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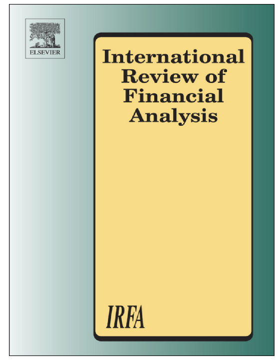

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Stock Price Default Boundary: A Black-Cox Model Approach

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Abstract

In this paper, we incorporate the information from Credit Default Swap (CDS) and options markets to extract the relative default boundary at the stock price level. We propose a reduced-form Black-Cox Model (BCM) with a Deterministic Linear Function (DLF) to extract default information from the CDS and options market to gauge the default boundaries. Using S&P 500 index, CDS, and options data from 2002 to 2017, we extract default boundaries for S&P 500 index via the Unscented Kalman Filter (UKF). Our results suggest that our method performs well when compared with the historical mean relative default boundaries and the recent Unit Recovery Claim (URC)-based default boundaries.

Keywords: Credit Default Swap; Default Boundary; Implied Volatility; Options; Unscented Kalman Filter

JEL classification: C11, C12, C13, G11, G12

1 Introduction

The literature on corporate default typically relies on structural models which assume that the firm defaults once its asset value falls below a certain threshold (e.g., the debt value). In such structural models, the equity value of the firm should be zero if the default event occurs and afterward, and hence it can be assumed that no default boundary exists at the stock price level. However, identifying potential default boundaries of firms within such a framework is very important (Carr and Wu, 2011). On the one hand, instead of being a continuous process, the firm's equity value may jump from a strictly positive value before the default happens, to a much lower value afterward. On the other hand, firms may strategically default, as debt holders may induce or force bankruptcy well before the asset value falls below the debt value (i.e., equity value completely vanishes).

Structure models assume the default barrier exists at the asset level which means that structural models assume that the market value of a firm's assets reflects its economic distress or prosperity. Firm's asset value can be used as a state variable that fully captures the firm's default risk (Davydenko, 2012). For example, Merton (1974) suggests that the default barrier is the par value of debt, but it is unrealistic to assume that the firm can only default at the maturity of the debt. Black and Cox (1976) allow the firm to default before the debt's maturity. Once the asset value drops below a specific default level, then, the equity value is equivalent to a down-and-out call option (Brockman and Turtle, 2003). However, the default level is still on an asset level related to the par value of debt and left for model calibration or other advanced sets. Collin-Dufresn et al., (2001) combine the stochastic interest rate in the structural model to produce a relatively stable leverage ratio and a non-constant default boundary.

Moody's KMV model defines the default barrier as the sum of short-term debt and half of the long-term debt, which is usually between 0.5 and 1 of total debt. Leland (1994) and Leland and Toft (1996) calculate the default boundary as a portion of the par value of debt, absorbing information about debt structure with coupon rate, asset volatility, recovery rate, and other variables related to market fractions.

In this paper, we extract default boundaries at the stock price level from both CDS and options markets. Recent studies explore the relationship between CDS and options market. For example, Carr and Wu (2010) jointly price CDS and individual stock' options. They also bridge CDS and Deep OTM American puts with the URC. The default boundaries estimated in our models based on stock prices. The stock price reflects the future expected profits as well as equity holder preferred recovery claims near default (Edmans 2011; Favara et al., 2012), while the market value of assets which reflects the market's expectations about the firm in structural models normally cannot be observed directly. In our models, K_d/S is defined as the relative default boundary, where K_d is the default-level strike price and S is the stock price. Specifically, we use a reduced-form Black-Cox (1976) model (or the Unit Recovery Claim, URC, theory) together with a Deterministic Linear Function (DLF) to extract this information from the CDS and OTM-Put implied volatility surface.

As defaulting companies tend to be small and with low credit ratings, individual firm-level trading data are usually unreliable due to a lack of trading volumes in both underlying stock and corresponding options. The low liquidity in the stock market will incur the high cost of a delta-hedging strategy, increasing the bid-ask spread in its corresponding option contracts. Even though the market makers in the options market provide the bid and ask quotes on deep OTM puts, the extremely high bid-ask spread makes the implied volatility inferred from the mid-price unreliable. Therefore, we construct a CDX index with nationwide CDS data as well as options for all S&P 500 stocks to infer a market-level default boundary, which is compared to a firm-level default boundary afterward. We model the following processes into a state-space model, where log-normal of CDS-inferred volatility and relative default boundary are two hidden states; 1-year default probability (or URC value) and zero value of DLF are two measurements applied. Due to the nonlinear relationship of measurement, we apply the Unscented Kalman filter (UKF) to capture this feature. With the optimal evolving speed, we estimate four auxiliary parameters related to the covariance matrix of error in states and measurements by maximizing the likelihood function of two measurements.

This study has twofold motivations. First, traditional structural models use explicit or endogenous-generated default levels to match historical default probability. This usually assumes that the likelihood function of the observed equity value is maximized. However, this assumption is not empirically supported (Davydenko, 2012), while our proposed method can avoid this problem. Second, compared with structural models, we extract default boundary information based on market data at the stock price level rather than at asset values. Carr and Wu (2010, 2011) connect the relationship between CDS and options markets (specifically for OTM put options) and point out the possible default price existing at the stock level (Da Fonseca and Gottschalk, 2014; Zhou, 2018).

Some interesting findings emerge from our approach. First, by considering the real relative default level for bankruptcy companies, the mean relative default boundaries inferred from the reduced-form Black Cox model can provide closer estimates compared with those estimated from URC theory. Specifically, the reduced-form Black-Cox Model together with 1-year CDX (5-year CDX) suggests that the mean relative default boundary on the market over our sample period is 0.23 (0.32). Alternatively, the URC-based model together with 1-year CDX (5-year CDX), the mean relative default boundary is 0.57 (0.68). Second, after analyzing the sample of bankruptcy companies between 2002 and 2017, the mean relative default boundary and its 95% confidence level are 13.4% ([8%, 18.8%]) and 24.1% ([13.6%, 34.7%]), by using the mean stock price before 1-week (1month) of the default-level strike as K_d separately and 1-year-before-default stock price as S. Hence, the average relative default boundary is mostly reliable using the reduced-form Black-Cox model and 1-year default probability inferred from 1-year CDX.

The structure of this paper is as follows. Section 2 presents the methodology. Section 3 discusses the data set. Section 4 presents the UKF estimated CIV and relative default boundary at the S&P 500 index level, as well as the dynamics of the historical relative default boundary. Section 5 concludes.

2 Methodology

This section presents the main methods used in this paper: The Black-Cox model and URC theory together with UKF.

2.1 Black-Cox Model

The first passage time model (such as the Black-Cox model, 1976) deals with the problem in the Merton model (Bielecki and Rutkowski, 2013). Merton (1974) assumes that the default event can only happen at the maturity of the firm-issued zero-coupon bond without considering the asset path before maturity. However, due to safety clauses present in issued firm debt, creditors can liquidate their bonds if they observe the firm's assets are below some safety level. Hence, assuming the asset value and the default boundary evolve following some specific stochastic processes, the firm defaults at the first time of these two processes across each other. we assume X^1 (asset value) and X^2 (default barrier) satisfy a Geometric Brownian Motion:

$$dX_t^i = \mu_i X_t^i dt + \sigma_i X_t^i dW_t^i \tag{1}$$

with $X_t^i > 0, i = 1,2$, where W^i is independent standard Brownian motion with respect to a natural filtration F. We also assume $X_0^1 > X_0^2$ (default will not happen at the initial time). The default time τ is the first hitting time from X_t^1 to X_t^2:

$$\tau = \inf\{t \geq 0 : X_t^1 \leq X_t^2\}. \tag{2}$$

Defining a log-ratio process between asset price and debt, $Y_t = \ln(X_t^1 / X_t^2)$, We obtain a solution for Y at any time point t.

$$Y_t = Y_0 + \nu t + \sigma W_t, \quad \text{for } t \geq 0 \tag{3}$$

where $\nu = \mu_1 - \mu_2 - \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2, \sigma^2 = \sigma_1^2 + \sigma_2^2, Y_0 = ln(X^1/X^2) > 0,$ and W_t is a standard Brownian motion. The default time τ is then classified by

$$\tau = \inf\{t \geq 0 : Y_t \leq 0\}. \tag{4}$$

Let Y be given by the above equation. Then τ has an inverse Gaussian distribution under P, i.e., for any t < T, on the set {τ > t}, the default probability is given by

$$P(\tau \leq T|F_t) = \Phi(\frac{-Y_t - \nu(T-t)}{\sigma\sqrt{T-t}}) + e^{-2\nu Y_t/\sigma^2}\Phi(\frac{-Y_t + \nu(T-t)}{\sigma\sqrt{T-t}}). \tag{5}$$

Like the Merton model, the Black-Cox model is always widely used as a structural model, with inputs of asset value and asset volatility. Focusing on the reduced-form model, we use stock price instead of asset value and stock default level instead of default level on liability. The intuition is that asset value is not that transparent compared with the stock price. Although it is difficult to observe whether asset value is below some creditor-believed safety level, it is much easier to observe the stock price, which is publicly available.

After setting current time t as the initial time, we assume X^1 is stock price S_t, X^2 is default level K, the interest rate is r and the dividend is viewed as q. The volatility of S_t is σ, the volatility of K is σ_k as 0, maturity is T, recovery rate is R with a constant 0.4 and s is CDS spread. According to the Black-Cox model, we simplify the model into a more straightforward version combining the information in the CDS market.

$$Y_t = ln(\frac{S_t}{K}), \nu = r - q - \frac{1}{2}\sigma^2 \tag{6}$$

$$P(\tau \leq T|F_t) = N\left(\frac{\ln\left(\frac{K}{S_t}\right)-\left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) + e^{\frac{r-q-\frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2}\cdot\ln(\frac{K}{S_t})} \cdot N(\frac{\ln\left(\frac{K}{S_t}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}) \tag{7}$$

2.2 Unit Recovery Claim

According to the URC theory proposed by Carr and Wu (2011) and considering the characteristic of the Deep OTM American put option (low probability in early exercise), the value of the Deep OTM American put is approximate to that of the Deep OTM European put.

$$URC^{CDS} = \frac{s}{1-R}\frac{1-e^{-(r+\frac{s}{1-R})(T-t)}}{r+\frac{s}{1-R}} \tag{8}$$

$$URC^{put} = \frac{Put^{American}}{K} \approx \frac{Put^{European}}{K}$$

$$= \frac{Ke^{-r(T-t)} \cdot N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) - S_t e^{-qT} \cdot N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)}{K}$$

$$= e^{-r(T-t)} \cdot N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) - \frac{e^{-qT}}{\frac{K}{S_t}} \cdot N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{9}$$

Under the assumption of Carr and Wu (2011), URC values inferred from the two markets are the same.

$$URC^{CDS} = URC^{put} \tag{10}$$

2.3 Unscented Kalman Filter for Hidden Default Level and Corresponding CDS Implied Volatility (CIV)

We use the reduced-form Black-Cox model to connect CDS, default-level strike price and CDS-inferred implied volatility. URC theory proposed by Carr and Wu (2011) is the alternative model for a robust check. Data for CDS and Deep OTM options are noisy, due to the low liquidity like comparable high bid-ask spread. Assuming all prices as correct, we use the UKF to obtain the hidden default-level strike price and its corresponding CIV from the CDS spread and Deep OTM put observations.

To combine information from the Options market, we apply a DLF from Bernales and Guidolin (2014), which connects the relationship between implied volatility, strike price and maturity. The DLF model from Bernales and Guidolin (2014) can provide a better fitting on volatility surface compared with other alternative models. At each observation date, we select options with K/S less than 0.8 for index options, as the OTM level. The rationale for applying a low OTM level rather than 0.9 is because the index option is much more liquid than the individual stock option. we run the deterministic linear regression on log-normal of option implied volatility with factors related to moneyness and maturity. Under the assumption of continuous dividends, M_i as the time-adjusted moneyness can be transformed into a simpler form, combining the pure moneyness (K/S) and maturity.

$$\ln(\sigma_{t,i}) = \beta_0 + \beta_1 M_i + \beta_2 M_i^2 + \beta_3 \tau_i + \beta_4 (M_i * \tau_i) + \tag{11}$$

$$M_i = \frac{ln(K/S) - (r-q)\cdot\tau_i}{\sqrt{\tau_i}}$$

The obtained coefficients at each observation date, $[\hat{\beta}_{0,t}, \hat{\beta}_{1,t}, \hat{\beta}_{2,t}, \hat{\beta}_{3,t}, \hat{\beta}_{4,t}]$ are used as control variables for the UKF. Due to the positive default level (K/S), we transform K/S into $ln(K/S)$ as the first hidden state, which only allows the positive value. To keep a positive volatility, we use $ln(\sigma_t)$ as another hidden state.

Hence, the two hidden states are structured into a vector (X):

$$X_t = [ln(\sigma_t),\ ln(K/S)]^T \tag{12}$$

The two observed measurement equations are as follows: the first is the expected function value equaling 0 and the second is the observed 1-year conditional default probability calculated from the CDX market (or URC value calculated by CDX spread). Due to the higher market liquidity of 5-year CDS compared with 1-year CDS, we also use 5-year CDX as an alternative.

Based on the reduced-form Black-Cox model, we obtain the two measurement equations as follows.

$$0 = E(-\,ln(\sigma_{t,i}) + \hat{\beta}_{0,t} + \hat{\beta}_{1,t}M_i + \hat{\beta}_{2,t}M_i^2 + \hat{\beta}_{3,t}\tau_i + \hat{\beta}_{4,t}(M_i * \tau_i)) \tag{13}$$

$$1 - e^{-\frac{s}{1-R}*(T-t)} = N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) + e^{\frac{r-q-\frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2}\cdot\ln(\frac{K}{S_t})} \cdot N(\frac{\ln\left(\frac{K}{S_t}\right) + \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}) \tag{14}$$

Based on the URC theory proposed by Carr and Wu (2011), we propose the following two measurement equations as a robust check.

$$0 = E(-\,ln(\sigma_{t,i}) + \hat{\beta}_{0,t} + \hat{\beta}_{1,t}M_i + \hat{\beta}_{2,t}M_i^2 + \hat{\beta}_{3,t}\tau_i + \hat{\beta}_{4,t}(M_i * \tau_i)) \tag{15}$$

$$\frac{s}{1-R}\frac{1-e^{-(r+\frac{s}{1-R})(T-t)}}{r+\frac{s}{1-R}} = e^{-r(T-t)} \cdot N\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) - \frac{e^{-qT}}{\frac{K}{S_t}} \cdot N(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r-q+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}) \tag{16}$$

At each observation date, $[ln(\sigma_t),\ ln(K/S)]$ capture the default level and the corresponding implied volatility at this default-level strike price in two markets, respectively. As the pricing of CDS and options do not rely on states' dynamics, we model their relationship into a state-space model. Within the state-space model, the covariance matrix of error in states and measurements are hidden, and the observations in function and 1-year conditional default probability are also measurements with an error.

To simplify the transformation, we model the stochastic process of two states as a random walk, due to its feature of no-determined drift.

$$X_t = X_{t-1} + \sqrt{\Sigma_x} * \varepsilon_t \tag{17}$$

$$\Sigma_x = \begin{bmatrix} \sigma_1^2 \cdot \Delta t & 0 \\ 0 & \sigma_2^2 \cdot \Delta t \end{bmatrix}$$

where ε_t is a 2*1 vector that contains two standard random numbers, with zero mean and variance of one. Furthermore, we simplify the covariance matrix of X with diagonal values, of which the movements in two hidden states are independent with different variations and Δt =7/365 meaning the sampling frequency. The state propagation equation is left for a random walk setting mostly due to its no determinant drift or other movement pattern.

We also define two measurements equations on 1-year default probability observed from the CDS market (or URC^{CDS} value) and another measurement equation connecting implied volatility and strike level observed from the option market. The movements in the two measurements are independent with normally distributed errors. $[\hat{\beta}_{0,t}, \hat{\beta}_{1,t}, \hat{\beta}_{2,t}, \hat{\beta}_{3,t}, \hat{\beta}_{4,t}]$ work as the controlling variables in the second measurement equation.

$$y_t = h(X_t) + \sqrt{\Sigma_y} * \epsilon_t \tag{18}$$

$$\Sigma_y = \begin{bmatrix} \sigma_3^2 & 0 \\ 0 & \sigma_4^2 \end{bmatrix}$$

where $y_t \in \mathbb{R}^2$ is a two-dimension measurement: for the reduced-form Black-Cox model, it is 0 and 1-year default probability inferred from CDX spread; for URC theory, it is 0 and URC^{CDS}. During the measurement's propagation equation, we use the log-normal of CDS-inferred volatility to guarantee its positive feature. Similar to the covariance matrix in states error, we also assume independent and identical errors in two measurement equations.

For the regular linear state-space model, Kalman (1960) provides an excellent method to measure the time series for the hidden states, together with the hidden variance and covariance matrix. However, the measurement equations are nonlinear. Hence, we use the UKF to deal with it (Julier and Uhlmann, 1997).

There are four auxiliary parameters for the covariance of states error and measurements error for estimation. As pointed by Carr and Wu (2016), a large magnitude of covariance in state-propagation compared with that in measurement-propagation will result in quick movement of states capturing the variation in measurements. We use the quasi-maximum likelihood method to maximize the likelihood function, which allows me to obtain the optimal evolving speed and four auxiliary parameters.

We construct the log-likelihood value assuming the normally distributed errors. For the estimation process, we have only two measurements and assign the same weights to both measurements. We calculate the likelihood value on Measurement 1 (l_t^1) and that on measurement 2 (l_t^2), and maximize the sum of both likelihood values of measurement 1 and measurement 2 over the total time period to estimate the four auxiliary parameters together with the time-varied states variables.

$$\Theta \equiv \arg max_{\Theta} \sum_{t=1}^{N} (l_t^1 + l_t^2) \qquad (19)$$

where Θ denotes four model parameters including $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ and the N = 835 means the number of weeks over the whole sample.

3 Data

Synthetic CDX is constructed by all American companies' CDS with Markit-implied rating above BBB from 2002-01-02 to 2017-12-29. The individual companies' CDS data is accessed from Markit in WRDS. CDS spread is selected under MR and MR14 clause, which reduces the pricing error with a fixed recovery rate. We choose every Wednesday as the weekly observation, due to its highest liquidity among one week. We obtain 835 weekly observations for the full sample. On each Wednesday, we only select companies with a rating above BBB as investment level. For each maturity, we make the average of all available CDS as CDX spread for specific maturities.

[Insert Figure 1 about here]

Figure 1 shows the daily number of companies with a rating higher than BBB during the period between 2002 and 2017. The number of companies with investment level ratings increases from 200 in the early 2000s to the highest at 800. The number decreases to about 600 companies from 2010 to 2016. Since the middle of 2016, the number of available observations decreases to around 400 companies. CDS market booms with the bear market but shrinks with the bull market.

[Insert Figure 2 about here]

Figure 2 shows the time series pattern for constructed CDX with different maturities. During 2002 and 2008-2009, both short-term and long-term CDX cluster at a high level, of which the average is close to 0.02. the short-term CDX is often lower than those with long-term ones.

Options data for the S&P 500 index are also obtained from Option Metrics in

WRDS for a sample from 2002-01-01 to 2017-12-29. Maturities of options contracts are selected with a minimum of 8 days. The data includes closing bid/ask quotes, volume, strike prices, expiration dates, Greeks (i.e., Delta, Gamma, and Vega), and implied volatility (mid-quote). Several exclusionary criteria are applied to option observations. Firstly, options will be eliminated if violating basic no-arbitrage conditions. Secondly, options with zero open interest are excluded. As pointed out by Carr and Wu (2020), IV of the ITM option contract is more unreliable than that of the corresponding OTM option. Due to the reason that the default boundary and its corresponding implied volatility is in the left tail of the volatility smile, we only choose the OTM put option with moneyness (K/S) lower than 0.8 and the implied volatilities of the OTM puts.

Applying DLF, we use the whole past 1 week' OTM implied volatilities as the weekly observation by considering enough options. Table 1 shows the summary statistics for deterministic IVS model coefficients estimated by OLS for S&P 500 OTM put Options. The average weekly OTM IV observation is 1761, and the average R square is as high as 0.982. All coefficients are significantly different from zero, especially for the constant term. As pointed by Bernales and Guidolin (2014), β_0 is the implied volatility level; β_1 captures the smile slope; β_2 captures the curvature of smile level, while β_3 captures the slope in the term structure. Finally, β_4 explain the possible relationship between moneyness and maturity.

[Insert Table 1 about here]

One of the measurements is the 1-year default probability. Due to the high liquidity of 5-year CDS, we also construct a 1-year default probability by using a 5-year CDX spread as a robust check. During the late 2000s financial crisis, these two 1-year default probabilities converge to each other, as shown in Figure 3.

[Insert Figure 3 about here]

Default events are obtained from credit events, Markit from WRDS between Jan-1st 2002 to Dec-31th, 2017. Obtaining the stocks which only go bankrupt, we match the tickers with CRSP daily stock to obtain their corresponding daily stock prices. To avoid the extreme stock price fluctuation around the default date, we use the average stock price of the previous one week and one month separately before default, instead of the default-date stock price. The rating information is Markit's implied credit rating.

4 Pricing Performance and State Dynamics Analysis on the S&P 500 Index Level

We firstly examine the pricing performance on observed 1-year default probability together with option implied volatility. After that, we analyze the time series of two extracted states (CDS-inferred implied volatility and relative default strike price).

4.1 Pricing Performance

Table 2 reports the summary statistics for calibration error in two measurements, including 1-year default probability and the value for a DLF. First, errors in 1-year default probability are similar in two models, including the reduced-form Black-Cox model and URC theory, and that inferred from 1-year CDX and 5-year CDX. However, the error in 1-year default probability inferred from 5-year CDS is as double as that inferred from 1-year CDS spread. This matches the relative magnitude between the 1-year default probability inferred from 5-year CDX and that from 1-year CDX. Second, the error term of function value is relatively smaller in URC theory compared to that in the reduced-form Black-Cox model. The relative default strike price in URC theory is much higher than that in the reduced-form Black-Cox model. Hence, the option implied volatility is much more reliable at the strike price in URC theory compared with that in the reduced-form Black-Cox model. Comparing the magnitude between errors in 1-year default probability and DLF value, noise is much higher in OTM implied volatility surface than CDX-inferred 1-year default probability. Hence, that is the reason why the model cannot fully capture the variation in deep OTM strike price.

[Insert Table 2 about here]

Figure 4 shows the UKF-fitted 1-year default probabilities inferred from 1-year CDX and 5-year CDX separately. Compared with Figure 3, both the reduced-form Black-Cox model and URC theory can capture the dynamic patterns in time series of 1-year default probability. For example, the high values of 1-year default probability during the two financial distress periods including 2002-2003 and 2008-2010, are fully captured by these two models estimated via the UKF.

[Insert Figure 4 about here]

Figure 5 shows the time series of error in the deterministic linear function, as the second measurement equation. During the late 2000s financial crisis, the error is the highest in either two models or two versions of 1-year default probability. This is consistent with arguments by Drechsler (2013), who explores the impact of uncertainty on asset price and volatility risk premium. When confronting the late 2000s financial crisis, the demand of deep OTM put option increases dramatically, which reduces its low-level liquidity into a more serious level; then, the bid-ask spread of this kind of option also dramatically rises to a mountaintop level; this results in really high implied volatility and its high uncertainty. Moreover, errors in function value are much more symmetric for URC theory compared with that in the reduced-form Black-Cox model.

The error in function value (Black-Cox model + 1-year CDX) is consistently positive since 2010, which results in a relatively high measurement error shown in table 2. The errors in function value of (URC + 1-year CDX) and (Black-Cox model + 5-year CDX) are much more symmetric and of small magnitude, consistent with the results in table 2.

[Insert Figure 5 about here]

When pricing the CDS spread and OTM-Put implied volatility surface, our model depends on only current states (log-normal of CIV and log-normal of moneyness, K/S), instead of any fixed parameters requiring further model setting on their future dynamics. Hence, this will not result in model re-calibration risk in our model setting. During the model estimation process, we introduce four auxiliary parameters related to the error's covariance matrix for both states and measurements. These four auxiliary parameters are estimated via the maximum likelihood method.

4.2 CDS-inferred Implied Volatility and Relative Default Strike Price

Table 3 reports the summary statistics for UKF-estimated CIV and default level. There are four combinations, which are two models (reduced-form Black-Cox model, and, URC theory) and two versions of 1-year default probability (1-year CDX and 5-year CDX). Firstly, comparing the CIVs under four combinations, those calculated from the URC theory proposed by Carr and Wu (2011) are much smaller than those inferred from the reduced-form Black-Cox model. The average CIV inferred from Carr and Wu's URC theory is 30% compared with the average CIV of 48% from the Black-Cox model. Moreover, those CIVs calculated from the Black-Cox model have a higher standard deviation, which is consistent with the intuition that, the deeper OTM volatility has a higher standard deviation. Second, comparing the relative strike level, this ratio from the Black-Cox Model (i.e., BCM) is much smaller than those inferred from Carr and Wu's URC theory. For example, with the combination of BCM and 1-year CDX (5-year CDX), this ratio is averagely at 0.23 (0.32); while using URC theory and 1-year CDX (5-year CDX), the ratio is relatively high at 0.57 (0.68). The systematic collapse-level ratio of around 0.6 is too high. From a mathematical perspective, the URC theory suffers from modeling bias which results in a high relative strike level. From an empirical perspective, Carr and Wu (2011) show that URC (CDS) is a little higher than URC (put). Hence, the reduced-form Black-Cox model is better at extracting a comparably reliable default level.

[Insert Table 3 about here]

Figure 6 plots the time series on the extracted CIV. For the first pair (URC+1-year CDX) and (URC+5-year CDX) proposed by Carr and Wu (2011), the CIVs share a similar pattern, with high peaks in the early 2000s recession and late 2008s financial crisis. Moreover, CIV calculated from 5-year CDX is less fluctuated compared with that from 1-year CDX. Second, comparing the pair (Black-Cox model + 1-year CDX) and (Black-Cox model + 5-year CDX), the CIVs are much higher but less volatile than those calculated from Carr and Wu's URC theory, sharing a similar pattern during this sample period. Those CIVs from two methods with 1-year CDX are at similar values around 0.6 during the 2008s financial crisis. The average implied volatility increases to a high peak, which increases the absolute value of delta in Deep OTM put options. Hence, the model biases in URC theory (the difference between $URC^{CDS}$and URC^{put}) is significantly reduced, due to its high variance risk premium approaching credit risk premium; this will result in a similar magnitude of CIV and relative default boundary from the two models. According to Figure 7, the difference in estimated relative default boundary in these two models significantly decreases during the late 2008s financial crisis.

[Insert Figure 6 about here]

Figure 7 plots the time series of the UKF-estimated relative default boundary. For the first case (URC+1-year CDX), the relative default boundary begins at about 0.75 and decreases steadily to around 0.5, during the period between 2002 to 2018; in the second case (URC+5-year CDX), the ratio shares a similar pattern in the first case, but with lower standard deviation and a little high value. For the third case (BCM+1-year CDX), the relative default boundary decreases from 0.45 (2002) to 0.15 (2013), and then moves less volatile till 2017; in the fourth case (BCM+5-year CDX), this ratio also shares a similar pattern with the third case, but with a much higher value. According to Figure 7, the relative default boundaries estimated from either URC or BCM are decreasing before the late 2008s financial crisis, while staying in a relatively stable condition since 2011. Moreover, the magnitude of the relative default boundary is much lower inferred from BCM than those calculated from the URC theory.

[Insert Figure 7 about here]

4.3 The Relative Default Strike Price on Individual Stock Level

Table 4 reports the summary statistics of bankruptcy events for individual companies, during the period from 2002 to 2017. The rating is a Markit-implied rating, observed at least one year ago. Matching ticker information in CRSP daily stock, we report stocks that have a valid stock price within 1 month before default. 68.75% (11/16) companies in the sample are CCC rating, 12.5% (2/16) companies are BB rating, and 18.75% (3/16) companies are B rating. Another significant phenomenon is the stock price at 1 year before default that 25% (4/16) of companies in the sample have a low stock price below $5 per share (see, e.g., Carr and Wu, 2011); 31.25% (5/16) companies are at a low stock price, between 5 and 10. Hence, a little more than half of the samples are below a low stock price ($5), at the time of 1 year before default. When time moves into 1 week before default, the average stock price this week (viewed as default price in 1 week) is mostly below $5 per share, and half of them are below $1 per share. When analyzing the mean relative default level, we find this ratio is 0.1340 (0.2413), calculated by using default-1-week-before mean stock price (default-1-month-before mean stock price) divided by 1-year-before stock price separately.

According to Table 4, all default companies in our sample have all entered the junk-level credit rating, when it is 1 year before the default. In other words, the market has realized the high possibility for these companies to face severe financial distress or economic distress before approaching bankruptcy. Bankruptcy does not occur in a short-term period. This is also consistent with the argument from Davydenko (2012) that, many distressed companies avoid bankruptcy for years. About half of the bankruptcy companies in the sample are related to the late 2008s financial crisis. Hence, systematic default risk plays an important role in individual stock default events. Then, a systematic relative default boundary is possible to work as a reliable indicator for individual stock's default.

[Insert Table 4 about here]

To show how the relative default price level evolves before default, we obtain the mean $K^{Default}$/S and its 95% confidence level. S is the average stock price in a specific month. To obtain the positive low/upbound of 95% confidence level, we firstly obtain the mean μ_t and standard deviation σ_t at each date, which is assumed as monthly observations. The 95% confidence level for mean K/S is $[\mu_t - 1.96 \cdot \frac{\sigma_t}{\sqrt{N-1}}, \mu_t + 1.96 \cdot \frac{\sigma_t}{\sqrt{N-1}}]$.

Figure 8 shows the dynamics for the mean relative default strike price together with its 95% confidence interval during the period between the default date and 1.5 years before it, where the default strike price is the mean of the last week's stock price before default. Between 18-month and 12-month, there is only a very soft fluctuation. Between 1 year and 0.5 years, the relative default strike price level increases by approximately 50%. This ratio increases dramatically till default. When it is one month before default, this ratio is averagely around 0.5.

Figure 9 shows the dynamics for the mean relative default strike price together with its 95% confidence interval during the period between the default date and 1.5 years before it, where the default strike price is the mean of the last month's stock price before default. Compared with Figure 8, the relative default strike level is much higher. Between 1.5 years and 1 year, this ratio is relatively stable. Between 1 year and 0.5 years, this ratio increases by about 1/3. Over the last 6 months, this ratio increases dramatically to around 0.6, until one month before the default.

[Insert Figure 8-9 about here]

5 Concluding Remarks

We incorporate the information from CDS and options markets to extract the relative default boundary at the stock price level. First, we transform the traditional Black-Cox model into reduced-form, assuming defaults occur once the stock price drops below the default boundary. Both BCM and URC can provide a relationship between CDS-inferred default probability, default boundary, and implied stock volatility. Moreover, DLF on OTM-put implied volatility surface can connect implied volatility and default boundary directly (Bernales and Guidolin, 2014). Second, we apply the Unscented Kalman Filter (UKF) to extract the time series of CDS-inferred volatility and relative default boundary.

We construct a Credit Default Swap Index (CDX) with all American investment-rating firm-level CDS data as well as options for S&P 500 index to infer a market-level default boundary, which is compared to the historical firm-level average default boundary. Specifically, 0 is the value for DLF as the first measurement; 1-year default probability (URC value) is inferred from 1-year or 5-year Credit Default Swap Index (CDX) as the second measurement; Log-normal of CDS-inferred volatility and relative default boundary are hidden states. Then, we filter the bankruptcy companies between 2002 and 2017, and analyze the dynamics of the relative default boundary before default.

By considering the real relative default level for bankruptcy companies, the ones (between 0.2 and 0.3) inferred from the reduced form Black Cox model can provide closer estimates compared with those (between 0.5 and 0.6) estimated from URC theory. Carr and Wu (2011) suggest that the potential bias between URC^{put} and URC^{CDS} due to maturity mismatch between CDS and option and option characteristics. In this paper, we use 1-year CDX to mitigate the maturity mismatch between CDS and option while searching the implied default boundary and implied volatility from 1-year implied volatility smile is used to mitigate the bias caused by choosing options with fixed strikes. Besides, one reason for different results from these two models can be their assumptions: the reduced-form Black-Cox model assumes stock price follows a diffusive process while the URC theory assumes stock price follows a process containing both diffusive and jump parts. Another reason for the difference is the bridge: the reduced-form Black-Cox model uses the 1-year default probability as the bridge; URC theory uses an artificial tool (i.e., the URC) as the bridge. However, 1-year default probability is more transparent compared with man-made URC products.

The properties of the default boundary are fundamental to the behavior of risky debt and have implications for corporate financing decisions. Further research can explore default boundaries in a specific industry which may provide detailed information about possible industry-level default, as CBOE has offered cash-settled options on 11 industries from the S&P 500 index since Feb 2019. Different industries have different levels of cash flow and leverage. Defaults will happen in low-rating firms or the same industry, if the increased credit risk is mainly caused by the dropping economy.

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Table 1. Summary Statistics for Deterministic IVS model coefficients

The table reports the coefficients of the deterministic IVS model estimated by OLS on S&P 500 OTM put Options (see equation 11). On each Wednesday, we use the past 5 days' IV observations to run the regression of lognormal volatility against moneyness (M_i and M_i^2), maturity (τ_i), and interacted terms ($M_i * \tau_i$). There are 835 weeks in total. #Obs is the number of IV observations each week.

Variable	Mean	S.D.	Min	0.25	Median	0.75	Max	Skewness	Kurtosis
β0	-1.542	0.273	-2.170	-1.698	-1.555	-1.409	-0.[illegible]71	0.671	1.255
β1	-0.700	0.200	-1.440	-0.798	-0.639	-0.57[illegible]	-0.248	-0.952	0.745
β2	-0.058	0.044	-0.273	-0.068	-0.043	-0.036	0.075	-1.726	4.056
β3	0.044	0.077	-0.240	-0.005	0.045	0.091	0.364	0.014	0.889
β4	0.061	0.104	-0.153	-0.016	0.03[illegible]	0.127	0.552	0.944	0.873
R2	0.982	0.010	0.894	0.979	0.98[illegible]	0.988	0.996		
#Obs	1761	1667	100	320	1300	2797	6445		

Table 2. UKF-fitted Error in Measurement Equations

Entries report the error's summary statistics in measurement equations using UKF as the estimation method. Two measurement equations are for probability (PB) and deterministic IVS function (Function) separately; two 1-year market-level default probabilities are extracted from 1-year CDX and 5-year CDX; two models connecting CDS and options markets are Carr and Wu's URC theory (CW) and Black and Cox model (BCM).

Variable	Mean	S.D.	Min	0.25	Median	0.75	Max
Error of PB-CW, y1	0.0077	0.0077	0.0017	0.0029	0.0049	0.0092	0.0426
Error of PB-CW, y5	0.0143	0.0064	0.0067	0.0102	0.0124	0.0168	0.0416
Error of PB-BCM, y1	0.0078	0.0077	0.0017	0.0030	0.0049	0.0093	0.0428
Error of PB-BCM, y5	0.0144	0.0064	0.0069	0.0102	0.0124	0.0169	0.0419
Error of Function-CW, y1	0.0006	0.0331	-0.0971	-0.0184	-0.0038	0.0136	0.1838
Error of Function-CW, y5	-0.0001	0.0498	-0.1443	-0.0305	-0.0093	0.0168	0.2750
Error of Function-BCM, y1	0.0438	0.0568	-0.1797	0.0113	0.0489	0.0805	0.2679
Error of Function-BCM, y5	0.0032	0.0555	-0.1211	-0.0324	-0.0055	0.0305	0.2743

Table 3. UKF-fitted CIV and Default Level

Entries report the hidden states' summary statistics estimated by UKF, which are CDS-inferred stock volatility (CIV) and relative default-level strike price (Default Level). Two 1-year market-level default probabilities are extracted from 1-year CDX and 5-year CDX; two models connecting CDS and options markets are Carr and Wu's URC theory (CW) and the Black and Cox model (BCM).

Variable	Mean	S.D.	Min	0.25	Median	0.75	Max
CIV of CW, y1	0.3235	0.0746	0.1818	0.2660	0.3[illegible]71	0.3667	0.5624
CIV of CW, y5	0.2898	0.0677	0.1709	0.2492	0.280[illegible]	0.3242	0.5071
CIV of BCM, y1	0.5167	0.1154	0.3289	0.3960	0.5524	0.6106	0.6944
CIV of BCM, y5	0.4401	0.0875	0.2848	0.3812	0.4319	0.5185	0.6068
Default Level of CW, y1	0.5652	0.0964	0.4034	0.4836	0.5381	0.6555	0.7725
Default Level of CW, y5	0.6768	0.0673	0.5492	0.6290	0.6825	0.7129	0.8303
Default Level of BCM, y1	0.2299	0.0963	0.1177	0.1401	0.1970	0.3303	0.4135
Default Level of BCM, y5	0.3184	0.0746	0.2062	0.2600	0.3033	0.3770	0.4812

Table 4. Bankruptcy Events

Entries report the bankruptcy firms during the period between 2002 and 2017. DFL.week means the ratio of the average stock price in 1 week before default divided by the stock price 1 year before default; DFL.month means the ratio related to the average stock price in 1 month before default. Price.1year, Price.1week, and Price.1month represent the stock price at 1 year before default, an average of that in 1 week before default, and an average of that in 1 month before default, respectively.

Ticker	Default date	DFL.week	DFL.month	Price.1year	Price.1week	Price.1month	Rating
ABK	20101108	0.3769	0.6126	1.3130	0.4949	0.8044	CCC
AMR	20111129	0.1644	0.2472	8.2113	1.3500	2.0300	CCC
AV	20170119		0.8015	14.7519		11.8233	CCC
CEM	20090318	0.0187	0.0462	7.9045	0.1480	0.3652	CCC
CIT	20091101	0.1609	0.2007	5.1904	0.8350	1.0417	BB
CPN	20051220		0.2790	3.4214		0.9545	BB
DAL	20050914	0.2393	0.3074	3.9773	0.9517	1.2227	B
DCN	20060303	0.0920	0.2201	15.6391	1.4383	3.4427	B
DPH	20051008	0.2573	0.3575	9.1705	2.3600	3.2783	B
EK	20120119	0.1019	0.1063	5.5841	0.5692	0.5934	CCC
GM	20090601	0.0501	0.0672	20.8186	1.0420	1.3982	CCC
LEH	20080915	0.1113	0.2266	55.8114	6.2117	12.6482	CCC
NT	20090114	0.0225	0.0193	15.1238	0.3400	0.2912	CCC
OSG	20121114	0.0940	0.1438	13.5739	1.2760	1.9525	CCC
RSH	20150205	0.1179	0.1403	2.3395	0.2758	0.3283	CCC

WM	20080927	0.0685	0.0852	434.9854	29.7809	37.0422	CCC
Mean		0.1340	0.2413				

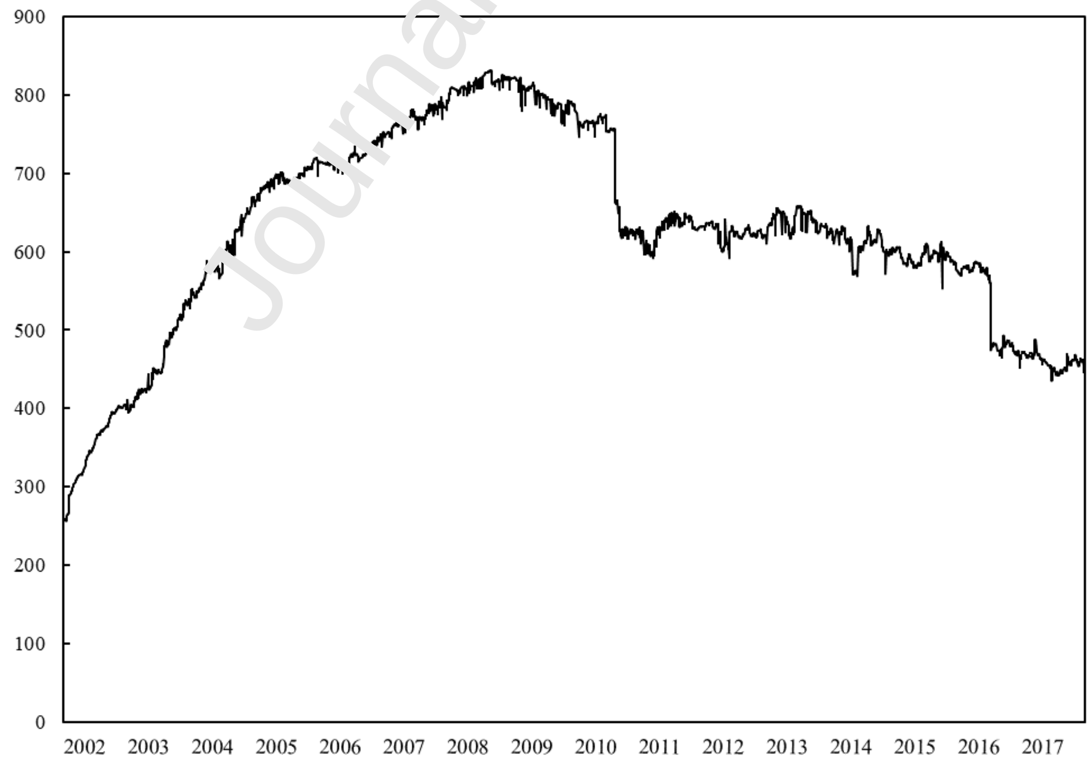

Figure 1. Number of Companies with Rating above BBB

The plots show the time series of the number of companies with Markit-implied rating above BBB, over the sample period from 2002 to 2017.

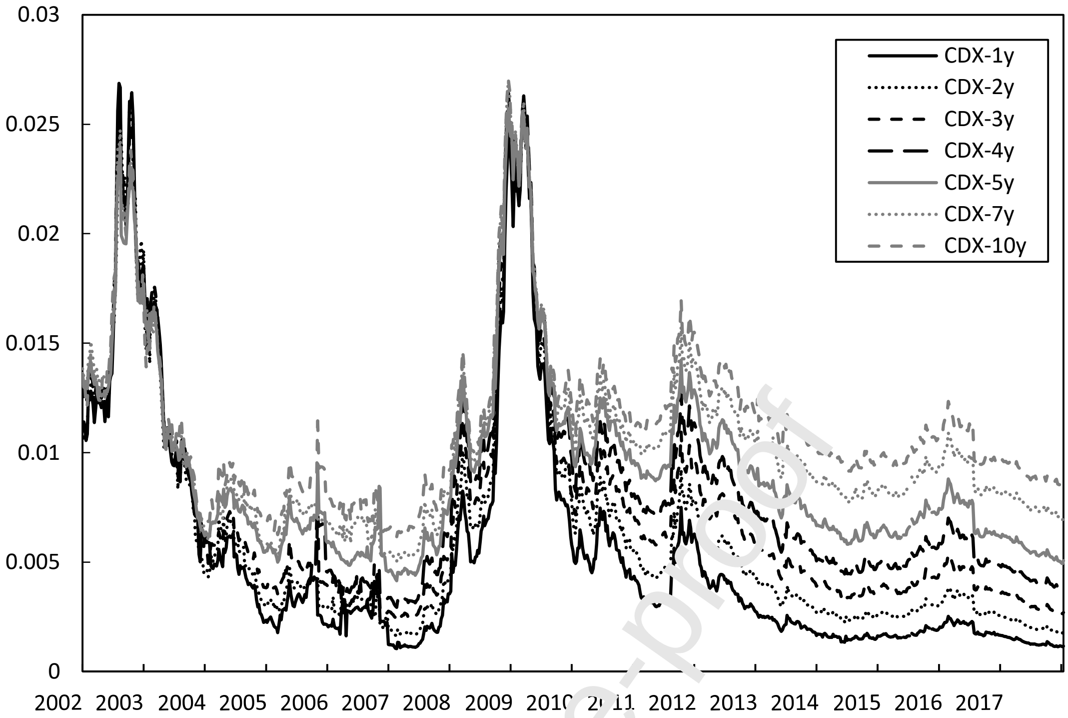

Figure 2. CDX for Different Maturities

The plots show the time series of different-maturity CDX between 2002 and 2017, including 1-year, 2-year, 3-year, 4-year, 5-year, 7-year, and 10-year CDX.

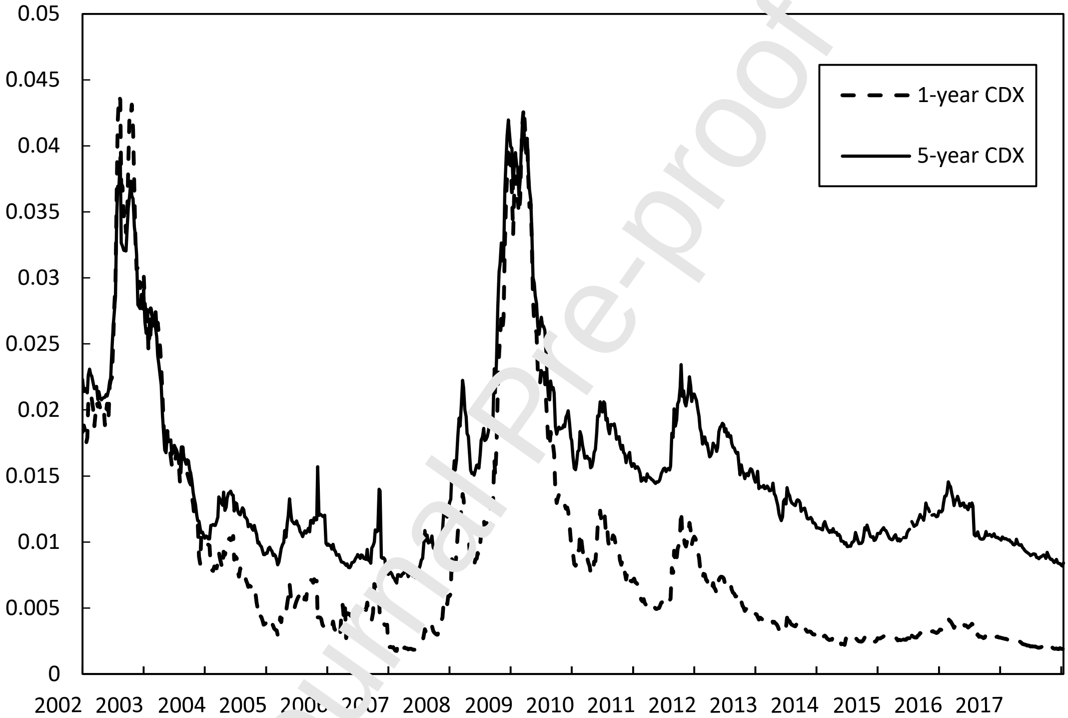

Figure 3. Expected 1-year Market-level Default Probability

The plots show the time series of expected 1-year market-level default probability between 2002 and 2017, inferred from 1-year CDX and 5-year CDX, respectively.

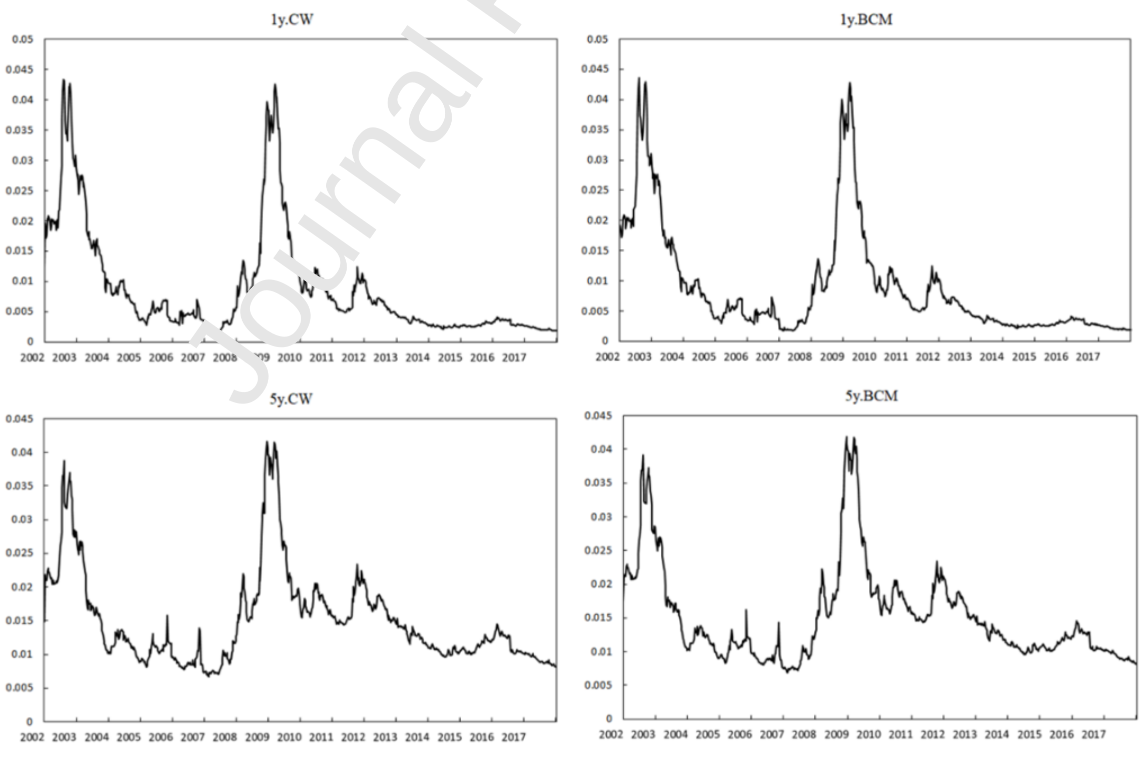

Figure 4. UKF fitted 1-year Default Probability

These plots show the time series of fitted 1-year default probability between 2002 and 2017, inferred from two

CDXs (1-year, 5-year) and two models (Carr and Wu's URC, and Black & Cox model).

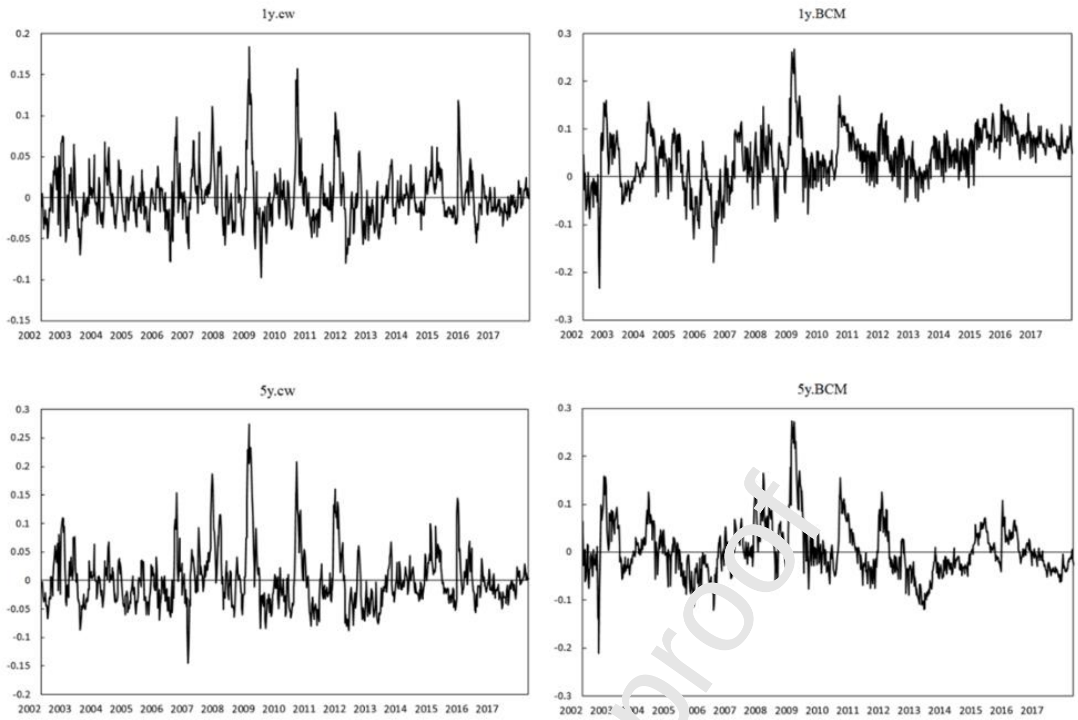

Figure 5. UKF-fitted Measurement Equation

The plots show the time series of fitted value of the measurement equation (deterministic IVS equation) between 2002 and 2017, inferred from two CDXs (1-year, and 5-year) and two models (Carr and Wu's URC, and Black & Cox model).

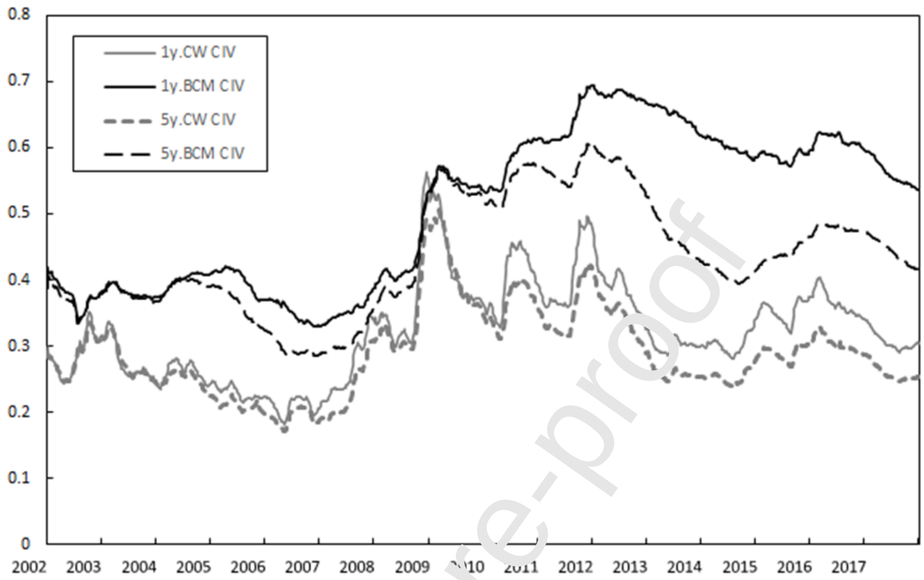

Figure 6. UKF-fitted Default-level CIV

These plots show the time series of fitted default-level CDS-inferred implied volatility (CIV) between 2002 and 2017, inferred from two CDXs (1-year, and 5-year) and two models (Carr and Wu's URC, and Black & Cox model).

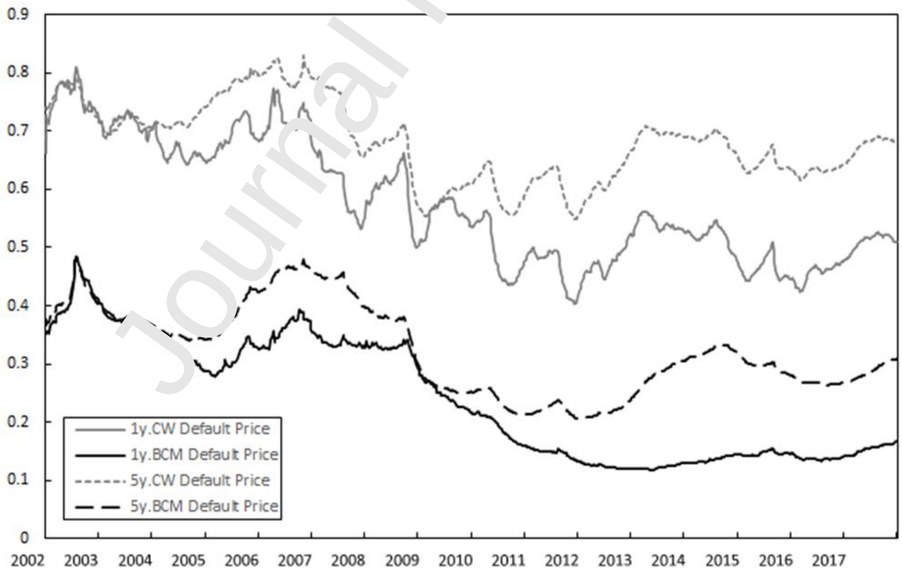

Figure 7. UKF-fitted Relative Default-level Strike Price

These plots show the time series of fitted relative default-level strike prices between 2002 and 2017, inferred from two CDXs (1-year, and 5-year) and two models (Carr and Wu's URC, and Black & Cox model).

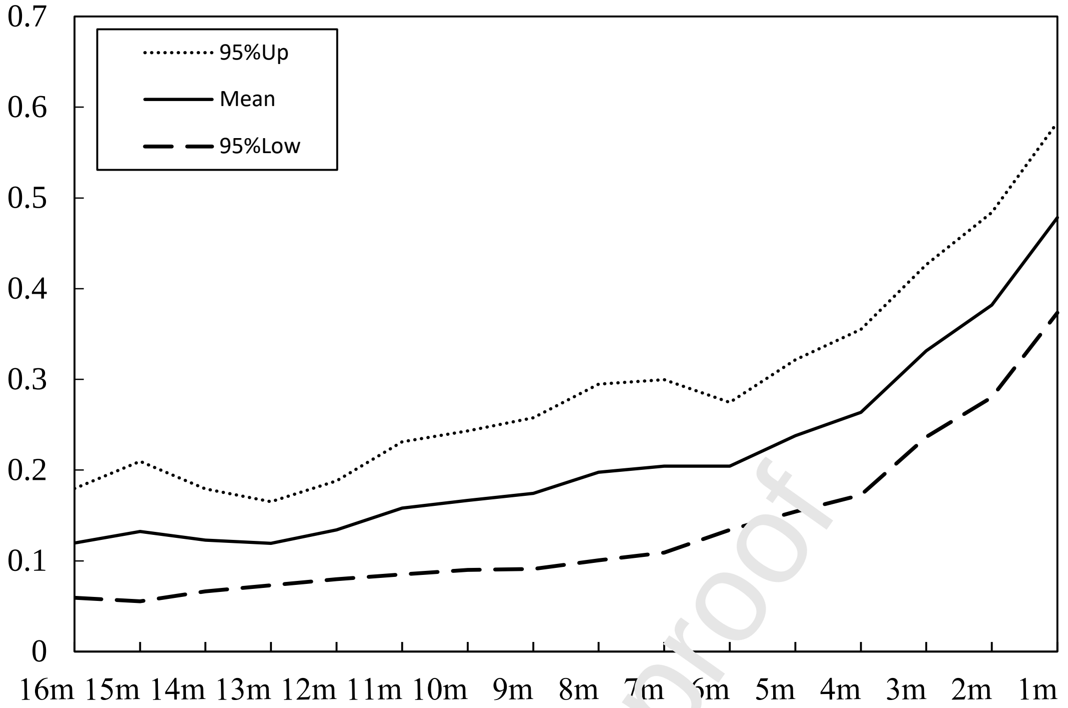

Figure 8. Dynamics of Relative Strike Level (1-week mean before default)

These plots show the time series of UKF estimated relative default boundary, which is calculated with a 1-month average stock price divided by 1-week mean stock price before default.

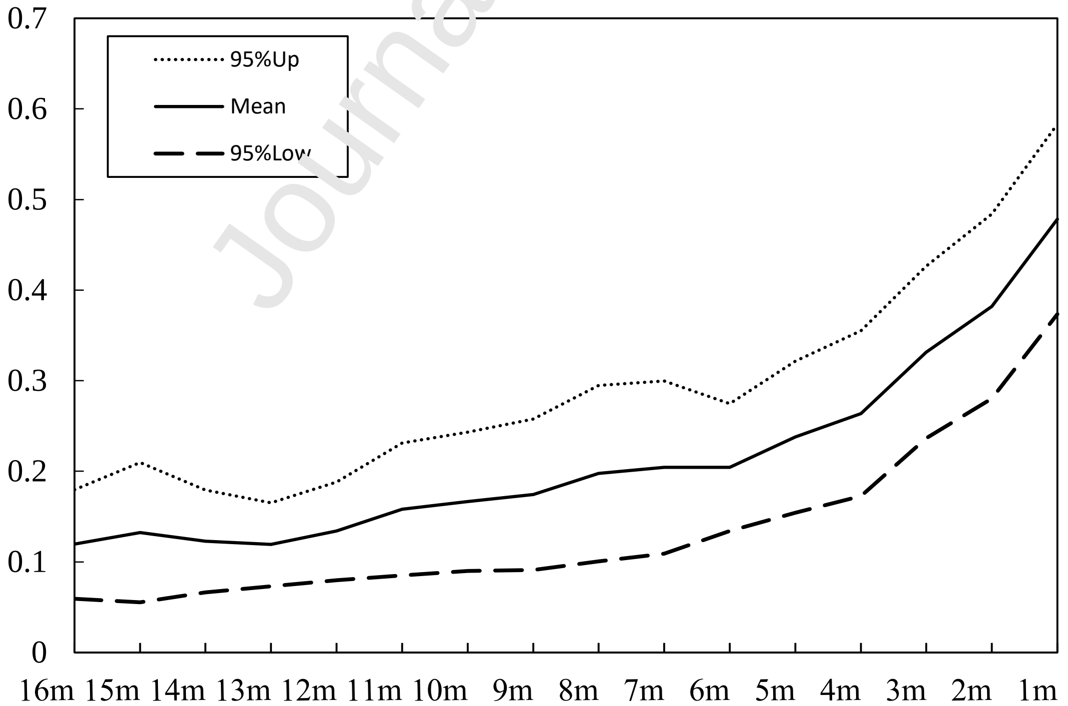

Figure 9. Dynamics of Relative Strike Level (1-month mean before default)

These plots show the time series of UKF estimated relative default boundary, which is calculated with a 1-month average stock price divided by 1-month mean stock price before default.

Highlights

- We extract the relative default boundary at the stock price level.
- We propose a reduced-form Black-Cox Model with a Deterministic Linear Function.
- Default information from the CDS and options market are used to gauge the default boundaries.
- Our method outperforms the historical mean relative default boundaries and the Unit Recovery Claim default boundaries.