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Failure of Fatou type theorems for solutions to PDE of *p*-Laplace type in domains with flat boundaries

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ABSTRACT

Let $\mathbb{R}^n$ denote Euclidean n space and given k a positive integer let $\Lambda_k \subset \mathbb{R}^n$, $1 \leq k < n-1, n \geq 3$, be a k-dimensional plane with $0 \in \Lambda_k$. If $n-k < p < \infty$, we first study the Martin boundary problem for solutions to the p-Laplace equation (called p-harmonic functions) in $\mathbb{R}^n \setminus \Lambda_k$ relative to $\{0\}$. We then use the results from our study to extend the work of Wolff on the failure of Fatou type theorems for p-harmonic functions in $\mathbb{R}^2_+$ to p-harmonic functions in $\mathbb{R}^n \setminus \Lambda_k$ when $n-k < p < \infty$. Finally, we discuss generalizations of our work to solutions of p-Laplace type PDE (called $\mathcal{A}$-harmonic functions).

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1. Introduction

In 1984 Wolff brilliantly used ideas from harmonic analysis and PDE to prove that the Fatou theorem fails for p-harmonic functions when $2 < p < \infty$. He proved

Theorem 1.1 ([1, Theorem 1]). *If $2 < p < \infty$ then there exist bounded weak solutions $\hat{u}$ of the p-Laplace equation:*

$$\mathcal{L}_p \hat{u} := \nabla \cdot \left(|\nabla \hat{u}|^{p-2} \nabla \hat{u} \right) = 0 \tag{1.1}$$

in $\mathbb{R}^2_+ = \{x = (x_1, x_2) : x_2 > 0\}$, for which $\{t \in \mathbb{R} : \lim_{y \to 0} \hat{u}(t+iy) \text{ exists}\}$ has Lebesgue measure zero. Also there exist positive bounded weak solutions of $\mathcal{L}_p \hat{v} = 0$ such that $\{t \in \mathbb{R} : \limsup_{y \to 0} \hat{v}(t+iy) > 0\}$ has Lebesgue measure 0.

The key to his proof and the only obstacle in extending Theorem 1.1 to $1 < p \neq 2 < \infty$ was the validity of the following theorem, stated as Lemma 1 in [1].

Theorem 1.2 ([1, Lemma 1]). *If $2 < p < \infty$ there exists a bounded Lipschitz function Φ on the closure of $\mathbb{R}^2_+$ with $\Phi(z+1) = \Phi(z)$ for $z \in \mathbb{R}^2_+$, $\mathcal{L}_p \Phi = 0$ weakly on $\mathbb{R}^2_+$, $\int_{(0,1)\times(0,\infty)} |\nabla \Phi|^p \, dxdy < \infty$, and*

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$$\lim_{y\to\infty} \Phi(t+iy) = 0 \quad \text{uniformly for} \quad t \in \mathbb{R}, \quad \text{but} \quad \int_0^1 \Phi(s)ds \neq 0. \tag{1.2}$$

Theorem 1.2 was later proved for $1 < p < 2$, by the second author of this article in [2] (so Theorem 1.1 is valid for $1 < p \neq 2 < \infty$). Wolff notes that Theorems 1.1 and 1.2, generalize to $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) : x_n > 0\}$ simply by defining $\hat{u}, \hat{v}, \Phi,$ to be constant in the additional coordinate directions. Wolff remarks above the statement of his Lemma 1, that Theorem 1.1 "should generalize to other domains but the arguments are easiest in a half space since $\mathcal{L}_p$ behaves nicely under Euclidean operations". In fact Wolff made extensive use in his argument of the fact that $\Phi(Nz + z_0)$, $z = x + iy \in \mathbb{R}^2_+$, N a positive integer, $z_0 \in \mathbb{R}^2_+$, is p-harmonic in $\mathbb{R}^2_+$, and $1/N$ periodic in x, with Lipschitz norm $\approx N$ on $\mathbb{R} = \partial\mathbb{R}^2_+$. Also he used functional analysis-PDE arguments, involving the Fredholm alternative and perturbation of certain p-harmonic functions to get Φ satisfying (1.2) when $2 < p < \infty$.

Building on a work of Varpanen in [3], we managed to obtain analogues of Theorems 1.1 and 1.2 for $1 < p \neq 2 < \infty$, in the unit disk of $\mathbb{R}^2$ in [4]. In fact we gave two proofs of these theorems when $p > 2$. One proof used the exact values of exponents in the Martin boundary problem for p-harmonic functions, for $p > 2$, in $\mathbb{R}^2_+$ relative to $\{0\}$, as well as, boundary Harnack inequalities for certain p-harmonic functions. This proof seemed conceptually simpler and more straight forward to us than the other proof, so we dubbed it 'a hands on proof'. As a warm up for this proof we first gave, in Lemma 3.1 of [4], a 'hands on example' of a Φ for which Theorem 1.2 is valid. In this paper we use a similar argument to prove an analogue of Theorems 1.1 and 1.2 for p-harmonic functions in domains whose complements in $\mathbb{R}^n$, are k-dimensional planes where $1 \leq k < n - 1$. To be more specific we need some definitions and notations.

1.1. Definitions and notations

Let $n \geq 2$ and denote points in Euclidean n-space $\mathbb{R}^n$ by $y = (y_1, ..., y_n)$. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. We write $e_m, 1 \leq m \leq n,$ for the point in $\mathbb{R}^n$ with 1 in the m-th coordinate and 0 elsewhere. Let $\bar{E}, \partial E,$ and $\operatorname{diam}(E)$ be the closure, boundary, and diameter of the set $E \subset \mathbb{R}^n$ respectively. We define $d(y, E)$ to be the distance from $y \in \mathbb{R}^n$ to E. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$ and let $|y| = \langle y, y \rangle^{1/2}$ be the Euclidean norm of y. For $z \in \mathbb{R}^n$ and $r > 0$, put

$$B(z, r) = \{y \in \mathbb{R}^n : |z - y| < r\}.$$

Let dy denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$ and let $\mathcal{H}^\lambda, 0 < \lambda \leq n,$ denote the λ-dimensional *Hausdorff measure* on $\mathbb{R}^n$ defined by

$$\mathcal{H}^\lambda(E) = \lim_{\delta\to 0} \inf \left\{ \sum_j r_j^\lambda; \quad E \subset \bigcup_j B(x_j, r_j), \quad r_j \leq \delta \right\}$$

where the infimum is taken over all possible δ-covering $\{B(x_j, r_j)\}$ of E. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions h with distributional gradient $\nabla h = (h_{y_1}, ..., h_{y_n})$, both of which are q-th power integrable on O. Let

$$||h||_{1,q} = ||h||_q + ||\ |\nabla h|\ ||_q$$

be the norm in $W^{1,q}(O)$ where $||\cdot||_q$ is the usual Lebesgue q norm of functions in the Lebesgue space $L^q(O)$. Let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Definition 1.3. For fixed p with $1 < p < \infty$, given a compact set E and open set O with $E \subset O$, define the p-capacity of E relative to O by

$$\mathcal{C}_p(E, O) := \inf\left\{\int_O |\nabla h|^p dx : h \in W_0^{1,p}(O) \text{ with } h \geq 1 \text{ on } E\right\}.$$

Definition 1.4. If p is fixed, $1 < p < \infty$, then $\hat{u}$ *is said to be p-harmonic in an open set* O provided $\hat{u} \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int \langle |\nabla \hat{u}|^{p-2} \nabla \hat{u}(y), \nabla \theta(y) \rangle \ dy = 0 \quad \text{whenever} \quad \theta \in W_0^{1,p}(G). \tag{1.3}$$

We say that $\hat{u}$ is a p *sub-solution* (p *super-solution*) in O if $\hat{u} \in W^{1,p}(G)$ whenever G is as above and (1.3) holds with $=$ replaced by $\leq$ ($\geq$) whenever $\theta \in W_0^{1,p}(G)$ with $\theta \geq 0$. Here $\nabla \cdot$ denotes the divergence operator.

Definition 1.5. Given $1 \leq k \leq n-2, n \geq 3$, let $\Lambda_k \subset \mathbb{R}^n$ be a k-dimensional plane. If p is fixed, $n-k < p < \infty$, and $z \in \Lambda_k$, then u is said to be a p-Martin function for Λ_k, relative to $\{z\}$, provided $u > 0$ is p-harmonic in $\mathbb{R}^n \setminus \Lambda_k$ and $u(x) \to 0$ as $x \to \infty$, $x \in \mathbb{R}^n \setminus \Lambda_k$. Also u is continuous in $\mathbb{R}^n \setminus \{z\}$ with $u \equiv 0$ on $\Lambda_k \setminus \{z\}$. A p-Martin function is defined similarly when $k = n-1$, $z \in \Lambda_k$ only relative to a component of $\mathbb{R}^n \setminus \Lambda_k$.

Existence of u for $1 \leq k \leq n-2, p > n-k$, is shown in Lemma 8.2 of [5]. For existence of u when $k = n-1$ and $p > 2$, see Subsection 5.1 in [6]. Also (see (4.1) in Section 4),

$$\begin{aligned} &(a)\ u \text{ is unique up to constant multiples,} \\ &(b)\ \text{there exists } \sigma = \sigma(p, n, k) > 0 \text{ such that } \ u(z + tx) = t^{-\sigma} u(z + x) \\ &\quad \text{whenever } t > 0. \end{aligned} \tag{1.4}$$

To make our 'hands on' argument work, when $1 \leq k \leq n-2$ and $p > n-k$, we need to show in (1.4) (b) that $\sigma < k$ when $n-k < p$. In estimating σ and in statement of our theorems, we assume that

$$z = 0 \text{ and } \Lambda_k = \{(x_1, ..., x_k, 0, ..., 0) \in \mathbb{R}^n : x_i \in \mathbb{R}, 1 \leq i \leq k\}.$$

This assumption is permissible since p-harmonic functions are invariant under translation and rotation. Moreover with a slight abuse of notation we write $\mathbb{R}^k$ for Λ_k and $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. We now state our first result.

Theorem A. *Let k, n be fixed positive integers with $1 \le k \le n-2$ and $p > n-k$. Let $x' = (x_1, x_2, \ldots, x_k), x'' = (x_{k+1}, \ldots, x_n)$, and $x = (x', x'')$. Put*

$$\hat{u}(x) = |x''|^{\beta}|x|^{-\gamma} = |x''|^{\beta} r^{-\gamma} \text{ where } \beta = \frac{p+k-n}{p-1}, \quad r = |x|, \text{ and } \gamma > \beta > 0. \tag{1.5}$$

Let $\lambda = \gamma - \beta$. If

$$\lambda > \max\left(\frac{(p+k-n)(k+p-2)}{(p-1)(2p-n+k-2)}, \frac{k}{p-1}\right) =: \chi = \chi(p,n,k), \tag{1.6}$$

then $\hat{u}$ is a p-subsolution on $\mathbb{R}^n \backslash \mathbb{R}^k$ and $\chi < k$, while if

$$\lambda < \min\left(\frac{(p+k-n)(k+p-2)}{(p-1)(2p-n+k-2)}, \frac{k}{p-1}\right) =: \breve{\chi} = \breve{\chi}(p,n,k), \tag{1.7}$$

then $\hat{u}$ is a p-supersolution on $\mathbb{R}^n \backslash \mathbb{R}^k$.

Remark 1.6. We note that Llorente, Manfredi, Troy, and Wu in [7] proved (1.6), (1.7), when $k = n-1$ and $2 < p < \infty$. We shall use this result throughout Section 6.

In order to state our second result, we need to introduce some notations. Given $\tau > 0$ and $y' \in \Lambda_k = \mathbb{R}^k$, let

$$Q_\tau(y') = Q_\tau^{(k)}(y') := \{z' \in \mathbb{R}^k : |z'_i - y'_i| < \tau/2, \text{ when } 1 \le i \le k\}. \tag{1.8}$$

Armed with Theorem A and Remark 1.6, our second result generalizes the work of Wolff [1] and our earlier work in [4] when the boundary is a low dimensional plane.

Theorem B. Let k, n be positive integers with either (i) $1 \le k \le n-2$ and $p > n-k$, or (ii) $k = n-1$ and $p > 2$. In case (i) there exists a p-harmonic function Ψ on $\mathbb{R}^n \backslash \mathbb{R}^k$, that is continuous on $\mathbb{R}^n$, with

$$\begin{aligned}
&(a) \ \Psi \text{ Lipschitz on } \mathbb{R}^k \text{ and } \int_{Q_{1/2}(0)\times\mathbb{R}^{n-k}} |\nabla\Psi|^p \, dx < \infty, \\
&(b) \ \Psi(x + e_i) = \Psi(x) \text{ for } 1 \le i \le k, \text{ whenever } x \in \mathbb{R}^n, \\
&(c) \ \lim_{x'' \in \mathbb{R}^{n-k} \to \infty} \Psi(x', x'') = 0 \text{ uniformly for } x' \in \bar{Q}_{1/2}(0), \\
&(d) \ \int_{Q_{1/2}(0)} \Psi(x', 0) \quad d\mathcal{H}^k x' \neq 0.
\end{aligned} \tag{1.9}$$

In case (ii) there exists a p-harmonic function Ψ on $\mathbb{R}^n_+$ that is continuous on the closure of $\mathbb{R}^n_+$, satisfying (1.9) when $k = n-1$ with $x'' = x_n > 0$.

Theorem A and the technique in proving Theorem B are also easily seen to imply the following corollary.

Corollary 1.7. *Let χ and $\breve{\chi}$ be as in Theorem A. Let k, n be positive integers with $1 \le k \le n-2$, and p fixed, $p > n-k$. Let $0 \le \omega_p(B(0,r) \cap \mathbb{R}^k, \cdot) \le 1$, denote the unique bounded p-harmonic function on $\mathbb{R}^n \backslash \mathbb{R}^k$ which is 1 on $B(0,r) \cap \mathbb{R}^k$ and 0 on $\mathbb{R}^k \backslash \bar{B}(0,r)$. There exists $c = c(p,n,k) \ge 1$ so that if $0 < r < 1/2$, then*

$$c^{-1}\ r^{\chi} \leq \omega_p(B(0,r)\cap\mathbb{R}^k, e_n) \leq c r^{\breve{\chi}}.$$

Corollary 1.7 was proved for $p > 2$ and $k = n-1$ in [7] (see also [8]) using the analogue of Theorem A (see Remark 1.6).

We can use the gist of Wolff's argument and Theorem B to show the failure of a Fatou's theorem for p-harmonic functions vanishing on low dimensional planes.

Theorem C. *Let k, n be positive integers with either (i) $1 \leq k \leq n-2$ with $p > n-k$, or (ii) $k = n-1$ with $p > 2$. In case (i) there exists bounded p-harmonic functions $\hat{u}, \hat{v}$ in $\mathbb{R}^n\backslash\mathbb{R}^k$ with the following properties. Suppose $\zeta : \mathbb{R}^{n-k}\backslash\{0\} \to \mathbb{R}^{n-k}\backslash\{0\}$ is continuous with $\lim_{x''\to 0}\zeta(x'') = (0,..,0)$. Then*

$$\{x' \in \mathbb{R}^k : \lim_{x''\to 0} \hat{u}(x', \zeta(x''))\quad \text{exists}\} \subset D_1 \quad \text{where} \quad \mathcal{H}^k(D_1) = 0.$$

Also

$$\{x' \in \mathbb{R}^k : \limsup_{x''\to 0}\ \hat{v}(x', \zeta(x'')) > 0\} \subset D_2 \quad \text{where} \quad \mathcal{H}^k(D_2) = 0.$$

The Borel sets D_1 and D_2 are independent of the choice of ζ. In case (ii) there exists bounded p-harmonic functions $\hat{u}, \hat{v}$ in $\mathbb{R}^n_+$ with the above properties when $k = n-1$ and $x'' = x_n > 0$.

Remark 1.8. To get Theorem C for $\hat{v}$ in case (ii) it suffices to just prove existence of limit 0 for $\mathcal{H}^{n-1}$ almost every $x' \in \mathbb{R}^{n-1}$ when $\zeta(x'') = (0,...,0,x_n)$, thanks to Harnack's inequality for positive p-harmonic functions (see (2.1) (c)). Also in case (i), one could just prove this theorem for $k = 1, p > n-1$, since the general case would then follow from extending these functions to $\mathbb{R}^{n+k-1}\backslash\mathbb{R}^k$ for $1 < k$ by defining them to be constant in the other added $k-1$ coordinate directions. However our approach yields a larger and arguably more interesting variety of examples.

We also note that when $1 < p \leq n-k$

$$\mathcal{C}_p(\mathbb{R}^k \cap \bar{B}(0,R), B(0,2R)) = 0 \tag{1.10}$$

for every $R > 0$ (see [9], p. 43]) and consequentially neither Theorem B or Theorem C has an analogue in case (i) when $1 < p \leq n-k$. For the reader's convenience a proof of this statement is given in Remark 5.6 after the proof of Theorem C in section 5.

Finally in Section 6 we consider partial analogues of Theorem 1.1, Theorem 1.2, Theorems A-C, for solutions to a more general class of PDE's modeled on the p-Laplacian, which are called $\mathcal{A}$-harmonic functions (see Definition 6.2 in Section 6). $\mathcal{A}$-harmonic functions share with p-harmonic functions the properties used in the proof of Theorems A-C, so originally we hoped to prove these theorems with p-harmonic replaced by $\mathcal{A}$-harmonic. However preliminary investigations using maple and hand calculations, indicated that this class would not in general yield the necessary estimates on exponents of an $\mathcal{A}$-harmonic Martin function for p in the required ranges. For this reason we relegated our discussion of $\mathcal{A}$-harmonic functions to Section 6 and made this discussion more or less self contained. Subsections of Section 6 include

(1) A definition of $\mathcal{A}$-harmonic functions and listing of their basic properties.
(2) Statement of two Propositions concerning validity of Theorems B - C for $\mathcal{A}$-harmonic operators sufficiently near the p-Laplace operator.
(3) Estimates of $\mathcal{A}$-harmonic Martin exponents when $1 \leq k \leq n-2$ and $p > n-k$ for $n > 3$, in $\mathbb{R}^n \setminus \mathbb{R}^k$ (see subsection 6.3) and when $k = n-1$ and $p > 2$ for $n \geq 2$ in $\mathbb{R}^n_+$ (see subsection 6.4). These estimates give partial analogues of Theorems 1.1 and 1.2 for a subclass of $\mathcal{A}$-harmonic operators, only slightly more general than the p-Laplace operator. Still this was an interesting subclass for us to highlight computational difficulties in showing $\sigma < k$ for the exponent in (1.4) of an $\mathcal{A}$-harmonic Martin function. Moreover, in the baseline $n = 2$ and $p > 2$ case we obtained a rather surprising result (see Subsection 6.5 for more details).

As for the plan of this paper in Section 2 we introduce and state some lemmas listing basic estimates for p-harmonic functions. Statements and references for proofs of these lemmas are made so that we can essentially say 'ditto' in our discussion of $\mathcal{A}$-harmonic functions. In Section 3 we prove Theorem A. In Section 4 we use Theorem A to prove Theorem B. In Section 5 we indicate the changes in Wolff's main lemmas for applications and prove Theorem C. In Section 6 we introduce $\mathcal{A}$-harmonic functions and proceed as outlined above.

2. Definition and basic estimates for *p*-harmonic functions

In this section we first introduce some more notation and then state some fundamental estimates for p-harmonic functions. Concerning constants, unless otherwise stated, in Sections 2–6, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on p, n, k. In general throughout this paper, $c(a_1, a_2, ..., a_m)$ denotes a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on $a_1, ..., a_m$. Also $A \approx B$ means A/B is bounded above and below by positive constants whose dependence will be stated. We also let $\max_E \hat{v}$, $\min_E \hat{v}$ denote the essential supremum and infimum of $\hat{v}$ (with respect to Lebesgue n-measure) whenever $E \subset \mathbb{R}^n$ and $\hat{v}$ is defined on E.

Next we state some basic lemmas for p-harmonic functions.

Lemma 2.1. *For fixed $p, 1 < p < \infty$, suppose $\hat{v}$ is a p-subsolution and $\hat{h}$ is a p-supersolution in the open set O with* $\max(\hat{v} - \hat{h}, 0) \in W_0^{1,p}(O)$. *Then* $\max_O(\hat{v} - \hat{h}) \leq 0$.

Proof. A proof of this lemma can be found in [9, Lemma 3.18]. □

Lemma 2.2. *For fixed $p, 1 < p < \infty$, let $\hat{v}$ be p-harmonic in $B(z_0, 4\rho)$ for some $\rho > 0$ and $z_0 \in \mathbb{R}^n$. Then there exists $c = c(p, n)$ with*

$$\begin{aligned}
&(a)\ \max_{B(z_0,\rho/2)} \hat{v} - \min_{B(z_0,\rho/2)} \hat{v} \le c\left(\rho^{p-n}\int_{B(z_0,\rho)} |\nabla \hat{v}|^p\, dx\right)^{1/p} \le c^2\ (\max_{B(z_0,2\rho)} \hat{v} - \min_{B(z_0,2\rho)} \hat{v}).\\
&\textit{Furthermore, there exists } \tilde{\beta} = \tilde{\beta}(p,n) \in (0,1) \textit{ such that if } s \le \rho, \textit{ then}\\
&(b)\ \max_{B(z_0,s)} \hat{v} - \min_{B(z_0,s)} \hat{v} \le c\left(\frac{s}{\rho}\right)^{\tilde{\beta}} \left(\max_{B(z_0,2\rho)} \hat{v} - \min_{B(z_0,2\rho)} \hat{v}\right).\\
&(c)\ \textit{If } \hat{v} \ge 0 \textit{ in } B(z_0,4\rho), \textit{ then } \max_{B(z_0,2\rho)} \hat{v} \le c \min_{B(z_0,2\rho)} \hat{v}.
\end{aligned} \tag{2.1}$$

Proof. Lemma 2.2 is well known. A proof of this lemma, using Moser iteration of positive solutions to PDE of p-Laplace type, can be found in [10] or Chapter 6 in [9]. (2.1) (c) is called Harnack's inequality. □

Lemma 2.3. *Let k be a positive integer $1 \le k \le n-1$, p fixed with $n-k < p < \infty$ and $\rho > 0$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}, z_0 \in \mathbb{R}^k$, and put $\Omega = (\mathbb{R}^n \backslash \mathbb{R}^k) \cap B(z_0, 4\rho)$ when $1 \le k \le n-2$ while $\Omega = \mathbb{R}^n_+ \cap B(z_0,4\rho)$ when $k = n-1$. Let $\zeta \in C_0^\infty(B(z_0,4\rho))$ with $\zeta \equiv 1$ on $B(z_0,3\rho)$ and $|\nabla \zeta| \le c(n)\rho^{-1}$. Suppose $\hat{v}$ is p-harmonic in Ω, $\hat{h} \in W^{1,p}(\Omega)$, and $(\hat{v} - \hat{h})\zeta \in W_0^{1,p}(\Omega)$.*

If $\hat{h}$ is continuous on $\partial\Omega \cap B(z_0,4\rho)$, then $\hat{v}$ has a continuous extension to $\bar{\Omega} \cap B(z_0,4\rho)$, also denoted $\hat{v}$, with $\hat{v} \equiv \hat{h}$ on $\partial\Omega \cap B(z_0,4\rho)$. If

$$|\hat{h}(z) - \hat{h}(w)| \le M'|z-w|^{\hat{\sigma}} \quad \text{whenever} \quad z, w \in \partial\Omega \cap B(z_0, 3\rho),$$

for some $\hat{\sigma} \in (0,1]$, and $1 \le M' < \infty$, then there exists $\hat{\sigma}_1 \in (0,1], c \ge 1$, depending only on $\hat{\sigma}, n,$ and p, such that

$$|\hat{v}(z) - \hat{v}(w)| \le 8M'\rho^{\hat{\sigma}} + (|z-w|/2\rho)^{\hat{\sigma}_1} \max_{\Omega \cap \bar{B}(z_0,2\rho)} |\hat{v}| \tag{2.2}$$

whenever $z, w \in \Omega \cap B(z_0,\rho)$.

If $\hat{h} \equiv 0$ on $\partial\Omega$, $\hat{v} \ge 0$ in $B(z_0,4\rho)$, $\hat{c} \ge 1$, and $z_1 \in \Omega \cap B(z_0,3\rho)$, with $\hat{c}\ d(z_1, \partial\Omega) \ge \rho$, then there exists $\tilde{c}$, depending only on $\hat{c}, n,$ and p, such that

$$(+) \qquad \max_{B(z_0,2\rho)} \hat{v} \le \tilde{c}\left(\rho^{p-n}\int_{B(z_0,3\rho)} |\nabla\hat{v}|^p\, dx\right) \le (\tilde{c})^2\ \hat{v}(z_1)^{1/p}. \tag{2.3}$$

Furthermore, using (2.2), it follows for $z, w \in \bar{\Omega} \cap B(z_0, 2\rho)$ that

$$(++) \quad |\hat{v}(z) - \hat{v}(w)| \le c\ \hat{v}(z_1)\left(\frac{|z-w|}{\rho}\right)^{\hat{\sigma}_1}.$$

Proof. Continuity of $\hat{v}$ given continuity of $\hat{h}$ in $\bar{\Omega}$ follows from Corollary 6.36 in [9]. This Corollary and the Hölder continuity estimate on h above, are then used in Theorem 6.44 of [9] to prove an inequality analogous to (2.3). Proofs involve Wiener type estimates (in terms of p-capacity) for p-harmonic subsolutions that vanish on $\partial\Omega \cap B(z_0,3\rho)$. □

Lemma 2.4. *Let $p, \hat{v}, z_0, \rho,$ be as in Lemma 2.2. Then $\hat{v}$ has a representative locally in $W^{1,p}(B(z_0, 4\rho))$, with Hölder continuous partial derivatives in $B(z_0, 4\rho)$ (also denoted $\hat{v}$), and there exist $\hat{\gamma} \in (0, 1]$ and $c \geq 1$, depending only on p, n, such that if $z, w \in B(z_0, \rho/2)$, then*

$$(\hat{a})\ c^{-1}\,|\nabla\hat{v}(z) - \nabla\hat{v}(w)| \leq (|z-w|/\rho)^{\hat{\gamma}} \max_{B(z_0,\,\rho)} |\nabla\hat{v}| \leq\ c\ \rho^{-1}\ (|z-w|/\rho)^{\hat{\gamma}} \max_{B(z_0,\,2\rho)} |\hat{v}|.$$

Also $\hat{v}$ has distributional second partials with

$$(\hat{b})\quad \int_{B(z_0,\,\rho)\cap\{\nabla\hat{v}\neq 0\}} |\nabla\hat{v}|^{p-2} \left(\sum_{i,j=1}^{n} |\hat{v}_{x_i x_j}|^2(z)\right) dx \leq c\ \rho^{n-p-2} \max_{B(z_0,\,2\rho)} |\hat{v}|. \tag{2.4}$$

Proof. For a proof of (2.4) $(\hat{a}), (\hat{b})$, see Theorem 1 and Proposition 1 in [11]. □

In the proof of Theorems B and C, we need the following boundary Harnack inequalities.

Lemma 2.5. *Let $k, n, p, z_0, \Omega,$ be as in Lemma 2.3. Suppose $\hat{u}$, $\hat{v}$, are non-negative p-harmonic functions in Ω with $\hat{u} = \hat{v} \equiv 0$ on $\partial\Omega \cap B(z_0, 4\rho)$. There exists $c = c(p, n, k)$ and $\beta = \beta(p, n, k)$ such that if $y, z \in \Omega \cap B(z_0, \rho)$, then*

$$\left|\frac{\hat{u}(z)}{\hat{v}(z)} - \frac{\hat{u}(y)}{\hat{v}(y)}\right| \leq c\,\frac{\hat{u}(z)}{\hat{v}(z)}\left(\frac{|y-z|}{\rho}\right)^{\beta}. \tag{2.5}$$

Proof. For a proof of Lemma 2.5 when $k = n-1, p > 1$, see Theorem 1 in [6] and when $1 \leq k \leq n-2$ see Theorems 1.9 and 1.10 in [5]. □

Lemma 2.6. *Let $k, n, p, z_0, \rho,$ be as in Lemma 2.3. Let $G = \mathbb{R}^n \backslash (\mathbb{R}^k \cup \bar{B}(0, \rho))$ when $k < n-1$ and $G = \mathbb{R}^n_+ \backslash \bar{B}(0, \rho)$ when $k = n-1$. Suppose $\hat{u}$, $\hat{v}$, are non-negative p-harmonic functions in G with continuous boundary values and $\hat{u} = \hat{v} \equiv 0$ on $\partial G \cap \mathbb{R}^k$ when $k \leq n-1$. Moreover $\hat{u}(x) + \hat{v}(x) \to 0$* uniformly for $x \in G$ as $|x| \to \infty$. *There exists $c = c(p, n, k)$ and $\beta' = \beta'(p, n, k)$ such that if $y, z \in G \backslash B(0, 2\rho)$, then*

$$\left|\frac{\hat{u}(z)}{\hat{v}(z)} - \frac{\hat{u}(y)}{\hat{v}(y)}\right| \leq c\,\frac{\hat{u}(z)}{\hat{v}(z)}\left(\frac{\rho}{\min(|y|, |z|)}\right)^{\beta'}. \tag{2.6}$$

Proof. For the proof of Lemma 2.6 when $k = n-1, p > 1$, see Theorem 2 in [6]. For a proof of Lemma 2.6 when $1 \leq k \leq n-2$, see Theorem 1.13 in [5]. In both cases the proof of (2.6) is given only when $G = \Omega \backslash \bar{B}(0, \rho)$ and Ω is a bounded domain. However, the proof is essentially the same in either case for G as above. □

For fixed k with $1 \leq k \leq n-1$ for $n \geq 2$, and $y' \in \mathbb{R}^k$, let $Q_\tau(y') = Q_\tau^{(k)}(y')$, be as in (1.8). Let

$$S(y', \tau) = S^{(k)}(y', \tau) := \{(x', x'') \in Q_\tau^{(k)}(y') \times (\mathbb{R}^{n-k} \backslash \{0\})\}$$

when $1 \leq k \leq n-2$ and

$$S(y', \tau) := \{(x', x_n) : x' \in Q_\tau^{n-1}(y'), x_n > 0\}$$

when $k = n-1$. For short we write $S(\tau)$ when $y' = 0 \in \mathbb{R}^k$. If $1 \le k \le n-2$ with $p > n-k$, let $R^{1,p}(S(\tau))$ denote the Riesz space of equivalence classes of functions F with distributional derivatives on $\mathbb{R}^n \backslash \mathbb{R}^k$ and $F(z + \tau e_i) = F(z), 1 \le i \le k$, when $z \in \mathbb{R}^n \backslash \mathbb{R}^k$. Also

$$||F||_* = ||F||_{*,p} = \left(\int_{S(\tau)} |\nabla F|^p \, dx \right)^{1/p} < \infty. \tag{2.7}$$

If $k = n-1$ and $p > 2$, define $R^{1,p}(S(\tau))$ similarly, only with $\mathbb{R}^n \backslash \mathbb{R}^k$ replaced by $\mathbb{R}^n_+$. Next let $R_0^{1,p}(S(\tau))$ denote functions in $R^{1,p}(S(\tau))$ which can be approximated arbitrarily closely in the norm of $R^{1,p}(S(\tau))$ by functions in this space which are infinitely differentiable and vanish in an open neighborhood of $\mathbb{R}^k$. Using a variational argument as in [12] it can be shown that given $F \in R^{1,p}(S(\tau))$, there exists a unique p-harmonic function $\hat{v}$ on $\mathbb{R}^n \backslash \mathbb{R}^k$ when $1 \le k \le n-2, p > n-k$, and on $\mathbb{R}^n_+$ when $k = n-1, p > 2$, with $\hat{v}(z + \tau e_i) = \hat{v}(z), 1 \le i \le k$, for $z \in \mathbb{R}^n \backslash \mathbb{R}^k$ or $\mathbb{R}^n_+$. Moreover $\hat{v} - F \in R_0^{1,p}(S(\tau))$. In fact the usual minimization argument yields that $||\hat{v}||_{*,p}$ has minimum norm among all functions h in $R^{1,p}(S(\tau))$ with $h - F \in R_0^{1,p}(S(\tau))$. Uniqueness of $\hat{v}$ is a consequence of the maximum principle in Lemma 2.1. Next we state

Lemma 2.7. *Let $p, n, k, \tau, F, \hat{v}$, be as above. Given $t > 0$, let $Z(t) = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x''| = t\}$ when $1 \le k \le n-2$, and $Z(t) = \{(x', x_n') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x_n = t\}$ when $k = n-1$. There exists $\delta = \delta(p, n, k) \in (0, 1)$ and $\xi \in \mathbb{R}$ such that*

$$|\hat{v}(z) - \xi| \le \liminf_{t \to 0} \left(\max_{Z(t)} \hat{v} - \min_{Z(t)} \hat{v} \right) \left(\frac{\tau}{|z''|} \right)^\delta \tag{2.8}$$

whenever either $z = (z', z'') \in \mathbb{R}^n \backslash \mathbb{R}^k$ or $z \in \mathbb{R}^n_+$ with $z_n = |z''|$.

Proof. Fix $t > 0$ and first suppose that $1 \le k \le n-2, p > 2$. We note that $\hat{v}$ restricted to $D(t) = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x''| > t\}$ is a p-harmonic solution to a certain calculus of variations minimization problem. Moreover $\tilde{v}(x) = \min(\hat{v}(x), \max_{Z(t)} v)$ for $x \in D(t)$, belongs to the class of possible minimizers and

$$\int_{D(t)} |\nabla \tilde{v}|^p dx \le \int_{D(t)} |\nabla \hat{v}|^p dx$$

so by uniqueness of the minimizer, $\hat{v} \le \max_{Z(t)} v$ on $D(t)$. Similarly $\hat{v} \ge \min_{Z(t)} v$ on $D(t)$. Thus the term in (2.8) involving $\hat{v}$ is decreasing as a function of t. (2.8) follows from this fact, and an argument using $\tau e_i, 1 \le i \le k$, periodicity of $\hat{v}$ together with Harnack's inequality (see Lemma 1.3 in [1] for a similar argument). Note that

$$\xi = \lim_{t \to \infty} \max_{Z(t)} \hat{v} = \lim_{t \to \infty} \min_{Z(t)} \hat{v}.$$

A similar argument applies if $k = n-1$ and $p > 2$. We omit the details. □

Remark 2.8. If $k = n-1$ and $p > 2$, Harnack's inequality actually yields the stronger decay rate:

$$|\hat{v}(z) - \xi| \le \liminf_{t \to 0} \left(\max_{Z(t)} \hat{v} - \min_{Z(t)} \hat{v} \right) \; e^{-\delta z_n/\tau}.$$

Moreover this decay rate can also be obtained when $1 \le k \le n-2$ for $p > k$, using rotational invariance of $\hat{v}(x) = \hat{v}(x', x'')$ in the x'' variable. However, we can only prove (2.8) for the larger class of $\mathcal{A} = \nabla f-$harmonic functions discussed in Section 6 and this inequality is all we need in the proof of Theorem C.

3. Proof of Theorem A

In this section we prove Theorem A. To do so let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ for $1 \le k \le n-2$ and set $t = |x'|$, $s = |x''|$. We begin by deriving the p-Laplace equation for a smooth rotationally invariant function, $u(x) = u(s, t)$, in $\mathbb{R}^n \backslash \mathbb{R}^k$.

3.1. The p-Laplace equation in s, t

We compute

$$\nabla u = \left(\frac{u_t}{t} x', \frac{u_s}{s} x'' \right) \quad \text{and} \quad |\nabla u| = \sqrt{u_t^2 + u_s^2}. \tag{3.1}$$

Now we show that the p-Laplacian of $u(x)$, is the $(s,\, t)$ p-Laplace plus another term:

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = \nabla_{s,t} \cdot (|\nabla_{s,t} u(s,t)|^{p-2} \nabla_{s,t} u(s,t)) + \sqrt{u_t^2 + u_s^2}^{\,p-2} \left((k-1)\frac{u_t}{t} + (n-k-1)\frac{u_s}{s} \right). \tag{3.2}$$

This is the calculation

$$\begin{aligned}
&\nabla \cdot \sqrt{u_t^2 + u_s^2}^{\,p-2} \left(\frac{u_t}{t} x', \frac{u_s}{s} x'' \right) \\
&\quad = (p-2)\sqrt{u_t^2 + u_s^2}^{\,p-4} \left\langle \left((u_t u_{tt} + u_s u_{st})\frac{x'}{t}, (u_t u_{ts} + u_s u_{ss})\frac{x''}{s} \right), \left(\frac{u_t}{t} x', \frac{u_s}{s} x'' \right) \right\rangle \\
&\qquad + \sqrt{u_t^2 + u_s^2}^{\,p-2} \nabla \cdot \left(\frac{u_t}{t} x', \frac{u_s}{s} x'' \right) \\
&\quad = (p-2)\sqrt{u_t^2 + u_s^2}^{\,p-4} (u_t^2 u_{tt} + 2u_t u_s u_{ts} + u_s^2 u_{ss}) \\
&\qquad + \sqrt{u_t^2 + u_s^2}^{\,p-2} \left(u_{tt} + u_{ss} - \frac{u_t}{t} - \frac{u_s}{s} + k\frac{u_t}{t} + (n-k)\frac{u_s}{s} \right) \\
&\quad = (p-2)\sqrt{u_t^2 + u_s^2}^{\,p-4} (u_t^2 u_{tt} + 2u_t u_s u_{ts} + u_s^2 u_{ss}) \\
&\qquad + \sqrt{u_t^2 + u_s^2}^{\,p-2} \left(u_{tt} + u_{ss} + (k-1)\frac{u_t}{t} + (n-k-1)\frac{u_s}{s} \right) \\
&\quad = \sqrt{u_t^2 + u_s^2}^{\,p-4} \Bigg((p-2)(u_t^2 u_{tt} + 2u_t u_s u_{ts} + u_s^2 u_{ss}) \\
&\qquad + (u_t^2 + u_s^2)\left(u_{tt} + u_{ss} + (k-1)\frac{u_t}{t} + (n-k-1)\frac{u_s}{s} \right) \Bigg).
\end{aligned}$$

To find solutions, subsolutions, supersolutions in terms of (s, t) we need only study when the next display is 0, positive, negative

$$(p-2)(u_t^2 u_{tt} + 2u_t u_s u_{ts} + u_s^2 u_{ss}) + (u_t^2 + u_s^2)(u_{tt} + u_{ss} + (k-1)\frac{u_t}{t} + (n-k-1)\frac{u_s}{s}). \tag{3.3}$$

3.2. Particular forms for u(s, t) and proof of *Theorem A*

Proof of Theorem A. Let $r = \sqrt{s^2+t^2}$ and for Martin p-harmonic sub or super solutions we consider functions with the form

$$u(x) = u(s,t) = s^\beta r^{-(\lambda+\beta)}, \quad \text{for} \quad x \in \mathbb{R}^n \backslash \mathbb{R}^k, \tag{3.4}$$

where $\beta = \frac{p-n+k}{p-1}$ and $\lambda > 0$. Our choice of β and the form of u is motivated by (1.4) and by the boundary Harnack inequality in Lemma 2.5. Indeed $|x''|^\beta$, is a solution to the p-Laplace equation in $\mathbb{R}^n \backslash \mathbb{R}^k$ with continuous boundary value 0 on $\mathbb{R}^n \backslash \mathbb{R}^k$. So the Martin p-harmonic function relative to 0 for $\mathbb{R}^n \backslash \mathbb{R}^k$ is homogeneous in r with negative exponent and by Lemma 2.5 this function is bounded above and below at $x \in \partial B(0,1) \backslash \mathbb{R}^k$ by $|x''|^\beta$. Ratio constants depend only on p, n, k and the value of the Martin function at e_n.

Now for the derivatives of u we get

$$\begin{aligned} u_t &= -(\lambda+\beta)\frac{t}{r^2}u, \\ u_s &= u\frac{\beta}{s} - (\lambda+\beta)u\frac{s}{r^2} = \frac{-\lambda s^2 + \beta t^2}{sr^2}u. \end{aligned} \tag{3.5}$$

Consequently

$$u_s^2 + u_t^2 = u^2\left(\frac{(\lambda+\beta)^2 t^2 s^2}{s^2 r^4} + \frac{(-\lambda s^2 + \beta t^2)^2}{s^2 r^4}\right). \tag{3.6}$$

Now the cross term $2\lambda\beta s^2 t^2$ in the numerator cancels, leaving $(\lambda^2 + \beta^2)s^2t^2 + \lambda^2 s^4 + \beta^2 t^4$ which factors into $(\lambda^2 s^2 + \beta^2 t^2)(s^2 + t^2) = (\lambda^2 s^2 + \beta^2 t^2)r^2$. Altogether this gives

$$|\nabla u(s,t)|^2 = \frac{u^2}{s^2 r^2}(\lambda^2 s^2 + \beta^2 t^2). \tag{3.7}$$

Writing the second derivatives in terms of u we have

$$\begin{aligned} u_{tt} &= \frac{u}{r^4}(\lambda+\beta)((\lambda+\beta+1)t^2 - s^2), \\ u_{ts} &= \frac{ut}{sr^4}(\lambda+\beta)((2+\lambda)s^2 - \beta t^2), \\ u_{ss} &= \frac{u}{s^2 r^4}((\lambda^2+\lambda)s^4 - (\lambda + 3\beta + 2\lambda\beta)s^2 t^2 + (\beta^2 - \beta)t^4). \end{aligned} \tag{3.8}$$

Looking at equation (3.3) we see that we can factor out $\frac{u^3}{s^4 r^8}$ in the first term, the rest of the first term factors

$$(p-2)\frac{u^3}{s^4r^8}(\lambda^3(\lambda+1)s^4+\beta^2(2\lambda^2-\beta+\lambda)s^2t^2+\beta^3(\beta-1)t^4)r^4. \tag{3.9}$$

In the second term we can factor out $\frac{u^3}{s^4r^6}$ and the rest of it factors

$$\frac{u^3}{s^4r^6}(\lambda^2s^2+\beta^2t^2)((\lambda^2+(2-n)\lambda-\beta k)s^2+\beta(\beta+n-k-2)t^2)r^2. \tag{3.10}$$

Canceling the r's and adding these two terms we get

$$\frac{u^3}{s^4r^4}(A\lambda^2s^4+B\beta s^2t^2+C\beta^3t^4) \tag{3.11}$$

where

$$\begin{aligned} A(\lambda,\beta) &= (p-2)(\lambda^2+\lambda)+\lambda^2+(2-n)\lambda-\beta k=(p-1)\lambda^2+(p-n)\lambda-\beta k,\\ B(\lambda,\beta) &= \beta(2\lambda^2-\beta+\lambda)(p-2)+\lambda^2(\beta+n-k-2)+\beta(\lambda^2+(2-n)\lambda-\beta k)\\ &= (2\beta(p-1)+n-k-2)\lambda^2+\beta(p-n)\lambda-\beta^2(p-2+k),\\ C(\lambda,\beta) &= (p-2)(\beta-1)+\beta+n-k-2=(p-1)\beta-(p-n+k). \end{aligned} \tag{3.12}$$

We are using $\beta=\frac{p-n+k}{p-1}$ so that $C=0$, and in equation (3.11) we can factor out an s^2. Thus to determine solutions, subsolutions, supersolutions we just need to know when the following equation is 0, positive, negative for all (s, t)

$$A\lambda^2s^2+B\beta t^2. \tag{3.13}$$

For this we need $A,B=0$, $A,B>0$, $A,B<0$ respectively. Now A, B are quadratics in λ with positive leading coefficient. Using the quadratic formula, the discriminant is a perfect square, A has roots $-\beta$ and $\frac{k}{p-1}$ while B has roots $-\beta$ and $\beta\frac{p-2+k}{2p-n+k-2}$. Therefore

$$\begin{aligned} A(\lambda,\beta) &= (p-1)(\lambda+\beta)\left(\lambda-\frac{k}{p-1}\right),\\ B(\lambda,\beta) &= (2\beta(p-1)+n-k-2)(\lambda+\beta)(\lambda-\frac{\beta(p-2+k)}{2p-n+k-2}), \end{aligned} \tag{3.14}$$

In view of these facts, (3.14), and (3.13), we conclude (1.6), (1.7) of Theorem A. To show $\chi<k, \chi$ as in (1.6), it suffices to show

$$k>\frac{(p+k-n)(k+p-2)}{(p-1)(2p-n+k-2)}=\beta\frac{k+p-2}{2p-n+k-2}. \tag{3.15}$$

Since $\beta<1,$ it is easily seen that (3.15) is true for $p\geq n$. To prove that this inequality holds for $2<p<n,$ we gather terms in k to get that (3.15) is valid if

$$I=(p-2)\left[k^2+k(2p-n-2)+n-p\right]>0. \tag{3.16}$$

Since $n>p>n-k$ and $p>2,$ it follows from (3.16) that

$$\begin{aligned} I &> (p-2)\left[k^2+k(p+n-k-n-2)+n-p\right]\\ &= (p-2)\left[k(p-2)+n-p\right]>0. \end{aligned} \tag{3.17}$$

Thus $\chi<k$. □

Remark 3.1. When $p = n$ the quadratics A, B have the common roots $\lambda = \pm\beta$. This checks, when $\lambda = -\beta$ as we already know the solution $u = s^{\beta}$ in $\mathbb{R}^n_+$ and that the n-Laplacian is invariant under an inversion.

4. Proof of Theorem B

Proof. We prove Theorem B only when $1 \le k \le n-2$, since the proof when $k = n-1$ and $p > 2$ is essentially the same. Recall from Section 1 that if $p > n-k$ with $1 \le k \le n-2$, then u is said to be a p-harmonic Martin function for $\mathbb{R}^n \backslash \mathbb{R}^k$ relative to 0 provided $u > 0$ is p-harmonic in $\mathbb{R}^n \backslash \mathbb{R}^k, u(x) \to 0,$ as $x \to \infty, x \in \mathbb{R}^n \backslash \mathbb{R}^k$, and u has continuous boundary value 0 on $\mathbb{R}^k \backslash \{0\}$. To briefly outline the proof of (1.4), suppose v is another p-harmonic Martin function relative to 0. Applying (2.6) of Lemma 2.6 to u, v and letting $\rho \to 0$ it follows that $u/v \equiv$ a constant in $\mathbb{R}^n \backslash \mathbb{R}^k$. Since p-harmonic functions are invariant under dilation we deduce that if $t > 0$, and $u(e_n) = 1$, then

$$u(tx) = u(te_n)u(x) \quad \text{whenever} \quad x \in \mathbb{R}^n \backslash \mathbb{R}^k. \tag{4.1}$$

Differentiating (4.1) with respect to t (permissible by (2.4) $(\hat{a})$ of Lemma 2.4) and evaluating at $t = 1$ we see that

$$\langle x, \nabla u(x) \rangle = \langle e_n, \nabla u(e_n) \rangle u(x) \quad \text{whenever} \quad x \in \mathbb{R}^n \backslash \mathbb{R}^k.$$

If we put $\rho = |x|, x/|x| = \omega \in \mathbb{S}^{n-1}$, in this identity we obtain that

$$\rho \ (u)_{\rho}(\rho\omega) = \langle e_n, \nabla u(e_n) \rangle u(\rho\omega).$$

Dividing this equality by $\rho u(\rho\omega)$, integrating with respect to ρ, and exponentiating, we find for $r > 0$ and $\omega \in \mathbb{S}^{n-1}$ that

$$u(r\omega) = r^{-\sigma} u(\omega) \quad \text{where} \quad \sigma = -\langle e_1, \nabla u(e_1) \rangle. \tag{4.2}$$

For fixed positive integer $1 \le k \le n-2$ and $p > n-k$, let $\hat{u}$ be the p-subsolution defined in (1.5) where λ is chosen so that $\chi < \lambda < k$ where χ is as in Theorem A. From Lemma 2.5 and the fact (mentioned earlier) that $|x''|^{\beta}$ is p-harmonic in $\mathbb{R}^n \backslash \mathbb{R}^k$ we see that $u/v \approx 1$ on $\partial B(0,1) \cap (\mathbb{R}^n \backslash \mathbb{R}^k)$. Comparing boundary values of $\hat{u}, u$ and using the fact that $\hat{u}(x) + u(x) \to 0$ as $x \to \infty, x \in \mathbb{R}^n \backslash \mathbb{R}^k$, we deduce from the boundary maximum principle in Lemma 2.1 that $\hat{u} \le cu$ in $\mathbb{R}^n \backslash (\mathbb{R}^k \cup B(0,1))$ where $c = c(p, n.k)$. Letting $x \to \infty$ we get

$$\sigma \le \lambda < k. \tag{4.3}$$

Next given $0 < t < 10^{-10n}$, let $a(\cdot)$ be a C^{∞} smooth function on $\mathbb{R}$ with compact support in $(-t,t), 0 \le a \le 1, \ a \equiv 1$ on $(-t/2, t/2)$, and $|\nabla a| \le 10^5/t$. Let $f(x) = \prod_{i=1}^n a(x_i), x \in \mathbb{R}^n$, and for fixed $p > n-k$, let $\tilde{v}$ be the unique p-harmonic function on $\mathbb{R}^n \backslash \mathbb{R}^k$ with $0 \le \tilde{v} \le 1$ satisfying

$$\int_{\mathbb{R}^n \backslash \mathbb{R}^k} |\nabla \tilde{v}|^p dx \le \int_{\mathbb{R}^n \backslash \mathbb{R}^k} |\nabla f|^p dx \le c t^{n-p} \tag{4.4}$$

where $c = c(p,n,k)$ and $(\tilde{v} - f)\zeta \in W_0^{1,p}(B(0,\rho) \backslash \mathbb{R}^k)$ whenever $0 < \rho < \infty$. Here ζ is as in Lemma 2.3 (for a fixed ρ). Once again existence and uniqueness of $\tilde{v}$ follows with slight modification from the usual calculus of variations argument for bounded domains (see [12]). We claim that there exist $\beta_* = \beta_*(p,n,k) \in (0,1]$ and $c = c(p,n,k,t) \geq 1$ such that if $x, y \in B(0,\rho) \cap (\mathbb{R}^n \setminus \mathbb{R}^k)$, then

$$|\tilde{v}(x) - \tilde{v}(y)| \leq c\left(\frac{|x-y|}{\rho}\right)^{\beta_*} \quad \text{and} \quad \tilde{v}(x) \leq \left(\frac{4nt}{|x|}\right)^{\beta_*}. \tag{4.5}$$

The left hand inequality in (4.5) follows from Lemma 2.3. To prove the right hand inequality in (4.5) let $\tilde{v}_j, j = 4n+1, 4n+2, \ldots,$ be the p-harmonic function in $B(0,j)\setminus\mathbb{R}^k$ with continuous boundary values $\tilde{v}_j = \tilde{v}$ on $B(0,j) \cap \mathbb{R}^k$ and $\tilde{v}_j = 0$ on $\partial B(0,j)$. Observe from the boundary maximum principle in Lemma 2.1 and $0 \leq \tilde{v}_j \leq 1$, that $\max_{\partial B(0,r)} \tilde{v}_j$ is nonincreasing for $r \in (2nt, j)$. Also using uniqueness of $\tilde{v}$ in the calculus of variations minimizing argument it follows that $\tilde{v}_j \to \tilde{v}$ uniformly on compact subsets of $\mathbb{R}^n$. Thus $\max_{\partial B(0,r)} \tilde{v}$ is also non increasing as a function of r. Using this fact and Harnack's inequality in Lemma 2.2 (c) applied to $\max_{\partial B(0,r)} \tilde{v} - \tilde{v}$, and (2.3) $(++)$ we deduce the existence of $\theta \in (0,1)$ with

$$\max_{\partial B(0,2r)} \tilde{v} \leq \theta \max_{\partial B(0,r)} \tilde{v} \quad \text{whenever} \quad r > 2nt. \tag{4.6}$$

Iterating this inequality we get the right hand inequality in (4.5).

Next we show that

$$\tilde{v}(e_n) \approx t^{\sigma} \tag{4.7}$$

where σ is as in (4.2) and the proportionality constants depend only on p, n, k. To prove (4.7), put $\tilde{u} = t^{\sigma} u(x), x \in \mathbb{R}^n$. Then from Harnack's inequality and (2.3) $(++)$ of Lemma 2.3 with $\hat{v} = 1 - \tilde{v}$, we find that $\tilde{v}(te_n) \approx 1$. In view of the boundary values of $\tilde{v}$, $\tilde{u}$, and $\tilde{v}(te_n) \approx \tilde{u}(te_n) = 1$, as well as Harnack's inequality in (2.1) (c), we see that first Lemma 2.5 can be applied to get

$$\tilde{u}/\tilde{v} \approx 1 \tag{4.8}$$

on $\partial B(0, 2nt)\setminus\mathbb{R}^k$ where ratio constants depend only on k, n, p. Second from (4.5) for $\tilde{v}$, and $\sigma > 0$ we find that $\tilde{u}(z), \tilde{v}(z) \to 0$ as $z \to \infty$ in $\mathbb{R}^n\setminus\mathbb{R}^k$ and thereupon from Lemma 2.1 that (4.8) holds in $(\mathbb{R}^n\setminus\mathbb{R}^k)\setminus B(0,2nt)$.

Since $\tilde{u}(e_n) = t^{\sigma}$ we conclude from (4.8) that (4.7) is true.

Let $\tilde{a}$ denote the one periodic extension of $a|_{[-1/2,1/2]}$ to $\mathbb{R}$. That is $\tilde{a}(r+1) = \tilde{a}(r)$ for $r \in \mathbb{R}$ and $\tilde{a} = a$ on $[-1/2, 1/2]$. Also let $\tilde{\Psi}$ be the p-harmonic function on $\mathbb{R}^n\setminus\mathbb{R}^k$ with continuous boundary values on $\mathbb{R}^k$ and

$$\begin{aligned}
&(a)\ \tilde{\Psi}(x + e_i) = \tilde{\Psi}(x) \quad \text{for} \quad 1 \leq i \leq k, \quad \text{whenever} \quad x \in \mathbb{R}^n\setminus\mathbb{R}^k, \\
&(b)\ \tilde{\Psi}(x) - \prod_{i=1}^{n} \tilde{a}(x_i) \in R_0^{1,p}(S(1)) \quad \text{and} \quad 0 \leq \tilde{\Psi} \leq 1 \ \text{in} \ \mathbb{R}^n\setminus\mathbb{R}^k, \\
&(c)\ \int_{S(1)} |\nabla \tilde{\Psi}|^p dxdy \leq c\ t^{n-p} < \infty, \ \text{where} \ c = c(p,n,k), \\
&(d)\ \lim_{x'' \to \infty,\, x'' \in \mathbb{R}^{n-k}} \tilde{\Psi}(x', x'') = \xi \ \ \text{a constant, uniformly for} \ \ x' \in \mathbb{R}^k.
\end{aligned} \tag{4.9}$$

Existence of $\tilde{\Psi}$ satisfying $(a)-(d)$ of (4.9) follows from the discussions after (2.7) and (2.8). Comparing boundary values of $\tilde{v}$, $\tilde{\Psi}$, we see that $\tilde{v} \leq \tilde{\Psi}$ on $\mathbb{R}^k$. Using this fact and Lemma 2.1 we find in view of (4.5) that

$$\tilde{v} \leq \tilde{\Psi} \quad \text{in} \quad \mathbb{R}^n \backslash \mathbb{R}^k. \tag{4.10}$$

Let $\hat{e}$ be that point in $\mathbb{R}^{n-k}$ with $\hat{e} = (\hat{e}_1, ..., \hat{e}_{n-k})$ where $\hat{e}_i = 0, 1 \leq i \leq n-k-1$, and $\hat{e}_{n-k} = 1$. From (4.10), (4.8), and Harnack's inequality for $\tilde{v}$, we have

$$\int_{Q^{(k)}_{1/2}(0)} \tilde{\Psi}(x', \hat{e}) \; d\mathcal{H}^k x' \geq \int_{Q^{(k)}_{1/2}(0)} \tilde{v}(x', \hat{e}) \; d\mathcal{H}^k x' \approx t^\sigma. \tag{4.11}$$

Also from (4.9)(b), the definition of $\tilde{a}$, and continuity of $\tilde{\Psi}$ in $\mathbb{R}^n$ we obtain

$$\int_{Q^{(k)}_{1/2}(0)} \tilde{\Psi}(x', s\hat{e}) \; d\mathcal{H}^k x' \leq 4^k t^k \;, \tag{4.12}$$

for small $s > 0$. Thus there exists $c = c(p, n, k) \geq 1$ with

$$\int_{Q^{(k)}_{1/2}(0)} \tilde{\Psi}(x', s\hat{e}) \; d\mathcal{H}^k x' \leq c t^{k-\sigma} \int_{Q^{(k)}_{1/2}(0)} \tilde{\Psi}(x', \hat{e}) \; d\mathcal{H}^k x' \;. \tag{4.13}$$

We conclude for $t > 0$, small enough that Theorem B is valid with $\Psi(x) = \Psi(x', x'')$ one of the functions, $\tilde{\Psi}(x', x'' + \hat{e}) - \xi$ or $\tilde{\Psi}(x', x'' + s\hat{e}) - \xi$ when $x \in \mathbb{R}^n$. □

The proof of Theorem B in case (ii), i.e., when $k = n-1, p > 2$, and in $\mathbb{R}^n_+$ is essentially the same as in case (ii), only in this case one uses Remark 1.6. Thus we omit the details.

Remark 4.1. We remark that Corollary 1.7 follows from (4.8) and Lemma 2.1.

5. Proof of Theorem C

In this section we indicate the changes in Wolff's argument that are necessary to show that Theorem B implies Theorem C. Unless otherwise stated, we let $c \geq 1$ denote a positive constant which may depend on p, n, k, and the Lipschitz norm of $\Psi|_{\mathbb{R}^k}$. Recall the definition of $f \in R^{1,p}(S(1))$ and $||f||_{*,p}$ in (2.7) when k, n are fixed positive integers with $1 \leq k \leq n-2$ or $k = n-1$. Given $h \in R^{1,p}(S(1))$ we note that h has a trace on $\mathbb{R}^k$ which is well defined $\mathcal{H}^k$ almost everywhere. Let $h|_{\mathbb{R}^k}$ denote this trace and extend h to $\mathbb{R}^k$ by setting $h(x', 0) = h|_{\mathbb{R}^k}(x', 0)$ for $x' \in \mathbb{R}^k$. Also let $||h|_{\mathbb{R}^k}||_\infty$ and $\breve{||h|_{\mathbb{R}^k}||} = |||\nabla h|_{\mathbb{R}^k}|||_\infty$, denote respectively the ∞ and Lipschitz norms of $h|_{\mathbb{R}^k}$. Let $\hat{h} \in R^{1,p}(S(1))$, be the p-harmonic function on either (i) $\mathbb{R}^n \backslash \mathbb{R}^k$ (when $1 \leq k \leq n-2, p > n-k$) or (ii) $\mathbb{R}^n_+$ (when $k = n-1$ and $p > 2$), with $\hat{h} - h \in R^{1,p}_0(S(1))$. Throughout this section $Q^{(k)}_r(x') = Q_r(x')$ when $x' \in \mathbb{R}^k$ and $r > 0$. Also proofs of Lemmas will only be given in case (i), as the proof in case (ii) is essentially the same.

We first state an analogue of Lemma 1.4 in [1].

Lemma 5.1. *Suppose $\bar{u}, \bar{v}$ are p-harmonic in either (i) $\mathbb{R}^n \backslash \mathbb{R}^k$ when $1 \leq k \leq n-2$ and $p > n-k$, or (ii) $\mathbb{R}^n_+$ when $k = n-1$ and $p > 2$, with continuous boundary values. Also assume that $\bar{u}, \bar{v} \in R^{1,p}(S(1))$ with $||\bar{u}|_{\mathbb{R}^k}||_\infty + ||\bar{v}|_{\mathbb{R}^k}||_\infty < \infty$ and for some $z' \in \mathbb{R}^k, 0 < \gamma \leq 1/4$, that $\bar{u}|_{\mathbb{R}^k} \leq \bar{v}|_{\mathbb{R}^k}$ on $Q_{2\gamma}(z')$. Let $0 < t \leq 1/2$ and if (i) holds put*

$$E(t)=\{x=(x',x'')\text{ with }\bar u(x)-\bar v(x)>0\quad\text{and}\quad x\in Q_\gamma(z')\times\{x'':|x''|\le t\}\}$$

while if (ii) holds replace x'' in the above display by x_n where $x_n>0$. Then there exists $c=c(p,n,k,\gamma)$ such that

$$\int_{E(t)}|\nabla(\bar u-\bar v)^+|\ dx\le c\ t^{(n-k)(p-1)/p}(||\bar u||_{p,*}+||\bar v||_{p,*})^\alpha\quad\left[\max_{S(1)}(\bar u-\bar v)^+\right]^{1-\alpha}\tag{5.1}$$

where $\alpha=1-2/p$ and $a^+=\max(a,0)$.

Proof. We note that since $p>2$, then

$$(|\nabla\bar u|^{p-2}\nabla\bar u-|\nabla\bar v|^{p-2}\nabla\bar v)\cdot\nabla(\bar u-\bar v)\ge c^{-1}(|\nabla\bar u|+|\nabla\bar v|)^{p-2}|\nabla\bar u-\nabla\bar v|^2\tag{5.2}$$

and

$$||\nabla\bar u|^{p-2}\nabla\bar u-|\nabla\bar v|^{p-2}\nabla\bar v|\le c(|\nabla\bar u|+|\nabla\bar v|)^{p-2}|\nabla\bar u-\nabla\bar v|\tag{5.3}$$

on S (1). Let $0\le\theta\le1\in C_0^\infty(Q_{2\gamma}(z')\times\{x'':|x''|\le1\})$ with $\theta\equiv1$ on $Q_\gamma(z')\times\{x'':|x''|\le t\}$, and $|\nabla\theta|\le c\gamma^{-1}$. If $a^+=\max(a,0)$, then $\theta^2(\bar u-\bar v)^+$ can be used as a test function in the definition of p-harmonicity for $\bar u,\bar v$. Doing this, using (5.2), (5.3), and a standard Caccioppoli type argument we obtain

$$\begin{aligned}\int\limits_{E(t)}|\nabla\bar u-\nabla\bar v|^pdx&\le\int\theta^2|\nabla(\bar u-\bar v)^+|^pdx\\&\le c\gamma^{-2}\int_{S(1)\cap B(z',2n)}\left[(\bar u-\bar v)^+\right]^2(|\nabla\bar u|+|\nabla\bar v|)^{p-2}dx\\&\le c\gamma^{-2}\ (||\bar u||_{*,p}+||\bar v||_{*,p})^{p-2}\quad\left[\max_{S(1)}(\bar u-\bar v)^+\right]^2\end{aligned}\tag{5.4}$$

where $c=c(p,n,k)$ in (5.1)-(5.4). □

Next we state an analogue of Lemma 1.6 in [1] which the authors view as Wolff's main lemma for applications.

Lemma 5.2. *Let k,n be positive integers with either (i) $1\le k\le n-2$ and $p>n-k$ or (ii) $k=n-1$ and $p>2$ fixed. Let $\epsilon\in(0,1)$ and $1\le M<\infty$. Then there are constants $A=A(p,n,k,\epsilon,M)>0$ and $\nu_0=\nu_0(p,n,k,\epsilon,M)<\infty$, such that if $\nu>\nu_0>100$ is a positive integer, f, $g\in R^{1,p}(S(1))$, $q\in R^{1,p}(S(\nu^{-1}))$, and if*

$$\max(||f|_{\mathbb R^k}||_\infty,||g|_{\mathbb R^k}||_\infty,||q|_{\mathbb R^k}||_\infty,||f|_{\mathbb R^k}\breve{||},||g|_{\mathbb R^k}\breve{||},\nu^{-1}||q|_{\mathbb R^k}\breve{||})\le M,\tag{5.5}$$

then for $x=(x',x'')\in S(1)$, $\alpha=1-2/p$, and $1\le k\le n-2,p>n-k$,

$$|\widehat{fq+g}(x)-f(x',0)\ \hat q(x)-g(x',0)|<\epsilon\quad\text{if}\quad|x''|\le A\nu^{-\alpha}.\tag{5.6}$$

If, in addition, $\hat q(y)\to0$, uniformly as $y\to\infty,y\in S(1)$, then

$$|\widehat{qf+g}\ \ (x',x'')-g(x',0)|<2\epsilon\quad\text{if}\quad|x''|=A\nu^{-\alpha},\tag{5.7}$$

and

$$|\widehat{qf+g}(x)-\hat{g}(x)|<3\epsilon \quad \text{if} \quad |x''|\geq A\nu^{-\alpha}. \tag{5.8}$$

If $k=n-1, p>2,$ replace x'' by $x_n, x_n>0$ in (5.6)-(5.8).

Proof. We note that our proof scheme is similar to Wolff's but details are somewhat different. The first step in the proof of (5.6) is to show for given $\beta\in(0,10^{-4}),$ that (5.6) holds for some $A=A(p,n,k,\epsilon,M,\beta)>0$ with $\beta\nu^{-1}\leq|x''|\leq A\nu^{-\alpha}$ provided $\nu\geq\nu_0(p,n,k,\epsilon,M)$. To do this let

$$J(x)=\widehat{fq+g}(x)-f(x',0)\ \hat{q}(x)-g(x',0), \quad \text{for} \quad x\in S(1).$$

Now suppose that

$$|J(z)|>\epsilon \tag{5.9}$$

where $z=(z',z'')$ with $z'\in Q_1(0),\ |z''|=t,$ and $\beta\nu^{-1}\leq t<1/4$. Let

$$H=\widehat{fq+g} \quad \text{and} \quad K=f(z',0)\hat{q}+g(z',0).$$

Then H and K are both p-harmonic in $\mathbb{R}^n\backslash\mathbb{R}^k$ when $1\leq k\leq n-2$ and in $\mathbb{R}^n_+$ when $k=n-1$ with continuous boundary values. Also from (5.5) and $H,K\in R^{1,p}(S(1)),$ we deduce that

$$|H(x',0)-K(x',0)|\leq(M^2+M)|x'-z'| \quad \text{for} \quad x'\in\mathbb{R}^k. \tag{5.10}$$

From translation invariance of p-harmonic functions and the maximum principle for p-harmonic functions in Lemma 2.1 we see for $x\in\mathbb{R}^n$ and $1\leq j\leq k$ that

$$\hat{q}(x+ie_j/\nu)=\hat{q}(x) \quad \text{for every integer } i. \tag{5.11}$$

From (5.11), (5.10), Lemma 2.1, and translation invariance of p-harmonic functions it follows that if

$$\Lambda=\left\{\sum_{j=1}^{k} i_je_j/\nu, \quad \text{where} \quad i_j, 1\leq j\leq k, \quad \text{are integers}\right\}$$

then for $\hat{\zeta}\in\Lambda,$

$$\begin{aligned}|H(z'+\hat{\zeta},z'')-H(z',z'')|&+|K(z'+\hat{\zeta},z'')-K(z',z'')|\\ &\leq\max_{\mathbb{R}^k}\ |H(x'+\hat{\zeta},0)-H(x',0)|\\ &\leq(M^2+M)|\hat{\zeta}|.\end{aligned} \tag{5.12}$$

Next we note from Lemma 2.2 (b) and $H,K\in R^{1,p}(S(1)),$ that there exists $\rho=\rho(p,n,k,\epsilon,M)\in(0,1/2),$ such that if $x\in B(z+\hat{\zeta},\rho t),$ then

$$|H(x)-H(z+\hat{\zeta})|+|K(x)-K(z+\hat{\zeta})|<\epsilon/1000 \tag{5.13}$$

whenever $\hat{\zeta}\in\Lambda$. Let $\gamma=\min(\frac{\epsilon}{10^3n(M^2+M)},1/4)$. Then from (5.10) we observe that

$$|H-K|\leq\epsilon/100 \quad \text{on} \quad Q_{2\gamma}(z'). \tag{5.14}$$

Also without loss of generality assume that

$$J(z) = (H-K)(z) > \epsilon. \tag{5.15}$$

Let $\tilde{\Lambda} = \{\hat{\zeta} \in \Lambda : z' + \hat{\zeta} \in Q_\gamma(z')\}$ and suppose $\gamma \geq 10^3 \nu^{-1}$. Put

$$W(t) := Q_\gamma(z') \cap \{x' : \min_{\hat{\zeta} \in \tilde{\Lambda}} |x' - (z' + \hat{\zeta})| < \rho t/(10n^2)\}.$$

Then either $W(t) = Q_\gamma(z')$ or

$$Q_{\rho t}(z' + \hat{\zeta}) \cap Q_\gamma(z') \neq \emptyset \text{ for } \approx (\gamma\nu)^k \text{ points } \hat{\zeta} \in \tilde{\Lambda}, \tag{5.16}$$

where proportionality constants depend only on k. From (5.16) and $\beta/\nu \leq t \leq 1/4$, we conclude in either case that

$$1 \leq c(p, n, k, \epsilon, M, \beta) \; \mathcal{H}^k(W(t)). \tag{5.17}$$

Also if $y' \in W(t)$ and $k \leq n-2$, we see from (5.12)-(5.15), the definition of γ, $\nu^{-1} \leq 10^{-3}\gamma$, that there is a Borel set $F(t, y') \subset \{(y', x'') \in \mathbb{R}^n : |x''| = t\}$ satisfying

$$t^{n-k-1} \leq c(p, n, k, \epsilon, M) \mathcal{H}^{n-k-1}(F(t, y')), \tag{5.18}$$

and the property that if $(y', w'') \in F(t, y')$, then $(y', w'') \in B(z + \hat{\zeta}, \rho t)$ for some $\hat{\zeta} \in \tilde{\Lambda}$, with

$$\begin{aligned} (H-K)(y', w'') &\geq (H-K)(z + \hat{\zeta}/\nu, z'') - \epsilon/100 \\ &\geq (H-K)(z) - 2\epsilon/100 \\ &\geq 98\epsilon/100. \end{aligned} \tag{5.19}$$

Let $G(t, y') = \{\omega \in \mathbb{S}^{n-k-1} : t\omega \in F(t, y')\}$ and $\bar{u} = H$, $\bar{v} = K + \epsilon/4$. From (5.14) we see that $\bar{u} - \bar{v} < 0$ on $Q_{2\gamma}(z')$. Using this observation, (5.18), (5.19), and continuity of H, K in $\mathbb{R}^n$ it follows that for some $\tilde{c} = \tilde{c}(p, n, k, \epsilon, M) \geq 1$,

$$\begin{aligned} 1 &\leq \tilde{c} \; \left[\int_{G(t,y')} (\bar{u} - \bar{v})^+(t\omega) d\mathcal{H}^{n-k-1}\omega \right]^p \\ &\leq \tilde{c} \; \left[\int_{G(t,y')} \int_0^t |\nabla(\bar{u} - \bar{v})^+(r\omega)| dr d\mathcal{H}^{n-k-1}\omega \right]^p. \end{aligned} \tag{5.20}$$

Let $\hat{E}(t, y')$ be the set of points consisting of line segments with one endpoint y' and the other endpoint in $G(t, y')$. Switching to polar coordinates we see from (5.20) and Hölder's inequality applied to $|\nabla(\bar{u} - \bar{v})^+(r\omega)| r^{(n-k-1)/p}$ and $r^{(-n+k+1)/p}$, that

$$\left[\int_{G(t,y')} \int_0^t |\nabla(\bar{u} - \bar{v})^+(r\omega)| dr d\mathcal{H}^{n-k-1}\omega \right]^p \leq c \; t^{p+k-n} \int_{\hat{E}(t,y')} |\nabla(\bar{u} - \bar{v})^+|^p d\mathcal{H}^{n-k} \tag{5.21}$$

where $c = c(p, n, k, \epsilon, M)$. Using (5.21), (5.20), (5.17), and integrating over $y' \in W(t)$ we get

$$1 \leq \bar{c}(p, n, k, \epsilon, M, \beta) \; t^{p+k-n} \int_{E(t)} |\nabla(\bar{u} - \bar{v})^+|^p dx \tag{5.22}$$

where $E(t)$ is as in Lemma 5.1. Applying Lemma 5.1 and using Lemma 2.1, (5.5), we arrive at

$$1 \leq \ c(p,n,k,\epsilon,M,\beta)t^{p+k-n} \ \left(||H||_{*,p} + ||K||_{*,p}||\right)^{p-2} \tag{5.23}$$

Let

$$h(x) = \left[f(x',0)q(x',0) + g(x',0)\right] \ \phi(|x''|),$$

when $x \in \mathbb{R}^n$ where $\phi \in C_0^\infty(-2/\nu, 2/\nu)$ with $\phi \equiv 1$ on $(-1/\nu, 1/\nu)$ and $|\phi'| \leq 1000 \ \nu$. Then $h - H \in W_0^{1,p}(S(1))$, so

$$||H||_{*,p}^{p-2} \leq \left(\int_{S(1)} |\nabla h|^p dx\right)^{\alpha} \leq c(p,n,k)(M^2+M)^k \nu^{\alpha(p+k-n)}. \tag{5.24}$$

Likewise one gets the same estimate for $||K||_{*,p}^{p-2}$ as for $||H||_{*,p}^{p-2}$ in (5.24). Using these estimates in (5.23) we conclude that $c(p,n,k,\epsilon,M,\beta) \ t > \nu^{-\alpha}$ provided $\nu \geq \nu_0(p,n,k,\epsilon,M,\beta)$.

To complete the proof of (5.6) it remains to fix $\beta = \beta(p,n,k,\epsilon,M)$ and show (5.6) holds for $0 < t \leq \beta/\nu$. To do this we apply (2.2) of Lemma 2.3 with $\hat{v} = \widehat{qf+g}$, $\hat{q}$, and with $\rho = \beta^{1/2}\nu^{-1}$, $\hat{\sigma} = 1$, $M' = (M^2+M)\nu$, to get for $|x''| < \beta\nu^{-1}$,

$$\begin{aligned} |J(x)| = |J(x) - J(x',0)| &\leq c(M)\left(\nu \ (\beta^{1/2}\nu^{-1}) + \left(\frac{\beta\nu^{-1}}{\beta^{1/2}\nu^{-1}}\right)^{\hat{\sigma}_1}\right) \\ &\leq c'(M) \ \beta^{\hat{\sigma}_1/2}. \end{aligned} \tag{5.25}$$

Choosing $\beta = \beta(p,n,k,\epsilon,M) > 0$ small enough and then fixing β we obtain (5.6) from (5.25) for $t < \beta\nu^{-1}$.

To prove (5.7) we note from (5.5) and (2.8) of Lemma 2.7 that

$$|\hat{q}(x)| \leq 2M(|x''|\nu)^{-\delta} \tag{5.26}$$

since $\hat{q}(x) \to 0$ as $|x''| \to \infty$. Choosing $|x''| = A\nu^{-\alpha}$ and ν_0, still larger if necessary we get (5.7) from (5.26). To prove (5.8) observe from (2.8) of Lemma 2.7 with $\hat{v} = \hat{g}$ and $\rho = A\nu^{-\alpha/2}$ that

$$|\hat{g}(x) - g(x',0)| < \epsilon \quad \text{when} \quad |x''| = A\nu^{-\alpha} \tag{5.27}$$

for $\nu_0 = \nu_0(p,n,k,\epsilon,M)$ large enough. Now (5.8) follows from (5.27) and Lemma 2.1. This finishes the proof of Lemma 5.2 when $1 \leq k \leq n-2$ and $p > n-k$. The same conclusion holds if $k = n-1$. However in this case the proof is somewhat simpler since $G(t,y')$ is a point and $F(t,y')$ is a line segment so one can write a version of (5.21) with a single integral. □

5.1. Lemmas on gap series

Throughout the rest of this section we write dx' for $d\mathcal{H}^k x'$ when $x' \in \mathbb{R}^k$. The examples in Theorem C will be constructed when either (*i*) $1 \leq k \leq n-2$ and $p > n-k$, or (*ii*) $k = n-1$ and $p > 2$, using Theorem B, as the uniform limit on compact subsets of $\mathbb{R}^n \backslash \mathbb{R}^k$ or $\mathbb{R}^n_+$ of a sequence of p-harmonic functions whose boundary values are partial sums of $a_jL_j(x')\Psi(T_jx',0)$, where Ψ is as in Theorem B, (L_j) is to be defined, and (T_j) is a sequence of positive integers satisfying

$$T_1 = 1,$$
$$T_{j+1} \text{ is an integer multiple of } T_j \text{ and } \quad T_{j+1} \geq 4T_j\big[\log(2+T_j)\big]^3 \quad \text{for} \quad j = 1,2,\ldots. \tag{5.28}$$

Lemma 5.2 will be used to make estimates on this sequence. Throughout the rest of this section we also assume, as we may, that if Ψ is as in Theorem B, then

$$||\Psi(x',0)||_\infty + ||\Psi(x',0)||_{\breve{}} \leq 1/2. \tag{5.29}$$

Also, let $\bar{b} = \int_{Q_{1/2}(1)} \Psi(x',0)dx'$ and let (a_j) be a sequence of real numbers with

$$\hat{\chi}^2 := \sum_{j=1}^{\infty} a_j^2 < \infty. \tag{5.30}$$

Lemma 5.3. *Let* $\psi(x') = \Psi(x',0) - \bar{b}$ *for* $x' \in \mathbb{R}^k$ *and put* $\psi_j(x') = \psi(T_jx')$ *for* $j = 1,2,\ldots$. *If*

$$s^*(x') := \sup_l \left|\sum_{j=1}^{l} a_j\psi_j(x')\right|, \quad \text{for} \quad x' \in \mathbb{R}^k,$$

then for $\lambda > 0$,

$$\lambda^2 \ \mathcal{H}^k(\{x' \in Q_{1/2}(0) : s^*(x') > \lambda\}) \leq c\ \hat{\chi}^2 \tag{5.31}$$

where $c = c(k) \geq 1$. *Consequently,*

$$s(x') := \lim_{l\to\infty} \sum_{j=1}^{l} a_j\psi_j(x') \text{ exists for } \mathcal{H}^k \text{ almost every } x' \in \mathbb{R}^k. \tag{5.32}$$

Proof. For $m = 1,2,\ldots$, let G_m denote the set of all open cubes Q in $\mathbb{R}^k$ with side length $r_m = 1/T_m$ and center at $r_m\tau = (r_m\tau_1, r_m\tau_2, \ldots r_m\tau_k)$ where $\tau_i, 1 \leq i \leq k$, are integers. If $Q \in G_m$ and $j > m$ then from (5.28) we see that $\mathcal{H}^k$ almost everywhere

$$Q = \cup\{Q' \in G_j : Q' \subset Q\}. \tag{5.33}$$

Also from (5.28), and the definition of (ψ_j), we see that if $m < j$ and $Q' \in G_j$, then

$$\int_{Q'} \psi_j dx' = 0.$$

Therefore, if $y' \in Q'$, then

$$\begin{aligned} \left|\int_{Q'} \psi_m\psi_j dx'\right| &= \left|\int_{Q'} (\psi_m - \psi_m(y'))\psi_j dx'\right| \\ &\leq k^{1/2}\ (T_m/T_j)\ \mathcal{H}^k(Q'). \end{aligned} \tag{5.34}$$

Summing over $Q' \subset Q$, and then over $Q \in G_m$ with $Q \subset Q_{1/2}(0)$, it follows from (5.34) that

$$\left|\int_{Q_{1/2}(0)}\psi_m\psi_j dx'\right| \le k^{1/2}\ T_m/T_j. \tag{5.35}$$

Now from (5.28) we observe that

$$T_m/T_j \le (j-1)^{-3}4^{m-j} \quad \text{for } m = 1,2,\dots \text{ and } j = m+1,\dots. \tag{5.36}$$

Let

$$s_l(x') = \sum_{j=1}^{l} a_j\psi_j(x'), \quad \text{for } l = 1,2,\dots, \text{ and } x' \in \mathbb{R}^k.$$

In view of (5.36), (5.35) we get

$$\begin{aligned}
\sup_l \int_{Q_{1/2}(0)} s_l^2 dx' &= \sup_l \sum_{m,j=1}^{l} a_m a_j \int_{Q_{1/2}(0)} \psi_j\psi_m dx' \\
&\le \hat{\chi}^2 + 2\sum_{m=1}^{\infty}|a_m| \left(\sum_{j=m+1}^{l}|a_j|\left|\int_{Q_{1/2}(0)}\psi_j\psi_m dx'\right|\right) \\
&\le \hat{\chi}^2\left[1 + 2\sum_{m=1}^{\infty}\left(\sum_{j=m+1}^{\infty}k^{1/2}(j-1)^{-3}4^{m-j}\right)\right] \\
&\le 3k^{1/2}\hat{\chi}^2.
\end{aligned} \tag{5.37}$$

Using (5.37) with s_l replaced by $s_l - s_i, i < l,$ and letting $l, i \to \infty$ through certain sequences we see from (5.30) that $\lim_{j\to\infty} s_j = s$ in the $||\cdot||_2$ norm of $Q_{1/2}(0)$. Moreover (5.37) is valid with s_l replaced by s.

Next let q be a large positive integer and set

$$\breve{s}\ (x') := \sup_{1\le l\le q} |s_l(x')| \quad \text{for} \quad x' \in Q_{1/2}(0).$$

We note from Cauchy's inequality that $s_l^2 \le \hat{\chi}^2\ l$. Thus if i is a positive integer and $s_l(x') > i\hat{\chi},$ then $i^2 < l$. With i fixed, i a positive integer, let K_i be those cubes $Q' \in \bigcup_{j=1}^{q} G_j$ with $Q' \in G_l,$ if and only if l is the smallest positive integer satisfying $|s_l(y')| > 8k^{1/2}i\hat{\chi}$ for some $y' \in Q'$. Clearly the cubes in K_i are disjoint. Note that if $Q' \in K_i,$ then $\{x' : \breve{s}\ (x') > 8k^{1/2}i\hat{\chi}\} \cap Q' \ne \emptyset$ and $(\cup_{m=1}^{64} G_m) \cap K_i = \emptyset$. Also if $Q' \in K_i \cap G_l,$ it follows from (5.28), (5.29), as in the last line of (5.37), that for $x' \in Q'$,

$$\begin{aligned}
(8k^{1/2}i + 1)\hat{\chi} \ge |s_l(x')| &\ge 8k^{1/2}\hat{\chi}i - |s_{l-1}(x') - s_{l-1}(y')| - k^{1/2}\hat{\chi} \\
&\ge 7k^{1/2}\hat{\chi}i - k^{1/2}\hat{\chi}\sum_{j=1}^{l-1} T_j/T_l \\
&\ge 6k^{1/2}\hat{\chi}i.
\end{aligned} \tag{5.38}$$

Next using (5.35), (5.36), we obtain

$$
\begin{aligned}
\int_{Q_{1/2}(0)} s_q^2 dx' &\geq \sum_{l=1}^{q} \sum_{Q' \in K_i \cap G_l} \int_{Q' \cap Q_{1/2}(0)} s_q^2 dx' \\
&\geq \sum_{l=1}^{q} \sum_{Q' \in K_i \cap G_l} \int_{Q' \cap Q_{1/2}(0)} \left[s_l^2 + 2(s_q - s_l) s_l \right] dx' \\
&= \sum_{l=1}^{q} \sum_{Q' \in K_i \cap G_l} \left(\int_{Q' \cap Q_{1/2}(0)} s_l^2 dx' - \sum\nolimits_{j=l+1}^{q} \sum\nolimits_{m=1}^{l} 2 a_j a_m \int_{Q' \cap Q_{1/2}(0)} \psi_j \psi_m dx' \right) \\
&\geq \sum_{l=1}^{q} \sum_{Q' \in K_i \cap G_l} \int_{Q' \cap Q_{1/2}(0)} s_l^2 dx' - 4 k^{1/2} \hat{\chi}^2 \sum\nolimits_{Q' \in K_i} \mathcal{H}^k (Q' \cap Q_{1/2}(0))
\end{aligned}
\tag{5.39}
$$

where the last inequality is proved once again using the same argument as in the last line of (5.37). From (5.37), (5.38), and (5.39) we conclude that

$$
\begin{aligned}
64 k \hat{\chi}^2 i^2 \mathcal{H}^k (\{x' \in Q_{1/2}(0) : \breve{s}\,(x') > 8 k^{1/2} \hat{\chi} i\}) &\leq 64 k \hat{\chi}^2 i^2 \sum\nolimits_{Q' \in K_i} \mathcal{H}^k (\bar{Q}' \cap Q_{1/2}(0)) \\
&\leq 2 \sum\nolimits_{l=1}^{q} \sum\nolimits_{Q' \in K_i \cap G_l} \int_{Q' \cap Q_{1/2}(0)} s_l^2 dx' \\
&\leq 2 \int_{Q_{1/2}(0)} s_q^2 dx' + 8 k^{1/2} \hat{\chi}^2 \\
&\leq 14 k^{1/2} \hat{\chi}^2.
\end{aligned}
\tag{5.40}
$$

Letting $q \to \infty$ in (5.40) and using the definition of sup we find after some elementary algebra that (5.31) is true. To prove (5.32) we can now use (5.31) and a standard argument. Indeed observe that if $r > 0$, then

$$
V = \{x' : \limsup_{l \to \infty} s_l(x') - \liminf_{l \to \infty} s_l(x') > r\} \subset \{x' : s^*(x') > r/2\}.
$$

Also V is unchanged if we put any finite number of $a_j = 0$. Using these observations and (5.31) we get $\mathcal{H}^k(V) = 0$. Since $r > 0$ is arbitrary we conclude from our earlier work that $s_l(x') \to s(x')$ as $l \to \infty$ for $\mathcal{H}^k$ almost every $x' \in \mathbb{R}^k$. □

5.2. Construction of examples

Let (T_j) be as in (5.28), $(a_j), \hat{\chi}$ as in (5.30) and $\bar{b}, \psi, (\psi_j), (s_j), s, s^*$, as in Lemma 5.3. Let $\tilde{\psi}_j = \psi_j + \bar{b}$ for $j = 1, 2, \ldots$. For our first example we choose $(a_j)_1^\infty$ in addition to (5.30) so that if

$$
d_m = \sum_{j=1}^{m} a_j, \quad \text{for} \quad m = 1, 2, \ldots,
$$

then

$$|d_m| \le C < \infty \quad \text{and} \quad \lim_{m\to\infty} d_m \quad \text{does not exist.} \tag{5.41}$$

Also put

$$\tilde{s}_l = \sum_{j=1}^{l} a_j \tilde{\psi}_j = s_l + \bar{b} \sum_{j=1}^{l} a_j \quad \text{when} \quad x' \in \mathbb{R}^k. \tag{5.42}$$

From (5.32), (5.41), and $\bar{b} \neq 0$, (thanks to Theorem B), we see that

$$\sum_{j=1}^{\infty} a_j \tilde{\psi}_j \text{ diverges } \mathcal{H}^k \text{ almost everywhere.} \tag{5.43}$$

Also we construct a sequence of functions (L_j) satisfying

$$\begin{aligned} &(a)\ L_1 \equiv 1 \quad \text{and} \quad L_j(x' + e_l) = L_j(x') \quad \text{for } x' \in \mathbb{R}^k, \quad 1 \le l \le k, \quad \text{and} \quad j = 1, 2, \ldots \\ &(b)\ 1/2 \le L_{j+1}/L_j \le 1 \quad \text{and} \quad ||L_{j+1}\breve{||} \le c^*(k) T_j, \quad \text{for} \quad j = 1, 2, \ldots. \end{aligned} \tag{5.44}$$

Moreover we shall make the construction so that

Lemma 5.4. *If $\sigma_m = \sum_{j=1}^m a_j L_j \tilde{\psi}_j$, $m = 1, 2, \ldots,$ then for some $0 < C' = C'(k, \bar{b}, \hat{\chi}) < \infty$, we have*

$$\sup_m |\sigma_m(x')| < C' \text{ for all } x' \in \mathbb{R}^k \quad \text{and} \quad \sum_{j=1}^{\infty} a_j L_j \tilde{\psi}_j \text{ diverges } \mathcal{H}^k \text{ almost every where.} \tag{5.45}$$

Proof. To construct (L_j) we proceed by induction: $L_1 \equiv 1$ and if L_j has been defined so that (5.44) is true for j let K_i be as defined above (5.38) with $q = \infty$ and G_j as defined after (5.32) with $m = j$. If $Q' \in G_j \cap (\cup_i K_i)$, put $L_{j+1} \equiv (1/2) L_j$ on $\bar{Q}'$. Let

$$\begin{aligned} \breve{E}\ &= \bigcup_i \{\bar{Q}' : Q' \in K_i \cap G_j\}, \\ F_1 &= \left\{ x' \in \mathbb{R}^k : d(x', \breve{E}\ ) \ge \frac{1}{4T_j} \right\}, \\ F_2 &= \left\{ x' \in \mathbb{R}^k : d(x', \breve{E}\ ) \ge \frac{1}{8T_j} \right\}. \end{aligned}$$

If $\breve{E} \neq \emptyset$, let $0 \le \theta \in C_0^\infty(Q_{1/2}(0))$ with $|\nabla\theta| \le c'(k)$ and $\int_{Q_{1/2}(0)} \theta(y')dy' \equiv 1$. Let χ_{F_2} denote the characteristic function of F_2 and let $\epsilon = \frac{1}{16k^{1/2}T_j}$. Set

$$\zeta_j(x') = \epsilon^{-k} \int_{\mathbb{R}^k} \theta\left(\frac{x' - y'}{\epsilon}\right) \chi_{F_2}(y')dy' \quad \text{when} \quad x' \in \mathbb{R}^k.$$

Then one easily verifies that

$$\zeta_j \equiv 0 \quad \text{on } \breve{E}\,, \quad \zeta_j \equiv 1 \quad \text{on } F_1, \quad 0 \le \zeta_j \le 1, \quad ||\zeta_j\breve{||} \le c'(k) T_j. \tag{5.46}$$

Put

$$L_{j+1} = (1/2)(\zeta_j + 1)L_j. \tag{5.47}$$

If $\breve{E} = \mathbb{R}^k$, let $L_{j+1} = L_j$. From (5.46), (5.47), the induction hypothesis, and the definition of (T_j) in (5.28) we see that

$$|\nabla L_{j+1}(x')| \leq (1/2)c'(k)T_j + c^*(k)T_{j-1} \leq c^*(k)T_j \quad \text{for} \quad c^*(k) = c'(k). \tag{5.48}$$

The rest of the induction hypothesis is also easily checked using (5.46), (5.47). Thus by induction we have defined $(L_j)_1^\infty$ satisfying (5.44).

To begin the proof of Lemma 5.4 we note that if $x' \in Q' \in K_i \cap G_j$ then $|L_{j+1}^{1/2}\, \tilde{s}_j|$ is uniformly bounded. Indeed from (5.29), (5.30), we have $|s_{l+1} - s_l| \leq \hat{\chi}$ so there exist indices $l_1 < l_2 < \ldots < l_{8i} \leq j$ with $L_{l_m+1}(x') = (1/2)L_{l_m}(x')$. Thus

$$|L_{j+1}^{1/2}(x')\tilde{s}_j(x')| \leq 2^{-4i}\left[(8ik^{1/2}+1)\hat{\chi} + C|\bar{b}|\right] = \tilde{C}. \tag{5.49}$$

To prove (5.45) we use (5.49) and following Wolff (see proof of Lemma 2.12 in [1]) integrate by parts to obtain

$$\begin{aligned} |\sigma_m(x')| &\leq |\sum_{l=1}^{m}(L_l(x') - L_{l+1}(x'))\tilde{s}_l(x')| + c(k)(|\bar{b}|C + \hat{\chi}) \\ &\leq \sum_{l=1}^{\infty}(L_l(x')^{1/2} - L_{l+1}(x')^{1/2})(L_l(x')^{1/2} + L_{l+1}(x')^{1/2})|\tilde{s}_l(x')| + c(k)(|\bar{b}|C + \hat{\chi}) \\ &\leq (\sqrt{2}+1)\tilde{C}\sum_{l=1}^{\infty}(L_l(x')^{1/2} - L_{l+1}(x')^{1/2}) + c(k)(|\bar{b}|C + \hat{\chi}) \leq C' \end{aligned} \tag{5.50}$$

where C' is as in (5.45). To prove the last statement in Lemma 5.4, given $r > 0$ and a cube $Q \subset \mathbb{R}^k$, let rQ denote the cube $\subset \mathbb{R}^k$ with the same center as Q and side length $= r$ times the side length of Q. If $x' \in Q_{1/2}(0)$, we see from the definition of $(L_j), (\zeta_j)$, that either $L_j(x') = L_m(x')$ for some integer m and $j \geq m$, or at least one of $(a), (b)$ is true where

$$\begin{aligned} &(a) \text{ For arbitrary large } m \text{ there is a } Q' \in \bigcup_{i=m}^{\infty} K_i \text{ with } x' \in \frac{5}{4}Q' \\ &(b) \text{ For some } i \text{ and arbitrary large } m \text{ there is a } Q' \in K_i \text{ with} \\ &\text{sidelength } \leq 1/T_m \text{ and } x' \in \frac{5}{4}Q'. \end{aligned} \tag{5.51}$$

Now from (5.40) and basic measure theory it follows that the set of all x' in $Q_{1/2}(0)$ for which either (5.51) (a) or (b) holds has $\mathcal{H}^k$ measure 0. Moreover if $L_j(x')$ is eventually constant then from (5.43) we deduce that $\sum_{j=1}^{\infty}(a_jL_j\tilde{\psi}_j)(x')$ diverges $\mathcal{H}^k$ almost everywhere. □

Lemma 5.4 will be used to construct $\hat{u}$ and for this we need the next lemma.

Lemma 5.5. *For $j = 1, 2, \ldots$, let $a_j = -\frac{1}{4j}$ and define $T_j, \psi_j, \tilde{\psi}_j, G_j$, as in Lemma 5.4 for $j = 1, 2, \ldots$ Also define $s_m, \tilde{s}_m$ relative to $(\psi_j), (\tilde{\psi}_j)$, and the current (a_j) as in (5.42). There is a choice of (L_j) satisfying (5.44) such that if $\tilde{\sigma}_m = 1 + \sum_{j=1}^{m} a_jL_j\tilde{\psi}_j$, then*

$$
\begin{aligned}
&(a')\ \tilde{\sigma}_m > 0 \quad \text{for} \quad m = 1, 2, \ldots \\
&(b')\ \sup\ \tilde{\sigma}_m(x') < c(k) < \infty \quad \text{for all} \quad x' \in \mathbb{R}^k, \\
&(c')\ \sigma(x') = \lim_{m \to \infty} \tilde{\sigma}_m(x') = 0 \quad \text{for } \mathcal{H}^k \text{ almost every} \quad x' \in \mathbb{R}^k.
\end{aligned}
\tag{5.52}
$$

Proof. Lemma 5.5 is essentially a k-dimensional version of Lemma 2.13 in [1]. To begin the proof let i, m be positive integers and for fixed m, let $\mathcal{K}_{im}$ for $i = 1, 2, \ldots,$ be the set of $Q \in G_m$ for which $\max_{\bar{Q}} \tilde{s}_m > i$ and Q is not contained in a $Q' \in \mathcal{K}_{im'}$ for some $m' < m$. Set $L_1 \equiv 1$ so that $\tilde{\sigma}_1 = 1 + a_1 \tilde{\psi}_1$. Next define $(L_m), (\tilde{\sigma}_m)$, and sets of cubes, $\mathcal{F}_{im}$, $\mathcal{H}_{im}$, by induction as follows: Suppose L_m, σ_m have been defined for $m \le j$. Assume also that $\mathcal{F}_{im}, \mathcal{H}_{im} \subset G_m$ have been defined for nonnegative integers $m < j$ and all positive integers i with $\mathcal{F}_{i0} = \emptyset = \mathcal{H}_{i0}$. If i is a positive integer and $Q \in G_j$, we put $Q \in \mathcal{F}_{ij}$ if $\min_{\bar{Q}} \tilde{\sigma}_j < 2^{-i}$ and this cube is not contained in any cube in $\mathcal{F}_{im}$ for some $m < j$. Moreover we put $Q \in \mathcal{H}_{ij}$ if $\min_{\bar{Q}} \tilde{s}_j < -\frac{2^i}{i+1}$ and $\max_{\bar{Q}} L_j > 2^{-i}$. We then define ζ_j, L_{j+1} as in (5.46), (5.47), only now

$$
\breve{E} = \left\{ \bar{Q}' : Q' \in \bigcup_i (\mathcal{F}_{ij} \cup \mathcal{H}_{ij} \cup \mathcal{K}_{ij}) \right\}. \tag{5.53}
$$

Arguing as in (5.48) we then get (5.44). With L_{j+1} defined we put $\tilde{\sigma}_{j+1} = \sum_{m=1}^{j+1} a_m L_m \tilde{\psi}_m$ and after that define $\mathcal{F}_{i(j+1)}, \mathcal{H}_{i(j+1)}$, for all positive integers i. By induction we conclude the definitions of $(L_j), (\tilde{\sigma}_j), (\mathcal{F}_{ij}), (\mathcal{H}_{ij})$.

From the definition of ζ_j, L_{j+1}, in (5.46), (5.47), as regards $\breve{E}$ in (5.53), and the same argument as in the proof of (5.49) in Lemma 5.4 we deduce that

$$
L_j(x')^{1/2} \max(\tilde{s}_j(x'), 0) \text{ is uniformly bounded for all } x' \text{ in } \mathbb{R}^k.
$$

Using this fact and arguing as in (5.50) we get (5.52) (b').

Next if $2^{-(m+1)} \le \min_Q \tilde{\sigma}_j < 2^{-m}$ and $L_{j+1} \le 2^{-m}$ on $Q \in G_j$, then from (5.29) and our choice of (a_l), we see that

$$
\tilde{\sigma}_{j+1} \ge \tilde{\sigma}_j - 2^{-(m+2)} \ge 2^{-(m+2)} \quad \text{on} \quad Q.
$$

Using this observation and the definition of ζ_j, L_{j+1}, in (5.46), (5.47), as regards $\breve{E}$ in (5.53), one can show by induction on m that for a positive integers l,

$$
\text{if } Q \in G_l \text{ and } \min_Q\ \tilde{\sigma}_l < 2^{-m} \text{ then } L_{l+1} < 2^{-m} \text{ on } Q. \tag{5.54}
$$

(5.54) implies (5.52) (a') since $\tilde{\sigma}_1 > 0$ and if $2^{-(m+1)} \le \tilde{\sigma}_l < 2^{-m}$ on $Q \in G_l$, then from (5.54), (5.29), and the definition of (a_j),

$$
\tilde{\sigma}_{l+1} > \tilde{\sigma}_l - 2^{-(m+2)} > 0 \quad \text{on} \quad Q.
$$

Thus (5.52) (a') holds.

It remains to prove (5.52) (c'). To do so we first claim that for $l = 1, 2, \ldots,$

$$
c(k)^{-1}\ L_{l+1}(x') \le L_{l+1}(y') \le c(k)\ L_{l+1}(x')) \tag{5.55}
$$

whenever $x, y \in \frac{5}{4}Q \in G_l$. Indeed from (5.46), (5.47), we have

$$\frac{L_{l+1}(y')}{L_{l+1}(x')} \leq \frac{\prod_{j=1}^{l}(1/2)(1+\zeta_j(y'))}{\prod_{j=1}^{l} 1/2} \leq \frac{\prod_{j=1}^{l}(1/2)(1+c'(k)T_j/T_l)}{\prod_{j=1}^{l} 1/2} \leq c(k).$$

The lower estimate is proved similarly.

To prove (5.52) (c') let E_m denote the set of all $x' \in \mathbb{R}^k$ for which there exist l_1 and l_2 positive integers with $l_1 < l_2$ satisfying

$$\tilde{s}_{l_2}(x') > -\frac{2^m}{2(m+1)} \quad \text{while} \quad \tilde{s}_{l_1}(x') < -\frac{2^m}{m+1} \quad .$$

Since $a_j = -(1/4)j^{-1}$ for $j = 1, 2, \ldots$ it follows that

$$\begin{aligned} \max\left[|s_{l_1}(x')|, |s_{l_2}(x')|\right] &= \max\left[|\tilde{s}_{l_1}(x') + (\bar{b}/4)\sum_{j=1}^{l_1} j^{-1}| \ , \ |\tilde{s}_{l_2}(x') + (\bar{b}/4)\sum_{j=1}^{l_2} j^{-1}|\right] \\ &\geq \frac{2^m \bar{b}}{8(m+1)} \end{aligned} \tag{5.56}$$

for $m \geq 100$. If we let

$$\Gamma := \{x' \in \mathbb{R}^k : x' \in E_m \text{ for infinitely many } m\} \cup \{x' \in \mathbb{R}^k : \limsup_{j\to\infty} \tilde{s}_j(x') > -\infty\}$$

then using (5.56) and (5.31) of Lemma 5.3 we arrive at

$$|\Gamma| = 0. \tag{5.57}$$

Next from induction on m and the definitions of $\mathcal{H}_{lm}, L_{l+1}$, it follows that if $\tilde{s}_l(y') < -\frac{2^m}{m+1}$, $y' \in \bar{Q} \in G_l$, then $L_{l+1}(y') \leq 2^{-m}$. Therefore if $y' \notin \Gamma$, then

$$\lim_{l\to\infty} \tilde{s}_l \ (y') = -\infty \quad \text{and} \quad \lim_{l\to\infty} (\tilde{s}_l \ L_{l+1})(y') = 0. \tag{5.58}$$

Now (5.58), (5.52) (a'), and

$$0 < \tilde{\sigma}_j(y') = 1 + \sum_{l \leq j}(L_l - L_{l+1})\tilde{s}_l(y') + s_j L_{j+1}(y') \tag{5.59}$$

imply that if $y' \notin \Gamma$, then it must be true that $\lim_{j\to\infty} \tilde{\sigma}_j(y')$ exists and is non-negative.

Suppose this limit is positive. In this case we observe from the definition of (a_l), (5.28), (5.29), (5.44), that if $\beta_l = T_l^{-1}(||\breve{\tilde{\sigma}_l}|| + ||\breve{\tilde{s}_l}||), l = 1, 2, \ldots$ then

$$\sup_l \beta_l \leq c(k) \quad \text{and} \quad \beta_l \to 0 \quad \text{as} \quad l \to \infty. \tag{5.60}$$

It follows from (5.60), (5.52) (a'), and the facts $\lim_{j\to\infty} \tilde{s}_j(y') = -\infty$, $\tilde{\sigma}(y') > 0$, that

$$\sup_j\{\max_{\frac{5}{4}\bar{Q}} \tilde{s}_j : y' \in Q \in G_j\} < \infty \quad \text{and} \quad \inf_j\{\min_{\frac{5}{4}\bar{Q}} \tilde{\sigma}_j : y' \in G_j\} > 0. \tag{5.61}$$

So y' belongs to at most a finite number of $\frac{5}{4}Q$ with $Q \in \mathcal{K}_{lm} \cup \mathcal{F}_{lm}$ for $l, m = 1, 2, \ldots$. Now if $y' \in \frac{5}{4}\bar{Q}$, $Q \in \mathcal{H}_{i(m+1)}$, then from (5.55) and $y' \notin \Gamma$, (5.60), we find for $i \geq i_0(y')$ and $m \geq m_0(y')$, sufficiently large that

$$c(k)L_i(y') \geq 2^{-m} \quad \text{and} \quad \tilde{s}_{i'}(y') < -\frac{2^{m-1}}{m} \quad \text{for} \quad i' \geq i. \tag{5.62}$$

Using (5.58), (5.62), we deduce the existence of an increasing sequence (i_l) for $l \geq l_0$ so that

$$L_{i_l}(y') = 2^{-l} \quad \text{and} \quad c(k)\tilde{s}_{i_{l+1}}(y') \leq -\frac{2^{-l}}{l+1}.$$

It then follows from (5.59) that $\tilde{\sigma}(y') = -\infty$ which contradicts $\tilde{\sigma}(y') > 0$. Thus $\tilde{\sigma}(y') = 0$ for $\mathcal{H}^k$ almost every $y' \in \mathbb{R}^k$ and the proof of Lemma 5.5 is complete. □

5.3. Final proof of Theorem C

To finish the proof of Theorem C we again follow Wolff in [1] and use Lemmas 5.2, 5.4, and 5.5 to construct examples. Let $T_1 = 1$ and by induction suppose $T_2, \ldots, T_l$ have been chosen, as in (5.28). Let $(a_j), (\psi_j), (\tilde{\psi}_j)$ be as in Lemmas 5.4, 5.5. First we define $\sigma_j, \tilde{\sigma}_j, 1 \leq j \leq l$, relative to these sequences, and after that $L_{j+1}, \sigma_{j+1}, \tilde{\sigma}_{j+1}$. Next we define T_{l+1} satisfying several conditions: First suppose (5.28) is valid for $j = l$. Let $g = \sigma_l$, or $\tilde{\sigma}_l, f = a_{l+1}L_{l+1}$, and define $q = \tilde{\psi}_{l+1}$ relative to T_{l+1}. Also suppose that

$$\max(||f||_\infty, ||g||_\infty, ||q||_\infty, ||f\breve{||}, ||g\breve{||}, T_{l+1}^{-1}||q\breve{||}) \leq M \tag{5.63}$$

where $M = M(T_1, \ldots, T_l)$ is a constant. Next apply Lemma 5.2 with M as in (5.5), $3\epsilon = 2^{-(l+1)}$, obtaining $A = A_l$ and $\nu_0 = \nu_0(p, n, k, l, M)$ so that (5.6)–(5.8) are valid. Choosing T_{l+1} still larger if necessary we may assume that $T_{l+1} > \nu_0$ and $A_l T_{l+1}^{-\alpha} < \frac{1}{2}A_{l-1}T_l^{-\alpha}$ where α is as in Lemma 5.2. By induction we now get (σ_l) or $(\tilde{\sigma}_l)$ as in Lemma 5.4 or Lemma 5.5. Moreover if $\sigma'_j \in \{\sigma_j, \tilde{\sigma}_j\}, j = 1, 2, \ldots$, then in case (i) of Lemma 5.2 we have

$$|\hat{\sigma}'_{j+1}(x) - \hat{\sigma}'_j(x)| < 2^{-(j+1)} \quad \text{when } |x''| > A_j T_{j+1}^{-\alpha}, \tag{5.64}$$

and

$$|\hat{\sigma}'_{j+1}(x) - \sigma'_j(x')| < 2^{-(j+1)} + |a_{j+1}| \quad \text{when } |x''| < A_j T_{j+1}^{-\alpha}. \tag{5.65}$$

From (5.64) we see that $(\hat{\sigma}'_{j+1})$ converges uniformly on compact subsets of $\mathbb{R}^n \setminus \mathbb{R}^k$ to a p-harmonic function $\hat{\sigma}'$ satisfying

$$|\hat{\sigma}'(x) - \hat{\sigma}'_l(x)| < 2^{-l} \quad \text{when } |x''| > A_l T_{l+1}^{-\alpha}. \tag{5.66}$$

Using (5.65)-(5.66), and the triangle inequality we also have for $A_{l+1}T_{l+2}^{-\alpha} < |x''| < A_l T_{l+1}^{-\alpha}$ that

$$\begin{aligned} |\hat{\sigma}'(x) - \sigma_l'(x')| &\leq |\hat{\sigma}'(x) - \hat{\sigma}_{l+1}'(x)| + |\hat{\sigma}_{l+1}'(x) - \hat{\sigma}_l'(x')| \\ &< 2^{-(l+1)} + 2^{-l} + |a_{l+1}|. \end{aligned} \tag{5.67}$$

From (5.67) and our choice of (a_l) we see for ζ as in Theorem C and (σ_l), as in Lemma 5.4 that $\lim_{x''\to 0} \hat{\sigma}'(x', \zeta(x''))$ does not exist for $\mathcal{H}^k$ almost every $x' \in \mathbb{R}^k$ while if $(\tilde{\sigma}_l)$ is as in Lemma 5.5, then $\hat{\sigma}' > 0$ on $\mathbb{R}^n \backslash \mathbb{R}^k$ and $\lim_{x''\to 0} \hat{\sigma}'(x', \zeta(x'')) = 0$ for $\mathcal{H}^k$ almost every $x' \in \mathbb{R}^k$. Moreover from uniform boundedness of $(\sigma_l), (\tilde{\sigma}_l)$, and the maximum principle for p-harmonic functions we deduce that $\hat{\sigma}'$ is bounded. To complete the proof of Theorem C in case (i), put $\hat{\sigma}' = \hat{u}$ or $\hat{\sigma}' = \hat{v}$ depending on whether (a_j) as in Lemma 5.4 or Lemma 5.5, respectively, was used to construct $\hat{\sigma}'$.

The same argument gives Theorem C in case (ii). This finishes the proof of Theorem C.

Remark 5.6. As mentioned in Remark 1.8, there is no analogue of Theorems B or C when $1 < p \leq n-k$. To prove this assertion for $\hat{u}$, given $0 < r < 1/10$, let

$$\phi(x) = \max\left[1 + \frac{\log(|x''|)}{\log(1/r)}\ ,0\right], \quad \text{for} \quad x \in \mathbb{R}^n \backslash \mathbb{R}^k.$$

If $y' \in \mathbb{R}^k$ and $\theta \in C_0^\infty(B(y', 1))$, then $\phi\theta$ can be used as a test function in the definition of p-harmonicity for $\hat{u}$ in (1.3). Thus

$$0 = \int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \theta dx + \int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \phi dx = I_1 + I_2. \tag{5.68}$$

To estimate I_1, I_2, we observe from uniform boundedness of $\hat{u}$ that $\hat{u}\phi\theta$ can also be used as a test function in (1.3). Doing this and using Hölder's inequality in a Caccioppoli type argument, we obtain

$$\int |\nabla \hat{u}|^p \ (\phi\theta)^p \ dx \ \leq \ c(p,n,k)||\hat{u}||_\infty^p \int |\nabla(\phi\theta)|^p dx. \tag{5.69}$$

Clearly, $0 \leq \phi \leq 1$ in $B(y', 1)$ and $\phi(\cdot, r) \to 1$ uniformly as $r \to 0$ on compact subsets of $B(y', 1) \backslash \{0\}$. Moreover

$$|\nabla \phi(x)| \leq |x''|^{-1} [\log(1/r)]^{-1} \quad \text{for} \quad x \in B(y', 1) \cap \{x : |x''| \geq r\}. \tag{5.70}$$

Using these inequalities in (5.69) and the Lebesgue monotone convergence theorem we get for $1 < p \leq n-k$,

$$\begin{aligned} \int |\nabla \hat{u}|^p \ \theta^p \ dx \ &\leq \ c'(p,n,k,\theta)||\hat{u}||_\infty^p \left[1 + \limsup_{r\to 0} \left\{ [\log(1/r)]^{-p} \int_r^1 \rho^{n-k-p-1} d\rho \right\}\right] \\ &\leq c''(p,n,k,\theta)||\hat{u}||_\infty^p \left[1 + \lim_{r\to 0} (\log(1/r))^{1-p}\ \right] \\ &= c''(p,n,k,\theta)||\hat{u}||_\infty^p \end{aligned} \tag{5.71}$$

Armed with (5.70), (5.71), and Hölder's inequality we can now estimate I_1 and I_2 in (5.68). We find that

$$I_2 \to \int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle dx \quad \text{and} \quad I_1 \to 0 \quad \text{as} \quad r \to 0.$$

We conclude from this inequality, (5.68), and (5.71) that $\hat{u}$ extends to a uniformly bounded p-harmonic function on $\mathbb{R}^n$. Using Liouville's theorem for bounded entire p-harmonic functions (an easy consequence of (2.1) (b)) we conclude that $\hat{u} =$ constant. Thus Theorem C does not have an analogue when $1 < p \leq n-k$ for $\hat{u}$. A similar argument yields that Theorem B does not have an analogue and that Theorem C does not have an analogue for $\hat{v}$.

6. $\mathcal{A}$-Harmonic functions

6.1. Definition and basic properties of $\mathcal{A}$-harmonic functions

In this subsection we introduce $\mathcal{A}$-harmonic functions and discuss their basic properties.

Definition 6.1. Let $p, \alpha \in (1, \infty)$ and

$$\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_n) \ : \ \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n,$$

be such that $\mathcal{A} = \mathcal{A}(\eta)$ has continuous partial derivatives in η_k for $k = 1, 2, ..., n$ on $\mathbb{R}^n \setminus \{0\}$. We say that the function $\mathcal{A}$ belongs to the class $M_p(\alpha)$ if the following conditions are satisfied whenever $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$:

(i) Ellipticity : $\alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^n \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta)\xi_i\xi_j$ and $\sum_{i=1}^n |\nabla \mathcal{A}_i(\eta)| \leq \alpha \ |\eta|^{p-2}$,

(ii) Homogeneity : $\mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\eta/|\eta|)$.

We put $\mathcal{A}(0) = 0$ and note that Definition 6.1 (i) and (ii) implies that

$$\begin{aligned} &(a') \ (|\eta| + |\eta'|)^{p-2} \ |\eta - \eta'|^2 \leq c(p, n, k, \alpha)\langle \mathcal{A}(\eta) - \mathcal{A}(\eta'), \eta - \eta' \rangle, \\ &(b') \ |\mathcal{A}(\eta) - \mathcal{A}(\eta')| \leq c(p, n, k, \alpha)(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|, \end{aligned} \tag{6.1}$$

whenever $\eta, \eta' \in \mathbb{R}^n \setminus \{0\}$.

Definition 6.2. Let $p \in (1, \infty)$ and let $\mathcal{A} \in M_p(\alpha)$ for some $\alpha \in (1, \infty)$. Given an open set O we say that u is $\mathcal{A}$-harmonic in O provided $u \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int \langle \mathcal{A}(\nabla u(y)), \nabla \theta(y) \rangle \ dy = 0 \quad \text{whenever} \quad \theta \in W_0^{1,p}(G). \tag{6.2}$$

We say that u is an $\mathcal{A}$-subsolution ($\mathcal{A}$-supersolution) in O if $u \in W^{1,p}(G)$ whenever G is as above and (6.2) holds with $=$ replaced by $\leq$ $(\geq)$ whenever $\theta \in W_0^{1,p}(G)$ with $\theta \geq 0$. As a short notation for (6.2) we write $\nabla \cdot \mathcal{A}(\nabla u) = 0$ in O.

Remark 6.3. We remark for $O, \mathcal{A}, p, u,$ as in Definition 6.2 that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is the composition of a translation and a dilation then

$$\hat{u}(z) = u(F(z)) \text{ is } \mathcal{A}\text{-harmonic in } \ F^{-1}(O).$$

Moreover, if $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ is the composition of a translation, a dilation, and a rotation then

$$\tilde{u}(z) = u(\tilde{F}(z)) \text{ is } \tilde{\mathcal{A}}\text{-harmonic in } \tilde{F}^{-1}(O) \text{ for some } \tilde{\mathcal{A}} \in M_p(\alpha).$$

We note that $\mathcal{A}$-harmonic PDEs have been studied in [9]. Also dimensional properties of the Radon measure associated with a positive $\mathcal{A}$-harmonic function u, vanishing on a portion of the boundary of O, have been studied in [4, 13–15], see also [6, 5]. As mentioned in the introduction our goal in this section is to discuss validity of Theorems B and C, for $\mathcal{A}$-harmonic functions. To this end we state

Lemma 6.4. *Lemmas 2.1–2.4 are valid with p-harmonic replaced by $\mathcal{A}$-harmonic. However constants may also depend on α.*

Proof. References for proofs of Lemmas 2.1–2.4 were purposely chosen to be references for proofs in the $\mathcal{A}$-harmonic setting. □

Next given $p, 1 < p < \infty$, suppose $f : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ satisfies:

$$\begin{aligned}
&(a) \quad f(t\eta) = t^p f(\eta) \quad \text{when} \quad t > 0 \quad \text{and} \quad \eta \in \mathbb{R}^n. \\
&(b) \quad \text{There exists } \hat{\alpha} \geq 1 \text{ such that if } \eta, \xi \in \mathbb{R}^n \setminus \{0\}, \text{ then} \\
&\qquad \hat{\alpha}^{-1} \, |\xi|^2 |\eta|^{p-2} \leq \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \eta_i \partial \eta_j}(\eta) \; \xi_i \; \xi_j \leq \hat{\alpha} \; |\xi|^2 |\eta|^{p-2}. \\
&(c) \quad \text{There exists } \alpha' \geq 1 \text{ such that for } \mathcal{H}^n - \text{almost every } \eta \in B(0,2) \setminus B(0,1/2), \\
&\qquad \sum_{i,j,k=1}^{n} \left| \frac{\partial^3 f}{\partial \eta_i \partial \eta_j \partial \eta_k}(\eta) \right| \leq \alpha'.
\end{aligned} \tag{6.3}$$

We note that $\mathcal{A} = \nabla f \in M_p(\alpha)$ for some $\alpha \in (1, \infty)$.

Lemma 6.5. *Lemmas 2.5 and 2.6, are valid when $k = n - 1$ and $p > 2$, or $k = 1$ and $p > n - 1$ with p-harmonic replaced by $\mathcal{A}$-harmonic whenever $A \in M_p(\alpha)$. Constants may also depend on α. If $1 < k \leq n - 2$ and $n \geq 3$ with $p > n - k$, Lemmas 2.5 and 2.6 are valid when $A = \nabla f$ and (6.3) holds. Constants may also depend on $\hat{\alpha}, \alpha'$.*

Proof. References given for Lemmas 2.5 and 2.6 provide proofs for Lemma 6.5. □

An $\mathcal{A}$-harmonic Martin function relative to $z \in \mathbb{R}^k$ is defined as in Definition 1.5 with p-harmonic replaced by $\mathcal{A}$-harmonic. Using Lemmas 6.4 and 6.5 one can now argue as at the beginning of Section 4 to show the existence of an $\mathcal{A}$-harmonic Martin function satisfying (1.4) $(a), (b)$, with p-harmonic replaced by $\mathcal{A} \in M_p(\alpha)$-harmonic when $k = n - 1$ with $p > 2$ and $k = 1$ with $p > n - 1$, or under the additional assumption that $\mathcal{A} = \nabla f, f$ as in (6.3), when $1 < k \leq n - 2$ with $p > n - k$.

Next for fixed $p, n, k, \alpha, \mathcal{A} = \nabla f$ as in (6.3), and either $1 \leq k \leq n - 2$ and $p > n - k$ or $k = n - 1$ and $p > 2$, we claim for given $F \in R^{1,p}(S(\tau))$, that there exists a unique $\mathcal{A} = \nabla f$-harmonic function $\hat{v}$ on $\mathbb{R}^n \setminus \mathbb{R}^k$ when $1 \leq k \leq n - 2$ and $p > n - k$ and on $\mathbb{R}^n_+$ when $k = n - 1$ and $p > 2$, with

$$\hat{v}(z+\tau e_i) = \hat{v}(z), 1 \le i \le k, \text{ for } z \in \mathbb{R}^n \backslash \mathbb{R}^k \text{ or } \mathbb{R}^n_+, \text{ satisfying } \hat{v} - F \in R_0^{1,p}(S(\tau)). \tag{6.4}$$

In fact the usual minimization argument yields that if h in $R^{1,p}(S(\tau))$ with $h - F \in R_0^{1,p}(S(\eta))$, then

$$\int_{S(\tau)} f(\nabla \hat{v}) dx \le \int_{S(\tau)} f(\nabla h) dx. \tag{6.5}$$

Also the same argument as in Lemma 2.7 yields

Lemma 6.6. *Let $p, n, k, \tau, F, f, \hat{v}$, be as above. Given $t > 0$, let $Z(t) = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x''| = t\}$ when $1 \le k \le n-2$, and $Z(t) = \{x \in \mathbb{R}^n_+ : x_n = t\}$ when $k = n-1$. There exists $\delta \in (0,1), c \ge 1$, depending on $p, n, k, \hat{\alpha}, \alpha' \in (0,1)$ and $\xi \in \mathbb{R}$ such that*

$$|\hat{v}(z) - \xi| \le \liminf_{t \to 0} \left(\max_{Z(t)} \hat{v} - \min_{Z(t)} \hat{v} \right) \left(\frac{\tau}{|z'''|} \right)^{\delta}. \tag{6.6}$$

Here $z''' = z''$ when $z = (z', z'') \in \mathbb{R}^n \backslash \mathbb{R}^k$ and $z''' = z_n$ when $z \in \mathbb{R}^{n-1}_+$.

Proof. From (6.6) we observe as in Lemma 2.7 that $\max_{Z(t)} \hat{v}$, $-\min_{Z(t)} \hat{v}$, are nonincreasing functions of t on $(0, \infty)$. This fact and Harnack's inequality for $\mathcal{A}$-harmonic functions imply Lemma 6.6. □

6.2. Theorems B and C for $\mathcal{A}=\nabla f$-harmonic functions

Finally, we state several modest propositions:

Proposition 6.7. *Fix p, n, k with either $1 \le k \le n-2$ and $p > n-k$, or $k = n-1$ and $p > 2$. Let u be the $\mathcal{A} = \nabla f$-harmonic Martin function in (1.4) relative to 0, where f is as in (6.3). If $\sigma < k$, then Theorem B, C are valid with p-harmonic replaced by $\mathcal{A}$-harmonic.*

Proof. Using Lemmas 6.4–6.6 we can give a proof of Theorem B in Proposition 6.7 by essentially copying the proof of Theorem B for p-harmonic functions. To get Theorem C we note that inequality (6.1) can be used in place of (5.2), (5.3) to obtain an analogue of Lemma 5.1 for $\mathcal{A} = \nabla f$-harmonic functions. Using this analogue, Theorem A for $\mathcal{A}$-harmonic functions, as well as Lemmas 6.4-6.6, one now obtains an analogue of Lemma 5.2 in the $\mathcal{A} = \nabla f$-harmonic setting. The rest of the proof of Theorem B follows from this analogue and lemmas on gap series in Subsection 5.1. □

Proposition 6.8. *Let p, n, k, f, u, σ, be as in Proposition 6.7. There exists $\epsilon > 0$ depending only on $p, n, k, \hat{\alpha}, \alpha'$ such that if*

$$\sum_{i,j=1}^{n} \left| \frac{\partial^2 f}{\partial \eta_i \partial \eta_j} - |\eta|^{p-4} ((p-2)|\eta_i \eta_j + \delta_{i,j} |\eta|^2) \right| < \epsilon \tag{6.7}$$

whenever $|\eta| = 1$, then $\sigma < k$.

Proof. If Proposition 6.8 is false there exists a sequence (u_j) of $\mathcal{A}_j = \nabla f_j$-harmonic Martin functions with $u_j(e_n) = 1$, that are $-\sigma_j$ homogeneous on $\mathbb{R}^n \backslash \{0\}$ where $\sigma_j \geq k$ for $j = 1, 2, \ldots$. Also f_j satisfies (6.3) for a fixed $\hat{\alpha}, \alpha'$ and (6.7) with ϵ replaced by $\epsilon_j, \epsilon_j \to 0$ as $j \to \infty$. To get a contradiction observe from Lemmas 2.1–2.3 for $\mathcal{A}$-harmonic functions, $u_j(e_n) = 1$, and $-\sigma_j$ homogeneity of each u_j that a subsequence of (u_j) say (u_{j_l}) converges uniformly on compact subsets of $\mathbb{R}^n \backslash \{0\}$ to a $-\tilde{\sigma}$ homogeneous function u on $\mathbb{R}^n \backslash \{0\}$ with $\tilde{\sigma} \geq k$. Also $u \geq 0$, $u(e_n) = 1$, and u is Hölder continuous on $\mathbb{R}^n \backslash \{0\}$ with $u = 0$ on either $\mathbb{R}^k$ or the complement of $\mathbb{R}^n_+$. Choosing subsequences of the subsequence if necessary we see from Lemma 2.4 that we may also assume ∇u_{j_l} converges uniformly on compact subsets of either $\mathbb{R}^n \backslash \mathbb{R}^k$ or $\mathbb{R}^n_+$ as $l \to \infty$ to ∇u. Moreover from (6.3) we observe that each component of (∇f_j) converges locally uniformly in $C^1(\mathbb{R}^n)$ as $j \to \infty$ to a component of $\nabla(|\eta|^p/p)$. An easy argument using these facts, then gives that u is a p-harmonic Martin function and thereupon that

$$-\langle \nabla u(e_n), e_n \rangle = \tilde{\sigma} = -\lim_{l \to \infty} \langle \nabla u_{j_l}, e_n \rangle = \lim_{l \to \infty} \sigma_{j_l} \geq k,$$

which is a contradiction to $\tilde{\sigma} < k$ as shown in (4.3). □

6.3. $\mathcal{A}=\nabla f$-subsolutions in $\mathbb{R}^n \setminus \mathbb{R}^k$ when $f(\eta)=p^{-1}(|\eta| + \langle a, \eta \rangle)^p$

For fixed $p, n \geq 2, a \in \mathbb{R}^n$ with $|a| < 1$, let $q(\eta) = |\eta| + \langle a, \eta \rangle$ for $\eta \in \mathbb{R}^n$. In this subsection we study the if part of Proposition 6.7 and dependence on ϵ in Proposition 6.8 when $\mathcal{A} = \nabla f$ and $f(\eta) = p^{-1} q^p(\eta)$ for $\eta \in \mathbb{R}^n$. To avoid confusion in calculations we write $Dq, D^2 q$ for ∇q and the n by n matrix of second derivatives of q with respect to η. We note that

$$\begin{aligned} q(\eta) &= |\eta| + \langle a, \eta \rangle, \\ Dq(\eta) &= \frac{\eta}{|\eta|} + a, \\ D^2 q(\eta) &= \frac{1}{|\eta|}\left(I - \frac{\eta}{|\eta|} \otimes \frac{\eta}{|\eta|} \right). \end{aligned}$$

Thus

$$Dq(\nabla u) D^2 u(x) \tilde{D}q(\nabla u) = \frac{\nabla u}{|\nabla u|} D^2 u \frac{\tilde{\nabla} u}{|\nabla u|} + 2 \frac{\nabla u}{|\nabla u|} D^2 u \tilde{a} + a D^2 u \tilde{a}$$

where $\tilde{D}q, \tilde{a}, \tilde{\nabla} u$, denotes the $n \times 1$ transpose of $Dq, a, \nabla u$, considered respectively as $n \times 1$ row matrices. Also

$$q \ \operatorname{trace}\big(D^2 q(\nabla u) D^2 u\big) = \left(1 + \frac{\langle a, \nabla u \rangle}{|\nabla u|}\right)\left(\Delta u - \frac{\nabla u}{|\nabla u|} D^2 u \frac{\tilde{\nabla} u}{|\nabla u|}\right).$$

Finally we arrive at

$$\begin{aligned}
q^{2-p}(\nabla u)\nabla\cdot(Df(\nabla u)) &= q^{2-p}(\nabla u)\ \nabla\cdot\left(D\left(\frac{1}{p}q^p\right)(\nabla u)\right)\\
&= (p-1)Dq(\nabla u)D^2u(x)\tilde{D}q(\nabla u) + q(\nabla u)\ \operatorname{trace}\left(D^2q(\nabla u)D^2u\right)\\
&= \left(p-2-\frac{\langle a,\nabla u\rangle}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|}D^2u\frac{\tilde{\nabla} u}{|\nabla u|}\\
&\quad + (p-1)\left(2\frac{\nabla u}{|\nabla u|}\ D^2u\ \tilde{a} + aD^2u\tilde{a}\right) + \left(1+\frac{\langle a,\nabla u\rangle}{|\nabla u|}\right)\Delta u.
\end{aligned} \tag{6.8}$$

As in Section 4 we rewrite (6.8) when $u = u(x', x''), x = (x', x'') \in \mathbb{R}^n$, with $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^{n-k}$. Also $t = |x'|$, $s = |x''|$, $r^2 = t^2 + s^2$, and $a = (a', a'')$. Similarly, $\nabla = (\nabla', \nabla'')$ but we will often write $\nabla' u$ or $\nabla'' u$, in which we regard each vector as $n \times 1$ row vectors. For example $\nabla' u(x) = \left(\frac{\partial u}{\partial x'_1}, \ldots, \frac{\partial u}{\partial x'_k}, 0, \ldots, 0\right)$. Likewise $\nabla' \times \nabla''$, $\nabla'' \times \nabla'$, $\nabla' \times \nabla'$, $\nabla'' \times \nabla'$, are $n \times n$ matrix operators. So $\nabla' \times \nabla''$ is the operator matrix whose i th row and j th column is $\frac{\partial^2}{\partial x'_i \partial x''_j}$ when $1 \le i \le k, k+1 \le j \le n$. All other entries are zero. Next the $k \times k$ and $n-k \times n-k$ identity matrices, denoted I', I'' are regarded as $n \times n$ matrices. So $I' = (\delta'_{ij}), I'' = (\delta''_{ij})$, where $\delta'_{ii} = 1$ for $1 \le i \le k$ and $\delta''_{ii} = 1$ for $k+1 \le i \le n$. Other entries in these matrices are zero. Finally $x' \otimes x', x'' \otimes x', x'' \otimes x'', x' \otimes x''$, are considered as $n \times n$ matrices. Using this notation it follows from the chain rule as in (3.1) - (3.3) that

$$\begin{aligned}
\nabla'' u &= \frac{u_s}{s}x'' \quad \text{and} \quad \langle \nabla'' u, a\rangle = \frac{u_s}{s}\langle x'', a''\rangle,\\
\nabla' u &= \frac{u_t}{t}x', \ \text{and}\ \langle a, \nabla' u\rangle = \frac{u_t}{t}\langle a', x'\rangle,\\
\nabla'\nabla' u &= \frac{u_t}{t}I' + \left(\frac{u_{tt}}{t^2} - \frac{u_t}{t^3}\right)x' \otimes x',\\
\nabla''\nabla' u &= \frac{u_{st}}{st}x'' \otimes x',\\
\nabla'\nabla'' u &= \frac{u_{st}}{st}x' \otimes x'',\\
\nabla''\nabla'' u &= \frac{u_s}{s}I'' + \left(\frac{u_{ss}}{s^2} - \frac{u_s}{s^3}\right)x'' \otimes x''.
\end{aligned}$$

Then

$$\nabla' u\nabla'\nabla' u\tilde{\nabla}' u = \frac{u_t^2}{t^2}\left(\frac{u_t}{t}t^2 - \frac{u_t}{t^3}t^4 + \frac{u_{tt}}{t^2}t^4\right) = u_t^2 u_{tt}.$$

Similarly $\nabla'' u\nabla''\nabla'' u\tilde{\nabla}'' u = u_s^2 u_{ss}$ and $\nabla' u\nabla''\nabla' u\tilde{\nabla}'' u = \frac{u_t}{t}\frac{u_s}{s}\frac{u_{st}}{st}t^2s^2 = u_s u_t u_{st}$. This gives

$$\nabla u D^2 u\tilde{\nabla} u = u_t^2 u_{tt} + 2u_t u_s u_{st} + u_s^2 u_{ss}. \tag{6.9}$$

Observe that

$$\begin{aligned}\nabla u D^2 u \tilde{a} &= \nabla' u \nabla' \nabla' u \tilde{a'} + \nabla'' u \nabla'' \nabla' u \tilde{a'} + \nabla' u \nabla' \nabla'' u \tilde{a''} + \nabla'' u \nabla'' \nabla'' u \tilde{a''} \\ &= \left(\frac{u_s u_{st}}{t} + \frac{u_{tt} u_t}{t}\right)\langle x', a'\rangle + \left(\frac{u_t u_{st}}{s} + \frac{u_{ss} u_s}{s}\right)\langle x'', a''\rangle.\end{aligned} \tag{6.10}$$

The next term is

$$\begin{aligned} a D^2 u \tilde{a} &= a' \nabla' \nabla' u \tilde{a'} + 2a'' \nabla'' \nabla' u \tilde{a'} + a'' \nabla'' u \tilde{a''} \\ &= \frac{u_t}{t}|a'|^2 + \left(\frac{u_{tt}}{t^2} - \frac{u_t}{t^3}\right)(x' \cdot a')^2 \\ &\quad + 2\langle x', a'\rangle\langle x'', a''\rangle \frac{u_{st}}{st} + \frac{u_s}{s}|a''|^2 + \left(\frac{u_{ss}}{s^2} - \frac{u_s}{s^3}\right)\langle x'', a''\rangle^2. \end{aligned} \tag{6.11}$$

Finally we calculate $\Delta u = \text{trace}(D^2 u) = \text{trace}(\nabla' \nabla' u) + \text{trace}(\nabla'' \nabla'' u)$:

$$\Delta u = \frac{u_t}{t}(k-1) + u_{tt} + \frac{u_s}{s}(n-k-1) + u_{ss}. \tag{6.12}$$

Substituting into (6.8) we get (where all derivatives on the right hand side are with respect to (s, t))

$$\begin{aligned} & q^{2-p}(\nabla u)\ \nabla \cdot \left(D\left(\frac{1}{p} q^p\right)(\nabla u)\right) \\ &= \left(p - 2 - \frac{\langle x', a'\rangle}{|\nabla u| t} u_t - \frac{\langle x'', a''\rangle}{|\nabla u| s} u_s\right) \frac{(u_t^2 u_{tt} + 2u_t u_s u_{st} + u_s^2 u_{ss})}{|\nabla u|^2} \\ &\quad + 2(p-1)\left[\left(\frac{u_s u_{st}}{|\nabla u| t} + \frac{u_{tt} u_t}{|\nabla u| t}\right)\langle x', a'\rangle + \left(\frac{u_t u_{st}}{|\nabla u| s} + \frac{u_{ss} u_s}{|\nabla u| s}\right)\langle x'', a''\rangle\right] \\ &\quad + (p-1)\left[\frac{u_t}{t}|a'|^2 + \left(\frac{u_{tt}}{t^2} - \frac{u_t}{t^3}\right)\langle x', a'\rangle^2 + 2\langle x', a'\rangle\langle x'', a''\rangle \frac{u_{st}}{st} + \frac{u_s}{s}|a''|^2 + \left(\frac{u_{ss}}{s^2} - \frac{u_s}{s^3}\right)\langle x'', a''\rangle^2\right] \\ &\quad + \left(1 + \frac{\langle x', a'\rangle u_t}{|\nabla u| t} + \frac{\langle x'', a''\rangle u_s}{|\nabla u| s}\right)\left(\frac{u_t}{t}(k-1) + u_{tt} + \frac{u_s}{s}(n-k-1) + u_{ss}\right). \end{aligned} \tag{6.13}$$

We use subsolutions of (3.3) when $1 \le k \le n-2$ and $p > n-k$ to study (6.13). For this purpose recall the notation in Section 3 and let $u = s^{\tilde{\beta}}/r^{(\tilde{\lambda}+\tilde{\beta})}$ where

$$\tilde{\beta} = (1+\delta)\beta, \quad \tilde{\lambda} = (1+\delta)\lambda, \quad \text{and} \quad \beta = \frac{p-n+k}{p-1}$$

with $\delta \ge 0$ and $\lambda \ge \chi$, (χ as in Theorem A). From (3.2), (3.3), (3.7), (3.11), with β, λ, replaced by $\tilde{\beta}, \tilde{\lambda}$, we deduce that

$$|\nabla u|^{2-p}\ \nabla \cdot (|\nabla u|^{p-2} \nabla u) = \frac{u\ (A(\tilde{\lambda}, \tilde{\beta})\tilde{\lambda}^2 s^4 + B(\tilde{\lambda}, \tilde{\beta}) s^2 t^2 + C(\tilde{\lambda}, \tilde{\beta})\tilde{\beta}^3 t^4)}{s^2 r^2 (\tilde{\lambda}^2 s^2 + \tilde{\beta}^2 t^2)} \tag{6.14}$$

Here $\tilde{A} = A(\tilde{\lambda}, \tilde{\beta}), \tilde{B} = B(\tilde{\lambda}, \tilde{\beta}), \tilde{C} = C(\tilde{\lambda}, \tilde{\beta})$, are defined as in (3.12) with λ, β replaced by $\tilde{\lambda}, \tilde{\beta}$. Using (6.14) we rewrite (6.13) with $u = s^{\tilde{\beta}}/r^{\tilde{\lambda}+\tilde{\beta}}$ as

$$q^{2-p}(\nabla u)\ \nabla\cdot(D\left(\frac{1}{p}q^p\right)(\nabla u)) = \frac{u(\tilde{A}\tilde{\lambda}^2 s^4 + \tilde{B}\tilde{\beta}s^2t^2 + \tilde{C}\tilde{\beta}^3 t^4)}{s^2r^2(\tilde{\lambda}^2 s^2 + \tilde{\beta}^2 t^2)} + E_1 + E_2 + E_3 + E_4 \tag{6.15}$$

where

$$E_1 = -\left(\frac{\langle x', a'\rangle}{|\nabla u| t}\ u_t + \frac{\langle x'', a''\rangle}{|\nabla u| s}\ u_s\right)\frac{(u_t^2u_{tt} + 2u_tu_su_{st} + u_s^2u_{ss})}{|\nabla u|^2}, \tag{6.16}$$

$$E_2 = 2(p-1)\left[\left(\frac{u_su_{st}}{|\nabla u|t} + \frac{u_{tt}u_t}{|\nabla u|t}\right)\langle x', a'\rangle + \left(\frac{u_tu_{st}}{|\nabla u|s} + \frac{u_{ss}u_s}{|\nabla u|s}\right)\langle x'', a''\rangle\right], \tag{6.17}$$

$$E_3 = (p-1)\left[\frac{u_t}{t}|a'|^2 + \left(\frac{u_{tt}}{t^2} - \frac{u_t}{t^3}\right)\langle x', a'\rangle^2 + 2\langle x', a'\rangle\langle x'', a''\rangle\frac{u_{st}}{st} + \frac{u_s}{s}|a''|^2 + \left(\frac{u_{ss}}{s^2} - \frac{u_s}{s^3}\right)\langle x'', a''\rangle^2\right], \tag{6.18}$$

and

$$E_4 = \left(\frac{\langle x', a'\rangle u_t}{|\nabla u| t} + \frac{\langle x'', a''\rangle u_s}{|\nabla u| s}\right)\left(\frac{u_t}{t}(k-1) + u_{tt} + \frac{u_s}{s}(n-k-1) + u_{ss}\right) \tag{6.19}$$

whenever $t = |x|'$ and $s = |x''|$. We use (6.14)-(6.19) to estimate $|a|$ in terms of δ so that u is an $\mathcal{A} = \nabla f$-subsolution in $\mathbb{R}^n\backslash\mathbb{R}^k$. From homogeneity of f it suffices to make this estimate when $|x|^2 = s^2 + t^2 = 1$. We first show that $\delta > 0$ is a necessary assumption in order for u to be a $\mathcal{A} = \nabla f$-subsolution or supersolution when $a \neq 0$. Indeed if $\delta = 0$, then $\tilde{C} = 0$ and using $0 < \beta < 1, x'' = s\omega'', s > 0$, we find that for fixed $p, n, k,$ as $s \to 0^+$,

$$q^{2-p}(\nabla u)\ \nabla\cdot(D\left(\frac{1}{p}q^p\right)(\nabla u)) = o(s^{\beta-2}) + E_1 + E_2 + E_3 + E_4. \tag{6.20}$$

The E terms are also $o(s^{\beta-2})$ as $s \to 0$, except for those containing u_{ss} or u_s/s. Using this observation and $u_s/|\nabla u| \to 1$ as $s \to 0$, we can continue the estimate in (6.20) to obtain

$$\begin{aligned} &s^{2-\beta}(E_1 + E_2 + E_3 + E_4)\\ &\quad = o(1) + \beta(p-1)|a''|^2 + \left[2(p-1)\beta(\beta-1) + \beta(n-k-1)\right]\langle\omega'', a''\rangle\\ &\qquad + (p-1)(\beta(\beta-1) - \beta)\langle\omega'', a''\rangle^2\\ &\quad = o(1) + \beta((p-1)|a''|^2 + (k+1-n)\langle\omega'', a''\rangle + (p-1)(\beta-2)\langle\omega'', a''\rangle^2). \end{aligned} \tag{6.21}$$

We first choose ω'' so that $\langle\omega'', a''\rangle = |a''|$. Then for this value of x'' we see for s small enough from (6.20), (6.21) that $\nabla\cdot(D\left(\frac{1}{p}q^p\right)(\nabla u)) < 0$ (since $\beta < 1$ and $k+1-n \leq 0$). On the other hand choosing ω'' so that $\langle\omega'', a''\rangle = 0$ we have $\nabla\cdot(D\left(\frac{1}{p}q^p\right)(\nabla u)) > 0$ for this value of x'' and s small enough. Thus u can never be a $\mathcal{A} = \nabla f$-subsolution or supersolution when $\delta = 0$ and $a'' \neq 0$.

If $\delta = 0$ and $a'' = 0$, the E terms are $o(s^{\beta-1})$ as $s \to 0$, except for terms in E_1, E_4 containing $u_{ss}, u_s/s,$ and a term in E_2 containing u_{st}. Using (6.14)-(6.19) it follows that

$$
\begin{aligned}
&s^{1-\beta}(E_1+E_2+E_3+E_4)\\
&\quad = o(1)+\langle \omega',a'\rangle\big[(\lambda+\beta)(\beta-1)-2(p-1)(\lambda+\beta)\beta-(\lambda+\beta)(n-k+\beta-2)\big]\\
&\quad = o(1)-\langle \omega',a'\rangle(\lambda+\beta)\big[2(p-1)\beta+n-k-1\big].
\end{aligned}
\tag{6.22}
$$

The last term in brackets of (6.22) is always positive so choosing ω' with $\langle\omega',a'\rangle = \pm\ |a'|$ and $s>0$ small enough we conclude that u cannot be either an $\mathcal{A}=\nabla f$-subsolution or supersolution when $\delta=0$ and $a\neq 0$.

Now suppose that $\delta>0, \lambda\geq\chi(p,n,k)$, and recall that $r^2=s^2+t^2=1$, $s=|x''|, t=|x'|$. Then from (3.12) we note that

$$
\begin{aligned}
A(\tilde\lambda,\tilde\beta) &= (p-1)\tilde\lambda^2+(p-n)\tilde\lambda-\tilde\beta k=(1+\delta)A(\lambda,\beta)+(p-1)\delta(1+\delta)\lambda^2\\
&\geq (p-1)\delta(1+\delta)\chi^2\\
B(\tilde\lambda,\tilde\beta) &= (2\tilde\beta(p-1)+n-k-2)\tilde\lambda^2+\tilde\beta(p-n)\tilde\lambda-\tilde\beta^2(p-2+k)\\
&= 2(\tilde\beta-\beta)(p-1)\tilde\lambda^2+(1+\delta)^2B(\lambda,\beta)\geq 0\\
C(\tilde\lambda,\tilde\beta) &= (p-1)\tilde\beta-(p-n+k)=(p-1)\delta\beta.
\end{aligned}
\tag{6.23}
$$

To get a ballpark estimate on $|a|$ from above we use (6.23) and either s^2 or $t^2\geq 1/2$ to first get

$$
\frac{u\ (\tilde A\tilde\lambda^2 s^4+\tilde B\tilde\beta s^2t^2+\tilde C\tilde\beta^3t^4)}{s^2r^2(\tilde\lambda^2s^2+\tilde\beta^2t^2)}\geq s^{\tilde\beta-2}\quad\frac{(1/4)(p-1)\delta(1+\delta)^3\min\{\chi^4,\beta^4\}}{\tilde\lambda^2+\tilde\beta^2}. \tag{6.24}
$$

Next from (3.5), (3.7), (3.9), (6.16), we get for $x\in\partial B(0,1)\backslash\mathbb{R}^k$

$$
\begin{aligned}
|E_1|+|E_2| &\leq 2(2p-1)|a|(|u_{tt}|+2|u_{st}|+|u_{ss}|)\\
&\leq 20(p-1)\ |a|\ s^{\tilde\beta-2}(\tilde\lambda+\tilde\beta+1)^2.
\end{aligned}
\tag{6.25}
$$

Similarly

$$
|E_3|\leq 10(p-1)|a|s^{\tilde\beta-2}(\tilde\lambda+\tilde\beta+1)^2, \tag{6.26}
$$

and

$$
|E_4|\leq 10|a|s^{\tilde\beta-2}(\tilde\lambda+\tilde\beta+1)^2(n+k). \tag{6.27}
$$

From (6.24)–(6.27) we conclude that if

$$
|a| < \quad \frac{(1/4)(p-1)\delta(1+\delta)^3\min\{\chi^4,\beta^4\}}{(\tilde\lambda^2+\tilde\beta^2)\Big[(30(p-1)+10(n+k))(\tilde\lambda+\tilde\beta+1)^2\Big]} \tag{6.28}
$$

then u is an $\mathcal{A}=\nabla f$-subsolution on $\mathbb{R}^n\backslash\mathbb{R}^k$.

Lemma 6.9. *Let $1\leq k\leq n-2, p>n-k$, and $f(\eta)=p^{-1}(|\eta|+\langle\eta,a\rangle)^p$ for $\eta\in\mathbb{R}^n$. Let v be the $\mathcal{A}=\nabla f$-harmonic Martin function for $\mathbb{R}^n\backslash\mathbb{R}^k$ with $v(e_n)=1$. If v has homogeneity $-\sigma$, and u as in (6.15) is an $\mathcal{A}$-subsolution with $\lambda=\chi(p,n,k)$, then $\sigma\leq(1+\delta)\ \chi(p,n,k)$.*

Proof. Let $\tilde{q}(\eta'') = |\eta''| + \langle a, \eta'' \rangle$ for $\eta'' \in \mathbb{R}^{n-k}$, and put

$$\tilde{h}(x'') = \sup\{\langle x'', \eta'' \rangle : \tilde{q}(\eta'') \leq 1\}, x'' \in \mathbb{R}^{n-k}.$$

$\tilde{h}$ is homogeneous 1 on $\mathbb{R}^{n-k}$ and in [5] (see also [13, 16] it is shown that if

$$\tilde{f}(\eta'') = p^{-1}\tilde{q}(\eta'')^p \quad \text{and} \quad \beta = \frac{p+k-n}{p-1}$$

then

$$\tilde{h}(x'') \approx |x''| \quad \text{on} \quad \mathbb{R}^{n-k} \quad \text{and} \quad \tilde{h}^{\beta} \quad \text{is} \quad \tilde{\mathcal{A}} = \nabla \tilde{f} - \text{harmonic on} \quad \mathbb{R}^{n-k} \setminus \{0\}. \tag{6.29}$$

Constants in (6.29) depend only on $p, n-k, \alpha', \hat{\alpha}$. Extend $\tilde{h}$ to $\mathbb{R}^n$ by setting $h(x) = \tilde{h}(x''), x = \langle x', x'' \rangle \in \mathbb{R}^n$, and observe that h^{β} is continuous on $\mathbb{R}^n$ as well as $\mathcal{A} = \nabla f$-harmonic on $\mathbb{R}^n \setminus \mathbb{R}^k$ with $h^{\beta} \equiv 0$ on $\mathbb{R}^k$. Also from Lemma 2.5 for $\mathcal{A} = \nabla f$-harmonic functions we deduce that $v/h^{\beta} \approx 1$ on $\partial B(0,1) \setminus \mathbb{R}^k$ with constants having the same dependence as those in (6.29). Using this deduction, (6.29), $\delta \geq 0$, and the definition of u, we find that

$$u \leq cv \quad \text{on} \quad \partial B(0,1) \setminus \mathbb{R}^k \quad \text{where} \quad c = c(\lambda, \delta, p, n, k, \alpha', \hat{\alpha}). \tag{6.30}$$

From (6.30), $u + v \to 0$ uniformly as $x \to \infty$, and the boundary maximum principle in Lemma 2.1 for $\mathcal{A} = \nabla f$-harmonic functions we conclude that $\sigma \leq \tilde{\lambda}$. Taking $\lambda = \chi$ we obtain Lemma 6.9. □

Remark 6.10. From Lemma 6.9, Proposition 6.7, and Assumption (6.28), we conclude Theorems B and C in the $\mathcal{A} = \nabla f$ setting when $f(\eta) = p^{-1}(|\eta| + \langle \eta, a \rangle)^p$ and $(1+\delta)\chi(p, n, k) < k$.

Since (6.28) is a rather awkward assumption for applications, we prove:

Lemma 6.11. *Assume $p > n-1, n \geq 3, k = 1,$ and choose δ so that $(1+\delta)\chi(p, n, 1) = 1 - \frac{(p-2)}{4(p-1)}$. If $p > n-1,$ and*

$$|a| \leq \frac{(p-2)\min\{(p+1-n)^4, 1\}}{100000(p-1)} \tag{6.31}$$

then (6.28) implies that u is an $\mathcal{A} = \nabla f$-subsolution in $\mathbb{R}^n \setminus \mathbb{R}$.

Proof. We note that if $p \geq n \geq 3, k = 1,$ then

$$1 \geq \beta = \frac{p+1-n}{p-1} \geq \chi = \beta \frac{p-1}{2p-n-1} \geq \beta/2 \tag{6.32}$$

and χ is nondecreasing on (n, ∞) as a function of p. Thus $\chi \leq 1/2$ so

$$\delta\beta \geq \chi\delta \geq 1/4 \quad \text{and} \quad (1+\delta)\beta \leq 2. \tag{6.33}$$

Using (6.33) and (6.32) in (6.28) and another ballpark estimate we get (6.31) and thereupon Lemma 6.11 when $p \geq n$. If $n-1 < p < n,$ we note that $\chi = (p-1)^{-1},$ so

$$\delta = (3/4)(p-2) \quad \text{and} \quad (1+\delta)\beta = (p+1-n)(1+\delta)\chi. \tag{6.34}$$

Using this information in (6.28) we get (6.31) and Lemma 6.11 for $n-1 < p < n$. $\square$

Corollary 6.12. *If a satisfies (6.31), then Theorems B and C are valid for* $1 \le k \le n-2$ *and* $p > n-k$, *in* $\mathbb{R}^n \setminus \mathbb{R}^k$ *and the* $\mathcal{A} = \nabla f$*-harmonic setting when* $f(\eta) = p^{-1}(|\eta| + \langle a, \eta \rangle)^p$ *for* $\eta \in \mathbb{R}^n$.

Proof. We first observe that Corollary 6.12 is implied by Lemma 6.11, Lemma 6.9, and Proposition 6.7 when $k=1$. As in the p-harmonic setting, Theorems B, C, for other values of k follow from the $k=1$ case by adding dummy variables (see Remark 1.8). $\square$

6.4. $\mathcal{A}=\nabla f$-subsolutions in $\mathbb{R}^n_+$ when $f(\eta)=p^{-1}(|\eta| + \langle a, \eta \rangle)^p$

In this subsection we continue our investigation of the if part of Proposition 6.7 and ϵ in Proposition 6.8 in $\mathbb{R}^n_+$ when $k = n-1$ and $p > 2$. We begin with

Lemma 6.13. *Let u be an* $\mathcal{A} = \nabla f$*-harmonic Martin function in* $\mathbb{R}^n_+$ *when* $p \ge 2$ *is fixed and f is as in (6.3). If* $p_1 > p$, *then u is an* $\mathcal{A}^{(1)} = \nabla(f^{p_1/p})$*-subsolution in* $\mathbb{R}^n_+$.

Proof. Given $m > n+2$ let $V_m \ge 0$ be the $\mathcal{A} = \nabla f$-harmonic function in

$$\Omega_m = \left[B(0,m) \setminus \bar{B}\left(\frac{2e_n}{m}, \frac{1}{m}\right) \right] \cap \mathbb{R}^n_+$$

with $V_m(e_n) = 1$, and continuous boundary values $V_m \equiv b_m$ on $\partial B\left(\frac{2e_m}{m}, \frac{1}{m}\right)$ while $V_m \equiv 0$ on $\mathbb{R}^n \setminus (B(0,m) \cap \mathbb{R}^n_+)$. Here $b_m > 0$ is a constant with $b_m \to \infty$ as $m \to \infty$. Once again using Lemmas 2.2–2.4 we find that (V_m) is uniformly bounded and locally Hölder continuous in $B(0,m) \setminus B(0,1)$ with Hölder exponent and bounds that are independent of m. Also there exists $\tau \in (0,1)$ with

$$\max_{B(0,s) \cap \Omega_m} V_m \le c(p, n, \alpha', \hat{\alpha})(1/s)^\tau \quad \text{whenever} \quad m > s \ge 2. \tag{6.35}$$

To briefly outline the proof of (6.35), it follows from Harnack's inequality and the above lemmas applied to $\max_{\partial B(0,1)} V_m - V_m$ that for some $\theta \in (0,1)$ (independent of m),

$$\max_{\partial B(0,2)} V_m \le \theta \max_{\partial B(0,1)} V_m. \tag{6.36}$$

Iterating (6.36) we get (6.35).

Next using (6.3) and arguing as in Lemma 4.4 of [13] when $2 \le p < n$ and as in [4] when $p \ge n$, it follows that

$$\text{for each } t \in (0, b_m), \text{ the set } \{x : V_m(x) > t\} \text{ is a convex open set.} \tag{6.37}$$

Using (6.35), (6.37), the above lemmas and Ascoli's theorem it follows that a subsequence of (V_m) converges uniformly to u on compact subsets of $\mathbb{R}^n_+$ and (6.37) is valid with V_m replaced by u whenever $t \in (0, \infty)$. We deduce first from homogeneity of u that $\nabla u \neq 0$ in $\mathbb{R}^n_+$ and thereupon from (6.3), Lemma 2.4, and a Schauder type argument that $f \in C^2(\mathbb{R}^n)$ and that u has locally Hölder continuous second partial derivatives in $\mathbb{R}^n_+$ with exponent depending only on $p, n, \alpha', \hat{\alpha}$. Let $q = p^{-1/p} f^{1/p}$. Given $t, 0 < t < 1$, let T denote the tangent plane to $y \in \{x : u(x) = t\}$. Since u has continuous second partials and $\{x : u(x) > t\}$ is convex we note from the maximum principle for $\mathcal{A}$-harmonic functions that $u|_{T \cap \mathbb{R}^n_+}$ has a relative maximum at y. From the second derivative test for maxima we conclude that if $z \in T, z \neq y$, and $\xi = (z - y)/|z - y|$, then $u_\xi(y) = 0, u_{\xi\xi}(y) \leq 0$. Next we choose an orthonormal basis, $\{\xi^{(1)}, \xi^{(2)}, ..., \xi^{(n)}\}$ for $\mathbb{R}^n$ so that $\xi^1 = \nabla u(y)/|\nabla u(y)|$ and with $\xi^{(i)}, 2 \leq i \leq n$, joining y to points in T. Thus

$$u_{\xi^{(j)}\xi^{(j)}}(y) \leq 0 \quad \text{for} \quad 2 \leq j \leq n. \tag{6.38}$$

Now each component of $\nabla q(\eta) = (q_{\eta_1}, ..., q_{\eta_n})$ is homogeneous of degree 0 on $\mathbb{R}^n$ so for $2 \leq i \leq n$,

$$\sum_{j=1}^n \xi_j^{(1)} q_{\eta_i \eta_j}(\xi^{(1)}) = 0. \tag{6.39}$$

Also from (6.3) $(a), (b)$, and

$$\begin{aligned} &p^{-1} f_{\eta_i \eta_j}(\nabla u(y)) \\ &\quad = q^{p-2}(\nabla u) \Big[(p-1)(q_{\eta_i}(\nabla u(y)) q_{\eta_j}(\nabla u(y)) + q(\nabla u(y)) q_{\eta_i \eta_j}(\nabla u(y)) \Big] \end{aligned} \tag{6.40}$$

we deduce first that if $(w_1, w_2, ..., w_n)$ is orthogonal to $\nabla q(\nabla u(y))$, then

$$c(p, n, \alpha', \hat{\alpha}) \sum_{i,j=1}^n q_{\eta_i \eta_j}(\nabla u(y)) w_i w_j \geq |w|^2 / |\nabla u(y)|. \tag{6.41}$$

The subspace, say Γ, generated by all such w has dimension $n - 1$. Also $\nabla u(y)$ is not in this subspace since $\langle \nabla q(\nabla u(y)), \nabla u(y) \rangle = q(\nabla u(y)) \neq 0$. We conclude from (6.41) and (6.39) that the $n \times n$ matrix $(q_{\eta_i \eta_j}(\nabla u(y)))$ is positive semidefinite and 0 is an eigenvalue of this matrix while $\nabla u(y)$ (or $\nabla q(\nabla u(y))$) is an eigenvector corresponding to 0. Next we note from (6.40) and $\mathcal{A} = \nabla f$-harmonicity of u that

$$0 = \sum\nolimits_{i,j=1}^n \Big[(p-1)(q_{\eta_i}(\nabla u(y)) q_{\eta_j}(\nabla u(y)) + q(\nabla u(y)) q_{\eta_i \eta_j}(\nabla u(y)) \Big] u_{x_i x_j}(y) \tag{6.42}$$

whenever $y \in \mathbb{R}^n_+$ so to prove Lemma 6.13 it suffices to show that

$$\operatorname{trace}\Big((q_{\eta_i \eta_j}(\nabla u(y))(u_{x_i x_j}(y)) \Big) = \sum_{i,j=1}^n q_{\eta_i \eta_j}(\nabla u(y)) \;\; u_{x_i x_j}(y) \leq 0 \tag{6.43}$$

whenever $y \in \mathbb{R}^n_+$. Since the trace of the product of two symmetric matrices is unchanged under an orthogonal transformation we may assume that $\nabla u(y) = |\nabla u(y)|(1, 0, ..., 0)$. Then from (6.38) we see that $(u_{x_i x_j}), 2 \leq i, j \leq n$, is a negative semidefinite matrix and from (6.39), (6.41) that $(q_{\eta_i \eta_j}(\nabla u(y)), 2 \leq i, j \leq n$, is positive semidefinite. Using this fact and the observation that the trace of the product of two positive semidefinite matrices is positive semidefinite, we get (after possibly another rotation) that (6.43) and thereupon Lemma 6.13 is true. □

Remark 6.14. Lemma 6.13 implies that if $-\lambda(p'), p' \geq p$, denotes the Martin exponent for a $\mathcal{A}' = \nabla(f^{p'/p})$-harmonic function, then $\lambda(p')$ is a nonincreasing function in $[p, \infty)$. Indeed from the boundary Harnack inequality in Lemma 2.5, it is easily seen that if u' denotes the $\mathcal{A}'$ Martin function with $u'(e_n) = 1$, and $x \in \partial B(0,1) \cap \mathbb{R}^n_+$, then

$$u(x)/x_n \approx u'(x)/x_n \tag{6.44}$$

where the proportionality constants depend on $p, p', n, \alpha', \hat{\alpha}$. Using this fact, homogeneity of u, u', and Lemma 2.1, we get $\lambda(p') \leq \lambda(p)$. We do not know if a similar inequality holds when k is fixed, $1 \leq k \leq n-2$ and $p > n-k$, in $\mathbb{R}^n \setminus \mathbb{R}^k$, even for p-harmonic Martin functions, although it is clear from drawing levels that the above proof fails.

Next we consider subsolutions of (6.13) in $\mathbb{R}^n_+$ when $p > 2$. We begin by mimicking the argument when $p > n-k$ and $1 \leq k \leq n-2$, with $\beta = 1$. Let

$$u(x', x'') = u(t, s) = s^{1+\delta} r^{-(1+\delta)(\lambda+1)} = x_n^{1+\delta} |x|^{-(1+\delta)(\lambda+1)} \tag{6.45}$$

where

$$x' = (x_1', \ldots, x_{n-1}'), \quad |x'| = t, \quad x'' = x_n = s \geq 0, \quad r = \sqrt{|x'|^2 + x_n^2} = (t^2 + s^2)^{1/2}.$$

Also $\delta \geq 0$ and $n \geq (1+\delta)\lambda \geq \chi(p, n, n-1)$ (χ as in (1.6) of Theorem A with $k = n-1$). With this understanding one can start by writing down the new version of (6.13) and then continue the argument to get (6.14) - (6.19) with $\tilde{\lambda} = (1+\delta)\lambda$ and $\tilde{\beta} = 1+\delta$. Also $r=1$ and $\tilde{A}, \tilde{B}, \tilde{C}$ are defined in the same way as A, B, C are defined in (3.12) (see also (6.23)) only with $\tilde{\lambda}, \tilde{\beta}, k$ replaced by $(1+\delta)\lambda, 1+\delta, n-1$, respectively. Next we investigate as in Subsection 6.3 whether u can be an $\mathcal{A} = \nabla f$-subsolution when $\delta = 0$ and $f(\eta) = p^{-1}(\eta + \langle a, \eta \rangle)^p$ for $\eta \in \mathbb{R}^n$. Indeed, if $\delta = 0$, then from the new version of (6.20) we have for fixed p, n, and uniformly for $s \in (0,1]$ that

$$q^{2-p}(\nabla u) \ \nabla \cdot \left(D\left(\frac{1}{p} q^p\right)(\nabla u)\right) = O(s) + E_1 + E_2 + E_3 + E_4. \tag{6.46}$$

To estimate the $E's$ we observe for $s \in (0,1]$ that

$$|u_t| + |u_{tt}| + |u_{ss}| = O(s)$$

while

$$u_s = |\nabla u| + O(s) \quad \text{and} \quad u_{st} = -(1+\lambda) + O(s).$$

Using these equalities and $x'' = s e_n$ and $x' = t\omega'$, we find for $x \in \partial B(0,1) \cap \mathbb{R}^n_+$, and $s \in (0,1]$ that

$$\begin{aligned} |E_1| + |E_4| &= O(s), \\ E_2 &= -2(p-1)(1+\lambda)\langle \omega', a' \rangle + O(s), \\ E_3 &= -2(p-1)(1+\lambda)(\omega', a')\langle e_n, a'' \rangle + O(s). \end{aligned} \tag{6.47}$$

Combining (6.46)–(6.47) and letting $s \to 0$ we arrive at

$$\lim_{s\to 0} q^{2-p}(\nabla u)\ \nabla\cdot\left(D\left(\frac{1}{p}q^p\right)(\nabla u)\right) = -2(p-1)(1+\lambda)\langle\omega', a'\rangle(1+\langle a'', e_n\rangle). \tag{6.48}$$

Since $|a''| < 1$ and we can choose ω' so that $\langle\omega', a'\rangle = \pm|a'|$, we conclude that u can be neither an $\mathcal{A} = \nabla f$-subsolution nor supersolution when $a' \neq 0$. If $a' = 0$ and $a'' \neq 0$ we need to make more detailed calculations. For this purpose we temporarily allow $p = 2$ in our calculations and note from (3.14), (6.15) that if $|x'| = t, s = x'' = x_n > 0$, then at $x = (x', x'') \in \partial B(0,1) \cap \mathbb{R}^n_+$,

$$s^{-1}(1+\lambda)^{-1}q^{2-p}(\nabla u)\ \nabla\cdot\left(D\left(\frac{1}{p}q^p\right)(\nabla u)\right) = \tilde{G} + s^{-1}(\lambda+1)^{-1}(E_1+E_2+E_3+E_4) \tag{6.49}$$

where

$$\begin{aligned}\tilde{G} &= \frac{(p-1)\left(\lambda - \frac{n-1}{p-1}\right)\lambda^2 s^2 + (2p-3)\left(\lambda - \frac{(p+n-3)}{2p-3}\right)t^2}{\lambda^2 s^2 + t^2} \\ &= \frac{(p-1)\left(\lambda - \frac{n-1}{p-1}\right)\lambda^2 w + (2p-3)\left(\lambda - \frac{(p+n-3)}{2p-3}\right)(1-w)}{(\lambda^2-1)w+1} =: G(w)\end{aligned} \tag{6.50}$$

where we have put $s^2 = w = 1 - t^2$ in the last equality. Also from (6.16) and (3.5), (3.7), (3.9) with $\beta = \tilde{\beta} = 1$, $\tilde{\lambda} = \lambda$ and $\langle a'', e_n\rangle = b$ we find that

$$\begin{aligned}s^{-1}(\lambda+1)^{-1}E_1 &= -\frac{b(-\lambda s^2 + t^2)}{\sqrt{\lambda^2 s^2 + t^2}}\frac{(\lambda^3 s^2 + (2\lambda-1)t^2)}{\lambda^2 s^2 + t^2} \\ &= \frac{b[(\lambda+1)w - 1]\left[(\lambda^3 - 2\lambda + 1)w + 2\lambda - 1\right]}{\left[(\lambda^2-1)w+1\right]^{3/2}} =: F_1(w).\end{aligned} \tag{6.51}$$

From (6.17) and (3.8) we calculate

$$\begin{aligned}&s^{-1}(\lambda+1)^{-1}E_2 \\ &\quad= 2(p-1)b\frac{(1+\lambda)t^2 - (1+\lambda)(3+\lambda)s^2t^2 + ((3+\lambda)s^2 - 3)(1-(1+\lambda)s^2)}{\sqrt{\lambda^2 s^2 + t^2}} \\ &\quad= 2(p-1)b\frac{[\lambda - 2 - w(\lambda+2)(\lambda-1)]}{\sqrt{1 + w(\lambda^2-1)}} =: F_2(w).\end{aligned} \tag{6.52}$$

From (6.18) we have

$$s^{-1}(1+\lambda)^{-1}E_3 = (p-1)b^2\left[-3 + s^2(3+\lambda)\right] = (p-1)b^2[-3 + w(3+\lambda)] =: F_3(w). \tag{6.53}$$

Finally from (6.19), (3.5), (3.8) we arrive at

$$\begin{aligned} s^{-1}(1+\lambda)^{-1}E_4 &= \frac{b(1-(\lambda+1)s^2)}{\sqrt{\lambda^2 s^2+t^2}}((\lambda+2)t^2-s^2)+((3+\lambda)s^2-3)-(n-2)) \\ &= \frac{b(1-(\lambda+1)s^2)}{\sqrt{\lambda^2 s^2+t^2}}(\lambda+1-n) \\ &= \left(\frac{b(1-(\lambda+1)w)}{\sqrt{1+w(\lambda^2-1)}}\right)(\lambda+1-n) =: F_4(w) \end{aligned} \tag{6.54}$$

Armed with (6.50)–(6.54) we first search for $\mathcal{A}=\nabla f$-subsolutions in the baseline $n=2, p=2, \lambda=1$ case. In this case, $G(w)=0=F_4(w)$ for $w\in[0,1]$, and

$$\begin{aligned} F_1(w)+F_2(w)+F_3(w) &= b(2w-1)-2b+(4w-3)b^2 \\ &= b[2w-3+(4w-3)b]>0 \end{aligned} \tag{6.55}$$

whenever $-1<b<0$ and $0\le w\le 1$, while the reverse inequality holds when $0<b<1$. We conclude from (6.55)

Corollary 6.15. *Theorems B and C are valid in $\mathbb{R}^n_+$ and the $\mathcal{A}=\nabla f$-harmonic setting whenever $k=n-1, p>2, n\ge 2, b\in(-1,0)$ and $f(\eta)=p^{-1}(|\eta|+b\ \eta_n)^p$ for $\eta\in\mathbb{R}^n$.*

Proof. From a continuity argument we deduce for given $b\in(-1,0)$ the existence of a positive $\lambda=\lambda(b)<1$ for which (6.55) remains positive for $w\in[0,1]$. From this observation, Remark 6.14, and Proposition 6.7 we obtain Theorems B and C first in $\mathbb{R}^2_+$ and then by adding dummy variables in $\mathbb{R}^n_+, n\ge 3$. □

Next we ask for what values of $p,n,b,$ (for $b\in(0,1)$) is u in (6.49) an $\mathcal{A}=\nabla f$-subsolution on $\mathbb{R}^n_+$? To partially answer this question first put $p=2, n\ge 3, \lambda=n-1$ in (6.50)-(6.56). Then again $G\equiv F_4\equiv 0$. Evaluating $F_1,F_2,F_3,$ at $w=0$ we have

$$\begin{aligned} s^{-1}(1+\lambda)^{-1}q(\nabla u)\ \nabla\cdot\left(D\left(\frac{1}{2}q^2\right)(\nabla u)\right) &= -b(2n-3)+2b(n-3)-3b^2 \\ &= -3b-3b^2<0 \end{aligned} \tag{6.56}$$

when $b\in(0,1)$. From (6.56) and a continuity argument we conclude that u in (6.49) is not an $\mathcal{A}=\nabla f$-subsolution for $p>2$ and $\lambda<n-1$ provided p,λ are sufficiently near 2, $n-1$ respectively. On the other hand, we prove

Lemma 6.16. *Given $b\in(0,1)$, and $p>1$ with $\frac{p-2}{p-1}>2b-b^2$. There exists $n'=n'(b)$, a positive integer, such that if $n\ge n'(b)$, then u in (6.49) is a $\mathcal{A}=\nabla f$-subsolution on $\mathbb{R}^n_+$ for some $\lambda<n-1$.*

Proof. For fixed $n\ge 3$ let $\lambda=n-1$ in the definition of G and the F's. Then

$$G(w)=\frac{(p-2)\left(\left[(n-1)^3-2n+3\right]w+(2n-3)\right)}{n(n-2)w+1} \tag{6.57}$$

and

$$\frac{dG}{dw} = \frac{(p-2)(-n^3+4n^2-5n+2)}{(n(n-2)w+1)^2} < 0 \tag{6.58}$$

when $w \in [0,1]$. Also,

$$(p-1)^{-1}F_2(w) = 2b\frac{[(n-3-w(n+1)(n-2)]}{(n(n-2)w+1)^{1/2}} \tag{6.59}$$

and

$$\begin{aligned}(p-1)^{-1}\frac{dF_2}{dw} &= \frac{-2b(n+1)(n-2)(n(n-2)w+1) - bn(n-2)[n-3-w(n+1)(n-2)]}{(n(n-2)w+1)^{3/2}} \\ &= \frac{-b(n+1)n(n-2)^2 w - b(n-2)(n^2-n+2)}{(n(n-2)w+1)^{3/2}} < 0\end{aligned} \tag{6.60}$$

for $w \in [0,1]$ and $n = 3, 4, \ldots$.

Finally, $(p-1)^{-1}dF_3/dw = (n+2)b^2$. This inequality and (6.59) and (6.60) imply for given $b \in (0,1)$ that there exists a positive integer $n_0 = n_0(b)$ such that

$$dF_2/dw + dF_3/dw < 0 \quad \text{for} \quad w \in [0,1] \quad \text{and} \quad n \geq n_0. \tag{6.61}$$

To prove this assertion suppose $M \geq 1$ is a positive number to be defined. If $n(n-2)w \leq M$, then from (6.59) and (6.60) we see for $n \geq 3$ at $w \in [0,1]$ that

$$(p-1)^{-1}dF_2/dw \leq \frac{-b(n-2)(n^2-n+1)}{(M+1)^{3/2}} \tag{6.62}$$

while if $n(n-2)w \geq M$ and $w \in [0,1]$, we have

$$\begin{aligned}(p-1)^{-1}dF_2/dw &\leq \frac{-b(n+1)n(n-2)^2 w}{(n(n-2)w(1+1/M))^{3/2}} \\ &\leq -b(n+1)n^{-1/2}(n-2)^{1/2}(1+1/M)^{-3/2}.\end{aligned} \tag{6.63}$$

Define M by

$$(1+1/M)^{-3/2} = (3/4) + (1/4)b.$$

With M now defined we see from (6.63) that there exists $n_1 = n_1(b)$, a positive integer such that if $n \geq n_1, w \in [0,1]$, and $n(n-2)w \geq M$, then

$$\begin{aligned}(p-1)^{-1}[dF_2/dw + dF_3/dw] &\leq -b[(3/4) + (1/4)b](n+1)n^{-1/2}(n-2)^{1/2} + b^2(n+2) \\ &\leq -(1/2)(b-b^2)(n+2) < 0.\end{aligned} \tag{6.64}$$

Next we see from (6.62) that there exists $n_2(b) \geq n_1(b)$ for which (6.64) remains valid when $n \geq n_2(b), w \in [0,1]$, and $0 \leq n(n-2)w \leq M$.

With assertion (6.61) now proved, we note from (6.59) and (6.60) that for $w \in [0, 1]$

$$
\begin{aligned}
-F_1(w) &= b\frac{(1-nw)\left[((n-1)^3-2n+3)w+(2n-3)\right]}{(n(n-2)w+1)^{3/2}} \\
&\leq \frac{b}{(p-2)}G(w) \\
&< \frac{1}{(p-1)(2-b)}G(w)
\end{aligned} \tag{6.65}
$$

where we have used the fact that $p-2 > (p-1)(2b-b^2)$. Also, clearly $F_1(w) \geq 0$ on $[1/n, 1]$. From this fact, (6.57), (6.58), and (6.61) we obtain for $n \geq n_2(b)$ and $w \in [1/n, 1]$, that

$$
\begin{aligned}
G(w)+F_1(w)+F_2(w)+F_3(w) &\geq G(1)+F_2(1)+F_3(1) \\
&= \left[p-2+(p-1)(-2b+b^2)\right](n-1) > 0
\end{aligned} \tag{6.66}
$$

Next from (6.65), (6.61), and (6.57), (6.58) we have for $w \in [0, 1/n]$

$$
\begin{aligned}
G(w)+F_1(w)+F_2(w)+F_3(w) &\geq F_2(1/n)+F_3(1/n)+\frac{(p-1)(2-b)-1}{(p-1)(2-b)}G(1) \\
&\geq -4(p-1)+\frac{(p-2)((p-1)(2-b)-1)}{(p-1)(2-b)}(n-1) > 0
\end{aligned} \tag{6.67}
$$

provided $n \geq n_3$ and $n_3 = n_3(b)$ is chosen large enough. (6.67), (6.66), and a continuity argument imply Lemma 6.16. □

Lemma 6.16 implies

Corollary 6.17. *Given $b \in (0, 1)$ and p with $\frac{p-2}{p-1} > 2b - b^2$. There exists a positive integer $n' = n'(b)$ such that Theorems B and C are valid in $\mathbb{R}^n_+$ and the $\mathcal{A} = \nabla f$-harmonic setting for $f(\eta) = p^{-1}(|\eta| + b\eta_n)^p$ for $\eta \in \mathbb{R}^n$, when $n > n'(b)$.*

If $\delta > 0$ in (6.45) we can easily get an analogue of (6.28) when $k = n-1, n \geq 2, p > 2$, in $\mathbb{R}^n_+$. In fact we can just copy the proof given for Lemma 6.11 with $\beta = 1$ and λ with $\chi(p, n, n-1) < \lambda < n-1$. Using this notation we get first (6.23) with $\tilde{A}, \tilde{B}, \tilde{C}$ defined as in (3.12) and with λ replaced by $\lambda(1+\delta)$ and β by $1+\delta$. After that we simply copy the proof from (6.23) - (6.28) (once again with $\tilde{\beta}, \tilde{\lambda}$ replaced by $1+\delta, \lambda(1+\delta)$) to conclude that if (6.28) holds then u is an $\mathcal{A}$ subsolution. in $\mathbb{R}^n_+$. Using the analogue of (6.28) one gets Lemma 6.13 for a $\mathcal{A} = \nabla f$-harmonic Martin function in $\mathbb{R}^n_+$. After that copying the argument in the $p > n$ case of Lemma 6.11, we get first

Lemma 6.18. *Assume $p > 2, n = 2$, and choose δ so that $(1+\delta)\chi(p, 2, 1) = 1 - \frac{(p-2)}{4(p-1)}$. If*

$$
|a| \leq \frac{p-2}{100000(p-1)} \tag{6.68}
$$

then u is an $\mathcal{A} = \nabla f$-subsolution in $\mathbb{R}^2_+$.

After this lemma we obtain once again

Corollary 6.19. *If a satisfies (6.68), then Theorems B and C are valid for $p > 2$ in $\mathbb{R}^n_+$ and the $\mathcal{A} = \nabla f$ setting for $f(\eta) = p^{-1}(|\eta| + \langle a, \eta\rangle)^p$ for $\eta \in \mathbb{R}^2$.*

Proof. The proof follows in $\mathbb{R}^2_+$ from Lemma 6.18 and the analogue of Lemma 6.9 in $\mathbb{R}^2_+$. To get a proof in $\mathbb{R}^n_+, n > 2,$ extend the solution in $\mathbb{R}^2_+$ to $\mathbb{R}^n_+$ by adding dummy variables. □

6.5. Final remarks

We began our investigation of the exponent for $\mathcal{A} = \nabla f$-harmonic Martin functions in $\mathbb{R}^2_+$ when $p = 2$ with

$$f(\eta) = 2^{-1}q(\eta)^2 \quad \text{for} \quad \eta \in \mathbb{R}^2$$

and $q(0) = 0$, q smooth, 1-homogeneous on $\mathbb{R}^2 \setminus \{0\}$. We assumed an $\mathcal{A}$-harmonic Martin function, u, on $\mathbb{R}^2_+$, to have the form:

$$u(x_1, x_2) = \frac{x_2}{l(x_1, x_2)^{1+\lambda}}$$

where $\lambda > 0$, $l > 0$ is smooth, 1-homogeneous on $\mathbb{R}^2 \setminus \{0\}$ which was the form dictated by (1.4) and the boundary Harnack inequalities in Lemma 2.5 for $\mathcal{A}$-harmonic functions. Using $\mathcal{A}$-harmonicity of u and the homogeneities, we wrote down a fully nonlinear second order differential equation for l involving $q, q_{\eta_1}, q_{\eta_2}, q_{\eta_1\eta_2}$ evaluated at $(-\lambda x_2 l_{x_1}(x_1, x_2), (l - \lambda x_2 l_{x_2})(x_1, x_2))$. Again taking $q(\eta) = |\eta| + \langle a, \eta\rangle$ and $x_1^2 + x_2^2 = 1$, we obtained upon letting $x_2 \to 0^+$ the necessary condition

$$\langle \nabla q(0, 1), \nabla l(x, 0)\rangle \le 0, \quad \text{for} \quad -1 \le x \le 1, \tag{6.69}$$

for u to be an $\mathcal{A}$-subsolution on $\mathbb{R}^2_+$ while the reverse inequality was necessary for u to be an $\mathcal{A}$-supersolution. (6.69) for example, showed that if $l(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, and $a = (b, 0)$, then u could not be a $\mathcal{A}$-subsolution or supersolution on $\mathbb{R}^2_+$. We also let $x_2 = 1$ in our differential equation for l and obtained a rather complicated equation for $l_{x_1}(0, 1), l_{x_2}(0, 1)$, which however greatly simplified if $l_{x_1}(0, 1) = 0$. Assuming this equality we were able to check without much difficulty that u was a Martin subsolution for some $0 < \lambda < 2$ at (0,1) if $l(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ and $a = (0, b)$ with $b < 0$. Next we asked Maple to calculate and plot the graph of

$$x_2 = \nabla \cdot \left(D\left(\frac{1}{2}q^2\right)(\nabla u)\right)(x_1^2, 1 - x_1^2) \qquad \text{for } x_1 \in (0, 1)$$

when $\langle a, e_1\rangle \ne 0$ and for several different choices of l other than $l = \sqrt{x_1^2 + x_2^2}$. Maple had a difficult time with this. However when $l = \sqrt{x_1^2 + x_2^2}$ and $a = (0, b)$, Maple plots gave strong indications that u was a Martin subsolution when $b < 0$ for some $0 < \lambda < 1$ and a Martin supersolution for some $\lambda > 1$ when $b > 0$. This result went against our intuition, as it did not seem to depend on uniform ellipticity constants for f in (6.3)(b). However thanks to Maple we eventually obtained (6.55) and Corollary 6.15.

Finally the ballpark estimates given for $|a|$ in Lemmas 6.11 and 6.18 could definitely be improved on by a more serious estimate of the E's. Also we note that the Martin exponent for $p > 1$ and a p-harmonic Martin function on $\mathbb{R}^2_+$ is [17, 18]

$$(1/3)\left(p - 3 - 2\sqrt{p^2 - 3p + 3}\right)/(p - 1)$$

and the Martin function can be written down more or less explicitly. Using this fact and arguing as in the proof of (6.28), (6.68), with u replaced by the Martin function, one should be able to get a better estimate in terms of $|a|$ for the exponent of an $A = \nabla f$-harmonic Martin function on $\mathbb{R}^2_+$ when $p > 2$ and $f(\eta) = p^{-1}(|\eta|| + \langle a, \eta \rangle)^p$.

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