF	?	Open4.11.1	?
FFT	Y	Th4.10.1	$\mathbb{Z}_4$ or large

Table 2.1: Summary of results.

& Parker's in [5].

Section 4.10 establishes the Tits alternative and some largeness properties of groups in the FFT case. First, largeness is obtained for some of these groups, using the shift extension mentioned above. Some congruence conditions are then proven for the remaining groups using [34] again; then, assuming these conditions, a series of lemmas using isomorphisms of the $G_n(w)$ from [5] allow us to show that the rest are either $\mathbb{Z}_4$ or, again, large. The computer software GAP [15] was used extensively in this section to form conjectures that later turned out to be lemmas and a theorem.

In Section 4.11, results are summarised in an open problem that, if solved, will complete the proof of the Tits' alternative for cyclically presented groups of positive word length four.

Main Theorem 2.0.1. *For any case except FTF, the Tits alternative holds for $G_n(x_0x_jx_kx_l)$. In the FTF case, the Tits alternative holds for $G_n(x_0x_jx_kx_l)$ except possibly when both $(n, j, k, l) = 1$ and $\gamma = 1$.*

Chapter 5 is a miscellaneous theorem classifying the finiteness of the abelianisations for positive word length four. Similar work can be found in [14, Lemma 2.2] for positive word length three, as well as [10], and [33, Theorem 4] for the non-positive length three case.

Chapter 6 is much more empirical in that it contains tables of data on the cyclically presented groups of positive word length four relators. These focus mainly on the FTF case. This is similar to the work in [9] and [25].

The Chapters 4, 5, and 6 form a basis for the joint paper [21] I have submitted with my supervisor, Prof. G. Williams; it is still under review at the time of writing.

There is an Appendix, A, on the GAP code I used throughout.

CHAPTER 3

Preliminaries.

All congruences are taken modulo n unless stated otherwise.

3.1 Cyclically Presented Groups.

Let $n \in \mathbb{N}$ and let F_n be the free group of rank n with basis $(x_i)_{i \in \overline{0,n-1}}$. Define $\theta : F_n \to F_n$ by $\theta(x_i) = x_{i+1}$, taking the subscripts modulo n. Given a word $w \in F_n$, define the *cyclic (group) presentation* by

$$\mathcal{P}_n(w) = \langle x_0, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w) \rangle;$$

the group $G_n(w)$ the presentation determines is called a *cyclically presented group*. Some background on cyclically presented groups is given in [34].

In [14], the Tits alternative was almost fully established for the case when $w = x_0 x_k x_l$, where $k, l \in \{0, \ldots, n-1\}$ are the subscripts for the generators in the defining word, where n is the number of generators of the defining presentation. They were introduced in [10, Section 4] and studied further in [25]. Groups of Fibonacci type, $G_n(x_0 x_m x_k^{-1})$, were introduced in [23]; see [36] for a survey; here, as above, $m, k \in \{0, \ldots, n-1\}$ are subscripts of generators in the defining word and n is the number of generators in the defining presentation.

We shall consider the case when $w = x_0 x_j x_k x_l$. Unless stated otherwise, $G = G_n(w)$. We shall use the notation $G_n(j, k, l)$ for $G_n(x_0 x_j x_k x_l)$.

3.2 The Tits Alternative.

A class $\mathcal{G}$ of groups satisfies the *Tits Alternative* if for any G in $\mathcal{G}$ either G has a free, non-abelian subgroup or G has a solvable subgroup of finite index. The name comes from the person who first proved this property holds, Jacque Tits, for the class of finitely presented linear groups [31].

3.3 Largeness, SQ-Universality, and Free Subgroups of Rank Two.

A *large* group is a group with a finite index subgroup that maps onto the free group F_2 of rank 2. A group that maps onto a large group is large. A group G is *SQ-universal* if every countable group can be embedded in a quotient group of G. A group that maps onto an SQ-universal group is SQ-universal. Every large group is SQ-universal (since F_2 is SQ-universal [18] and SQ-universality is closed under taking finite-index subgroups [27]). The Higman group is described in [34] as SQ-universal but not large. Every SQ-universal group has a free subgroup of rank 2 but the group $G_7(x_0x_1x_3)$ is described in [13] (and again, explicitly, in [25, Example 3.8]) as having a free subgroup of rank 2 and yet is not SQ-universal.

A free product $H * K$ (with H, K non-trivial) is large if and only if either H and K have non-trivial finite homomorphic images $\bar{H}, \bar{K}$ such that $\{|\bar{H}|, |\bar{K}|\} \neq \{2\}$ or either H or K is large [30]. If an amalgamated free product $G = H *_L K$ is such that $[H : L] \geq 2, [K : L] \geq 2, \{[H : L], [K : L]\} \neq \{2\}$, then G contains a free subgroup of rank 2 [3]; if, further, L is finite then G is SQ-universal [24].

3.4 Primary Divisor.

Bogley & Parker define in [5] the *primary divisor* as (n, j, k, l). We may assume for the Tits alternative that the primary divisor is 1 since [12] (and [34, Theorem 1]) implies the following.

Theorem 3.4.1. *If the primary divisor is $d > 1$, then*

$$G_n(x_0x_jx_kx_l) = *_{i=1}^{d} G_{n/d}(x_0x_{j/d}x_{k/d}x_{l/d}),$$

which is large.

We can conclude this because the positive word length four case maps onto $\mathbb{Z}_4$ (by sending each of the letters in the word to a single generator) and hence each factor of the free product above has a finite homomorphic image of order greater than two.

3.5 The (A), (B), (C) Conditions.

A group presentation is *aspherical* if its second homotopy module is trivial. The work of Bogley and Parker in [5] classified the case when $w = x_0x_jx_kx_l$ by finiteness & asphericity (as defined here), with two unresolved cases, using the congruences

(A) $2k \equiv 0$ or $2j \equiv 2l$,

(B) $k \equiv 2j$ or $k \equiv 2l$ or $j + l \equiv 2k$ or $j + l \equiv 0$, and

(C) $l \equiv j + k$ or $l \equiv j - k$,

all modulo n. That is to say that they proved exactly which of the groups are finite and, with two exceptions, they proved exactly which of the groups are aspherical. For the details on what each of (A), (B), and (C) imply in these situations, see [5, Table 2].

It is pleasant to notice that in each of (A), (B), and (C), there is symmetry in swapping j with l.

Note that if (A) & (C) are true, then both conditions in (A) are true and both conditions of (C) are true; if, further, (B) is true, then all conditions of (B) are true too.

For details on what each of (A), (B), and (C) imply, see Sections 4.2, 4.3, and 4.4, respectively.

3.6 Deficiency.

The *deficiency of a presentation* $P = \langle X \mid R \rangle$ is given by $\operatorname{def}(P) = |X| - |R|$, whereas the *deficiency of a group* is given by

$$\operatorname{def}(G) = \max\{\operatorname{def}(P) \mid P \text{ is a presentation of } G\}.$$

It is well known that if G is a group and $\operatorname{def}(G) \geq 1$, then G is infinite. A result by Baumslag & Pride in [2] states that if $\operatorname{def}(G) \geq 2$, then G is large. Let P be a presentation of a group G. Then if $\operatorname{def}(P) = 0$ and P is aspherical, $\operatorname{def}(G) = 0$ by [32, p. 478].

3.7 Shift Extension.

By automorphisms, the cyclic group $\mathbb{Z}_n$ of order n acts on $G_n(w)$, so there is a *shift extension*

$$E_n(w) = G_n(w) \rtimes_\theta \mathbb{Z}_n,$$

which so happens to admit a two-relator, two-generator presentation $E_n(w) \cong \langle a, x \mid a^n, W\rangle$, where W is just w after the substitutions $x_i = a^i x a^{-i}$, so the shift $\theta \in \mathrm{Aut}(G)$, the same θ as in Section 3.1 that sends x_i to x_{i+1}, arises via conjugation by a in $E_n(w)$. Thus in our case we have

$$E_n(x_0x_jx_kx_l) \cong \langle a, x \mid a^n, xa^jxa^{k-j}xa^{l-k}xa^{-l}\rangle. \tag{3.7.1}$$

Note that [5, Lemma 5.2] states that, for (C) true, the extension (3.7.1) of G has the presentation

$$\langle a, z \mid a^n, z^2a^{k-2p}z^2a^{-k-2p}\rangle, \tag{3.7.2}$$

where $p = j$ if $l \equiv j + k$, and $p = -l$ if $l \equiv j - k$.

3.8 Small Cancellation Conditions.

If any group H has a presentation that satisfies the small cancellation conditions $C(4)$ and $T(4)$, then a result by Collins in [11] states that H contains a free subgroup of rank two if and only if H is not isomorphic to one of:

(i) $\langle a, b \mid ab = ba\rangle \cong \mathbb{Z} \times \mathbb{Z}$,

(ii) $\langle a, b \mid a = bab\rangle$ or $\langle a, b \mid b = aba\rangle$,

(iii) $\langle a, b \mid a^2b^2\rangle$ or $\langle a, b \mid a^2b^{-2}\rangle$,

(iv) $\langle a, b \mid a^4, b^4, (ab)^2\rangle$ or $\langle a, b \mid a^4, b^4, (ab^{-1})^2\rangle$,

(v) $\langle a, b \mid (ab)^2, (ab^{-1})^2\rangle$,

(vi) $\langle a \mid\rangle \cong \mathbb{Z}$,

(vii) $\langle a \mid a^s\rangle \cong \mathbb{Z}_s$,

$(viii)$ $\langle a, b \mid a^2, b^2\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong D_\infty$.

3.9 Bogley & Parker Isomorphisms.

In the beginning of [5, Section 3], four bijective transformations $\sigma, \tau, \theta_F, u$ are considered on the set Φ_n of positive words of length four in the generators $(x_i)_{i \in \overline{0,n-1}}$ of the free group F_n. They are given by

$$\begin{aligned} \sigma(x_i x_j x_k x_l) &= x_i x_l x_k x_j, \\ \tau(x_i x_j x_k x_l) &= x_l x_i x_j x_k, \\ \theta_F(x_i x_j x_k x_l) &= x_{i+1} x_{j+1} x_{k+1} x_{l+1}, \quad \text{and} \\ u(x_i x_j x_k x_l) &= x_{ui} x_{uj} x_{uk} x_{ul}, \end{aligned} \tag{3.9.1}$$

where u is an element of the group $\mathbb{Z}_n^*$ of units modulo n.

For $w = x_0 x_j x_k x_l$ and $c \in \Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$, we have $G_n(w) \cong G_n(c(w))$ via an isomorphism that preserves shift dynamics by [5, Corollary 3.3], where D_4 is the dihedral group of 8 elements generated by σ and τ, computed directly in the proof of [5, Lemma 3.1], and the semidirect product consists of $\mathbb{Z}_n^*$, which corresponds to u, acting naturally by multiplication on $\mathbb{Z}_n$, which corresponds to θ_F.

3.10 Polynomial Associated With a Word.

Let a_i be the exponent-sum of x_i for a word w in the x_i with $0 \leq i < n$. (For example, if $w = x_0 x_1 x_0 x_1^{-1}$, then $a_0 = 2$ and $a_1 = 0$.) Then the *polynomial associated with* w is

$$f(x) = \sum_{i=0}^{n-1} a_i x^i.$$

CHAPTER 4

Towards a Tits Alternative.

The main result of this chapter is a proof that the Tits alternative (defined in Section 3.2) holds for cyclically presented groups of positive word length four, except possibly the FTF case for primary divisor (defined in Section 3.4) equal to one and $\gamma = 1$ (where γ is as defined in Section 4.1).

To do this we begin by showing some results about a useful number and the shift extension (3.7.1) that will be used throughout this chapter. Next we consider when each of the conditions (A), (B), and (C) are true in isolation from the truth values of the other conditions. The theorems we gather from this have applications when considering each of the individual cases given by the potential truth values we can assign to each of (A), (B), and (C). With these results established, we inspect the TFF and FFF cases, which, as we shall see, can be handled together. The rest of the sections of this chapter cover the remaining cases. At the end of these sections we will establish that the Tits alternative does indeed hold for cyclically presented groups of positive word length four, with the possible exception of the FTF case, which is left open.

The ideas from Chapter 3 are used throughout; we will make of note of them as they arise.

4.1 The Secondary Divisor and the Shift Extension.

Bogley & Parker define $\gamma = (n, k-2j, l-2k+j, k-2l, j+l)$ as the *secondary divisor* in [5]. (The primary divisor is simply (n, j, k, l).)

Consider the shift extension (3.7.1). Here we will analyse the introduction of different generators (by means of Tietze transformations) then quotient out by particular powers of the generator a (defined below) in order to map $E_n(w)$ onto large groups (by using the results of Section 3.3):

$$\begin{aligned} E_n(j,k,l) &= \langle a, x \mid a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \rangle \\ &\cong \langle a, u \mid a^n, u^2 a^{k-2j} ua^{l-k-j} ua^{-l-j} \rangle \text{ (by } u = xa^j) \\ &\cong \langle a, u \mid a^n, ua^{2j-k} u^2 a^{l+j-2k} ua^{-k+j-l} \rangle \text{ (by } u = xa^{k-j}) \\ &\cong \langle a, u \mid a^n, ua^{j-l+k} ua^{-j-l+2k} u^2 a^{k-2l} \rangle \text{ (by } u = xa^{l-k}). \\ &\cong \langle a, u \mid a^n, ua^{j+l} ua^{k-j+l} ua^{-k+2l} u \rangle \text{ (by } u = xa^{-l}). \end{aligned}$$

From the first substitution, E maps onto $\mathbb{Z}_{M_0} * \mathbb{Z}_4$, where $M_0 = (n, k-2j, -(j+l) - (k-2l), -l-j)$; from the second, E maps onto $\mathbb{Z}_{M_1} * \mathbb{Z}_4$, where $M_1 = (n, -(k-2j), l - 2k + l, -(k-2j) - (j+l))$; from the third, E maps onto $\mathbb{Z}_{M_2} * \mathbb{Z}_4$, where $M_2 = (n, -(l - 2k + j) - (k-2j), -(l-2k+j), k-2l)$; and from the fourth, E maps onto $\mathbb{Z}_{M_3} * \mathbb{Z}_4$, where $M_3 = (n, j+l, -(l-2k+j) - (k-2l), -(k-2l))$. But now γ divides all of M_0, M_1, M_2 and M_3.

Thus we have the following. (See Section 3.3.)

Theorem 4.1.1. *If $M_0 > 1, M_1 > 1, M_2 > 1$, or $M_3 > 1$, then G is large. In particular, if $\gamma > 1$, then G is large.*

This is already quite powerful.

Example 4.1.2. *If n is even and $G = G_n(1,2,5)$ (i.e., $G = G_n(x_0x_1x_2x_5)$), then G is large, since $\gamma = (n, 0, 2, -4, 6) = 2 > 1$.*

Theorem 4.1.3. *Let $n \geq 2$, $0 \leq j,k,l < n$, and $0 \leq j', k', l' < n$. Consider the vectors*

$$v_1 = ((k-2j) \pmod n, (l-k-j) \pmod n, (-l-j) \pmod n),$$
$$v_2 = ((j+l-2k) \pmod n, (j-k-l) \pmod n, (2j-k) \pmod n),$$
$$v_3 = ((k-2l) \pmod n, (j+k-l) \pmod n, (2k-j-l) \pmod n),$$
$$v_4 = ((l+j) \pmod n, (l+k-j) \pmod n, (2l-k) \pmod n),$$

$$v_1' = ((k'-2j') \pmod n, (l'-k'-j') \pmod n, (-l'-j') \pmod n),$$
$$v_2' = ((j'+l'-2k') \pmod n, (j'-k'-l') \pmod n, (2j'-k') \pmod n),$$
$$v_3' = ((k'-2l') \pmod n, (j'+k'-l') \pmod n, (2k'-j'-l') \pmod n),$$
$$v_4' = ((l'+j') \pmod n, (l'+k'-j') \pmod n, (2l'-k') \pmod n).$$

If $v_t = v_s'$ for some $1 \le s,t \le 4$ then $E_n(j,k,l) \cong E_n(j',k',l')$.

Proof. Cyclically permute the relators in the different presentations of $E_n(J,K,L)$ above so that u^2 is the prefix of the said relators. Then compare them. □

The following is now immediate.

Corollary 4.1.4. *Let $n \ge 2$, $0 \le j,k,l < n$, $0 \le j',k',l' < n$ and suppose $G_n(j,k,l)$ contains a non-abelian free subgroup. If $v_t = v_s'$ for some $1 \le s,t \le 4$, then $G_n(j',k',l')$ contains a non-abelian free subgroup.*

For the following sections, recall (A), (B), and (C) from Section 3.5. We study them in isolation first simply because it is possible and fruitful to do so and to prevent repetition in the sections handling the cases (*e.g.* TTT).

4.2 If (A) is true.

Recall Section 3.5: (A) is the congruence conditions $2k \equiv 0$ or $2j \equiv 2l$

Suppose (A) is true. Since the defining word for G is $x_0x_jx_kx_l$, we can cyclically permute the word to $x_jx_kx_lx_0$. Now subtract j from all the subscripts, giving

$$x_0x_{k-j}x_{l-j}x_{n-j}.$$

Define $J = k-j, K = l-j, L = n-j$ to get

$$x_0x_Jx_Kx_L.$$

But now $2j \equiv 2l$ if and only if $2(l-j) \equiv 0$ if and only if $2K \equiv 0$. Thus, without loss of generality, we may assume $2k \equiv 0$.

4.3 If (B) is true.

Recall Section 3.5: (B) is the congruence conditions $k \equiv 2j$ or $k \equiv 2l$ or $j+l \equiv 2k$ or $j+l \equiv 0$.

By [5, Lemma 5.3], if n, j, k, l satisfy (B), then there exists $c \in \Gamma_n$ such that $c(x_0 x_j x_k x_l) = x_0 x_{j'} x_{k'} x_{l'}$, where:

(a) $k' \equiv 2j'$,

(b) $\gcd(n, j', k', l') = \gcd(n, j, k, l)$,

(c) $\gcd(n, k'-2j', l'-2k'+j', k'-2l', j'+l') = \gcd(n, k-2j, l-2k+j, k-2l, j+1)$,

(d) the parameters (n', j', k', l') satisfy
(A) if and only if (n, j, k, l) satisfy (A), and

(e) the parameters (n', j', k', l') satisfy
(C) if and only if (n, j, k, l) satisfy (C).

Part (a) is established by setting c equal to: the identity if $k \equiv 2j$; σ if $k \equiv 2l$; $\tau \circ \theta_F^j$ if $j+l \equiv 0$; and $\theta_F^{-k} \circ \sigma \circ \tau^2$ if $j+l \equiv 2k$. Here σ, τ, and θ_F are from (3.9.1). The parameters (n, j', k', l') are given, respectively, by (n, j, k, l), (n, l, k, j), $(n, j, 2j, j+k)$, and $(n, j-k, -k, l-k)$; the parts (b)-(e) are established through direct computation.

Remark 4.3.1. *We may thus assume, by [5, Corollary 3.3], that if (B) is true then $k \equiv 2j$.*

If $k \equiv 2j$, [5, Lemma 5.4] states that: the shift extension $E = G \rtimes_\theta \mathbb{Z}_n$ takes the form $\langle a, u \mid a^n, u^3 \alpha u \beta \rangle$, where $\alpha = a^{1-3j}$ and $\beta = a^{-l-j}$ in $\langle a \rangle \cong \mathbb{Z}_n$, by the substitution $u = xa^j$ into the presentation (3.7.1) for $E_n(w)$; the condition (A) is true if and only if $\alpha = \beta^{\pm 1}$ (by direct computation); if (A) is true, then the element $\alpha \in \langle a \rangle \cong \mathbb{Z}_n$ has order n/γ, and this holds because we can write $\gamma = (n, j+l)$ and the order of $\alpha = \beta^{\pm 1}$ is $n/(n, j+l)$; and the condition (C) is true if and only if both $\alpha = 1$ and $\beta = 1$ (by direct computation).

4.4 If (C) is true.

Recall Section 3.5: (C) is congruence conditions $l \equiv j+k$ or $l \equiv j-k$.

The following observations for the two conditions of when (C) is true allow us to apply Theorem 4.1.1.

Remark 4.4.1. *For $l \equiv j+k$,*

$$\begin{aligned}(n, k-2j, k+2j) &= (n, k-2j, 2j-k, -2j-k, 2j+k)\\ &= (n, k-2j, j+k-2k+j, k-2(j+k), j+(j+k))\\ &= (n, k-2j, l-2k+j, k-2l, j+l)\\ &= \gamma.\end{aligned}$$

For $j \equiv l+k$, similarly, we have

$$\begin{aligned}(n, k-2l, k+2l) &= (n, -k-2l, 2l-k, k-2l, 2l+k)\\ &= (n, k-2(l+k), l-2k+(l+k), k-2l, (l+k)+l)\\ &= (n, k-2j, l-2k+j, k-2l, j+l)\\ &= \gamma.\end{aligned}$$

The following is immediate from Theorem 4.1.1.

Corollary 4.4.2. *Let $G = G_n(x_0x_jx_kx_l)$.*

1. *If $l \equiv j+k$ and $d = (n, k-2j, k+2j) \geq 2$, then G is large.*

2. *If $j \equiv l+k$ and $d = (n, k-2l, k+2l) \geq 2$, then G is large.*

It remains to consider the cases when $l \equiv j+k$ & $\gamma = 1$ and when $l \equiv j-k$ & $\gamma = 1$. These are handled in Section 4.6, Section 4.8, Section 4.9, and Section 4.10.

Lemma 4.4.3. *Suppose (C) is true. Let $p = j$ if $l \equiv j+k$, and $p = -l$ if $j \equiv l+k$. Then $G \cong G_n(p, k, k+p)$.*

Proof. Let $l \equiv j+k$. Then clearly $G = G_n(p, k, k+p)$. Let $j \equiv l+k$. Then $G_n(j,k,l) = G_n(x_0x_{k-p}x_kx_{-p}) = G_n(x_px_kx_{k+p}x_0) = G_n(x_0x_px_kx_{k+p})$ by cyclically permuting. □

4.5 TFF and FFF.

Here we consider when (B) and (C) are both false. The next theorem also covers most of the cases when both (A) and (C) are both true.

Theorem 4.5.1. *Suppose either*

1. *both (A) and (C) are true and suppose either $k \not\equiv 0$ or $j \not\equiv l$, or*

2. *both (B) and (C) are false.*

Then G has a free subgroup of rank two.

Proof. According to [5, Lemma 6.2 and Theorem 6.3], the presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is aspherical and C(4)-T(4) if either both (A) and (C) are true or both (B) and (C) are false. If (A) and (C) are true, C(4)-T(4) is shown by considering length two subwords of w up to cyclic shifts and showing that neither of these subwords is a piece, which is both necessary and sufficient by [5, Lemma 2.1], whose proof relies on the fact that w is length four in that C(4) is satisfied exactly when each piece has length one, and if each piece is of length one, then T(4) is satisfied as the relators are all positive (see [17] in the proof of [5, Lemma 2.1]); and C(4)-T(4) presentations are aspherical. If (B) and (C) are false, C(4)-T(4) is shown by [5, Lemma 2.1], which is applicable since, modulo n, the numbers $j, k-j, l-k, -l$ are pairwise distinct.

The same section states contrapositively that if $\mathcal{P}_n(x_0x_jx_kx_l)$ is aspherical and either $k \not\equiv 0$ or $j \not\equiv l$, then $G = G_n(x_0x_jx_kx_l)$ is torsion-free.[1] The deficiency of G is zero by 3.6. Now G has a free subgroup of rank two since G cannot be isomorphic to any of the presentations Collins listed[2] as these groups either have positive deficiency or contain a torsion element.

We have $k \equiv 0$ and $j \equiv l$ only when both (A) and (C) are true. □

The cases when (A) and (C) are true are covered in Section 4.6 and Section 4.8.

4.6 TTT.

We consider when all of (A), (B), and (C) are true.

Theorem 4.6.1. *Let $G = G_n(x_0x_jx_kx_l)$. Suppose $(n, j, k, l) = 1$ and $\gamma = 1$. If (A), (B), and (C) are true, then all possible cases are as follows:*

1. *$j \equiv k \equiv l \equiv 0$, in which case $n = 1$ and $G \cong G_1(x_0^4) \cong \mathbb{Z}_4$;*

2. *$j \equiv \frac{\varepsilon n}{4}$, $k \equiv \frac{n}{2}$, $l \equiv \frac{(\varepsilon+2)n}{4}$, where $\varepsilon = \pm 1$, in which case $n = 4$ and $G \cong G_4(x_0x_1x_2x_3) \cong F_3$*

[1] Bogley & Parker's proof of this states that if $\mathcal{P}_n(x_0x_jx_kx_l)$ is aspherical and G is torsion-free, then [20, Theorem 3] tells us that the relator w or one of its shifts must be a proper power in the free group F_n with basis $(x_i)_{i\in\overline{0,n-1}}$, which in turn implies $k \equiv 0$ and $j \equiv l$.

[2] See 3.8.

In particular, G is large when $n \neq 1$ and finite when $n = 1$. Hence the Tits alternative holds for G when (A), (B), and (C) are all true.

Proof. Note that by Section 3.5 above, all conditions of (A), of (B), and of (C) are true.

Since $2k \equiv 0$, either $k \equiv 0$ or $k \equiv \frac{n}{2}$. In the former case, since $l \equiv j + k$, we have $l \equiv j$, which in turn gives $2j \equiv 0$ as $j + l \equiv 0$. Then we have two cases: $j \equiv k \equiv l \equiv 0$, which satisfy all conditions and imply $G \cong G_1(x_0^4) \cong \mathbb{Z}_4$; or $k \equiv 0$ and $j \equiv l \equiv \frac{n}{2}$, which imply that $n = \gamma = 1$, a contradiction.

In the latter case, since $l \equiv j \pm k$, we have $l \equiv j \pm \frac{n}{2}$, which in turn gives either $j \equiv \frac{n}{4}$ or $\frac{3n}{4}$ (because $2j \equiv k$) and either $l \equiv \frac{3n}{4}$ or $\frac{n}{4}$, respectively, which satisfy all the conditions, so now $(n, j, k, l) = 1$ implies $n = 4$ and $G \cong G_4(x_0x_1x_2x_3) \cong F_3$ is large. □

4.7 TTF.

We consider the case when (A) and (B) are true but (C) is false.

If $\gamma = 1$, then G is finite by [5, Theorem 8.1]; and if $\gamma > 1$, then G is large by Theorem 4.1.1. We record this as follows:

Corollary 4.7.1. *The Tits Alternative holds for cyclically presented groups of positive word length four when (A) and (B) are true together with when (C) is false; in particular, if $\gamma = 1$, then G is finite, and if $\gamma > 1$, G is large.*

4.8 TFT.

This section focuses on when (A) is true, (B) is false, and (C) is true.

We make use of the following theorem.

Theorem 4.8.1. *[34, Theorem 18] For the group given by the presentation*

$$H = \langle a, b \mid a^p, b^q, (a^\mu b^\nu)^r \rangle,$$

where $1 \leq \mu \leq p-1$ and $1 \leq \nu \leq q-1$,

(a) if $(\mu, p) = 1$, $(\nu, q) = 1$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then H is large; if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then H is infinite and solvable; and if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, then H is finite.

(b) if $(\mu, p) > 1$ or $(\nu, q) > 1$, then H is large unless either

(i) $p = 2, r = 2$, and $(\nu, q) = 2$; or

(ii) $q = 2, r = 2$, and $(\mu, p) = 2$;

in which case H is infnite and solvable.

Theorem 4.8.2. *Let (A) & (C) be true and (B) be false.*

1. *If $n \geq 3$, then G is large.*

2. *If $n = 2$, then G is infinite and solvable.*

Proof. Note that [5, Lemma 5.2] implies that the extension E of G in this case is of the form

$$\langle a, z \mid a^n, (z^2 a^{k-2p})^2 \rangle,$$

for either $p = j$ or $p = -l$. These presentations map onto $\langle a, z \mid a^n, z^m, (z^2 a^{k-2p})^2 \rangle$, which, by Theorem 4.8.1, is large for $1 \leq 2 \leq m - 1$, $k - 2p \equiv 0$, $(k - 2p, n) = 1$, $(2, m) = 1$, and $\frac{1}{n} + \frac{1}{m} + \frac{1}{2} < 1$, so, letting, say, $m = 7$, we have G is large for $n \geq 3$.

By Section 3.5, we may assume through the results of Section 4.4 that since both conditions of (C) are true, we have $(n, k - 2j, k + 2j) = 1$ and $(n, k - 2l, k + 2l) = 1$, and by the above, either: (K) either $(k - 2j, n) > 1$ or $k \equiv 2j$, or (L) either $(k + 2l, n) > 1$ or $k \equiv -2l$. But by Section 3.5 again, $(n, k - 2j, k + 2j) = 1$ and $(n, k - 2l, k + 2l) = 1$, as well as $(k + 2l, n) = 1$, are each equivalent to $(k - 2j, n) = 1$, and if (B) is false, then neither $k \equiv 2j$ nor $k \equiv -2l$, so neither (K) nor (L) holds.

We may thus assume that $n = 2$. A systematic search through the parameters j, k, l when $n = 2$ gives two presentations in the TFT case, both of which are isomorphic to $G_2(x_0^2 x_1^2)$ (the Baumslag-Solitar group $\mathrm{BS}(1, -1)$), which is Collins' group (iii) of [11] of Section 3.8 since the TFT case satisfies the C(4)-T(4) small cancellation conditions by [5, Theorem 6.3], so therefore $G_2(x_0^2 x_1^2)$ does not have a free subgroup of rank 2. However, it is solvable, since the abelian subgroup generated by $x_0 x_1$ and x_0^2 has index 2.[3][4]

□

[3] I would like to thank the Mathematics Stack Exchange user Derek Holt for this observation. It can be verified via GAP easily.

[4] In particular, we can say $K = G_2(x_0^2 x_1^2)$ is "virtually abelian", which is to say that K has a abelian subgroup of finite index. I would like to thank Prof. Laura Ciobanu for this observation.

4.9 FTT.

We now consider when (A) is false but both (B) and (C) are true.

Theorem 4.9.1. *[5, Theorem 7.2(ii),(iv)] If (A) is false and (C) is true, then $G = \mathbb{Z}_4$ if and only if both $(n, 2k) = 1$ and either $l \equiv j + k$ & $(n, j) = 1$ or $l \equiv j - k$ & $(n, l) = 1$.*

Theorem 4.9.2. *Let (A) be false, (B) be true, and (C) be true. Suppose $\gamma = 1$. Suppose $(n, j, k, l) = 1$. Then $G = \mathbb{Z}_4$.*

Proof. We may assume that $k \equiv 2j$ by Section 4.3.

Let $d = (n, 2k)$.

For when $l \equiv j+k$, let $\theta_1 = (n, j)$. We have $\theta_1 \mid j$, so $\theta_1 \mid k$, meaning $\theta_1 \mid (n, k-2j, k+2j) = 1$. Thus $\theta_1 = 1$. Furthermore, by Section 4.4, we have $(n, k - 2j, k + 2j) = 1$, which is true if and only if $(n, 0, 4j) = 1$, but the latter implies $d = 1$, so Theorem 4.9.1 applies.

For when $l \equiv j - k$, let $\theta_2 = (n, l)$. We have that $\theta_2 \mid j - k \equiv -j$ implies $\theta_2 \mid k$, which implies $\theta_2 \mid (n, k - 2l, k + 2l) = 1$, *i.e.*, $\theta_2 = 1$. We have, by Section 4.4, $(n, k - 2l, k + 2l) = 1$, which is true if and only if $(n, 3k - 2l, 2j - k) = 1$ if and only if $(n, 4j, 0) = 1$, but then $d = 1$. Apply Theorem 4.9.1.

□

4.10 FFT.

Here we consider when both (A) and (B) are false and (C) is true. I prove the following in this section.

Theorem 4.10.1. *Suppose (A) and (B) are false and (C) is true. Let*

$$p = \begin{cases} j & : l \equiv j + k \\ -l & : j \equiv l + k. \end{cases}$$

Suppose $\gamma = 1$ and the primary divisor is one. Then if $(n, p) = 1$ and $(n, 2k) = 1$, then $G \cong \mathbb{Z}_4$; otherwise, G is large.

This is done using three lemmata.

Lemma 4.10.2. *Suppose (A) and (B) are false while (C) is true. Let p be as above. If G is not large, then one of the following holds:*

1. *$(n, p) = 1$ and $(n, 2k) = 1$, in which case $G \cong \mathbb{Z}_2$;*
2. *$G \cong G_n(1, J, J + 1)$, where $(n, 4) = 2$ and $(n, J) = 2$;*
3. *$G \cong G_n(J, 1, J + 1)$, where $(n, 4) = 2$ and $(n, J) = 2$.*

Proof. Suppose $G = G_n(j, k, l)$ is not large.

By (C), [5, Lemma 5.2] gives that $E = E_n(j, k, l)$ has as a presentation $E \cong \langle a, z \mid a^n, z^2a^{k-2p}z^2a^{-k-2p}\rangle$. Now E maps onto $\langle a, z \mid a^2, z^4\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_4$ if (n, k) is even; but that group is large, a contradiction. Thus (n, k) is odd. If $(n, 4p) > 2$, then (by adjoining the relator z^2) we have that E maps onto $\langle a, z \mid a^{(n,4p)}, z^2\rangle$, which is large, a contradiction. Thus $(n, 4p) \leq 2$. Moreover, for any $q \geq 1$, the group E maps onto $G(q) = \langle a, z \mid a^{(n,2k)}, (z^2a^{k-2p})^2, z^q\rangle$. Suppose $k - 2p \equiv 0 \pmod{(n, 2k)}$. Then $Q(4) \cong \mathbb{Z}_{(n,2k)} * \mathbb{Z}_4$, which is large if $(n, 2k) > 1$. Now suppose $k - 2p \not\equiv 0 \pmod{(n, 2k)}$. Then $Q(7)$ is large if $(n, 2k) > 2$ by [1]. Therefore, we may assume $(n, 2k) \leq 2$.

Let $(n, 2k) = 1$. Then $(n, 2p) = 1$, so $(n, p) = 1$, so then $G \cong \mathbb{Z}_4$ by [5, Theorem 7.2]. Hence (1). Thus we may assume $(n, 2k) = 2$, so that $(n, 4p) = 2$ and $(n, k) = 1$; this implies $(n, 4) = 2$. By Lemma 4.4.3, $G = G_n(p, k, k+p)$. Now $G \cong G_n(J, 1, J+1)$ since $(n, k) = 1$, where $J \equiv pk^{-1}$ (by [5, Section 3]). Then $(n, J) = (n, pk^{-1}) = (n, p)$ is either 1 or 2. The latter case gives item (3) on the list. If $(n, J) = 1$, then $G_n(x_0x_Jx_1x_{J+1}) \cong G_n(x_0x_1x_{J^{-1}}x_{J^{-1}+1}$, and replacing J^{-1} by J then gives item (2) on the list. $\square$

Items (2) and (3) of Lemma 4.10.2 in the next couple of lemmata.

Lemma 4.10.3. *Let $n \geq 4$ be even and J, odd. Then $G_n(1, J, J + 1)$ is large.*

Proof. Let $G = G_n(1, J, J + 1)$. Then

$$G \cong \langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1} \mid y_i = x_ix_{i+1}, y_iy_{i+1}(1 \leq i < n)\rangle.$$

Note that $y_0 = y_J^{-1} = y_{2J} = y_{3J}^{-1} = \cdots = y_{(n-2)J} = y_{(n-1)J}^{-1}$; *i.e.*, $y_i = y_0^{(-1)^i}$ since J is odd. Thus

$$
\begin{aligned}
G &\cong \langle x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \mid y_i = x_i x_{i+1}, y_i = y_0^{(-1)^i} (0 \le i < n) \rangle \\
&\cong \langle x_0, \dots, x_{n-1}, y \mid y^{(-1)^i} = x_i x_{i+1} (0 \le i < n) \rangle \text{ (by writing } y = y_0 \text{ and eliminating } y_1 \dots, y_{n-1})\rangle \\
&\cong \langle x_0, \dots, x_{n-1}, y \mid x_{2u} x_{2u+1} = y, x_{2u+1} x_{2u+2} = y^{-1} (0 \le u < n) \rangle \\
&\cong \langle x_0, \dots, x_{n-1}, y \mid x_{2u} x_{2u+1} = y, x_{2u+1} = y^{-1} x_{2u+2}^{-1} (0 \le u < n) \rangle \\
&\cong \langle x_0, x_2, \dots, x_{n-2}, y \mid x_{2u} y^{-1} x_{2u+2}^{-1} = y (0 \le u < n) \rangle \text{ (by eliminating } x_1, x_3, \dots, x_{n-1}) \\
&\cong \langle x_0, x_2, \dots, x_{n-2}, y \mid x_{2u+2} = y^{-1} x_{2u} y^{-1} (0 \le u < n) \rangle.
\end{aligned}
$$

Now we can eliminate $x_2, x_4, \dots, x_{n-4}, x_{n-2}$ and write $x = x_0$ to get $G \cong \langle x, y \mid y^{-n/2} x y^{-n/2} \rangle$. Kill $y^{n/2}$. Then G maps onto $\langle x, y \mid y^{n/2} \rangle \cong \mathbb{Z} * \mathbb{Z}_{n/2}$. But $n \ge 4$, so G is large.

□

Lemma 4.10.4. *Let $n \ge 4$, $(n, 4) = 2$, $(n, J) = 2$. Then $G_n(J, 1, J+1)$ is large.*

Proof. Suppose $n = 2m$ for $m \ge 3$ is odd, $J = 2q$, $(m, q) = 1$. Let $G = G_n(J, 1, J+1)$. Then

$$
\begin{aligned}
G &\cong \langle x_0, \dots, x_{2m-1} \mid x_i x_{i+2q} x_{i+1} x_{i+2q+1} (0 \le i < 2m) \rangle \\
&\cong \langle x_0, \dots, x_{2m-1}, y_0, \dots, y_{2m-1} \mid y_i y_{i+1}, y_i = x_i x_{i+2q} (0 \le i < 2m) \rangle
\end{aligned}
$$

Then $y_i = y_0^{(-1)^i}$ for every $0 \le i < 2m$, so you can eliminate $y_1, \dots, y_{2m-1}$ and write $y = y_0$. Now

$$
\begin{aligned}
G &\cong \langle x_0, \dots, x_{2m-1}, y \mid y^{(-1)^i} = x_i x_{i+2q} (0 \le i < 2m) \rangle \\
&\cong \langle x_0, \dots, x_{2m-1}, y \mid y = x_{2u} x_{2u+2q}, y^{-1} = x_{2u+1} x_{2(u+q)+1} (0 \le u < m) \rangle \\
&\cong \langle a_0, \dots, a_{2m-1}, b_0, \dots, b_{2m-1}, y \mid y = a_u a_{u+q}, y^{-1} = b_u b_{u+q} (0 \le u < m) \rangle,
\end{aligned}
$$

by letting $a_u = x_{2u}$ and $b_u = x_{2u+1}$ for $0 \le u < m$, where the subscripts are now modulo m. For $0 \le u < m$, multiply the subscripts by $q^{-1} \pmod m$ and set $v = uq^{-1} \pmod m$. Then

$$
G \cong \langle a_0, \dots, a_{2m-1}, b_0, \dots, b_{2m-1}, y \mid y = a_v a_{v+1}, y^{-1} = b_v b_{v+1} (0 \le v < m) \rangle.
$$

Write $a = a_0$, $b = b_0$ and eliminate $a_1, \dots, a_{m-2}, a_{m-1}$ and $b_1, \dots, b_{m-2}, b_{m-1}$. Then

$$
\begin{aligned}
G &\cong \langle a, b, y \mid a = y^{-(m-1)/2} a^{-1} y^{(m+1)/2}, b = y^{-(m-1)/2} b^{-1} y^{(m+1)/2} \rangle \\
&\cong \langle a, b, y \mid a y^{(m-1)/2} a = y^{(m+1)/2}, b y^{(m-1)/2} b = y^{(m+1)/2} \rangle \\
&\cong \langle a, b, y \mid (a y^{(m-1)/2})^2 = y^m, (b y^{(m-1)/2})^2 = y^m \rangle,
\end{aligned}
$$

which maps onto $Q = \langle a, y \mid (ay^{(m-1)/2})^2, y^m, a^7 \rangle$, which is large for all $m \geq 3$ by [1].

□

Theorem 4.10.1 now follows from Lemmata 4.10.2, 4.10.3, 4.10.4.

4.11 FTF.

No progress, so far, has been made regarding the Tits alternative in the FTF case (beyond the preliminary results above, such as those about the secondary divisor γ, and the calculations in Chapter 6); we leave the following as an open problem.

Open Problem 4.11.1. *Suppose (A) and (C) are false, $k \equiv 2j$ (so (B) is true), $(n, j, k, l) = 1$, and $\gamma = 1$ for $G = G_n(j, k, l)$. Does G satisfy the Tits alternative?*

This is a similar situation to that of the Tits alternative for cyclically presented groups of positive word length three. See [14].

4.12 Conclusion.

By pulling together the results of this Chapter, from Section 4.1 to Section 4.11, we have proven the following.

Main Theorem 4.12.1. *Suppose Bogley & Parker's (A), (B), and (C) are not false, true, and false, respectively; that is, suppose any case except FTF. Then the Tits alternative holds for $G_n(x_0x_jx_kx_l)$. In the FTF case, the Tits alternative holds for $G_n(x_0x_jx_kx_l)$ except possibly when both $(n, j, k, l) = 1$ and $\gamma = 1$.*

CHAPTER 5

Infinite Abelianisation.

In Theorem 5.0.2 we classify the groups $G_n(j, k, l)$ that have infinite abelianisation. To do this we use the following number theoretic characterisation of when a cyclically presented group has infinite abelianisation.

Theorem 5.0.1. *[22] If $G_n(w)$ is a cyclically presented group and f is the polynomial associated with w, then*

$$|G_n(w)^{\mathrm{ab}}| = \left| \prod_{\xi^n=1} f(\xi) \right|,$$

where the left hand side is infinity if and only if the right hand side is zero.

Theorem 5.0.2. *For $w = x_0 x_j x_k x_l$ such that $(n, j, k, l) = 1$, we have $|G_n(w)^{\mathrm{ab}}|$ is infinite if and only if both n is even and $\{j, k, l\}$ corresponds to the multiset $\{odd, odd, even\}$.*

Proof. Let $(n, j, k, l) = 1$.

We have $|G_n(w)^{\mathrm{ab}}|$ is infinite if and only if

$$\left| \prod_{\xi^n=1} f(\xi) \right| = 0$$

if and only if $f(\xi) = 0$ for some $\xi^n = 1$, where $f(t) = 1 + t^j + t^k + t^l$ is the polynomial associated with w.

If n is even and $\{j, k, l\} = \{\text{odd}, \text{odd}, \text{even}\}$, then $f(-1) = 0$, so $|G_n(w)^{\mathrm{ab}}|$ is infinite.

If $f(\xi) = 0$ for some $\xi^n = 1$, then $|\xi| = 1$ implies that $\bar{\xi} = \xi^{-1}$, so that $f(\xi) = 0$ and $f(\xi^{-1}) = f(\bar{\xi}) = 0$ imply

$$1 + J + K + L = 0$$

and

$$1 + J^{-1} + K^{-1} + L^{-1} = 0,$$

where $J = \xi^j, K = \xi^k, L = \xi^l$. We have

$$\begin{aligned}1 &= JJ^{-1} \\ &= (-K - L - 1)(-K^{-1} - L^{-1} - 1) \\ &= 3 + KL^{-1} + K^{-1}L + K + K^{-1} + L + L^{-1},\end{aligned}$$

which implies

$$(K + L)(1 + L)(1 + K) = 0;$$

similarly, we have

$$(L + J)(1 + J)(1 + L) = 0 \quad \text{and}$$
$$(J + K)(1 + K)(1 + J) = 0.$$

Notice that if none of J, K, L is -1, then $J = K = L = 0$, a contradiction, and so the system of nonlinear equations has solutions (J, K, L) of the form $(-1, K, -K)$, $(J, -1, -J)$, and $(J, -J, -1)$.

If $(J, K, L) = (-1, K, -K)$, then since $J = -1$, n is even. We have $\xi^k = K = -L = JL = \xi^{j+l}$, so that $k \equiv j + l \pmod{\operatorname{ord}(\xi)}$. Notice that since $J = -1$, we have $2 \mid \operatorname{ord}(\xi)$ and so $k \equiv j + l \pmod 2$, which implies that $\{j, k, l\}$ corresponds to the multiset $\{\text{odd}, \text{odd}, \text{even}\}$.

Similar arguments apply for (J, K, L) equal to $(J, -1, -J)$ and to $(J, -J, -1)$. $\square$

Corollary 5.0.3. *For $w = x_0x_jx_kx_l$ such that $(n, j, k, l) = 1$, we have $|G_n(w)^{\mathrm{ab}}|$ is infinite if and only if both n is even and $l \equiv j + k \pmod 2$.*

CHAPTER 6

A study of the groups with few generators.

In this Chapter we perform a computational analysis of the groups $G_n(j,k,l)$ for small values of n. We obtain results concerning the structure of the group and its abelianisation and the existence of a non-abelian free subgroup. This data was gathered using the computer programme GAP [15]; the code is in Appendix A. The software KBMAG was used [19]. We also investigate some structural aspects of the groups, based on the tables.

As noted in the introduction, what I do here is similar to the work on different words in [9] and [25].

6.1 Generic Tables up to $n = 8$.

Here tables are constructed for low values of n, making a note of the number of groups up to isomorphism, the ABC values, the value of γ, $|G|$, asphericity, the deficiency of the presentation after an application of GAP's TzGoGo command (a command which simplifies the group presentation), the existence of free subgroup of rank two, the abelian invariants, and why groups with shared abelian invariants are distinct.

(Please see Tables 6.1, 6.2, 6.3, 6.4 and 6.5.)

6.2 Tables from $n = 9$ up to $n = 18$ for the FTF case.

Here we restrict our focus to the FTF case. We keep track of γ values, the existence of free subgroups of rank two and the methods of discovering them, the abelian invariants, and why

n	$N(n)$	$[n, j, k, l]$ representatives	ABC	γ	$\|G\|$	$A(G)$	$D(G)$	$FS(G)$	Abelian Invariants [15]	Reason Distinct
1	1	$[1, 0, 0, 0] \cong \mathbb{Z}_4$	TTT	1	4	Y	$0, \mu$	N	4	
2	3	$[2, 0, 0, 1] \cong \mathbb{Z}_8$	TTF	1	8	N	$0, \mu$	N	8	
		$[2, 0, 1, 1] \cong \mathrm{BS}(1, -1)$	TFT	1	∞	Y	1	N	$0, 2$	
		$[2, 1, 0, 1] \cong \mathbb{Z}_2 * \mathbb{Z}$, [35]	TTT	2	∞	Y	1	Y		
3	5	$[3, 0, 0, 1] \cong \mathbb{Z}_{28}$	TTF	1	28	N	$0, \mu$	Y	$4, 7$	
		$[3, 0, 1, 1]$	FFT	1	∞	N	0	Y	$2, 2, 4$	†
		$[3, 1, 0, 1]$	TFT	1	∞	Y	$0, \mu$	Y		
		$[3, 0, 1, 2] \cong \mathbb{Z}_4$, [35]	FTT	1	4	N	$0, \mu$	N	4	†
		$[3, 1, 0, 2] \cong [12, 1]$	TTF	1	12	N	$0, \mu$	N		
4	9	$[4, 0, 0, 1] \cong \mathbb{Z}_{80}$	TTF	1	80	N	$0, \mu$	N	$5, 16$	
		$[4, 0, 1, 1]$	FFT	1	∞	N	1	Y	$0, 2, 2, 2$	††
		$[4, 1, 0, 1]$	TFT	2	∞	Y	$0, \mu$	Y		
		$[4, 0, 1, 2] \cong [80, 3]$	TTF	1	80	N	$0, \mu$	N	16	††
		$[4, 1, 0, 2] \cong [80, 1]$	TTF	1	80	N	$0, \mu$	N		
		$[4, 0, 1, 3]$	FFT	1	∞	N	1	Y	$0, 4$	††
		$[4, 1, 0, 3] \cong \mathbb{Z} * \mathbb{Z}_4$	TTF	2	∞	N	1	Y		
		$[4, 1, 2, 3] \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, [35]	TTT	2	∞	Y	3	Y	$0, 0, 0$	††
		$[4, 1, 3, 2]$	FFT	1	∞	N	1	Y		

$A(G)$: Aspherical?
$FS(G)$: Free Subgroup of rank two?
$D(G)$: Deficiency of a presentation of G after TzGoGo.[15]
$N(n)$: The number of isomorphism classes for n.
Discovered using the KBMAG software.[15]
μ: Theoretical maximum.
†: The derived subgroups have different abelianisation.[15]
††: The index two subgroups are different.[15]

Table 6.1: The structure of groups for $n \leq 4$.

n	$N(n)$	$[n, j, k, l]$ representatives	ABC	γ	$\|G\|$	$A(G)$	$D(G)$	$FS(G)$	Abelian Invariants [15]	Reason Distinct
5	8	$[5, 0, 0, 1] \cong \mathbb{Z}_{244}$	TTF	1	244	N	$0, \mu$	N	$4, 61$	
		$[5, 0, 1, 1]$	FFT	1	∞	N	0	Y	$2, 2, 2, 2, 4$	†
		$[5, 1, 0, 1]$	FTF	1	∞	Y	$0, \mu$	Y		
		$[5, 0, 1, 2] \cong [220, 9]$	FTF	1	220	Y	$0, \mu$	N	$4, 11$	†
		$[5, 1, 0, 2]$	TFF	1	∞	N	0	Y		
		$[5, 0, 1, 4] \cong \mathbb{Z}_4$, [35]	FFT	1	4	N	$0, \mu$	N	4	
		$[5, 1, 0, 4] \cong [20, 1]$	TTF	1	20	N	$0, \mu$	N		
		$[5, 1, 3, 2]$	FFF	1	∞	Y	$0, \mu$	Y		

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Discovered using the KBMAG software.[15]
μ: Theoretical maximum.
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††: The index two subgroups are different.[15]

Table 6.2: The structure of groups for $n = 5$.

n	$N(n)$	$[n, j, k, l]$ representatives	ABC	γ	$\|G\|$	$A(G)$	$D(G)$	$FS(G)$	Abelian Invariants [15]	Reason Distinct
6	21	$[6, 0, 0, 1] \cong \mathbb{Z}_{728}$	TTF	1	728	N	$0, \mu$	N	$7, 8, 13$	
		$[6, 0, 1, 1]$	FFT	1	∞	N	0	Y	$0, 2, 2, 2, 2, 2$	††
		$[6, 1, 0, 1]$	TFT	2	∞	Y	$0, \mu$	Y		
		$[6, 0, 1, 2]$	FTF	1	4088448	N	$0, \mu$	N	$7, 8$	
		$[6, 0, 2, 3]$	TFF	1	∞	Y	$0, \mu$	Y		†
		$[6, 1, 0, 2]$	TFF	1	∞	Y	$0, \mu$	Y		
		$[6, 2, 0, 3] \cong [728, 2]$	TTF	1	728	N	$0, \mu$	N		
		$[6, 0, 1, 3]$	TFF	1	∞	Y	$0, \mu$	Y	$0, 2, 7$	††
		$[6, 1, 0, 3] \cong \mathbb{Z} * \mathbb{Z}_{14}$	TTF	2	∞	N	1	Y		
		$[6, 0, 1, 4]$	FFF	1	∞	Y	$0, \mu$	Y	$4, 4, 8$	†
		$[6, 1, 0, 4]$	TFF	1	∞	Y	$0, \mu$	Y		
		$[6, 0, 1, 5]$	FTF	1	∞	Y	1	Y	$0, 2, 3$	††
		$[6, 1, 0, 5]$	TTF	2	∞	N	1	Y		
		$[6, 1, 2, 3] \cong \mathbb{Z} * \mathbb{Z}_2$[35]	FTT	2	∞	N	1	Y	$0, 2$	††
		$[6, 1, 3, 2]$	TFF	1	∞	Y	$0, \mu$	Y		
		$[6, 1, 5, 2]$	FFT	1	∞	N	1	Y		
		$[6, 1, 2, 4] \cong [24, 1]$	TTF	1	24	N	$0, \mu$	N	8	††
		$[6, 1, 4, 2]$	FTF	1	4088448	N	$0, \mu$	N		
		$[6, 1, 3, 4]$	TFT	1	∞	Y	3	Y	$0, 0, 0, 2$	††
		$[6, 1, 4, 3]$	FFT	2	∞	N	1	Y		
		$[6, 2, 5, 3]$	FFT	1	∞	N	1	Y		

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Discovered using the KBMAG software.[15]
μ: Theoretical maximum.
†: The derived subgroups have different abelianisation.[15]
††: The index two subgroups are different.[15]

Table 6.3: The structure of groups for $n = 6$.

n	$N(n)$	$[n, j, k, l]$ representatives	ABC	γ	$\|G\|$	$A(G)$	$D(G)$	$FS(G)$	Abelian Invariants [15]	Reason Distinct
7	11	$[7, 0, 0, 1] \cong \mathbb{Z}_{2188}$	TTF	1	2188	N	$0, \mu$	N	$4, 547$	
		$[7, 0, 1, 1]$	FFT	1	∞	N	0	Y	$2, 2, 2, 2, 2, 2, 4$	†
		$[7, 1, 0, 1]$	TFT	1	∞	Y	$0, \mu$	Y		
		$[7, 0, 1, 2]$	FTF	1	∞	Y	$0, \mu$	?	$4, 29$	†
		$[7, 1, 0, 2]$	TFF	1	∞	Y	$0, \mu$	Y		
		$[7, 0, 1, 3]$	FFF	1	∞	Y	$0, \mu$	Y	$4, 71$	††
		$[7, 1, 0, 3]$	TFF	1	∞	Y	$0, \mu$	Y		
		$[7, 0, 1, 6] \cong \mathbb{Z}_4$[35]	FFT	1	4	N	$0, \mu$	N	4	†
		$[7, 1, 0, 6] \cong [28, 1]$	TTF	1	28	N	$0, \mu$	N		
		$[7, 1, 3, 2]$	FFF	1	∞	Y	$0, \mu$	Y		
		$[7, 1, 2, 4]$	FTF	1	∞	Y	$0, \mu$	Y	$2, 2, 2, 4$	

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$N(n)$: The number of isomorphism classes for n.
Discovered using the KBMAG software.[15]
μ: Theoretical maximum.
†: The derived subgroups have different abelianisation.[15]
††: The index two subgroups are different.[15]

Table 6.4: The structure of groups for $n = 7$.

n	$N(n)$	$[n, j, k, l]$ representatives	ABC	γ	$\lvert G\rvert$	$A(G)$	$D(G)$	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
8	25	$[8, 0, 0, 1] \cong \mathbb{Z}_{6560}$	TTF	1	6560	N	0	N		$5, 32, 41$	
		$[8, 0, 1, 1]$	FFT	1	∞	N	$0, \mu$	Y		$0, 2, 2, 2, 2, 2, 2, 2$	††
		$[8, 1, 0, 1]$	TFT	2	∞	Y	$0, \mu$	Y			
		$[8, 0, 1, 2]$	FTF	1	∞	Y	$0, \mu$	$Y\#$	$G.1, G.5$	$3, 3, 32$	†
		$[8, 1, 0, 2]$	TFF	1	∞	Y	$0, \mu$	Y			
		$[8, 0, 1, 3]$	FFF	1	∞	Y	$0, \mu$	Y		$0, 3, 3, 8$	††
		$[8, 1, 0, 3]$	TFF	2	∞	Y	$0, \mu$	Y			
		$[8, 0, 1, 4]$	TFF	1	∞	Y	$0, \mu$	Y		$5, 32$	†
		$[8, 1, 0, 4]$	TTF	1	6560	N	$0, \mu$	N			
		$[8, 0, 1, 5]$	FFF	1	∞	Y	$0, \mu$	Y		$0, 4, 4, 4$	††
		$[8, 1, 0, 5]$	TFF	2	∞	Y	$0, \mu$	Y			
		$[8, 0, 1, 6]$	FFF	1	∞	Y	$0, \mu$	Y		$17, 32$	††
		$[8, 1, 0, 6]$	TFF	1	∞	Y	$0, \mu$	Y			
		$[8, 0, 1, 7]$	FFT	1	∞	N	1	Y		$0, 8$	††
		$[8, 1, 0, 7]$	TTF	2	∞	N	1	Y			
		$[8, 1, 4, 3]$	TFF	2	∞	Y	$0, \mu$	Y			
		$[8, 1, 5, 2]$	FFF	1	∞	Y	$0, \mu$	Y			
		$[8, 1, 2, 3]$ $\cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$[35]	FTT	2	∞	N	3	Y		$0, 0, 0$	††
		$[8, 1, 3, 2]$	FFF	1	∞	N	0	Y			
		$[8, 1, 3, 6]$	FFT	1	∞	N	1	Y			
		$[8, 1, 2, 4]$	FTF	1	∞	Y	$0, \mu$	$Y\#$	$G.1, G.2$	32	†
		$[8, 1, 3, 5]$	TTF	1	6560	N	$0, \mu$	N			
		$[8, 1, 5, 3]$	FFF	1	∞	Y	$0, \mu$	Y			
		$[8, 1, 4, 5]$	TFT	2	∞	Y	4	Y		$0, 0, 0, 0, 0$	††
		$[8, 4, 1, 5]$	FFT	1	∞	N	1	Y			

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$N(n)$: The number of isomorphism classes for n.
\# Discovered using the KBMAG software.[15]
μ: Theoretical maximum.
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Table 6.5: The structure of groups for $n = 8$.

groups with shared abelian invariants are distinct.

(Please see Tables 6.6, 6.7, 6.8, 6.9, and 6.10.)

6.3 Tables for FTF case with prime n from $n = 19$.

This section is also restricted to the FTF case, but this time, we inspect the case when n is prime up to 109, making a note of γ, the existence of free subgroups of rank two, the methods used to find the free subgroups of rank two, the abelian invariants, and why groups with shared abelian invariants are distinct. One table is displayed for prime n greater than 37, with lower and upper bounds on the number of groups and exact numbers where possible.

(Please see Tables 6.11, 6.12, 6.13, 6.14, 6.15, and 6.16.)

6.4 Tables for FTF case with n as prime powers.

Here we look at when n is a prime power greater than 18 for the FTF case. The tables make a note of the value of γ, the existence of a free subgroup of rank two (and why), the abelian invariants, and the reason groups with shared abelian invariants are distinct.

(Please see Tables 6.17, 6.18, 6.19, 6.20, and 6.21.)

6.5 The structure of particular families of groups.

Theorem 6.5.1. *Let $n = 2t, t \geq 1, G = G_n(1, 0, n-1)$. Then*

$$\begin{aligned} G &\cong G_n(1,2,1) \\ &\cong \langle 0, 1 \mid (01)^n \rangle \\ &\cong \mathbb{Z}_n * \mathbb{Z}. \end{aligned}$$

Proof. The defining presentation for $G_n(1, 0, n-1)$ has relators $x_i x_{i+1} x_i x_{i+n-1}$ for $i \in \overline{0, n-1}$. Adding 1 to each of the subscripts modulo n (so, applying Bogley & Parkers's θ; see 3.9.1), these become $x_{i+1}x_{i+2}x_{i+1}x_i$; cylically permuting then gives $R_i := x_i x_{i+1} x_{i+2} x_{i+1}$; that is, R_i is the i'th relator. Hence

$$G_n(1, 0, n-1) \cong G_n(1, 2, 1).$$

n	$M(n)$	$[n, j, k, l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
9	4	$[9, 0, 1, 2]$	1	?		$[4, 127]$	
		$[9, 1, 2, 4]$	1	$Y\#$	$G.1, G.8$	$[4, 19]$	†
		$[9, 1, 2, 5]$	1	?			
		$[9, 1, 3, 6]$	1	?		$[4, 7]$	
10	5	$[10, 0, 1, 2]$	1	$Y\#$	$G.1, G.5$	$[8, 11, 11]$	
		$[10, 1, 2, 4]$	1	?		$[8, 11]$	†
		$[10, 1, 3, 5]$	1	$Y\#$	$G.1, G.6$		
		$[10, 1, 4, 2]$	1	?			
		$[10, 1, 2, 5]$	2	Y		$[0, 2, 11]$	
11	4	$[11, 0, 1, 2]$	1	$Y\#$	$G.1, G.5$	$[4, 23, 23]$	
		$[11, 1, 2, 4]$	1	?		$[4, 23]$	†
		$[11, 1, 2, 6]$	1	$Y\#$	$G.1, G.9$		
		$[11, 1, 2, 5]$	1	$Y\#$	$G.1, G.9$	$[4, 89]$	
12	8	$[12, 0, 1, 2]$	1	$Y\#$	$G.1, G.9$	$[7, 16, 37]$	
		$[12, 1, 2, 4]$	1	$Y\#$	$G.1, G.9$	$[13, 16]$	†
		$[12, 1, 4, 2]$	1	$Y\#$	$G.1, G.9$		
		$[12, 1, 2, 5]$	2	Y		$[0, 4, 13]$	
		$[12, 1, 2, 6]$	1	$Y\#$	$G.1, G.9$	$[7, 16]$	
		$[12, 1, 3, 5]$	1	$Y\#$	$G.1, G.9$	$[5, 5, 16]$	
		$[12, 1, 4, 8]$	1	$Y\#$	$G.1, G.9$	$[5, 16]$	†
		$[12, 1, 8, 4]$	1	$Y\#$	$G.1, G.9$		
13	5	$[13, 0, 1, 2]$	1	$Y\#$	$G.1, G.9$	$[4, 2003]$	
		$[13, 1, 2, 4]$	1	$Y\#$	$G.1, G.9$	$[4, 79]$	†
		$[13, 1, 2, 7]$	1	$Y\#$	$G.1, G.9$		
		$[13, 1, 2, 5]$	1	$Y\#$	$G.1, G.9$	$[4, 157]$	
		$[13, 1, 2, 6]$	1	?		$[4, 53]$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
Discovered using the KBMAG software.[19]
†: The derived subgroups have different abelianisation.[15]
††: The index two subgroups are different.[15]
† † †: The index three subgroups are different.[15]

Table 6.6: The structure of FTF groups for $n = 9$ to $n = 13$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
14	8	$[14,0,1,2]$	1	$Y\#$	$G.1, G.9$	$[8,29,71]$	
		$[14,1,2,4]$	1	$Y\#$	$G.1, G.9$	$[4,4,4,8]$	† † †
		$[14,1,4,2]$	1	$Y\#$	$G.1, G.9$		
		$[14,1,2,5]$	2	Y		$[0,2,4,4,4]$	
		$[14,1,2,6]$	1	?		$[8,29]$	†
		$[14,1,8,4]$	1	$Y\#$	$G.1, G.9$		
		$[14,1,2,7]$	2	Y		$[0,2,29]$	
		$[14,1,3,5]$	1	?		$[8,43]$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
Discovered using the KBMAG software.[19]
†: The derived subgroups have different abelianisation.[15]
††: The index two subgroups are different.[15]
† † †: The index three subgroups are different.[15]

Table 6.7: The structure of FTF groups for $n = 14$.

Now consider $G \cong G_n(1,2,1)$. The defining presentation is now

$$\langle x_0, x_1, \ldots, x_{n-1} \mid x_0x_1x_2x_1, x_1x_2x_3x_2, \ldots, x_{n-1}x_0x_1x_0 \rangle.$$

Note that R_0 gives $x_2 = x_1^{-1}x_0^{-1}x_1^{-1}$, so that, in turn, R_1 is $x_1(x_1^{-1}x_0^{-1}x_1^{-1})x_3(x_1^{-1}x_0^{-1}x_1^{-1})$. Thus $x_3 = x_1(x_0x_1)^2$.

We proceed by induction, showing that for $0 \leq m \leq n/2 - 2$,

$$R_{2m} : x_{2m+2} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2m+1},$$
$$R_{2m+1} : x_{2m+3} = x_1(x_0x_1)^{2m+2}.$$

The base cases $m = 0$ and $m = 1$ (*i.e.*, R_0 and R_1 above) hold.

Assume the cases when $0 \leq m \leq r$.

Consider when $m = r + 1$:

$$R_{2(r+1)} = R_{2r+2} : x_{2r+2}x_{2r+3}x_{2r+4}x_{2r+3},$$

so now

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
15	11	$[15,0,1,2]$	1	$Y\#$	$G.1, G.9$	$[4,11,751]$	
		$[15,1,2,4]$	1	$Y\#$	$G.1, G.9$	$[4,211]$	††
		$[15,1,2,8]$	1	$Y\#$	$G.1, G.9$		
		$[15,1,2,5]$	1	$Y\#$	$G.1, G.9$	$[4,11,31]$	
		$[15,1,2,6]$	1	$Y\#$	$G.1, G.9$	$[4,61]$	†
		$[15,1,5,10]$	1	$Y\#$	$G.1, G.9$		
		$[15,1,10,5]$	1	$Y\#$	$G,1, G.9$		
		$[15,1,2,7]$	1	$Y\#$	$G.1, G.9$	$[2,2,2,2,4,11]$	
		$[15,1,3,9]$	1	$Y\#$	$G.1, G.9$	$[4,7,31]$	
		$[15,1,6,3]$	1	?		$[4,7,11]$	
		$[15,1,10,4]$	1	$Y\#$	$G.1, G.9$	$[2,2,2,2,4,7]$	
16	8	$[16,0,1,2]$	1	$Y\#$	$G.1, G.9$	$[3,3,64,113]$	
		$[16,1,2,4]$	1	$Y\#$	$G.1, G.9$	$[17,64]$	?
		$[16,1,2,6]$	1	$Y\#$	$G.1, G.9$		
		$[16,1,3,5]$	1	$Y\#$	$G.1, G.9$		
		$[16,1,4,2]$	1	$Y\#$	$G.1, G.9$		
		$[16,1,2,5]$	2	Y		$[0,16,17]$	
		$[16,1,2,7]$	2	Y		$[0,0,0,17]$	
		$[16,1,2,8]$	1	$Y\#$	$G.1, G.9$	$[3,3,64]$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

Discovered using the KBMAG software.[19]

†: The derived subgroups have different abelianisation.[15]

††: The index two subgroups are different.[15]

† † †: The index three subgroups are different.[15]

Table 6.8: The structure of FTF groups for $n = 15$ to $n = 16$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
17	7	$[17,0,1,2]$	1	$Y\#$	$G.1, G.9$	$4,137,239$	
		$[17,1,2,4]$	1	$Y\#$	$G.1, G.9$	$4,443$	††
		$[17,1,2,9]$	1	?	?		
		$[17,1,2,5]$	1	$Y\#$	$G.1, G.9$	$4,953$	
		$[17,1,2,6]$	1	?		$4,103$	
		$[17,1,2,7]$	1	$Y\#$	$G.1, G.9$	$4,307$	
		$[17,1,2,8]$	1	?	?	$4,647$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.9: The structure of FTF groups for $n = 17$.

$$\begin{aligned}
x_{2r+4} &= x_{2r+3}^{-1} x_{2r+2}^{-1} z_{2r+3}^{-1} \\
&= (x_1(x_0x_1)^{2r+2})^{-1}(x_1^{-1}(x_0^{-1}x_1^{-1})^{2r+1})^{-1}(x_1(x_0x_1)^{2r+2})^{-1} \quad \text{(by induction)} \\
&= (x_1(x_0x_1)^{2r+2})^{-1}\left[(x_1x_0)^{2r+1}x_1\right]((x_1^{-1}x_0^{-1})^{2r+2}x_1^{-1}) \\
&= ((x_1^{-1}x_0^{-1})^{2r+2}x_1^{-1})\underbrace{\left[((x_1x_0)^{2r+1}x_1)(x_1^{-1}(x_0^{-1}x_1^{-1})^{2r+1}x_0^{-1}x_1^{-1}\right]}_{=x_0^{-1}x_1^{-1}} \\
&= x_1^{-1}(x_0^{-1}x_1^{-1})^{2r+3},
\end{aligned}$$

and

$$R_{2(r+1)+1} = R_{2r+3} : x_{2r+3}x_{2r+4}x_{2r+5}x_{2r+4}$$

implies

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
18	14	$[18,0,1,2]$	1	?	?	$7,8,37,127$	
		$[18,1,2,4]$	1	?	?	$8,19,19$	††
		$[18,1,4,2]$	1	?	?		
		$[18,1,2,5]$	2	Y	N/a	$0,2,19,19$	
		$[18,1,2,6]$	1	?	?	$7,8,19$	††
		$[18,1,2,8]$	1	?	?		
		$[18,1,2,7]$	2	Y	?	$0,2,9,19$	
		$[18,1,2,9]$	2	Y	?	$0,2,127$	
		$[18,1,3,5]$	1	?	?	$8,19,37$	
		$[18,1,4,7]$	2	Y	?	$0,2,7,19$	
		$[18,1,5,9]$	1	?	?	$8,127$	
		$[18,1,6,12]$	1	?	?	$7,8,13$	
		$[18,1,11,3]$	1	?	?	$8,109$	
		$[18,2,5,8]$	1	?	?	$7,8,73$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.10: The structure of FTF groups for $n = 18$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
19	8	$[19,0,1,2]$	1	?	?	$4, 130873$	
		$[19,1,2,4]$	1	?	?	$4, 1103$	††
		$[19,1,2,10]$	1	?	?		
		$[19,1,2,5]$	1	?	?	$4, 2243$	
		$[19,1,2,6]$	1	?	?	$4, 571$	
		$[19,1,2,7]$	1	?	?	$4, 647$	
		$[19,1,2,8]$	1	?	?	$4, 191$	
		$[19,1,2,9]$	1	?	?	$4, 1559$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.11: The structure of FTF groups for $n = 19$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
23	10	$[23,0,1,2]$	1	?	?	$4, 2095853$	
		$[23,1,2,4]$	1	?	?	$4, 47, 139$	††
		$[23,1,2,9]$	1	?	?		
		$[23,1,2,12]$	1	?	?		
		$[23,1,2,5]$	1	?	?	$4, 47, 277$	
		$[23,1,2,6]$	1	?	?	$4, 2393$	
		$[23,1,2,7]$	1	?	?	$4, 47, 47$	
		$[23,1,2,8]$	1	?	?	$4, 691$	
		$[23,1,2,10]$	1	?	?	$4, 3037$	
		$[23,1,2,11]$	1	?	?	$4, 6763$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.12: The structure of FTF groups for $n = 23$.

n	$M(n)$	$[n, j, k, l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
29	13	$[29, 0, 1, 2]$	1	?	?	$4, 8353, 16067$	
		$[29, 1, 2, 4]$	1	?	?	$4, 59, 1567$	††
		$[29, 1, 2, 15]$	1	?	?		
		$[29, 1, 2, 5]$	1	?	?	$4, 185369$	
		$[29, 1, 2, 6]$	1	?	?	$4, 20533$	
		$[29, 1, 2, 7]$	1	?	?	$4, 18097$	
		$[29, 1, 2, 8]$	1	?	?	$4, 54289$	
		$[29, 1, 2, 9]$	1	?	?	$4, 59, 1103$	
		$[29, 1, 2, 10]$	1	?	?	$4, 59, 59$	
		$[29, 1, 2, 11]$	1	?	?	$4, 82129$	
		$[29, 1, 2, 12]$	1	?	?	$4, 36599$	
		$[29, 1, 2, 13]$	1	?	?	$4, 3, 59, 233$	
		$[29, 1, 2, 14]$	1	?	?	$4, 59, 1741$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

††: The index two subgroups are different.[15]

Discovered using the KBMAG software.[19]

Table 6.13: The structure of FTF groups for $n = 29$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
31	14	$[31,0,1,2]$	1	?	?	$4,373,1439393$	
		$[31,1,2,4]$	1	?	?	$4,225061$	††
		$[31,1,2,16]$	1	?	?		
		$[31,1,2,5]$	1	?	?	$4,446401$	
		$[31,1,2,6]$	1	?	?	$2,2,2,2,2,4,1427$	
		$[31,1,2,7]$	1	?	?	$4,34721$	
		$[31,1,2,8]$	1	?	?	$4,311,311$	
		$[31,1,2,9]$	1	?	?	$4,115879$	
		$[31,1,2,10]$	1	?	?	$2,2,2,2,2,4,2357$	
		$[31,1,2,11]$	1	?	?	$2,2,2,2,2,4,5023$	
		$[31,1,2,12]$	1	?	?	$4,6263$	
		$[31,1,2,13]$	1	?	?	$4,43711$	
		$[31,1,2,14]$	1	?	?	$4,69317$	
		$[31,1,2,15]$	1	?	?	$4,5,5,5,2543$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

††: The index two subgroups are different.[15]

Discovered using the KBMAG software.[19]

Table 6.14: The structure of FTF groups for $n = 31$.

n	$M(n)$	$[n, j, k, l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
37	17	$[37, 0, 1, 2]$	1	?	?	$4, 149, 230603167$	
		$[37, 1, 2, 4]$	1	?	?	$4, 3183703$	††
		$[37, 1, 2, 19]$	1	?	?		
		$[37, 1, 2, 5]$	1	?	?	$4, 6357859$	
		$[37, 1, 2, 6]$	1	?	?	$4, 396937$	
		$[37, 1, 2, 7]$	1	?	?	$4, 262553$	
		$[37, 1, 2, 8]$	1	?	?	$4, 223, 1259$	
		$[37, 1, 2, 9]$	1	?	?	$4, 1215673$	
		$[37, 1, 2, 10]$	1	?	?	$4, 960151$	
		$[37, 1, 2, 11]$	1	?	?	$4, 223, 3331$	
		$[37, 1, 2, 12]$	1	?	?	$4, 919081$	
		$[37, 1, 2, 13]$	1	?	?	$4, 3118139$	
		$[37, 1, 2, 14]$	1	?	?	$4, 32783$	
		$[37, 1, 2, 15]$	1	?	?	$4, 149, 6143$	
		$[37, 1, 2, 16]$	1	?	?	$4, 1211603$	
		$[37, 1, 2, 17]$	1	?	?	$4, 577349$	
		$[37, 1, 2, 18]$	1	?	?	$4, 4295479$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

††: The index two subgroups are different.[15]

Discovered using the KBMAG software.[19]

Table 6.15: The structure of FTF groups for $n = 37$.

n	$M(n)$	$L(n)$	Exact
41			$19 \dagger\dagger, *$
43			$20 \dagger\dagger, *$
47			$22 \dagger\dagger, *$
53			$25 \dagger\dagger, *$
59			$28 \dagger\dagger, *$
61			$29 \dagger\dagger, *$
67			$32 \dagger\dagger, *$
71			$34 \dagger\dagger, *$
73			$35 \dagger\dagger, *$
79			$38 \dagger\dagger, *$
83			$40 \dagger\dagger, *$
89			$43 \dagger\dagger, *$
97			$47 \dagger\dagger, *$
101			$49 \dagger\dagger, *$
103	50	49	$*$
107	52	51	$*$
109	53	52	$*$

$M(n)$: Maximum number of FTF isomorphism classes.[15]
$L(n)$: Minimum number of FTF isomorphism classes.[15]
Exact: Exact number of FTF isomorphism classes.[15]
$*$: The majority of the abelian invariants are distinct.[15]
$\dagger\dagger$: The index two subgroups are different.[15]

Table 6.16: The number of isomorphism classes for FTF groups for prime n with $41 \leq n \leq 109$.

n	$M(n)$	$[n, j, k, l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
25	13	$[25, 0, 1, 2]$	1	?	?	$4, 11, 151, 5051$	
		$[25, 1, 2, 4]$	1	?	?	$4, 15901$	††
		$[25, 1, 2, 13]$	1	?	?		
		$[25, 1, 2, 5]$	1	?	?	$4, 11, 2851$	
		$[25, 1, 2, 6]$	1	?	?	$4, 3251$	
		$[25, 1, 2, 7]$	1	?	?	$4, 11, 401$	
		$[25, 1, 2, 8]$	1	?	?	$4, 101, 101$	
		$[25, 1, 2, 9]$	1	?	?	$4, 19001$	
		$[25, 1, 2, 10]$	1	?	?	$4, 11, 101$	
		$[25, 1, 2, 11]$	1	?	?	$4, 4751$	
		$[25, 1, 2, 12]$	1	?	?	$4, 11, 2251$	
		$[25, 1, 5, 15]$	1	?	?	$4, 61, 101$	
		$[25, 1, 6, 11]$	1	?	?	$4, 61, 151$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

††: The index two subgroups are different.[15]

Discovered using the KBMAG software.[19]

Table 6.17: The structure of FTF groups for $n = 25$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
27	16	$[27,0,1,2]$	1	?	?	$4,127,163,1621$	
		$[27,1,2,4]$	1	?	?	$4,19,1999$	††
		$[27,1,2,14]$	1	?	?		
		$[27,1,2,5]$	1	?	?	$4,19,4051$	
		$[27,1,2,6]$	1	?	?	$4,19,379$	
		$[27,1,2,7]$	1	?	?	$4,19,487$	
		$[27,1,2,8]$	1	?	?	$4,109,163$	
		$[27,1,2,9]$	1	?	?	$4,127,163$	
		$[27,1,2,10]$	1	?	?	$4,2269$	
		$[27,1,2,11]$	1	?	?	$4,109,127$	
		$[27,1,2,12]$	1	?	?	$4,12097$	
		$[27,1,2,13]$	1	?	?	$4,19,2593$	
		$[27,1,3,15]$	1	?	?	$4,7,6211$	
		$[27,1,4,7]$	1	?	?	$4,7,3079$	
		$[27,1,6,3]$	1	?	?	$4,7,271$	
		$[27,1,9,18]$	1	?	?	$4,7,19,37$	

$M(n)$: Maximum number of FTF isomorphism classes.

$FS(G)$: Free Subgroup of rank two?

††: The index two subgroups are different.[15]

Discovered using the KBMAG software.[19]

Table 6.18: The structure of FTF groups for $n = 27$.

n	$M(n)$	$[n, j, k, l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
32	20	$[32, 0, 1, 2]$	1	?	?	$3, 3, 113, 128, 32993$	
		$[32, 1, 2, 4]$	1	?	?	$17, 128, 641$	††
		$[32, 1, 4, 2]$	1	?	?		
		$[32, 1, 2, 5]$	2	Y		$0, 17, 32, 641$	
		$[32, 1, 2, 6]$	1	?	?	$17, 97, 128$	
		$[32, 1, 2, 7]$	2	Y		$0, 0, 0, 17, 193$	
		$[32, 1, 2, 8]$	1	?	?	$3, 3, 128, 257$	
		$[32, 1, 2, 9]$	2	Y		$0, 32, 1217$	
		$[32, 1, 2, 10]$	1	?	?	$3, 3, 128, 257$	
		$[32, 1, 2, 11]$	2	Y		$0, 0, 0, 17, 97$	
		$[32, 1, 2, 12]$	1	?	?	$17, 128, 673$	
		$[32, 1, 2, 13]$	2	Y		$0, 17, 32, 97$	
		$[32, 1, 2, 14]$	1	?	?	$17, 128, 449$	
		$[32, 1, 2, 15]$	2	Y		$0, 0, 0, 2113$	
		$[32, 1, 2, 16]$	1	?	?	$3, 3, 113, 128$	
		$[32, 1, 3, 5]$	1	?	?	$17, 128, 929$	
		$[32, 1, 4, 18]$	1	?	?	$17, 97, 128$	
		$[32, 1, 5, 9]$	1	?	?	$5, 128, 577$	?
		$[32, 1, 8, 20]$	1	?	?		
		$[32, 1, 19, 5]$	1	?	?	$17, 128, 193$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.19: The structure of FTF groups for $n = 32$.

n	$M(n)$	$[n,j,k,l]$ representatives	γ	$FS(G)$	Generators used for $FS(G)$	Abelian Invariants [15]	Reason Distinct
49	26	$[49,0,1,2]$	1	?	?	$4,29,1801927,2693237$	
		$[49,1,2,4]$	1	?	?	$2,2,2,4,491,163171$	††
		$[49,1,2,25]$	1	?	?		
		$[49,1,2,5]$	1	?	?	$2,2,2,4,160240193$	
		$[49,1,2,6]$	1	?	?	$4,44002883$	
		$[49,1,2,7]$	1	?	?	$4,29,197,2647$	
		$[49,1,2,8]$	1	?	?	$4,1471,35869$	
		$[49,1,2,9]$	1	?	?	$4,29,197,34301$	
		$[49,1,2,10]$	1	?	?	$4,197,313993$	
		$[49,1,2,11]$	1	?	?	$2,2,2,4,10409561$	
		$[49,1,2,12]$	1	?	?	$2,2,2,4,8722099$	
		$[49,1,2,13]$	1	?	?	$4,1373,135241$	
		$[49,1,2,14]$	1	?	?	$4,29,3808967$	
		$[49,1,2,15]$	1	?	?	$4,3137,10487$	
		$[49,1,2,16]$	1	?	?	$4,29,3172751$	
		$[49,1,2,17]$	1	?	?	$4,197,2817599$	
		$[49,1,2,18]$	1	?	?	$2,2,2,4,120737$	
		$[49,1,2,19]$	1	?	?	$2,2,2,4,19203493$	
		$[49,1,2,20]$	1	?	?	$4,197,491,1667$	
		$[49,1,2,21]$	1	?	?	$4,29,197,2549$	
		$[49,1,2,22]$	1	?	?	$4,1667,18719$	
		$[49,1,2,23]$	1	?	?	$4,29,1996751$	
		$[49,1,2,24]$	1	?	?	$4,30577,32341$	
		$[49,1,7,28]$	1	?	?	$4,547,63211$	
		$[49,1,8,15]$	1	?	?	$4,547,119267$	
		$[49,1,15,29]$	1	?	?	$4,197,491,547$	

$M(n)$: Maximum number of FTF isomorphism classes.
$FS(G)$: Free Subgroup of rank two?
††: The index two subgroups are different.[15]
Discovered using the KBMAG software.[19]

Table 6.20: The structure of FTF groups for $n = 49$.

n	$M(n)$	$L(n)$	Exact
64	44	41	$*$
81	52	51	$*$

$M(n)$: Maximum number of FTF isomorphism classes for n.[15]
$L(n)$: Minimum number of FTF isomorphism classes.[15]
Exact: Exact number of FTF isomorphism classes.[15]
$*$: The majority of the abelian invariants are distinct.[15]
$\dagger\dagger$: The index two subgroups are different.[15]

Table 6.21: The number of isomorphism classes of the FTF groups for prime powers n with $64 \leq n \leq 81$.

$$\begin{aligned}
x_{2r+5} &= x_{2(r+1)+3} \\
&= x_{2r+4}^{-1}x_{2r+3}^{-1}x_{2r+4}^{-1} \\
&= x_1(x_0x_1)^{2r+3}\left[x_1(x_0x_1)^{2r+2}\right]^{-1}x_1(x_0x_1)^{2r+3} \\
&= x_1(x_0x_1)^{2r+3}\underbrace{\left[(x_1^{-1}x_0^{-1})^{2r+2}x_1^{-1}\right]x_1(x_0x_1)^{2r+3}}_{=x_0x_1} \\
&= x_1(x_0x_1)^{2r+4} \\
&= x_1(x_0x_1)^{2(r+1)+2}.
\end{aligned}$$

Thus all generators can be written in terms of x_0 and x_1.

Now consider

$$\begin{aligned}
G_n(1,2,1) &= \langle x_i\,(0 \leq i < n) | x_ix_{i+1}x_{i+2}x_{i+1}(0 \leq i < n)\rangle \\
&= \left\langle x_i\,(0 \leq i < n) \middle| \begin{array}{c} x_ix_{i+1}x_{i+2}x_{i+1}(0 \leq i < n), \\ x_{2m+2} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2m+1}, x_{2m+3} = x_1(x_0x_1)^{2m+2}(0 \leq m \leq n/2-2) \end{array} \right\rangle \\
&= \left\langle x_i\,(0 \leq i < n) \middle| \begin{array}{c} x_{2r}x_{2r+1}x_{2r+2}x_{2r+1}, x_{2r+1}x_{2r+2}x_{2r+3}x_{2r+2}, (0 \leq r < n/2), \\ x_{2m+2} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2m+1}, x_{2m+3} = x_1(x_0x_1)^{2m+2}(0 \leq m < n/2-2) \end{array} \right\rangle \\
&= \left\langle x_i\,(0 \leq i < n) \middle| \begin{array}{c} x_{2r}x_{2r+1}x_{2r+2}x_{2r+1}, x_{2r+1}x_{2r+2}x_{2r+3}x_{2r+2}, (0 \leq r < n/2-1), \\ x_{n-2}x_{n-1}x_0x_{n-1}, x_{n-1}x_0x_1x_0, \\ x_{2s} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2s-1}, x_{2s+1} = x_1(x_0x_1)^{2s}(1 \leq s \leq n/2-1) \end{array} \right\rangle
\end{aligned}$$

$$= \left\langle x_i\,(0 \le i < n) \;\middle|\; \begin{array}{c} x_1^{-1}(x_0^{-1}x_1^{-1})^{2r-1} \cdot x_1(x_0x_1)^{2r} \cdot x_1^{-1}(x_0^{-1}x_1^{-1})^{2r+1} \cdot x_1(x_0x_1)^{2r}, \\ x_1(x_0x_1)^{2r} \cdot x_1^{-1}(x_0^{-1}x_1^{-1})^{2k+1} \cdot x_1(x_0x_1)^{2r+2} \cdot (x_0^{-1}x_1^{-1})^{2k+1}, (0 \le r < n/2-1) \\ x_{n-2}x_{n-1}x_0x_{n-1}, x_{n-1}x_0x_1x_0, \\ x_{2s} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2s-1}, x_{2s+1} = x_1(x_0x_1)^{2s}(1 \le s \le n/2-1) \end{array} \right.$$

$$= \left\langle x_i\,(0 \le i < n) \;\middle|\; \begin{array}{c} x_{n-2}x_{n-1}x_0x_{n-1}, x_{n-1}x_0x_1x_0, \\ x_{2s} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2s-1}, x_{2s+1} = x_1(x_0x_1)^{2s}(1 \le s \le n/2-1) \end{array} \right\rangle$$

$$= \left\langle x_i\,(0 \le i < n) \;\middle|\; \begin{array}{c} x_1^{-1}(x_0^{-1}x_1^{-1})^{n-3} \cdot x_1(x_0x_1)^{n-2} \cdot x_0 \cdot x_1(x_0x_1)^{n-2}, x_1(x_0x_1)^{n-2} \cdot x_0x_1x_0, \\ x_{2s} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2s-1}, x_{2s+1} = x_1(x_0x_1)^{2s}(1 \le s \le n/2-1) \end{array} \right\rangle$$

$$= \left\langle x_i\,(0 \le i < n) \;\middle|\; \begin{array}{c} (x_0x_1)^n, (x_1x_0)^n, \\ x_{2s} = x_1^{-1}(x_0^{-1}x_1^{-1})^{2s-1}, x_{2s+1} = x_1(x_0x_1)^{2s}(1 \le s \le n/2-1) \end{array} \right\rangle$$

$$= \langle x_0, x_1 \mid (x_0x_1)^n, (x_1x_0)^n \rangle$$

$$= \langle x_0, x_1 \mid (x_0x_1)^n \rangle$$

$$\cong \mathbb{Z}_n * \mathbb{Z}.$$

□

Lemma 6.5.2. *For $n \ge 1$,*

$$\begin{aligned} G_n(1,2,3) &= G_n(x_0x_1x_2x_3) \\ &\cong \mathbb{Z}_{4/(n,4)} * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(n,4)-1 \text{ copies of } \mathbb{Z}.} \end{aligned}$$

In particular,

$$G \cong \begin{cases} \mathbb{Z} * \mathbb{Z} * \mathbb{Z} & n \equiv 0 \pmod 4 \\ \mathbb{Z}_4 & n \equiv 1,3 \pmod 4 \\ \mathbb{Z}_2 * \mathbb{Z} & n \equiv 2 \pmod 4. \end{cases}$$

Proof. See [35, Theorem C]. □

Lemma 6.5.3. *The group $G_n(0,1,1)$ is large for $n > 2$.*

Proof. Observe that, by adding relators x_i^2 for all i to the presentation for $G = G_n(x_0^2x_1^2)$, G maps onto

$$G_n(x_0^2) \cong \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2,}_{n \text{ times,}}$$

which is large for $n > 2$. □

Lemma 6.5.4. *For even $n > 2$, $G_n(n/2, 1, 1 + n/2)$ is large.*

Proof. We have for $G_n(n/2, 1, n/2 + 1)$ the relators $x_0 x_{n/2} x_1 x_{n/2+1}$, so

$$\begin{aligned}
E_n(n/2, 1, n/2 + 1) &\cong \langle x, t \mid t^n, xt^{n/2}xt^{-n/2}txt^{-1}t^{n/2+1}xt^{-n/2-1}\rangle \\
&\cong \langle x, t \mid t^n, xt^{n/2}xt^{n/2+1}xt^{n/2}xt^{n/2-1}\rangle \\
&\cong \langle x, t, y \mid t^n, xt^{n/2}xt^{n/2+1}xt^{n/2}xt^{n/2-1}, y = xt^{n/2}\rangle \\
&\cong \langle x, t, y \mid t^n, xt^{n/2}xt^{n/2+1}xt^{n/2}xt^{n/2-1}, x = yt^{n/2}\rangle \\
&\cong \langle t, y \mid t^n, yt^{n/2}t^{n/2}yt^{n/2}t^{n/2+1}yt^{n/2}t^{n/2}yt^{n/2}t^{n/2-1}\rangle \\
&\cong \langle t, y \mid t^n, yytyyt^{-1}\rangle \\
&\cong \langle t, y \mid t^n, y^2ty^2t^{-1}\rangle \\
&\cong E_n(0, 1, 1)
\end{aligned}$$

Thus $G_n(0, 1, 1)$ is large if and only if $G_n(n/2, 1, n/2 + 1)$ is large.

But $G_n(0, 1, 1)$ is large if $n > 2$.

□

Theorem 6.5.5. *If $p > 2$ is prime, then*

$$G_p(1, 2, 4)^{ab} \cong G_p(1, 2, (p + 1)/2)^{ab}.$$

Proof. Since $p > 2$ is prime, $(2, p) = 1$, so we can use Bogley & Parker's u (see 3.9.1) to get

$$G_p(1, 2, (p + 1)/2)^{\mathrm{ab}} \cong G_p(2, 4, 1)^{\mathrm{ab}},$$

the relators of which look like, besides the commutators,

$$x_i x_{i+2} x_{i+4} x_{i+1},$$

which we can commute to give

$$x_i x_{i+1} x_{i+2} x_{i+4},$$

but then we have $G_p(1, 2, 4)^{\mathrm{ab}}$.

□

Theorem 6.5.6. *Let $G = G_n(j, k, l)$ be finite. Then there is a homomorphism from $G_n(j, k, l)^{ab}$ onto $\mathbb{Z}_4$. Thus G has an abelian invariant divisible by four.*

Proof. By mapping each generator in G to a fixed generator, G maps onto $\mathbb{Z}_4$. But then G^{ab} maps onto $\mathbb{Z}_4$ as a quotient of G. □

For prime $n > 2$, see OEIS A067076 (see [28], the list of numbers a such that $2a+3$ is prime.

Theorem 6.5.7. *Let (A) and (C) be false while (B) is true. Assume $(n, j, k, l) = 1$. Then, for prime $n > 2$, there is at most $(n-3)/2$ isomorphism classes of $G = G_n(j, k, l)$.*

Proof. Let $n = p > 2$ be prime. Since (B) is true, we may assume that $k \equiv 2j$ by Section 4.3. Then $G = G_p(j, 2j, l)$, so either $(j, p) = 1$ or $(j, p) = p$; but if $(j, p) = p$, then $j = p$ and $(l, p) = 1$, so $G = G_p(0, 0, l) = G_p(0, 0, 1)$, meaning (A) is true, a contradiction.

So assume $(j, p) = 1$, so that j has a multiplicative inverse modulo p. Thus $G_p(j, 2j, l) \cong G_p(1, 2, L)$, where $L = lj^{-1} \pmod p$ by Bogley & Parker's u (see 3.9.1).

Consider $G_p(1, 2, L)$ for $0 \leq L < p$. Then, so far, there is at most p isomorphism classes here. But $L \neq 1$ because, otherwise, (A) is true. This brings us down to $p-1$ values of L. But $L \neq 3$ and $L \neq p-1$ since, otherwise, (C) is true; now there is at most $p-3$ values of L.

Consider the relators for $G_p(1, 2, L)$. They are of the form

$$x_i x_{i+1} x_{i+2} x_{i+L},$$

which we can invert then substitute $y_i = x_i^{-1}$ for each i to get

$$y_{i+L} y_{i+2} y_{i+1} y_i.$$

Permuting then gives

$$y_{i+2} y_{i+1} y_i y_{i+L},$$

so that negating the subscripts gives

$$y_{-i-2} y_{-i-1} y_{-i} y_{-i-L},$$

in which we can substitute $t = -i$ to get

$$y_{t-2} y_{t-1} y_t y_{t-L},$$

so that then adding two to the subscripts via Bogley & Parker's θ (see 3.9.1) gives

$$y_t y_{t+1} y_{t+2} y_{t+2-L}.$$

But this defines $G_p(1,2,p+2-L)$. Therefore $G_p(1,2,L) \cong G_p(1,2,p+2-L)$.

Therefore, the number of isomorphism classes is halved, leaving us with $(p-3)/2$ isomorphism classes. □

Lemma 6.5.8. *For prime $p > 2$, we have*

$$G_p(1,2,2) \cong G_p(0,1,2).$$

Proof. The defining word of $G_p(1,2,2)$ is $x_0x_1x_2x_2$. We can subtract two from each of the subscripts by using θ. Then we have $x_{-2}x_{-1}x_0x_0$. Permuting cyclically gives $x_0x_{-2}x_{-1}x_0$. Apply σ: $x_0x_0x_{-1}x_{-2}$. Since $(p,p-1)=1$, we can multiply the subscripts by $p-1$ using u, which yields $x_0x_0x_1x_2$. But this implies $G_p(1,2,2) \cong G_p(0,1,2)$. □

Conjecture 6.5.9. *The groups $G_p(1,2,4)$ and $G_p(2,4,1)$ are not isomorphic for prime p.*

The evidence gathered so far indicates that the two groups in the conjecture have unique index two subgroups.

Another conjecture, based on the tables, is the following.

Conjecture 6.5.10. *Let (A) and (C) be false while (B) is true. Suppose p is prime and $(p^2,j,k,l)=1$. Then there is at most $(p^2+p-4)/2$ groups $G_{p^2}(j,k,l)$ up to isomorphism.*

CHAPTER 7

Acknowledgements.

I would like to thank: the Department of Mathematical Sciences at the University of Essex for the funding, Prof. G. Williams for being an excellent supervisor, Prof. C. Saker for some suggestions on wording things, the Mathematics Stack Exchange community, and my friends & family for their support.

APPENDIX A

Appendix: GAP Code.

A.1 The Groups.

This GAP code produces the group $G_n(j,k,l)$ by means of its defining presentation.

```
Gnjkl:=function(n,j,k,l)

      local F, rels, i, I, gens;
      F:=FreeGroup(n);
      gens:=GeneratorsOfGroup(F);
      rels:=[];
      for i in [1.. n] do
            I:=gens[1+(i mod n)]*gens[1+(j+i) mod n]*gens[1+((k+i)
               mod n)]*gens[1+((l+i) mod n)];
            Add(rels, I);
      od;

      return F/rels;

end;
```

A.2 The Bogley & Parker Isomorphisms.

These functions are Bogley & Parker isomorphisms, implemented by means of converting the vector $G = [n, j, k, l]$ into the form $[n, 0, j, k, l, \text{unit}]$ to keep track of the last unit modulo n used by u and to inserting the value $i = 0$ for the isomorphisms to act on, then coverting the vector back into the form $[n, J, K, L]$ after the isomorphisms are applied.

```
ConvertToL:=function(G)

      local L;

      L:=[G[1], 0, G[2], G[3], G[4], 1]; #Here L[6] is "one".

      return L;

end;

sigma:=function(L)

      local K;

      K:=[L[1], L[2], L[5], L[4], L[3], L[6]];

      return K;

end;

tau:=function(L)

      local K;

      K:=[L[1], L[5], L[2], L[3], L[4], L[6]];
```

```
        return K;

end;

theta:=function(L)

        local K;

        K:=[L[1],
        (L[2]+1) mod L[1],
        (L[3]+1) mod L[1],
        (L[4]+1) mod L[1],
        (L[5]+1) mod L[1],
        L[6]];

        return K;

end;




UnitsOfL:=function(L)

        local j, i, n, K, unitlist, coprime;

        coprime:=[];

        unitlist:=[];

        n:=L[1];

        for i in [0..n] do
```

```
            if GcdInt(n, i)=1 then
                  Add(coprime, i);
            fi;
      od;

      for j in coprime do
            K:=[L[1], L[2], L[3], L[4], L[5], j];
            Add(unitlist, K);
      od;

      return unitlist;

end;


unitL6:=function(L)

      local K, n, i, unitmodn;

      K:=[];

      n:=L[1];

      unitmodn:=L[6];

      K[1]:=n;

      K[6]:=1;

      for i in [2..5] do
            K[i]:=(L[i]*unitmodn) mod n;
      od;
```

```
        return K;

end;

ConvertLBack:=function(L)

        local G;

        G:=[L[1], L[3], L[4], L[5]];

        return G;

end;
```

A.3 (A), (B), (C) Conditions.

This GAP code is fairly straightforward: it describes functions that compute the truth values of (A), (B), and (C), along with a function that outputs the vector of truth values of (A), (B), and (C), in that order.

```
A:=function(n,j,k,l)

        local answer;
        answer:=false;
        if (2*k) mod n=0 or (2*(j -l)) mod n =0 then
                answer:=true;
        fi;
        return answer;

end;

B:=function(n,j,k,l)
```

```
      local answer;
      answer:=false;
      if (k -2*j) mod n=0 or (k -2*l) mod n =0 or (j+l -2*k) mod n
         =0 or (j+l) mod n =0 then
            answer:=true;
      fi;
      return answer;

end;

C:=function(n,j,k,l)

      local answer;
      answer:=false;
      if (j+k -l) mod n=0 or (j -k -l) mod n =0 then
            answer:=true;
      fi;
      return answer;

end;

ABC:=function(n,j,k,l)

      local L;
      L:=[];
      L[1]:=A(n,j,k,l);
      L[2]:=B(n,j,k,l);
      L[3]:=C(n,j,k,l);
      return L;

end;
```

A.4 The Bogley & Parker Isomorphism Classes.

Given a vector $G = [n, j, k, l]$, the function IsoClass applies the Bogley & Parker isomorphism functions from above until all possible vectors corresponding to the groups in the Bogley & Parker isomorphism class are computed.

```
IsoClass:=function(G)

      local L, U, UandSClass, u, F, S, v, ThetaClass, s,t,m,h,
         FinalIsomorphismClassOfL, x, y, n, FinalIsoClassOfG, i, g,
          UandSClass2;

      L:=ConvertToL(G);

      U:=UnitsOfL(L);

      UandSClass:=[];

      for u in U do
            F:=unitL6(u);
            AddSet(UandSClass, F);
      od;

      UandSClass2:=UandSClass;

      for v in UandSClass2 do
            S:=sigma(v);
            AddSet(UandSClass, S);
      od;

      ThetaClass:=UandSClass;

      for s in UandSClass do
```

```
        t:=s;
        for m in [1..(G[1])] do
              t:=theta(t);
              for h in [2..5] do
                    if t[h]=0 then
                    AddSet(ThetaClass, t);
                    fi;
              od;
        od;
od;

FinalIsomorphismClassOfL:=[];

for x in ThetaClass do
      y:=x;
      for n in [0..3] do
            y:=tau(y);
            if y[2]=0 then
                  AddSet(FinalIsomorphismClassOfL, y);
            fi;
      od;
od;

FinalIsoClassOfG:=[];

for i in FinalIsomorphismClassOfL do
      g:=ConvertLBack(i);
      AddSet(FinalIsoClassOfG, g);
od;

return FinalIsoClassOfG;
```

```
end;
```

A.5 The Secondary Divisor.

This one simply computes the value of γ given $G = [n, j, k, l]$.

```
gamma:=function(G)

      local g;

      g:=GcdInt(G[1], GcdInt(G[3]-2*G[2], GcdInt(G[4]-2*G[3]+G[2],
         GcdInt(G[3]-2*G[4], G[2]+1))));

      return g;

end;
```

A.6 Vectors For Extensions.

Here the vectors from Theorem 4.1.3 are computed for $[n, j, k, l]$ compared to those for $[n, jd, kd, ld]$.

```
VectorsForExtensions:=function(n,j,k,l,jd,kd,ld)

      local v, v1, v2, v3, v4, vd, vd1, vd2, vd3, vd4, s, t;

      v:=[];
            v1:=[(k-2*j) mod n, (l-k-j) mod n, (-l-j) mod n];
            v2:=[(j+l-2*k) mod n, (j-k-l) mod n, (2*j-k) mod n];
            v3:=[(k-2*l) mod n, (j+k-l) mod n, (2*k-j-l) mod n];
            v4:=[(j+l) mod n, (l+k-j) mod n, (2*l-k) mod n];
      v:=[v1,v2,v3,v4];
```

```
    vd:=[];
        vd1:=[(kd-2*jd) mod n, (ld-kd-jd) mod n, (-ld-jd) mod n
          ];
        vd2:=[(jd+ld-2*kd) mod n, (jd-kd-ld) mod n, (2*jd-kd)
          mod n];
        vd3:=[(kd-2*ld) mod n, (jd+kd-ld) mod n, (2*kd-jd-ld)
          mod n];
        vd4:=[(jd+ld) mod n, (ld+kd-jd) mod n, (2*ld-kd) mod n
          ];
    vd:=[vd1,vd2,vd3,vd4];

    for s in [1..4] do
        for t in [1..4] do
            if v[s]=vd[t] then
                Print("For␣s=",s,"␣and␣t=",t, ",␣we␣have␣",
                    v[s], "=", vd[t], "\n");
            fi;
        od;
    od;

end;
```

A.7 The Extensions.

Here GAP outputs the extension of the group corresponding to $G = [n, j, k, l]$ using the presentation 3.7.1.

```
Enjkl:=function(n,j,k,l)

    local F, rels;

    F:=FreeGroup(2);
```

```
        rels:=[F.1^n, F.2*F.1^j*F.1*F.2*F.1^(k-j)*F.2*F.1^(l-k)*F.2*F
            .1^(-l)];

        return F/rels;

end;
```

A.8 The Search Function.

This uses GAP to produce γ value and the abelian invariants for all vectors with n value t and (A),(B), and (C) truth values A, B, C, respectively, then counts the number of distinct vectors as well as the number of distinct abelian invariants.

```
SearchFunction:=function(A,B,C,t)

        local n, nclasses1, nAbInvs1, j, k, l, G, c, I, H, AbInv,
            nclasses, nAbInvs, d, f, T, S;

        for n in [t] do
                nclasses1:=[[]];
                nAbInvs1:=[[]];
                for j in [0..n -1] do
                        for k in [0..n -1] do
                                for l in [0..n -1] do
                                        if GcdInt(GcdInt(n,j),GcdInt(k,l))=1
                                            then
                                                if ABC(n,j,k,l)=[A, B, C] then
                                                        G:=[n,j,k,l];
                                                        if ForAll(nclasses1, c ->
                                                            not G in c) then
                                                                I:=IsoClass(G);
                                                                H:=Gnjkl(n,j,k,l);
```

```
                                                AbInv:=
                                                   AbelianInvariants
                                                   (H);
                                                Print( G, "␣has␣
                                                   AbelianInvariants
                                                   ␣", AbInv, ",␣
                                                   with␣gamma=",
                                                   gamma(G), "\n");
                                                Add(nclasses1, I);
                                                AddSet(nAbInvs1,
                                                   AbInv);
                                          fi;
                                    fi;
                              fi;
                        od;
                  od;
            od;
            nclasses:=[];
            for d in nclasses1 do
                  if not d=[] then
                        Add(nclasses, d);
                  fi;
            od;
            S:=Size(nclasses);

            nAbInvs:=[];
            for f in nAbInvs1 do
                  if not f=[] then
                        Add(nAbInvs, f);
                  fi;
            od;
            T:=Size(nAbInvs);
```

```
            Print("There␣are␣", S, "␣B&P␣classes␣for␣n=", n, "␣and␣
               ABC=", A, B, C, "␣with␣", T, "␣distinct␣sets␣of␣
               AbelianInvariants", "\n");
      od;

end;
```

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