

Real Hamiltonian forms of affine Toda field theories: spectral aspects

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Abstract

The paper is devoted to real Hamiltonian forms of 2-dimensional Toda field theories related to exceptional simple Lie algebras, and to the spectral theory of the associated Lax operators. Real Hamiltonian forms are a special type of “reductions” of Hamiltonian systems, similar to real forms of semi-simple Lie algebras. Examples of real Hamiltonian forms of affine Toda field theories related to exceptional complex untwisted affine Kac-Moody algebras are studied. Along with the associated Lax representations, we also formulate the relevant Riemann-Hilbert problems and derive the minimal sets of scattering data that determine uniquely the scattering matrices and the potentials of the Lax operators.

1 Introduction

Affine Toda field theories [44, 7, 8, 46, 47] received considerable attention of the Mathematical physics community in the past three decades and are one of the best understood integrable massive field theories at classical and quantum levels in $1 + 1$ dimensions [3]. The interest in Toda field theories was inspired by the work of A. Zamolodchikov [53] on deformation of conformal field theories preserving integrability. It was shown in [53] that the resultant theory is characterised by eight masses related to the Cartan matrix of E_8 , and by integrals of motion with spins given by the exponents of E_8 modulo its Coxeter number.

Affine Toda field theories (ATFT) are integrable models of real scalar field $\mathbf{q} = (q_1, \dots, q_n)$, in one space dimension with exponential interactions. The Lagrangian of the theory is given by

$$\mathcal{L}[\mathbf{q}] = \frac{1}{2}(\partial_\mu \mathbf{q} \cdot \partial^\mu \mathbf{q}) - \frac{m^2}{\beta^2} \sum_{k=0}^n n_k (e^{\beta(\alpha_k \cdot \mathbf{q})} - 1), \quad (1)$$

where the field $\mathbf{q}(x, t)$ is an n -dimensional vector. This corresponds to equations of motion of the form

$$\frac{\partial^2 \mathbf{q}}{\partial x \partial t} = \sum_{j=0}^r n_j \alpha_j e^{-(\alpha_j, \mathbf{q}(x, t))}, \quad (2)$$

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Each ATFT is associated to a (finite) simple Lie algebra \mathfrak{g} [2, 3]. Here n is the rank of \mathfrak{g} , α_k s ($k = 1, \dots, n$) are the simple roots of \mathfrak{g} and α_0 is the minimal root. Thus, (1) are described by the extended Dynkin diagrams of the associated affine algebra $\hat{\mathfrak{g}}$. The fields q_k can be rescaled so that β only appears in \mathcal{L} through a common factor $1/\beta^2$ (expanding in powers of β^2 is equivalent in Quantum theory to expanding in \hbar) [7, 8].

Non-simply laced theories are obtained from the affine Toda theories based on simply laced algebras by folding the corresponding Dynkin diagrams. The same process, called classical “reduction”, provides solutions of a non-simply laced theory from the classical solutions with special symmetries of the parent simply laced theory.

One of the important features of affine Toda field theories is their integrability by the inverse scattering method (ISM) [17, 45]. The starting point is the existence of the so-called Lax operator (see (4) below). The interpretation of the ISM as a generalized Fourier transforms [1, 18, 33] allows one to study all the fundamental properties of the corresponding nonlinear evolutionary equations (NLEE’s): i) the description of the class of NLEE related to a given Lax operator $L(\lambda)$ and solvable by the ISM; ii) derivation of the infinite family of integrals of motion; iii) their hierarchy of Hamiltonian structures [32]; and iv) description of the gauge equivalent systems [24, 37, 38].

Real Hamiltonian forms (RHF) are another type of “reductions” of Hamiltonian systems. The extraction of RHF’s is similar to the obtaining a real forms of a semi-simple Lie algebra. The Killing form for the later is indefinite in general (it is negatively-definite for the compact real forms). So one should not be surprised of getting RHF’s with indefinite kinetic energy quadratic form. Of course this is an obstacle for their quantization.

The purpose of this paper is to outline the spectral theory of the Lax operators of real Hamiltonian forms of affine Toda field theories related to complex untwisted exceptional affine Lie algebras.

The structure of the paper is as follows: In Section 2 we provide a brief summary of the Lax representation of ATFTs and all necessary Lie algebraic background knowledge. As to the root systems of the exceptional algebras we are following the conventions in [6]. In Section 3 first we describe the general method for constructing RHF’s. Then we briefly describe the sets of admissible roots of the exceptional Lie algebras, and the explicit constructions of their RHF’s. Section 4 is devoted to the spectral theory of the Lax operators of ATFTs with \mathbb{Z}_h -reductions, where h is the Coxeter number of \mathfrak{g} . We first provide the general construction of the fundamental analytic solutions $\xi_\nu(x, t, \lambda)$ (FAS) of the Lax operator and show their relevance to the Riemann-Hilbert problem (RHP) on a set of $2h_g$ rays l_ν closing angles π/h_g . Then we introduce the asymptotics of the FAS $\xi_\nu(x, t, \lambda)$ for $x \rightarrow \pm\infty$ and $\lambda \in l_\nu$ which determine the scattering data of L . This Section ends with a theorem specifying the minimal sets of scattering data for generic choice of the simple Lie algebra \mathfrak{g} . Section 5 contains the specific data, concerning each of the exceptional Lie algebras that allows one to determine the minimal set of scattering data for each case. The paper ends up with Conclusions.

2 Affine Toda field theories: preliminaries

To each simple Lie algebra \mathfrak{g} one can relate a Toda field theory in $1 + 1$ dimensions. It allows a Lax representation in a zero-curvature form:

$$[L, M] = 0, \tag{3}$$

where L and M are first order ordinary differential (Lax) operators:

$$L\psi \equiv \left(i \frac{d}{dx} - iq_x(x, t) - \lambda J_0 \right) \psi(x, t, \lambda) = 0, \quad (4)$$

$$M\psi \equiv \left(i \frac{d}{dt} - \frac{1}{\lambda} I(x, t) \right) \psi(x, t, \lambda) = 0. \quad (5)$$

whose potentials take values in \mathfrak{g} . Here also $q(x, t) \in \mathfrak{h}$ - the Cartan subalgebra of \mathfrak{g} , $\mathbf{q}(x, t) = (q_1, \dots, q_r)$ is its dual r -component vector ($r = \text{rank } \mathfrak{g}$). The potentials of the Lax operators are chosen as follows

$$J_0 = \sum_{\alpha \in \pi} E_\alpha, \quad I(x, t) = \sum_{\alpha \in \pi} e^{-\langle \alpha, \mathbf{q}(x, t) \rangle} E_{-\alpha}. \quad (6)$$

Here $\pi_{\mathfrak{g}}$ stands for the set of admissible roots of \mathfrak{g} , i.e. $\pi_{\mathfrak{g}} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$, with $\alpha_1, \dots, \alpha_r$ being the simple roots of \mathfrak{g} and α_0 being the minimal root of \mathfrak{g} . The corresponding Toda field theory is known as affine Toda field theory (ATFT). The Dynkin graph corresponding to the set of admissible roots $\pi_{\mathfrak{g}} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ of \mathfrak{g} is called extended Dynkin diagram (EDD). The equations of motion are of the form:

$$\frac{\partial^2 \mathbf{q}}{\partial x \partial t} = \sum_{j=0}^r n_j \alpha_j e^{-\langle \alpha_j, \mathbf{q}(x, t) \rangle}, \quad (7)$$

where n_j are the minimal positive integer coefficients n_k that provide the decomposition of the α_0 over the simple roots of \mathfrak{g} :

$$-\alpha_0 = \sum_{k=1}^r n_k \alpha_k. \quad (8)$$

It is well known that ATFT models are an infinite-dimensional Hamiltonian system. The (canonical) Hamiltonian structure is given by:

$$H_{\mathfrak{g}} = \int_{-\infty}^{\infty} dx \mathcal{H}_{\mathfrak{g}}(x, t), \quad \mathcal{H}_{\mathfrak{g}}(x, t) = \frac{1}{2} (\mathbf{p}(x, t), \mathbf{p}(x, t)) + \sum_{k=0}^r n_k (e^{-\langle \mathbf{q}(x, t), \alpha_k \rangle} - 1), \quad (9)$$

$$\Omega_{\mathfrak{g}} = \int_{-\infty}^{\infty} dx \omega_{\mathfrak{g}}(x, t), \quad \omega_{\mathfrak{g}}(x, t) = (\delta \mathbf{p}(x, t) \wedge \delta \mathbf{q}(x, t)), \quad (10)$$

where $H_{\mathfrak{g}}$ is the canonical Hamilton function and $\Omega_{\mathfrak{g}}$ is the canonical symplectic structure. Here also $\mathbf{p} = d\mathbf{q}/dt$ are the canonical momenta and coordinates satisfying canonical Poisson brackets:

$$\{q_k(x, t), p_j(y, t)\} = \delta_{jk} \delta(x - y). \quad (11)$$

The infinite-dimensional phase space $\mathcal{M} = \{\mathbf{q}(x, t), \mathbf{p}(x, t)\}$ is spanned by the canonical coordinates and momenta.

3 Real Hamiltonian forms and affine Toda field theories

The Lax representations of the ATFT models widely discussed in the literature (see e.g. [43, 44, 46, 42] and the references therein) are related mostly to the normal real form of the Lie algebra \mathfrak{g} , see [39]. Here we will study real Hamiltonian forms of ATFT models. The notion of real Hamiltonian forms was introduced in [25] and used to study reductions of ATFTs in [23]. After a brief outline of the basic theory (following [25]), in this Section we will describe the real Hamiltonian forms for ATFT models related to exceptional untwisted complex Kac-Moody algebras [41, 48, 49].

3.1 Real Hamiltonian forms

The starting point in the construction of real Hamiltonian forms (RHF) is the complexification of the field involved in the Hamiltonian of the model. First for dynamical variables we consider complex-valued fields

$$\mathbf{q}^{\mathbb{C}} = \mathbf{q}^0 + i\mathbf{q}^1, \quad \mathbf{p}^{\mathbb{C}} = \mathbf{p}^0 + i\mathbf{p}^1.$$

Next we introduce an involution \mathcal{C} acting on the phase space $\mathcal{M} \equiv \{q_k(x), p_k(x)\}_{k=1}^n$ as follows:

$$\begin{aligned} 1) \quad & \mathcal{C}(F(p_k, q_k)) = F(\mathcal{C}(p_k), \mathcal{C}(q_k)), \\ 2) \quad & \mathcal{C}(\{F(p_k, q_k), G(p_k, q_k)\}) = \{\mathcal{C}(F), \mathcal{C}(G)\}, \\ 3) \quad & \mathcal{C}(H(p_k, q_k)) = H(p_k, q_k). \end{aligned} \tag{12}$$

Here $F(p_k, q_k)$, $G(p_k, q_k)$ and the Hamiltonian $H(p_k, q_k)$ are functionals on \mathcal{M} depending analytically on the fields $q_k(x, t)$ and $p_k(x, t)$.

The complexification of the ATFT is rather straightforward. The resulting complex ATFT (CATFT) can be written down as standard Hamiltonian system with twice as many fields $\mathbf{q}^a(x, t)$, $\mathbf{p}^a(x, t)$, $a = 0, 1$:

$$\mathbf{p}^{\mathbb{C}}(x, t) = \mathbf{p}^0(x, t) + i\mathbf{p}^1(x, t), \quad \mathbf{q}^{\mathbb{C}}(x, t) = \mathbf{q}^0(x, t) + i\mathbf{q}^1(x, t), \tag{13}$$

$$\{q_k^0(x, t), p_j^0(y, t)\} = -\{q_k^1(x, t), p_j^1(y, t)\} = \delta_{kj}\delta(x - y). \tag{14}$$

The densities of the corresponding Hamiltonian and symplectic form equal

$$\begin{aligned} \mathcal{H}_{\text{ATFT}}^{\mathbb{C}} &\equiv \text{Re } \mathcal{H}_{\text{ATFT}}(\mathbf{p}^0 + i\mathbf{p}^1, \mathbf{q}^0 + i\mathbf{q}^1) \\ &= \frac{1}{2}(\mathbf{p}^0, \mathbf{p}^0) - \frac{1}{2}(\mathbf{p}^1, \mathbf{p}^1) + \sum_{k=0}^r e^{-(\mathbf{q}^0, \alpha_k)} \cos((\mathbf{q}^1, \alpha_k)), \end{aligned} \tag{15}$$

$$\omega^{\mathbb{C}} = (d\mathbf{p}^0 \wedge id\mathbf{q}^0) - (d\mathbf{p}^1 \wedge d\mathbf{q}^1). \tag{16}$$

The family of RHF then are obtained from the CATFT by imposing an invariance condition with respect to the involution $\tilde{\mathcal{C}} \equiv \mathcal{C} \circ *$ where by $*$ we denote the complex conjugation. The involution $\tilde{\mathcal{C}}$ splits the phase space $\mathcal{M}^{\mathbb{C}}$ into a direct sum $\mathcal{M}^{\mathbb{C}} \equiv \mathcal{M}_+^{\mathbb{C}} \oplus \mathcal{M}_-^{\mathbb{C}}$ where

$$\begin{aligned} \mathcal{M}_+^{\mathbb{C}} &= \mathcal{M}_0 \oplus i\mathcal{M}_1, & \mathcal{M}_-^{\mathbb{C}} &= \mathcal{M}_0 \oplus -i\mathcal{M}_1, \\ \mathcal{C}(\mathbf{q}^+ + i\mathbf{q}^-) &= (\mathbf{q}^+ - i\mathbf{q}^-), & \mathcal{C}(\mathbf{p}^+ + i\mathbf{p}^-) &= (\mathbf{p}^+ - i\mathbf{p}^-). \end{aligned} \tag{17}$$

Each involution \mathcal{C} induces an involution (involutive automorphism) $\mathcal{C}^{\#}$ also of \mathfrak{g} . Below we will choose $\mathcal{C}^{\#}$ in a special way, which is related to the Coxeter automorphism defined below in its dihedral form (26), see [6, 39, 9]. Indeed, we will define $\mathcal{C}^{\#}$ using the reflection S_1 with respect to the set of black roots of the Dinkin dyagram (see Figure 1). More precisely:

$$\begin{aligned} \mathcal{C}^{\#}\alpha_j &= \alpha_j \quad \text{for } \alpha_j \in W_g, & \mathcal{C}^{\#}\beta_j &= -\beta_j \quad \text{for } \beta_j \in B_g, \\ \mathbf{p}^0(x, t) &= \sum_{\alpha_j \in W_g} \mathbf{p}_j(x, t)\alpha_j, & \mathbf{p}^1(x, t) &= \sum_{\beta_j \in B_g} \mathbf{p}_j(x, t)\beta_j, \\ \mathbf{q}^0(x, t) &= \sum_{\alpha_j \in W_g} \mathbf{q}_j(x, t)\alpha_j, & \mathbf{q}^1(x, t) &= \sum_{\beta_j \in B_g} \mathbf{q}_j(x, t)\beta_j, \end{aligned} \tag{18}$$

where W_g (resp. B_g) is the set of white roots (resp. black roots) of the Dinkin diagram.

Thus to each involution \mathcal{C} one can relate a RHF of the ATFT. Note that $\mathcal{C}^\#$ preserves the system of admissible roots of \mathfrak{g} and the extended Dynkin diagrams of \mathfrak{g} studied in [42]. For a sake of brevity below we will skip the dependence of \mathbf{p} and \mathbf{q} on x and t . Indeed, the condition 3) in (12) requires that:

$$(\mathcal{C}(\mathbf{q}), \alpha) = (\mathbf{q}, \mathcal{C}^\#(\alpha)), \quad \alpha \in \pi_{\mathfrak{g}}, \quad (19)$$

and therefore we must have $\mathcal{C}(\pi_{\mathfrak{g}}) = \pi_{\mathfrak{g}}$. As a result:

$$\begin{aligned} \mathcal{C}(\mathbf{p}^0(x, t), \alpha_j) &= (\mathbf{p}^0(x, t), \alpha_j), & \mathcal{C}(\mathbf{p}^1(x, t), \beta_j) &= -(\mathbf{p}^1(x, t), \beta_j), \\ (\mathbf{p}^0(x, t), \beta_j) &= 0, & (\mathbf{p}^1(x, t), \alpha_j) &= 0, \\ \mathcal{C}(\mathbf{q}^0(x, t), \alpha_j) &= (\mathbf{q}^0(x, t), \alpha_j), & \mathcal{C}(\mathbf{q}^1(x, t), \beta_j) &= -(\mathbf{q}^1(x, t), \beta_j), \\ (\mathbf{q}^0(x, t), \beta_j) &= 0, & (\mathbf{q}^1(x, t), \alpha_j) &= 0, \end{aligned} \quad (20)$$

where $\alpha_j \in W_g$, and $\beta_j \in B_g$. Then applying the ideas of [43] we obtain the following result for the RHF of the ATFT related to \mathfrak{g} :

$$\mathcal{H}_{\mathfrak{g}}^{\mathbb{R}} = \frac{1}{2}(\mathbf{p}^+, \mathbf{p}^+) - \frac{1}{2}(\mathbf{p}^-, \mathbf{p}^-) + \sum_{\alpha_k \in W_g} n'_k (e^{-(\mathbf{q}^+, \alpha_k)} - 1) + \sum_{\beta_k \in B_g} n''_k (\cos(\mathbf{q}^-, \beta_k) - 1), \quad (21)$$

$$\omega_{\mathfrak{g}}^{\mathbb{R}} = (\delta\mathbf{p}^+ \wedge \delta\mathbf{q}^+) - (\delta\mathbf{p}^- \wedge \delta\mathbf{q}^-), \quad (22)$$

The Hamiltonian along with the terms related to the simple roots, contains also the minimal root $-\alpha_0$. It is well known that the maximal root α_0 can be expanded as sum of the simple roots with integer nonnegative coefficients:

$$\alpha_0 = \sum_{j=1}^r n_j \alpha_j = \sum_{\alpha_k \in W_g} n'_k \alpha_k + \sum_{\beta_j \in B_g} n''_k \beta_j. \quad (23)$$

In the second expression above we have separated the terms with the white and black roots.

The RHF of ATFT are more general integrable systems than the models described in [14, 15, 16, 42] which involve only the fields \mathbf{q}^+ , \mathbf{p}^+ invariant with respect to \mathcal{C} .

3.2 Affine Toda field theories related to $E_6^{(1)}$

The set of admissible roots for this algebra is

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), & \alpha_2 &= e_1 + e_2, \\ \alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, & \alpha_5 &= e_4 - e_3, & \alpha_6 &= e_5 - e_4, \\ \alpha_0 &= -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8). \end{aligned} \quad (24)$$

Here $\alpha_1, \dots, \alpha_6$ form the set of simple roots of E_6 and α_0 is the minimal root of the algebra. The extended Dynkin diagram of $E_6^{(1)}$ is shown on Figure 1. This is the standard definition of the root system of E_6 embedded into the 8-dimensional Euclidean space \mathbb{E}^8 . The root space \mathbb{E}_6 of the algebra E_6 is the 6-dimensional subspace of \mathbb{E}^8 orthogonal to the vectors $e_7 + e_8$ and $e_6 + e_7 + 2e_8$. Thus any vector \mathbf{q} belonging to \mathbb{E}_6 has only 6 independent coordinates and can be written as:

$$\mathbf{q} = \sum_{k=1}^5 q_k e_k + q_6 e'_6, \quad e'_6 = \frac{1}{\sqrt{3}}(e_6 + e_7 - e_8). \quad (25)$$

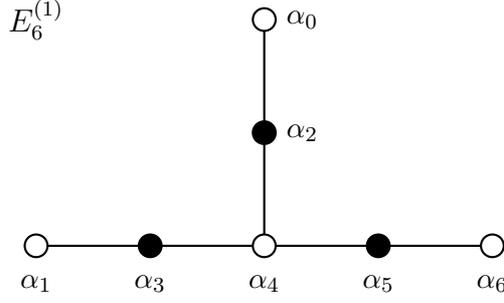


Figure 1: The extended Dynkin diagram of the complex untwisted affine Kac-Moody algebra $E_6^{(1)}$. The white roots are invariant with respect to the automorphism S_2 in (26).

The fundamental weights of E_6 are

$$\begin{aligned}\omega_1 &= \frac{2}{3}(e_8 - e_7 - e_6), & \omega_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8); \\ \omega_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5) + \frac{5}{6}(e_8 - e_7 - e_6), & \omega_4 &= e_3 + e_4 + e_5 - e_6 - e_7 + e_8, \\ \omega_5 &= e_4 + e_5 + \frac{2}{3}(e_8 - e_7 - e_6), & \omega_6 &= e_5 + \frac{1}{3}(e_8 - e_7 - e_6).\end{aligned}$$

We will use the dihedral realization of the Coxeter automorphism for $E_6^{(1)}$:

$$\text{Cox}(E_6^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5}, \quad S_2 = S_{\alpha_1} \circ S_{\alpha_4} \circ S_{\alpha_6}. \quad (26)$$

If we require an invariance of the Hamiltonian and the symplectic form (21) with respect to S_1 , and restrict on the set of admissible roots β_k , then we get a real Hamiltonian form of the $E_6^{(1)}$ ATFT, described by:

$$\mathcal{H}_{E_6}^{\mathbb{R}} = \frac{1}{2}(\mathbf{p}^+, \mathbf{p}^+) - \frac{1}{2}(\mathbf{p}^-, \mathbf{p}^-) + \sum_{k=0,1,4,6} n'_k (e^{-\langle \mathbf{q}^+, \alpha_k \rangle} - 1) + \sum_{k=2,3,5} n''_k (\cos(\langle \mathbf{q}^-, \beta_k \rangle) - 1), \quad (27)$$

where $n'_1 = n'_6 = 1$, $n'_4 = 3$ and $n''_2 = n''_3 = n''_5 = 2$ [6], and

$$\omega_{E_6^{(1)}}^{\mathbb{R}} = (\delta \mathbf{p}^+ \wedge \delta \mathbf{q}^+) - (\delta \mathbf{p}^- \wedge \delta \mathbf{q}^-). \quad (28)$$

3.3 Affine Toda field theories related to $E_7^{(1)}$

The extended root system of this algebra is given by

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), & \alpha_2 &= e_1 + e_2, \\ \alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, & \alpha_5 &= e_4 - e_3, \\ \alpha_6 &= e_5 - e_4, & \alpha_7 &= e_6 - e_5, & \alpha_0 &= e_7 - e_8,\end{aligned} \quad (29)$$

where $\alpha_1, \dots, \alpha_7$ form the set of simple roots of E_7 and α_0 is the minimal root of the algebra. The extended Dynkin diagram for $E_7^{(1)}$ is shown on Figure 2. This is the standard definition of the root system of E_7 embedded into the 8-dimensional Euclidean space \mathbb{E}^8 . The root space

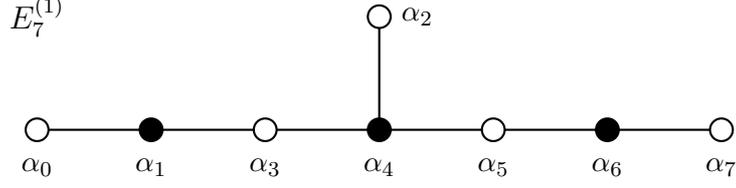


Figure 2: The extended Dynkin diagram of the complex untwisted affine Kac-Moody algebra $E_7^{(1)}$. The white roots are invariant with respect to the automorphism S_1 in (31).

\mathbb{E}_7 of the algebra E_7 is the 7-dimensional subspace of \mathbb{E}^8 orthogonal to the vector $e_7 + e_8$. Thus any vector \vec{q} belonging to \mathbb{E}_7 has 7 independent coordinates and can be written as:

$$\vec{q} = \sum_{k=1}^6 q_k e_k + q_7 e'_7, \quad e'_7 = \frac{1}{\sqrt{2}}(e_7 - e_8). \quad (30)$$

Using the dihedral realization of the Coxeter automorphism for $E_7^{(1)}$:

$$\text{Cox}(E_7^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_1} \circ S_{\alpha_4} \circ S_{\alpha_6}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5} \circ S_{\alpha_7}, \quad (31)$$

we impose an invariance of the Hamiltonian (21) with respect to the automorphism S_1 in (31). Then the root space \mathbb{E}_7 will split into direct sums: $\mathbb{E}_7 = \mathbb{E}_{7,+} \oplus \mathbb{E}_{7,-}$, with

$$\alpha_2, \alpha_3, \alpha_5, \alpha_7 \in \mathbb{E}^{7,+}, \quad \alpha_1, \alpha_4, \alpha_6 \in \mathbb{E}^{7,-}.$$

If we again require an invariance of the Hamiltonian and the symplectic form (21) with respect to S_1 from (31), and restrict on the set of admissible roots β_k , then we get a real Hamiltonian form of the $E_7^{(1)}$ ATFT, described by:

$$\mathcal{H}_{E_7}^{\mathbb{R}} = \frac{1}{2}(\mathbf{p}^+, \mathbf{p}^+) - \frac{1}{2}(\mathbf{p}^-, \mathbf{p}^-) + \sum_{k=0,2,3,5,7} n'_k (e^{-(\mathbf{q}^+, \alpha_k)} - 1) + \sum_{k=1,4,6} n''_k (\cos(\mathbf{q}^-, \beta_k) - 1), \quad (32)$$

where $n'_1 = n'_6 = 2$, $n'_4 = 4$ and $n''_2 = 2$, $n''_3 = n''_5 = 3$, $n''_7 = 1$ [6], and

$$\omega_{E_7^{(1)}}^{\mathbb{R}} = (\delta \mathbf{p}^+ \wedge \delta \mathbf{q}^+) - (\delta \mathbf{p}^- \wedge \delta \mathbf{q}^-). \quad (33)$$

3.4 Affine Toda field theories related to $E_8^{(1)}$

The set of admissible roots for this algebra is

$$\begin{aligned} \alpha_0 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) & \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\ \alpha_2 &= e_1 + e_2, & \alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, \\ \alpha_5 &= e_4 - e_3, & \alpha_6 &= e_5 - e_4, & \alpha_7 &= e_6 - e_5 & \alpha_8 &= e_7 - e_6. \end{aligned} \quad (34)$$

where $\alpha_1, \dots, \alpha_8$ form the set of simple roots of E_8 and α_0 is the minimal root of the algebra. If we exclude α_8 from (34), we will get the set of positive roots (29) form E_7 . The extended Dynkin diagram for $E_8^{(1)}$ is shown on Figure 3.

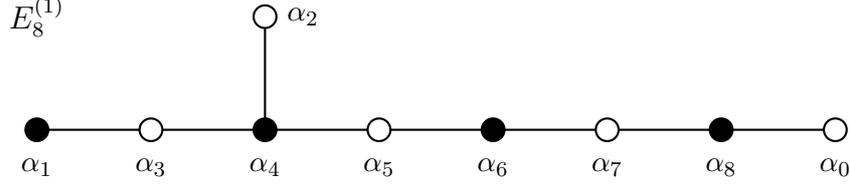


Figure 3: The extended Dynkin diagram of the complex untwisted affine Kac-Moody algebra $E_8^{(1)}$. The white roots are invariant with respect to the automorphism S_2 in (35).

The fundamental weights of E_8 are

$$\begin{aligned} \omega_1 &= 2e_8, & \omega_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8), \\ \omega_3 &= \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + 7e_8), & \omega_4 &= e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8, \\ \omega_5 &= e_4 + e_5 + e_6 + e_7 + 4e_8, & \omega_6 &= e_5 + e_6 + e_7 + 3e_8, \\ \omega_7 &= e_6 + e_7 + 2e_8, & \omega_8 &= e_7 + e_8 = -\alpha_0. \end{aligned}$$

We will use the dihedral realization of the Coxeter automorphism for $E_8^{(1)}$:

$$\text{Cox}(E_8^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_1} \circ S_{\alpha_4} \circ S_{\alpha_6} \circ S_{\alpha_8}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_3} \circ S_{\alpha_5} \circ S_{\alpha_7}. \quad (35)$$

We impose an invariance of the Hamiltonian (21) with respect to the automorphism S_1 in (31). Then the root space \mathbb{E}_8 will split into direct sums: $\mathbb{E}_8 = \mathbb{E}_{8,+} \oplus \mathbb{E}_{8,-}$, with

$$\alpha_2, \alpha_3, \alpha_5, \alpha_7 \in \mathbb{E}_{8,+}, \quad \alpha_1, \alpha_4, \alpha_6, \alpha_8 \in \mathbb{E}_{8,-}.$$

The invariance of the Hamiltonian and the symplectic form (21) with respect to S_1 from (35), and restricting on the set of admissible roots β_k , will result in a real Hamiltonian form of the $E_8^{(1)}$ ATFT, described by:

$$\mathcal{H}_{E_8}^{\mathbb{R}} = \frac{1}{2}(\mathbf{p}^+, \mathbf{p}^+) - \frac{1}{2}(\mathbf{p}^-, \mathbf{p}^-) + \sum_{k=0,2,3,5,7} n'_k (e^{-\langle \mathbf{q}^+, \alpha_k \rangle} - 1) + \sum_{k=1,4,6,8} n''_k (\cos(\langle \mathbf{q}^-, \beta_k \rangle) - 1), \quad (36)$$

where $n''_1 = n''_8 = 2$, $n''_4 = 6$, $n''_6 = 4$ and $n'_2 = n'_7 = 3$, $n'_3 = 4$, $n'_5 = 5$ [6], and

$$\omega_{E_8^{(1)}}^{\mathbb{R}} = (\delta \mathbf{p}^+ \wedge \delta \mathbf{q}^+) - (\delta \mathbf{p}^- \wedge \delta \mathbf{q}^-). \quad (37)$$

3.5 Affine Toda field theories related to $F_4^{(1)}$

The extended roots of this algebra are

$$\begin{aligned} \alpha_1 &= e_2 - e_3, & \alpha_2 &= e_3 - e_4, & \alpha_3 &= e_4, \\ \alpha_4 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4), & \alpha_0 &= -e_1 - e_2, \end{aligned} \quad (38)$$

with α_0 being the minimal root. The extended Dynkin diagram of $F_4^{(1)}$ is shown on Figure 4. The fundamental weights of F_4 are

$$\begin{aligned} \omega_1 &= e_1 + e_2, & \omega_2 &= 2e_1 + e_2 + e_3, \\ \omega_3 &= \frac{1}{2}(3e_1 + e_2 + e_3 + e_4), & \omega_4 &= e_1. \end{aligned} \quad (39)$$



Figure 4: The extended Dynkin diagram of the complex untwisted affine Kac-Moody algebra $F_4^{(1)}$. The white roots are invariant with respect to the automorphism S_2 in (40).

Using the dihedral realisation of the Coxeter automorphism for $F_4^{(1)}$:

$$\text{Cox}(F_4^{(1)}) = S_1 \circ S_2, \quad S_1 = S_{\alpha_1} \circ S_{\alpha_3}, \quad S_2 = S_{\alpha_2} \circ S_{\alpha_4}, \quad (40)$$

and imposing an invariance of the Hamiltonian (21) with respect to the automorphism S_1 in (40), will extract a real Hamiltonian form of $F_4^{(1)}$ ATFT of the form

$$\mathcal{H}_{F_4^{(1)}}^{\mathbb{R}} = \frac{1}{2} \sum_{k=1}^4 ((p_k^+)^2 - (p_k^-)^2) + \sum_{k=0,2,4} n'_k (e^{-\langle \mathbf{q}^+, \alpha_k \rangle} - 1) + \sum_{k=1,3} n''_k (\cos(\langle \mathbf{q}^-, \beta_k \rangle) - 1), \quad (41)$$

where $n'_2 = 3$, $n'_4 = 2$ and $n''_1 = 2$, $n''_3 = 4$. In addition

$$\omega_{F_4^{(1)}}^{\mathbb{R}} = (\delta \mathbf{p}^+ \wedge \delta \mathbf{q}^+) - (\delta \mathbf{p}^- \wedge \delta \mathbf{q}^-). \quad (42)$$

Then the root space \mathbb{E}_4 will split into direct sums: $\mathbb{E}_4 = \mathbb{E}_{4,+} \oplus \mathbb{E}_{4,-}$, with $\alpha_2, \alpha_4 \in \mathbb{E}_{4,+}$ and $\alpha_1, \alpha_3 \in \mathbb{E}_{4,-}$.

3.6 Affine Toda field theories related to $G_2^{(1)}$

The set of admissible roots for this algebra is given by

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = -e_1 + e_2 + e_3, \quad \alpha_0 = -e_1 - e_2 + 2e_3. \quad (43)$$

The fundamental weights of G_2 are

$$\omega_1 = e_1 - e_2 + 2e_3, \quad \omega_2 = -e_1 - e_2 + 2e_3.$$

The extended Dynkin diagram of $G_2^{(1)}$ is shown on Figure 5.

Let us take $C_1 = S_{\alpha_2}$. The invariance of the Hamiltonian (21) with respect to the automorphism C_1 in will give a real Hamiltonian form of $G_2^{(1)}$ ATFT described by

$$\mathcal{H}_{G_2^{(1)}}^{\mathbb{R}} = \frac{1}{2} \sum_{k=1}^r ((p_k^+)^2 - (p_k^-)^2) + (e^{-\langle \mathbf{q}^+, \alpha_0 \rangle} - 1) + 3(e^{-\langle \mathbf{q}^+, \alpha_1 \rangle} - 1) + 2(\cos(\langle \mathbf{q}^-, \alpha_2 \rangle) - 1), \quad (44)$$

$$\omega_{G_2^{(1)}}^{\mathbb{R}} = (\delta \mathbf{p}^+ \wedge \delta \mathbf{q}^+) - (\delta \mathbf{p}^- \wedge \delta \mathbf{q}^-), \quad (45)$$

The root space \mathbb{E}_2 will split into direct sums: $\mathbb{E}_2 = \mathbb{E}_{2,+} \oplus \mathbb{E}_{2,-}$, with $\alpha_0, \alpha_1 \in \mathbb{E}_{2,+}$ and $\alpha_2 \in \mathbb{E}_{2,-}$.

4 On the spectral properties of the Lax operators with \mathbb{Z}_h -reduction

4.1 General theory

The operator L (4) is very convenient for deriving the 2-dimensional TFT. However this formulation is not convenient for constructing the fundamental analytic solutions (FAS) of L . The

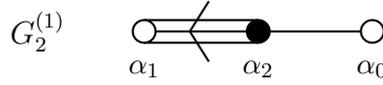


Figure 5: The extended Dynkin diagram of the complex untwisted affine Kac-Moody algebra $G_2^{(1)}$.

first step we will do will be to apply on L a similarity transformation

$$\tilde{L}\tilde{\psi} = g_0^{-1}Lg_0\tilde{\psi} = i\frac{\partial\tilde{\psi}}{\partial x} - \left(i\sum_{k=0}^6 q_k \mathcal{E}_k + \lambda\tilde{J}_0 \right) \tilde{\psi} = 0 \quad (46)$$

where $\tilde{J}_0 \in \mathfrak{h}$; i.e.

$$\tilde{J}_0 = \sum_{k=1}^6 y_k H_{\alpha_k} = \sum_{k=1}^6 z_k H_{\omega_k}$$

where y_j and z_j must be expressed in terms of $\omega = \exp(2\pi i/h_g)$.

Let us somewhat simplify the notations in the two Lax operators

$$\begin{aligned} L\psi &\equiv \frac{\partial\psi}{\partial x} + (\vec{q}_x - \lambda J_0)\psi(x, t, \lambda) = 0, \\ \tilde{L}\tilde{\psi} &\equiv \frac{\partial\tilde{\psi}}{\partial x} + (Q(x, t) - \lambda\tilde{J})\tilde{\psi}(x, t, \lambda) = 0, \end{aligned} \quad (47)$$

where $J_0 = \sum_{\alpha \in \delta_0} E_\alpha$, δ_0 is the set of simple roots α_j and the minimal root α_0 , and $\tilde{J} \in \mathfrak{h}$ and $\omega = \exp(2\pi i/h)$.

The spectral theory of the Lax operators with \mathbb{Z}_h Mikhailov reduction groups [43] for the classical series of simple Lie algebras are by now well developed, see [33, 34, 35]. Here we will formulate them taking into account when necessary the peculiarities of the exceptional algebras.

In [22] the problem for the interrelation between L and \tilde{L} was solved by using the Chevallie basis for $A_5^{(1)}$ [13]. Indeed, for the classical series Chevallie basis for the typical representations are well known so it is not difficult to diagonalize J_0 and then evaluate $\alpha_j(\tilde{J}_0)$. The same procedure for the exceptional algebras (except G_2) is rather more complicated. Indeed, the bases for the typical representations of F_4 , E_6 and E_7 are well known, see [40] where one can find all root-vectors of F_4 as 26×26 matrices. For E_6 and E_7 the simple roots have been given in [40]; it is possible that all root vectors for all exceptional algebras have been evaluated in the framework of MAGMA project. So in general one can find J_0 for the typical representations of all exceptional algebras. Then the construction of \tilde{J}_0 becomes the next task. The eigenvalues of J_0 and the eigenvectors may be calculated, though the latter would be rather involved expressions. The next challenge would be to constructing out of the properly normalized eigenvectors g_0 and to ensure that it is an element of the corresponding exceptional group.

The easiest way to get a realization of the Coxeter automorphism C_g is to represent it as an element of the Cartan subgroup:

$$C_g = \exp \left(\frac{2\pi i}{h_g} \sum_{j=1}^r H_{\omega_j} \right), \quad (48)$$

where h_g is the Coxeter number and ω_j are the fundamental weights of the algebra \mathfrak{g} . Indeed, using the Cartan-Weyl commutation relations [39] one can easily check that

$$C_g J_0 C_g^{-1} = \omega J_0, \quad \omega = \exp\left(\frac{2\pi i}{h_g}\right). \quad (49)$$

This construction is compatible with the Lax operator L , since C_g commutes with \vec{q}_x and therefore

$$C_g L(\lambda) C_g^{-1} = L(\omega\lambda). \quad (50)$$

Treating the operator \tilde{L} requires the use of the Coxeter automorphism \tilde{C}_g realized as element of the Weyl group \mathcal{W}_g . We already mentioned the difficulties involved with the standard approach based on the use of the Chevallie basis. Therefore, treating the exceptional algebras we will use a different approach. Our idea is to work only in the Euclidean space \mathbb{E}_r in which the root systems are embedded. Indeed, there we can construct \tilde{C}_g using its dihedral form [39, 6, 9]:

$$\tilde{C}_g = S_w \circ S_b, \quad S_w = \prod_{\alpha \in W_g} S_\alpha, \quad S_b = \prod_{\beta \in B_g} S_\beta, \quad (51)$$

where W_g (respectively B_g) are the sets of ‘white’ roots (resp. ‘black’ roots) of the Dinkin diagram of the algebra \mathfrak{g} . Note that all elements of W_g (resp. B_g) are all orthogonal, so S_w and S_b are Weyl reflections, i.e. $S_w^2 = \mathbb{1}$ and $S_b^2 = \mathbb{1}$. Then $C_g^{h_g} = \mathbb{1}$, where h_g is the Coxeter number of the algebra \mathfrak{g} . Thus we will find the realization of the Coxeter automorphism as element of the Weyl group \mathcal{W}_g of the algebra.

The next step should be to construct the Cartan subalgebra element \tilde{J}_0 . In fact we will replace this step, and instead of constructing \tilde{J}_0 we will construct the vector $\vec{v}_g \in \mathbb{E}_r$ which is dual to \tilde{J}_0 . Then we will use the fact that $\alpha_j(\tilde{J}_0) = (\alpha_j, \vec{v}_g)$. This is all we need in order to construct the continuous spectrum and the spectral data for \tilde{L} . On the other hand, since \tilde{L} is related to L by the similarity transformation $\tilde{L} = g_0 L g_0^{-1}$ then both operators will have the same spectral data.

The construction of the vector \vec{v}_g is based on its basic property of invariance under the \mathbb{Z}_h Mikhailov reduction group; like in (50) we must have:

$$\tilde{C}_g \tilde{L}(\lambda) \tilde{C}_g^{-1} = \tilde{L}(\omega\lambda), \quad \text{i.e.} \quad \tilde{C}_g(\vec{v}_g) = \omega \vec{v}_g. \quad (52)$$

Such invariant vectors can be obtained by taking weighted averaged action by \tilde{C}_g :

$$\vec{v}_0 = \sum_{s=0}^{h_g-1} \omega^{-s} \tilde{C}_g^s(\vec{v}_g). \quad (53)$$

Obviously there is an arbitrariness in the choice of \vec{v}_g : it can be a root, or fundamental weight of the algebra \mathfrak{g} . At the same time we can use the fact that the spectral parameter λ can be redefined, so this arbitrariness can be taken care of.

4.2 The FAS of the Lax operators L

Here we just outline the procedure of constructing the fundamental analytic solutions (FAS) of L [4, 5, 34, 33]. First we have to determine the regions of analyticity. For smooth potentials $Q(x)$ that fall off fast enough for $x \rightarrow \pm\infty$ these regions are the $2h$ sectors Ω_ν , separated by the

rays l_ν on which $\text{Re } \lambda(\alpha, \vec{v}_g) = 0$, where by α is a root of \mathfrak{g} . The rays l_ν and the sectors Ω_ν are given by:

$$l_\nu: \arg(\lambda) = \frac{\pi(\nu-1)}{h_g}, \quad \Omega_\nu: \frac{\pi(\nu-1)}{h_g} \leq \arg(\lambda) \leq \frac{\nu}{h_g}, \quad \nu = 1, \dots, 2h, \quad (54)$$

and close angles equal to π/h .

The main result is that in each of the sectors Ω_ν we can construct fundamental analytic solutions (FAS) $\xi_\nu(x, t, \lambda)$ of the Lax operator \mathcal{L} :

$$\begin{aligned} \mathcal{L}\xi_\nu(x, t, \lambda) &\equiv i\frac{\partial \xi_\nu}{\partial x} + Q_x \xi_\nu(x, \lambda) - \lambda[\tilde{J}, \xi_\nu] = 0, \\ Q_x &= \sum_{j=1}^r q_{j,x} \mathcal{O}_j, \quad \mathcal{O}_j = \sum_{s=0}^{h_g-1} C_g^s(E_{\alpha_j}). \end{aligned} \quad (55)$$

The asymptotics of $\xi^\nu(x, \lambda)$ and $\xi^{\nu-1}(x, \lambda)$ along the ray l_ν can be written in the form [20, 33]:

$$\begin{aligned} \lim_{x \rightarrow -\infty} e^{-\lambda \tilde{J} x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda \tilde{J} x} &= S_\nu^+(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow -\infty} e^{-\lambda \tilde{J} x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda \tilde{J} x} &= S_\nu^-(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow \infty} e^{-\lambda \tilde{J} x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda \tilde{J} x} &= T_\nu^- D_\nu^+(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow \infty} e^{-\lambda \tilde{J} x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda \tilde{J} x} &= T_\nu^+ D_\nu^-(\lambda), \quad \lambda \in l_\nu, \end{aligned} \quad (56)$$

where the matrices S_ν^+ , T_ν^+ (resp. S_ν^- , T_ν^-) are upper-triangular (resp. lower-triangular) with respect to the ν -ordering. They provide the Gauss decomposition of the scattering matrix with respect to the ν -ordering, i.e.:

$$\begin{aligned} T_\nu(\lambda) &= T_\nu^-(\lambda) D_\nu^+(\lambda) \hat{S}_\nu^+(\lambda), \quad \lambda \in l_\nu e^{i0}, \\ &= T_\nu^+(\lambda) D_\nu^-(\lambda) \hat{S}_\nu^-(\lambda), \quad \lambda \in l_\nu e^{-i0}. \end{aligned} \quad (57)$$

More careful analysis shows [33] that in fact $T_\nu(\lambda)$ belongs to a subgroup \mathcal{G}_ν of $SL(N, \mathbb{C})$. In particular we can choose λ in such a way that the simple roots of \mathfrak{g} will be related to the rays l_0 and l_1 ; more precisely the ‘black’ roots will be related to l_0 while the ‘white’ roots will be related to l_1 . In particular this means that with l_0 and l_1 we can relate subalgebras of \mathfrak{g} which will be direct sums of several copies of $sl(2)$. In addition we can introduce the analogs of the reflection coefficients $\rho_{\nu,j}^\pm$ and $\tau_{\nu,j}^\pm$ with $\nu = 0, 1$ as follows:

$$\begin{aligned} S_0^\pm(\lambda) &= \exp\left(\sum_{\alpha \in B_g} \tau_{0,\alpha}^\pm(\lambda) E_{\pm\alpha}\right), \quad T_0^\mp(\lambda) = \exp\left(\sum_{\alpha \in B_g} \rho_{0,\alpha}^\pm(\lambda) E_{\mp\alpha}\right), \\ S_1^\pm(\lambda) &= \exp\left(\sum_{\alpha \in W_g} \tau_{0,\alpha}^\pm(\lambda) E_{\pm\alpha}\right), \quad T_1^\mp(\lambda) = \exp\left(\sum_{\alpha \in W_g} \rho_{0,\alpha}^\pm(\lambda) E_{\mp\alpha}\right), \\ D_0^\pm(\lambda) &= \exp\left(\sum_{j=1}^r \frac{2d_{0,j}^\pm(\lambda)}{(\alpha_j, \alpha_j)} H_{\alpha_j}\right), \quad D_1^\pm(\lambda) = \exp\left(\sum_{j=1}^r \frac{2d_{0,j}^\pm(\lambda)}{(\alpha_j, \alpha_j)} H_{\alpha_j}\right). \end{aligned} \quad (58)$$

Next the \mathbb{Z}_h -symmetry allows us to extend these scattering data to the other rays as well. Indeed applying the Coxeter automorphism we get first the relations between FAS:

$$\begin{aligned} C_g(\xi^\nu(x, \lambda)) &= \xi^{\nu+2}(x, \lambda\omega), \quad C_g(T_\nu^\pm(\lambda)) = T_{\nu+2}(\lambda\omega), \\ C_g(S_\nu^\pm(\lambda)) &= S_{\nu+2}^\pm(\lambda\omega), \quad C_g(D_\nu^\pm(\lambda)) = D_{\nu+2}^\pm(\lambda\omega), \end{aligned} \quad (59)$$

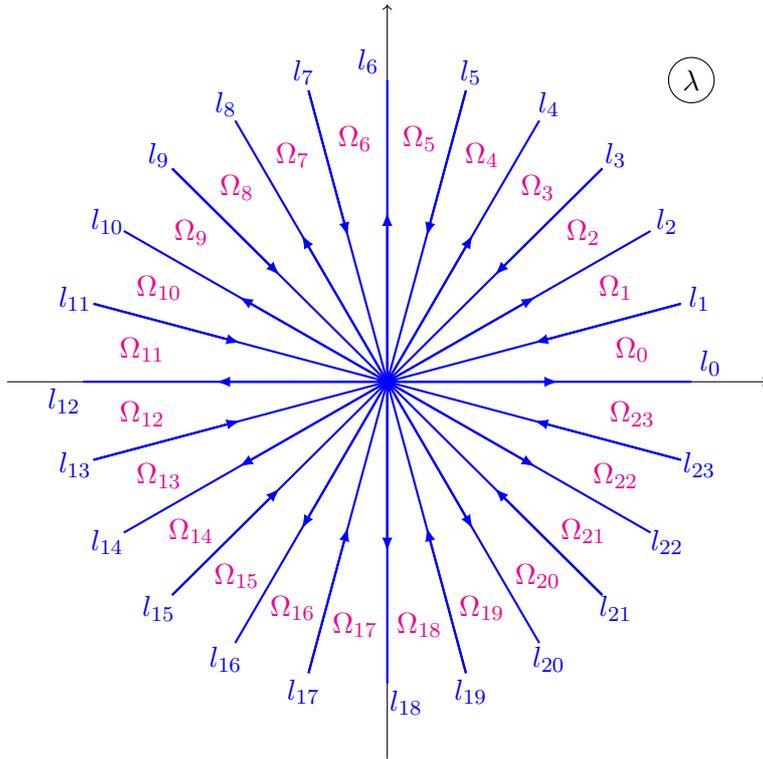


Figure 6: The continuous spectrum of the Lax operators related to the algebras E_6 and F_4 whose Coxeter numbers equal 12. Obviously the continuous spectra of for E_8 , E_7 and G_2 will have different number of rays equal to 60, 36 and 12 respectively.

where the index $\nu + 2$ should be taken modulo $2h$. Consequently we can view as independent only the data on two of the rays, e.g. on l_1 and l_0 ; all the rest will be recovered using (59).

4.3 The inverse scattering problem and the Riemann-Hilbert problem

The next important step is the possibility to reduce the solution of the inverse scattering problem (ISP) for the generalized Zakharov-Shabat system to a (local) RHP. More precisely, we have:

$$\begin{aligned} \xi^\nu(x, t, \lambda) &= \xi^{\nu-1}(x, t, \lambda)G_\nu(x, t, \lambda), & \lambda \in l_\nu, \\ G_\nu(x, t, \lambda) &= e^{\lambda \tilde{J}x - \lambda^{-1} \tilde{I}t} G_{0,\nu}(\lambda) e^{-\lambda \tilde{J}x + \lambda^{-1} \tilde{I}t}, & G_{0,\nu}(\lambda) = \hat{S}_\nu^- S_\nu^+(\lambda) \Big|_{t=0}. \end{aligned} \quad (60)$$

where $\tilde{I} = g_0^{-1} I(x, t) g_0$. The collection of all relations (60) for $\nu = 1, 2, \dots, 2h$ together with

$$\lim_{\lambda \rightarrow \infty} \xi^\nu(x, t, \lambda) = \mathbb{1}, \quad (61)$$

can be viewed as a local RHP posed on the collection of rays $\Sigma \equiv \{l_\nu\}_{\nu=1}^{2h}$ with canonical normalization. Rather straightforwardly we can prove that if $\xi^\nu(x, \lambda)$ is a solution of the RHP (60), (61) then $\chi^\nu(x, \lambda) = \xi^\nu(x, \lambda) e^{-\lambda U_1 x}$ is a FAS of L with potential

$$Q_x(x, t) = \lim_{\lambda \rightarrow \infty} \lambda \left(\tilde{J} - \xi^\nu(x, t, \lambda) \tilde{J} \hat{\xi}^\nu(x, t, \lambda) \right). \quad (62)$$

For the classical Lie algebras the analyticity properties of $d_j^\pm(\lambda)$ allow one to reconstruct them from the sewing function $G(\lambda)$ (60) and from the and from the locations of their zeroes and poles [33, 32]. The idea was to consider the relations (57) in the j -th fundamental representation of \mathfrak{g} and take the matrix elements between the highest $|\omega_j^+\rangle$ (resp. lowest $|\omega_j^-\rangle$) weight vectors. This can be done also for exceptional groups with the results

$$\begin{aligned} d_{\nu,j}^+(\lambda) - d_{\nu,j}^-(\lambda) &= -\ln\langle\omega_j^+|\hat{T}_\nu^+T_\nu^-(\lambda)|\omega_j^+\rangle \\ &= -\ln\langle\omega_j^-|\hat{S}_\nu^-S_\nu^+(\lambda)|\omega_j^+\rangle. \end{aligned} \quad (63)$$

If \mathfrak{g} is one of the classical Lie algebras, then the right hand sides of (63) can be related to the principal upper and lower minors of the scattering matrix $T(\lambda)$ in the typical representation. For the exceptional algebras the fundamental representations can not so easy be related to the typical (or lowest dimensional) one. So recovering the generating functionals of integrals of motion from their analyticity properties is an open problem.

5 The minimal sets of scattering data

5.1 The minimal sets of scattering data from the RHP

As a consequence of the results in Section 4.3 we conclude that the following theorem holds true:

Theorem 1. *Let us assume that the Lax operator \mathcal{L} (55) is constructed along the ideas in Section 4 above, i.e. it is related to a simple Lie algebra \mathfrak{g} and possesses \mathbb{Z}_h symmetry where h is the Coxeter number of \mathfrak{g} . Let us also assume that its potential Q_x is such that \mathcal{L} has no discrete eigenvalues. Then its FAS $\xi^\nu(x, t, \lambda)$ in the sector Ω_ν will provide a regular solution of the Riemann-Hilbert Problem (60). Let us denote by δ_0 and δ_1 the subsets of roots of the algebra \mathfrak{g} which satisfy the equations:*

$$\begin{aligned} \alpha \in \delta_0 & \quad \text{if} \quad \text{Im} \lambda(\alpha, \vec{v}_g) = 0 & \quad \text{for} \quad \lambda \in l_0, \\ \alpha \in \delta_1 & \quad \text{if} \quad \text{Im} \lambda(\alpha, \vec{v}_g) = 0 & \quad \text{for} \quad \lambda \in l_1. \end{aligned} \quad (64)$$

Let also

$$\begin{aligned} \mathcal{T}_1 &= \{\tau_{0,\alpha}^\pm(\lambda), \quad \alpha \in \delta_0, \lambda \in l_0\} \cup \{\tau_{1,\alpha}^\pm(\lambda), \quad \alpha \in \delta_1, \lambda \in l_1\}; \\ \mathcal{T}_2 &= \{\rho_{0,\alpha}^\pm(\lambda), \quad \alpha \in \delta_0, \lambda \in l_0\} \cup \{\rho_{1,\alpha}^\pm(\lambda), \quad \alpha \in \delta_1, \lambda \in l_1\}. \end{aligned} \quad (65)$$

Then: A) \mathcal{T}_1 (resp. \mathcal{T}_2) provide a minimal set of scattering data which allow one to recover:

1. the Gauss factors $S_\nu^\pm(\lambda)$ for $\nu = 0, 1$ (resp. $T_\nu^\pm(\lambda)$ for $\nu = 0, 1$);
2. the Gauss factors $D_\nu^\pm(\lambda)$ for $\nu = 0, 1$ (resp. $T_\nu^\pm(\lambda)$ for $\nu = 0, 1$);
3. the sewing functions $G_\nu(x, t, \lambda)$ and the scattering matrices $T_\nu(\lambda)$ for each ray l_ν , $\nu = 0, \dots, 2h_g - 1$;

B) \mathcal{T}_1 (resp. \mathcal{T}_2) allows one to construct the regular solution $\xi^\nu(x, t, \lambda)$ of the RHP;

C) \mathcal{T}_1 (resp. \mathcal{T}_2) allows one to recover the potential Q_x of the Lax operator.

Proof. A) Given \mathcal{T}_1 (respectively \mathcal{T}_2) and using (59) we recover the Gauss factors, i.e., the group-valued functions $S_\nu^\pm(\lambda, t)$ (respectively $T_\nu^\pm(\lambda, t)$) for $\nu = 0, 1$. Next, using the \mathbb{Z}_h -symmetry we construct $S_\nu^\pm(\lambda, t)$ (respectively $T_\nu^\pm(\lambda, t)$) for $\nu = 2, \dots, 2h_g - 1$, see eq. (59).

The next step consists in recovering the functionals of the integrals of motion $d_{\nu,j}^\pm(\lambda)$. The relations (63) provide in principle this possibility. Indeed, the sets \mathcal{T}_1 and \mathcal{T}_1 then we know the Gauss factors $S_\nu^\pm(\lambda, t)$ and $T_\nu^\pm(\lambda, t)$ not only in the typical representations, but also in all fundamental representations. Of course there are serious difficulties in evaluating the right hand sides of (63). However there is no doubt that they are determined uniquely by any of the sets \mathcal{T}_1 and \mathcal{T}_1 . As a result we have determined all Gauss factors in eq. (58), i.e. we obtained all sewing functions $G_\nu(x, t, \lambda)$ and the scattering matrices $T_\nu(t, \lambda)$.

B) Since the operator \mathcal{L} has no discrete eigenvalues, then the corresponding FAS satisfy a regular RHP. Therefore the sewing functions $G_\nu(x, t, \lambda)$ determine uniquely the solution $\xi_\nu^\pm(\lambda, t)$ of the RHP.

C) Since the solution $\xi_\nu^\pm(\lambda, t)$ of the RHP is uniquely determined then the corresponding potential of \mathcal{L} is determined from eq. (62). □

5.2 Minimal sets of scattering data for the exceptional algebras

Here we will formulate the results about the spectral data for the Lax operators related to the exceptional algebras. The calculations were performed by Maple. This allowed us to calculate the corresponding vectors \vec{v}_g as well as the scalar products (\vec{v}_g, α_j) and to establish for each of the exceptional algebras the results in Subsection 4.2. In particular we found that the subsets of roots δ_0 and δ_1 are given by the sets of ‘black’ and ‘white’ roots, i.e:

$$\delta_0 \equiv B_g, \quad \delta_1 \equiv W_g. \quad (66)$$

Below we list, besides the sets of roots δ_0 and δ_1 , also Coxeter automorphism as a linear operator (matrix) acting on the Euclidean space \mathbb{E}^r .

5.2.1 $\mathfrak{g} \simeq E_6^{(1)}$

Let $\varepsilon_6 = \frac{1}{\sqrt{3}}(-e_6 - e_7 + e_8)$. Then the set of admissible roots (24) can be rewritten as:

$$\begin{aligned} \alpha_0 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + \sqrt{3}\varepsilon_6), & \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3}\varepsilon_6), \\ \alpha_2 &= e_1 + e_2, & \alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, & \alpha_5 &= e_4 - e_3, & \alpha_6 &= e_5 - e_4. \end{aligned} \quad (67)$$

There are two rays (54), corresponding to the Coxeter automorphism of $E_6^{(1)}$. Roots related to the rays l_0 and l_1 :

$$\begin{aligned} l_0 &= \arg \lambda = 0, & B_g &\equiv \{\alpha_1, \alpha_4, \alpha_6\}; \\ l_1 &= \arg \lambda = \frac{\pi}{12}, & W_g &\equiv \{-\alpha_2, -\alpha_3, -\alpha_5\}; \end{aligned} \quad (68)$$

The Coxeter number of $E_6^{(1)}$ is $h = 12$ and the exponents are 1, 4, 5, 7, 8, 11.

The Coxeter automorphism as a linear operator on the dual Euclidean space \mathbb{E}_6 :

$$C_{E_6} = \frac{1}{4} \begin{pmatrix} -3 & -1 & -1 & -1 & -1 & \sqrt{3} \\ -1 & 1 & -3 & 1 & 1 & -\sqrt{3} \\ 1 & -1 & -1 & -1 & 3 & \sqrt{3} \\ 1 & 3 & -1 & -1 & -1 & \sqrt{3} \\ 1 & -1 & -1 & 3 & -1 & \sqrt{3} \\ -\sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix}, \quad C_{E_6}^{12} = \mathbb{1}_6, \quad \det C_{E_6} = 1. \quad (69)$$

Here e_j , $j = 1, \dots, 5$ and ε_6 is the basis of \mathbb{E}_6 . The characteristic polynomial of C_{E_6} reads:

$$P_{E_6} = z^6 + z^5 - z^3 + z + 1. \quad (70)$$

5.2.2 $\mathfrak{g} \simeq E_7^{(1)}$

Introducing the notation

$$\varepsilon_7 = -\frac{e_7 - e_8}{\sqrt{2}},$$

the set of admissible roots (29) can be rewritten as:

$$\begin{aligned} \alpha_0 &= -\sqrt{2}\varepsilon_7 = e_7 - e_8, & \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}\varepsilon_7), \\ \alpha_2 &= e_1 + e_2, & \alpha_2 &= e_1 + e_2, & \alpha_3 &= e_2 - e_1, & \alpha_4 &= e_3 - e_2, \\ \alpha_5 &= e_4 - e_3, & \alpha_6 &= e_5 - e_4, & \alpha_7 &= e_6 - e_5. \end{aligned} \quad (71)$$

Roots related to the rays l_0 and l_1 are respectively:

$$\begin{aligned} l_0 &= \arg \lambda = 0, & B_g &\equiv \{\alpha_1, \alpha_4, \alpha_6\}; \\ l_1 &= \arg \lambda = \frac{\pi}{18}, & W_g &\equiv \{-\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}; \end{aligned} \quad (72)$$

The Coxeter number for $E_7^{(1)}$ is $h = 18$ and the exponents are 1, 5, 7, 9, 11, 13, 17. The Coxeter automorphism acts as a linear operator on the space \mathbb{E}_7 as:

$$C_{E_7} = \frac{1}{4} \begin{pmatrix} -3 & -1 & 1 & 1 & 1 & 1 & \sqrt{2} \\ -1 & 1 & -1 & 3 & -1 & -1 & -\sqrt{2} \\ -1 & -1 & -3 & -1 & -1 & -1 & -\sqrt{2} \\ -1 & 1 & -1 & -1 & -1 & 3 & -\sqrt{2} \\ -1 & 1 & -1 & -1 & 3 & -1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}, \quad C_{E_7}^9 = -\mathbb{1}_7, \quad \det C_{E_7} = -1. \quad (73)$$

Here e_j , $j = 1, \dots, 6$ and ε_7 forms the basis in the dual Euclidean space \mathbb{E}_7 . The characteristic polynomial of C_{E_7} is:

$$P_{E_7} = z^7 + z^8 - z^4 - z^3 + z + 1 \quad (74)$$

5.2.3 $\mathfrak{g} \simeq E_8^{(1)}$

The set of admissible roots is given by (34). Roots related to the rays l_0 and l_1 are:

$$\begin{aligned} l_0 &= \arg \lambda = 0, & B_g &\equiv \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\}; \\ l_1 &= \arg \lambda = \frac{\pi}{30}, & W_g &\equiv \{-\alpha_2, -\alpha_3, -\alpha_5, -\alpha_7\}; \end{aligned} \quad (75)$$

The Coxeter number for $E_8^{(1)}$ is $h = 30$ and the exponents are 1, 7, 11, 13, 17, 19, 23, 29. The action of the Coxeter automorphism as a linear operator on the dual Euclidean space \mathbb{E}_7 is given by

$$C_{E_8} = \frac{1}{4} \begin{pmatrix} -3 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 3 & -1 & -1 & -1 & 1 \\ -1 & -3 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 3 & -1 & 1 \\ -1 & 1 & 3 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 3 & 1 \\ -1 & 1 & -1 & -1 & 3 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}, \quad C_{E_8}^{15} = -\mathbb{1}_8, \quad \det C_{E_8} = -1. \quad (76)$$

Here e_j , $j = 1, \dots, 6$ and e_7 form up the basis of \mathbb{E}_7 . The characteristic polynomial of C_{E_8} :

$$P_{E_8} = z^8 + z^7 - z^5 - z^4 - z^3 + z + 1. \quad (77)$$

5.2.4 $\mathfrak{g} \simeq F_4^{(1)}$

The set of admissible roots is given by (38). Roots related to the rays l_0 and l_1 are as follows:

$$\begin{aligned} l_0 &= \arg \lambda = 0, & B_g &\equiv \{\alpha_1, \alpha_3\}; \\ l_1 &= \arg \lambda = \frac{\pi}{12}, & W_g &\equiv \{-\alpha_2, -\alpha_4\}; \end{aligned} \quad (78)$$

The Coxeter number of $F_4^{(1)}$ is $h = 12$ and the exponents are 1, 5, 7, 11. The Coxeter automorphism acts as a linear operator on the dual space \mathbb{E}_4 via:

$$C_{F_4} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C_{F_4}^6 = -\mathbb{1}_4, \quad \det C_{F_4} = 1. \quad (79)$$

Here e_j , $j = 1, \dots, 4$ is the basis of \mathbb{E}_4 . The characteristic polynomial of C_{F_4} :

$$P_{F_4} = z^4 - z^2 + 1. \quad (80)$$

5.2.5 $\mathfrak{g} \simeq G_2^{(1)}$

This comes to be both the simplest and the most peculiar case. The peculiarity is in the fact, that though the rank of G_2 equals 2, its root system is traditionally written as a set of vectors in the 3-dimensional space \mathbb{E}_3 which are all orthogonal to the vector $e_1 + e_2 + e_3$. The Coxeter automorphism here is the composition of the Weyl reflection $C_{G_2} = S_{\alpha_1} S_{\alpha_2}$. One can check that C_{G_2} is expressed as the following linear operator in \mathbb{E}_3 :

$$C_{G_2} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}, \quad (81)$$

which has the following properties:

$$C_{G_2}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_{G_2}^4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_{G_2}^6 = \mathbb{1}_3. \quad (82)$$

The characteristic polynomial is

$$P_{G_2} = z^3 - 2z^2 + 2z - 1 = (z - 1)(z^2 - z + 1). \quad (83)$$

Another way to approach the problem is to project the root system of G_2 onto the 2-dimensional plane orthogonal to the vector $e_1 + e_2 + e_3$. Indeed, let

$$\varepsilon_1 = \frac{1}{\sqrt{2}}(e_2 - e_1), \quad \varepsilon_2 = \frac{1}{\sqrt{2}}(e_3 - e_2).$$

Then the set of admissible roots (43) can be rewritten as:

$$\begin{aligned} \alpha_0 &= \sqrt{2}(\varepsilon_1 + 2\varepsilon_2) = -e_1 - e_2 + 2e_3, & \alpha_1 &= \sqrt{2}\varepsilon_1 = e_2 - e_1, \\ \alpha_2 &= \sqrt{2}(\varepsilon_2 - \varepsilon_1) = e_1 - 2e_2 + e_3, \end{aligned} \quad (84)$$

Roots related to the rays l_0 and l_1 are:

$$\begin{aligned} l_0 &= \arg \lambda = 0, & B_g &\equiv \{\alpha_1\}; \\ l_1 &= \arg \lambda = \frac{\pi}{12}, & W_g &\equiv \{-\alpha_2\}; \end{aligned} \quad (85)$$

The Coxeter number for $G_2^{(1)}$ is $h = 6$ and the exponents are 1, 5. This Coxeter automorphism induces an action on the dual space \mathbb{E}_2 with a matrix:

$$C'_{G_2} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad C'_{G_2}{}^3 = -\mathbb{1}_2, \quad \det C'_{G_2} = 1. \quad (86)$$

Here $\{\varepsilon_1, \varepsilon_2\}$ form up a basis in \mathbb{E}_2 . Finally, the characteristic polynomial of C_{G_2} reads:

$$P_{G_2} = z^2 - z + 1. \quad (87)$$

6 Conclusions

We presented here real Hamiltonian forms of affine Toda field theories related exceptional untwisted complex Kac-Moody algebras. We established that the special properties of these models allow us to relate the construction of the RHF to the study of \mathbb{Z}_h symmetries of the extended Dynkin diagrams (with h being the Coxeter number of \mathfrak{g}). The general construction is illustrated by several examples of such models related to the exceptional Kac-Moody algebras $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $D_4^{(1)}$ and $G_2^{(1)}$. We used the ‘‘dihedral realisation’’ of the Coxeter automorphism $\text{Cox}(\mathfrak{g}) = S_1 \circ S_2$, where S_1, S_2 are \mathbb{Z}_2 -automorphisms of \mathfrak{g} . One of these automorphisms extracts the RHF and the other one acts as additional \mathbb{Z}_2 -reduction on it.

Each of the real Hamiltonian forms obtained above has its dual one. Indeed, we could define the involution \mathcal{C} using the Weyl reflection S_2 . The consideration is quite analogous: simply we interchange the black roots with the white ones. The reformulation of all the above results is rather obvious. Note that the dual real Hamiltonian forms of 2DTFT *are not* equivalent to the ones obtained above. Indeed, it is easy to check that in all cases treated above the number of the white roots is different from the number of black roots.

Some additional problems are natural extensions to the results presented here:

- The complete classification of all nonequivalent RHF of ATFT.

- The description of the hierarchy of Hamiltonian structures of RHF of ATFT (for a review of the infinite-dimensional cases see e.g. [13, 20] and the references therein) It is also an open problem to construct the RHF for ATFT using some of its higher Hamiltonian structures.
- The extension of the dressing Zakharov-Shabat method [52] to the above classes of Lax operators is also an open problem. One of the difficulties is due to the fact that the \mathbb{Z}_h reductions requires dressing factors with $2h$ pole singularities [19]. This makes the relevant linear algebraic equations rather involved [54].
- Another open problem is to study types of boundary conditions and boundary effects of ATFT's and their RHF [10, 11]. This includes types of boundary defects [12].
- The last and technically more involved problem is to solve the inverse scattering problem for the Lax operator and thus prove the complete integrability of all these models. The ideas of [1, 32] about the interpretation of the inverse scattering method as a generalized Fourier transform holds true also for the \mathbb{Z}_h reduces Lax operators [34, 33, 50, 35, 51]. This may allow one to derive the action-angle variables for these classes of NLEE.

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