THE SANDPILE & SMITH GROUPS OF CERTAIN CLASSES OF GRAPHS

Baha Tamimi

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Department of Mathematical Sciences
University of Essex

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To my mother, family, and friends.

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Abstract

The Smith group and the sandpile group are graph invariants. In this thesis we compute these groups for a variety of classes of graphs. We provide background materials in Chapter 1. In Chapter 2, on integral circulant graphs, Theorem A provides the Smith group of multiple copies of these graphs. For integral circulant graphs of prime power order, Theorem B presents a new graph construction that is isomorphic to them. For a subset of these graphs, the Smith group of the Kronecker product of a graph with an all ones matrix is presented in Theorem C. The sandpile group of another subset is given in Theorem D; Conjecture 2.3.6 proposes a generalisation. Theorem E provides the maximum number of integral circulant graphs of order n with d components where d divides n. In Chapter 3, we focus on graphs with at most four distinct eigenvalues, including a number of symmetric graphs. Theorems F, G gives the sandpile group structure of a Kronecker product of the complement of a 6-cycle with an all ones matrix and its complement. Theorems H,I gives the sandpile group of modified complete bipartite graphs. Chapter 4 is about the sandpile group of some classes of threshold graph, and other graph constructions that were selected using Maple code. They are covered in Theorems J, K, L. In Chapter 5, we consider graphs with sandpile group of small rank, giving a condition for a graph join between a path and a complete graph of order two to have non-cyclic sandpile group in Theorem M, and discussing when it is cyclic in Conjecture 5.2.18. Finally, as many theorems that we prove were conjectured by running experiments in Maple, in Chapter 6 we describe the computational approach we used and provide code and examples to facilitate future research.

Chapter 1

Introduction

The main purpose of this chapter is to introduce standard definitions, theorems, and ideas from algebraic graph theory and combinatorial group theory. They are relevant to our research on both the sandpile and the Smith groups. There are no new results in this chapter. Proofs are introduced to help familiarise the reader with the subject and the techniques used.

Let Γ be an undirected graph of order n, with a vertex set $V(\Gamma)$, and an edge multiset $E(\Gamma)$ such that

$$V(\Gamma) = \{v_0, v_1, \ldots, v_{n-1}\},$$

$$E(\Gamma) = \{v_i v_j : \text{ whenever } v_i \text{ is adjacent to } v_j,\ v_i, v_j \in V(\Gamma)\}.$$

We call Γ a *multigraph* if there are multiple edges between two vertices or if there is at least one *loop* (an edge where the two endpoints are the same vertex). Otherwise we call Γ a *simple graph*. Throughout this thesis Γ will be a simple graph unless specified otherwise.

1.1 Graph spectra

The graph Γ can be represented as a matrix. Two of the most common representations are the adjacency and Laplacian matrices.

The *adjacency matrix* of Γ is an $n \times n$ matrix such that

$$A(\Gamma) = (a_{ij}), \text{ where } a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \text{ for } v_i, v_j \in V(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

The *Laplacian matrix* of Γ is

$$L(\Gamma) = D(\Gamma) - A(\Gamma),$$

where $D(\Gamma)$ is the *degree matrix* of Γ,

$$D(\Gamma) = \mathrm{diag}(\deg(v_0), \deg(v_1), \ldots, \deg(v_{n-1})).$$

Matrix associated objects such as an eigenvalue of a matrix or the *characteristic polynomial* are also relevant to the study of graph matrices. An *eigenvalue of a graph* is defined as an eigenvalue of the adjacency matrix of a graph. If we would like to talk about an eigenvalue of the Laplacian matrix of a graph, we will refer to it as the *Laplacian eigenvalue of a graph*. The same logic applies to other objects related to matrices such as the *rank* and the *spectrum of a matrix*.

It is well known that the eigenvalues of a real symmetric matrix are real numbers, which is the case for both $A(\Gamma)$ and $L(\Gamma)$ (as Γ is an undirected graph).

And if we know the Laplacian eigenvalues of Γ, we can calculate the Laplacian eigenvalues of $\overline{\Gamma}$ as in the following theorem.

Theorem 1.1.1. [50, Corollary 1.3.7] *Let Γ be a graph of order n and let $\overline{\Gamma}$ be its complement graph, and suppose λ_j, μ_j, $j \in \{0, 1, \ldots, n-1\}$ are the Laplacian eigenvalues of Γ, $\overline{\Gamma}$ respectively, then*

$$\mu_j = \begin{cases} \lambda_0 = 0, & \text{if } j = 0 \\ n - \lambda_{n-j}, & \text{for } j \in \{1, 2, \ldots, n-1\}. \end{cases}$$

Theorem 1.1.1 is also mentioned in [10, Section 1.3.2].

Theorem 1.1.2. [20, Theorem 2.1.2, page 25] *Let Γ be a regular graph of order n and degree k, and suppose k, λ_j, $j \in \{1, 2, \ldots, n-1\}$ are the eigenvalues of Γ, then the eigenvalues of $\overline{\Gamma}$ are $n-1-k$, $-\lambda_j-1$, $j \in \{1, 2, \ldots, n-1\}$.*

We call a graph an *integral graph*, if and only if all the eigenvalues of its adjacency matrix are integers. Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $A(\Gamma)$ and that Γ is an integral and regular graph. Then using the definition of the Laplacian matrix $L(\Gamma)$, the eigenvalues of $L(\Gamma)$ are all integers as follows

$$\deg(\Gamma) - \lambda_0, \deg(\Gamma) - \lambda_1, \ldots, \deg(\Gamma) - \lambda_{n-1}.$$

Another thing we can observe as a consequence of Theorem 1.1.1, $\overline{\Gamma}$ is integral if and only if Γ is integral.

1.2 Graph operations

Graph operations are methods of constructing new graphs from existing ones. This includes the following commonly used operations.

Let Γ_1 and Γ_2 be graphs with disjoint vertex sets $V(\Gamma_1)$, $V(\Gamma_2)$ and edge sets $E(\Gamma_1)$, $E(\Gamma_2)$. The *graph*

union $\Gamma = \Gamma_1 \cup \Gamma_2$ is a graph with

$$V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2), \text{ and } E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2).$$

The *graph join* $\Gamma = \Gamma_1 \bigtriangledown \Gamma_2$ is a graph with

$$V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2), \text{ and } E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2) \cup \big(V(\Gamma_1) \times V(\Gamma_2)\big),$$

where $V(\Gamma_1) \times V(\Gamma_2)$ is the Cartesian product of $V(\Gamma_1), V(\Gamma_2)$. Notice that $\Gamma_1 \bigtriangledown \Gamma_2 = \overline{\overline{\Gamma_1} \cup \overline{\Gamma_2}}$.

The *Cartesian graph product* $\Gamma = \Gamma_1 \square \Gamma_2$ is a graph with

$$V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2),$$

$$((u_0, u_1), (v_0, v_1)) \in E(\Gamma) \text{ when either } u_0 = v_0 \text{ and } u_1 \text{ is adjacent to } v_1$$

$$\text{or } u_1 = v_1 \text{ and } u_0 \text{ is adjacent to } v_0. \text{ Note that } A(\Gamma) = A(\Gamma_1) \otimes I + I \otimes A(\Gamma_2).$$

The *Kronecker graph product* $\Gamma = \Gamma_1 \otimes \Gamma_2$ is a graph with

$$V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2),$$

$$((u_0, u_1), (v_0, v_1)) \in E(\Gamma) \text{ when } u_0 = v_0 \text{ and } u_1 = v_1. \text{ Note that } A(\Gamma) = A(\Gamma_1) \otimes A(\Gamma_2).$$

The Kronecker graph product is also known as the *tensor graph product*, and the *weak direct graph product*.

The *strong graph product* $\Gamma = \Gamma_1 \boxtimes \Gamma_2$ is a graph with

$$V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2), \text{ and } E(\Gamma) = E(\Gamma_1 \square \Gamma_2) \cup E(\Gamma_1 \otimes \Gamma_2).$$

A useful feature of graphs constructed using graph operations is how easy it is to determine their eigenvalues as we will see in Theorems 1.2.1 - 1.2.5.

Theorem 1.2.1. [50, Theorem 1.4.4] *The Laplacian Spectrum of a union of graphs* $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$ *is*

$$\mathrm{Spec}(L(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n)) = \bigcup_{i=1}^{k} \mathrm{Spec}(L(\Gamma_i)).$$

Theorem 1.2.2. [50, Theorem 1.4.5] *Let* Γ_1, Γ_2 *be graphs such that* $n_1 = |V(\Gamma_1)|, n_2 = |V(\Gamma_2)|$. *And let* $\mathrm{Spec}(L(\Gamma_1)) = \{0 = \lambda_0, \lambda_1, \ldots, \lambda_{n_1-1}\}$, $\mathrm{Spec}(L(\Gamma_2)) = \{0 = \mu_0, \mu_1, \ldots, \mu_{n_2-1}\}$. *The join graph of* Γ_1 *and*

Γ_2 *has the following Laplacian spectrum*

$$\mathrm{Spec}(L(\Gamma_1 \bigtriangledown \Gamma_2)) = \{0, \lambda_1 + n_2, \lambda_2 + n_2, \ldots, \lambda_{n_1-1} + n_2, \mu_1 + n_1, \mu_2 + n_1, \ldots, \mu_{n_2-1} + n_1, n_1 + n_2\}.$$

Theorem 1.2.3. [10, Section 1.4.6, page 10] *Let Γ_1, Γ_2 be graphs, and let λ and μ be Laplacian eigenvalues of Γ_1 and Γ_2 respectively, then the Laplacian eigenvalues of their Cartesian product $\Gamma_1 \square \Gamma_2$ are of the form $\lambda + \mu$.*

Theorem 1.2.4. [10, Section 1.4.7, page 10] *Let Γ_1, Γ_2 be graphs, and let λ and μ be eigenvalues of Γ_1 and Γ_2 respectively, then the eigenvalues of their Kronecker product $\Gamma_1 \otimes \Gamma_2$ are of the form $\lambda\mu$.*

Theorem 1.2.5. [10, Section 1.4.8, page 11] *Let Γ_1, Γ_2 be graphs, and let λ and μ be eigenvalues of Γ_1 and Γ_2 respectively, then the eigenvalues of their strong product $\Gamma_1 \boxtimes \Gamma_2$ are of the form $(\lambda + 1)(\mu + 1) - 1$.*

Definition 1.2.6 (Induced subgraph)**.** *Let Γ be a graph with vertex set V and edge set E. Let Γ' be a graph with vertex set $W \subset V$, and edge set $F = \{xy : xy \in E \text{ and } x, y \in W\}$. Then we call Γ' an* induced subgraph *of Γ.*

Definition 1.2.7 (Graph component)**.** *For a graph Γ, we call a maximal connected subgraph, a* component *of Γ.*

A well known graph family that can be constructed using graph operations is the threshold graph.

Definition 1.2.8 (Threshold graph)**.** *A graph is called* decomposable *if any induced subgraph is a join or union of two graphs. If we can construct a graph by starting with an empty set and for each new vertex added we either perform a join or union operation with all other vertices, then we call this graph a* threshold graph*. This graph is also known as* $1-$decomposable graph *and* degree maximal graph*.*

Threshold graphs are integral graphs, we can see this is true based on how it can be constructed. Define a graph Γ_0 that has no vertices and no edges. Each time we add a vertex to Γ_j we increment the label to $j + 1$. We start by adding the first vertex to Γ_0, which gives us Γ_1 a K_1 graph. Γ_1 now has one eigenvalue 0. We then continue adding one vertex at a time to Γ_i and either performing a union or a join operation between Γ_i and the new vertex K_1. Theorems 1.2.2, 1.2.1 shows that the resulting eigenvalues of Γ_{i+1} will continue to be integral, and by Theorem 1.1.1 we can see that its complement is also integral.

Threshold graphs were introduced by Chvátal & Hammer (see [16]), and they proved the following theorem that helps in identifying if a graph is threshold graphs or not.

Theorem 1.2.9. [16, Theorem 1, page 8] *Let Γ be a graph, then the following statements are equivalent,*

1. *Γ is a threshold graph.*

2. *Γ does not contain an induced subgraph of order 4 that is isomorphic to either C_4, P_4, or $K_2 \cup K_2$.*

3. *There is an ordering of the vertices of Γ, $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and a partition of V with subsets P and Q such that*

- *for all $v_j \in P$, v_j is adjacent to all v_i whenever $i < j$.*
- *for all $v_j \in Q$, v_j is not adjacent to any v_i whenever $i < j$.*

Definition 1.2.10 (Graph power)**.** *Let Γ be a graph, then the k-th power of Γ, denoted by Γ^k is a graph that has $V(\Gamma^k) = V(\Gamma)$, such that each for each $x, y \in \Gamma^k$, x is adjacent to y if their distance in Γ is at most k.*

1.3 Tiling operations

Definition 1.3.1 (Edge tiling)**.** *Let Γ_1, Γ_2 be connected graphs, and select an edge from each graph $vu \in E(\Gamma_1)$ and $wz \in E(\Gamma_2)$. Concatenate the vertices v, w into one vertex x and also concatenate u, z into one vertex y. Note that the resulting graph has two copies of the edge xy, so we remove one of them. We call the resulting graph $(\Gamma_1 \cup \Gamma_2)_{\phi'(vu,wz)}$ an* edge tiling *of Γ_1 and Γ_2.*

Definition 1.3.2 (Vertex tiling)**.** *Let Γ_1, Γ_2 be connected graphs, and select a vertex from each graph $x \in \Gamma_1$ and $y \in \Gamma_2$. Let $\Theta(x), \Theta(y)$ be the sets of edges incident to x, y respectively, and denote their sets of neighbours as $N(x)$, $N(y)$. The* vertex tiling *of x, y in Γ_1, Γ_2 results in a new graph $\Gamma = (\Gamma_1 \cup \Gamma_2)_{\phi(x,y)}$, where*

$$V(\Gamma) = (V(\Gamma_1)\backslash\{x\}) \cup (V(\Gamma_2)\backslash\{y\}) \cup \{z\},$$
$$E(\Gamma) = (E(\Gamma_1)\backslash\Theta(x)) \cup (E(\Gamma_2)\backslash\Theta(y))) \cup \{(z, v) : v \in N(x) \cup N(y)\}.$$

Definition 1.3.3 (Planar graphs and their dual graphs)**.** *A graph is* a planar graph *if it can be drawn on a plane without any edge crossing another. Suppose Γ is a planar graph, and let Γ' be a graph that has a vertex for each face in Γ and an edge whenever two faces share the same edge. We call Γ' the* dual graph *of Γ.*

The chain cyclic graph

In [41], a family of planar graphs is introduced, called the *chain cyclic* graph and it is defined as follows.

Consider an edge tiling of two connected planar graphs Γ_1, Γ_2 and a cycle C in the middle in the following way:

Tile Γ_2 and C by selecting an edge e in Γ_2 and the edge $v_{n-1}v_n$ in C. Then tile the resulted graph from Γ_2 and C with Γ_1 by selecting an edge $t_i = v_iv_{i+1}, i \in \{1, 2, .., n-2\}$ in C and an edge h in Γ_1. If we have a list of cycles tiled as a chain we use the notation $\mathrm{CH}(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n$ represent the order of each cycle. This is simpler than expressing this construction as $(((C_{a_1} \cup C_{a_2})_{\phi'(e_1,f_2)} \cup C_{a_3})_{\phi'(e_2,f_3)} \cup \cdots \cup C_{a_n})_{\phi'(e_{n-1},f_n)}$.

1.4 Graphs with symmetric properties

Suppose that we have two graphs Γ_1, Γ_2, and define the bijection $f : V(\Gamma_1) \to V(\Gamma_2)$, such that for all $x, y \in V(\Gamma_1)$, $x \sim y$ if and only if $f(y) \sim f(y)$, $f(x), f(y) \in V(\Gamma_2)$. We say Γ_1 is *isomorphic* to Γ_2 ($\Gamma_1 \cong \Gamma_2$), where

f is an *isomorphism* from Γ_1 to Γ_2.

If g is an isomorphism from a graph Γ to itself (i.e. $g : V(\Gamma) \to V(\Gamma)$) we call g an *automorphism* of Γ. We can think of g as a permutation of $V(\Gamma)$. Then the set of all automorphisms of Γ forms a permutation group, we call this group the *automorphism group* of Γ, denoted by $\mathrm{Aut}(\Gamma)$.

The following graph families are defined using their automorphism group. For more details on these graphs are related families we recommend Godsil and Royle book (see [30]).

Definition 1.4.1 (Vertex transitive graphs)**.** *A graph Γ is* vertex transitive *if for all $v, u \in V(\Gamma)$, there exist $g \in \mathrm{Aut}(\Gamma)$ such that $g(v) = u$.*

In Chapter 3, we will investigate a well known vertex transitive family of graphs called Kneser graphs.

Definition 1.4.2 (Cayley graphs)**.** *Let G be a group and let S be subset of G. The* Cayley digraph $\mathrm{CayD}(G, S) = (V, A)$ *is defined as follows: The set of vertices corresponds to the elements in G i.e. $V = G$. The set of arcs (directed edges) $A = \{\{g, gs\} \mid g \in G, s \in S\}$. If S is* symmetric *($s \in S \Rightarrow s^{-1} \in S$), by replacing the arcs $\{g, gs\}, \{gs, g\}$ in A with an edge $\{g, gs\}$ for all $g \in G, s \in S$ we get an undirected Cayley graph denoted by $\mathrm{Cay}(G, S)$. Note that if the identity element $e \in S$, then $\{g, g\} \in E(\mathrm{CayD}(G, S))$ is a digraph loop, this will be is replaced with a loop in $\mathrm{Cay}(G, S)$.*

Since we are only working with undirected graphs we will assume that S is a symmetric set and that $e \notin S$, unless otherwise stated.

Definition 1.4.3 (Circulant graphs)**.** *Let $A = (a_{ij})$ be an $n \times n$ matrix, and suppose that its entries satisfy the equation $a_{i,j} = a_{1,j-i+1}$, where $\{j-i+1 \bmod n\} = \{1, 2, ..., n\}$. We call the matrix A a* circulant matrix *denoted by $\mathrm{circ}_n(a_0, a_1, \ldots, a_{n-1})$, where $a_i = a_{0,i}$. A graph is a* circulant graph *if it has a circulant adjacency matrix.*

All circulant graphs are Cayley graphs with cyclic group ($\mathrm{Cay}(\mathbb{Z}_n, S)$), and all Cayley graphs are vertex transitive graphs. The converse is not true as for example, the Petersen graph can be shown to not be a Cayley graph (see for more examples [47]).

Definition 1.4.4 (Strongly regular graphs)**.** *The graph $\Gamma = \mathrm{srg}(n, k, a, c)$ is a* strongly regular *graph if*

- *Γ is of order n.*
- *Γ is regular of degree k.*
- *$\forall x, y \in V(\Gamma)$, that are adjacent, the number of adjacent vertices to both of them is exactly a.*
- *$\forall x, y \in V(\Gamma)$, that are not adjacent, the number of adjacent vertices to both of them is exactly c.*

The complement of $\mathrm{srg}(n, k, a, c)$ is also a strongly regular graph of the form $\mathrm{srg}(n, n-k-1, n-2-2k+c, n-2k+a)$. Strongly regular graphs are well known class of graphs. For general information on them we recommend Godsil & Royle book (see [30, Chapter 10]).

The parameters n, k, a, and c are not independent, they must satisfy the following condition

$$(n-k-1)c = k(k-a-1).$$

The eigenvalues of $\mathrm{srg}(n,k,a,c)$ are well understood. Let $\Delta = (a-c)^2 + 4(k-c)$, then the eigenvalues of $\mathrm{srg}(n,k,a,c)$ have the following form:

$$\lambda_0 = k, \lambda_1 = \frac{(a-c)+\sqrt{\Delta}}{2}, \lambda_2 = \frac{(a-c)-\sqrt{\Delta}}{2},$$

with multiplicities

$$m_0 = 1, m_1 = \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right), m_2 = \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right),$$

respectively. As m_i, $i \in \{0,1,2\}$ must be integer values, we can see that we have another condition to satisfy in order to construct a valid strongly regular graph. Furthermore, a strongly regular graph $\Gamma = \mathrm{srg}(n,k,a,c)$ with $m_1 = m_2$ is called a *conference graph*. Otherwise, Γ is an integral graph, with $(\lambda_1 - \lambda_2)^2$ being a perfect square (see [30, Lemma 10.3.3, page 222]).

Let $q = p^t \equiv 1 \mod 4$, where p is a prime number. The *Paley graph* $Paley(q)$ is a graph with the element of the finite field $\mathbb{F}_q$ representing its vertex set, such that two vertices a, b are adjacent if and only if $b-a$, $a-b$ are square numbers in $\mathbb{F}_q$. The strongly regular graph $\mathrm{srg}(q,(q-1)/2,(q-5)/4,(q-1)/4)$ is a Paley graph. It is also a conference graph as its eigenvalues are $(q-1)/2$, $(\sqrt{q}-1)/2$, $-(\sqrt{q}+1)/2$ with multiplicities 1, $(q-1)/2$, $(q-1)/2$ respectively. If q is a prime, then this Paley graph is a circulant graph $\mathrm{Cay}(\mathbb{Z}_q, S)$, where S are the squares $(\mathrm{mod}\ q)$.

Definition 1.4.5 (Distance regular graphs)**.** *A graph Γ that is connected, regular of degree k and diameter d is* distance regular *if and only if for all $i \leqslant j \leqslant d$ there is an array of numbers (known as Γ's* intersection array*) $[b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d]$ such that for any pair $x, y \in V(\Gamma)$,*

- *b_j is equal to the number of neighbours of y at distance $j+1$ from x.*
- *c_j is equal to the number of neighbours of y at distance $j-1$ from x.*

Definition 1.4.6 (Walk regular graphs)**.** *Let Γ be a graph, we say that Γ is a* walk-regular graph *if the number of closed walks from $v \in V(\Gamma)$ to itself of any length is independent of the choice of v.*

This class of graphs includes *vertex transitive graphs* and *distance regular graphs*. Note that it will also includes *strongly regular graphs* as they are distance regular.

Definition 1.4.7 (Strongly walk regular graphs)**.** *Let Γ be a graph of order n. Γ is a* strongly l-walk regular *graph with parameters (a_l, c_l, k_l) if*

- $\forall x, y \in V(\Gamma)$, *that are adjacent, the number* a_l *of walks of length* l *from* x *to* y *is constant independent of* x *and* y.
- $\forall x, y \in V(\Gamma)$, *that are not adjacent, the number* c_l *of walks of length* l *from* x *to* y *is constant independent of* x *and* y.
- $\forall x \in V(\Gamma)$, *the number* k_l *of closed walks of length* l, *is a constant independent of* x.

We denote strongly l-walk regular graph with $\mathrm{swrg}_l(a_l, c_l, k_l)$. This class of graphs generalises *strongly regular graphs* as we can think of them as strongly 2-walk regular graphs.

1.5 The matrix tree theorem

A *tree* is a connected acyclic graph. A *spanning tree* of a graph Γ, is a subgraph that includes all of $V(\Gamma)$ in its set of vertices and a minimal subset of $E(\Gamma)$ in its edges set enough to be a tree.

By calculating any cofactor of the Laplacian matrix of Γ as in the following theorem, we get the number of the spanning trees of Γ (which is also known as the complexity of Γ).

Theorem 1.5.1. (The Matrix Tree Theorem [30, Theorem 13.2.1, page 282]). *Let* Γ *be a graph, the number of spanning trees is equal to the absolute value of any cofactor of* Γ*'s Laplacian matrix.*

Another well known method to calculate the number of the spanning trees is the deletion-contraction recurrence below.

Theorem 1.5.2. [30, Equation 13.2, page 282] *Let* Γ *be a graph, then*

$$\tau(\Gamma) = \tau(\Gamma - x) + \tau(\Gamma/x),$$

where $\tau(\Gamma)$ *is the number of spanning trees,* $\Gamma - x$ *is obtained by removing an edge* x, Γ/x *is obtained by contracting an edge* x *and removing any loop that might occur.*

We can also calculate the number of spanning trees of Γ using its Laplacian eigenvalues.

Theorem 1.5.3. ([30, Lemma 13.2.4, page 284]). *Let* Γ *be a graph with Laplacian eigenvalues* $0 = \lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, *then the number of spanning trees is equal to* $\frac{1}{n}\prod_{i=1}^{n} \lambda_i$.

As an example of calculating the number of spanning trees, we will do the calculation on $K_m \bigtriangledown P_n$. Note that the sandpile group (and the number of spanning trees) has been determined for this graph in [60], which we will discuss in detail in Chapter 5.

Example 1.5.4.

$$\tau(K_m \bigtriangledown P_n) = (m+n)^{m-1} \prod_{j=1}^{n-1} (m + 2 + 2\cos(\frac{\pi j}{n}).$$

Proof. The Laplacian eigenvalues for K_m, P_n are

$$\{0, \underbrace{m, m, \ldots, m}_{m-1}\}, \text{ and}$$

$$\{0, 2 + 2\cos(\frac{\pi j}{n})\ (j \in \{1, 2, \ldots, n-1\})\}$$

respectively. Applying Theorem 1.2.2 we get

$$\{0, \underbrace{m+n, m+n, \ldots, m+n}_{m-1}\}, m + 2 + 2\cos(\frac{\pi j}{n})\ (j \in \{1, 2, \ldots, n-1\}), m+n\},$$

and by Theorem 1.5.3

$$\tau(K_m \bigtriangledown P_n) = (m+n)^{m-1} \prod_{j=1}^{n-1} (m + 2 + 2\cos(\frac{\pi j}{n}).$$

□

This is further simplified in [60, Theorem 1.1] to

$$\tau(K_m \bigtriangledown P_n) = \frac{1}{t}\left(\frac{m+2+t}{2}\right)^n - \frac{1}{t}\left(\frac{m+2-t}{2}\right)^n, t = \sqrt{m^2 + 4m}.$$

$R_{t,q}$ is an integer sequence satisfying the recurrence

$$R_{t,q} = tR_{t,q-1} - R_{t,q-2} \text{ and } R_{t,0} = 0, R_{t,1} = 1 \quad (q, t \in \mathbb{Z}).$$

Next we calculate the number of spanning trees of $K_1 \bigtriangledown P_n$ using Theorem 1.5.2.

Example 1.5.5. $\tau(K_1 \bigtriangledown P_n) = R_{3,n} = F_{2n}$, $(n > 0)$ *where F_n is the nth Fibonacci number.*

Proof. Let v_0 be the vertex in K_1, and $v_1, v_2, \ldots, v_n$ be the vertices in P_n ordered from left to right, and let H_i denote the graph $(K_1 \bigtriangledown P_i)/(v_{i-1}v_i)$, $i \geqslant 2$ (see Figure 1.1). Now we have

$$\begin{aligned}\tau(K_1 \bigtriangledown P_n) &= \tau((K_1 \bigtriangledown P_n) - v_{n-1}v_n) + \tau(H_n) \ \text{ (Theorem 1.5.2)}\\ &= \tau(K_1 \bigtriangledown P_{n-1}) + \tau(H_n).\end{aligned}$$

Which we can express as

$$\tau(H_n) = \tau(K_1 \bigtriangledown P_n) - \tau(K_1 \bigtriangledown P_{n-1}).$$

On the other hand, by applying Theorem 1.5.2 on one of the edges v_0v_{n-1} in H_n and removing loops we get

$$\tau(H_n) = \tau(H_n - v_0v_{n-1}) + \tau(H_n/(v_0v_{n-1}))$$

$$= \tau(K_1 \bigtriangledown P_{n-1}) + \tau(H_{n-1}).$$

Combining these observations,

$$\begin{aligned}
\tau(K_1 \bigtriangledown P_n) &= \tau(K_1 \bigtriangledown P_{n-1}) + \tau(H_n) \\
&= \tau(K_1 \bigtriangledown P_{n-1}) + \tau(K_1 \bigtriangledown P_{n-1}) + \tau(H_{n-1}) \\
&= \tau(K_1 \bigtriangledown P_{n-1}) + \tau(K_1 \bigtriangledown P_{n-1}) + \tau(K_1 \bigtriangledown P_{n-1}) - \tau(K_1 \bigtriangledown P_{n-2}) \\
&= 3\,\tau(K_1 \bigtriangledown P_{n-1}) - \tau(K_1 \bigtriangledown P_{n-2}).
\end{aligned}$$

$\tau(K_1 \bigtriangledown P_n)$ has the same form as $R_{3,n}$ and satisfies it as $\tau(K_1) = R_{3,0} = 0, \tau(K_1 \bigtriangledown P_1) = R_{3,1} = 1$. To show that this is equivalent to F_{2n}, checking initial cases, $F_0 = 0, F_2 = 1$. For higher values of $n > 2$

$$\begin{aligned}
F_{2n} &= F_{2n-1} + F_{2n-2} \\
&= (F_{2n-2} + F_{2n-3}) + F_{2n-2} \\
&= (F_{2n-2} + (F_{2n-2} - F_{2n-4})) + F_{2n-2} \\
&= 3F_{2n-2} - F_{2n-4} \\
&= 3F_{2(n-1)} - F_{2(n-2)}.
\end{aligned}$$

Therefore $R_{3,n} = F_{2n}$. □

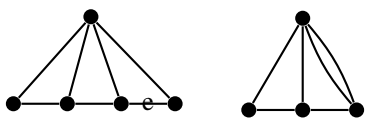

Figure 1.1: $K_1 \bigtriangledown P_4$ and $H_1 = (K_1 \bigtriangledown P_4)/e$ from left to right.

1.6 The Smith normal form

We start with the following well known theorem by Smith. For more information on this subject see for example [52, Section 11.2].

Theorem 1.6.1. (Smith Theorem). *Let $M_{n\times n}$ be an integer matrix of rank s, there exist two invertible matrices U and T such that*

$$M = UQT$$

where

$$Q = \operatorname{diag}(q_0, q_1, \ldots, q_s, \underbrace{0, 0, \ldots, 0}_{n-s}), \text{ and } q_0|q_1|\ldots|q_s, q_i \geqslant 0, 0 \leqslant i \leqslant s, 0 \leqslant s \leqslant n.$$

We call Q the *Smith normal form* of M denoted by $\mathrm{SNF}(M)$, where $q_0, q_1, \ldots, q_s$ are the *invariant factors* of M.

The invariant factors $q_0, q_1, \ldots, q_s$ satisfy the relation

$$q_i = \frac{d_i(M)}{d_{i-1}(M)},$$

where $d_0(M) = 1, d_i(M)$ is the greatest common divisor of all $i \times i$ minors of M $(i \in \{0, 1, \ldots, s\})$.

This relation is hardly practical to calculate the Smith normal form of a matrix. One alternative is to use an algorithm that is similar to the row operations in Gaussian elimination without using division in order to preserve the integer equivalent class. These operations include:

- interchanging two rows or two columns.
- multiplying a row or column by -1.
- adding a row to another or a column to another multiple times $R_i \to R_i + aR_j$, $C_i \to C_i + aC_j$.

The following are examples of calculating the Smith normal form.

Example 1.6.2. *Let J_m be the $m \times m$ all ones matrix, and let $t > 0$. The Smith normal form of tJ_m is*

$$\mathrm{diag}(t, 0, 0, \ldots, 0).$$

Proof.

$$tJ_m = \begin{pmatrix} t & t & t & \cdots & t \\ t & t & t & \cdots & t \\ t & t & t & \cdots & t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \cdots & t \end{pmatrix}$$

By applying the following operations for $(i = 2, 3, \ldots t)$

$$R_i \leftarrow R_i - R_1,$$

$$C_i \leftarrow C_i + C_1,$$

we get $\mathrm{diag}(t, 0, \ldots, 0, 0)$ as required. □

Example 1.6.3. *The Smith normal form of $A(K_m)$ is*

$$\mathrm{diag}(1, 1, \ldots, 1, m-1).$$

Proof. The complete graph adjacency matrix is as follows

$$A(K_m) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

By applying the following operations for $(i = 2, 3, \ldots m)$

$$R_i \leftarrow R_1 - R_i,$$

$$C_1 \leftarrow C_1 + C_i,$$

$$R_1 \leftarrow R_1 - R_i,$$

and rearranging rows and columns will result in $\mathrm{diag}(1, 1, \ldots, 1, m-1)$ as required. □

Example 1.6.4. *The Smith normal form of* $A(K\underbrace{m, m, \ldots, m}_{q})$ *is*

$$\mathrm{diag}(\underbrace{1, 1, \ldots, 1, q-1}_{q}, \underbrace{0, 0, \ldots, 0}_{(m-1)q}).$$

Proof. The complete k-partite graph $K\underbrace{m, m, \ldots, m}_{q}$ has an adjacency matrix

$$A(K\underbrace{m, m, \ldots, m}_{q}) = \begin{pmatrix} 0_m & J_m & J_m & \cdots & J_m \\ J_m & 0_m & J_m & \cdots & J_m \\ J_m & J_m & 0_m & \cdots & J_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_m & J_m & J_m & \cdots & 0_m \end{pmatrix}$$

where 0_m, J_m are the $m \times m$ all zeros and all ones matrices respectively. As we have the sequence of operations for calculating $\mathrm{SNF}(A(K_q))$ (see Example 1.6.3). We can reuse each operation for row or column say i by applying it on all rows or columns from $m(i-1)+1$ up to mi in $A(K\underbrace{m, m, \ldots, m}_{q})$. After applying all operations we

should have the block matrix

$$\begin{pmatrix} J_m & \cdots & 0_m & 0_m & 0_m \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_m & \cdots & J_m & 0_m & 0_m \\ 0_m & \cdots & 0_m & J_m & 0_m \\ 0_m & \cdots & 0_m & 0_m & (q-1)J_m \end{pmatrix}$$

As all the sub-matrices on the diagonal are of the form tJ_m and the rest are 0_m, applying the operations in Example 1.6.2 and rearranging rows and columns will give us $\operatorname{diag}(\underbrace{1, 1, \ldots, 1, q-1}_{q}, \underbrace{0, 0, \ldots, 0}_{(m-1)q})$. □

The following theorem provides a useful connection between the eigenvalues of a matrix and its Smith normal form.

Theorem 1.6.5. [10, Proposition 13.8.1] *Let λ be an integral eigenvalue of an integral matrix M and has a geometric multiplicity s, then λ divides at least s invariant factors in the Smith normal form of M.*

1.7 Group presentations and Tietze transformations

Let G be a group, and let S be a subset of G. A *word* is any product of the form $t_1 t_2 \cdots t_n$, where $t_i \in S \cup S^{-1}$. If a word does not have any $t_i = {t_{i+1}}^{-1}$ for all $i \in \{1, 2, \ldots, n\}$ then it is a *reduced word*.

A group G is called *free* if there is a generating set S of G such that every non trivial element of G is defined by a unique reduced word in S. In this case G is also called *free on S, generated by S* and S is a *free basis* of G. We can use the notation F_S for the free group G.

$P = \langle S \mid R \rangle = \langle x_0, x_1, \ldots, x_{n-1} \mid w_0, w_1, \ldots, w_{m-1} \rangle$ is a *group presentation* of G, where $x_i \in S$ are the *generators* of G and $w_j \in R, R \subseteq F_S$ are the *relators* (words on those generators x_i). If both S and R are finite then P is a *finitely presented group*. Let N be the normal closure of R in F_S (the smallest normal subgroup containing R), then $G = F_S/N$.

As we will be focusing on abelian groups we use the superscript ab to make it explicit. So,

$$P = \langle x_0, x_1, \ldots, x_{n-1} \mid w_0, w_1, \ldots, w_{m-1} \rangle^{ab}$$

is a group presentation for an abelian group. To give some examples, for one generator we have $\langle x \mid \rangle \cong \mathbb{Z}$, $\langle x \mid x^5 \rangle \cong \mathbb{Z}_5$. For more than one generator, having a group presentation defined explicitly for abelian group, makes it simpler to define,

$$\langle x, y \mid x^2, y^2, (xy)^2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \langle x, y \mid x^2, y^2 \rangle^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

One advantage of using a group presentation is we can apply Tietze transformations. They are a set of ma-

nipulations on generators and relators of the presentation of a group while preserving the group structure. This technique used in combinatorial group theory for more information see for example [36, Chapter 4].

Tietze transformations

Let G be a group and let $P = \langle S \mid R \rangle$ be a group presentation of G. There are four transformations to obtain another presentation of G from P.

- Adding a relator: $\langle S \mid R \rangle = \langle S \mid R \cup \{r\} \rangle$, $r \in N \backslash R$.
- Removing a relator: $\langle S \mid R \rangle = \langle S \mid R \backslash \{r\} \rangle$, $r \in \overline{R \backslash \{r\}} \cap R$.
- Adding a generator: $\langle S \mid R \rangle = \langle S \cup \{g\} \mid R \cup \{g^{-1}r\} \rangle$, $r \in F_S, g \notin S$.
- Removing a generator: $\langle S \mid R \rangle = \langle S \backslash \{g\} \mid R \backslash \{g^{-1}r\} \rangle$, $g \in S$, r is a word that does not contain g and $g^{-1}r$ is the only word in R that contains g.

These transformations are called *Tietze transformations*.

Define a matrix $B = (b_{ij})_{n \times m}$, where b_{ij} represent the exponent of x_i in a relation w_j, we call B the *relation matrix* for P. For the abelian case, P is a presentation for the group $\mathbb{Z}^n / B\mathbb{Z}^m$. This way using Tietze transformations, we have another method for abelianization (other than the Smith normal form as we describe in section 1.8).

Tietze transformations proofs writing convention

The convention we use in presenting Tietze transformations proofs to improve their readability is as follows

- Any change to the set of generators or an individual relator in a presentation is double underlined in the next presentation.
- Any generator or relator added is added to the end of presentation.
- The order of a generator or a relator in a presentation is maintained in the next presentation.
- If we have an expression of relators, then expanding a term is added to the right of the original expression.

The next example shows how Tietze transformations proof works in the convention we provided above.

Example 1.7.1. *The complete graph K_n has Laplacian $circ_n(n-1, -1, -1, \ldots, -1)$ and the Smith form of this is given by the abelianization of the cyclically presented group $G = G_n(x_0^{n-1} x_1^{-1} \ldots x_{n-1}^{-1})$ which is equal to $\mathbb{Z} \oplus \mathbb{Z}_n^{n-2}$.*

Proof. In this proof all x indices calculations are modulus n (we will encounter similar statements in other Tietze transformations proofs given how we tend to express the relators in our group presentation).

$$\begin{aligned}
G_n &= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^{n-1} x_{i+1}^{-1} x_{i+2}^{-1} \ldots x_{i+(n-1)}^{-1} \ (i = 0, 1, \ldots, n-1) \rangle^{ab} \\
&= \langle x_0, x_1, \ldots, x_{n-1} \mid \underline{\underline{x_i^n x_i^{-1} x_{i+1}^{-1} x_{i+2}^{-1} \ldots x_{i+(n-1)}^{-1}}} \ (i = 0, 1, \ldots, n-1) \rangle^{ab} \\
&= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n x_i^{-1} x_{i+1}^{-1} \ldots x_{i+(n-1)}^{-1} \ (i = \underline{1}, \ldots, n-1), \underline{\underline{x_0^n x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1}}} \rangle^{ab}.
\end{aligned}$$

Given that $\{x_i, x_{i+1}, \ldots x_{i+n-1}\} = \{x_0, x_1, \ldots x_{n-1}\}$ $(i = 0, 1, \ldots n-1)$

$$\begin{aligned}
G_n &= \langle x_0, x_1, \ldots, x_{n-1} \mid \underline{\underline{x_i^n x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1}}} \ (i = 1, \ldots, n-1), x_0^n x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1} \rangle^{ab} \\
&= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1} \ (i = 1, \ldots, n-1), \underline{\underline{x_0^{-n} = x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1}}} \rangle^{ab}.
\end{aligned}$$

By replacing the expression on the left with the one on the right in $x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1} = x_0^{-n}$, we get

$$G_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \underline{\underline{x_i^n x_0^{-n}}} \ (i = 1, \ldots, n-1), x_0^{-n} = x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1} \rangle^{ab}.$$

Let $y_i = x_0^{-1} x_i$ $(i = 1, 2, \ldots n-1)$

$$\begin{aligned}
G_n = \langle & \underline{\underline{x_0, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}}} \mid x_i^n x_0^{-n}, x_0^{-n} = x_0^{-1} x_1^{-1} \ldots x_{n-1}^{-1}, \\
& \underline{\underline{y_i = x_0^{-1} x_i}} \ (i = 1, \ldots, n-1) \rangle^{ab}.
\end{aligned}$$

Eliminate $x_i = y_i x_0$ $(i \in \{1, 2, \ldots n-1\}$

$$\begin{aligned}
G_n &= \langle \underline{\underline{x_0, y_1, y_2 \ldots, y_{n-1}}} \mid \underline{\underline{(y_i x_0)^n x_0^{-n}}} \ (i = 1, \ldots, n-1), \underline{\underline{x_0^{-n} = x_0^{-1} (x_0 y_1)^{-1} \ldots (x_0 y_{n-1})^{-1}}} \rangle^{ab} \\
&= \langle x_0, y_1, y_2, \ldots, y_{n-1} \mid \underline{\underline{y_i^n}} \ (i = 1, \ldots n-1), \underline{\underline{1 = y_1^{-1} y_2^{-1} \ldots y_{n-1}^{-1}}} \rangle^{ab}.
\end{aligned}$$

Eliminate $y_1 = y_2^{-1} \ldots y_{n-1}^{-1}$

$$\begin{aligned}
G_n &= \langle \underline{\underline{x_0, y_2, \ldots, y_{n-1}}} \mid y_i^n \ (i = 2, \ldots n-1) \rangle^{ab} \\
&= \langle x_0 \mid \rangle^{ab} \oplus \langle y_2 \mid y_2^n \rangle^{ab} \oplus \langle y_3 \mid y_3^n \rangle^{ab} \oplus \oplus \langle y_{n-1} \mid y_{n-1}^n \rangle^{ab} \\
&\cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n}_{n-2}.
\end{aligned}$$

□

1.8 Application of the Smith normal form to group theory

Let G be a group, x be a vector of generators of G, and suppose G has a relation an $n \times n$ matrix M, where $Mx = 0$. Then by Smith Theorem we can determine the canonical generators of G by calculating the Smith normal form of M. Given that $M = UQT$, $Q = \operatorname{diag}(q_0, q_1, \ldots, q_s, \underbrace{0, 0, \ldots, 0}_{n-s})$, and $Mx = 0$, we can see that the new generators are Tx as $Q(Tx) = 0$. G can be thought of as the group $\mathbb{Z}^n / M\mathbb{Z}^n$, hence, the abelianization

of $\mathbb{Z}^n/M\mathbb{Z}^n$ is

$$\mathbb{Z}^n/M\mathbb{Z}^n \cong \mathbb{Z}_{q_0} \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_s} \oplus \mathbb{Z}^{n-s}.$$

The Smith normal form and Tietze Transformations can produce different abelianizations of $\mathbb{Z}^n/M\mathbb{Z}^n$. The following known lemma (see for example [29, page 47]) can be used to show that both resulting groups are isomorphic.

Lemma 1.8.1. $\mathbb{Z}_a \oplus \mathbb{Z}_b \cong \mathbb{Z}_{\gcd(a,b)} \oplus \mathbb{Z}_{\operatorname{lcm}(a,b)}$, *where* $\gcd$ *and* lcm *are the greatest common divisor and the least common multiple respectively.*

1.9 The Smith group

The Smith group is defined as follows. For a graph Γ, we can think of the adjacency matrix $A = A(\Gamma)$ as a linear map $A\colon \mathbb{Z}^{|V(\Gamma)|} \to \mathbb{Z}^{|V(\Gamma)|}$. The cokernel of A is $\mathbb{Z}^{|V(\Gamma)|}/\operatorname{Image}(A) \cong \operatorname{Sm}(\Gamma) \oplus \mathbb{Z}^t$, $t \geqslant 0$, we call the finite group $\operatorname{Sm}(\Gamma)$ the *Smith group of* Γ. Note that it is also common to define the Smith group as the cokernel of A and not only its torsion group. We chose to avoid this.

The Smith group was introduced by Rushanan in [57], where he also calculated the Smith group for strongly regular graphs and the Smith normal form of other matrices. This group has also been determined for a variety of circulant graphs (see [65]), and the Paley graph (see [12]).

1.10 The sandpile model

The frequency of certain events in nature appears to follow a *power distribution* where larger events are exponentially less frequent than smaller ones. For example, the Gutenberg–Richter law, shows that the frequency of the earthquakes is inversely proportional to their magnitude. Another example is Zipf's law, where in a data set, the frequency of an object is inversely proportional to its rank. Theses laws are characterised as a $1/f$ (or cf^{-t}) *noise*, or *fractal noise*. One attempt to explain the emergence of $1/f$ noise in such complex systems was introduced by Bak, Tang, and Wiesenfeld (see [2]), through the concept of *self organised criticality*, which can be thought of as a system that reaches *critical* states, where these states are unstable and lead to *avalanches* of different sizes and durations. In a way that is consistent with a power law distribution. The main model they used to demonstrate self organised criticality, was the *sandpile model*, which we will explain next.

Let Γ be a connected graph of order n with no loops, and select one vertex x to be *the sink*. We start by assigning a non negative number of *sand grains* $s(v)$ to every vertex $v \in V(\Gamma)\backslash\{x\}$, such that the vector

$$s = (s(v_0), s(v_1), \ldots, s(v_{n-1}))$$

represents these grains as a *sand grains configuration*. A vertex is *unstable* if it has sand grains greater than or

equal to its degree, otherwise it is *stable*. The *topple rule* on an unstable vertex, is to move one sand grain to each neighbour it has. Every vertex $v \in V(\Gamma)\backslash\{x\}$ must *topple* if it becomes unstable, one at a time in some order. An event where at least one vertex starts to topple is referred to as an *avalanche*. Given that Γ has a sink x that will remove any sand grain it receives, every avalanche that happens will eventually reach a configuration (regardless of the toppling order) where there isn't a single unstable vertex, for which Γ has a *stable* configuration. The remaining number of sand grains for a vertex v at a stable state is

$$s'(v) = -t(v)d(v) + \sum\nolimits_{(v,u)\in V(\Gamma)} t(u), \text{ where } t(v) \text{ is the number of times } v \text{ topples.}$$

Which means that the stable configuration s' can be defined as

$$s' = s - L(\Gamma)t,$$

where $t = (t(v_0), t(v_1), \ldots, t(v_{n-1})$, and $L(\Gamma)$ is the *Laplacian Matrix* of Γ. A *critical* configuration is a stable configuration that is reachable from an initial configuration

$$b = (b_0, b_1, \ldots, b_{n-1}),\ b_i \geqslant \deg(v_i), v_i \in V(\Gamma)\backslash\{x\}.$$

Adding two critical configurations will give us a configuration that reaches a critical configuration after it stabilizes. The set of critical configurations along side addition, form an abelian group which we call the *sandpile group*.

The sandpile model, is also related to the chip firing game introduced by Biggs (see [7]). For a more general background on this subject we recommend the following books by Corry & Perkinson [19], and Klivans [37].

1.11 The sandpile group

Let Γ be a connected graph. The Laplacian matrix $L = L(\Gamma)$ can be thought of as a linear map $L\colon \mathbb{Z}^{|V(\Gamma)|} \to \mathbb{Z}^{|V(\Gamma)|}$. The cokernel of L is $\mathbb{Z}^{|V(\Gamma)|}/\operatorname{Image}(L) \cong \operatorname{Sp}(\Gamma) \oplus \mathbb{Z}$, we call the finite group $\operatorname{Sp}(\Gamma)$ the *sandpile group of* Γ. The sandpile group appeared under different names in the literature such as the *component group*, the *Picard group*, the *Jacobian group* and the *critical group*.

The order of the sandpile group is known. Biggs showed that it is equal to the number of spanning trees (see [7, Theorem 6.2]). He also calculated the sandpile group of the wheel graph, and a cyclic subgroup of the sandpile group of strongly regular graphs [7, Theorems 9.2 & 10.2]. The sandpile group of complete multipartite graphs is as follows.

Theorem 1.11.1. [34, Theorem 1] *The sandpile group of* $K_{n_1,n_2,\ldots,n_k}$, $k > 2$, $n_i \geqslant 2$, $i \in \{1, 2, \ldots, k\}$ *is isomorphic to*

$$\bigoplus_{i=1}^{k} \mathbb{Z}_{N_i}{}^{n_i-2} \oplus \mathbb{Z}_g \oplus \mathbb{Z}_h \oplus \bigoplus_{i=3}^{k} \mathbb{Z}_{a_i n},$$

where

$$n = n_1 + n_2 + \cdots + n_k,$$

$$N_i = n - n_i,$$

$a_1, a_2, \ldots, a_k$ *are the invariant factors of* $\mathrm{SNF}(\mathrm{diag}(N_1, N_2, \ldots, N_k))$*, such that* $a_i | a_{i+1}$,

$$g = \gcd(k-1, N_1, N_2, \ldots, N_k),$$

$$h = a_1 a_2 / g.$$

Notice that the condition $n_i \geqslant 2$ is not explicitly made in [34, Theorem 1]. We believe it is necessary as the resulting sandpile group does not match calculations when any $n_i = 1$. For the special case $K_{\underbrace{m, m, \ldots, m}_{q}}$ we have the following corollary.

Corollary 1.11.2. [34, Corollary 5] *The sandpile group of* $K_{\underbrace{m, m, \ldots, m}_{q}}$ *is isomorphic to*

$$\mathbb{Z}_{(q-1)m}{}^{q(m-2)} \oplus \mathbb{Z}_{\gcd(q-1,m)} \oplus \mathbb{Z}_{\frac{((q-1)m)^2}{\gcd(q-1,m)}} \oplus \mathbb{Z}_{(q-1)qm^2}{}^{q-2}.$$

In general calculating the sandpile group for any graph is not easy. It has been determined for many infinite families of graphs including the complete graph and cycle graph (see [18]). For the threshold graph family which happens to be integral graphs, the sandpile group of a subclass has been determined (see [15]). We will discuss them in more details in Chapter 4.

Graphs with symmetric properties

The sandpile group for several graph families with symmetric properties has been studied. For vertex transitive graphs such as the Kneser graph $K(n, 2)$, the sandpile is calculated in [27]. As well as Paley graphs (see [12]). For Cayley graphs we see in [5] that Theorems 1.3, 6.2 and Problem 1.9, in [54] implies the following. If the underlying group G of a Cayley graph Γ is abelian, then there is a surjection $\mathrm{Sp}(\Gamma) \twoheadrightarrow G$, where $\mathrm{Sp}(\Gamma)$ is the sandpile group of Γ. Another result on Cayley graphs in [24], it is shown that if the underlying group is D_n, then the sandpile group is a direct product of two or three cyclic groups. For circulant graphs (which are a subset of Cayley graphs), the sandpile group is known for the square cycle (see [33]).

A list of known vertex transitive graphs is available on Prof. A. E. Brouwer website (see [9]). We present here their sandpile group and their Laplacian spectrum, this was calculated using Maple.

Notice that all of these sandpile groups are non-cyclic and that even though Clebsch and Shrikhande have the same eigenvalues set (with different multiplicities) their Smith Groups are quite different. We also notice that the

Table 1.1: The sandpile group of small strongly regular graphs

Graph	Laplacian Spectrum	Sandpile Group
Clebsch	$\{0^1, 4^{10}, 8^5\}$	$\mathbb{Z}_4^4 \oplus \mathbb{Z}_8^1 \oplus \mathbb{Z}_{32}^4$
Shrikhande	$\{0^1, 4^6, 8^9\}$	$\mathbb{Z}_2^1 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{32}^4$
Heawood	$\{0^1, 6^1, (3+\sqrt{2})^6, (3-\sqrt{2})^6\}$	$\mathbb{Z}_7^4 \oplus \mathbb{Z}_{21}^1$
Mobius Kantor	$\{0^1, 2^3, 4^3, 6^1, (3+\sqrt{3})^4, (3-\sqrt{3})^4\}$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_{12}^1 \oplus \mathbb{Z}_{48}^2$
Pappus	$\{0^1, 3^4, 6^1, (3+\sqrt{3})^6, (3-\sqrt{3})^6\}$	$\mathbb{Z}_2^1 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{18}^2 \oplus \mathbb{Z}_{54}^1$
Desargues	$\{0^1, 1^4, 2^5, 4^5, 5^4, 6^1\}$	$\mathbb{Z}_4^1 \oplus \mathbb{Z}_8^1 \oplus \mathbb{Z}_{40}^2 \oplus \mathbb{Z}_{120}^1$
Coxeter	$\{0^1, (4+\sqrt{2})^6, (4-\sqrt{2})^6, 1^8, 4^7\}$	$\mathbb{Z}_4^1 \oplus \mathbb{Z}_{56}^5$
Dyck	$\{0^1, 2^9, 4^9, 6^1, (3+\sqrt{5})^6, (3-\sqrt{5})^6\}$	$\mathbb{Z}_2^1 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{32}^3 \oplus \mathbb{Z}_{96}^1$
Hoffman Singleton	$\{0^1, 5^{28}, 10^{21}\}$	$\mathbb{Z}_5^8 \oplus \mathbb{Z}_{10}^1 \oplus \mathbb{Z}_{50}^{19}$

number of distinct invariant factors of the sandpile group of the graphs above seems to be bounded by the number of distinct eigenvalues.

Calculating the sandpile group using other graphs' sandpile groups

Suppose we know the sandpile group of a graph Γ. In some cases we can use this information to determine the sandpile group of another related graph that was constructed using Γ. In [18] Cori and Rossin introduced the two cases of this in theorems on planar graphs and on graphs with an articulation point as we will explain below.

Theorem 1.11.3. [18, Theorem 2] *Suppose we have a planar graph Γ and let Γ' be its dual graph. The sandpile group of Γ is isomorphic to the sandpile group of Γ'.*

If a graph Γ has a vertex v such that removing it results in a disconnected graph, we call v *an articulation point of* Γ.

Theorem 1.11.4. [18, Proposition 1.2] *Suppose we have two connected graphs Γ_1 and Γ_2, and let x be a vertex in Γ_1 and y be a vertex in Γ_2. The sandpile group of the graph resulting from tiling x and y vertices $(\Gamma_1 \cup \Gamma_2)_{\phi(x,y)}$ is isomorphic to the direct product of the sandpile groups of Γ_1 and Γ_2.*

Theorem 1.11.4 was also presented by Krepkiy later in [40].

Graphs with cyclic sandpile group

There are general results on the probability of a random graph being cyclic, as well as examples of graph families with cyclic sandpile group which were introduced in [41, 4]. We will discuss this topic in more detail in Chapter 5.

Theorem 1.11.3 tells us that as a planar graph and its dual share the same sandpile group.

In [41], a family of planar graphs, called the chain cyclic graphs (for definition see Section 1.3) are shown to have cyclic sandpile group.

Theorem 1.11.5. [41, Theorem 4] *The sandpile group of* $\mathrm{CH}(a_1, a_2, \ldots, a_n)$ *is cyclic.*

The order of $a_1, a_2, \ldots, a_n$ is irrelevant, as a consequence of Theorem 1.11.6.

Theorem 1.11.6. [41, Theorem 3] *Suppose we have two different chain cycle graphs $H_i = \mathrm{CH}(a_1, a_2, \ldots, a_n)$ and $H_j = \mathrm{CH}(b_1, b_2, \ldots, b_n)$, where $b_1, b_2, \ldots, b_n$ is a permutation of $a_1, a_2, \ldots, a_n$, then the sandpile group of H_i is isomorphic to H_j.*

1.12 Research motivation

When studying the sandpile and Smith groups for a graph Γ, we are generally interested in the following questions:

1. What is their group structure?
2. What is their rank?
3. Is their group structure similar to other graphs of similar graph structure?

In this thesis, we focus more on the sandpile group and as we go through existing results in the literature, we observe that many sandpile groups contains recurrence sequences. For example, the sandpile group of the wheel graph $C_n \bigtriangledown K_1$ is

$$\mathrm{Sp}(C_n \bigtriangledown K_1) = \begin{cases} {\mathbb{Z}_{L_n}}^2 & \text{if } n \text{ is odd;} \\ \mathbb{Z}_5 \oplus {\mathbb{Z}_{F_n}}^2 & \text{if } n \text{ is even.} \end{cases}$$

Where L_n, F_n are the Lucas and Fibonacci numbers respectively.

The wheel graph can be thought of as the graph product $C_p \bigtriangledown \overline{K_1}$, so it is natural to consider $C_p \bigtriangledown \overline{K_n}$. Here, we present a new conjecture for the graph $C_p \bigtriangledown \overline{K_2}$. This has been verified for $p \leqslant 60$ using Maple.

Conjecture 1.12.1. *Let $A_{t,q}$ be a sequence defined as $A_{t,q} = tA_{t,q-1} - A_{t,q-2}$, $A_{t,0} = 0, A_{t,1} = 1$. And let $B_{t,q}$ be a similar sequence defined as $B_{t,q} = tB_{t,q-1} - B_{t,q-2}$, $B_{t,0} = -1, B_{t,1} = 1$. The sandpile group of the graph $C_p \bigtriangledown \overline{K_2}$ is isomorphic to*

$$\begin{cases} {\mathbb{Z}_{B_{(4,(p+1)/2)}}}^2 \oplus \mathbb{Z}_p & \textit{if } n \equiv 1, 3 \mod 4 \\ {\mathbb{Z}_{B_{(14,(p+2)/4)}}}^2 \oplus \mathbb{Z}_{6p} & \textit{if } n \equiv 0 \mod 4 \\ {\mathbb{Z}_{8A_{(14,p/4)}}}^2 \oplus \mathbb{Z}_{3p/2} & \textit{if } n \equiv 2 \mod 4 \end{cases}$$

We notice that the recurrence sequences we get are not the same as the wheel graph. Another result with recurrence sequences of a wheel-like graph is presented in [53].

Other known results that contains recurrence sequences includes the graphs $K_m \bigtriangledown P_n$, $P_m \bigtriangledown P_n$ (see [60]), $K_m \otimes C_n$ (see [63]), $P_4 \otimes C_n$ (see [14]), $C_4 \otimes C_n$ (see [64]), $K_m \otimes P_n$ (see [42]) and the $3 \times n$ twisted bracelets (see [59]).

None of the graphs above are regular. However, there is reason to believe that it is common to find recurrence sequences in the sandpile group of regular graphs as well. For example, the sandpile group of the square cycle $\mathrm{Sp}({C^2}_n) = \mathbb{Z}_{\gcd(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{\mathrm{lcm}(n,F_n)}$ (see [33]). Also, when we look at the number of spanning trees (the order of their sandpile group) of a variety of circulant graphs we can find recurrence sequences (see [56, 67]).

All of this makes it reasonable to wonder which graphs have a sandpile group that does not contain a recurrence sequence? (and which do not?) Integral graphs seems to be a good candidate to explore the following reasons. All of the graphs that had recurrence sequences that we have looked at were non-integral graphs. And given that Theorem 1.6.5 explains the relationship between the eigenvalues of a graph and its group structure, it is reasonable to expect to see a simple description (not having recurrence sequences) of the sandpile group. We see this indeed for example in the cases of the complete multipartite graphs and complete graphs Cartesian products (see [34]).

Chapter 2

Integral Circulant Graphs and their Smith and Sandpile Groups

In this chapter, we examine integral circulant graphs. Mainly, we characterise the structure of circulant graphs of prime power order (see Theorem B), and determine for some of them the Smith and sandpile groups (Theorems A, C, and D, and Conjecture 2.3.6). Finally, on the number of integral circulant graphs we introduce Theorems 2.4.2, 2.4.3 and Theorem E.

2.1 Introduction

A circulant graph (Definition 1.4.3) is a Cayley graph on a cyclic group $\mathrm{Cay}(\mathbb{Z}_n, S)$, where S is a set such that $S \subseteq \mathbb{Z}_n$. It is easy to test whether a circulant graph is connected or not as in the next theorem.

Theorem 2.1.1. (See [8, Proposition 1]) *A circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$*,* $S = \{s_1, s_2, \ldots, s_m\}$ *is connected if and only if* $\gcd(n, s_1, s_2, \ldots, s_m) = 1$.

We use this as a way to identify which graphs are connected circulant graphs that we could calculate their sandpile group. $\mathrm{Cay}(\mathbb{Z}_n, S)$ has $\gcd(n, s_1, s_2, \ldots, s_m)$ isomorphic components. A standard reference for this claim comes as a corollary of Theorem 2.1.1. We present a proof in Theorem 2.1.2.

Theorem 2.1.2. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_n, S)$*,* $S = \{s_1, s_2, \ldots, s_l\}$*, such that* $\gcd(n, s_1, s_2, \ldots, s_l) = 1$*. Suppose we have* $\mathrm{Cay}(\mathbb{Z}_{rn}, T)$*,* $r \geqslant 1$*, then* $\underbrace{\Gamma \cup \Gamma \cup \cdots \cup \Gamma}_{r} \cong \mathrm{Cay}(\mathbb{Z}_{rn}, T)$ *if and only if* $T = \{rs_1, rs_2, \ldots, rs_l\}$.

Proof. Examining $\underbrace{\Gamma \cup \Gamma \cup \cdots \cup \Gamma}_{r}$, as Γ is a circulant, for all $u_i \in V(\Gamma)$, u_i is adjacent to u_{i-j}, and u_{i+j}, $i, j \in S$. In order to distinguish between different Γ in $\underbrace{\Gamma \cup \Gamma \cup \cdots \cup \Gamma}_{r}$, we label them from left to right as Γ_t, $t \in \{0, 1, \ldots, r-1\}$. We also relabel each vertex $u_i \in V(\Gamma_t)$ to v_{ir+t}. We can see that,

$$u_i = v_{ir+t} \sim v_{(i-j)r+t} = u_{i-j},$$

$$u_i = v_{ir+t} \sim v_{(i+j)r+t} = u_{i+j}.$$

This shows that $\underbrace{\Gamma \cup \Gamma \cup \cdots \cup \Gamma}_{r}$ is a circulant graph. The neighbours of v_0 are $\{v_{ri} : i \in S\}$, which implies that $\underbrace{\Gamma \cup \Gamma \cup \cdots \cup \Gamma}_{r}$ is the circulant graph $\mathrm{Cay}(\mathbb{Z}_{rn}, rS)$. Conversely, suppose that $T = \{rs_1, rs_2, \ldots, rs_l\}$, then $\gcd(rn, rs_1, rs_2, \ldots, rs_l) = r$. Partition $V(\mathrm{Cay}(\mathbb{Z}_{rn}, T))$ into $P(i) = \{v_k : 0 \leqslant k \leqslant rn-1, k \equiv i \pmod{r}\}$, $i \in \{0, 1, \ldots, r-1\}$. It is clear that $P(i)$ is a uniform partition as each $P(i)$ has the same order. Define the map $f : P(i) \to \Gamma$, $f(v_{ir+k}) = u_i$, $v_l \in V(\mathrm{Cay}(\mathbb{Z}_{rn}, T))$, $u_i \in V(\Gamma)$. f is an isomorphism as

$$f(v_{ir+k}) = u_i \sim u_{i-j} = f(v_{(i-j)r+k}),$$

$$f(v_{ir+k}) = u_i \sim u_{i+j} = f(v_{(i+j)r+k}).$$

□

As we are mainly focusing on the Smith and sandpile group of integral graphs, we will see that for circulant graphs, the eigenvalues can be defined using a subset of the divisors of n (see Theorem 2.1.7).

We can calculate the eigenvalues of circulant graphs in general as follows:

Theorem 2.1.3. (See [6, page 16]) *The eigenvalues of a circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$ *are*

$$\lambda_k = \sum_{s \in S} \omega^{sk}, \quad 0 \leqslant k \leqslant n-1,$$

where $\omega = e^{(2\pi i)/n}, i = \sqrt{-1}$. *Note that* $\omega^k,\ 0 \leqslant k \leqslant n-1$ *are the nth roots of unity.*

In [61, pages 154-156] So showed that an integral circulant graph of order n can be defined by constructing the set S using a subset of the set of divisors of n, as we will describe.

Let $D(n)$ be the set of all positive divisors of n, and let $S_n(d)$, $d \in D(n)$ be

$$S_n(d) = \{k \ : \ 1 \leqslant k \leqslant n, \gcd(k, n) = d\}.$$

A nice property of $S_n(d)$ is the following.

Proposition 2.1.4. (See [61, page 155]) *If* $d \mid n$ *then* $S_n(d) = dS_{n/d}(1)$.

Now we can define an integral circulant graph using $S_n(d)$.

Theorem 2.1.5. (See [61, Theorem 7.1 on page 157]) *The circulant graph* $\Gamma = \mathrm{Cay}(\mathbb{Z}_n, S)$ *is integral if and only if*

$$S = \bigcup_{d \in D} S_n(d), \text{ where } D \subseteq D(n).$$

Given the definition of $S_n(d)$, $s \in S$ if and only if $d = \gcd(s, n) \in D$. $S_n(d)$ is symmetric, this is due to the fact that $\gcd(k, n) = \gcd(n - k, n)$. As a result S is always symmetric which explains why $\mathrm{Cay}(\mathbb{Z}_n, S)$ is always an undirected Cayley graph.

As Theorem 2.1.5 shows, we can define an integral circulant graph Γ of order n using a set of divisors $D \subseteq D(n)$. We will denote Γ with $\mathrm{ICG}(n, D)$.

Note that if we want to allow loops, then we must have $n \in D$. This way $S_n(n) = \{0\}$ the identity element. If $n \in D$, then the adjacency matrix $A(\mathrm{ICG}(n, D))$ will have $a_{ii} = 1, i \in \{0, 1, \ldots n - 1\}$. We consider simple integral circulant graphs throughout this chapter, unless otherwise stated.

We can easily see that $|S_n(1)| = \phi(n)$, which is Euler's totient function. And by combining Proposition 2.1.4 and Theorem 2.1.5 the degree is easy to calculate

$$\deg(\Gamma) = |S| = \sum_{d \in D} d\phi(n/d).$$

Ramanujan's sum is used in calculating the eigenvalues of integral circulant graphs and is defined as follows

$$c(k, m) = \sum_{\substack{0 \leqslant j \leqslant m-1 \\ \gcd(j,m)=1}} \omega^{jk} = \sum_{s \in S_m(1)} \omega^{sk},\ \omega = e^{(2\pi i)/m}, i = \sqrt{-1}.$$

It is well known (see for example [39, Lemma 2.5, page 196]) that it is equivalent to the following integral sum

$$c(k, m) = \mu(t(k, m)) \frac{\phi(m)}{\phi(t(k, m))},$$

where $t(k, n) = n/\gcd(k, n)$, μ is the Möbius function

$$\mu(m) = \begin{cases} 0, & \text{if } m \text{ has at least one repeated prime factor} \\ 1, & \text{if } m = 1 \\ (-1)^r, & \text{if } m \text{ is a product of } r \text{ distinct prime numbers,} \end{cases}$$

and ϕ is Euler's totient function.

Theorem 2.1.6. (See [61, Theorem 5.1 on page 156]) *The eigenvalues of the integral circulant graph* $\mathrm{ICG}(n, \{d\})$, $d \in D(n)$ *are*

$$\lambda_k = c(k, n/d), \quad 0 \leqslant k \leqslant n - 1.$$

In the proof of Theorem 16 in [38, page 11], the following formula is presented. Note that integral circulant graphs are also known as *gcd-graphs*.

Corollary 2.1.7. *The eigenvalues of an integral circulant graph* $\mathrm{ICG}(n, D)$*,* $D \subseteq D(n)$ *are*

$$\lambda_k = \sum_{d \in D} c(k, n/d), \quad 0 \leqslant k \leqslant n-1.$$

2.1.1 So's conjecture

As we can define an integral circulant graph of order n using $D \subseteq D(n)$, it is natural to ask, if two different subsets of divisors (equivalently two sets) can result in two non isomorphic graphs. This is what So conjectured.

So's Conjecture 2.1.8. (See [61, Conjecture 7.3]) *Let* $\Gamma_1 = \mathrm{Cay}(\mathbb{Z}_n, S_1)$*,* $\Gamma_2 = \mathrm{Cay}(\mathbb{Z}_n, S_2)$ *be two integral circulant graphs, if* $S_1 \neq S_2$ *then the spectrum of* Γ_1 *and* Γ_2 *are different, therefore they are not isomorphic.*

So also showed that integral circulant graphs of order $n = p^r$ satisfy this conjecture (see Proposition 2.1.12, below). He also mentioned that $n = pq$ (where p, q are prime numbers) satisfy this conjecture. A partial result is proved by J.W. Sander and T. Sander (see [58, Theorem 1.2]), which includes $n = p^r$, $n = pq$. We state their result in Theorem 2.1.10, below.

First, we define the product of non empty integer sets $A_1, A_2, \ldots, A_t$ as follows

$$\prod_{j=1}^{t} A_j = \{a_1 a_2 \ldots a_t : a_j \in A_j \ (1 \leqslant j \leqslant t)\}.$$

Let p be a prime number, $\mathcal{D}$ be a set of positive integers and define the set $\mathcal{D}_p = \{p^{e_p(d)} \ : \ d \in \mathcal{D}\}$ where $e_p(d)$ is the p-adic order of d. We call $\mathcal{D}$ a *multiplicative divisor set* if and only if $\mathcal{D} = \prod_{p \in \mathbb{P}} \mathcal{D}_p$, where $\mathbb{P}$ is the set of all prime numbers. Note that in the case of a prime number q that does not divide d, then we have $\mathcal{D}_q = \{1\}$.

Examples of multiplicative divisor sets:

Example 2.1.9. *Let* $A_1 = \{1, 2, 3, 6\}$*,* $A_2 = \{2, 4\}$*,* $A_3 = \{1, 3, 6\}$ *be subsets of* $D(12) = \{1, 2, 3, 4, 6\}$*,*

A_1 *is a multiplicative divisor set as* $\mathcal{D}_2 = \{2^{e_2(d)} \ : \ d \in A_1\} = \{1, 2\}$*,*

$\mathcal{D}_3 = \{3^{e_3(d)} \ : \ d \in A_1\} = \{1, 3\}$*, and* $\mathcal{D}_2\mathcal{D}_3 = \{1, 2, 3, 6\}$*.*

A_2 *is a multiplicative divisor set as* $\mathcal{D}_2 = \{2^{e_2(d)} \ : \ d \in A_2\} = \{2, 4\}$*,*

$\mathcal{D}_3 = \{3^{e_3(d)} \ : \ d \in A_2\} = \{1\}$*, and* $\mathcal{D}_2\mathcal{D}_3 = \{2, 4\}$*.*

A_3 *is not a multiplicative divisor set as* $\mathcal{D}_2 = \{2^{e_2(d)} \ : \ d \in A_3\} = \{1, 2\}$*,*

$\mathcal{D}_3 = \{3^{e_3(d)} \ : \ d \in A_3\} = \{1, 3\}$*, and* $\mathcal{D}_2\mathcal{D}_3 = \{1, 2, 3, 6\}$*.*

Let $\mathcal{D} \subseteq D(n)$ be a multiplicative divisor set, and let $\Gamma = \mathrm{ICG}(n, D)$. We call Γ a *multiplicative divisor graph*. We will use $\mathrm{MDG}(n, D)$ to denote a multiplicative divisor graph of $\mathcal{D}$ of order n.

The *spectral vector* of an integral circulant graph $\mathrm{ICG}(n, D)$ is a vector of this form

$$\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}),$$

$\lambda_j,\ (0 \leqslant j \leqslant n-1)$ are the eigenvalues of Γ ordered as in Theorem 2.1.7.

Theorem 2.1.10. (See [58, Theorem 1.2]) *Let $\Gamma_1 = \mathrm{MDG}(n, \mathcal{D})$, $\Gamma_2 = \mathrm{MDG}(n, \mathcal{E})$ be multiplicative divisor graphs. Then Γ_1 and Γ_2 are isomorphic graphs if and only if $\vec{\lambda_1} = \vec{\lambda_2}$.*

Theorem 2.1.10 proves So's Conjecture for a subset of integral circulant graphs. It takes into account graphs that contain loops. Counting the number of integral graphs gives us a better understanding of how many graphs So's Conjecture remains open for. As there are $2^{|D(n)-1|}$ subsets of $D(n)\backslash\{n\}$ (not counting graphs with loops), we have

Proposition 2.1.11. (See [61, Corollary 7.2]) *The number of integral circulant graphs of order n is at most $2^{|D(n)-1|}$.*

If So's Conjecture is correct, then Proposition 2.1.11 represents the exact number of integral circulant graphs of order n (up to isomorphism). The next result shows it is exact for integral circulant graphs of prime power order.

Proposition 2.1.12. (See [61, page 157]) *The number of integral circulant graphs of order p^r is 2^r (up to isomorphism).*

This is consistent with Proposition 2.1.11 as $|D(2^r)| = r + 1$.

Counting the number of multiplicative divisor graphs including cases where we can have loops, we have

Proposition 2.1.13. (See [58, Proposition 1.1]) *Let $n \geqslant 1$ be an integer with prime factorization $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. The number of multiplicative divisor graphs of order n is $2^{(k_1+1)+(k_2+1)+\cdots+(k_r+1)}$.*

As one may expect, the number of multiplicative divisor graphs is much smaller than the number of integral circulant graphs as it is clear that $2^{(k_1+1)+(k_2+1)+\cdots+(k_r+1)} < 2^{(k_1+1)(k_2+1)\cdots(k_r+1)}$ for large enough values of r.

2.2 Integral circulant graphs defined using graph operations

Considering the graph operations complement, join, union, it is obvious that the complement of a circulant graph is circulant, and as the join of two graphs is the complement of their union it is reducible to these operations. Theorem 2.1.2 shows that circulant graphs can be constructed from a union of circulant graphs. Therefore, we will focus on how graph operations on integral circulant graphs changes the resulting set of divisors.

Let $X = \mathrm{ICG}(n, D)$, the set of divisors of the complement of X and a union of X is described in the next two propositions.

Proposition 2.2.1. *Let $X = \mathrm{ICG}(n, D)$, $D \subseteq D(n)\backslash\{n\}$, then the complement $\overline{X}$ has divisors set equal to $D(n)\backslash(D \cup \{n\})$.*

Proof. This follows from Theorem 2.1.5, as the set of X contains only the elements that correspond to the divisors in D. Any divisor not included is in the complement graph. □

Next we can show how the set of divisors changes under a union operation on $\mathrm{ICG}(n, D)$. The converse of the next theorem is equivalent to Theorem 2.2 in [3].

Theorem 2.2.2. *Let $X = \mathrm{ICG}(n, D)$, $D \subseteq D(n)\backslash\{n\}$, where X is a connected graph. Suppose we also have $Y = \mathrm{ICG}(rn, E)$, $r \geqslant 1$. If and only if $Y = \underbrace{X \cup X \cup \cdots \cup X}_{r}$ then $E = rD$.*

Proof. As both X, Y are circulant graphs we denote their sets by S, T respectively. Suppose $Y = \underbrace{X \cup X \cup \cdots \cup X}_{r}$, Let $X_j = X$, $j \in \{0, 1, \ldots, r-1\}$ be a labeling based on the order they appear in Y. Recalling the map in Theorem 2.1.2, a vertex $u_i \in V(X_j)$ maps into $v_{ir+j} \in V(Y)$. The set of indices of the neighbours of v_0 is equal to T, and as $u_0 \to v_0$, it should be is enough to consider the graph X_0. The set of indices of the neighbours of u_0 is equal to S. By Theorem 2.1.5, $S = \bigcup_{d \in D} S_n(d)$, $T = \bigcup_{e \in E} S_{(rn)}(e)$. As $u_i \to v_{id}$ we can see that

$$r \bigcup_{d \in D} S_n(d) = \bigcup_{rd \in rD} S_{(nr)}(rd) = \bigcup_{e \in E} S_{(nr)}((e)) = T.$$

Conversely, let $S = \{s_1, s_2, \ldots, s_l\}$, $1 \leqslant l \leqslant n-1$, if $E = rD$, then as $T = \{rs_1, rs_2, \ldots, rs_l\}$, $\gcd(rs_1, rs_2, \ldots, rs_l) = r$. By Theorem 2.1.2 Y is a union of r components isomorphic to X. □

2.2.1 The Smith group

We know the Smith group of some integral circulant graphs, for example the Smith group of the complete graph K_n (which is equivalent to $\mathrm{ICG}(n, D(n)/\{n\})$) is $\mathbb{Z}_{n-1}$ (see [65, Corollary 3.3 on page 26]).

Given Theorem 2.2.2 explains the relation between $\mathrm{ICG}(n, D)$ and $\mathrm{ICG}(rn, E)$, $D \subseteq D(n)\backslash\{n\}$, we can easily calculate the Smith group of $\mathrm{ICG}(rn, E)$ given $\mathrm{ICG}(n, D)$ as in the next proposition.

Theorem A. *Let $X = \mathrm{ICG}(n, D)$, $D \subseteq D(n)\backslash\{n\}$. Suppose we also have $Y = \mathrm{ICG}(rn, rD)$, $r \geqslant 1$. Also let G be the Smith group of X, then the Smith group of Y is the group G^r.*

Proof. Theorem 2.2.2 shows that $Y = \underbrace{X \cup X \cup \cdots \cup X}_{r}$, which implies that the adjacency matrix of Y is equivalent to the Kronecker product $I_r \otimes A(X)$. This is a block matrix where only the diagonal has $A(X)$ sub-matrices and zeros elsewhere, which tells us that the $\mathrm{SNF}(A(Y))$ has r copies of $\mathrm{SNF}(A(X))$. □

2.3 Integral circulant graphs of prime power order

Theorem 2.1.5 tells us how we can construct integral circulant graphs of order n using a subset of $D(n)$. For graphs of order $n = p^r$, where p is a prime number and $r \geqslant 1$ we show that this class of graphs is isomorphic to a

new family of graphs that can be defined recursively (see Theorem B).

Let $\mathcal{X}_p^r$ be a set of graphs, such that

$$\mathcal{X}_p^1 = \Big\{K_p\Big\},$$

$$\mathcal{X}_p^r = \Big\{\underbrace{\overline{X_i \cup X_i \cup \cdots \cup X_i}}_{p}, \underbrace{\overline{\overline{X_i} \cup \overline{X_i} \cup \cdots \cup \overline{X_i}}}_{p} : X_i \in \mathcal{X}_p^{r-1}\Big\}.$$

Notice that all the graphs in $\mathcal{X}_p^r$ are regular as K_p is regular and

$$\deg(\underbrace{\overline{X_i \cup X_i \cup \cdots \cup X_i}}_{p}) = p|X_i| - 1 - \deg(X_i),$$

$$\deg(\underbrace{\overline{\overline{X_i} \cup \overline{X_i} \cup \cdots \cup \overline{X_i}}}_{p}) = p|X_i| - 1 - (|X_i| - 1 - \deg(X_i)) = (p-1)|X_i| + \deg(X_i).$$

We denote the set of complement graphs of $\mathcal{X}_p^r$ by $\overline{\mathcal{X}_p^r}$. $\overline{\mathcal{X}_p^r}$ can also be defined as

$$\overline{\mathcal{X}_p^1} = \Big\{\overline{K_p}\Big\},$$

$$\overline{\mathcal{X}_p^r} = \Big\{\underbrace{X_i \cup X_i \cup \cdots \cup X_i}_{p}, \underbrace{\overline{X_i} \cup \overline{X_i} \cup \cdots \cup \overline{X_i}}_{p} : X_i \in \overline{\mathcal{X}_p^{r-1}}\Big\}.$$

They are also regular and their degree is obtained using the following

$$\deg(\underbrace{X_i \cup X_i \cup \cdots \cup X_i}_{p}) = \deg(X_i),$$

$$\deg(\underbrace{\overline{X_i} \cup \overline{X_i} \cup \cdots \cup \overline{X_i}}_{p}) = |X_i| - 1 - \deg(X_i).$$

Theorem B. *Let T be the set of integral circulant graph of order p^r, then T is equivalent to $\mathcal{X}_p^r \cup \overline{\mathcal{X}_p^r}$. In other words, for any $t \in T$, $x \in \mathcal{X}_p^r \cup \overline{\mathcal{X}_p^r}$, if $\deg(t) = \deg(x)$, then $t \cong x$.*

In order to prove this, we need to show that the graphs in $\mathcal{X}_p^r$ or its complement are circulant graphs. Also, we show that they are pairwise non-isomorphic (as $\deg(X_1) \neq \deg(X_2), X_1, X_2 \in \mathcal{X}_p^r$), and integral graphs, and are 2^r in total.

Proposition 2.3.1. *For all graphs $X \in \mathcal{X}_p^r$, $p \geqslant 2$. X is a connected graph and the graph $\overline{X}$ is a disconnected graph.*

Proof. For $\mathcal{X}_p^1 = \Big\{K_p\Big\}$ the claim holds. Suppose all graphs $X_i \in \mathcal{X}_p^{r-1}$ are connected. By definition of $\mathcal{X}_p^r$, in both cases $\underbrace{\overline{X_i \cup X_i \cup \cdots \cup X_i}}_{p}$, $\underbrace{\overline{\overline{X_i} \cup \overline{X_i} \cup \cdots \cup \overline{X_i}}}_{p}$, $X_i \in \mathcal{X}_p^{r-1}$, are the complement of a union of either X_i or $\overline{X_i}$. A union of graphs is a disconnected graph and its complement is a connected graph. □

Proposition 2.3.2. *Each graph in $\mathcal{X}_p^r$ is a circulant graph.*

Proof. The complement of a circulant graph is circulant. Therefore it is sufficient to show that $\underbrace{X_i \cup X_i \cup \cdots \cup X_i}_{p}$, $X_i \in \mathcal{X}_p^{r-1}$ is circulant. This follows by the induction hypothesis on r. For the base case $r = 1$, K_p is a circulant graph. Let set of graphs in $\mathcal{X}_p^{r-1}$ be circulant graphs, then by applying Theorem 2.1.2 we can see that each graph in the new set $\mathcal{X}_p^r$ is a circulant graph. □

Proposition 2.3.3. *Each graph in $\mathcal{X}_p^r$ is an integral graph.*

Proof. This follows by the induction hypothesis on r. For the base case $r = 1$, K_p is an integral graph. Assuming that the set of graphs in $\mathcal{X}_p^{r-1}$ are all integral. Theorems 1.2.1 and 1.1.1 tell us that the resulting graphs in $\mathcal{X}_p^r$ will be integral graphs. □

Proposition 2.3.4. *Let $X_1, X_2 \in \mathcal{X}_p^r$, $X_1 \neq X_2$, then $\deg(X_1) \neq \deg(X_2)$, which implies that they are pairwise non-isomorphic.*

Proof. Given the definition of $\mathcal{X}_p^n$, for any $X \in \mathcal{X}_p^n$, we have the following bound on $\deg(X)$

$$p^{n-1}(p-1) \leqslant \deg(X) \leqslant p^n - 1.$$

By induction on r, the set $\mathcal{X}_p = \{K_p\}$ contains one graph with degree $p-1$, so the base case is valid. Suppose that $\deg(W_1) \neq \deg(W_2)$, for all $W_1, W_2 \in \mathcal{X}_p^k$, $k \leqslant r-1$. Given the definition of $\mathcal{X}_p^r$, we only need to confirm that for any two graphs $Y_1, Y_2 \in \mathcal{X}_p^{r-1}$, the following is true

$$\deg(\overline{\underbrace{Y_1 \cup Y_1 \cup \cdots \cup Y_1}_{p}}) \neq \deg(\overline{\underbrace{Y_2 \cup Y_2 \cup \cdots \cup Y_2}_{p}}), \text{ and}$$

$$\deg(\overline{\underbrace{Y_1 \cup Y_1 \cup \cdots \cup Y_1}_{p}}) \neq \deg(\overline{\underbrace{\overline{Y_2} \cup \overline{Y_2} \cup \cdots \cup \overline{Y_2}}_{p}}).$$

The first part is easy as

$$\deg(\overline{\underbrace{Y_1 \cup Y_1 \cup \cdots \cup Y_1}_{p}}) = p^r - 1 - \deg(Y_1) \neq \deg(\overline{\underbrace{Y_2 \cup Y_2 \cup \cdots \cup Y_2}_{p}}) = p^r - 1 - \deg(Y_2),$$

given that $\deg(Y_1) \neq \deg(Y_2)$.

As for the second part, suppose

$$\deg(\overline{\underbrace{Y_1 \cup Y_1 \cup \cdots \cup Y_1}_{p}}) = p^r - 1 - \deg(Y_1) = \deg(\overline{\underbrace{\overline{Y_2} \cup \overline{Y_2} \cup \cdots \cup \overline{Y_2}}_{p}}) = p^r - 1 - (p^{r-1} - 1 - \deg(Y_2)).$$

We know Y_2 cannot be isomorphic to $\overline{Y_1}$ given Proposition 2.3.1. Therefore, the assumption tells us that $\deg(Y_1) =$

$p^{r-1} - 1 - \deg(Y_2)$. Suppose $Y_1 = \overline{\underbrace{Z_1 \cup Z_1 \cup \cdots \cup Z_1}_{p}}$, $Z_1 \in \mathcal{X}_p^{r-2}$, then

$$\begin{aligned}\deg(Y_1) = \deg(\overline{\underbrace{Z_1 \cup Z_1 \cup \cdots \cup Z_1}_{p}}) &= p^{r-1} - 1 - \deg(Z_1) \\ &= p^{r-1} - 1 - \deg(Y_2).\end{aligned}$$

This implies that $\deg(Y_2) = \deg(Z_1)$, which is a contradiction as the following cannot be true

$$p^{r-2} - 1 \geqslant \deg(Z_1) \geqslant p^{r-2}(p-1).$$

Otherwise, if $Y_1 = \overline{\underbrace{\overline{Z_1} \cup \overline{Z_1} \cup \cdots \cup \overline{Z_1}}_{p}}$, $Z_1 \in \mathcal{X}_p^{r-2}$, then

$$\begin{aligned}\deg(Y_1) = \deg(\overline{\underbrace{\overline{Z_1} \cup \overline{Z_1} \cup \cdots \cup \overline{Z_1}}_{p}}) &= p^{r-1} - 1 - (p^{r-2} - 1 - \deg(Z_1)) \\ &= p^{r-1} - 1 - \deg(Y_2).\end{aligned}$$

Hence, $\deg(Y_2) = p^{r-2} - 1 - \deg(Z_1)$, which also cannot be true, as we get a similar contradiction

$$p^{r-2} - 1 \geqslant \deg(Y_2) \geqslant p^{r-2}(p-1).$$

Therefore,

$$\deg(\overline{\underbrace{Y_1 \cup Y_1 \cup \cdots \cup Y_1}_{p}}) \neq \deg(\overline{\underbrace{\overline{Y_2} \cup \overline{Y_2} \cup \cdots \cup \overline{Y_2}}_{p}}).$$

□

Now we get to Theorem B proof.

Proof of Theorem B. For all graphs in $\mathcal{X}_p^r$, and $\overline{\mathcal{X}_p^r}$ we have: Proposition 2.3.1 shows that $\mathcal{X}_p^{r-1}$ are connected graphs, and $\overline{\mathcal{X}_p^r}$ are disconnected graphs. Propositions 2.3.2 show that they are circulant graphs. Propositions 2.3.3 show that they are integral graphs. Proposition 2.1.12 shows there are 2^r integral circulant graphs of order p^r. $\mathcal{X}_p^r \cap \overline{\mathcal{X}_p^r} = \varnothing$ (Proposition 2.3.4, 2.3.1), and by counting them we get $|\mathcal{X}_p^r| + |\overline{\mathcal{X}_p^r}| = 2^r$. □

2.3.1 The Smith group

Suppose we have the following graphs $\mathrm{ICG}(p^r, D)$, $\mathrm{ICG}(p^{r+1}, D)$, where $p \geqslant 2$ is a prime number, $r > 1$, $D \subseteq D(p^{r-1})$. In this section we show that their Smith groups are isomorphic in Theorem C.

Lemma 2.3.5. *Let $p \geqslant 2$ be a prime number, and let $\mathrm{ICG}(p^r, D)$, where $k, r \in \mathbb{Z}$, $0 \leqslant k \leqslant r-1$, $D \subseteq D(p^{r-1})$ and the largest divisor in D is p^k then the adjacency matrix of $\mathrm{ICG}(p^{r+1}, D)$ is the Kronecker product*

of $A(\mathrm{ICG}(p^r, D))$ and J_p, where J_p is a $p \times p$ all ones matrix.

Proof. Let $A = A(\mathrm{ICG}(p^{r+1}, D))$, $B = A(\mathrm{ICG}(p^r, D))$, then

$$A = circ_{p^{r+1}}(v), v = (v_0, v_1, \ldots, v_{p^{r+1}-1}), B = circ_{p^r}(u), u = (u_0, u_1, \ldots, u_{p^r-1}),$$

where

$$v_i = \begin{cases} 1, & \text{if } \gcd(i, p^{r+1}) = p^d, d \in D, \\ 0, & \text{otherwise} \end{cases}, u_j = \begin{cases} 1, & \text{if } \gcd(i, p^r) = p^d, d \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $v_0 = v_{p^r} = \cdots = v_{(p-1)p^r} = u_0 = 0$ as $\gcd(tp^r, p^{r+1}) = p^r$ which is greater any element in D. Therefore, by partitioning v into p partitions

$$P_0 = (v_0, v_1, \ldots v_{p^r-1}), P_1 = (v_{p^r}, v_{p^r+1}, \ldots v_{2p^r-1}), \ldots, P_{p-1} = (v_{(p-1)p^r}, v_{(p-1)p^r+1}, \ldots v_{p^{r+1}-1}),$$

we can see that for any $P_t, t \in \{0, 1, \ldots, p-1\}$, $P_t = u$. As the elements in A satisfy $a_{x,y} = a_{0,y-x} = v_{y-x} = u_{y-x \bmod p^r}, x, y \in \{0, 1, \ldots, p^{r+1}-1\}$ it is easy to see that A is a block matrix of B, which is equivalent to the Kronecker product of B and J_p, where J_p is a $p \times p$ all ones matrix. □

Theorem C. *Let p be a prime number such that $p \geqslant 2$, $D \subseteq D(p^{r-2})$, and let G be the Smith group of $\mathrm{ICG}(p^r, D)$, then the Smith group of $\mathrm{ICG}(p^{r+1}, D)$ is G.*

Proof. Using Lemma 2.3.5, we can see that $A(\mathrm{ICG}(p^{r+1}, D)) = A(\mathrm{ICG}(p^r, D)) \otimes J_p$. We can now eliminate the rows and columns $\{p^r, p^r + 1, \ldots, p^{r+1} - 1\}$, using the sub-matrix $A(\mathrm{ICG}(p^r, D))$. Then we can see that the Smith normal form of $A(\mathrm{ICG}(p^{r+1}, D))$ contains the elements in $\mathrm{SNF}(A(\mathrm{ICG}(p^r, D)))$ and $(p-1)p^r$ added zeros. □

2.3.2 The sandpile group

Let $\Gamma = \mathrm{ICG}(p^r, D)$, if Γ is connected then $1 \in D$ as otherwise $p \mid \gcd(D)$, this is easy to see using Theorem 2.1.1. As we noted in the introduction, the sandpile group of complete multipartite graphs has been determined (see Theorem 1.11.1). Therefore, while we examine the sandpile group of integral circulant graphs we would like to avoid cases where they are isomorphic to a complete multipartite graph. Considering possible values of D where this could be the case, if $D = D(p^m)$, $m \leqslant r-1$, then as $\mathrm{ICG}(p^{m+1}, D) = K_m$, Lemma 2.3.5 tells us that $\mathrm{ICG}(p^r, D)$ is $K_m \otimes J_{p^{r-m-1}}$ which is a complete multipartite graph.

Given all of the above, it is reasonable to consider $\mathrm{ICG}(p^r, \{1, p^k\})$, $2 \leqslant k \leqslant r-2$, $r \geqslant 4$. We will first present a general conjecture.

Conjecture 2.3.6. *Let $\Gamma = \mathrm{ICG}(p^r, \{1, p^k\})$, where $p \geqslant 2$ is a prime number, $2 \leqslant k \leqslant r-2$, $r \geqslant 4$. The sandpile*

group of Γ *is*

$$\begin{aligned}(\mathbb{Z}_{p-1})^{p^k-p} \oplus \mathbb{Z}_a \oplus (\mathbb{Z}_{ap^{r-k-1}})^{p^{k+1}(p^{r-k-1}-2)} \\ \oplus (\mathbb{Z}_{ap^{2(r-k-1)}})^{p^k-1} \oplus \mathbb{Z}_{abp^{2(r-k-1)}} \\ \oplus (\mathbb{Z}_{abp^{2r-2k-1}})^{p^k(p-2)} \oplus (\mathbb{Z}_{abp^{2(r-k)}})^{p} \\ \oplus (\mathbb{Z}_{abp^{2r-k-1}})^{p^k-2p} \oplus \mathbb{Z}_{abp^{2(r-1)}} \\ \oplus (\mathbb{Z}_{abp^{2r-1}})^{p-2},\end{aligned}$$

where $a = p^{k+1} - p^k + (p-1)$, $b = p^k - p^{k-1} + 1$.

Conjecture 2.3.6 has been verified using Maple for $p \in \{3, 5\}, k \in \{2, 3\}$, and $k + 2 \leqslant r \leqslant 5$. In the case of $p = 2$, we have Theorem D for $k = 2$, and we are able to go a bit higher, with k up to 5 and r up to 8. Therefore, as we checked $p = 2$ for larger values of k and r, we present the case when $p = 2$ as a separate conjecture, even though it fits Conjecture 2.3.6.

Conjecture 2.3.7. *Let* $\Gamma = \mathrm{ICG}(2^r, \{1, 2^k\})$, $2 \leqslant k \leqslant r-2$, $r \geqslant 4$. *The sandpile group of* Γ *is*

$$\begin{aligned}\mathbb{Z}_a \oplus (\mathbb{Z}_{a2^{r-k-1}})^{2^{k+1}(2^{r-k-1}-2)} \oplus (\mathbb{Z}_{a2^{2(r-k-1)}})^{2^k-1} \\ \oplus \mathbb{Z}_{ab2^{2(r-k-1)}} \oplus (\mathbb{Z}_{ab2^{2(r-k)}})^2 \\ \oplus (\mathbb{Z}_{ab2^{2r-k-1}})^{2^k-4} \oplus \mathbb{Z}_{ab2^{2(r-1)}},\end{aligned}$$

where $a = 2^k + 1$, $b = 2^{k-1} + 1$.

As $\mathrm{ICG}(2^r, \{1, 2^2\}) = \mathrm{Cay}(\mathbb{Z}_{2^r}, S)$. Using Theorem 2.1.5,

$$\begin{aligned}S &= S_{2^r}(1) \cup S_{2^r}(2^2) \\ &= \{k \mid 1 \leqslant k \leqslant n-1, \gcd(k,n) = 1\} \cup \{k \mid 1 \leqslant k \leqslant n-1, \gcd(k,n) = 2^2\} \\ &= \{1, 3, \ldots, 2^{r-1}\} \cup \{4, 12, \ldots, 2^{r-4}\}.\end{aligned}$$

Therefore, to calculate the sandpile group of $\mathrm{ICG}(2^r, \{1, 2^2\})$, we can use Tietze transformations on

$$\mathrm{Cay}(\mathbb{Z}_{2^r}, \{1, 3, \ldots, 2^{r-1}\} \cup \{4, 12, \ldots, 2^{r-4}\}).$$

For a reminder of Tietze transformation rules and convention used below, see Section 1.7.

Theorem D. *Let* $S = \{1, 3, \ldots, 2^{r-1}\} \cup \{4, 12, \ldots, 2^{r-4}\}$, *and let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_{2^r}, S)$. *The Laplacian matrix of* Γ *is* $circ_{2^r}(5(2^{r-3}), c_1, \ldots, c_{2^r-1})$, $c_i = -1$ *if* $i \in S$, $c_i = 0$ *otherwise. The Smith form of this is given by the*

abelianization of the cyclically presented group $G = G_{2^r}(x_0^{5(2^{r-3})}x_0^{c_0}x_1^{c_1}\cdots x_{2^r-1}^{c_{2^r-1}})$ *which is isomorphic to*

$$\mathbb{Z}\oplus\mathbb{Z}_5\oplus(\mathbb{Z}_{5(2^{r-3})})^{2^r-16}\oplus(\mathbb{Z}_{5(2^{2(r-3)})})^3\oplus\mathbb{Z}_{15(2^{2(r-3)})}\oplus(\mathbb{Z}_{15(2^{2(r-2)})})^2\oplus\mathbb{Z}_{15(2^{2(r-1)})}.$$

Proof. Remark: In this proof, for a set of generators with indices say $0,1,\ldots,l-1$ any calculation using their indices is modulo l.

$$G_{2^r} = \langle x_0,x_1,\ldots,x_{2^{r-1}} \mid x_i^{5(2^{r-3})}(x_{i+1}x_{i+3}\cdots x_{i+2^r-1})^{-1}(x_{i+4}x_{i+12}\cdots x_{i+2^r-4})^{-1}=1$$
$$(i\in\{0,1,\ldots,2^r-1\})\rangle^{ab}.$$

Given that $\{x_{i+1},x_{i+3},\ldots,x_{i+2^r-1}\}=\{x_1,x_3,\ldots,x_{2^r-1}\}$ $(i\in\{0,2,\ldots,2^r-2\})$,

and $\{x_{i+1},x_{i+3},\ldots,x_{i+2^r-1}\}=\{x_0,x_2,\ldots,x_{2^r-2}\}$ $(i\in\{1,3,\ldots,2^r-1\})$.

$$G_{2^r} = \langle x_0,x_1,\ldots,x_{2^{r-1}} \mid \underline{\underline{x_{2i}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_{2i+4}x_{2i+12}\cdots x_{2i+2^r-4})^{-1}}},$$
$$\underline{\underline{x_{2i+1}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_{2i+5}x_{2i+13}\cdots x_{2i+2^r-3})^{-1}}}$$
$$(i\in\{0,1,\ldots,2^{r-1}-1\})\rangle^{ab}.$$

Also, we notice that $\{x_{2i+4},x_{2i+12},\ldots,x_{2i+2^r-4}\}=\{x_4,x_{12},\ldots,x_{2^r-4}\}$ $(i\in\{0,4,\ldots,2^r-4\})$,

$\{x_{2i+4},x_{2i+12},\ldots,x_{2i+2^r-4}\}=\{x_6,x_{14},\ldots,x_{2^r-2}\}$ $(i\in\{1,5,\ldots,2^r-3\})$,

$\{x_{2i+4},x_{2i+12},\ldots,x_{2i+2^r-4}\}=\{x_8,x_{16},\ldots,x_0\}$ $(i\in\{2,6,\ldots,2^r-2\})$,

$\{x_{2i+4},x_{2i+12},\ldots,x_{2i+2^r-4}\}=\{x_{10},x_{18},\ldots,x_2\}$ $(i\in\{3,7,\ldots,2^r-1\})$,

and $\{x_{2i+5},x_{2i+13},\ldots,x_{2i+2^r-3}\}=\{x_5,x_{13},\ldots,x_{2^r-3}\}$ $(i\in\{0,4,\ldots,2^r-4\})$,

$\{x_{2i+5},x_{2i+13},\ldots,x_{2i+2^r-3}\}=\{x_7,x_{15},\ldots,x_{2^r-1}\}$ $(i\in\{1,5,\ldots,2^r-3\})$,

$\{x_{2i+5},x_{2i+13},\ldots,x_{2i+2^r-3}\}=\{x_9,x_{17},\ldots,x_1\}$ $(i\in\{2,6,\ldots,2^r-2\})$,

$\{x_{2i+5},x_{2i+13},\ldots,x_{2i+2^r-3}\}=\{x_{11},x_{19},\ldots,x_3\}$ $(i\in\{3,7,\ldots,2^r-1\})$,

$$G_{2^r} = \langle x_0,x_1,\ldots,x_{2^{r-1}} \mid \underline{\underline{x_{8i}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_4x_{12}\cdots x_{2^r-4})^{-1}}},$$
$$\underline{\underline{x_{8i+1}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_5x_{13}\cdots x_{2^r-3})^{-1}}},$$
$$\underline{\underline{x_{8i+2}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_6x_{14}\cdots x_{2^r-2})^{-1}}},$$
$$\underline{\underline{x_{8i+3}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_7x_{15}\cdots x_{2^r-1})^{-1}}},$$
$$\underline{\underline{x_{8i+4}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_8x_{16}\cdots x_0)^{-1}}},$$
$$\underline{\underline{x_{8i+5}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_9x_{17}\cdots x_1)^{-1}}},$$
$$\underline{\underline{x_{8i+6}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_{10}x_{18}\cdots x_2)^{-1}}},$$
$$\underline{\underline{x_{8i+7}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_{11}x_{19}\cdots x_3)^{-1}}}$$

$$(i \in \{0,1,\dots,2^{r-3}-1\})\rangle^{ab}.$$

Let $z_j = x_j x_{j+8} \cdots x_{j+2^r-8}$ $(j \in \{0,1,\dots,7\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \dots, x_{2^{r-1}}, z_0, z_1, \dots z_7}} \mid x_{8i}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_4x_{12}\cdots x_{2^r-4})^{-1},$$

$$x_{8i+1}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_5x_{13}\cdots x_{2^r-3})^{-1},$$

$$x_{8i+2}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_6x_{14}\cdots x_{2^r-2})^{-1},$$

$$x_{8i+3}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_7x_{15}\cdots x_{2^r-1})^{-1},$$

$$x_{8i+4}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_8x_{16}\cdots x_0)^{-1},$$

$$x_{8i+5}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_9x_{17}\cdots x_1)^{-1},$$

$$x_{8i+6}^{5(2^{r-3})}(x_1x_3\cdots x_{2^r-1})^{-1}(x_{10}x_{18}\cdots x_2)^{-1},$$

$$x_{8i+7}^{5(2^{r-3})}(x_0x_2\cdots x_{2^r-2})^{-1}(x_{11}x_{19}\cdots x_3)^{-1} \ (i \in \{0,1,\dots,2^{r-3}-1\}),$$

$$\underline{\underline{z_j = x_j x_{j+8} \cdots x_{j+2^r-8}}} \ (j \in \{0,1,\dots,7\})\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_{2^{r-1}}, z_0, z_1, \dots z_7 \mid$$

$$\underline{\underline{x_{8i}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_4^{-1}}}, \underline{\underline{x_{8i+1}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_5^{-1}}},$$

$$\underline{\underline{x_{8i+2}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_6^{-1}}}, \underline{\underline{x_{8i+3}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_7^{-1}}},$$

$$\underline{\underline{x_{8i+4}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_0^{-1}}}, \underline{\underline{x_{8i+5}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_1^{-1}}},$$

$$\underline{\underline{x_{8i+6}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_2^{-1}}}, \underline{\underline{x_{8i+7}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_3^{-1}}} \ (i \in \{0,1,\dots,2^{r-3}-1\}),$$

$$z_j = x_j x_{j+8} \cdots x_{j+2^r-8} \ (j \in \{0,1,\dots,7\})\rangle^{ab}.$$

Let $y_0 = z_0z_2z_4z_6, y_1 = z_1z_3z_5z_7$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \dots, x_{2^{r-1}}, z_0, z_1, \dots z_7, y_0, y_1}} \mid$$

$$x_{8i}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_4^{-1}, x_{8i+1}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_5^{-1},$$

$$x_{8i+2}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_6^{-1}, x_{8i+3}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_7^{-1},$$

$$x_{8i+4}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_0^{-1}, x_{8i+5}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_1^{-1},$$

$$x_{8i+6}^{5(2^{r-3})}(z_1z_3z_5z_7)^{-1}z_2^{-1}, x_{8i+7}^{5(2^{r-3})}(z_0z_2z_4z_6)^{-1}z_3^{-1}$$

$$(i \in \{0,1,\dots,2^{r-3}-1\}),$$

$$z_j = x_j x_{j+8} \cdots x_{j+2^r-8} \ (j \in \{0,1,\dots,7\}),$$

$$\underline{\underline{y_0 = z_0z_2z_4z_6, y_1 = z_1z_3z_5z_7}}\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_{2^{r-1}}, z_0, z_1, \dots z_7, y_0, y_1 \mid \underline{\underline{x_{8i}^{5(2^{r-3})}y_1^{-1}z_4^{-1}}}, \underline{\underline{x_{8i+1}^{5(2^{r-3})}y_0^{-1}z_5^{-1}}},$$

$$\underline{\underline{x_{8i+2}^{5(2^{r-3})}y_1^{-1}z_6^{-1}}}, \underline{\underline{x_{8i+3}^{5(2^{r-3})}y_0^{-1}z_7^{-1}}},$$

$$\underline{\underline{x_{8i+4}^{5(2^{r-3})}y_1^{-1}z_0^{-1}}}, \underline{\underline{x_{8i+5}^{5(2^{r-3})}y_0^{-1}z_1^{-1}}},$$

$$\underline{\underline{x_{8i+6}^{5(2^{r-3})}y_1^{-1}z_2^{-1}}}, \underline{\underline{x_{8i+7}^{5(2^{r-3})}y_0^{-1}z_3^{-1}}} \ (i \in \{0, 1, \ldots, 2^{r-3}-1\}),$$

$$z_j = x_j x_{j+8} \cdots x_{j+2^r-8} \ (j \in \{0, 1, \ldots, 7\}),$$

$$y_0 = z_0 z_2 z_4 z_6, y_1 = z_1 z_3 z_5 z_7 \rangle^{ab}.$$

Eliminate $z_{2k+4} = x_{2k}^{5(2^{r-3})}y_1^{-1}, z_{2k+5} = x_{2i+1}^{5(2^{r-3})}y_0^{-1}$ $(k \in \{0, 1, 2, 3\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_{2^{r-1}}, y_0, y_1}} \mid$$

$$\underline{\underline{x_{8i}^{5(2^{r-3})}y_1^{-1}(x_0^{5(2^{r-3})}y_1^{-1})^{-1}}}, \underline{\underline{x_{8i+1}^{5(2^{r-3})}y_0^{-1}(x_1^{5(2^{r-3})}y_0^{-1})^{-1}}},$$

$$\underline{\underline{x_{8i+2}^{5(2^{r-3})}y_1^{-1}(x_2^{5(2^{r-3})}y_1^{-1})^{-1}}}, \underline{\underline{x_{8i+3}^{5(2^{r-3})}y_0^{-1}(x_3^{5(2^{r-3})}y_0^{-1})^{-1}}},$$

$$\underline{\underline{x_{8i+4}^{5(2^{r-3})}y_1^{-1}(x_4^{5(2^{r-3})}y_1^{-1})^{-1}}}, \underline{\underline{x_{8i+5}^{5(2^{r-3})}y_0^{-1}(x_5^{5(2^{r-3})}y_0^{-1})^{-1}}},$$

$$\underline{\underline{x_{8i+6}^{5(2^{r-3})}y_1^{-1}(x_6^{5(2^{r-3})}y_1^{-1})^{-1}}}, \underline{\underline{x_{8i+7}^{5(2^{r-3})}y_0^{-1}(x_7^{5(2^{r-3})}y_0^{-1})^{-1}}}$$

$$(i \in \{\underline{1}, 2, \ldots, 2^{r-3}-1\}),$$

$$\underline{\underline{(x_{2k}^{5(2^{r-3})}y_1^{-1}) = x_{2k+4}x_{2k+12}\cdots x_{2k+2^r-4}}},$$

$$\underline{\underline{(x_{2k+1}^{5(2^{r-3})}y_0^{-1}) = x_{2k+5}x_{2k+13}\cdots x_{2k+2^r-3}}} \ (k \in \{0, 1, 2, 3\}),$$

$$\underline{\underline{y_0 = (x_4^{5(2^{r-3})}y_1^{-1})(x_6^{5(2^{r-3})}y_1^{-1})(x_0^{5(2^{r-3})}y_1^{-1})(x_2^{5(2^{r-3})}y_1^{-1})}},$$

$$\underline{\underline{y_1 = (x_5^{5(2^{r-3})}y_0^{-1})(x_7^{5(2^{r-3})}y_0^{-1})(x_1^{5(2^{r-3})}y_0^{-1})(x_3^{5(2^{r-3})}y_0^{-1})}}\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_{2^{r-1}}, y_0, y_1 \mid$$

$$\underline{\underline{(x_{8i+j}x_j^{-1})^{5(2^{r-3})}}} \ (i \in \{1, 2, \ldots, 2^{r-3}-1\}, j \in \{0, 1, \ldots, 7\}),$$

$$\underline{\underline{(x_{2k}^{5(2^{r-3})}y_1^{-1}) = x_{2k+4}x_{2k+12}\cdots x_{2k+2^r-4}}},$$

$$\underline{\underline{(x_{2k+1}^{5(2^{r-3})}y_0^{-1}) = x_{2k+5}x_{2k+13}\cdots x_{2k+2^r-3}}} \ (k \in \{0, 1, 2, 3\}),$$

$$\underline{\underline{y_0 = (x_0x_2x_4x_6)^{5(2^{r-3})}y_1^{-4}}}, \underline{\underline{y_1 = (x_1x_3x_5x_7)^{5(2^{r-3})}y_0^{-4}}}\rangle^{ab}.$$

Let $a_{8i+j} = x_{8i+j}x_j^{-1}$ $(i \in \{1, 2, \ldots, 2^{r-3}-1\}, j \in \{0, 1, \ldots, 7\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_{2^{r-1}}, y_0, y_1, a_8, a_9, \ldots, a_{2^r-1}}} \mid (x_{8i+j}x_j^{-1})^{5(2^{r-3})},$$

$$(x_{2k}^{5(2^{r-3})}y_1^{-1}) = x_{2k+4}x_{2k+12}\cdots x_{2k+2^r-4},$$

$$(x_{2k+1}^{5(2^{r-3})}y_0^{-1}) = x_{2k+5}x_{2k+13}\cdots x_{2k+2^r-3} \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0x_2x_4x_6)^{5(2^{r-3})}y_1^{-4}, y_1 = (x_1x_3x_5x_7)^{5(2^{r-3})}y_0^{-4}$$

$$\underline{\underline{a_{8i+j} = x_{8i+j}x_j^{-1}}}(i \in \{1, 2, \ldots, 2^{r-3}-1\}, j \in \{0, 1, \ldots, 7\})\rangle^{ab}.$$

Eliminate $x_{8i+j} = a_{8i+j}x_j$ $(i \in \{1, 2, \ldots, 2^{r-3} - 1\}, j \in \{0, 1, \ldots, 7\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_7, y_0, y_1, a_8, a_9, \ldots, a_{2^r-1}}} \mid \underline{\underline{a_i^{5(2^{r-3})}}} (i \in \{8, 9, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(x_{2k}^{5(2^{r-3})} y_1^{-1}) = x_{2k+4}^{2^{r-3}} (a_{2k+12} a_{2k+20} \cdots a_{2k+2^r-4})}},$$

$$\underline{\underline{(x_{2k+1}^{5(2^{r-3})} y_0^{-1}) = x_{2k+5}^{2^{r-3}} (a_{2k+13} a_{2k+21} \cdots a_{2k+2^r-3})}} \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0 x_2 x_4 x_6)^{5(2^{r-3})} y_1^{-4}, y_1 = (x_1 x_3 x_5 x_7)^{5(2^{r-3})} y_0^{-4} \rangle^{ab}.$$

Let $b_j = a_{8+j} a_{16+j} \cdots a_{2^r-8+j}$ $(j \in \{0, 1, \ldots, 7\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_7, y_0, y_1, a_8, a_9, \ldots, a_{2^r-1}, b_0, b_1, \ldots, b_7}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{8, 9, \ldots, 2^r - 1\}),$$

$$(x_{2k}^{5(2^{r-3})} y_1^{-1}) = x_{2k+4}^{2^{r-3}} (a_{2k+12} a_{2k+20} \cdots a_{2k+2^r-4}),$$

$$(x_{2k+1}^{5(2^{r-3})} y_0^{-1}) = x_{2k+5}^{2^{r-3}} (a_{2k+13} a_{2k+21} \cdots a_{2k+2^r-3}) \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0 x_2 x_4 x_6)^{5(2^{r-3})} y_1^{-4}, y_1 = (x_1 x_3 x_5 x_7)^{5(2^{r-3})} y_0^{-4},$$

$$\underline{\underline{b_j = a_{8+j} a_{16+j} \cdots a_{2^r-8+j}}} \ (j \in \{0, 1, \ldots, 7\}) \rangle^{ab}.$$

Eliminate $a_{8+j} = b_j (a_{16+j} \cdots a_{2^r-8+j})^{-1}$ $(j \in \{0, 1, \ldots, 7\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_7, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, b_0, b_1, \ldots, b_7}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{\underline{\underline{16}}, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(b_j (a_{16+j} \cdots a_{2^r-8+j})^{-1})^{5(2^{r-3})}}} \ (j \in \{0, 1, \ldots, 7\}),$$

$$\underline{\underline{(x_{2k}^{5(2^{r-3})} y_1^{-1}) = x_{2k+4}^{2^{r-3}} b_{2k+4}}},$$

$$\underline{\underline{(x_{2k+1}^{5(2^{r-3})} y_0^{-1}) = x_{2k+5}^{2^{r-3}} b_{2k+5}}} \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0 x_2 x_4 x_6)^{5(2^{r-3})} y_1^{-4}, y_1 = (x_1 x_3 x_5 x_7)^{5(2^{r-3})} y_0^{-4} \rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_7, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, b_0, b_1, \ldots, b_7 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{b_j^{5(2^{r-3})}}} \ (j \in \{0, 1, \ldots, 7\}), (x_{2k}^{5(2^{r-3})} y_1^{-1}) = x_{2k+4}^{2^{r-3}} b_{2k+4},$$

$$(x_{2k+1}^{5(2^{r-3})} y_0^{-1}) = x_{2k+5}^{2^{r-3}} b_{2k+5} \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0 x_2 x_4 x_6)^{5(2^{r-3})} y_1^{-4}, y_1 = (x_1 x_3 x_5 x_7)^{5(2^{r-3})} y_0^{-4} \rangle^{ab}.$$

Eliminate $b_{2k+4} = (x_{2k}^5 x_{2k+4}^{-1})^{2^{r-3}} y_1^{-1}, b_{2k+5} = (x_{2k+1}^5 x_{2k+5}^{-1})^{2^{r-3}} y_0^{-1}$ $(k \in \{0, 1, 2, 3\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_7, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{((x_{2k}^5 x_{2k+4}^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((x_{2k+1}^5 x_{2k+5}^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}}} \ (k \in \{0, 1, 2, 3\}),$$

$$y_0 = (x_0x_2x_4x_6)^{5(2^{r-3})}y_1^{-4}, y_1 = (x_1x_3x_5x_7)^{5(2^{r-3})}y_0^{-4}\rangle^{ab}.$$

Let $c_k = x_k^5x_{k+4}^{-1}$ $(k \in 0,1,2,3)$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, \ldots, x_7, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, c_2, c_3}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$((x_{2k}^5x_{2k+4}^{-1})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}, ((x_{2k+1}^5x_{2k+5}^{-1})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})},$$

$$y_0 = (x_0x_2x_4x_6)^{5(2^{r-3})}y_1^{-4}, y_1 = (x_1x_3x_5x_7)^{5(2^{r-3})}y_0^{-4},$$

$$\underline{\underline{c_k = x_k^5x_{k+4}^{-1}}} \ (k \in 0,1,2,3)\rangle^{ab}.$$

Eliminate $x_{k+4} = x_k^5c_k^{-1}$ $(k \in 0,1,2,3)$

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, x_2, x_3, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, c_2, c_3}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{((x_{2k}^5(x_{2k}^5c_{2k}^{-1})^{-1})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((x_{2k+1}^5(x_{2k+1}^5c_{2k+1}^{-1})^{-1})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{(((x_{2k}^5c_{2k}^{-1})^5x_{2k}^{-1})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{(((x_{2k+1}^5c_{2k+1}^{-1})^5x_{2k+1}^{-1})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}} \ (k \in \{0,1\}),$$

$$\underline{\underline{y_0 = (x_0x_2(x_0^5c_0^{-1})(x_2^5c_2^{-1}))^{5(2^{r-3})}y_1^{-4}}},$$

$$\underline{\underline{y_1 = (x_1x_3(x_1^5c_1^{-1})(x_3^5c_3^{-1}))^{5(2^{r-3})}y_0^{-4}}}\rangle^{ab}$$

$$= \langle x_0, x_1, x_2, x_3, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, c_2, c_3 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(c_{2k}^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{(c_{2k+1}^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{((x_{2k}^{24}c_{2k}^{-5})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((x_{2k+1}^{24}c_{2k+1}^{-5})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}} \ (k \in \{0,1\}),$$

$$\underline{\underline{y_0 = (x_0^6x_2^6c_0^{-1}c_2^{-1})^{5(2^{r-3})}y_1^{-4}}}, \underline{\underline{y_1 = (x_1^6x_3^6c_1^{-1}c_3^{-1})^{5(2^{r-3})}y_0^{-4}}}\rangle^{ab}.$$

Multiply $(c_{2k}^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}$ into $((x_{2k}^{24}c_{2k}^{-5})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}$,

and $(c_{2k+1}^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}$ into $((x_{2k+1}^{24}c_{2k+1}^{-5})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle x_0, x_1, x_2, x_3, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, c_2, c_3 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(c_{2k}^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{(c_{2k+1}^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{(x_{2k}^4c_{2k}^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(x_{2k+1}^4c_{2k+1}^{-1})^{15(2^{2r-5})}}} \ (k \in \{0,1\}),$$

$$y_0 = (x_0^6x_2^6c_0^{-1}c_2^{-1})^{5(2^{r-3})}y_1^{-4}, y_1 = (x_1^6x_3^6c_1^{-1}c_3^{-1})^{5(2^{r-3})}y_0^{-4}\rangle^{ab}.$$

Let $d_k = c_k c_{k+2}, e_k = x_k c_{k+2}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, x_2, x_3, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, c_2, c_3, d_0, d_1, e_0, e_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_{2k}^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, (c_{2k+1}^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$

$$(x_{2k}^4 c_{2k}^{-1})^{15(2^{2r-5})}, (x_{2k+1}^4 c_{2k+1}^{-1})^{15(2^{2r-5})} \ (k \in \{0,1\}),$$

$$y_0 = (x_0^6 x_2^6 c_0^{-1} c_2^{-1})^{5(2^{r-3})} y_1^{-4}, y_1 = (x_1^6 x_3^6 c_1^{-1} c_3^{-1})^{5(2^{r-3})} y_0^{-4},$$

$$\underline{\underline{d_k = c_k c_{k+2}, e_k = x_k c_{k+2}}} \rangle^{ab}.$$

Eliminate $c_{k+2} = d_k c_k^{-1}, x_{k+2} = e_k x_k^{-1}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, d_0, d_1, e_0, e_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}, \underline{\underline{(x_0^4 c_0^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(x_1^4 c_1^{-1})^{15(2^{2r-5})}}},$$

$$\underline{\underline{((d_0 c_0^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((d_1 c_1^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{((e_0 x_0^{-1})^4 (d_0 c_0^{-1})^{-1})^{15(2^{2r-5})}}}, \underline{\underline{((e_1 x_1^{-1})^4 (d_1 c_1^{-1})^{-1})^{15(2^{2r-5})}}},$$

$$\underline{\underline{y_0 = (x_0^6 (e_k x_0^{-1})^6 c_0^{-1} (d_0 c_0^{-1})^{-1})^{5(2^{r-3})} y_1^{-4}}},$$

$$\underline{\underline{y_1 = (x_1^6 (e_1 x_1^{-1})^6 c_1^{-1} (d_1 c_1^{-1})^{-1})^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}.$$

Multiply $(c_0^{2^{r-3}} y_1^{-1})^{-5(2^{r-3})}$ into $((d_0 c_0^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}$,

$(c_1^{2^{r-3}} y_0^{-1})^{-5(2^{r-3})}$ into $((e_1 x_1^{-1})^4 (d_1 c_1^{-1})^{-1})^{15(2^{2r-5})}$,

and $(x_0^4 c_0^{-1})^{15(2^{2r-5})}$ into $((e_0 x_0^{-1})^4 (d_0 c_0^{-1})^{-1})^{15(2^{2r-5})}$,

$(x_1^4 c_1^{-1})^{15(2^{2r-5})}$ into $((d_1 c_1^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}$,

$$G_{2^r} = \langle x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, d_0, d_1, e_0, e_1 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}, (x_0^4 c_0^{-1})^{15(2^{2r-5})}, (x_1^4 c_1^{-1})^{15(2^{2r-5})},$$

$$\underline{\underline{(d_0 c_0^{-2})^{5(2^{2(r-3)})}}}, \underline{\underline{(d_1 c_1^{-2})^{5(2^{2(r-3)})}}}, \underline{\underline{(e_0^4 d_0^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(e_1^4 d_1^{-1})^{15(2^{2r-5})}}},$$

$$\underline{\underline{y_0 = (e_0^6 d_0^{-1})^{5(2^{r-3})} y_1^{-4}}}, \underline{\underline{y_1 = (e_1^6 d_1^{-1})^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}.$$

Let $f_k = d_k c_k^{-2}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, d_0, d_1, e_0, e_1, f_0, f_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}, (x_0^4 c_0^{-1})^{15(2^{2r-5})}, (x_1^4 c_1^{-1})^{15(2^{2r-5})},$$

$$(d_0c_0^{-2})^{5(2^{2(r-3)})}, (d_1c_1^{-2})^{5(2^{2(r-3)})}, (e_0^4d_0^{-1})^{15(2^{2r-5})}, (e_1^4d_1^{-1})^{15(2^{2r-5})},$$

$$y_0 = (e_0^6d_0^{-1})^{5(2^{r-3})}y_1^{-4}, y_1 = (e_1^6d_1^{-1})^{5(2^{r-3})}y_0^{-4}, \underline{\underline{f_k = d_kc_k^{-2}}}\ (k \in \{0,1\})\rangle^{ab}.$$

Eliminate $d_k = f_kc_k^2$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, e_0, e_1, f_0, f_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}, (x_0^4c_0^{-1})^{15(2^{2r-5})}, (x_1^4c_1^{-1})^{15(2^{2r-5})},$$

$$\underline{\underline{((f_0c_0^2)c_0^{-2})^{5(2^{2(r-3)})}}}, \underline{\underline{((f_1c_1^2)c_1^{-2})^{5(2^{2(r-3)})}}},$$

$$\underline{\underline{(e_0^4(f_0c_0^2)^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(e_1^4(f_1c_1^2)^{-1})^{15(2^{2r-5})}}},$$

$$\underline{\underline{y_0 = (e_0^6(f_0c_0^2)^{-1})^{5(2^{r-3})}y_1^{-4}}}, \underline{\underline{y_1 = (e_1^6(f_1c_1^2)^{-1})^{5(2^{r-3})}y_0^{-4}}}\rangle^{ab}$$

$$= \langle x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, e_0, e_1, f_0, f_1 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}, (x_0^4c_0^{-1})^{15(2^{2r-5})}, (x_1^4c_1^{-1})^{15(2^{2r-5})},$$

$$\underline{\underline{f_0^{5(2^{2(r-3)})}}}, \underline{\underline{f_1^{5(2^{2(r-3)})}}}, \underline{\underline{(e_0^2c_0^{-1})^{15(2^{2(r-2)})}}}, \underline{\underline{(e_1^2c_1^{-1})^{15(2^{2(r-2)})}}},$$

$$y_0 = (e_0^6(f_0c_0^2)^{-1})^{5(2^{r-3})}y_1^{-4}, y_1 = (e_1^6(f_1c_1^2)^{-1})^{5(2^{r-3})}y_0^{-4}\rangle^{ab}.$$

Let $g_k = e_k^2c_k^{-1}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, c_0, c_1, e_0, e_1, f_0, f_1, g_0, g_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(c_0^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}, (c_1^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}, (x_0^4c_0^{-1})^{15(2^{2r-5})}, (x_1^4c_1^{-1})^{15(2^{2r-5})},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, (e_0^2c_0^{-1})^{15(2^{2(r-3)})}, (e_1^2c_1^{-1})^{15(2^{2(r-2)})},$$

$$y_0 = (e_0^6(f_0c_0^2)^{-1})^{5(2^{r-2})}y_1^{-4}, y_1 = (e_1^6(f_1c_1^2)^{-1})^{5(2^{r-3})}y_0^{-4},$$

$$\underline{\underline{g_k = e_k^2c_k^{-1}}}\ (k \in \{0,1\})\rangle^{ab}.$$

Eliminate $c_k = e_k^2g_k^{-1}$ $(k \in \{0,1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, e_0, e_1, f_0, f_1, g_0, g_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{((e_0^2g_0^{-1})^{2^{r-3}}y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((e_1^2g_1^{-1})^{2^{r-3}}y_0^{-1})^{5(2^{r-3})}}},$$

$$\underline{\underline{(x_0^4(e_0^2g_0^{-1})^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(x_1^4(e_1^2g_1^{-1})^{-1})^{15(2^{2r-5})}}},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, \underline{\underline{(e_0^2(e_0^2g_0^{-1})^{-1})^{15(2^{2(r-2)})}}}, \underline{\underline{(e_1^2(e_1^2g_1^{-1})^{-1})^{15(2^{2(r-2)})}}},$$

$$\underline{\underline{y_0 = (e_0^6(f_0(e_0^2g_0^{-1})^2)^{-1})^{5(2^{r-3})}y_1^{-4}}}, \underline{\underline{y_1 = (e_1^6(f_1(e_1^2g_1^{-1})^2)^{-1})^{5(2^{r-3})}y_0^{-4}}}\rangle^{ab}$$

$$= \langle x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, e_0, e_1, f_0, f_1, g_0, g_1 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$((e_0^2 g_0^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, ((e_1^2 g_1^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$(x_0^4 (e_0^2 g_0^{-1})^{-1})^{15(2^{2r-5})}, (x_1^4 (e_1^2 g_1^{-1})^{-1})^{15(2^{2r-5})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, \underline{\underline{g_0^{15(2^{2(r-2)})}}}, \underline{\underline{g_1^{15(2^{2(r-2)})}}},$$
$$\underline{\underline{y_0 = (f_0^{-1} e_0^2 g_0^2)^{5(2^{r-3})} y_1^{-4}}}, \underline{\underline{y_1 = (f_1^{-1} e_1^2 g_1^2)^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}.$$

Let $h_k = e_k g_k$ $(k \in \{0, 1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, e_0, e_1, f_0, f_1, g_0, g_1, h_0, h_1}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$((e_0^2 g_0^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, ((e_1^2 g_1^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$(x_0^4 (e_0^2 g_0^{-1})^{-1})^{15(2^{2r-5})}, (x_1^4 (e_1^2 g_1^{-1})^{-1})^{15(2^{2r-5})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$y_0 = (f_0^{-1} e_0^2 g_0^2)^{5(2^{r-3})} y_1^{-4}, y_1 = (f_1^{-1} e_1^2 g_1^2)^{5(2^{r-3})} y_0^{-4},$$
$$\underline{\underline{h_k = e_k g_k \ (k \in \{0, 1\})}} \rangle^{ab}.$$

Eliminate $e_k = h_k g_k^{-1}$ $(k \in \{0, 1\})$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\underline{\underline{(((h_0 g_0^{-1})^2 g_0^{-1})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{(((h_1 g_1^{-1})^2 g_1^{-1})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}}},$$
$$\underline{\underline{(x_0^4 ((h_0 g_0^{-1})^2 g_0^{-1})^{-1})^{15(2^{2r-5})}}}, \underline{\underline{(x_1^4 ((h_1 g_1^{-1})^2 g_1^{-1})^{-1})^{15(2^{2r-5})}}},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$\underline{\underline{y_0 = (f_0^{-1} (h_0 g_0^{-1})^2 g_0^2)^{5(2^{r-3})} y_1^{-4}}}, \underline{\underline{y_1 = (f_1^{-1} (h_1 g_1^{-1})^2 g_1^2)^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}$$

$$= \langle x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1 \mid,$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\})$$
$$\underline{\underline{((h_0^2 g_0^{-3})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}}}, \underline{\underline{((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}}},$$
$$\underline{\underline{(x_0^4 h_0^{-2} g_0^3)^{15(2^{2r-5})}}}, \underline{\underline{(x_1^4 h_1^{-2} g_1^3)^{15(2^{2r-5})}}},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$\underline{\underline{y_0 = (f_0^{-1} h_0^2)^{5(2^{r-3})} y_1^{-4}}}, \underline{\underline{y_1 = (f_1^{-1} h_1^2)^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}$$

$$= \langle x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1 \mid,$$

$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\})$$
$$((h_0^2 g_0^{-3})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, ((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$\underline{\underline{(x_0^2 h_0^{-1})^{15(2^{2(r-2)})}}}, \underline{\underline{(x_1^2 h_1^{-1})^{15(2^{2(r-2)})}}},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$y_0 = (f_0^{-1} h_0^2)^{5(2^{r-3})} y_1^{-4}, y_1 = (f_1^{-1} h_1^2)^{5(2^{r-3})} y_0^{-4} \rangle^{ab}.$$

Let $s = y_0^{-1} y_1$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, y_1, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s}} \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\})$$
$$((h_0^2 g_0^{-3})^{2^{r-3}} y_1^{-1})^{5(2^{r-3})}, ((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$(x_0^2 h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2 h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$y_0 = (f_0^{-1} h_0^2)^{5(2^{r-3})} y_1^{-4}, y_1 = (f_1^{-1} h_1^2)^{5(2^{r-3})} y_0^{-4}, \underline{\underline{s = y_0^{-1} y_1}} \rangle^{ab}.$$

Eliminate $y_1 = s y_0$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s}} \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\underline{\underline{((h_0^2 g_0^{-3})^{2^{r-3}} (s y_0)^{-1})^{5(2^{r-3})}}}, ((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$(x_0^2 h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2 h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$y_0 = (f_0^{-1} h_0^2)^{5(2^{r-3})} (s y_0)^{-4}, \underline{\underline{(s y_0) = (f_1^{-1} h_1^2)^{5(2^{r-3})} y_0^{-4}}} \rangle^{ab}$$

$$= \langle x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\underline{\underline{((h_0^2 g_0^{-3})^{2^{r-3}} (s y_0)^{-1})^{5(2^{r-3})}}}, ((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})},$$
$$(x_0^2 h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2 h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$
$$y_0^5 = s^{-4} (f_0^{-1} h_0^2)^{5(2^{r-3})}, \underline{\underline{y_0^5 = s^{-1} (f_1^{-1} h_1^2)^{5(2^{r-3})}}} \rangle^{ab}.$$

Substitute $y_0^5 = s^{-4} (f_0^{-1} h_0^2)^{5(2^{r-3})}$ with $((h_0^2 g_0^{-3})^{2^{r-3}} (s y_0)^{-1})^{5(2^{r-3})}$,

and $y_0^5 = s^{-1} (f_1^{-1} h_1^2)^{5(2^{r-3})}$ into $((h_1^2 g_1^{-3})^{2^{r-3}} y_0^{-1})^{5(2^{r-3})}$,

$$G_{2^r} = \langle x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{((h_0^2g_0^{-3})^{5(2^{r-3})}(s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})})^{-1}s^{-5})^{(2^{r-3})}}},$$

$$\underline{\underline{((h_1^2g_1^{-3})^{5(2^{r-3})}(s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})})^{-1})^{(2^{r-3})}}},$$

$$(x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$

$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}\rangle^{ab}$$

$$= \langle x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}}}, \underline{\underline{g_1^{15(2^{2(r-3)})} = s^{2^{r-3}}}},$$

$$(x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$

$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}\rangle^{ab}.$$

Substitute $s^{2^{r-3}} = g_0^{15(2^{2(r-3)})}$ with $g_1^{15(2^{2(r-3)})} = s^{2^{r-3}}$,

$$G_{2^r} = \langle x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, \underline{\underline{g_1^{15(2^{2(r-3)})} = g_0^{15(2^{2(r-3)})}}},$$

$$(x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$

$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}\rangle^{ab}.$$

Let $t = g_0^{-1}g_1$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, g_1, h_0, h_1, s, t}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, g_1^{15(2^{2(r-3)})} = g_0^{15(2^{2(r-3)})},$$

$$(x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$

$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, g_1^{15(2^{2(r-2)})},$$

$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}, \underline{\underline{t = g_0^{-1}g_1}}\rangle^{ab}.$$

Eliminate $g_1 = tg_0$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, h_0, h_1, s, t}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, \underline{\underline{(tg_0)^{15(2^{2(r-3)})} = g_0^{15(2^{2(r-3)})}}},$$

$$(x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{(tg_0)^{15(2^{2(r-2)})}}},$$
$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}\rangle^{ab}$$
$$= \langle x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, h_0, h_1, s, t \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, \underline{\underline{t^{15(2^{2(r-3)})}}}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})}\rangle^{ab}.$$

Let $u_0 = f_0^{-1}f_1, u_1 = h_0^{-1}h_1, u_2 = x_0^{-1}x_1$,

$$G_{2^r} = \langle \underline{\underline{x_0, x_1, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, f_1, g_0, h_0, h_1, s, t, u_0, u_1, u_2}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (x_1^2h_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, f_1^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, y_0^5 = s^{-1}(f_1^{-1}h_1^2)^{5(2^{r-3})},$$
$$\underline{\underline{u_0 = f_0^{-1}f_1}}, \underline{\underline{u_1 = h_0^{-1}h_1, u_2 = x_0^{-1}x_1}}\rangle^{ab}.$$

Eliminate $f_1 = u_0f_0, h_1 = h_0u_1, x_1 = x_0u_2$,

$$G_{2^r} = \langle \underline{\underline{x_0, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})},$$
$$\underline{\underline{((u_2x_0)^2(u_1h_0)^{-1})^{15(2^{2(r-2)})}}}, f_0^{5(2^{2(r-3)})}, \underline{\underline{(u_0f_0)^{5(2^{2(r-3)})}}}, g_0^{15(2^{2(r-2)})},$$
$$y_0^5 = s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, \underline{\underline{y_0^5 = s^{-1}((u_0f_0)^{-1}(u_1h_0)^2)^{5(2^{r-3})}}}\rangle^{ab}$$
$$= \langle x_0, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, \underline{\underline{(u_2^2u_1^{-1})^{15(2^{2(r-2)})}}},$$
$$f_0^{5(2^{2(r-3)})}, \underline{\underline{u_0^{5(2^{2(r-3)})}}}, g_0^{15(2^{2(r-2)})},$$
$$\underline{\underline{y_0^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}}}, \underline{\underline{y_0^{-5}s^{-1}((u_0f_0)^{-1}(u_1h_0)^2)^{5(2^{r-3})}}}\rangle^{ab}.$$

Multiply $(y_0^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})})^{-1}$ into $y_0^{-5}s^{-1}((u_0f_0)^{-1}(u_1h_0)^2)^{5(2^{r-3})}$,

$$G_{2^r} = \langle x_0, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (u_2^2u_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, y_0^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, \underline{\underline{s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}}}\rangle^{ab}.$$

We notice that $1 = (g_0^{15(2^{2(r-2)})})^3 = (s^{2^{r-3}})^{12} = (y_0^{-5}(f_0^{-1}h_0^2)^{5(2^{r-3})})^{3(2^{r-3})} = (y_0^{-1}h_0^{(2^{r-2})})^{15(2^{r-3})}$

$= (y_0^{-1}x_0^{(2^{r-1})})^{15(2^{r-3})}$, therefore we add $w_1 = y_0^{-1}x_0^{(2^{r-1})}$ as a generator,

$$G_{2^r} = \langle \underline{\underline{x_0, y_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2, w_1}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (u_2^2u_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, y_0^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})},$$
$$\underline{\underline{w_1 = y_0^{-1}x_0^{(2^{r-1})}}}\rangle^{ab}.$$

Eliminate $y_0 = w_1^{-1}x_0^{(2^{r-1})}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2, w_1}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (u_2^2u_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$\underline{\underline{(w_1^{-1}x_0^{(2^{r-1})})^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}}}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}\rangle^{ab}.$$

Let $w_2 = x_0^2h_0^{-1}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, h_0, s, t, u_0, u_1, u_2, w_1, w_2}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, (x_0^2h_0^{-1})^{15(2^{2(r-2)})}, (u_2^2u_1^{-1})^{15(2^{2(r-2)})},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$(w_1^{-1}x_0^{(2^{r-1})})^{-5}s^{-4}(f_0^{-1}h_0^2)^{5(2^{r-3})}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}, \underline{\underline{w_2 = x_0^2h_0^{-1}}}\rangle^{ab}.$$

Eliminate $h_0 = x_0^2w_2^{-1}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_1, w_2}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, \underline{\underline{(x_0^2(x_0^2w_2^{-1})^{-1})^{15(2^{2(r-2)})}}},$$
$$(u_2^2u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$\underline{\underline{w_1^{-5}s^{-4}(f_0^{-1}x_0^{-4}(x_0^2w_2^{-1})^2)^{5(2^{r-3})}}}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}\rangle^{ab}$$

$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_1, w_2 \mid$$

$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, \underline{\underline{w_2^{15(2^{2(r-2)})}}},$$
$$(u_2^2 u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$\underline{\underline{w_1^{-5} s^{-4} (f_0^{-1} w_2^{-2})^{5(2^{r-3})}}}, s^3 (u_0^{-1} u_1^2)^{5(2^{r-3})} \rangle^{ab}.$$

Multiply $s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}$ into $w_1^{-5}s^{-4}(f_0^{-1}w_2^{-2})^{5(2^{r-3})}$,

$$G_{2^r} = \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_1, w_2 \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (u_2^2 u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{s^{-1}(w_1^{-1}(u_0^{-1}u_1^2 f_0^{-1} w_2^{-2})^{2^{r-3}})^5}}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})} \rangle^{ab}.$$

Let $w_3 = w_1(u_0 u_2^{-2} f_0 w_2^2)^{2^{r-3}}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_1, w_2, w_3}} \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (u_2^2 u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, s^{-1}(w_1^{-1}(u_0^{-1}u_1^2 f_0^{-1} w_2^{-2})^{2^{r-3}})^5, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})},$$
$$\underline{\underline{w_3 = w_1(u_0 u_2^{-2} f_0 w_2^2)^{2^{r-3}}}} \rangle^{ab}.$$

Eliminate $w_1 = w_3(u_0^{-1}u_1^2 f_0^{-1} w_2^{-2})^{2^{r-3}}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_2, w_3}} \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (u_2^2 u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})},$$
$$\underline{\underline{s^{-1}((w_3(u_0^{-1}u_1^2 f_0^{-1} w_2^{-2})^{2^{r-3}})^{-1}(u_0^{-1}u_1^2 f_0^{-1} w_2^{-2})^{2^{r-3}})^5}},$$
$$s^3(u_0^{-1}u_1^2)^{5(2^{r-3})} \rangle^{ab}$$
$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_2, w_3 \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (u_2^2 u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{s^{-1}w_3^{-5}}}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})} \rangle^{ab}.$$

Let $w_4 = u_2^2 u_1^{-1}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_1, u_2, w_2, w_3, w_4}} \mid$$
$$a_i^{5(2^{r-3})}(i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (u_2^2u_1^{-1})^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, s^{-1}w_3^{-5}, s^3(u_0^{-1}u_1^2)^{5(2^{r-3})}, \underline{\underline{w_4 = u_2^2u_1^{-1}}}\rangle^{ab}.$$

Eliminate $u_1 = u_2^2w_4^{-1}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, \underline{\underline{(u_2^2(u_2^2w_4^{-1})^{-1})^{15(2^{2(r-2)})}}},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, s^{-1}w_3^{-5}, \underline{\underline{s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}}}\rangle^{ab}$$
$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, \underline{\underline{w_4^{15(2^{2(r-2)})}}},$$
$$f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, s^{-1}w_3^{-5}, \underline{\underline{s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}}}\rangle^{ab}.$$

Multiply $(s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})})^2$ into $s^{-1}w_3^{-5}$,

$$G_{2^r} = \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{(s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})})^2s^{-1}w_3^{-5}}},$$
$$s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}\rangle^{ab}$$
$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{(sw_3^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})})^5}},$$
$$s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}\rangle^{ab}.$$

Let $\alpha_0 = sw_3^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4, \alpha_0}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, (sw_3^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})})^5,$$
$$s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}, \underline{\underline{\alpha_0 = sw_3^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})}}}\rangle^{ab}.$$

Eliminate $w_3 = s\alpha_0^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})}$,

$$
\begin{aligned}
G_{2^r} = \langle & \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4, \alpha_0}} \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}), \\
& g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, \\
& u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \\
& \underline{\underline{(s(s\alpha_0^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})})^{-1}(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-2})})^5}}, \\
& s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})} \rangle^{ab} \\
= \langle & x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4, \alpha_0 \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}), \\
& g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, \\
& f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \underline{\underline{\alpha_0^5}}, s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})} \rangle^{ab}.
\end{aligned}
$$

As $1 = (g_0^{15(2^{2(r-2)})}) = (s^{2^{r-3}})^4$,

we can substitute $s^2 = g_0^{15(2^{2(r-3)})}$ with $s^3(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}$,

$$
\begin{aligned}
G_{2^r} = \langle & x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4, \alpha_0 \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}), \\
& g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, \\
& f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5, \underline{\underline{s(g_0^{15(2^{2(r-3)})})(u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}}} \rangle^{ab} \\
= \langle & x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, s, t, u_0, u_2, w_2, w_3, w_4, \alpha_0 \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}), \\
& g_0^{15(2^{2(r-3)})} = s^{2^{r-3}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, \\
& f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \\
& \alpha_0^5, \underline{\underline{s(g_0^{3(2^{r-3})}u_0^{-1}(u_2^2w_4^{-1})^2)^{5(2^{r-3})}}} \rangle^{ab}.
\end{aligned}
$$

Eliminate $s = (g_0^{3(2^{r-3})}u_0^{-1}(u_2^2w_4^{-1})^2)^{-5(2^{r-3})}$,

$$
\begin{aligned}
G_{2^r} = \langle & \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, w_4, \alpha_0}} \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}) \\
& \underline{\underline{g_0^{15(2^{2(r-3)})} = ((g_0^{3(2^{r-3})}u_0^{-1}(u_2^2w_4^{-1})^2)^{-5(2^{r-3})})^{2^{r-3}}}}, t^{15(2^{2(r-3)})}, \\
& w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5 \rangle^{ab} \\
= \langle & x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, w_4, \alpha_0 \mid \\
& a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),
\end{aligned}
$$

$$\underline{\underline{(g_0^3 w_4^2 u_2^{-4})^{5(2^{2(r-3)})}}}, t^{15(2^{2(r-3)})},$$

$$w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5\rangle^{ab}.$$

Let $\alpha_1 = g_0 w_4 u_2^{-2}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, w_4, \alpha_0, \alpha_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(g_0^3 w_4^2 u_2^{-4})^{5(2^{2(r-3)})}, t^{15(2^{2(r-3)})},$$

$$w_2^{15(2^{2(r-2)})}, w_4^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5,$$

$$\underline{\underline{\alpha_1 = g_0 w_4 u_2^{-2}}}\rangle^{ab}.$$

Eliminate $w_4 = \alpha_1 g_0^{-1} u_2^2$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(g_0^3 (\alpha_1 g_0^{-1} u_2^2)^2 u_2^{-4})^{5(2^{2(r-3)})}}}, t^{15(2^{2(r-3)})},$$

$$w_2^{15(2^{2(r-2)})}, \underline{\underline{(\alpha_1 g_0^{-1} u_2^2)^{15(2^{2(r-2)})}}}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5\rangle^{ab}$$

$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1 \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{(g_0 \alpha_1^2)^{5(2^{2(r-3)})}}}, t^{15(2^{2(r-3)})},$$

$$w_2^{15(2^{2(r-2)})}, \underline{\underline{(\alpha_1 u_2^2)^{15(2^{2(r-2)})}}}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5\rangle^{ab}.$$

Let $\alpha_2 = g_0 \alpha_1^2$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, g_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1, \alpha_2}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$(g_0 \alpha_1^2)^{5(2^{2(r-3)})}, t^{15(2^{2(r-3)})},$$

$$w_2^{15(2^{2(r-2)})}, (\alpha_1 u_2^2)^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, g_0^{15(2^{2(r-2)})}, \alpha_0^5,$$

$$\underline{\underline{\alpha_2 = g_0 \alpha_1^2}}\rangle^{ab}.$$

Eliminate $g_0 = \alpha_2 \alpha_1^{-2}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1, \alpha_2}} \mid$$

$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$

$$\underline{\underline{((\alpha_2 \alpha_1^{-2}) \alpha_1^2)^{5(2^{2(r-3)})}}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (\alpha_1 u_2^2)^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$

$$u_0^{5(2^{2(r-3)})}, (\alpha_2 \alpha_1^{-2})^{15(2^{2(r-2)})}, \alpha_0^5, \rangle^{ab}$$

$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1, \alpha_2 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\underline{\underline{\alpha_2^{5(2^{2(r-3)})}}}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (\alpha_1 u_2^2)^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, \alpha_1^{15(2^{2r-3})}, \alpha_0^5 \rangle^{ab}.$$

Let $\alpha_3 = \alpha_1 u_2^2$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_1, \alpha_2, \alpha_3}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\alpha_2^{5(2^{2(r-3)})}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, (\alpha_1 u_2^2)^{15(2^{2(r-2)})}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, \alpha_1^{15(2^{2r-3})}, \alpha_0^5, \underline{\underline{\alpha_3 = \alpha_1 u_2^2}} \rangle^{ab}.$$

Eliminate $\alpha_1 = \alpha_3 u_2^{-2}$,

$$G_{2^r} = \langle \underline{\underline{x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_2, \alpha_3}} \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}),$$
$$\alpha_2^{5(2^{2(r-3)})}, t^{15(2^{2(r-3)})}, w_2^{15(2^{2(r-2)})}, \underline{\underline{((\alpha_3 u_2^{-2}) u_2^2)^{15(2^{2(r-2)})}}}, f_0^{5(2^{2(r-3)})},$$
$$u_0^{5(2^{2(r-3)})}, \underline{\underline{(\alpha_3 u_2^{-2})^{15(2^{2r-3})}}}, \alpha_0^5 \rangle^{ab}$$

$$= \langle x_0, a_{16}, a_{17}, \ldots, a_{2^r-1}, f_0, t, u_0, u_2, w_2, w_3, \alpha_0, \alpha_2, \alpha_3 \mid$$
$$a_i^{5(2^{r-3})} (i \in \{16, 17, \ldots, 2^r - 1\}), \alpha_2^{5(2^{2(r-3)})}, t^{15(2^{2(r-3)})},$$
$$w_2^{15(2^{2(r-2)})}, \underline{\underline{\alpha_3^{15(2^{2(r-2)})}}}, f_0^{5(2^{2(r-3)})}, u_0^{5(2^{2(r-3)})}, \underline{\underline{u_2^{15(2^{2(r-1)})}}}, \alpha_0^5 \rangle^{ab}$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_5 \oplus (\mathbb{Z}_{5(2^{r-3})})^{2^r-16} \oplus (\mathbb{Z}_{5(2^{2(r-3)})})^3 \oplus \mathbb{Z}_{15(2^{2(r-3)})} \oplus (\mathbb{Z}_{15(2^{2(r-2)})})^2 \oplus \mathbb{Z}_{15(2^{2(r-1)})}.$$

□

2.4 Results related to So's conjecture

We mentioned earlier two counting Propositions 2.1.11, 2.1.13 related to counting integral circulant graphs and multiplicative divisor graphs. We follow on these results in this section.

2.4.1 Lunar partitions and the number of connected integral circulant graphs

To start, we will present lunar arithmetic as well as the number of lunar partitions and show how they are related to the number of connected integral circulant graphs. The article by Applegate, LeBurn and Sloane (see [1]) is our main reference on lunar arithmetic, which we recommend for further reading on this subject.

Lunar arithmetic, previously known as dismal arithmetic (see [51, Sequence (A087097)]), is a simplified form of arithmetic that uses $\min$, $\max$ functions for each pair of digits instead of using multiplication or addition in ordinary arithmetic. Other than that, it looks similar to ordinary arithmetic except that there are no carries (as a result of taking the $\max$ of two digits instead of adding them).

Let $\mathcal{B}$ be the set $\{0, 1, \ldots, b-1\}$, where $b \geqslant 2$ is the number base. And define the following binary operations

$$x \mathbin{+_b} y := \max(x, y), \; x \mathbin{\times_b} y := \min(x, y), \quad \text{for } x, y \in \mathcal{B}.$$

Let $\mathcal{B}[X]$ be the semiring of polynomials $\sum_{i=0}^{t-1} z_i X^i$, $z_i \in \mathcal{B}$, then we call any element in $\mathcal{B}[X]$ a *lunar number*. For two lunar numbers $Z[X] := \sum_{i=0}^{t-1} z_i X^i$, $M[X] := \sum_{i=0}^{s-1} m_i X^i$, $z_i, m_i \in \mathcal{B}$, their *lunar sum* is defined as follows

$$Z[X] \mathbin{+_b} M[X] := \sum_{i=0}^{\max(t,s)-1} (z_i \mathbin{+_b} m_i) X^i,$$

and their *lunar product* is

$$Z[X] \mathbin{\times_b} M[X] := \sum_{i=0}^{t+s-2} q_i X^i,$$

where

$$\begin{aligned}
q_0 &:= z_0 \mathbin{\times_b} m_0, \\
q_1 &:= (z_0 \mathbin{\times_b} m_1) \mathbin{+_b} (z_1 \mathbin{\times_b} m_0), \\
q_2 &:= (z_0 \mathbin{\times_b} m_2) \mathbin{+_b} (z_1 \mathbin{\times_b} m_1) \mathbin{+_b} (z_2 \mathbin{\times_b} m_0), \\
&\ldots.
\end{aligned}$$

Notice that in both lunar addition $+_b$ and multiplication $\times_b$ we calculate the digits similar to these operations in ordinary arithmetic on polynomials. Also, these operations satisfy the commutative and associative laws as well as $\times_b$ distribution over $+_b$ (see [1, Theorem 1]). The lunar number $Z[X] := \sum_{i=0}^{t-1} z_i X^i$ results in the integer $z = \sum_{i=0}^{t-1} z_i b^i$ when evaluated for a base $X = b$. To simplify working with lunar numbers, we will write the number $\sum_{i=0}^{t-1} z_i b^i$ as $(z_{t-1} z_{t-2}, \ldots z_1 z_0)_b$. We do not define subtraction and division as both operations $\times_b$ and $+_b$ are non-injective surjective operations. For example, $82 +_{10} 13 = 83 +_{10} 71 = 83$ and $32 \times_{10} 14 = 52 \times_{10} 13 = 132$. This explains why lunar numbers only form a semiring.

Notice that for addition as we always take the maximum digit, if for $m := (m_{t-1} m_{t-2}, \ldots m_1 m_0)_b$, $z := (z_{t-1} z_{t-2}, \ldots z_1 z_0)_b$, $m_i \leqslant z_i$ for all i then $m +_b z = z$. In this case, we say m is *dominated by* z or write $m \ll_b z$. Suppose we have a set of all numbers dominated by z denoted by $\mathcal{Q}(z)$, then $\mathcal{Q}(z)$ together with $\ll_b$ form a poset (partially ordered set) as, for all $v, u, w \in \mathcal{Q}(z)$, we have

- $v \ll_b v$.

- $v \ll_b u$ and $u \ll_b v \implies v = u$.
- $v \ll_b u$ and $u \ll_b w \implies v \ll_b w$.

A partition of an integer z is a way of writing z as a sum of positive numbers. When considering a lunar analogue to integer partition, we have to apply restrictions on uniqueness as any $z +_b z = z$. We define $p_b(z)$ as the number of ways we can write z as a sum of distinct digits (number of lunar partitions) as follows

$$z = v_0 +_b v_1 +_b \ldots +_b v_{l-1},$$

where $l \geqslant 1$, the order of v_i does not matter and they should be distinct positive integers. This implies that any permutation of the digits of z results in the same $p_b(z)$ and we can ignore any 0 digit in z. For the special case $z = (0)_b$, we have $p_b(0) = 1$. The next theorem gives us a formula to calculate $p_b(z)$.

Theorem 2.4.1. [1, Theorem 23] *Let $z = (z_0 z_1 \ldots z_{l-1})_b$, $z_i \in \{0, 1, \ldots, b-1\}$ be a lunar number, then the number of lunar partitions of z is*

$$p_b(z) = \frac{1}{2} \sum_{T \subseteq \{0,1,\ldots,l-1\}} (-1)^{|T|} 2^{\prod_j (z_j + \epsilon_j)},$$

where $\epsilon_j = 0$ if $j \in T$, otherwise $\epsilon_j = 1$.

Proposition 2.1.11 is about the number of integral circulant graphs which includes disconnected graphs. When we count only connected integral circulant graphs of order n we find that it has the same count as $p_b(z)$ for some lunar number z.

Theorem 2.4.2. *Let $n = p_0^{r_0} p_1^{r_1} \ldots p_{t-1}^{r_{t-1}}$ be a prime factorization of n. The number of connected integral circulant graphs of order n is at most*

$$\frac{1}{2} \sum_{T \subseteq \{0,1,\ldots,t-1\}} (-1)^{|T|} 2^{\prod_j (r_j + \epsilon_j)},$$

where $\epsilon_j = 0$ if $j \in T$, otherwise $\epsilon_j = 1$.

Proof. It is known that the set of divisors $D(n)$ along side the division operation $|$ forms a poset. Suppose that $d \in D(n)$, $d = p_0^{i_0} p_1^{i_1} \ldots p_{t-1}^{i_{t-1}}$, $i_j \in \{0, 1, \ldots, r_j\}$, $j \in \{0, 1, \ldots, t-1\}$. Let $b \geqslant \max(r_0, r_1, \ldots, r_{t-1}) + 1$, and define the lunar number $z = (r_0, r_1, \ldots, r_{t-1})_b$. The poset $\mathcal{Q}(z)$ has a one to one map to $D(n)$ as d maps into $v = (r_0 - i_0, r_1 - i_1, \ldots, r_{t-1} - i_{t-1})_b$, $v \in \mathcal{Q}(z)$. Suppose we have the lunar partition

$$z = v_0 +_b v_1 +_b \ldots +_b v_{l-1},\ v_k \in \mathcal{Q}(z), k \leqslant l-1,$$

then each digit in z must satisfy

$$r_j = \max(v_0 : j, v_1 : j, \ldots, v_{l-1} : j), \quad \text{where } v_k : j \text{ is the } j\text{th digit in } v_k.$$

This is equal to

$$r_j = \max(r_j - e_{p_j}(d_0), r_j - e_{p_j}(d_1), \ldots, r_j - e_{p_j}(d_{l-1})), \text{ where } e_{p_j}(d_k) \text{ is the } p_j\text{-adic order of } d_k \in D(n),$$

$$r_j = r_j - \min(e_{p_j}(d_0), e_{p_j}(d_1), \ldots, e_{p_j}(d_{l-1})) = r_j - 0.$$

As all digits r_j expect at least one $e_{p_j}(d_k) = 0$, we can clearly see that this is exactly requiring $\gcd(d_0, d_1, \ldots, d_{l-1}) = 1$, and by Theorem 2.1.1, we can see that $\mathrm{ICG}(n, \{d_0, d_1, \ldots, d_{l-1}\})$ is a connected circulant graph. Hence, $p_b(z)$ is the maximum number of connected integral circulant graphs of order n. □

Next, we extend the arguments in Theorem 2.4.2 to count the maximum number of integral circulant graph with components of any $d \in D(n) \backslash \{n\}$.

Theorem E. *Let $n = p_0^{r_0} p_1^{r_1} \ldots p_{t-1}^{r_{t-1}}$ be a prime factorization of n, and let $e = p_0^{s_0} p_1^{s_1} \ldots p_{t-1}^{s_{t-1}} \in D(n)$. The number of integral circulant graphs of order n with e components is at most $p_b(z_e)$, where $z_e = (r_0 - s_0, r_1 - s_1, \ldots, r_{t-1} - s_{t-1})_b$.*

Proof. The proof in Theorem 2.4.2 gives us the maximum number of integral circulant graphs with $e = 1$ components. We know by Theorem 2.1.2 that e is the number of components of an integral circulant graph, and we can easily see that by replacing z with z_e in Theorem 2.4.2 with $e > 1$ we have for each digit

$$(r_j - s_j) = (r_j - s_j) - \min(e_{p_j}(d_0), e_{p_j}(d_1), \ldots, e_{p_j}(d_{l-1})) = (r_j - s_j) - 0, \text{ where } d_k \in D(n).$$

Which is the same as

$$r_j = r_j - \min(s_j + e_{p_j}(d_0), s_j + e_{p_j}(d_1), \ldots, s_j + e_{p_j}(d_{l-1})) = r_j - s_j.$$

Therefore, asking for each $f_k \in D \subset D(n)$ to have $\gcd(f_0, f_1, \ldots, f_{l-1}) = e$ is the same as asking $(f_k/e) \in D \subset D(n/e)$ to have $\gcd(f_0/e, f_1/e, \ldots, f_{l-1}/e) = 1$, which gives us the desired result. □

This gives us a nice way to break down Proposition 2.1.11 on the maximum number of integral circulant graphs of order n as

$$\sum_{d \in D(n) \backslash n} p_b(z_d) = 2^{|D(n)-1|}.$$

2.4.2 Counting multiplicative divisor graphs

When we count multiplicative divisor graphs as in Proposition 2.1.13, we notice it gives the number of multiplicative divisor graphs including graphs that have loops. It also includes multiple cases where the resulting graph is equivalent to the null graphs. By only counting simple graphs we come up with this refined result.

Theorem 2.4.3. *Let $n \geqslant 1$ be an integer with prime factorization $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$. The number of multiplicative divisor graphs of order n is as follows:*

$$\begin{cases} 1 + (2^{k_1+1} - 1)(2^{k_2+1} - 1) \dots (2^{k_r+1} - 1), & \textit{including graphs that have loops} \\ 1 + (2^{k_1+1} - 1)(2^{k_2+1} - 1) \dots (2^{k_r+1} - 1) - (2^{k_1+k_2+\dots+k_r}), & \textit{otherwise} \end{cases}$$

Proof. A circulant graph $\mathrm{ICG}(n, D)$ is a null graph if and only if D is an empty set. This happens to be the case when at least one $\mathcal{D}_{p_i}$ is empty. The power set $\mathcal{P}(\mathcal{D}_{p_i})$ contains only one empty set. Therefore $\prod_{i=1}^{r}(|\mathcal{P}(\mathcal{D}_{p_i})| - 1)$ is the number of multiplicative divisor graphs of order n without the null graph. We count one for exactly one null graph. For the second part, $\mathrm{ICG}(n, D)$ contains a loop if and only if $n \in D$, this is the case when $p^{k_i} \in \mathcal{D}_{p_i}$. Exactly half of the sets in $\mathcal{P}(\mathcal{D}_{p_i})$ contains p^{k_i} as a member. By subtracting the product of all these subsets we get the number of simple multiplicative divisor graphs of order n. □

2.5 Future work

The sandpile group of integral circulant graphs is challenging to calculate, even when considering only integral circulant graphs of prime power order of the form $\mathrm{ICG}(p^r, \{1, p^k\})$. It seems possible to prove Conjecture 2.3.7 or even Conjecture 2.3.6 using Tietze transformations as we did in Theorem D. However, we expect the complexity will rise quickly as soon as we consider more divisors or examine a more general structure like multiplicative divisor graphs. The Smith group of $\mathrm{ICG}(rn, rD)$ given an integral circulant graph $\mathrm{ICG}(n, D)$ with group G is G^r by Theorem A. $\mathrm{ICG}(rn, rD)$ clearly consists of r components that are isomorphic to $\mathrm{ICG}(n, D)$. Therefore, we only need to focus on the Smith group of connected integral circulant graphs. Theorem C, seems like a useful tool to understand the Smith group of integral circulant graphs of prime power in general. For example, the graph $\mathrm{ICG}(4, \{1\})$ is the cycle C_4. Theorem C tells us that its Smith group is the same as $\mathrm{ICG}(2, \{1\}) = K_2$, which has a trivial group. The same applies to all graphs of the form $\mathrm{ICG}(2^r, \{1\})$, $r \geqslant 1$. By Theorem A, we can also conclude that the Smith group for all graphs of the form $\mathrm{ICG}(2^r, \{2^{r-2}\})$, $r \geqslant 2$ is also trivial. Finally, Theorem B gives us another way of viewing integral circulant graphs of prime power order. Having some understanding of how the sandpile and Smith groups change under graph union, join and complement operations could make it easy to determine these groups for this class of graphs.

Chapter 3

Integral Regular Graphs with at most Four Distinct Eigenvalues and their Sandpile Group

In this chapter, we study the sandpile group of regular graphs with, at most, four eigenvalues. We present a partial result on the sandpile group of strongly regular graphs (see Theorem 3.2.1), this happens to fit with an existing result on Kneser graphs $K(n, 2)$. We obtain new results on the sandpile groups of circulant graphs of four distinct eigenvalues (see Theorems F, G, H,I, and 3.3.6).

3.1 Introduction

Theorem 1.6.5 explains the divisibility relationship between a matrix with integral eigenvalues and its Smith normal form. Here we consider graphs with up to four distinct integral eigenvalues to try to understand what their sandpile group looks like.

The only graph with one distinct eigenvalue is the null graph. For two distinct eigenvalues graphs the complete graph is the only graph. As the sandpile group of these graphs are already understood we will start with connected regular graphs with 3 distinct eigenvalues. It is known that such graphs are in fact strongly regular graphs (see for example [30, Lemma 10.2.1, page 220]).

3.1.1 Strongly regular graphs

Strongly regular graphs $\mathrm{srg}(n, k, a, c)$ (Definition 1.4.4) are regular graphs of order n and degree k with the number of common neighbours of any pair of vertices is a if they are adjacent or c otherwise.

The sandpile group is known for a number of strongly regular graphs, including the Kneser graph $K(n, 2)$,

Paley graph (see [27], and [49]), and a few others (see [25], and [26]).

We observe going through the literature, given what is known about strongly regular circulant graphs, that their sandpile group structure is well understood as we show next.

3.1.2 Strongly regular circulant graphs

Theorem 3.1.1 below is a classification of circulant graphs that are distance regular (Definition 1.4.5). This is useful as distance regular graphs generalises strongly regular graphs.

Theorem 3.1.1. [49, Theorem 1.2] *For any circulant graph H of order n, H is distance regular if and only if it is isomorphic to either C_n, K_n, $K_{\underbrace{m,m,\ldots,m}_{q}}$, $K_{m,m} - mK_2$ ($n = 2m$, m is odd) or the Paley graph $Paley(n)$, (where n is prime and $n \equiv 1 \mod 4$).*

Note that $K_{m,m} - mK_2$ is a complete bipartite graph with a perfect matching removed, this graph is also known as the Crown graph and its sandpile group has been determined in [45, Theorem 14, page 9].

The only distance regular graphs in Theorem 3.1.1 with diameter 2 (strongly regular) are $K_{\underbrace{m,m,\ldots,m}_{q}}$ and the Paley graph $Paley(n)$, The sandpile group of both of these graph families has been determined in [34], [12]. We introduced the sandpile group of $K_{\underbrace{m,m,\ldots,m}_{q}}$ in Corollary 1.11.2. For the Paley graph, we have the following result.

Theorem 3.1.2. [12, Theorem 4.1] *Let $q = p^t \equiv 1 \mod 4$, where p is a prime number. And let $\mathrm{SNF}(L(Paley(q)))$ be the Smith normal form of the Laplacian of the Paley Graph of order q. The number of non-zero invariants in $\mathrm{SNF}(L(Paley(q)))$ that p^s divides, is given by $f(t,s)$, where*

$$f(t,s) = \begin{cases} \displaystyle\sum_{i=0}^{\min(s,t-s)} \frac{t}{t-i}\binom{t-i}{i}\binom{t-2i}{s-i}(-p)^i\left(\frac{p+1}{2}\right)^{t-2i} & \text{if } 0 \leqslant s < t \\ \left(\dfrac{p+1}{2}\right)^t - 2 & \text{if } s = t. \end{cases}$$

3.1.3 Regular graphs with four distinct eigenvalues

Let Γ be a connected regular graph, and suppose it has the four distinct eigenvalues k, λ_1, λ_2, and λ_3 with multiplicities 1, m_1, m_2, and m_3 respectively. If an eigenvalue of Γ is a rational number then it is an integer. On the other hand, if either $a + \sqrt{b}$ or $a - \sqrt{b}$ is an eigenvalue then so is the other with the same multiplicity (see [21, Corollaries 2.3, 2.4]). In [22, pages 143-146], it is also shown that graphs with four distinct eigenvalues are walk regular graphs (Definition 1.4.6). This places more restrictions on their possible eigenvalues as we will see next.

Theorem 3.1.3. [21, Theorem 3.3] *Let Γ be a connected walk regular graph of order n with degree k, and eigenvalues $k, \lambda_1, \lambda_2, \ldots, \lambda_t$. Suppose there are two distinct simple eigenvalues k and λ_j. Then, n is even and the graph Γ has a vertex set that can be partitioned into two sets of size $n/2$, for which all vertices in one class have*

the same degree $((k+\lambda_j)/2$ *and all the vertices in the other class have the degree* $(k-\lambda_j)/2)$*. Also,* n *is a divisor of both*

$$\prod_{i\neq j}(k-\lambda_i)+\prod_{i\neq j}(\lambda_j-\lambda_i),\ \textit{and}\ \prod_{i\neq j}(k-\lambda_i)-\prod_{i\neq j}(\lambda_j-\lambda_i).$$

The class of strongly walk regular graphs (Definition 1.4.7) generalises strongly regular graphs and it has been shown to include some regular graphs with 4 distinct eigenvalues as explained in the next theorem.

Theorem 3.1.4. [23, Theorem 3.4] *Let* Γ *be a connected strongly* l*-walk regular graph,* $l>1$*, then* Γ *has at most four distinct eigenvalues. If* l *is even, then* Γ *has either* 2 *or* 3 *distinct eigenvalues, which implies that* Γ *is either the complete graph or a strongly regular graph.*

In Section 3.3, we introduce results on the sandpile group of regular graphs with 4 eigenvalues. We know they are walk regular, but they do not seem to be strongly walk regular graphs in general. The only two instances we found that are strongly walk regular graphs are, the strongly 3-walk regular graph $\mathrm{swrg}_3(10,6,6)$ which has the adjacency matrix $\begin{pmatrix} A(K_4) & I_{anti} \\ I_{anti} & A(K_4)\end{pmatrix}$. Note that I_{anti} is an anti-diagonal matrix defined as

$$I_{anti}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The second instance is the strongly 3-walk regular graph $\mathrm{swrg}_3(39,38,18)$ presented in Figure 3.1.

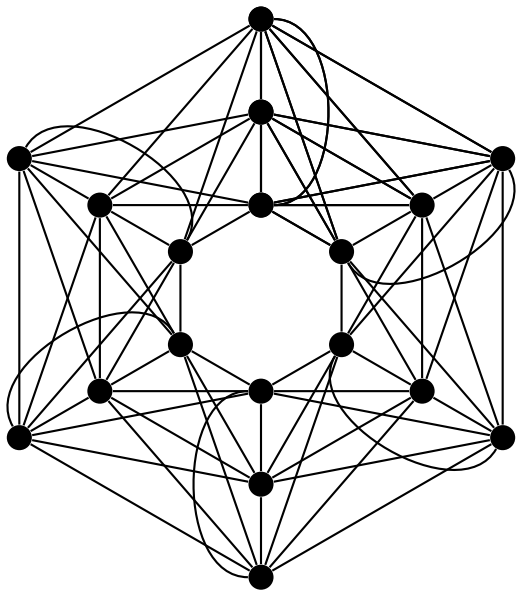

Figure 3.1: $\mathrm{swrg}_3(39,38,18)\cong\overline{\overline{C_6}\otimes\mathcal{J}_3}\cong\mathrm{Cay}(\mathbb{Z}_{18},\{1,5,6,7,11,12,13,17\})$.

3.2 The sandpile group of strongly regular graphs

Given that strongly regular graphs have three eigenvalues, here we use Theorem 1.6.5 to work out the sandpile group of a subset of these graphs. Theorem 3.2.1 shows that with some restrictions on the eigenvalues of a strongly regular graph, we can determine the four possibilities of its sandpile group.

Theorem 3.2.1. *Let Γ be a strongly regular graph of order ab, and suppose that the eigenvalues of the Laplacian of Γ are all integral as follows $\lambda_2 = bq, \lambda_1 = ap, 0$, with multiplicities $m_2, m_1, 1$, respectively, where $m_2 > m_1 > 1$, $\gcd(\lambda_2, \lambda_1) = 1$, and p, q being prime numbers. The sandpile group of Γ is isomorphic to one of four possibilities*

1. $\mathbb{Z}_q \oplus \mathbb{Z}_{\lambda_2}^{m_2-m_1-2} \oplus \mathbb{Z}_{p\lambda_2} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-1}$
2. $\mathbb{Z}_q \oplus \mathbb{Z}_{\lambda_2}^{m_2-m_1-1} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{p\lambda_1\lambda_2}$
3. $\mathbb{Z}_{\lambda_2}^{m_2-m_1-2} \oplus \mathbb{Z}_{p\lambda_2} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{q\lambda_1\lambda_2}$
4. $\mathbb{Z}_{\lambda_2}^{m_2-m_1-1} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{qp\lambda_1\lambda_2}$.

Proof. The Smith normal form of Γ can be written as $\mathrm{diag}(a_0, a_1, \ldots, a_{l-1}, 0)$, $l = ab-1$ the multiset of elements $D = \{a_0, a_1, \ldots, a_{l-1}\}$ defines the structure of the sandpile group. By Theorem 1.6.5, λ_1 divides 0 and $m_1 - 1$ elements in D, also λ_2 divides 0 and $m_2 - 1$ elements in D. The SNF satisfies the property $a_0|a_1|\ldots|a_{l-1}|0$, and given that $m_1 < m_2$ and $\gcd(\lambda_1, \lambda_2) = 1$, the SNF must have $m_1 - 1$ elements that are divisible by $\lambda_1\lambda_2$, and $m_2 - m_1 - 1$ further elements divisible by λ_2. Which we can write as,

$$\lambda_1 | a_{l-m_1+1}, a_{l-m_1+2}, \ldots, a_{l-1},$$

$$\lambda_2 | a_{l-m_2+1}, a_{l-m_2+2}, \ldots, a_{l-1}.$$

So,

$$a_{l-m_1+1+i} = \lambda_1\lambda_2 d_i, \qquad i = 0, 1, \ldots, m_1 - 2,$$

$$a_{l-m_2+1+i} = \lambda_2 b_i, \qquad i = 0, 1, \ldots, m_2 - m_1 - 1.$$

Given that, the Smith normal form of the Laplacian of Γ looks as follows:

$$\mathrm{SNF}(L(\Gamma)) = \mathrm{diag}(a_0, a_1, \ldots, a_{k_1}, \lambda_2 b_0, \lambda_2 b_1, \ldots, \lambda_2 b_{k_2}, \lambda_1\lambda_2 d_0, \lambda_1\lambda_2 d_1, \ldots, \lambda_1\lambda_2 d_{k_3}, 0),$$

where $k_1 = ab - m_2 - 1$, $k_2 = m_2 - m_1 - 1$, and $k_3 = m_1 - 2$. On the other hand, the order of the sandpile group of Γ is equal the number of spanning trees which by Theorem 1.5.1 is

$$\frac{1}{ab}\lambda_1^{m_1}\lambda_2^{m_2}.$$

Which is also equal to the product of the non-zero elements of the diagonal of the Smith normal form of the Laplacian of Γ, i.e.

$$\frac{1}{ab}\lambda_1^{m_1}\lambda_2^{m_2} = (a_0a_1\dots a_{k_1})(b_0b_1\dots b_{k_2})(d_0d_1\dots d_{k_3}){\lambda_1}^{m_1-1}{\lambda_2}^{m_2-1}.$$

By cancelling the eigenvalues on the right side we get

$$\frac{1}{ab}\lambda_1\lambda_2 = (a_0a_1\dots a_{k_1})(b_0b_1\dots b_{k_2})(d_0d_1\dots d_{k_3})$$

and as

$$\frac{1}{ab}\lambda_1\lambda_2 = \frac{ap\ bq}{ab} = pq$$

we get

$$(a_0a_1\dots a_{k_1})(b_0b_1\dots b_{k_2})(d_0d_1\dots d_{k_3}) = pq. \tag{3.1}$$

Suppose that $p|a_{k_1-1}$, then by the divisibility property of the Smith normal form p has to divide a_{k_1} which implies that $p^2|(a_1\dots a_{k_1})$ which is greater than pq. This is a contradiction. Therefore p can only divide a_{k_1} and the product $a_1a_2\dots a_{k_1-1}$ must be equal to 1. Similarly, suppose $p|b_{k_2-1}$, then p has to divide b_{k_2} which implies that $p^2|(b_1\dots b_{k_2})$ which does not divide pq which is a contradiction. Also, we get another contradiction if $p|d_{k_3-1}$, as then p has to divide d_{k_3} which implies that $p^2|(d_1\dots d_{k_3})$ which is greater than pq. Therefore $p|b_{k_2}$, $b_1b_2\dots b_{k_2-1} = 1$ and $p|d_{k_3}$, $d_1d_2\dots d_{k_3-1} = 1$. p can only divide either a_{k_1}, b_{k_2}, or d_{k_3} as otherwise it would contradict Equation (3.1).

This narrows down the possible values for a_{k_1}, b_{k_2} to either p, q or 1, while d_{k_3} can be p, q, pq or 1.

If $p|a_{k_1}$ then $p|\lambda_2$ (looking at the SNF), This is a contradiction as $\lambda_2 = bq$, $q, p \geqslant 3$. Also, if $q|b_{k_2}$ then $\lambda_2 q|\lambda_1\lambda_2 \implies q|\lambda_1 \implies q|pa$. This is also a contradiction.

Therefore the remaining possibilities are $a_{k_1} = q$, or 1, $b_{k_2} = p$, or 1, and $d_{k_3} = p, q, pq$, or 1. □

Theorem 3.2.1 has quite restrictive conditions for the strongly regular graphs it can be applied to. We show later in this section that it is applicable to the Kneser graph $K(n, 2)$ for special values of n. For other strongly regular graphs, it is not clear if there is an infinite family of strongly regular graphs that would satisfy the conditions in Theorem 3.2.1. However, by restricting our search to strongly regular graphs with integral eigenvalues $\lambda_2 = bq, \lambda_1 = ap, 0$, such that $\gcd(\lambda_1, \lambda_2) = 1$, where p, q are distinct prime numbers, we found a list of strongly regular graphs of order up to 650, that are not Kneser graphs (see Table 3.1). The full list of strongly regular graphs that this list was built on is available in the strongly regular graphs and tables section in [9].

The Kneser graph $K(n, k)$ is a vertex transitive graph (Definition 1.4.1) that has a vertex set of k-subsets of $\{1, 2, \dots, n\}$. If two subsets are disjoint, then there is an edge between the two vertices that represent them. The order of this graph is $\binom{n}{k}$ and it is a regular graph of degree $\binom{n-k}{k}$. (see [30, Chapter 7]).

Graph	a	b	p	q	Laplacian Spectrum
$\mathrm{srg}(36,21,10,15)$	9	4	5	3	$\{0^1, 20^{27}, 27^8\}$
$\mathrm{srg}(85,30,11,10)$	5	17	5	2	$\{0^1, 34^{50}, 25^{34}\}$
$\mathrm{srg}(91,66,45,55)$	7	13	11	5	$\{0^1, 65^{77}, 77^{13}\}$
$\mathrm{srg}(105,52,29,22)$	21	5	2	11	$\{0^1, 55^{84}, 42^{20}\}$
$\mathrm{srg}(136,105,78,91)$	17	8	7	13	$\{0^1, 104^{119}, 119^{16}\}$
$\mathrm{srg}(205,68,15,26)$	41	5	2	13	$\{0^1, 65^{164}, 82^{40}\}$
$\mathrm{srg}(217,66,15,22)$	7	31	11	2	$\{0^1, 62^{154}, 77^{62}\}$
$\mathrm{srg}(217,88,39,33)$	7	31	11	3	$\{0^1, 93^{154}, 77^{62}\}$
$\mathrm{srg}(253,140,87,65)$	11	23	5	13	$\{0^1, 143^{230}, 115^{22}\}$
$\mathrm{srg}(341,70,15,14)$	31	11	2	7	$\{0^1, 77^{186}, 62^{154}\}$
$\mathrm{srg}(533,132,31,33)$	13	41	11	3	$\{0^1, 123^{286}, 143^{246}\}$
$\mathrm{srg}(595,198,81,58)$	85	7	2	29	$\{0^1, 203^{510}, 170^{84}\}$
$\mathrm{srg}(616,205,90,57)$	56	11	19	3	$\{0^1, 209^{560}, 168^{55}\}$

Table 3.1: Examples of strongly regular graphs that fits Theorem 3.2.1 parameters.

Let $l = \binom{n}{k}$ and $r = \binom{n-k}{k}$. Suppose μ is an eigenvalue of $A(K(n,k))$, then $\lambda = r - \mu$ is a Laplacian eigenvalue of $K(n,k)$ as the Laplacian matrix $L(K(n,k)) = rI_l - A(K(n,k))$. The eigenvalues of $A(K(n,k))$ can be found in [30, Theorem 9.4.3, page 200].

The Laplacian eigenvalues of $K(n,k)$ are

$$\lambda_j = \binom{n-k}{k} + (-1)^{j+1}\binom{n-k-j}{k-j}, j = 0, 1, \ldots k,$$

with multiplicity

$$m(\lambda_0) = 1, m(\lambda_j) = \binom{n}{j} - \binom{n}{j-1}, j = 1, \ldots k.$$

For $k = 2$,

$$\lambda_0 = \binom{n-2}{2} - \binom{n-2}{2} = 0 \tag{3.2}$$

$$\lambda_1 = \binom{n-2}{2} + \binom{n-3}{1} = \frac{(n-2)(n-3) + 2(n-3)}{2} = \frac{n(n-3)}{2} \tag{3.3}$$

$$\lambda_2 = \binom{n-2}{2} - \binom{n-4}{0} = \frac{(n-2)(n-3) - 2}{2} = \frac{(n-1)(n-4)}{2} \tag{3.4}$$

$$m_0 = m(\lambda_0) = 1$$

$$m_1 = m(\lambda_1) = \binom{n}{1} - \binom{n}{0} = n - 1 \tag{3.5}$$

$$m_2 = m(\lambda_2) = \binom{n}{2} - \binom{n}{1} = \frac{n(n-1) - 2n}{2} = \frac{n(n-3)}{2}. \tag{3.6}$$

A prime number p is called a *Sophie Germain prime* if $2p + 1$ is also a prime (for more information see for example [55, page 329]). It is conjectured that there are infinitely many such primes. In fact there is an analog conjecture to the first Hardy–Littlewood conjecture on twin primes, which we present next.

Conjecture 3.2.2. (See [11, Conjecture 3.6, page 11]) *Let $p < N$ be a Sophie Germain prime number such that*

$2p+1$ *is also a prime number, then the total number of such primes is*

$$2C_2 \int_2^N \frac{dx}{\log x \log 2x} \sim \frac{2C_2 N}{(\log N)^2},$$

where

$$C_2 = \prod_{\substack{p \text{ is a prime} \\ p \geqslant 3}} (1-(p-1)^{-2})$$

is the twin prime constant.

In the next corollary, we show that for the Kneser graph $K(n,2)$, if $q=(n-4)/4$ is a Sophie Germain prime (with $p=(n-3)$ being a prime) then we can apply Theorem 3.2.1 to calculate the sandpile group of $K(n,2)$.

Corollary 3.2.3. *Let $p=n-3$, $q=(n-4)/2$, be prime numbers such that $p,q \geqslant 3$. The eigenvalues of the Laplacian of $K(n,2)$ are all integral as defined in Equations (3.2) - (3.6). Suppose that $\gcd(\lambda_1,\lambda_2)=1$, then the sandpile group of $K(n,2)$ is isomorphic to one of four possibilities*

1. $\mathbb{Z}_q \oplus \mathbb{Z}_{\lambda_2}^{m_2-m_1-2} \oplus \mathbb{Z}_{p\lambda_2} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-1}$
2. $\mathbb{Z}_q \oplus \mathbb{Z}_{\lambda_2}^{m_2-m_1-1} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{p\lambda_1\lambda_2}$
3. $\mathbb{Z}_{\lambda_2}^{m_2-m_1-2} \oplus \mathbb{Z}_{p\lambda_2} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{q\lambda_1\lambda_2}$
4. $\mathbb{Z}_{\lambda_2}^{m_2-m_1-1} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-2} \oplus \mathbb{Z}_{qp\lambda_1\lambda_2}$.

Proof. We have $p=n-3$, $q=(n-4)/2$, and as they are both prime numbers $\gcd(p,q)=1$. Let $a=n/2$, and $b=(n-1)$, then $\gcd(\lambda_2,\lambda_1)=1$. We can see this is true as $\lambda_1=bq, \lambda_2=ap$, so $\gcd(\lambda_2,\lambda_1)=\gcd(ap,bq)$, and given that $\gcd(p,q)=1$, and $a=q+2$, $b=p+2$, $\gcd(ap,bq)=\gcd(a,b)=\gcd(n/2,n-1)=1$. Now using Theorem 3.2.1 we get the desired result for the sandpile group. □

Computational experiments using Maple seems to indicate the cases $2,3,4$ cannot happen, this has been verified up to $p=59$, $q=29$, $n=62$.

After we obtained Corollary 3.2.3 we learnt that it has been superseded in 2018 by a general proof of the sandpile group $K(n,2)$ which appeared in [27]. We present their result below.

Theorem 3.2.4. (See [27, Theorem 4.1]) *The sandpile group of the Kneser graph $K(n,2)$ is*

$$\begin{cases} \mathbb{Z}_{n-4} \oplus \mathbb{Z}_{\frac{(n-1)(n-4)}{2}}{}^{\frac{n(n-5)}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-3)(n-4)}{4}} \oplus \mathbb{Z}_{\frac{n(n-1)(n-3)(n-4)}{4}}{}^{n-2} & \text{if } n \text{ is odd.} \\ \mathbb{Z}_{\frac{n-4}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-4)}{2}}{}^{\frac{n(n-5)}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-3)(n-4)}{2}} \oplus \mathbb{Z}_{\frac{n(n-1)(n-3)(n-4)}{4}}{}^{n-2} & \text{if } n \text{ is even.} \end{cases}$$

Given how q is defined in Corollary 3.2.3, we know that n has to be even. Considering case 1, and by substi-

tuting the values of λ_1, λ_2, m_1, and m_2 in Equations (3.2) - (3.6), we have,

$$\begin{aligned}
&\mathbb{Z}_q \oplus \mathbb{Z}_{\lambda_2}^{m_2-m_1-2} \oplus \mathbb{Z}_{p\lambda_2} \oplus \mathbb{Z}_{\lambda_1\lambda_2}{}^{m_1-1} \\
&= \mathbb{Z}_{\frac{n-4}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-4)}{2}}{}^{\left(\frac{n(n-3)}{2}-(n-1)-2\right)} \oplus \mathbb{Z}_{(n-3)\left(\frac{(n-1)(n-4)}{2}\right)} \oplus \mathbb{Z}_{\left(\frac{n(n-3)}{2}\right)\left(\frac{(n-1)(n-4)}{2}\right)}{}^{((n-1)-1)} \\
&= \mathbb{Z}_{\frac{n-4}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-4)}{2}}{}^{\frac{n(n-5)}{2}} \oplus \mathbb{Z}_{\frac{(n-1)(n-3)(n-4)}{2}} \oplus \mathbb{Z}_{\frac{n(n-1)(n-3)(n-4)}{4}}{}^{n-2},
\end{aligned}$$

which is the even case of Theorem 3.2.4.

3.3 The sandpile group of regular graphs with four distinct eigenvalues

Here we introduce graphs with 4 distinct eigenvalues that happen to be circulant. We will mainly focus on their sandpile group.

3.3.1 Kronecker product constructions

In [21, page 148], as examples of regular graphs with 4 distinct eigenvalues, the following graphs are introduced. $C_5 \otimes \mathcal{J}_t$, and $\overline{C_5 \otimes \mathcal{J}_t}$, $t \geqslant 2$, where C_5 is a cycle of order 5, $\mathcal{J}_t$ is a multigraph which has all ones adjacency matrix J_t. It also mentions the general construction for these graphs is $\Gamma \otimes \mathcal{J}_t$ and $\overline{\overline{\Gamma} \otimes \mathcal{J}_t}$, for a graph Γ of order r. Where $\overline{\overline{\Gamma} \otimes \mathcal{J}_t}$ has an adjacency matrix equivalent to $(A(\Gamma) + I_r) \otimes J_t - I_{rt}$. We can see this is true as the graph $\overline{\Gamma} \otimes J_t$ has an adjacency matrix

$$\begin{pmatrix} A(\overline{\Gamma}) & A(\overline{\Gamma}) & \cdots & A(\overline{\Gamma}) \\ A(\overline{\Gamma}) & A(\overline{\Gamma}) & \cdots & A(\overline{\Gamma}) \\ \vdots & \vdots & \ddots & \vdots \\ A(\overline{\Gamma}) & A(\overline{\Gamma}) & \cdots & A(\overline{\Gamma}) \end{pmatrix}$$

and the adjacency matrix of its complement is as follows

$$\begin{pmatrix} A(\Gamma) & A(\Gamma) + I_r & \cdots & A(\Gamma) + I_r \\ A(\Gamma) + I_r & A(\Gamma) & \cdots & A(\Gamma) + I_r \\ \vdots & \vdots & \ddots & \vdots \\ A(\Gamma) + I_r & A(\Gamma) + I_r & \cdots & A(\Gamma) \end{pmatrix}$$

which is equivalent to $(A(\Gamma) + I_r) \otimes J_t - I_{rt}$.

Given that $\overline{\overline{C_r} \otimes \mathcal{J}_t}$ is the following circulant graph

$$C(r,t) = \mathrm{Cay}(\mathbb{Z}_{rt}, \{1, rt-1\} \cup \{k, k+1, k+2 \mid k \in \{r-1, 2r-1, \ldots, (t-1)r-1\}\}).$$

See Figure 3.1 for an illustration of what the graph looks like in the special case $\Gamma = C_6$ and $r = 3$.

$\overline{C_r} \otimes \mathcal{J}_t$ is the complement of $C(r,t)$ which is the following circulant

$$\overline{C(r,t)} = \mathrm{Cay}(\mathbb{Z}_{rt}, \{k+3, k+4, \ldots, k+r-1 \mid k \in \{-1, r-1, 2r-1, \ldots, (t-1)r-1\}\}).$$

For $r = 6$, as the degrees of $C(r,t)$, $\overline{C(r,t)}$ are $3t-1$, $(r-3)t$ respectively. $C(6,t)$, $\overline{C(6,t)}$ have degrees $3t-1$, $3t$, which makes them close in terms of the number of edges, these two graphs happen to have integral eigenvalues as we show next.

The eigenvalues of a cycle C_r are

$$2, 2\cos(2\pi j/r),\ (j \in \{1, 2, \ldots, r-1\}).$$

And by Theorem 1.1.2, the eigenvalues of its complement $\overline{C_r}$ are

$$n-3, -2\cos(2\pi j/r) - 1,\ (j \in \{1, 2, \ldots, r-1\}).$$

It is also well known that the eigenvalues of J_t are $t, 0$ with multiplicities $1, t-1$ (see for example [32, Example 1.3.23]).

Given this and Theorem 1.2.4, we have the following:

- The eigenvalues of $C_r \otimes \mathcal{J}_t$ are 0 with multiplicity $r(t-1)$, $2t$ with multiplicity 1, and

$$2t\cos(2\pi j/r),\ (j \in \{1, 2, \ldots, r-1\})).$$

- The eigenvalues of $\overline{C_r} \otimes \mathcal{J}_t$ are 0 with multiplicity $r(t-1)$, $(r-3)t$ with multiplicity 1, and

$$-t(2\cos(2\pi j/r) + 1),\ (j \in \{1, 2, \ldots, r-1\})).$$

- The eigenvalues of $\overline{\overline{C_r} \otimes \mathcal{J}_t}$ are -1 with multiplicity $r(t-1)$, $(r-2)t-1$ with multiplicity 1, and

$$2t\cos(2\pi j/r),\ (j \in \{1, 2, \ldots, r-1\})) \text{ (Theorem 1.1.2).}$$

For the case when $r = 6$, we have four distinct integral eigenvalues for both $\overline{C_6} \otimes \mathcal{J}_t$, $\overline{\overline{C_6} \otimes \mathcal{J}_t}$ as

- The eigenvalues of $C_6 \otimes \mathcal{J}_t$ are $2t$, $-2t$, t, $-t$, and 0 with multiplicities 1, 1, 2, 2, and $6(t-1)$ respectively.

- The eigenvalues of $\overline{C_6} \otimes \mathcal{J}_t$ are $3t$, t, $-2t$, 0 with multiplicities 1, 1, 2, and $6t-4$ respectively.

- The eigenvalues of $\overline{\overline{C_6} \otimes \mathcal{J}_t}$ are $3t-1$, $-t-1$, $2t-1$, and -1 with multiplicities 1, 1, 2, and $6t-4$ respectively.

The Smith group of $\Gamma \otimes \mathcal{J}_t$ is easy to calculate.

Theorem 3.3.1. *Let Γ be a graph of order n with Smith group G, then the Smith group of $\Gamma \otimes \mathcal{J}_t$ is G.*

Proof. $A(\Gamma \otimes J_t) = A(\Gamma) \otimes J_t$. Therefore, we can now eliminate the rows and columns $\{n, n+1, \ldots, tn-1\}$, using the sub-matrix $A(\Gamma)$. Then we can see that the Smith normal form of $A(\Gamma \otimes J_t)$ contains the elements in $\mathrm{SNF}(A(\Gamma))$ and $(t-1)n$ zeros. □

As for what the sandpile group of these graphs look like, we choose to do the Tietze transformations on $C(6,t)$ in the next theorem. For a reminder of Tietze transformation rules and convention used below, see Section 1.7.

Theorem F. *The sandpile group of $C(6,t)$ is ${\mathbb{Z}_{3t}}^{6t-8} \oplus {\mathbb{Z}_{3t^2}}^2 \oplus \mathbb{Z}_{6t^2}$.*

Proof. As $L(C(6,t))$ is equivalent to $3tI_{6t} - ((A(\Gamma)+I_6) \otimes J_t - I_{6t})$ we have,

$$P = \langle x_0, x_1, \ldots, x_{6t-1} \mid x_{it+j}^{3t}((x_{it}x_{it+1}\ldots x_{it+t-1})(x_{(i+1)t}\ldots x_{(i+1)t+t-1})(x_{(i+5)t}\ldots x_{(i+5)t+t-1}))^{-1}$$
$$(i \in \{0,1,\ldots,5\}, j \in \{0,1,\ldots,t-1\})\rangle^{ab}.$$

Notice that x indices are modulo $6t$.

Let $y_i = x_{it}x_{it+1}\ldots x_{it+t-1},\ i \in \{0,1,\ldots,5\}$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5}} \mid \underline{\underline{x_{it+j}^{3t}(y_iy_{i+1}y_{i+5})^{-1}}}\ (i \in \{0,1,\ldots,5\}, j \in \{0,1,\ldots,t-1\}),$$
$$\underline{\underline{y_i = x_{it}x_{it+1}\ldots x_{it+t-1}}}\rangle^{ab}.$$

Let $z_i = y_iy_{i+1}y_{i+5},\ i \in \{0,1,\ldots,5\}$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5, z_0, z_1, \ldots, z_5}} \mid \underline{\underline{x_{it+j}^{3t}z_i^{-1}}}\ (i \in \{0,1,\ldots,5\}, j \in \{0,1,\ldots,t-1\}),$$
$$y_i = x_{it}x_{it+1}\ldots x_{it+t-1}, \underline{\underline{z_i = y_iy_{i+1}y_{i+5}}}\rangle^{ab}.$$

Eliminate $z_i = x_{it}^{3t},\ i \in \{0,1,\ldots,5\}$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5}} \mid \underline{\underline{x_{it+j}^{3t}x_{it}^{-3t}}}\ (i \in \{0,1,\ldots,5\}, j \in \{\underline{1}, 2, \ldots, t-1\}),$$
$$y_i = x_{it}x_{it+1}\ldots x_{it+t-1}, \underline{\underline{x_{it}^{3t} = y_iy_{i+1}y_{i+5}}}\rangle^{ab}.$$

Let $w_{it+j} = x_{it+j}x_{it}^{-1}$ $(i \in \{0,1,\ldots,5\}, j \in \{1,2,\ldots,t-1\})$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5, w_1, w_2, \ldots, w_{t-1}, w_{t+1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+1}, w_{5t+2}, \ldots, w_{6t-1}}}$$
$$\mid x_{it+j}^{3t}x_{it}^{-3t}\ (i \in \{0,1,\ldots,5\}, j \in \{1,2,\ldots,t-1\}),$$
$$y_i = x_{it}x_{it+1}\ldots x_{it+t-1}, x_{it}^{3t} = y_iy_{i+1}y_{i+5}, \underline{\underline{w_{it+j} = x_{it+j}x_{it}^{-1}}}\rangle^{ab}.$$

Eliminate $x_{it+j} = x_{it}w_{it+j},\ (i \in \{0,1,\ldots,5\}, j \in \{1,2,\ldots,t-1\})$

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, \ldots, y_5, w_1, w_2, \ldots, w_{t-1}, w_{t+1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+1}, w_{5t+2}, \ldots, w_{6t-1}}} \mid$$

$$\underline{\underline{w_{it+j}^{3t}}} \ (i \in \{0, 1, \ldots, 5\}, j \in \{1, 2, \ldots, t-1\}),$$

$$\underline{\underline{y_i = x_{it}(x_{it}w_{it+1})(x_{it}w_{it+2}) \ldots (x_{it}w_{it+t-1})}}, x_{it}^{3t} = y_i y_{i+1} y_{i+5} \rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, y_0, y_1, \ldots, y_5, w_1, w_2, \ldots, w_{t-1}, w_{t+1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+1}, w_{5t+2}, \ldots, w_{6t-1} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{1, 2, \ldots, t-1\}),$$

$$\underline{\underline{y_i = x_{it}^t(w_{it+1}w_{it+2} \ldots w_{it+t-1})}}, x_{it}^{3t} = y_i y_{i+1} y_{i+5} \rangle^{ab}.$$

Eliminate $w_{it+1} = y_i x_{it}^{-t}(w_{it+2}w_{it+3} \ldots w_{it+t-1})^{-1}, \ (i \in \{0, 1, \ldots, 5\})$

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, \ldots, y_5, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}}} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{\underline{2}, 3, \ldots, t-1\}),$$

$$\underline{\underline{(y_i x_{it}^{-t}(w_{it+2}w_{it+3} \ldots w_{it+t-1})^{-1})^{3t}}}, x_{it}^{3t} = y_i y_{i+1} y_{i+5} \rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, y_0, y_1, \ldots, y_5, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{2, 3, \ldots, t-1\}), \underline{\underline{(y_i^{-1}x_{it}^t)^{3t}}}, x_{it}^{3t} = y_i y_{i+1} y_{i+5} \rangle^{ab}.$$

Eliminate $y_5 = x_{5t}^{3t}(y_0 y_4)^{-1}$

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, y_2, y_3, y_4, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}}} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{2, 3, \ldots, t-1\}), \underline{\underline{(y_k^{-1}x_{kt}^t)^{3t}}} \ (k \in \{0, 1, 2, 3, 4\}),$$

$$\underline{\underline{((x_{5t}^{3t}(y_0 y_4)^{-1})^{-1}x_{kt}^t)^{3t}}}, \underline{\underline{x_0^{3t}(y_0 y_1(x_{5t}^{3t}(y_0 y_4)^{-1}))^{-1}}}, x_t^{3t}(y_1 y_2 y_0)^{-1},$$

$$x_{2t}^{3t}(y_2 y_3 y_1)^{-1}, x_{3t}^{3t}(y_3 y_4 y_2)^{-1}, \underline{\underline{x_{4t}^{3t}(y_4(x_{5t}^{3t}(y_0 y_4)^{-1})y_3)^{-1}}} \rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, y_0, y_1, y_2, y_3, y_4, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{2, 3, \ldots, t-1\}), (y_k^{-1}x_{kt}^t)^{3t} \ (k \in \{0, 1, 2, 3, 4\}), \underline{\underline{(x_{5t}^{-2t}y_0 y_4)^{3t}}},$$

$$\underline{\underline{(x_0 x_{5t}^{-1})^{3t}y_1^{-1}y_4}}, x_t^{3t}(y_1 y_2 y_0)^{-1}, x_{2t}^{3t}(y_2 y_3 y_1)^{-1}, x_{3t}^{3t}(y_3 y_4 y_2)^{-1}, \underline{\underline{(x_{4t}x_{5t}^{-1})^{3t}y_0 y_3^{-1}}} \rangle^{ab}.$$

Eliminate $y_2 = x_{2t}^{3t}(y_3 y_1)^{-1}$

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, y_3, y_4, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}}} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{2, 3, \ldots, t-1\}), (y_k^{-1}x_{kt}^t)^{3t} \ \underline{\underline{(k \in \{0, 1, 3, 4\})}},$$

$$\underline{\underline{((x_{2t}^{3t}(y_3 y_1)^{-1})^{-1}x_{2t}^{3t})^{3t}}}, (x_{5t}^{-2t}y_0 y_4)^{3t}, (x_0 x_{5t}^{-1})^{3t}y_1^{-1}y_4, \underline{\underline{x_t^{3t}(y_1(x_{2t}^{3t}(y_3 y_1)^{-1})y_0)^{-1}}},$$

$$\underline{\underline{x_{3t}^{3t}(y_3 y_4(x_{2t}^{3t}(y_3 y_1)^{-1}))^{-1}}}, (x_{4t}x_{5t}^{-1})^{3t}y_0 y_3^{-1} \rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, y_0, y_1, y_3, y_4, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1} \mid$$

$$w_{it+j}^{3t} \ (i \in \{0, 1, \ldots, 5\}, j \in \{2, 3, \ldots, t-1\}), (y_k^{-1}x_{kt}^t)^{3t} \ \underline{\underline{(k \in \{0, 1, 3, 4\})}}, \underline{\underline{(x_{2t}^{-2t}y_3 y_1)^{3t}}},$$

$$(x_{5t}^{-2t}y_0y_4)^{3t}, (x_0x_{5t}^{-1})^{3t}y_1^{-1}y_4, \underline{\underline{(x_tx_{2t}^{-1})^{3t}y_3y_0^{-1}}}, \underline{\underline{(x_{3t}x_{2t}^{-1})^{3t}y_4^{-1}y_1}}, (x_{4t}x_{5t}^{-1})^{3t}y_0y_3^{-1}\rangle^{ab}.$$

Eliminate $y_4 = y_1(x_0^{-1}x_{5t})^{3t}, y_3 = (x_t^{-1}x_{2t})^{3t}y_0$,

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), (y_k^{-1}x_{kt}^t)^{3t}\ \underline{\underline{(k \in \{0,1\})}},$$

$$\underline{\underline{(((x_t^{-1}x_{2t})^{3t}y_0)^{-1}x_{3t}^t)^{3t}}}, \underline{\underline{((y_1(x_0^{-1}x_{5t})^{3t})^{-1}x_{4t}^t)^{3t}}}, \underline{\underline{(x_{2t}^{-2t}((x_t^{-1}x_{2t})^{3t}y_0)y_1)^{3t}}},$$

$$\underline{\underline{(x_{5t}^{-2t}y_0(y_1(x_0^{-1}x_{5t})^{3t}))^{3t}}}, \underline{\underline{(x_{3t}x_{2t}^{-1})^{3t}(y_1(x_0^{-1}x_{5t})^{3t})^{-1}y_1}}, \underline{\underline{(x_{4t}x_{5t}^{-1})^{3t}y_0((x_t^{-1}x_{2t})^{3t}y_0)^{-1}}}\rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, y_0, y_1, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), (y_k^{-1}x_{kt}^t)^{3t}\ (k \in \{0,1\}),$$

$$\underline{\underline{(y_0^{-1}(x_t^3x_{2t}^{-3}x_{3t})^t)^{3t}}}, \underline{\underline{(y_1^{-1}(x_0^3x_{5t}^{-3}x_{4t})^t)^{3t}}}, \underline{\underline{((x_{2t}x_t^{-3})^ty_0y_1)^{3t}}}, \underline{\underline{((x_{5t}x_0^{-3})^ty_0y_1)^{3t}}},$$

$$\underline{\underline{(x_0x_{2t}^{-1}x_{3t}^{-1}x_{5t})^{3t}}}, \underline{\underline{(x_tx_{2t}^{-1}x_{4t}^{-1}x_{5t})^{3t}}}\rangle^{ab}.$$

Let $a_0 = y_0^{-1}x_0^t, a_1 = y_1^{-1}x_t^t$,

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, y_0, y_1, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), (y_k^{-1}x_{kt}^t)^{3t}\ (k \in \{0,1\}),$$

$$(y_0^{-1}(x_t^3x_{2t}^{-3}x_{3t})^t)^{3t}, (y_1^{-1}(x_0^3x_{5t}^{-3}x_{4t})^t)^{3t}, ((x_{2t}x_t^{-3})^ty_0y_1)^{3t}, ((x_{5t}x_0^{-3})^ty_0y_1)^{3t},$$

$$(x_0x_{2t}^{-1}x_{3t}^{-1}x_{5t})^{3t}, (x_tx_{2t}^{-1}x_{4t}^{-1}x_{5t})^{3t}, \underline{\underline{a_0 = y_0^{-1}x_0^t}}, \underline{\underline{a_1 = y_1^{-1}x_t^t}}\rangle^{ab}.$$

Eliminate $y_0 = a_0^{-1}x_0^t, y_1 = a_1^{-1}x_t^t$,

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), \underline{\underline{a_0^{3t}}}, \underline{\underline{a_1^{3t}}},$$

$$\underline{\underline{((a_0^{-1}x_0^t)^{-1}(x_t^3x_{2t}^{-3}x_{3t})^t)^{3t}}}, \underline{\underline{((a_1^{-1}x_t^t)^{-1}(x_0^3x_{5t}^{-3}x_{4t})^t)^{3t}}}, \underline{\underline{((x_{2t}x_t^{-3})^t(a_0^{-1}x_0^t)(a_1^{-1}x_t^t))^{3t}}},$$

$$\underline{\underline{((x_{5t}x_0^{-3})^t(a_0^{-1}x_0^t)(a_1^{-1}x_t^t))^{3t}}}, (x_0x_{2t}^{-1}x_{3t}^{-1}x_{5t})^{3t}, (x_tx_{2t}^{-1}x_{4t}^{-1}x_{5t})^{3t}\rangle^{ab}$$

$$= \langle x_0, x_t, \ldots, x_{6t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1 \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(x_0^{-1}x_t^3x_{2t}^{-3}x_{3t})^{3t^2}}},$$

$$\underline{\underline{(x_0^3x_t^{-1}x_{4t}x_{5t}^{-3})^{3t^2}}}, \underline{\underline{(x_0x_t^{-2}x_{2t})^{3t^2}}}, \underline{\underline{(x_0^{-2}x_tx_{5t})^{3t^2}}}, (x_0x_{2t}^{-1}x_{3t}^{-1}x_{5t})^{3t}, (x_tx_{2t}^{-1}x_{4t}^{-1}x_{5t})^{3t}\rangle^{ab}.$$

Let $b_k = x_{kt}x_{(k+3)t}\ (k \in \{0,1,2\})$,

$$P = \langle \underline{\underline{x_0, x_t, \ldots, x_{6t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, (x_0^{-1}x_t^3x_{2t}^{-3}x_{3t})^{3t^2}, (x_0^3x_t^{-1}x_{4t}x_{5t}^{-3})^{3t^2},$$

$$(x_0x_t^{-2}x_{2t})^{3t^2}, (x_0^{-2}x_tx_{5t})^{3t^2}, (x_0x_{2t}^{-1}x_{3t}^{-1}x_{5t})^{3t}, (x_tx_{2t}^{-1}x_{4t}^{-1}x_{5t})^{3t},$$

$$\underline{\underline{b_k = x_{kt}x_{(k+3)t}}}\ (k \in \{0,1,2\})\rangle^{ab}.$$

Eliminate $x_{(k+3)t} = b_k x_{kt}^{-1}$ $(k \in \{0,1,2\})$,

$$P = \langle \underline{\underline{x_0, x_t, x_{2t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(x_0^{-1}x_t^3x_{2t}^{-3}(b_0x_0^{-1}))^{3t^2}}},$$

$$\underline{\underline{(x_0^3x_t^{-1}(b_1x_t^{-1})(b_2x_{2t}^{-1})^{-3})^{3t^2}}}, (x_0x_t^{-2}x_{2t})^{3t^2}, \underline{\underline{(x_0^{-2}x_t(b_2x_{2t}^{-1}))^{3t^2}}}, \underline{\underline{(b_0b_2^{-1})^{3t}}}, \underline{\underline{(b_1b_2^{-1})^{3t}}}\rangle^{ab}$$

$$= \langle x_0, x_t, x_{2t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2 \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_0x_0^{-2}x_t^3x_{2t}^{-3})^{3t^2}}}, \underline{\underline{(b_1b_2^{-3}x_0^3x_t^{-2}x_{2t}^3)^{3t^2}}},$$

$$(x_0x_t^{-2}x_{2t})^{3t^2}, \underline{\underline{(b_2x_0^{-2}x_tx_{2t}^{-1})^{3t^2}}}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}\rangle^{ab}.$$

Let $c_0 = x_0x_t^{-2}x_{2t}$,

$$P = \langle \underline{\underline{x_0, x_t, x_{2t}, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, (b_0x_0^{-2}x_t^3x_{2t}^{-3})^{3t^2}, (b_1b_2^{-3}x_0^3x_t^{-2}x_{2t}^3)^{3t^2},$$

$$(x_0x_t^{-2}x_{2t})^{3t^2}, (b_2x_0^{-2}x_tx_{2t}^{-1})^{3t^2}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}, \underline{\underline{c_0 = x_0x_t^{-2}x_{2t}}}\rangle^{ab}.$$

Eliminate $x_{2t} = c_0x_0^{-1}x_t^2$,

$$P = \langle \underline{\underline{x_0, x_t, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_0x_0^{-2}x_t^3(c_0x_0^{-1}x_t^2)^{-3})^{3t^2}}},$$

$$\underline{\underline{(b_1b_2^{-3}x_0^3x_t^{-2}(c_0x_0^{-1}x_t^2)^3)^{3t^2}}}, \underline{\underline{c_0^{3t^2}}}, \underline{\underline{(b_2x_0^{-2}x_t(c_0x_0^{-1}x_t^2)^{-1})^{3t^2}}}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}\rangle^{ab}$$

$$= \langle x_0, x_t, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0 \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_0x_0x_t^{-3})^{3t^2}}},$$

$$\underline{\underline{(b_1b_2^{-3}x_t^4)^{3t^2}}}, c_0^{3t^2}, \underline{\underline{(b_2x_0^{-1}x_t^{-1})^{3t^2}}}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}\rangle^{ab}.$$

Let $c_1 = b_2x_0^{-1}x_t^{-1}$,

$$P = \langle \underline{\underline{x_0, x_t, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0, c_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, (b_0x_0x_t^{-3})^{3t^2},$$

$$(b_1b_2^{-3}x_t^4)^{3t^2}, c_0^{3t^2}, (b_2x_0^{-1}x_t^{-1})^{3t^2}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}, \underline{\underline{c_1 = b_2x_0^{-1}x_t^{-1}}}\rangle^{ab}.$$

Eliminate $x_t = b_2c_1^{-1}x_0^{-1}$,

$$P = \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0, c_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_0x_0(b_2c_1^{-1}x_0^{-1})^{-3})^{3t^2}}},$$

$$\underline{\underline{(b_1b_2^{-3}(b_2c_1^{-1}x_0^{-1})^4)^{3t^2}}}, c_0^{3t^2}, \underline{\underline{c_1^{3t^2}}}, (b_0b_2^{-1})^{3t}, (b_1b_2^{-1})^{3t}\rangle^{ab}$$

$$= \langle x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0, c_1 \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_0 b_2^{-3} x_0^4)^{3t^2}}},$$

$$\underline{\underline{(b_1 b_2 x_0^{-4})^{3t^2}}}, c_0^{3t^2}, c_1^{3t^2}, (b_0 b_2^{-1})^{3t}, (b_1 b_2^{-1})^{3t} \rangle^{ab}.$$

Let $d_k = b_k b_2^{-1}$ $(k \in \{0,1\})$,

$$P = \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_0, b_1, b_2, c_0, c_1, d_0, d_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, (b_0 b_2^{-3} x_0^4)^{3t^2},$$

$$(b_1 b_2 x_0^{-4})^{3t^2}, c_0^{3t^2}, c_1^{3t^2}, (b_0 b_2^{-1})^{3t}, (b_1 b_2^{-1})^{3t}, \underline{\underline{d_k = b_k b_2^{-1}}}\ (k \in \{0,1\}) \rangle^{ab}.$$

Eliminate $b_k = d_k b_2$ $(k \in \{0,1\})$,

$$P = \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_2, c_0, c_1, d_0, d_1}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{((d_0 b_2) b_2^{-3} x_0^4)^{3t^2}}},$$

$$\underline{\underline{((d_1 b_2) b_2 x_0^{-4})^{3t^2}}}, c_0^{3t^2}, c_1^{3t^2}, \underline{\underline{d_0^{3t}}}, \underline{\underline{d_1^{3t}}} \rangle^{ab}$$

$$= \langle x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_2, c_0, c_1, d_0, d_1 \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{(b_2^{-2} x_0^4)^{3t^2}}},$$

$$\underline{\underline{(b_2^2 x_0^{-4})^{3t^2}}}, c_0^{3t^2}, c_1^{3t^2}, d_0^{3t}, d_1^{3t} \rangle^{ab}.$$

Let $d_2 = b_2 x_0^{-2}$,

$$P = \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, b_2, c_0, c_1, d_0, d_1, d_2}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, (b_2^{-2} x_0^4)^{3t^2},$$

$$(b_2^2 x_0^{-4})^{3t^2}, c_0^{3t^2}, c_1^{3t^2}, d_0^{3t}, d_1^{3t}, \underline{\underline{d_2 = b_2 x_0^{-2}}} \rangle^{ab}.$$

Eliminate $b_2 = d_2 x_0^2$,

$$P = \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, c_0, c_1, d_0, d_1, d_2}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, \underline{\underline{d_2^{-6t^2}}},$$

$$\underline{\underline{d_2^{6t^2}}}, c_0^{3t^2}, c_1^{3t^2}, d_0^{3t}, d_1^{3t} \rangle^{ab}$$

$$= \langle \underline{\underline{x_0, w_2, \ldots, w_{t-1}, w_{t+2}, \ldots, w_{2t-1}, \ldots, w_{5t+2}, \ldots, w_{6t-1}, a_0, a_1, c_0, c_1, d_0, d_1, d_2}} \mid$$

$$w_{it+j}^{3t}\ (i \in \{0,1,\ldots,5\}, j \in \{2,3,\ldots,t-1\}), a_0^{3t}, a_1^{3t}, d_2^{6t^2}, c_0^{3t^2}, c_1^{3t^2}, d_0^{3t}, d_1^{3t} \rangle^{ab}.$$

□

As for $\overline{C(6,t)}$, we have the following theorem.

Theorem G. *The sandpile group of* $\overline{C(6,t)}$ *is* ${\mathbb{Z}_{3t}}^{6t-8} \oplus \mathbb{Z}_{3t^2} \oplus {\mathbb{Z}_{15t^2}}^2$.

Proof. The Laplacian matrix of $\overline{C(6,t)}$ is

$$L(\overline{C(6,t)}) = circ_{6t}(3t, 0, -1, -1, -1, 0, 0, 0, -1, -1, -1, 0, 0, 0, \ldots, -1, -1, -1, 0, 0, 0, -1, -1, -1, 0).$$

Hence,

$$P = \langle x_0, x_1, \ldots, x_{6t-1} \mid x_i^{3t}((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1}$$
$$(i = 0, 1, \ldots, 6t-1)\rangle^{ab}.$$

Notice that x indices are modulo $6t$.

$$P = \langle x_0, x_1, \ldots, x_{6t-1} \mid \underline{\underline{x_j^{3t}((x_2x_3x_4)(x_8x_9x_{10})\ldots(x_{6t-4}x_{6t-3}x_{6t-2}))^{-1}}},$$
$$\underline{\underline{x_{j+1}^{3t}((x_3x_4x_5)(x_9x_{10}x_{11})\ldots(x_{6t-3}x_{6t-2}x_{6t-1}))^{-1}}},$$
$$\underline{\underline{x_{j+2}^{3t}((x_4x_5x_6)(x_{10}x_{11}x_{12})\ldots(x_{6t-2}x_{6t-1}x_0))^{-1}}},$$
$$\underline{\underline{x_{j+3}^{3t}((x_5x_6x_7)(x_{11}x_{12}x_{13})\ldots(x_{6t-1}x_0x_1))^{-1}}},$$
$$\underline{\underline{x_{j+4}^{3t}((x_6x_7x_8)(x_{12}x_{13}x_{14})\ldots(x_0x_1x_2))^{-1}}},$$
$$\underline{\underline{x_{j+5}^{3t}((x_7x_8x_9)(x_{13}x_{14}x_{15})\ldots(x_1x_2x_3))^{-1}}}\ (j = 0, 6, \ldots, 6t-6)\rangle^{ab}.$$

Let $y_i = ((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1}$ $(i = 0, 1, \ldots, 5)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5}} \mid x_j^{3t}((x_2x_3x_4)(x_8x_9x_{10})\ldots(x_{6t-4}x_{6t-3}x_{6t-2}))^{-1},$$
$$x_{j+1}^{3t}((x_3x_4x_5)(x_9x_{10}x_{11})\ldots(x_{6t-3}x_{6t-2}x_{6t-1}))^{-1},$$
$$x_{j+2}^{3t}((x_4x_5x_6)(x_{10}x_{11}x_{12})\ldots(x_{6t-2}x_{6t-1}x_0))^{-1},$$
$$x_{j+3}^{3t}((x_5x_6x_7)(x_{11}x_{12}x_{13})\ldots(x_{6t-1}x_0x_1))^{-1},$$
$$x_{j+4}^{3t}((x_6x_7x_8)(x_{12}x_{13}x_{14})\ldots(x_0x_1x_2))^{-1}, x_{j+5}^{3t}((x_7x_8x_9)(x_{13}x_{14}x_{15})\ldots(x_1x_2x_3))^{-1},$$
$$\underline{\underline{y_i = ((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1}}}$$
$$(i = 0, 1, \ldots, 5, j = 0, 6, \ldots, 6t-6)\rangle^{ab}$$
$$= \langle x_0, x_1, \ldots, x_{6t-1}, y_0, y_1, \ldots, y_5 \mid \underline{\underline{x_j^{3t}y_0}}, \underline{\underline{x_{j+1}^{3t}y_1}}, \underline{\underline{x_{j+2}^{3t}y_2}}, \underline{\underline{x_{j+3}^{3t}y_3}}, \underline{\underline{x_{j+4}^{3t}y_4}}, \underline{\underline{x_{j+5}^{3t}y_5}},$$
$$y_i = ((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1}$$
$$(i = 0, 1, \ldots, 5, j = 0, 6, \ldots, 6t-6)\rangle^{ab}.$$

Eliminate $y_i = x_{0+i}^{-3t}$ $(i = 0, 1, \ldots, 5)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}}} \mid \underline{\underline{x_{j+i}^{3t}x_i^{-3t}}},$$
$$\underline{\underline{x_i^{-3t} = ((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1}}}$$

$$(i = 0, 1, \ldots, 5, j = 6, 12, \ldots, 6t - 6)\rangle^{ab}.$$

Let $s_{j+i-6} = x_{j+i}x_i^{-1}$ $(i = 0, 1, \ldots, 5, j = 6, 12, \ldots, 6t - 6)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{6t-1}, s_0, s_1, \ldots, s_{6t-7}}} \mid x_{j+i}^{3t}x_i^{-3t},$$

$$x_i^{-3t} = ((x_{i+2}x_{i+3}x_{i+4})(x_{i+8}x_{i+9}x_{i+10})\ldots(x_{i+6t-4}x_{i+6t-3}x_{i+6t-2}))^{-1},$$

$$\underline{\underline{s_{j+i-6} = x_{j+i}x_i^{-1}}}\ (i = 0, 1, \ldots, 5, j = 6, 12, \ldots, 6t - 6)\rangle^{ab}.$$

Eliminate $x_{j+i} = s_{j+i-6}x_i$ $(i = 0, 1, \ldots, 5, j = 6, 12, \ldots, 6t - 6)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_5, s_0, s_1, \ldots, s_{6t-7}}} \mid \underline{\underline{s_{j+i-6}^{3t}}}\ (i = 0, 1, \ldots, 5, j = 6, 12, \ldots, 6t - 6),$$

$$\underline{\underline{x_0^{3t}((x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1}}},$$

$$\underline{\underline{x_1^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1}}},$$

$$\underline{\underline{x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}s_0))^{-1}}},$$

$$\underline{\underline{x_3^{3t}((x_5x_0x_1)^t(s_5s_6s_7)(s_{11}s_{12}s_{13})\ldots(s_{6t-7}s_0s_1))^{-1}}},$$

$$\underline{\underline{x_4^{3t}((x_0x_1x_2)^t(s_6s_7s_8)(s_{12}s_{13}s_{14})\ldots(s_0s_1s_2))^{-1}}},$$

$$\underline{\underline{x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_1s_2s_3))^{-1}}}\rangle^{ab}.$$

Relabel s_{j+i-6}^{3t} as s_k^{3t}, $k = i + j - 6$,

$$P = \langle x_0, x_1, \ldots, x_5, s_0, s_1, \ldots, s_{6t-7} \mid \underline{\underline{s_k^{3t}}}\ (k = 0, 1, \ldots, 6t - 7),$$

$$x_0^{3t}((x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1},$$

$$x_1^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1},$$

$$x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}s_0))^{-1},$$

$$x_3^{3t}((x_5x_0x_1)^t(s_5s_6s_7)(s_{11}s_{12}s_{13})\ldots(s_{6t-7}s_0s_1))^{-1},$$

$$x_4^{3t}((x_0x_1x_2)^t(s_6s_7s_8)(s_{12}s_{13}s_{14})\ldots(s_0s_1s_2))^{-1},$$

$$x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_1s_2s_3))^{-1}\rangle^{ab}.$$

Eliminate $s_0 = x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}))^{-1}$,

$$s_1 = x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_2s_3))^{-1}$$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_5, s_2, s_3, \ldots, s_{6t-7}}} \mid s_k^{3t}\ (k = \underline{\underline{2}}, 3, \ldots, 6t - 7),$$

$$\underline{\underline{(x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}))^{-1})^{3t}}},$$

$$\underline{\underline{(x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_2s_3))^{-1})^{3t}}},$$

$$x_0^{3t}((x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1},$$

$$x_1^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1},$$

$$\underline{\underline{\Big(x_3^{3t}((x_5x_0x_1)^t(s_5s_6s_7)(s_{11}s_{12}s_{13})\cdot}}$$

$$\underline{\underline{\ldots(s_{6t-7}(x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}))^{-1})}}$$

$$\underline{\underline{\cdot(x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_2s_3))^{-1})s_{6t-7}))^{-1}\Big)}},$$

$$\underline{\underline{\Big(x_4^{3t}((x_0x_1x_2)^t(s_6s_7s_8)(s_{12}s_{13}s_{14})\cdot}}$$

$$\underline{\underline{\ldots((x_2^{3t}((x_4x_5x_0)^t(s_4s_5s_6)(s_{10}s_{11}s_{12})\ldots(s_{6t-8}s_{6t-7}))^{-1})}}$$

$$\underline{\underline{\cdot(x_5^{3t}((x_1x_2x_3)^t(s_7s_8s_9)(s_{13}s_{14}s_{15})\ldots(s_2s_3))^{-1})s_2))^{-1}\Big)}}\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_5, s_2, s_3, \ldots, s_{6t-7} \mid s_k^{3t}\ (k = 2, 3, \ldots, 6t-7),$$

$$\underline{\underline{(x_2^{3t}(x_4x_5x_0)^{-t})^{3t}}}, \underline{\underline{(x_5^{3t}(x_1x_2x_3)^{-t})^{3t}}},$$

$$x_0^{3t}((x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1},$$

$$x_1^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1},$$

$$\underline{\underline{\Big((x_3(x_2x_5)^{-1})^{3t}((x_5x_0x_1(x_4x_5x_0)^{-1}(x_1x_2x_3)^{-1})^t}}$$

$$\underline{\underline{\cdot(s_5s_6s_7(s_4s_5s_6)^{-1}(s_7s_8s_9)^{-1})(s_{11}s_{12}s_{13}(s_{10}s_{11}s_{12})^{-1}(s_{13}s_{14}s_{15})^{-1})\cdot}}$$

$$\underline{\underline{\ldots(s_{6t-7}(s_{6t-8}s_{6t-7})^{-1}(s_2s_3)^{-1}))^{-1}\Big)}},$$

$$\underline{\underline{\Big((x_4(x_2x_5)^{-1})^{3t}((x_0x_1x_2(x_4x_5x_0)^{-1}(x_1x_2x_3)^{-1})^t}}$$

$$\underline{\underline{\cdot(s_6s_7s_8(s_4s_5s_6)^{-1}(s_7s_8s_9)^{-1})(s_{10}s_{11}s_{12}(s_{10}s_{11}s_{12})^{-1}(s_{12}s_{13}s_{14})^{-1})\cdot}}$$

$$\underline{\underline{\ldots(s_2(s_{6t-8}s_{6t-7})^{-1}(s_2s_3)^{-1}))^{-1}\Big)}}\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_5, s_2, s_3, \ldots, s_{6t-7} \mid s_k^{3t}\ (k = 2, 3, \ldots, 6t-7),$$

$$(x_2^{3t}(x_4x_5x_0)^{-t})^{3t}, (x_5^{3t}(x_1x_2x_3)^{-t})^{3t},$$

$$x_0^{3t}((x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1},$$

$$x_1^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1},$$

$$\underline{\underline{(x_3(x_2x_5)^{-1})^{3t}(x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8})}},$$

$$\underline{\underline{(x_4(x_2x_5)^{-1})^{3t}((x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7})}}\rangle^{ab}.$$

Let $a = (s_2s_3s_4)(s_8s_9s_{10})\ldots(s_{6t-10}s_{6t-9}s_{6t-8}), b = (s_3s_4s_5)(s_9s_{10}s_{11})\ldots(s_{6t-9}s_{6t-8}s_{6t-7})$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_5, s_2, s_3, \ldots, s_{6t-7}, a, b}} \mid s_k^{3t}\ (k = 2, 3, \ldots, 6t-7),$$

$$(x_2^{3t}(x_4x_5x_0)^{-t})^{3t}, (x_5^{3t}(x_1x_2x_3)^{-t})^{3t}, \underline{\underline{x_0^{3t}((x_2x_3x_4)^t a)^{-1}}}, \underline{\underline{x_1^{3t}((x_3x_4x_5)^t b)^{-1}}},$$

$$\underline{\underline{(x_3(x_2x_5)^{-1})^{3t}(x_2x_3x_4)^t a}}, \underline{\underline{(x_4(x_2x_5)^{-1})^{3t}(x_3x_4x_5)^t b}},$$

$$\underline{\underline{a = (s_2s_3s_4)(s_8s_9s_{10})\dots(s_{6t-10}s_{6t-9}s_{6t-8})}},$$

$$\underline{\underline{b = (s_3s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7})}}\rangle^{ab}.$$

Eliminate $a = x_0^{3t}((x_2x_3x_4)^t)^{-1}, b = x_1^{3t}((x_3x_4x_5)^t)^{-1}$

$$P = \langle \underline{\underline{x_0, x_1, \dots, x_5, s_2, s_3, \dots, s_{6t-7}}} \mid s_k^{3t}\ (k = 2, 3, \dots, 6t-7),$$

$$(x_2^{3t}(x_4x_5x_0)^{-t})^{3t}, (x_5^{3t}(x_1x_2x_3)^{-t})^{3t}, \underline{\underline{(x_3(x_2x_5)^{-1})^{3t}(x_2x_3x_4)^t x_0^{3t}((x_2x_3x_4)^t)^{-1}}},$$

$$\underline{\underline{(x_4(x_2x_5)^{-1})^{3t}(x_3x_4x_5)^t x_1^{3t}((x_3x_4x_5)^t)^{-1}}},$$

$$\underline{\underline{x_0^{3t}((x_2x_3x_4)^t)^{-1} = (s_2s_3s_4)(s_8s_9s_{10})\dots(s_{6t-10}s_{6t-9}s_{6t-8})}},$$

$$\underline{\underline{x_1^{3t}((x_3x_4x_5)^t)^{-1} = (s_3s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7})}}\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_5, s_2, s_3, \dots, s_{6t-7} \mid s_k^{3t}\ (k = 2, 3, \dots, 6t-7),$$

$$(x_2^3(x_4x_5x_0)^{-1})^{3t^2}, (x_5^3(x_1x_2x_3)^{-1})^{3t^2}, \underline{\underline{(x_0x_3(x_2x_5)^{-1})^{3t}}}, \underline{\underline{(x_1x_4(x_2x_5)^{-1})^{3t}}},$$

$$\underline{\underline{x_0^{-3t}(x_2x_3x_4)^t(s_2s_3s_4)(s_8s_9s_{10})\dots(s_{6t-10}s_{6t-9}s_{6t-8})}},$$

$$\underline{\underline{x_1^{-3t}(x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7})}}\rangle^{ab}.$$

Eliminate $s_2 = (x_0^{-3t}(x_2x_3x_4)^t(s_3s_4)(s_8s_9s_{10})\dots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-1}$

$$P = \langle \underline{\underline{x_0, x_1, \dots, x_5, s_3, s_4, \dots, s_{6t-7}}} \mid s_k^{3t}\ (k = \underline{\underline{3}}, 4, \dots, 6t-7),$$

$$\underline{\underline{(x_0^{-3t}(x_2x_3x_4)^t(s_3s_4)(s_8s_9s_{10})\dots(s_{6t-10}s_{6t-9}s_{6t-8}))^{-3t}}},$$

$$(x_2^3(x_4x_5x_0)^{-1})^{3t^2}, (x_5^3(x_1x_2x_3)^{-1})^{3t^2}, (x_0x_3(x_2x_5)^{-1})^{3t}, (x_1x_4(x_2x_5)^{-1})^{3t},$$

$$x_1^{-3t}(x_3x_4x_5)^t(s_3s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7})\rangle^{ab}.$$

Eliminate $s_3 = (x_1^{-3t}(x_3x_4x_5)^t(s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-1}$

$$P = \langle \underline{\underline{x_0, x_1, \dots, x_5, s_4, s_5, \dots, s_{6t-7}}} \mid s_k^{3t}\ (k = \underline{\underline{4}}, 5, \dots, 6t-7),$$

$$\underline{\underline{(x_1^{-3t}(x_3x_4x_5)^t(s_4s_5)(s_9s_{10}s_{11})\dots(s_{6t-9}s_{6t-8}s_{6t-7}))^{-3t}}},$$

$$\underline{\underline{(x_0^{-3t}(x_2x_3x_4)^t)^{-3t}}},$$

$$(x_2^3(x_4x_5x_0)^{-1})^{3t^2}, (x_5^3(x_1x_2x_3)^{-1})^{3t^2}, (x_0x_3(x_2x_5)^{-1})^{3t}, (x_1x_4(x_2x_5)^{-1})^{3t}\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_5, s_4, s_5, \dots, s_{6t-7} \mid s_k^{3t}\ (k = 4, 5, \dots, 6t-7),$$

$$\underline{\underline{(x_5^3(x_1x_2x_3)^{-1})^{3t^2}}}, \underline{\underline{(x_0^3(x_2x_3x_4)^{-1})^{3t^2}}}, (x_1^3(x_3x_4x_5)^{-1})^{3t^2}, (x_2^3(x_4x_5x_0)^{-1})^{3t^2},$$

$$(x_0x_3(x_2x_5)^{-1})^{3t}, (x_1x_4(x_2x_5)^{-1})^{3t}\rangle^{ab}.$$

Let $v_i = x_i x_{i+3}$ $(i = 0, 1, 2)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_5, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2}} \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$(x_5^3(x_1x_2x_3)^{-1})^{3t^2}, (x_0^3(x_2x_3x_4)^{-1})^{3t^2}, (x_1^3(x_3x_4x_5)^{-1})^{3t^2}, (x_2^3(x_4x_5x_0)^{-1})^{3t^2},$$

$$(x_0x_3(x_2x_5)^{-1})^{3t}, (x_1x_4(x_2x_5)^{-1})^{3t}, \underline{\underline{v_i = x_ix_{i+3}}} \ (i = 0, 1, 2)\rangle^{ab}.$$

Eliminate $x_3 = v_0x_0^{-1}, x_4 = v_1x_1^{-1}$

$$P = \langle \underline{\underline{x_0, x_1, x_2, x_5, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2}} \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{(x_5^3(x_1x_2(v_0x_0^{-1}))^{-1})^{3t^2}}}, \underline{\underline{(x_0^3(x_2(v_0x_0^{-1})(v_1x_1^{-1}))^{-1})^{3t^2}}}, \underline{\underline{(x_1^3((v_0x_0^{-1})(v_1x_1^{-1})x_5)^{-1})^{3t^2}}},$$

$$\underline{\underline{(x_2^3((v_1x_1^{-1})x_5x_0)^{-1})^{3t^2}}}, \underline{\underline{(x_0(v_0x_0^{-1})(x_2x_5)^{-1})^{3t}}}, \underline{\underline{(x_1(v_1x_1^{-1})(x_2x_5)^{-1})^{3t}}}, v_2 = x_2x_5\rangle^{ab}$$

$$= \langle x_0, x_1, x_2, x_5, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2 \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{(x_5^4(x_1v_0v_2)^{-1}x_0)^{3t^2}}}, \underline{\underline{(x_0^4(x_2v_0v_1)^{-1}x_1)^{3t^2}}}, \underline{\underline{(x_1^4(x_5v_0v_1)^{-1}x_0)^{3t^2}}}, \underline{\underline{(x_2^4(x_0v_1v_2)^{-1}x_1)^{3t^2}}},$$

$$\underline{\underline{(v_0v_2^{-1})^{3t}}}, \underline{\underline{(v_1v_2^{-1})^{3t}}}, v_2 = x_2x_5\rangle^{ab}$$

$$= \langle x_0, x_1, x_2, x_5, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2 \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$(x_5^4(x_1v_0v_2\underline{\underline{(v_2v_2^{-1})}})^{-1}x_0)^{3t^2}, (x_0^4(x_2v_0v_1\underline{\underline{(v_2v_2^{-1})}})^{-1}x_1)^{3t^2}, (x_1^4(x_5v_0v_1\underline{\underline{(v_2v_2^{-1})}})^{-1}x_0)^{3t^2},$$

$$(x_2^4(x_0v_1v_2\underline{\underline{(v_2v_2^{-1})}})^{-1}x_1)^{3t^2}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}, v_2 = x_2x_5\rangle^{ab}.$$

As $(v_0v_2^{-1})^{3t} = (v_1v_2^{-1})^{3t} = 1$

$$P = \langle x_0, x_1, x_2, x_5, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2 \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{(x_5^4x_0x_1^{-1}v_2^{-2})^{3t^2}}}, \underline{\underline{(x_0^4x_1x_2^{-1}v_2^{-2})^{3t^2}}}, \underline{\underline{(x_1^4x_0x_5^{-1}v_2^{-2})^{3t^2}}}, \underline{\underline{(x_2^4x_1x_0^{-1}v_2^{-2})^{3t^2}}},$$

$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}, v_2 = x_2x_5\rangle^{ab}.$$

Eliminate $x_5 = v_2x_2^{-1}$

$$P = \langle \underline{\underline{x_0, x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2}} \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{((v_2x_2^{-1})^4x_0x_1^{-1}v_2^{-2})^{3t^2}}}, (x_0^4x_1x_2^{-1}v_2^{-2})^{3t^2}, \underline{\underline{(x_1^4x_0(v_2x_2^{-1})^{-1}v_2^{-2})^{3t^2}}},$$

$$(x_2^4x_1x_0^{-1}v_2^{-2})^{3t^2}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$

$$= \langle x_0, x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2 \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{(x_2^4x_0^{-1}x_1v_2^{-2})^{-3t^2}}}, (x_0^4x_1x_2^{-1}v_2^{-2})^{3t^2}, \underline{\underline{(x_1^4x_0x_2v_2^{-3})^{3t^2}}},$$

$$(x_2^4x_1x_0^{-1}v_2^{-2})^{3t^2}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$

$$= \langle x_0, x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2 \mid s_k^{3t} \ (k = 4, 5, \ldots, 6t-7),$$

$$(x_0^4x_1x_2^{-1}v_2^{-2})^{3t^2}, (x_1^4x_0x_2v_2^{-3})^{3t^2}, (x_2^4x_1x_0^{-1}v_2^{-2})^{3t^2},$$

$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}.$$

Let $w_0 = x_0x_1$

$$P = \langle \underline{\underline{x_0, x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$(x_0^4x_1x_2^{-1}v_2^{-2})^{3t^2}, (x_1^4x_0x_2v_2^{-3})^{3t^2}, (x_2^4x_1x_0^{-1}v_2^{-2})^{3t^2},$$
$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}, \underline{\underline{w_0 = x_0x_1}}\rangle^{ab}.$$

Eliminate $x_0 = x_1w_0^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{((x_1w_0^{-1})^4x_1x_2^{-1}v_2^{-2})^{3t^2}}}, \underline{\underline{(x_1^4(x_1w_0^{-1})x_2v_2^{-3})^{3t^2}}}, \underline{\underline{(x_2^4x_1(x_1w_0^{-1})^{-1}v_2^{-2})^{3t^2}}},$$
$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$
$$= \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{(x_1^5w_0^{-4}x_2^{-1}v_2^{-2})^{3t^2}}}, \underline{\underline{(x_1^5w_0^{-1}x_2v_2^{-3})^{3t^2}}}, \underline{\underline{(x_2^4w_0v_2^{-2})^{3t^2}}},$$
$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}.$$

Multiply $(x_2^4w_0v_2^{-2})^{3t^2}$ into $(x_1^5w_0^{-4}x_2^{-1}v_2^{-2})^{3t^2}$

$$P = \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{(x_1^5w_0^{-1}x_2v_2^{-3})^{3t^2}(w_0^{-3}x_2^{-2}v_2)^{3t^2}}}, (x_1^5w_0^{-1}x_2v_2^{-3})^{3t^2}, (x_2^4w_0v_2^{-2})^{3t^2},$$
$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$
$$= \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{(w_0^{-3}x_2^{-2}v_2)^{3t^2}}}, (x_1^5w_0^{-1}x_2v_2^{-3})^{3t^2}, (x_2^4w_0v_2^{-2})^{3t^2},$$
$$(v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}.$$

As $(v_2^{-2}x_2^4w_0)^{3t^2} \implies w_0^{3t^2} = (x_2^{-4}v_2^2)^{3t^2}$, we can substitute it into $(w_0^{-3}x_2^{-2}v_2)^{3t^2}, (x_1^5w_0^{-1}x_2v_2^{-3})^{3t^2}$

$$P = \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{((x_2^{-4}v_2^2)^{-3}x_2^{-2}v_2)^{3t^2}}}, \underline{\underline{(x_1^5(x_2^{-4}v_2^2)^{-1}x_2v_2^{-3})^{3t^2}}}, v_2^{-2}x_2^4w_0)^{3t^2}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$
$$= \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$\underline{\underline{(v_2^{-5}x_2^{10})^{3t^2}}}, \underline{\underline{(v_2^{-5}x_1^5x_2^5)^{3t^2}}}, v_2^{-2}x_2^4w_0)^{3t^2}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}.$$

Similarly $(v_2^{-5}x_1^5x_2^5)^{3t^2} \implies v_2^{15t^2} = x_2^{30t^2}$, so we can substitute it into $(v_2^{-5}x_1^5x_2^5)^{3t^2}, v_2^{-2}x_2^4w_0)^{3t^2}$

$$P = \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$(v_2^{-5}x_1^5x_2^5)^{3t^2}, \underline{\underline{((x_2^{-10})x_1^5x_2^5)^{3t^2}}}, \underline{\underline{(v_2^{-2}x_2^4w_0)^{3t^2}}}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}$$
$$= \langle x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$
$$(v_2x_2^{-2})^{15t^2}, \underline{\underline{(x_1x_2^{-1})^{15t^2}}}, \underline{\underline{(v_2^{-2}x_2^4w_0)^{3t^2}}}, (v_0v_2^{-1})^{3t}, (v_1v_2^{-1})^{3t}\rangle^{ab}.$$

Let $w_1 = x_1 x_2^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0, w_1}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$(v_2 x_2^{-2})^{15t^2}, (x_1 x_2^{-1})^{15t^2}, (v_2^{-2} x_2^4 w_0)^{3t^2}, (v_0 v_2^{-1})^{3t}, (v_1 v_2^{-1})^{3t}, \underline{\underline{w_1 = x_1 x_2^{-1}}} \rangle^{ab}.$$

Eliminate $x_1 = x_2 w_1$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0, w_1}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$(v_2 x_2^{-2})^{15t^2}, \underline{\underline{((x_2 w_1) x_2^{-1})^{15t^2}}}, (v_2^{-2} x_2^4 w_0)^{3t^2}, (v_0 v_2^{-1})^{3t}, (v_1 v_2^{-1})^{3t} \rangle^{ab}$$

$$= \langle x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0, w_1 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$(v_2 x_2^{-2})^{15t^2}, \underline{\underline{w_1^{15t^2}}}, (v_2^{-2} x_2^4 w_0)^{3t^2}, (v_0 v_2^{-1})^{3t}, (v_1 v_2^{-1})^{3t} \rangle^{ab}.$$

Let $w_2 = v_2 x_2^{-2}$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, v_2, w_0, w_1, w_2}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$(v_2 x_2^{-2})^{15t^2}, w_1^{15t^2}, (v_2^{-2} x_2^4 w_0)^{3t^2}, (v_0 v_2^{-1})^{3t}, (v_1 v_2^{-1})^{3t}, \underline{\underline{w_2 = v_2 x_2^{-2}}} \rangle^{ab}.$$

Eliminate $v_2 = w_2 x_2^2$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_0, w_1, w_2}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$\underline{\underline{w_2^{15t^2}}}, w_1^{15t^2}, \underline{\underline{((w_2 x_2^2)^{-2} x_2^4 w_0)^{3t^2}}}, \underline{\underline{(v_0 (w_2 x_2^2)^{-1})^{3t}}}, \underline{\underline{(v_1 (w_2 x_2^2)^{-1})^{3t}}} \rangle^{ab}$$

$$= \langle x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_0, w_1, w_2 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, \underline{\underline{(w_2^{-2} w_0)^{3t^2}}}, \underline{\underline{(v_0 w_2^{-1} x_2^{-2})^{3t}}}, \underline{\underline{(v_1 w_2^{-1} x_2^{-2})^{3t}}} \rangle^{ab}.$$

Let $u_0 = w_0 w_2^{-2}$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_0, w_1, w_2, u_0}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, (w_2^{-2} w_0)^{3t^2}, (v_0 w_2^{-1} x_2^{-2})^{3t}, (v_1 w_2^{-1} x_2^{-2})^{3t}, \underline{\underline{u_0 = w_0 w_2^{-2}}} \rangle^{ab}.$$

Eliminate $w_0 = u_0 w_2^2$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_1, w_2, u_0}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, \underline{\underline{(w_2^{-2} (u_0 w_2^2))^{3t^2}}}, (v_0 w_2^{-1} x_2^{-2})^{3t}, (v_1 w_2^{-1} x_2^{-2})^{3t} \rangle^{ab}$$

$$= \langle x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_1, w_2, u_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, \underline{\underline{u_0^{3t^2}}}, (v_0 w_2^{-1} x_2^{-2})^{3t}, (v_1 w_2^{-1} x_2^{-2})^{3t} \rangle^{ab}.$$

As $(v_0 w_2^{-1} x_2^{-2})^{3t} \implies v_0^{3t} = (w_2 x_2^2)^{3t}$, we can substitute it into $(v_1 w_2^{-1} x_2^{-2})^{3t}$

$$P = \langle x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_1, w_2, u_0 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, \underline{\underline{v_0^{3t} = (w_2 x_2^2)^{3t}}}, \underline{\underline{(v_1 v_0^{-1})^{3t}}} \rangle^{ab}.$$

Let $u_1 = v_0^{-1}v_1$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_0, v_1, w_1, w_2, u_0, u_1}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, v_0^{3t} = (w_2x_2^2)^{3t}, (v_1v_0^{-1})^{3t}, \underline{\underline{u_1 = v_0^{-1}v_1}}\rangle^{ab}.$$

Eliminate $v_0 = v_1u_1^{-1}$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_1, w_1, w_2, u_0, u_1}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, \underline{\underline{(v_1u_1^{-1})^{3t} = (w_2x_2^2)^{3t}}}, \underline{\underline{(v_1(v_1u_1^{-1})^{-1})^{3t}}}\rangle^{ab}$$

$$= \langle x_2, s_4, s_5, \ldots, s_{6t-7}, v_1, w_1, w_2, u_0, u_1 \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, \underline{\underline{(v_1^{-1}w_2x_2^2)^{3t}}}, \underline{\underline{u_1^{3t}}}\rangle^{ab}.$$

Let $u_2 = v_1^{-1}w_2x_2^2$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, v_1, w_1, w_2, u_0, u_1, u_2}} \mid s_k^{3t}\ (k = 4, 5, \ldots, 6t-7),$$

$$w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, \underline{\underline{(v_1^{-1}w_2x_2^2)^{3t}}}, u_1^{3t}, \underline{\underline{u_2 = v_1^{-1}w_2x_2^2}}\rangle^{ab}.$$

Eliminate $v_1 = u_2^{-1}w_2x_2^2$

$$P = \langle \underline{\underline{x_2, s_4, s_5, \ldots, s_{6t-7}, w_1, w_2, u_0, u_1, u_2}} \mid$$

$$s_k^{3t}\ (k = 4, 5, \ldots, 6t-7), w_2^{15t^2}, w_1^{15t^2}, u_0^{3t^2}, \underline{\underline{u_2^{3t}}}, u_1^{3t}\rangle^{ab}.$$

□

Theorems F, G gives us the following

$$\mathrm{Sp}(C(6,t)) = {\mathbb{Z}_{3t}}^{6t-8} \oplus {\mathbb{Z}_{3t^2}}^2 \oplus \mathbb{Z}_{6t^2},\ \mathrm{Sp}(\overline{C(6,t)}) = {\mathbb{Z}_{3t}}^{6t-8} \oplus \mathbb{Z}_{3t^2} \oplus {\mathbb{Z}_{15t^2}}^2,$$

which is an interesting case of a non-self complementary graph and its complement having sandpile groups with a lot of similarity in their structure.

3.3.2 Block matrix constructions

We can construct graphs using block matrix as their adjacency matrix. For example, the complete bipartite graph $K_{r,r}$ can be defined as a graph that has this adjacency matrix

$$A(K_{r,r}) = \begin{pmatrix} 0_r & J_r \\ J_r & 0_r \end{pmatrix}$$

where 0_r is a zero matrix and J_r is an all ones matrix.

It turned out that the circulant graph $H = \mathrm{Cay}(\mathbb{Z}_{4q}, \{1, 3, \ldots, 4q-1\} \cup \{2q\})$, $q \geqslant 2$ is isomor-

phic to the graph $T = (V, E)$ where $V = V(K_{2q,2q}) = \{v_0, v_1, \ldots, v_{4q-1}\}$, $E = E(K_{2q,2q}) \cup \{v_0v_{2q-1}, v_1v_{2q-2}, \ldots, v_{q-1}v_q, v_{2q}v_{4q-1}, v_{2q+1}v_{4q-2}, \ldots, v_{3q-1}v_{3q}\}$. We show this bijection in Proposition 3.3.2. The graph T has block adjacency matrix

$$A(T) = \begin{pmatrix} I_{anti} & J_{2q} \\ J_{2q} & I_{anti} \end{pmatrix}$$

Proposition 3.3.2. *The graph H is isomorphic to*

$$T = G\big(V(K_{2q,2q}), E(K_{2q,2q}) \cup \{v_0v_{2q-1}, v_1v_{2q-2}, \ldots, v_{q-1}v_q, v_{2q}v_{4q-1}, v_{2q+1}v_{4q-2}, \ldots, v_{3q-1}v_{3q}\}\big).$$

Proof. The following bijection $f\colon V(H) \to V(T)$ shows how to get graph T from H.

$$f(u_i) = \begin{cases} v_{4q-1-i} & i < 2q \text{ and } i \text{ is odd or } i \geqslant 2q \text{ and } i \text{ is even} \\ v_i & \text{otherwise.} \end{cases}$$

As H is a circulant graph, we can see that the edges are preserved by checking the vertices u_0, u_1. When we apply f to their incident edges, we see that u_0u_i are mapped into v_0v_{4q-1-i} for $i \in \{1, 3, \ldots, 2q-1\} \cup \{2q, 2q+2, \ldots, 4q-2\}$. u_1u_j are mapped into v_1v_j for $i \in \{0, 2, \ldots, 2q-2\} \cup \{2q+1, 2q+3, \ldots, 4q-1\}$ and u_1u_{2q} maps into v_1, v_{2q-1}. □

The following Proposition shows that T has four distinct integral Laplacian eigenvalues.

Proposition 3.3.3. *The graph T has the Laplacian eigenvalues $0, 4q, 2q, 2q+2$ with multiplicities $1, 1, 2q-2, 2q$ respectively.*

Proof. By looking at $A(T)$, we can clearly see that T is a join of two $2q$ sets of K_2. It is known that the Laplacian eigenvalues of K_2 are $0, 2$. Theorem **??** tells us that the Laplacian eigenvalues of $\underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q}$ are $0, 2$ both with multiplicity $2q$. Using Theorem 1.2.2, we can see that the Laplacian eigenvalues of

$$\underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q} \bigtriangledown \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q}$$

are $0, 4q, 2q, 2q+2$ with multiplicities $1, 1, 2q-2, 2q$ respectively. □

The sandpile group of H is calculated using Tietze transformations next.

Theorem H. *The Laplacian matrix of the graph H is*

$$circ_n(\underbrace{2q+1, -1, 0, -1, \ldots, 0, -1}_{2q}, \underbrace{-1, -1, 0, -1, 0, -1, \ldots, 0, -1}_{2q}).$$

The Smith form of this is given by the abelianization of the cyclically presented group $G = G_{4q}(x_0^{2q+1}x_1^{-1}x_3^{-1}\dots x_{2q-1}^{-1}x_{2q}^{-1}x_{2q+1}^{-1}x_{2q+3}^{-1}\dots x_{4q-1}^{-1})$ *which is isomorphic to*

$$\begin{cases} \mathbb{Z}_{(q+1)} \oplus \mathbb{Z}_{4(q+1)}{}^2 \oplus \mathbb{Z}_{4q(q+1)}{}^{2q-4} \oplus \mathbb{Z}_{4q^2(q+1)} & \textit{if } q \textit{ is even} \\ \mathbb{Z}_{2(q+1)}{}^2 \oplus \mathbb{Z}_{4(q+1)} \oplus \mathbb{Z}_{4q(q+1)}{}^{2q-4} \oplus \mathbb{Z}_{4q^2(q+1)} & \textit{if } q \textit{ is odd.} \end{cases}$$

Proof. Proposition 3.3.2 tells us that the circulant graph H is isomorphic to the graph

$$T = G\big(V(K_{2q,2q}), E(K_{2q,2q}) \cup \{v_0v_{2q-1}, v_1v_{2q-2}, \dots, v_{q-1}v_q, v_{2q}v_{4q-1}, v_{2q+1}v_{4q-2}, \dots, v_{3q-1}v_{3q}\}\big).$$

Therefore, the presentation $G_{4q}(x_0^{2q+1}x_1^{-1}x_3^{-1}\dots x_{2q-1}^{-1}x_{2q}^{-1}x_{2q+1}^{-1}x_{2q+3}^{-1}\dots x_{4q-1}^{-1})$ is isomorphic to the group presentation P below that has the relation matrix $L(T)$.

$$\begin{aligned} P = \langle & x_0, x_1, \dots, x_{2q-1}, y_0, y_1, \dots, y_{2q-1} \mid \\ & x_i^{2q+1}x_{2q-1-i}^{-1}(y_0y_1\dots y_{2q-1})^{-1} = 1, \\ & y_i^{2q+1}y_{2q-1-i}^{-1}(x_0x_1\dots x_{2q-1})^{-1} = 1 \ (i = 0, 1, \dots, 2q-1)\rangle^{ab}. \end{aligned}$$

Let $z = x_0(y_0y_1\dots y_{2q-1}), w = y_0(x_0x_1\dots x_{2q-1})$

$$\begin{aligned} P = \langle & \underline{\underline{x_0, x_1, \dots, x_{2q-1}, y_0, y_1, \dots, y_{2q-1}, z, w}} \mid x_i^{2q+1}x_{2q-1-i}^{-1}(y_0y_1\dots y_{2q-1})^{-1} = 1, \\ & y_i^{2q+1}y_{2q-1-i}^{-1}(x_0x_1\dots x_{2q-1})^{-1} \ (i = 0, 1, \dots, 2q-1), \\ & \underline{\underline{z = x_0(y_0y_1\dots y_{2q-1})}}, \underline{\underline{w = y_0(x_0x_1\dots x_{2q-1})}}\rangle^{ab}. \end{aligned}$$

Substitute $(y_0y_1\dots y_{2q-1})$ with zx_0^{-1} and $(x_0x_1\dots x_{2q-1})$ with wy_0^{-1}

$$\begin{aligned} P = \langle & x_0, x_1, \dots, x_{2q-1}, y_0, y_1, \dots, y_{2q-1}, z, w \mid \underline{\underline{x_i^{2q+1}x_{2q-1-i}^{-1}(zx_0^{-1})^{-1}}}, \\ & \underline{\underline{y_i^{2q+1}y_{2q-1-i}^{-1}(wy_0^{-1})^{-1}}} \ (i = 0, 1, \dots, 2q-1), \\ & z = x_0(y_0y_1\dots y_{2q-1}), w = y_0(x_0x_1\dots x_{2q-1})\rangle^{ab} \\ = \langle & x_0, x_1, \dots, x_{2q-1}, y_0, y_1, \dots, y_{2q-1}, z, w \mid x_i^{2q+1}x_{2q-1-i}^{-1}(zx_0^{-1})^{-1}, \underline{\underline{x_0^{2q+1}x_{2q-1}^{-1}(zx_0^{-1})^{-1}}}, \\ & \underline{\underline{x_{2q-1}^{2q+1}x_0^{-1}(zx_0^{-1})^{-1}}}, y_i^{2q+1}y_{2q-1-i}^{-1}(wy_0^{-1})^{-1}, \underline{\underline{y_0^{2q+1}y_{2q-1}^{-1}(wy_0^{-1})^{-1}}}, \\ & \underline{\underline{y_{2q-1}^{2q+1}y_0^{-1}(wy_0^{-1})^{-1}}} \ (i = \underline{\underline{1}}, 2, \dots, \underline{\underline{2q-2}}), z = x_0(y_0y_1\dots y_{2q-1}), w = y_0(x_0x_1\dots x_{2q-1})\rangle^{ab} \\ = \langle & x_0, x_1, \dots, x_{2q-1}, y_0, y_1, \dots, y_{2q-1}, z, w \mid \underline{\underline{x_i^{2q+1}x_{2q-1-i}^{-1}z^{-1}x_0}}, \underline{\underline{x_0^{2q+2}x_{2q-1}^{-1}z^{-1}}}, \underline{\underline{x_{2q-1}^{2q+1}z^{-1}}}, \\ & \underline{\underline{y_i^{2q+1}y_{2q-1-i}^{-1}w^{-1}y_0}}, \underline{\underline{y_0^{2q+2}y_{2q-1}^{-1}w^{-1}}}, \underline{\underline{y_{2q-1}^{2q+1}w^{-1}}} \ (i = 1, 2, \dots, 2q-2), \\ & z = x_0(y_0y_1\dots y_{2q-1}), w = y_0(x_0x_1\dots x_{2q-1})\rangle^{ab}. \end{aligned}$$

Eliminate $z = x_{2q-1}^{2q+1}, w = y_{2q-1}^{2q+1}$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{2q-1}, y_0, y_1, \ldots, y_{2q-1}}} \mid \underline{\underline{x_i^{2q+1} x_{2q-1-i}^{-1} (x_{2q-1}^{2q+1})^{-1} x_0}}, \underline{\underline{x_0^{2q+2} x_{2q-1}^{-1} (x_{2q-1}^{2q+1})^{-1}}},$$

$$\underline{\underline{y_i^{2q+1} y_{2q-1-i}^{-1} (y_{2q-1}^{2q+1})^{-1} y_0}}, \underline{\underline{y_0^{2q+2} y_{2q-1}^{-1} (y_{2q-1}^{2q+1})^{-1}}} \ (i = 1, 2, \ldots, 2q-2),$$

$$\underline{\underline{x_{2q-1}^{2q+1} = x_0 (y_0 y_1 \ldots y_{2q-1})}}, \underline{\underline{y_{2q-1}^{2q+1} = y_0 (x_0 x_1 \ldots x_{2q-1})}} \rangle^{ab}.$$

Let $s = y_0 y_1 \ldots y_{2q-1}, t = x_0 x_1 \ldots x_{2q-1}$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{2q-1}, y_0, y_1, \ldots, y_{2q-1}, s, t}} \mid x_i^{2q+1} x_{2q-1-i}^{-1} x_{2q-1}^{-(2q+1)} x_0, x_0^{2q+2} x_{2q-1}^{-(2q+2)},$$

$$y_i^{2q+1} y_{2q-1-i}^{-1} y_{2q-1}^{-(2q+1)} y_0, y_0^{2q+2} y_{2q-1}^{-(2q+2)} \ (i = 1, 2, \ldots, 2q-2),$$

$$x_{2q-1}^{2q+1} = x_0 (y_0 y_1 \ldots y_{2q-1}), y_{2q-1}^{2q+1} = y_0 (x_0 x_1 \ldots x_{2q-1}),$$

$$\underline{\underline{s = y_0 y_1 \ldots y_{2q-1}}}, \underline{\underline{t = x_0 x_1 \ldots x_{2q-1}}} \rangle^{ab}.$$

Substitute $(y_0 y_1 \ldots y_{2q-1})$ with s and $(x_0 x_1 \ldots x_{2q-1})$ with t

$$P = \langle x_0, x_1, \ldots, x_{2q-1}, y_0, y_1, \ldots, y_{2q-1}, s, t \mid x_i^{2q+1} x_{2q-1-i}^{-1} x_{2q-1}^{-(2q+1)} x_0, x_0^{2q+2} x_{2q-1}^{-(2q+2)},$$

$$y_i^{2q+1} y_{2q-1-i}^{-1} y_{2q-1}^{-(2q+1)} y_0, y_0^{2q+2} y_{2q-1}^{-(2q+2)} \ (i = 1, 2, \ldots, 2q-2),$$

$$\underline{\underline{x_{2q-1}^{2q+1} = x_0 s}}, \underline{\underline{y_{2q-1}^{2q+1} = y_0 t}}, s = y_0 y_1 \ldots y_{2q-1}, t = x_0 x_1 \ldots x_{2q-1} \rangle^{ab}.$$

Eliminate $x_0 = x_{2q-1}^{2q+1} s^{-1}, y_0 = y_{2q-1}^{2q+1} t^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{2q-1}, y_1, y_2, \ldots, y_{2q-1}, s, t}} \mid$$

$$\underline{\underline{x_i^{2q+1} x_{2q-1-i}^{-1} x_{2q-1}^{-(2q+1)} (x_{2q-1}^{2q+1} s^{-1})}}, \underline{\underline{(x_{2q-1}^{2q+1} s^{-1})^{2q+2} x_{2q-1}^{-(2q+2)}}},$$

$$\underline{\underline{y_i^{2q+1} y_{2q-1-i}^{-1} y_{2q-1}^{-(2q+1)} (y_{2q-1}^{2q+1} t^{-1})}}, \underline{\underline{(y_{2q-1}^{2q+1} t^{-1})^{2q+2} y_{2q-1}^{-(2q+2)}}} \ (i = 1, 2, \ldots, 2q-2),$$

$$\underline{\underline{s = (y_{2q-1}^{2q+1} t^{-1}) y_1 \ldots y_{2q-1}}}, \underline{\underline{t = (x_{2q-1}^{2q+1} s^{-1}) x_1 \ldots x_{2q-1}}} \rangle^{ab}$$

$$= \langle x_1, x_2, \ldots, x_{2q-1}, y_1, y_2, \ldots, y_{2q-1}, s, t \mid \underline{\underline{x_i^{2q+1} x_{2q-1-i}^{-1} s^{-1}}}, \underline{\underline{(x_{2q-1}^{2q} s^{-1})^{2q+2}}},$$

$$\underline{\underline{y_i^{2q+1} y_{2q-1-i}^{-1} t^{-1}}}, \underline{\underline{(y_{2q-1}^{2q} t^{-1})^{2q+2}}} \ (i = 1, 2, \ldots, 2q-2),$$

$$\underline{\underline{s = y_1 \ldots y_{2q-2} y_{2q-1}^{2q+2} t^{-1}}}, \underline{\underline{t = x_1 \ldots x_{2q-2} x_{2q-1}^{2q+2} s^{-1}}} \rangle^{ab}.$$

Eliminate $x_i = x_{2q-i-1}^{2q+1} s^{-1}, y_i = y_{2q-i-1}^{2q+1} t^{-1}$ for $\{i = 1, 2, \ldots, q-1\}$

$$P = \langle \underline{\underline{x_q, x_{q+1}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, s, t}} \mid \underline{\underline{(x_{2q-1-i}^{2q} s^{-1})^{2q+2}}},$$

$$\underline{\underline{(y_{2q-1-i}^{2q} t^{-1})^{2q+2}}} \ (i = q, q+1, \ldots, 2q-1),$$

$$\underline{\underline{s = (y_q y_{q+1} \ldots y_{2q-1})^{2q+2} t^{-q}}}, \underline{\underline{t = (x_q x_{q+1} \ldots x_{2q-1})^{2q+2} s^{-q}}} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, s, t \mid \underline{\underline{(x_i^{2q} s^{-1})^{2q+2}}}, \underline{\underline{(y_i^{2q} t^{-1})^{2q+2}}} \ (i = q, q+1, \ldots, 2q-1),$$

$$\underline{\underline{s = (y_q y_{q+1} \ldots y_{2q-1})^{2q+2} t^{-q}}}, \underline{\underline{t = (x_q x_{q+1} \ldots x_{2q-1})^{2q+2} s^{-q}}} \rangle^{ab}.$$

Eliminate $t = (x_q x_{q+1} \dots x_{2q-1})^{2q+2} s^{-q}$

$$P = \langle \underline{\underline{x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s}} \mid (x_i^{2q} s^{-1})^{2q+2},$$

$$\underline{\underline{(y_i^{2q}((x_q x_{q+1} \dots x_{2q-1})^{2q+2} s^{-q})^{-1})^{2q+2}}} \ (i = q, q+1, \dots, 2q-1),$$

$$\underline{\underline{s = (y_q y_{q+1} \dots y_{2q-1})^{2q+2}((x_q x_{q+1} \dots x_{2q-1})^{2q+2} s^{-q})^{-q}}} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s \mid \underline{\underline{(x_i^{2q} s^{-1})^{2q+2}}},$$

$$\underline{\underline{(y_i^{2q}(x_q x_{q+1} \dots x_{2q-1})^{-(2q+2)} s^q)^{2q+2}}} \ (i = q, q+1, \dots, 2q-1),$$

$$\underline{\underline{(y_q y_{q+1} \dots y_{2q-1})^{2q+2}(x_q x_{q+1} \dots x_{2q-1})^{-2q(q+1)} s^{q^2-1}}} \rangle^{ab}.$$

To simplify the relations $(y_i^{2q}(x_q x_{q+1} \dots x_{2q-1})^{-(2q+2)} s^q)^{2q+2}$ $(i = q, q+1, \dots, 2q-1)$, we take the relations $(x_i^{2q} s^{-1})^{2q+2}$ $(i = q, q+1, \dots, 2q-1)$ and multiply them to each of the relations above. Then we get

$$P = \langle x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s \mid (x_i^{2q} s^{-1})^{2q+2},$$

$$\underline{\underline{(y_i^{2q}(x_q x_{q+1} \dots x_{2q-1})^{-(2q+2)} s^q)^{2q+2}((x_q^{2q} s^{-1})(x_{q+1}^{2q} s^{-1}) \dots (x_{2q-1}^{2q} s^{-1}))^{2q+2}}}$$

$$(i = q, q+1, \dots, 2q-1), (y_q y_{q+1} \dots y_{2q-1})^{2q+2}(x_q x_{q+1} \dots x_{2q-1})^{-2q(q+1)} s^{q^2-1} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s \mid (x_i^{2q} s^{-1})^{2q+2},$$

$$\underline{\underline{(y_i^{2q}(x_q x_{q+1} \dots x_{2q-1})^{-2})^{2q+2}}} \ (i = q, q+1, \dots, 2q-1),$$

$$(y_q y_{q+1} \dots y_{2q-1})^{2q+2}(x_q x_{q+1} \dots x_{2q-1})^{-2q(q+1)} s^{q^2-1} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s \mid (x_i^{2q} s^{-1})^{2q+2},$$

$$\underline{\underline{(y_i^{q}(x_q x_{q+1} \dots x_{2q-1})^{-1})^{4(q+1)}}} \ (i = q, q+1, \dots, 2q-1),$$

$$((y_q y_{q+1} \dots y_{2q-1})^2(x_q x_{q+1} \dots x_{2q-1})^{-2q} s^{q-1})^{q+1} \rangle^{ab}.$$

Let $a = x_{2q-1}^{2q} s^{-1}$

$$P = \langle \underline{\underline{x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, s, a}} \mid (x_i^{2q} s^{-1})^{2q+2},$$

$$(y_i^{q}(x_q x_{q+1} \dots x_{2q-1})^{-1})^{4(q+1)} \ (i = q, q+1, \dots, 2q-1)$$

$$((y_q y_{q+1} \dots y_{2q-1})^2(x_q x_{q+1} \dots x_{2q-1})^{-2q} s^{q-1})^{q+1}, \underline{\underline{a = x_{2q-1}^{2q} s^{-1}}} \rangle^{ab}.$$

Eliminate $s = x_{2q-1}^{2q} a^{-1}$

$$P = \langle \underline{\underline{x_q, x_{q+1}, \dots, x_{2q-1}, y_q, y_{q+1}, \dots, y_{2q-1}, a}} \mid \underline{\underline{(x_i^{2q}(x_{2q-1}^{2q} a^{-1})^{-1})^{2q+2}}},$$

$$(y_i^{q}(x_q x_{q+1} \dots x_{2q-1})^{-1})^{4(q+1)} \ (i = q, q+1, \dots, 2q-1),$$

$$\underline{\underline{((y_q y_{q+1} \dots y_{2q-1})^2(x_q x_{q+1} \dots x_{2q-1})^{-2q}(x_{2q-1}^{2q} a^{-1})^{q-1})^{q+1}}} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a \mid \underline{\underline{((x_i x_{2q-1}^{-1})^{2q} a)^{2(q+1)}}}, \underline{\underline{a^{2(q+1)}}} \ (i = q, q+1, \ldots, \underline{\underline{2q-2}}),$$

$$(y_j^q (x_q x_{q+1} \ldots x_{2q-1})^{-1})^{4(q+1)} \ (j = q, q+1, \ldots, 2q-1),$$

$$\underline{\underline{((y_q y_{q+1} \ldots y_{2q-1})^2 (x_q x_{q+1} \ldots x_{2q-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}}} \rangle^{ab}$$

$$= \langle x_q, x_{q+1}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a \mid \underline{\underline{(x_i x_{2q-1}^{-1})^{4q(q+1)}}} \ (i = q, q+1, \ldots, 2q-2), a^{2(q+1)},$$

$$(y_j^q (x_q x_{q+1} \ldots x_{2q-1})^{-1})^{4(q+1)} \ (j = q, q+1, \ldots, 2q-1),$$

$$((y_q y_{q+1} \ldots y_{2q-1})^2 (x_q x_{q+1} \ldots x_{2q-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1} \rangle^{ab}.$$

Let $b = y_q^q (x_q x_{q+1} \ldots x_{2q-1})^{-1}$

$$P = \langle \underline{\underline{x_q, x_{q+1}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a, b}} \mid (x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q, q+1, \ldots, 2q-2), a^{2(q+1)},$$

$$(y_j^q (x_q x_{q+1} \ldots x_{2q-1})^{-1})^{4(q+1)} \ (j = q, q+1, \ldots, 2q-1),$$

$$((y_q y_{q+1} \ldots y_{2q-1})^2 (x_q x_{q+1} \ldots x_{2q-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}, \underline{\underline{b = y_q^q (x_q x_{q+1} \ldots x_{2q-1})^{-1}}} \rangle^{ab}.$$

Eliminate $x_q = y_q^q (x_{q+1} x_{q+2} \ldots x_{2q-1})^{-1} b^{-1}$

$$P = \langle \underline{\underline{x_{q+1}, x_{q+2}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a, b}} \mid (x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = \underline{\underline{q+1}}, q+2, \ldots, 2q-2),$$

$$a^{2(q+1)}, \underline{\underline{(y_q^q (x_{q+1} \ldots x_{2q-1})^{-1} b^{-1} x_{2q-1}^{-1})^{4q(q+1)}}},$$

$$\underline{\underline{(y_j^q ((y_q^q (x_{q+1} \ldots x_{2q-1})^{-1} b^{-1}) x_{q+1} \ldots x_{2q-1})^{-1})^{4(q+1)}}} \ (j = q, \ldots, 2q-1),$$

$$\underline{\underline{((y_q \ldots y_{2q-1})^2 ((y_q^q (x_{q+1} \ldots x_{2q-1})^{-1} b^{-1}) x_{q+1} \ldots x_{2q-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}}} \rangle^{ab}$$

$$= \langle x_{q+1}, x_{q+2}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a, b \mid (x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q+1, q+2, \ldots, 2q-2),$$

$$a^{2(q+1)}, \underline{\underline{(y_q^q (x_{q+1} \ldots x_{2q-1})^{-1} b^{-1} x_{2q-1}^{-1})^{4q(q+1)}}},$$

$$\underline{\underline{(y_j^q y_q^{-q} b)^{4(q+1)}}} \ (j = q, q+1, \ldots, 2q-1),$$

$$\underline{\underline{((y_q y_{q+1} \ldots y_{2q-1})^2 ((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}}} \rangle^{ab}$$

$$= \langle x_{q+1}, x_{q+2}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a, b \mid (x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q+1, q+2, \ldots, 2q-2),$$

$$a^{2(q+1)}, (y_q^q (x_{q+1} \ldots x_{2q-2})^{-1} x_{2q-1}^{-2})^{4q(q+1)},$$

$$\underline{\underline{(y_j y_q^{-1})^{4q(q+1)}}} \ (j = \underline{\underline{q+1}}, q+2, \ldots, 2q-1), \underline{\underline{b^{4(q+1)}}},$$

$$((y_q y_{q+1} \ldots y_{2q-1})^2 ((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1} \rangle^{ab}.$$

Let $c_j = y_j y_q^{-1}$, for $(j = q+1, q+2, \ldots, 2q-1)$

$$P = \langle x_{q+1}, x_{q+2}, \ldots, x_{2q-1}, y_q, y_{q+1}, \ldots, y_{2q-1}, a, b, c_{q+1}, c_{q+2}, \ldots, c_{2q-1} \mid$$

$$(x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q+1, q+2, \ldots, 2q-2), a^{2(q+1)}, (y_q^q (x_{q+1} \ldots x_{2q-2})^{-1} x_{2q-1}^{-2})^{4q(q+1)},$$

$$(y_j y_q^{-1})^{4q(q+1)} \ (j = q+1, q+2, \ldots, 2q-1), b^{4(q+1)},$$

$$((y_q y_{q+1} \dots y_{2q-1})^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}, \underline{\underline{c_j = y_j y_q^{-1}}}\rangle^{ab}.$$

Eliminate $y_j = c_j y_q$, for $(j = q+1, q+2, \dots, 2q-1)$

$$P = \langle \underline{\underline{x_{q+1}, x_{q+2}, \dots, x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-1}}} \mid$$

$$(x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, (y_q^q (x_{q+1} \dots x_{2q-2})^{-1} x_{2q-1}^{-2})^{4q(q+1)}, \underline{\underline{c_j^{4q(q+1)}}} \ (j = q+1, q+2, \dots, 2q-1), b^{4(q+1)}$$

$$\underline{\underline{((y_q^q c_{q+1} \dots c_{2q-1})^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}}}\rangle^{ab}.$$

Let $d_i = x_i x_{2q-1}^{-1}$, for $(i = q+1, q+2, \dots, 2q-2)$

$$P = \langle \underline{\underline{x_{q+1}, x_{q+2}, \dots, x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-1}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}}} \mid$$

$$(x_i x_{2q-1}^{-1})^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2), a^{2(q+1)}, (y_q^q (x_{q+1} \dots x_{2q-2})^{-1} x_{2q-1}^{-2})^{4q(q+1)},$$

$$c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-1), b^{4(q+1)}$$

$$((y_q^q c_{q+1} \dots c_{2q-1})^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}, \underline{\underline{d_i = x_i x_{2q-1}^{-1}}}\rangle^{ab}.$$

Eliminate $x_i = d_i x_q$, for $(i = q+1, q+2, \dots, 2q-2)$

$$P = \langle \underline{\underline{x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-1}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}}} \mid \underline{\underline{d_i^{4q(q+1)}}} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, \underline{\underline{(y_q^q (d_{q+1} \dots d_{2q-2})^{-1} x_{2q-1}^{-q})^{4q(q+1)}}}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-1), b^{4(q+1)},$$

$$((y_q^q c_{q+1} \dots c_{2q-1})^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}\rangle^{ab}$$

$$= \langle x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-1}, d_{q+1}, d_{q+2}, \dots, d_{2q-2} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, \underline{\underline{(y_q x_{2q-1}^{-1})^{4q^2(q+1)}}}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-1), b^{4(q+1)},$$

$$((y_q^q c_{q+1} \dots c_{2q-1})^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}\rangle^{ab}.$$

Let $g = c_{q+1} \dots c_{2q-1}$

$$P = \langle \underline{\underline{x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-1}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g}} \mid$$

$$d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, (y_q x_{2q-1}^{-1})^{4q^2(q+1)}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-1), b^{4(q+1)},$$

$$\underline{\underline{((y_q^q g)^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}}}, \underline{\underline{g = c_{q+1} \dots c_{2q-1}}}\rangle^{ab}.$$

Eliminate $c_{2q-1} = g(c_{q+1} \dots c_{2q-2})^{-1}$

$$P = \langle \underline{\underline{x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g}} \mid$$

$$d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, (y_q x_{2q-1}^{-1})^{4q^2(q+1)}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, \underline{\underline{2q-2}}), b^{4(q+1)},$$

$$\underline{\underline{(g(c_{q+1}\dots c_{2q-2})^{-1})^{4q(q+1)}}}, ((y_q^q g)^2((y_q^q b^{-1})^{-2q} x_{2q-1}^{2q(q-1)} a^{q-1})^{q+1}\rangle^{ab}$$

$$= \langle x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g \mid$$

$$d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, (y_q x_{2q-1}^{-1})^{4q^2(q+1)}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-2), b^{4(q+1)}, \underline{\underline{g^{4q(q+1)}}},$$

$$\underline{\underline{((y_q x_{2q-1}^{-1})^{-2q(q-1)} g^2 b^{2q} a^{q-1})^{q+1}}}\rangle^{ab}.$$

Let $f = y_q x_{2q-1}^{-1}$

$$P = \langle \underline{\underline{x_{2q-1}, y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g, f}} \mid$$

$$d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, (y_q x_{2q-1}^{-1})^{4q^2(q+1)}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-2), b^{4(q+1)}, g^{4q(q+1)},$$

$$((y_q x_{2q-1}^{-1})^{-2q(q-1)} g^2 b^{2q} a^{q-1})^{q+1}, \underline{\underline{f = y_q x_{2q-1}^{-1}}}\rangle^{ab}.$$

Eliminate $x_{2q-1} = y_q f^{-1}$

$$P = \langle \underline{\underline{y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g, f}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, \underline{\underline{f^{4q^2(q+1)}}}, c_j^{4q(q+1)} \ (j = q+1, q+2, \dots, 2q-2), b^{4(q+1)}, g^{4q(q+1)},$$

$$(f^{-2q(q-1)} g^2 b^{2q} a^{q-1})^{q+1}\rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g, f \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, f^{4q^2(q+1)}, \underline{\underline{c_i^{4q(q+1)}}}, b^{4(q+1)}, g^{4q(q+1)}, (f^{-2q(q-1)} g^2 b^{2q} a^{q-1})^{q+1}\rangle^{ab}.$$

Let $h = f^{-q(q-1)} g b^q$

$$P = \langle \underline{\underline{y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, g, f, h}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, g^{4q(q+1)}, (f^{-2q(q-1)} g^2 b^{2q} a^{q-1})^{q+1}, \underline{\underline{h = f^{-q(q-1)} g b^q}}\rangle^{ab}.$$

Eliminate $g = h f^{q(q-1)} b^{-q}$

$$P = \langle \underline{\underline{y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, \underline{\underline{(h f^{q(q-1)} b^{-q})^{4q(q+1)}}},$$

$$\underline{\underline{(f^{-2q(q-1)} (h f^{q(q-1)} b^{-q})^2 b^{2q} a^{q-1})^{q+1}}}\rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, (h f^{q(q-1)} b^{-q})^{4q(q+1)}, \underline{\underline{(h^2 a^{q-1})^{q+1}}}\rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$

$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, \underline{\underline{h^{4q(q+1)} (f^{4q^2(q+1)})^{q-1} (b^{4(q+1)})^{-2q^2}}}, (h^2 a^{q-1})^{q+1}\rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, \underline{\underline{h^{4q(q+1)}}}, (h^2 a^{q-1})^{q+1} \rangle^{ab}.$$

If $q = 2m+1,\ m = \{1, 2, \dots\}$, then

$$P = \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{(h^2 a^{2m})^{q+1}}} \rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{h^{2(q+1)} (a^{2(q+1)})^m}} \rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{h^{2(q+1)}}} \rangle^{ab}.$$

Otherwise if $q = 2m,\ m = \{1, 2, \dots\}$, then

$$P = \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{(h^2 a^{2m-1})^{q+1}}} \rangle^{ab}$$

$$= \langle y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{(h^2 a^{-1})^{q+1} (a^{2(q+1)})^m}} \rangle^{ab}.$$

Let $v = h^2 a^{-1}$

$$P = \langle \underline{\underline{y_q, a, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h, v}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$a^{2(q+1)}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{(h^2 a^{-1})^{q+1}}}, \underline{\underline{v = h^2 a^{-1}}} \rangle^{ab}.$$

Eliminate $a = h^2 v^{-1}$

$$P = \langle \underline{\underline{y_q, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h, v}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$\underline{\underline{(h^2 v^{-1})^{2(q+1)}}}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, h^{4q(q+1)}, \underline{\underline{v^{q+1}}} \rangle^{ab}$$

$$= \langle \underline{\underline{y_q, b, c_{q+1}, c_{q+2}, \dots, c_{2q-2}, d_{q+1}, d_{q+2}, \dots, d_{2q-2}, f, h, v}} \mid d_i^{4q(q+1)} \ (i = q+1, q+2, \dots, 2q-2),$$
$$\underline{\underline{h^{4(q+1)}}}, f^{4q^2(q+1)}, c_i^{4q(q+1)}, b^{4(q+1)}, v^{q+1} \rangle^{ab}.$$

□

The result in Theorem H is for a complete bipartite graph of even independent sets $K_{2q,2q}$ with a largest possible matching added to both of its vertex classes, which happens to be a perfect matching. If we try to do the same construction for $K_{2q-1,2q-1}$, we cannot have a perfect matching as the resulting graph will contain a loop, by avoiding adding a loop, we get $K_{2q-1,2q-1}$ with the largest possible matching added to it. This results in a semi-regular graph close enough to the "odd" case of Theorem H. Theorem I shows that the resulting sandpile

group structure is somewhat similar to Theorem H.

Let T_{odd} be a graph of order $n = 4q-2$, $q \geqslant 2$ where $V(T_{odd}) = G(V(K_{2q-1,2q-1})$, and $E(T_{odd}) = E(K_{2q-1,2q-1}) \cup \{v_0v_{2q-2}, v_1v_{2q-3}, \ldots, v_{q-2}v_q, v_{2q-1}v_{4q-3}, v_{2q}v_{4q-4}, \ldots, v_{3q-3}v_{3q-1}\})$, where $K_{2q-1,2q-1}$ is a complete bipartite graph. T_{odd} has the block adjacency matrix

$$A(T) = \begin{pmatrix} B & J_{2q-1} \\ J_{2q-1} & B \end{pmatrix}$$

where B is an anti-diagonal matrix defined as $I_{anti} - \text{diag}(\underbrace{0,0,\ldots,0}_{q-1}, 1, \underbrace{0,0,\ldots,0}_{q-1})$.

T_{odd} also has four distinct integral Laplacian eigenvalues. The following Proposition shows that.

Proposition 3.3.4. *The graph T_{odd} has the Laplacian eigenvalues $0, 4q-2, 2q-1, 2q+1$ with multiplicities $1, 1, 2q-2, 2q-2$ respectively.*

Proof. By looking at $A(T_{odd})$, we can clearly see that T is a join of two $K_1 \cup \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q-2}$ graphs. The Laplacian eigenvalues of K_2 are $0, 2$. Theorem **??** tells us that the Laplacian eigenvalues of $K_1 \cup \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q-2}$ are $0, 2$ with multiplicities $q, q-1$ respectively. Using Theorem 1.2.2, we can see that the Laplacian eigenvalues of $(K_1 \cup \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q-2}) \bigtriangledown (K_1 \cup \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{2q-2})$ are $0, 4q-2, 2q-1, 2q+1$ with multiplicities $1, 1, 2q-2, 2q-2$ respectively. □

Now, the sandpile group is as follows.

Theorem I. *The sandpile group of T_{odd} is $\mathbb{Z}_{2q+1} \oplus \mathbb{Z}_{(2q+1)(2q-1)}{}^{2q-4} \oplus \mathbb{Z}_{(2q+1)(2q-1)^2}$.*

Proof.

$$\begin{aligned} G_{4q} = \langle x_0, x_1, \ldots, x_{2q-2}, y_0, y_1, \ldots, y_{2q-2} \mid\ & x_i^{2q} x_{2q-2-i}^{-1} (y_0y_1 \ldots y_{2q-2})^{-1} = 1, \\ & y_i^{2q} y_{2q-2-i}^{-1} (x_0x_1 \ldots x_{2q-2})^{-1} = 1 \ (i = 0, 1, \ldots, q-2, q, \ldots, 2q-2), \\ & x_{q-1}^{2q-1} (y_0y_1 \ldots y_{q-1}y_{q+1} \ldots y_{2q-2})^{-1} = 1, y_{q-1}^{2q-1} (x_0x_1 \ldots x_{q-1}x_{q+1} \ldots x_{2q-2})^{-1} = 1 \rangle^{ab}. \end{aligned}$$

By multiplying $x_{q-1}^{2q-1} (y_0y_1 \ldots y_{q-1}y_{q+1} \ldots y_{2q-2})^{-1} = 1, y_{q-1}^{2q-1} (x_0x_1 \ldots x_{q-1}x_{q+1} \ldots x_{2q-2})^{-1} = 1$ with $x_{q-1}^{-1}x_{q-1}, y_{q-1}^{-1}y_{q-1}$ we can write G_{4q} as

$$\begin{aligned} G_{4q} = \langle x_0, x_1, \ldots, x_{2q-2}, y_0, y_1, \ldots, y_{2q-2} \mid\ & \underline{\underline{x_i^{2q} x_{2q-2-i}^{-1} (y_0y_1 \ldots y_{2q-2})^{-1} = 1}}, \\ & \underline{\underline{y_i^{2q} y_{2q-2-i}^{-1} (x_0x_1 \ldots x_{2q-2})^{-1} = 1}} \ (i = 0, 1, \ldots, 2q-2) \rangle^{ab}. \end{aligned}$$

Let $z = x_0(y_0y_1 \ldots y_{2q-2}), w = y_0(x_0x_1 \ldots x_{2q-2})$

$$\begin{aligned} G_{4q} = \langle \underline{\underline{x_0, x_1, \ldots, x_{2q-2}, y_0, y_1, \ldots, y_{2q-2}, z, w}} \mid\ & x_i^{2q} x_{2q-2-i}^{-1} (y_0y_1 \ldots y_{2q-2})^{-1}, \\ & y_i^{2q} y_{2q-2-i}^{-1} (x_0x_1 \ldots x_{2q-2})^{-1} \ (i = 0, 1, \ldots, 2q-2), \end{aligned}$$

$$\underline{\underline{z = x_0(y_0y_1\dots y_{2q-2})}}, \underline{\underline{w = y_0(x_0x_1\dots x_{2q-2})}}\rangle^{ab}.$$

Substitute $(y_0y_1\dots y_{2q-2})$ with zx_0^{-1} and $(x_0x_1\dots x_{2q-2})$ with wy_0^{-1}

$$G_{4q} = \langle x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2}, z, w \mid \underline{\underline{x_i^{2q}x_{2q-2-i}^{-1}x_0z^{-1}}}, \underline{\underline{y_i^{2q}y_{2q-2-i}^{-1}y_0w^{-1}}}\ (i = 0, 1, \dots, 2q-2),$$

$$z = x_0(y_0y_1\dots y_{2q-2}), w = y_0(x_0x_1\dots x_{2q-2})\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2}, z, w \mid x_i^{2q}x_{2q-2-i}^{-1}x_0z^{-1}, \underline{\underline{x_{2q-2}^{2q}z^{-1}}}, y_i^{2q}y_{2q-2-i}^{-1}y_0w^{-1}$$

$$(i = 0, 1, \dots, \underline{\underline{2q-3}}), \underline{\underline{y_{2q-2}^{2q}w^{-1}}}, z = x_0(y_0y_1\dots y_{2q-2}), w = y_0(x_0x_1\dots x_{2q-2})\rangle^{ab}.$$

Eliminate $z = x_{2q-2}^{2q}, w = y_{2q-2}^{2q}$

$$G_{4q} = \langle \underline{\underline{x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2}}} \mid \underline{\underline{x_i^{2q}x_{2q-2-i}^{-1}x_0(x_{2q-2}^{2q})^{-1}}}, \underline{\underline{y_i^{2q}y_{2q-2-i}^{-1}y_0(y_{2q-2}^{2q})^{-1}}}$$

$$(i = 0, 1, \dots, 2q-3), \underline{\underline{x_{2q-2}^{2q} = x_0(y_0y_1\dots y_{2q-2})}}, \underline{\underline{y_{2q-2}^{2q} = y_0(x_0x_1\dots x_{2q-2})}}\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2} \mid \underline{\underline{(x_ix_{2q-2}^{-1})^{2q}x_{2q-2-i}^{-1}x_0}}, \underline{\underline{(x_0x_{2q-2}^{-1})^{2q+1}}},$$

$$\underline{\underline{(y_iy_{2q-2}^{-1})^{2q}y_{2q-2-i}^{-1}y_0}}, \underline{\underline{(y_0y_{2q-2}^{-1})^{2q+1}}}\ (i = \underline{\underline{1}}, \dots, 2q-3),$$

$$x_{2q-2}^{2q} = x_0(y_0y_1\dots y_{2q-2}), y_{2q-2}^{2q} = y_0(x_0x_1\dots x_{2q-2})\rangle^{ab}.$$

Let $s = y_0y_1\dots y_{2q-2}, t = x_0x_1\dots x_{2q-2}$

$$G_{4q} = \langle \underline{\underline{x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2}, s, t}} \mid (x_ix_{2q-2}^{-1})^{2q}x_{2q-2-i}^{-1}x_0, (x_0x_{2q-2}^{-1})^{2q+1},$$

$$(y_iy_{2q-2}^{-1})^{2q}y_{2q-2-i}^{-1}y_0, \underline{\underline{(y_0y_{2q-2}^{-1})^{2q+1}}}\ (i = \underline{\underline{1}}, \dots, 2q-3),$$

$$x_{2q-2}^{2q} = x_0(y_0y_1\dots y_{2q-2}), y_{2q-2}^{2q} = y_0(x_0x_1\dots x_{2q-2}),$$

$$\underline{\underline{s = y_0y_1\dots y_{2q-2}}}, \underline{\underline{t = x_0x_1\dots x_{2q-2}}}\rangle^{ab}.$$

Substitute $(y_0y_1\dots y_{2q-2})$ with s and $(x_0x_1\dots x_{2q-2})$ with t

$$G_{4q} = \langle x_0, x_1, \dots, x_{2q-2}, y_0, y_1, \dots, y_{2q-2}, s, t \mid (x_ix_{2q-2}^{-1})^{2q}x_{2q-2-i}^{-1}x_0, (x_0x_{2q-2}^{-1})^{2q+1},$$

$$(y_iy_{2q-2}^{-1})^{2q}y_{2q-2-i}^{-1}y_0, (y_0y_{2q-2}^{-1})^{2q+1}\ (i = 1, \dots, 2q-3),$$

$$\underline{\underline{x_{2q-2}^{2q} = x_0s}}, \underline{\underline{y_{2q-2}^{2q} = y_0t}}, s = y_0y_1\dots y_{2q-2}, t = x_0x_1\dots x_{2q-2}\rangle^{ab}.$$

Eliminate $x_0 = x_{2q-2}^{2q}s^{-1}, y_0 = y_{2q-2}^{2q}t^{-1}$

$$G_{4q} = \langle \underline{\underline{x_1, \dots, x_{2q-2}, y_1, \dots, y_{2q-2}, s, t}} \mid \underline{\underline{(x_ix_{2q-2}^{-1})^{2q}x_{2q-2-i}^{-1}(x_{2q-2}^{2q}s^{-1})}}, \underline{\underline{((x_{2q-2}^{2q}s^{-1})x_{2q-2}^{-1})^{2q+1}}},$$

$$\underline{\underline{(y_iy_{2q-2}^{-1})^{2q}y_{2q-2-i}^{-1}(y_{2q-2}^{2q}t^{-1})}}, \underline{\underline{((y_{2q-2}^{2q}t^{-1})y_{2q-2}^{-1})^{2q+1}}}\ (i = 1, \dots, 2q-3),$$

$$\underline{\underline{s = (y_{2q-2}^{2q}t^{-1})y_1\dots y_{2q-2}}}, \underline{\underline{t = (x_{2q-2}^{2q}s^{-1})x_1\dots x_{2q-2}}}\rangle^{ab}$$

$$= \langle x_1, \dots, x_{2q-2}, y_1, \dots, y_{2q-2}, s, t \mid \underline{\underline{x_i^{2q}x_{2q-2-i}^{-1}s^{-1}}}, \underline{\underline{(x_{2q-2}^{2q-1}s^{-1})^{2q+1}}},$$

$$\underline{\underline{y_i^{2q}y_{2q-2-i}^{-1}t^{-1}}}, \underline{\underline{(y_{2q-2}^{2q-1}t^{-1})^{2q+1}}} \ (i = 1, \dots, 2q-3),$$

$$\underline{\underline{st = y_1 \dots y_{2q-3}y_{2q-2}^{2q+1}}}, \underline{\underline{st = x_1 \dots x_{2q-3}x_{2q-2}^{2q+1}}}\rangle^{ab}.$$

Eliminate $x_1 = x_{2q-3}^{2q}s^{-1}, y_1 = y_{2q-3}^{2q}t^{-1}$

$$G_{4q} = \langle \underline{\underline{x_2, \dots, x_{2q-2}, y_2, \dots, y_{2q-2}, s, t}} \mid x_i^{2q}x_{2q-2-i}^{-1}s^{-1}, (x_{2q-2}^{2q-1}s^{-1})^{2q+1}, \underline{\underline{(x_{2q-3}^{2q}s^{-1})^{2q}x_{2q-3}^{-1}s^{-1}}},$$

$$y_i^{2q}y_{2q-2-i}^{-1}t^{-1}, (y_{2q-2}^{2q-1}t^{-1})^{2q+1}, \underline{\underline{(y_{2q-3}^{2q}t^{-1})^{2q}y_{2q-3}^{-1}t^{-1}}} \ (i = 2, \dots, 2q-4),$$

$$\underline{\underline{st^2 = (y_2 \dots y_{2q-4})(y_{2q-3}y_{2q-2})^{2q+1}}}, \underline{\underline{ts^2 = (x_2 \dots x_{2q-4})(x_{2q-3}x_{2q-2})^{2q+1}}}\rangle^{ab}$$

$$= \langle x_2, \dots, x_{2q-2}, y_2, \dots, y_{2q-2}, s, t \mid x_i^{2q}x_{2q-2-i}^{-1}s^{-1}, (x_{2q-2}^{2q-1}s^{-1})^{2q+1}, \underline{\underline{(x_{2q-3}^{2q-1}s^{-1})^{2q+1}}},$$

$$y_i^{2q}y_{2q-2-i}^{-1}t^{-1}, (y_{2q-2}^{2q-1}t^{-1})^{2q+1}, \underline{\underline{(y_{2q-3}^{2q-1}t^{-1})^{2q+1}}} \ (i = 2, \dots, 2q-4),$$

$$st^2 = (y_2 \dots y_{2q-4})(y_{2q-3}y_{2q-2})^{2q+1}, ts^2 = (x_2 \dots x_{2q-4})(x_{2q-3}x_{2q-2})^{2q+1}\rangle^{ab}.$$

Eliminate $x_j = x_{2q-2-j}^{2q}s^{-1}, y_j = y_{2q-2-j}^{2q}t^{-1}$ $(j = 2, \dots, q-2)$

$$G_{4q} = \langle \underline{\underline{x_{q-1}, \dots, x_{2q-2}, y_{q-1}, \dots, y_{2q-2}, s, t}} \mid \underline{\underline{(x_i^{2q-1}s^{-1})^{2q+1}}}, \underline{\underline{(x_{q-1}^{2q-1}s^{-1})}},$$

$$\underline{\underline{(y_i^{2q-1}t^{-1})^{2q+1}}}, \underline{\underline{(y_{q-1}^{2q-1}t^{-1})}} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{st^{q-1} = y_{q-1}(y_q \dots y_{2q-2})^{2q+1}}}, \underline{\underline{ts^{q-1} = x_{q-1}(x_q \dots x_{2q-2})^{2q+1}}}\rangle^{ab}.$$

Eliminate $s = x_{q-1}^{2q-1}, t = y_{q-1}^{2q-1}$

$$G_{4q} = \langle \underline{\underline{x_{q-1}, \dots, x_{2q-2}, y_{q-1}, \dots, y_{2q-2}}} \mid \underline{\underline{(x_i^{2q-1}(x_{q-1}^{2q-1})^{-1})^{2q+1}}}, \underline{\underline{(y_i^{2q-1}(y_{q-1}^{2q-1})^{-1})^{2q+1}}} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{(x_{q-1}^{2q-1})(y_{q-1}^{2q-1})^{q-1} = y_{q-1}(y_q \dots y_{2q-2})^{2q+1}}}, \underline{\underline{(y_{q-1}^{2q-1})(x_{q-1}^{2q-1})^{q-1} = x_{q-1}(x_q \dots x_{2q-2})^{2q+1}}}\rangle^{ab}$$

$$= \langle x_{q-1}, \dots, x_{2q-2}, y_{q-1}, \dots, y_{2q-2} \mid \underline{\underline{(x_i x_{q-1}^{-1})^{(2q-1)(2q+1)}}}, (y_i y_{q-1}^{-1})^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{x_{q-1}^{2q-1}y_{q-1}^{q(2q-3)} = (y_q \dots y_{2q-2})^{2q+1}}}, \underline{\underline{y_{q-1}^{2q-1}x_{q-1}^{q(2q-3)} = (x_q \dots x_{2q-2})^{2q+1}}}\rangle^{ab}.$$

Let $a_i = x_i x_{q-1}, b_i = y_i y_{q-1}$ $(i = q, \dots, 2q-2)$

$$G_{4q} = \langle \underline{\underline{x_{q-1}, \dots, x_{2q-2}, y_{q-1}, \dots, y_{2q-2}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2}}} \mid (x_i x_{q-1}^{-1})^{(2q-1)(2q+1)},$$

$$(y_i y_{q-1}^{-1})^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2), x_{q-1}^{2q-1}y_{q-1}^{q(2q-3)} = (y_q \dots y_{2q-2})^{2q+1},$$

$$y_{q-1}^{2q-1}x_{q-1}^{q(2q-3)} = (x_q \dots x_{2q-2})^{2q+1}, \underline{\underline{a_i = x_i x_{q-1}}}, \underline{\underline{b_i = y_i y_{q-1}}}\rangle^{ab}.$$

Eliminate $x_i = a_i x_{q-1}, y_i = b_i y_{q-1}$ $(i = q, \dots, 2q-2)$

$$G_{4q} = \langle \underline{\underline{x_{q-1}, y_{q-1}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2}}} \mid \underline{\underline{a_i^{(2q-1)(2q+1)}}}, \underline{\underline{b_i^{(2q-1)(2q+1)}}} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{x_{q-1}^{2q-1}y_{q-1}^{q(2q-3)} = ((b_q y_{q-1}) \dots (b_{2q-2}y_{q-1}))^{2q+1}}},$$

$$\underline{\underline{y_{q-1}^{2q-1}x_{q-1}^{q(2q-3)} = ((a_q x_{q-1}) \dots (a_{2q-2}x_{q-1}))^{2q+1}}}\rangle^{ab}$$

$$= \langle x_{q-1}, y_{q-1}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2} \mid a_i^{(2q-1)(2q+1)}, b_i^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{(x_{q-1}y_{q-1}^{-1})^{2q-1} = (b_q \dots b_{2q-2})^{2q+1}}}, \underline{\underline{(x_{q-1}y_{q-1}^{-1})^{-(2q-1)} = (a_q \dots a_{2q-2})^{2q+1}}} \rangle^{ab}.$$

Let $c = x_{q-1}y_{q-1}^{-1}$

$$G_{4q} = \langle \underline{\underline{x_{q-1}, y_{q-1}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2}, c}} \mid a_i^{(2q-1)(2q+1)}, b_i^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2),$$

$$(x_{q-1}y_{q-1}^{-1})^{2q-1} = (b_q \dots b_{2q-2})^{2q+1}, (x_{q-1}y_{q-1}^{-1})^{-(2q-1)} = (a_q \dots a_{2q-2})^{2q+1}, \underline{\underline{c = x_{q-1}y_{q-1}^{-1}}} \rangle^{ab}.$$

Eliminate $x_{q-1} = cy_{q-1}$

$$G_{4q} = \langle \underline{\underline{y_{q-1}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2}, c}} \mid a_i^{(2q-1)(2q+1)}, b_i^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2),$$

$$\underline{\underline{c^{2q-1} = (b_q \dots b_{2q-2})^{2q+1}}}, \underline{\underline{c^{-(2q-1)} = (a_q \dots a_{2q-2})^{2q+1}}} \rangle^{ab}.$$

Let $d = b_q \dots b_{2q-2}, f = a_q \dots a_{2q-2}$

$$G_{4q} = \langle \underline{\underline{y_{q-1}, a_q, \dots, a_{2q-2}, b_q, \dots, b_{2q-2}, c, d, f}} \mid a_i^{(2q-1)(2q+1)}, b_i^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-2),$$

$$c^{2q-1} = (b_q \dots b_{2q-2})^{2q+1}, c^{-(2q-1)} = (a_q \dots a_{2q-2})^{2q+1}, \underline{\underline{d = b_q \dots b_{2q-2}}}, \underline{\underline{f = a_q \dots a_{2q-2}}} \rangle^{ab}.$$

Eliminate $b_{2q-2} = d(b_q \dots b_{2q-3})^{-1}, a_{2q-2} = f(a_q \dots a_{2q-3})^{-1}$

$$G_{4q} = \langle \underline{\underline{y_{q-1}, a_q, \dots, a_{2q-3}, b_q, \dots, b_{2q-3}, c, d, f}} \mid a_i^{(2q-1)(2q+1)}, \underline{\underline{(f(a_q \dots a_{2q-3})^{-1})^{(2q-1)(2q+1)}}},$$

$$b_i^{(2q-1)(2q+1)}, \underline{\underline{(d(b_q \dots b_{2q-3})^{-1})^{(2q-1)(2q+1)}}} \ (i = q, \dots, \underline{\underline{2q-3}}),$$

$$\underline{\underline{c^{2q-1} = d^{2q+1}}}, \underline{\underline{c^{-(2q-1)} = f^{2q+1}}} \rangle^{ab}$$

$$= \langle y_{q-1}, a_q, \dots, a_{2q-3}, b_q, \dots, b_{2q-3}, c, d, f \mid a_i^{(2q-1)(2q+1)}, \underline{\underline{f^{(2q-1)(2q+1)}}}, b_i^{(2q-1)(2q+1)},$$

$$\underline{\underline{d^{(2q-1)(2q+1)}}} \ (i = q, \dots, 2q-3), c^{2q-1} = d^{2q+1}, c^{-(2q-1)} = f^{2q+1} \rangle^{ab}.$$

As $f^{(2q-1)(2q+1)} = (f^{(2q+1)})^{(2q-1)} = (c^{-(2q-1)})^{(2q-1)} = (d^{-(2q+1)})^{(2q-1)} = d^{-(2q-1)(2q+1)} = 1$.

We can now eliminate $f^{(2q-1)(2q+1)}$. Let $t = df$

$$G_{4q} = \langle \underline{\underline{y_{q-1}, a_q, \dots, a_{2q-3}, b_q, \dots, b_{2q-3}, c, d, f, t}} \mid a_i^{(2q-1)(2q+1)},$$

$$b_i^{(2q-1)(2q+1)}, d^{(2q-1)(2q+1)} \ (i = q, \dots, 2q-3), c^{2q-1} = d^{2q+1}, c^{-(2q-1)} = f^{2q+1}, \underline{\underline{t = df}} \rangle^{ab}.$$

Eliminate $d = tf^{-1}$

$$G_{4q} = \langle \underline{\underline{y_{q-1}, a_q, \dots, a_{2q-3}, b_q, \dots, b_{2q-3}, c, f, t}} \mid a_i^{(2q-1)(2q+1)},$$

$$b_i^{(2q-1)(2q+1)}, \underline{\underline{(tf^{-1})^{(2q-1)(2q+1)}}} \ (i = q, \dots, 2q-3), \underline{\underline{c^{2q-1} = (tf^{-1})^{2q+1}}}, c^{-(2q-1)} = f^{2q+1} \rangle^{ab}.$$

Substitute c^{2q-1} with $f^{-(2q+1)}$ in $c^{2q-1} = (tf^{-1})^{2q+1}$

$$G_{4q} = \langle y_{q-1}, a_q, \dots, a_{2q-3}, b_q, \dots, b_{2q-3}, c, f, t \mid a_i^{(2q-1)(2q+1)},$$

$$b_i^{(2q-1)(2q+1)}, \underline{\underline{f^{-(2q-1)(2q+1)}}} \ (i = q, \dots, 2q-3), \underline{\underline{t^{2q+1}}}, c^{-(2q-1)} = f^{2q+1} \rangle^{ab}$$

$$= \langle y_{q-1}, t, a_q, \ldots, a_{2q-3}, b_q, \ldots, b_{2q-3} \mid a_i^{(2q-1)(2q+1)}, b_i^{(2q-1)(2q+1)}, t^{2q+1} \ (i = q, \ldots, 2q-3)\rangle^{ab}$$
$$\oplus \langle c, f \mid c^{2q-1} f^{2q+1}, f^{(2q-1)(2q+1)}\rangle^{ab}.$$

By converting $\langle c, f \mid c^{2q-1} f^{2q+1}, f^{(2q-1)(2q+1)}\rangle^{ab}$ into a matrix we get,

$$M = \begin{pmatrix} 2q-1 & 2q+1 \\ 0 & (2q-1)(2q+1) \end{pmatrix}.$$

As $\mathrm{SNF}(M) = \mathrm{diag}(l, \det(M)/l)$, where $l = \gcd(2q-1, 2q+1, 0, (2q-1)(2q+1))$.

We have, $\mathrm{SNF}(M) = \mathrm{diag}(1, (2q+1)(2q-1)^2)$.

□

Looking at another circulant graph $H = \mathrm{Cay}(\mathbb{Z}_{2q}, \{2, 4, \ldots, 2q-2\} \cup \{q\})$, where q is an odd integer. This graph is isomorphic to the following block construction. Let T be a graph with an adjacency matrix A

$$A = \begin{pmatrix} A(K_q) & I_{anti} \\ I_{anti} & A(K_q) \end{pmatrix},$$

where K_q is the complete graph of order q. We can see this isomorphism using the bijection $f\colon V(T) \to V(H)$.

$$f(u_i) = \begin{cases} v_{2i} & \text{if } i \in \{0, 1, \ldots, q-1\} \\ v_{5q-2(i-1)(\bmod 2q)} & \text{if } i \in \{q, q+1, \ldots, 2q-1\}. \end{cases}$$

The edges are preserved, by checking the vertex u_0, when we apply f to its incident edges, we see that u_0u_i are mapped into v_0v_{2i} for $i \in \{0, 1, \ldots, q-1\}$ and the last edge u_0u_{2q-1} is mapped into v_0v_q. T also has four distinct integral Laplacian eigenvalues as in the following Proposition.

Proposition 3.3.5. *The graph T has the Laplacian eigenvalues $0, 2, q+2, q$ with multiplicities $1, 1, q-1, q-1$ respectively.*

Proof. The graph K_q has a simple Laplacian eigenvalue 0, with the eigenvector $1_q = (1, 1, \ldots, 1)$, and the rest of its eigenvalues are q with multiplicity $q-1$, with eigenvectors $e_j,\ j \in \{1, 2, \ldots, q-1\}$, where e_i have all zeros except at j. The Laplacian of T is as follows

$$L = \begin{pmatrix} L(K_q) + I & -I_{anti} \\ -I_{anti} & L(K_q) + I \end{pmatrix}$$

where $L(K_q)$ is the Laplacian matrix of the complete graph K_q.

0 is an eigenvalue of L as

$$L1_{2q} = 0\ 1_{2q}.$$

2 is an eigenvalue of L as

$$L\begin{pmatrix} 1_q \\ -1_q \end{pmatrix} = 2\begin{pmatrix} 1_q \\ -1_q \end{pmatrix}.$$

$q+2$ is an eigenvalue of L with multiplicity $q-1$ as

$$L\begin{pmatrix} e_j \\ e_{q-j} \end{pmatrix} = (q+2)\begin{pmatrix} e_j \\ e_{q-j} \end{pmatrix} \text{ for } j \in \{1, 2, \ldots, q-1\}.$$

Finally, q is an eigenvalue of L with multiplicity $q-1$ as

$$L\begin{pmatrix} e_i \\ -e_{q-i} \end{pmatrix} = q\begin{pmatrix} e_i \\ -e_{q-i} \end{pmatrix} \text{ for } i \in \{1, 2, \ldots, q-1\}.$$

□

Now we calculate the sandpile group of H.

Theorem 3.3.6. *The sandpile group of H is $\mathbb{Z}_{q+2} \oplus \mathbb{Z}_{q(q+2)}{}^{q-2}$.*

Proof.

Notice that x, y indices below are modulo q.

$$G_{2q} = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid x_i^{q+1}(x_0x_1\ldots x_{q-1})^{-1}y_{q-1-i}^{-1},$$
$$y_i^{q+1}(y_0y_1\ldots y_{q-1})^{-1}x_{q-1-i}^{-1}\ (i = 0, 1, \ldots, q-1)\rangle^{ab}.$$

Let $z = x_0(y_0y_1\ldots y_{q-1}), w = y_0(x_0x_1\ldots x_{q-1})$,

and substitute $(y_0y_1\ldots y_{q-1})$ with zx_0^{-1} and $(x_0x_1\ldots x_{q-1})$ with wy_0^{-1}

$$G_{2q} = \langle \underline{\underline{x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1}, w, z}} \mid \underline{\underline{x_i^{q+1}y_{q-1-i}^{-1}y_0w^{-1}}}, \underline{\underline{y_i^{q+1}x_{q-1-i}^{-1}x_0z^{-1}}}(i = 0, 1, \ldots, q-1),$$
$$\underline{\underline{z = x_0(y_0y_1\ldots y_{q-1})}}, \underline{\underline{w = y_0(x_0x_1\ldots x_{q-1})}}\rangle^{ab}$$
$$= \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1}, w, z \mid x_i^{q+1}y_{q-1-i}^{-1}y_0w^{-1}, \underline{\underline{x_{q-1}^{q+1}w^{-1}}},$$
$$y_i^{q+1}x_{q-1-i}^{-1}x_0z^{-1}, \underline{\underline{y_{q-1}^{q+1}z^{-1}}}\ (i = 0, 1, \ldots, \underline{\underline{q-2}}),$$
$$z = x_0(y_0y_1\ldots y_{q-1}), w = y_0(x_0x_1\ldots x_{q-1})\rangle^{ab}.$$

Eliminate $w = x_{q-1}^{q+1}, z = y_{q-1}^{q+1}$

$$G_{2q} = \langle \underline{\underline{x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1}}} \mid \underline{\underline{x_i^{q+1} y_{q-1-i}^{-1} y_0 x_{q-1}^{-(q+1)}}}, \underline{\underline{y_i^{q+1} x_{q-1-i}^{-1} x_0 y_{q-1}^{-(q+1)}}} \ (i = 0, 1, \ldots, q-2),$$

$$\underline{\underline{y_{q-1}^{q+1} = x_0(y_0 y_1 \ldots y_{q-1})}}, \underline{\underline{x_{q-1}^{q+1} = y_0(x_0 x_1 \ldots x_{q-1})}} \rangle^{ab}.$$

Let $u_j = x_j x_{q-1}^{-1}, v_j = y_j y_{q-1}^{-1}$ $(j = 0, 1, \ldots, q-2)$

$$G_{2q} = \langle \underline{\underline{x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_0, v_1, \ldots, v_{q-2}}} \mid x_i^{q+1} y_{q-1-i}^{-1} y_0 x_{q-1}^{-(q+1)},$$

$$y_i^{q+1} x_{q-1-i}^{-1} x_0 y_{q-1}^{-(q+1)} \ (i = 0, 1, \ldots, q-2), y_{q-1}^{q+1} = x_0(y_0 y_1 \ldots y_{q-1}), x_{q-1}^{q+1} = y_0(x_0 x_1 \ldots x_{q-1}),$$

$$\underline{\underline{u_j = x_j x_{q-1}^{-1}}}, \underline{\underline{v_j = y_j y_{q-1}^{-1}}} \rangle^{ab}.$$

Eliminate $x_i = u_i x_{q-1}, y_i = v_i y_{q-1}$ $(i = 0, 1, \ldots, q-2)$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, y_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_0, v_1, \ldots, v_{q-2}}} \mid \underline{\underline{(u_i x_{q-1})^{q+1} (v_{q-1-i} y_{q-1})^{-1} (v_0 y_{q-1}) x_{q-1}^{-(q+1)}}},$$

$$\underline{\underline{(u_0 x_{q-1})^{q+1} y_{q-1}^{-1} (v_0 y_{q-1}) x_{q-1}^{-(q+1)}}}, \underline{\underline{(v_i y_{q-1})^{q+1} (u_{q-1-i} x_{q-1})^{-1} (u_0 x_{q-1}) y_{q-1}^{-(q+1)}}},$$

$$\underline{\underline{(v_0 y_{q-1})^{q+1} x_{q-1}^{-1} (u_0 x_{q-1}) y_{q-1}^{-(q+1)}}} \ (i = 1, \ldots, q-2),$$

$$\underline{\underline{y_{q-1}^{q+1} = (u_0 x_{q-1})((v_0 y_{q-1})(v_1 y_{q-1}) \ldots (v_{q-2} y_{q-1}) y_{q-1})}},$$

$$\underline{\underline{x_{q-1}^{q+1} = (v_0 y_{q-1})((u_i x_{q-1})(u_1 x_{q-1}) \ldots (u_{q-2} x_{q-1}) x_{q-1})}} \rangle^{ab}$$

$$= \langle x_{q-1}, y_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_0, v_1, \ldots, v_{q-2} \mid \underline{\underline{u_i^{q+1} v_{q-1-i}^{-1} v_0}}, \underline{\underline{u_0^{q+1} v_0}}, \underline{\underline{v_i^{q+1} u_{q-1-i}^{-1} u_0}},$$

$$\underline{\underline{v_0^{q+1} u_0}} \ (i = 1, \ldots, q-2), \underline{\underline{y_{q-1} = u_0 x_{q-1}(v_0 v_1 \ldots v_{q-2})}}, \underline{\underline{x_{q-1} = v_0 y_{q-1}(u_0 u_1 \ldots u_{q-2})}} \rangle^{ab}.$$

Eliminate $v_0 = u_0^{-(q+1)}$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, y_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_1, \ldots, v_{q-2}}} \mid \underline{\underline{u_i^{q+1} v_{q-1-i}^{-1} (u_0^{-(q+1)})}},$$

$$\underline{\underline{v_i^{q+1} u_{q-1-i}^{-1} u_0}}, \underline{\underline{(u_0^{-(q+1)})^{q+1} u_0}} (i = 1, \ldots, q-2),$$

$$\underline{\underline{y_{q-1} = u_0 x_{q-1}((u_0^{-(q+1)}) v_1 \ldots v_{q-2})}}, \underline{\underline{x_{q-1} = (u_0^{-(q+1)}) y_{q-1}(u_0 u_1 \ldots u_{q-2})}} \rangle^{ab}$$

$$= \langle x_{q-1}, y_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_1, \ldots, v_{q-2} \mid \underline{\underline{(u_i u_0^{-1})^{q+1} v_{q-1-i}^{-1}}}, \underline{\underline{v_i^{q+1} u_{q-1-i}^{-1} u_0}},$$

$$\underline{\underline{u_0^{-q(q+2)}}} \ (i = 1, \ldots, q-2),$$

$$\underline{\underline{y_{q-1} = u_0^{-q} x_{q-1}(v_1 \ldots v_{q-2})}}, \underline{\underline{x_{q-1} = u_0^{-q} y_{q-1}(u_1 \ldots u_{q-2})}} \rangle^{ab}.$$

Eliminate $y_{q-1} = u_0^{-q} x_{q-1}(v_1 \ldots v_{q-2})$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_1, \ldots, v_{q-2}}} \mid (u_i u_0^{-1})^{q+1} v_{q-1-i}^{-1}, v_i^{q+1} u_{q-1-i}^{-1} u_0, u_0^{-q(q+2)} \ (i = 1, \ldots, q-2),$$

$$\underline{\underline{x_{q-1} = u_0^{-q}(u_0^{-q} x_{q-1}(v_1 \ldots v_{q-2}))(u_1 \ldots u_{q-2})}} \rangle^{ab}$$

$$= \langle x_{q-1}, u_0, u_1, \ldots, u_{q-2}, v_1, \ldots, v_{q-2} \mid (u_i u_0^{-1})^{q+1} v_{q-1-i}^{-1}, v_i^{q+1} u_{q-1-i}^{-1} u_0, u_0^{-q(q+2)} \ (i = 1, \ldots, q-2),$$

$$\underline{\underline{u_0^{2q} = (v_1 \ldots v_{q-2})(u_1 \ldots u_{q-2})}} \rangle^{ab}.$$

Let $a_i = u_i u_0^{-1}$ $(i = 1, \dots, q-2)$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, u_0, u_1, \dots, u_{q-2}, v_1, \dots, v_{q-2}, a_1, \dots, a_{q-2}}} \mid (u_i u_0^{-1})^{q+1} v_{q-1-i}^{-1},$$

$$v_i^{q+1} u_{q-1-i}^{-1} u_0, u_0^{-q(q+2)} \ (i = 1, \dots, q-2), u_0^{2q} = (v_1 \dots v_{q-2})(u_1 \dots u_{q-2}), \underline{\underline{a_i = u_i u_0^{-1}}} \rangle^{ab}.$$

Eliminate $u_i = a_i u_0$ $(i = 1, \dots, q-2)$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, u_0, v_1, \dots, v_{q-2}, a_1, \dots, a_{q-2}}} \mid a_i^{q+1} v_{q-1-i}^{-1}, \underline{\underline{v_i^{q+1} (a_{q-1-i} u_0)^{-1} u_0}}, u_0^{-q(q+2)} \ (i = 1, \dots, q-2),$$

$$\underline{\underline{u_0^{2q} = (v_1 \dots v_{q-2})((a_1 u_0) \dots (a_{q-2} u_0))}} \rangle^{ab}$$

$$= \langle x_{q-1}, u_0, v_1, \dots, v_{q-2}, a_1, \dots, a_{q-2} \mid a_i^{q+1} v_{q-1-i}^{-1}, \underline{\underline{v_i^{q+1} a_{q-1-i}^{-1}}}, u_0^{-q(q+2)} \ (i = 1, \dots, q-2),$$

$$\underline{\underline{u_0^{q+2} = (v_1 \dots v_{q-2})(a_1 \dots a_{q-2})}} \rangle^{ab}.$$

Eliminate $v_{q-1-i} = a_i^{q+1}$ $(i = 1, \dots, q-2)$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, u_0, a_1, \dots, a_{q-2}}} \mid \underline{\underline{(a_{q-1-i}^{q+1})^{q+1} a_{q-1-i}^{-1}}}, u_0^{-q(q+2)} \ (i = 1, \dots, q-2),$$

$$\underline{\underline{u_0^{q+2} = (a_{q-2}^{q+1} \dots a_1^{q+1})(a_1 \dots a_{q-2})}} \rangle^{ab}$$

$$= \langle x_{q-1}, u_0, a_1, \dots, a_{q-2} \mid a_i^{q(q+2)}, u_0^{-q(q+2)} \ (i = 1, \dots, q-2), \underline{\underline{u_0^{q+2} = (a_1 \dots a_{q-2})^{q+2}}} \rangle^{ab}.$$

Let $b = (a_1 \dots a_{q-2}) u_0^{-1}$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, u_0, a_1, \dots, a_{q-2}, b}} \mid a_i^{q(q+2)}, u_0^{-q(q+2)} \ (i = 1, \dots, q-2), u_0^{q+2} = (a_1 \dots a_{q-2})^{q+2},$$

$$\underline{\underline{b = (a_1 \dots a_{q-2}) u_0^{-1}}} \rangle^{ab}.$$

Eliminate $u_0 = (a_1 \dots a_{q-2}) b^{-1}$

$$G_{2q} = \langle \underline{\underline{x_{q-1}, a_1, \dots, a_{q-2}, b}} \mid b^{q+2}, ((a_1 \dots a_{q-2}) b^{-1})^{-q(q+2)}, a_i^{q(q+2)} \ (i = 1, \dots, q-2) \rangle^{ab}.$$

□

3.4 Future work

The proof in Theorem F can be extended with some complexity to cover $\Gamma = \overline{\overline{C_r} \otimes \mathcal{J}_t}$, $L(\Gamma) = (3t-1)I_{rt} - ((A(C_r) + I_r) \otimes J_t - I_{rt})$ in general. It is expected to require an induction argument to resolve the relators introduced by C_r being generic. It is not very clear if the same would apply to Theorem G. The Smith group of the graphs presented in this chapter are worth investigating further as they are integral circulant graphs.

Chapter 4

The Sandpile Group of Graph Constructions

In this chapter, we study the sandpile group of different graph constructions. We present new results on the sandpile groups of some of these graphs (see Theorems J, K, L, 4.4.1 and Conjectures 4.2.3, 4.4.2, 4.4.3, and 4.5.1).

4.1 Introduction

The sandpile group of graph product has been studied for some graph classes including, the graphs $K_m \bigtriangledown P_n$, $K_m \otimes P_n$ (see [42]), $P_m \bigtriangledown P_n$ (see [60]), and $C_4 \otimes C_n$ (see [64]), to name a few. Related to graphs constructed by graph product is the threshold graph (Definition 1.2.8) which we expand on next.

The threshold graph

The sandpile group of the threshold graph has been studied by Christianson & Reiner (see [15]), where they provide Conjecture 4.1.2, and Theorem 4.1.3.

The *Ferrers diagram* is an array of points (or squares) of integer length in the plane that are left justified. They represent a weakly decreasing sequence $(a_0 \geqslant a_1 \geqslant \cdots \geqslant a_{n-1})$ such that each row i in the array has a number of points to represents the a_i in the sequence.

The threshold graph is integral and has an easy way to calculate its eigenvalues. Let Γ be a threshold graph of order n, with degree sequence $d_0 \geqslant d_1 \geqslant \cdots \geqslant d_{n-1}$ and Laplacian eigenvalues $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} = 0$. And suppose that $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_{n-1}$ are the column lengths of the Ferrers diagram of the degree sequence of Γ, then we have the following theorem,

Theorem 4.1.1. [48, Theorem 3] *For the threshold graph* Γ*, the following condition holds,*

$$\lambda_i = a_i,\ 0 \leqslant i \leqslant n-1.$$

Let s be the index of the smallest a which is are greater or equal to 2. Define the sequence b to be a reordering of $a_1, a_2, \ldots, a_s$, such that all the occurrences of a_1 are assigned to $b_1, b_2, \ldots, b_{l_{a_1}}$, where l_{a_1} is the number of occurrences of a_1. Then all the occurrences of a_s are assigned to $b_{1+l_{a_1}}, b_{2+l_{a_1}}, \ldots, b_{l_{a_1}+l_{a_s}}$, and do the same for the second largest, second smallest and so on. This way we ensure that we are assigning to the elements of b all occurrences of the largest a_i first, then all occurrences of the smallest, then the second largest, then the second smallest, etc. Now we construct H to be a graph with

$$V(H) = \{1, 2, \ldots, s\},$$

$$E(H) = \{(i, i+1) : b_i \neq b_{i+1} \text{ and } \gcd(b_i, b_{i+1}) \neq 1\}.$$

Let $(B_0, B_1, \ldots, B_{r-1})$ be a partition of $V(H)$ that represents the connected components in H, then we have the following,

Conjecture 4.1.2. [15, Conjecture 7] *For the threshold graph Γ, the sandpile group is isomorphic to*

$$\bigoplus_{i=0}^{r-1} \mathbb{Z}_{n_r}, \text{ where } n_r = \prod_{j \in B_r} b_j.$$

Theorem 4.1.3. [15, Theorem 3] *For the threshold graph Γ, and suppose that for each $i \in \{2, 3, \ldots, s-1\}$ there are two occurrences of b_i in the sequence b, then sandpile group is isomorphic to*

$$\bigoplus_{i=0}^{r-1} \mathbb{Z}_{n_r}, \text{ where } n_r = \prod_{j \in B_r} b_j.$$

4.2 The sandpile group of a threshold graph

$\overline{K_{n-q}} \bigtriangledown K_q$ is a graph product and also a threshold graph. We can see this is true as we can construct $\overline{K_{n-q}} \bigtriangledown K_q$ by starting with an empty vertex set, then performing $n-q$ union operations to include a new vertex one at a time, which gives us $\overline{K_{n-q}}$. Now we can join q vertices one at a time, which will eventually result in $\overline{K_{n-q}} \bigtriangledown K_q$.

We calculate the sandpile group of $\overline{K_{n-q}} \bigtriangledown K_q$ next using Tietze transformations as it gives us a more direct proof than Theorem 4.1.3. For a reminder of Tietze transformation rules and convention used below, see Section 1.7.

Proposition 4.2.1. *Let $n \geqslant q+2 \geqslant 4$, the sandpile group of $\overline{K_{n-q}} \bigtriangledown K_q$ is isomorphic to*

$$\begin{cases} \left(\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)}\right)^{q-2} \oplus {\mathbb{Z}_q}^{n-2q} \oplus \mathbb{Z}_{qn} & \text{if } n \geqslant 2q, \\ \left(\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)}\right)^{n-q-2} \oplus {\mathbb{Z}_n}^{2q-n} \oplus \mathbb{Z}_{qn} & \text{if } n < 2q. \end{cases}$$

Proof. The group presentation with relation matrix $L(\overline{K_{n-q}} \bigtriangledown K_q)$ is

$$G_n = \langle x_0, x_1, \ldots, x_{n-1} \mid (x_i^{n-1} x_{i+1}^{-1} x_{i+2}^{-1} \ldots x_{i+q-1}^{-1})(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1})$$

$$(i \in \{0, 1, \ldots q-1\}) \ (i \mod q),$$

$$x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} x_j^q \ (j \in \{q, q+1, \ldots, n-1\})\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_{n-1} \mid \underline{\underline{x_i^n (x_i^{-1} x_{i+1}^{-1} x_{i+2}^{-1} \ldots x_{i+q-1}^{-1})}}(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1})$$

$$(i \in \{0, 1, \ldots q-1\}) \ (i \mod q),$$

$$x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} x_j^q \ (j \in \{q, q+1, \ldots, n-1\})\rangle^{ab}.$$

Given that $\{x_i, x_{i+1}, \ldots x_{i+q-1}\} = \{x_0, x_1, \ldots x_{q-1}\}$ $(i \in \{0, 1, \ldots n-1\})$

$$G_n = \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n \underline{\underline{(x_0^{-1} x_1^{-1} x_2^{-1} \ldots x_{q-1}^{-1})}}(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} x_j^q \ (j \in \{q, q+1, \ldots, n-1\})\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n (x_0^{-1} x_1^{-1} x_2^{-1} \ldots x_{q-1}^{-1})(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} x_j^q \ (j \in \{q, q+1, \ldots, \underline{\underline{n-2}}\}), \underline{\underline{x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} x_{n-1}^q}}\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n (x_0^{-1} x_1^{-1} x_2^{-1} \ldots x_{q-1}^{-1})(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$(x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1}) x_j^q \ (j \in \{q, q+1, \ldots, n-2\}),$$

$$\underline{\underline{x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} = x_{n-1}^{-q}}}\rangle^{ab}.$$

Replace $(x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1})$ with x_{n-1}^{-q} in all relations

$$G_n = \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n \underline{\underline{x_{n-1}^{-q}}}(x_q^{-1} x_{q+1}^{-1} \ldots x_{n-1}^{-1}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$\underline{\underline{x_{n-1}^{-q}}} x_j^q \ (j \in \{q, q+1, \ldots, n-2\}), x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} = x_{n-1}^{-q}\rangle^{ab}$$

$$= \langle x_0, x_1, \ldots, x_{n-1} \mid x_i^n (x_q^{-1} x_{q+1}^{-1} \ldots x_{n-2}^{-1} \underline{\underline{x_{n-1}^{-(q+1)}}}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$x_{n-1}^{-q} x_j^q \ (j \in \{q, q+1, \ldots, n-2\}), x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} = x_{n-1}^{-q}\rangle^{ab}.$$

Let $y_k = x_{n-1}^{-1} y_k$ $(k \in \{0, 1, \ldots n-2\})$

$$G_n = \langle \underline{\underline{x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-2}}} \mid x_i^n (x_q^{-1} x_{q+1}^{-1} \ldots x_{n-2}^{-1} x_{n-1}^{-(q+1)}) \ (i \in \{0, 1, \ldots q-1\}),$$

$$x_{n-1}^{-q} x_j^q \ (j \in \{q, q+1, \ldots, n-2\}), x_0^{-1} x_1^{-1} \ldots x_{q-1}^{-1} = x_{n-1}^{-q},$$

$$\underline{\underline{y_k = x_k x_{n-1}^{-1} (k \in \{0, 1, \ldots n-2\})}}\rangle^{ab}.$$

Eliminate $x_k = x_{n-1} y_k$ $(k \in \{0, 1, \ldots n-2\})$

$$G_n = \langle \underline{\underline{x_{n-1}, y_0, y_1, \ldots, y_{n-2}}} \mid \underline{\underline{(y_i^n x_{n-1}^n)((y_q^{-1} x_{n-1}^{-1})(y_{q+1}^{-1} x_{n-1}^{-1}) \ldots (y_{n-2}^{-1} x_{n-1}^{-1}) x_{n-1}^{-(q+1)})}},$$

$$(i \in \{0, 1, \ldots q-1\}), \underline{\underline{y_j^q \ (j \in \{q, q+1, \ldots, n-2\})}},$$

$$\underline{\underline{(y_0^{-1}x_{n-1}^{-1})(y_1^{-1}x_{n-1}^{-1})\dots(y_{q-1}^{-1}x_{n-1}^{-1}) = x_{n-1}^{-q}}}\rangle^{ab}$$

$$= \langle x_{n-1}, y_0, y_1, \dots, y_{n-2} \mid \underline{\underline{y_i^n(y_q^{-1}y_{q+1}^{-1}\dots y_{n-2}^{-1})\ (i \in \{0,1,\dots q-1\})}},$$

$$\underline{\underline{y_j^q\ (j \in \{q, q+1, \dots, n-2\})}},$$

$$\underline{\underline{y_0^{-1}y_1^{-1}\dots y_{q-1}^{-1} = 1}}\rangle^{ab}$$

$$= \langle x_{n-1}, y_0, y_1, \dots, y_{n-2} \mid y_i^n(y_q^{-1}y_{q+1}^{-1}\dots y_{n-2}^{-1})\ (i \in \{\underline{1}, 2, \dots q-1\}),$$

$$\underline{\underline{y_0^n y_q^{-1}y_{q+1}^{-1}\dots y_{n-3}^{-1} = y_{n-2}}}, y_j^q\ (j \in \{q, q+1, \dots, n-2\}), y_0^{-1}y_1^{-1}\dots y_{q-1}^{-1} = 1\rangle^{ab}.$$

Eliminate $y_{n-2} = y_0^n y_q^{-1}y_{q+1}^{-1}\dots y_{n-3}^{-1}$

$$G_n = \langle \underline{\underline{x_{n-1}, y_0, y_1, \dots, y_{n-3}}} \mid \underline{\underline{y_i^n(y_q^{-1}y_{q+1}^{-1}\dots y_{n-3}^{-1}(y_0^n y_q^{-1}y_{q+1}^{-1}\dots y_{n-3}^{-1})^{-1})\ (i \in \{1,2,\dots q-1\})}},$$

$$y_j^q\ (j \in \{q, q+1, \dots, \underline{\underline{n-3}}\}), \underline{\underline{(y_0^n y_q^{-1}y_{q+1}^{-1}\dots y_{n-3}^{-1})^q}}, y_0^{-1}y_1^{-1}\dots y_{q-1}^{-1} = 1\rangle^{ab}$$

$$= \langle x_{n-1}, y_0, y_1, \dots, y_{n-3} \mid \underline{\underline{y_i^n y_0^{-n}\ (i \in \{1,2,\dots q-1\})}},$$

$$y_j^q\ (j \in \{q, q+1, \dots, n-3\}), \underline{\underline{y_0^{qn}}}, y_0^{-1}y_1^{-1}\dots y_{q-1}^{-1} = 1\rangle^{ab}.$$

Let $z_i = y_i y_0^{-1}\ (i \in \{1, 2, \dots q-1\})$

$$G_n = \langle \underline{\underline{x_{n-1}, y_0, y_1, \dots, y_{n-3}, z_1, z_2, \dots, z_{q-1}}} \mid y_i^n y_0^{-n}\ (i \in \{1,2,\dots q-1\}),$$

$$y_j^q\ (j \in \{q, q+1, \dots, n-3\}), y_0^{qn}, y_0^{-1}y_1^{-1}\dots y_{q-1}^{-1} = 1, \underline{\underline{z_i = y_i y_0^{-1}}}\rangle^{ab}.$$

Eliminate $y_i = z_i y_0\ (i \in \{1, 2, \dots q-1\})$

$$G_n = \langle \underline{\underline{x_{n-1}, y_0, y_q, y_{q+1}, \dots, y_{n-3}, z_1, z_2, \dots, z_{q-1}}} \mid \underline{\underline{z_i^n\ (i \in \{1,2,\dots q-1\})}},$$

$$y_j^q\ (j \in \{q, q+1, \dots, n-3\}), y_0^{qn},$$

$$\underline{\underline{y_0^{-1}(z_1y_0)^{-1}(z_2y_0)^{-1}\dots(z_{q-1}y_0)^{-1} = 1}}\rangle^{ab}$$

$$= \langle x_{n-1}, y_0, y_q, y_{q+1}, \dots, y_{n-3}, z_1, z_2, \dots, z_{q-1} \mid z_i^n\ (i \in \{1,2,\dots q-1\}),$$

$$y_j^q\ (j \in \{q, q+1, \dots, n-3\}), y_0^{qn}, \underline{\underline{y_0^{-q}z_1^{-1}z_2^{-1}\dots z_{q-1}^{-1} = 1}}\rangle^{ab}.$$

Eliminate $z_{q-1} = y_0^{-q}z_1^{-1}z_2^{-1}\dots z_{q-2}^{-1}$

$$G_n = \langle \underline{\underline{x_{n-1}, y_0, y_q, y_{q+1}, \dots, y_{n-3}, z_1, z_2, \dots, z_{q-2}}} \mid z_i^n\ (i \in \{1,2,\dots \underline{\underline{q-2}}\}),$$

$$\underline{\underline{(y_0^{-q}z_1^{-1}z_2^{-1}\dots z_{q-2}^{-1})^n}}, y_j^q\ (j \in \{q, q+1, \dots, n-3\}), y_0^{qn}\rangle^{ab}$$

$$= \langle x_{n-1}, y_0, y_q, y_{q+1}, \dots, y_{n-3}, z_1, z_2, \dots, z_{q-2} \mid z_i^n\ (i \in \{1,2,\dots q-2\}),$$

$$y_j^q\ (j \in \{q, q+1, \dots, n-3\}), y_0^{qn}\rangle^{ab}$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_{qn} \oplus \underbrace{\mathbb{Z}_q \oplus \mathbb{Z}_q \oplus \dots \oplus \mathbb{Z}_q}_{n-q-2} \oplus \underbrace{\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \dots \oplus \mathbb{Z}_n}_{q-2}.$$

We can now use Lemma 1.8.1 to show that the sandpile group $\mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-q-2} \oplus {\mathbb{Z}_n}^{q-2}$ is isomorphic to

$$\begin{cases} \left(\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)}\right)^{q-2} \oplus {\mathbb{Z}_q}^{n-2q} \oplus \mathbb{Z}_{qn} & \text{if } n \geqslant 2q, \\ \left(\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)}\right)^{n-q-2} \oplus {\mathbb{Z}_n}^{2q-n} \oplus \mathbb{Z}_{qn} & \text{if } n < 2q. \end{cases}$$

If $n \geqslant 2q$,

$$\begin{aligned} &\mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-q-2} \oplus {\mathbb{Z}_n}^{q-2} \\ &= \mathbb{Z}_{qn} \oplus ({\mathbb{Z}_q}^{n-2q} \oplus {\mathbb{Z}_q}^{q-2}) \oplus {\mathbb{Z}_n}^{q-2} \\ &= \mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-2q} \oplus ({\mathbb{Z}_q}^{q-2} \oplus {\mathbb{Z}_n}^{q-2}) \\ &= \mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-2q} \oplus (\mathbb{Z}_q \oplus \mathbb{Z}_n)^{q-2} \\ &\cong \mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-2q} \oplus (\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)})^{q-2} \end{aligned}$$

if $n < 2q$,

$$\begin{aligned} &\mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-q-2} \oplus {\mathbb{Z}_n}^{q-2} \\ &= \mathbb{Z}_{qn} \oplus {\mathbb{Z}_q}^{n-q-2} \oplus ({\mathbb{Z}_n}^{n-q-2} \oplus {\mathbb{Z}_n}^{2q-n}) \\ &= \mathbb{Z}_{qn} \oplus ({\mathbb{Z}_q}^{n-q-2} \oplus ({\mathbb{Z}_n}^{n-q-2}) \oplus {\mathbb{Z}_n}^{2q-n} \\ &= \mathbb{Z}_{qn} \oplus (\mathbb{Z}_q \oplus \mathbb{Z}_n)^{n-q-2} \oplus {\mathbb{Z}_n}^{2q-n} \\ &\cong \mathbb{Z}_{qn} \oplus (\mathbb{Z}_{\gcd(q,n)} \oplus \mathbb{Z}_{\mathrm{lcm}(q,n)})^{n-q-2} \oplus {\mathbb{Z}_n}^{2q-n} \end{aligned}$$

as required.

□

The sandpile group of $\overline{K_{n-q}} \bigtriangledown K_q$ is covered in Theorem 4.1.3. However, a similar construction to $\overline{K_{n-q}} \bigtriangledown K_q$ is presented in Theorem J which is not covered by Theorem 4.1.3. This comes down to the strict condition on the eigenvalues of the threshold graphs Theorem 4.1.3 covers.

The degree sequence of $\overline{K_{n-q}} \bigtriangledown K_q$ is $\underbrace{n-1, n-1, \ldots, n-1}_{q}, \underbrace{q, q, \ldots, q}_{n-q}$, which makes the column lengths of the Ferrers diagram as follows $\underbrace{n, n, \ldots, n}_{q}, \underbrace{q, q, \ldots, q}_{n-q-1}, 0$. Reordering a subset of these column lengths to create the sequence b following the rules above, we get $\underbrace{n, n, \ldots, n}_{q-1}, \underbrace{q, q, \ldots, q}_{n-q-1}$. We compare the sandpile group we get using Theorem J, and Proposition 4.2.1 in the next example and show how Theorem 1.11.1 does not work for this graph.

Example 4.2.2. *The sandpile group of* $\overline{K_6} \bigtriangledown K_4$

The degree sequence for $\overline{K_6} \bigtriangledown K_4$ is $9, 9, 9, 9, 4, 4, 4, 4, 4, 4$. Based on this we get the Ferrers diagram in Figure

4.1.

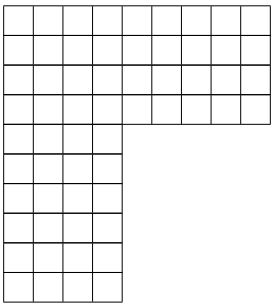

Figure 4.1: Ferrers diagram of degree sequence of $\overline{K_6} \bigtriangledown K_4$

Using Theorem 4.1.1, we can see that the eigenvalues are

$$a_0, a_1, \ldots, a_9 = 10, 10, 10, 10, 4, 4, 4, 4, 4, 0.$$

Now, by taking the columns $a_1, a_2, \ldots, a_{s=8}$ and reordering with a new label we get

$$b_1, b_2, \ldots, b_8 = 10, 10, 10, 4, 4, 4, 4, 4.$$

We use b to construct the graph H,

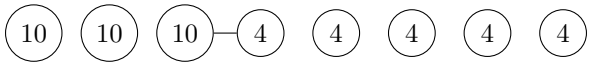

Figure 4.2: The graph H for $\overline{K_6} \bigtriangledown K_4$

Using Theorem 4.1.3, we get

$$\mathbb{Z}_{40} \oplus {\mathbb{Z}_{10}}^2 \oplus {\mathbb{Z}_4}^4.$$

Which is compatible with Proposition 4.2.1. We notice that $\overline{K_6} \bigtriangledown K_4$ can be viewed as the complete multipartite graph $K_{6,1,1,1,1}$. However when we try to apply Theorem 1.11.1, it does not seem to apply well as

$$n_1 = 6, n_2 = n_3 = n_4 = n_5 = 1, n = 10,$$

$$N_1 = 4, N_2 = N_3 = N_4 = N_5 = 9,$$

We create a diagonal matrix with N_i's and calculate its Smith normal form. So $\mathrm{diag}(4, 9, 9, 9, 9)$ has the Smith normal form $\mathrm{diag}(1, 9, 9, 9, 36)$. Hence,

$$g = \gcd(4, 4, 9, 9, 9, 9) = 1,$$

$$h = (1 \times 9)/1 = 9.$$

Which should give us

$$\mathbb{Z}_4^4 \oplus \mathbb{Z}_9^{-4} \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{10\times 9}^2 \oplus \mathbb{Z}_{10\times 36}$$

Which does not match the expected result. We conclude that $n_i \geqslant 2$ must be an implicit assumption in Theorem 1.11.1.

One way to go a step further to generalise $\overline{K_{n-q}} \bigtriangledown K_q$ is to consider what sandpile groups we get when we perform a join operation between a multipartite graph and another multipartite graph complement as follows $K\underbrace{p, p, \ldots, p}_{t} \bigtriangledown \overline{K\underbrace{q, q, \ldots, q}_{t}}$. This generalisation is not a threshold graph for $t > 1$, $p > 1$, $q > 1$. This is true because when we select two vertices from a partition, and another two vertices from another partition in $V(K\underbrace{p, p, \ldots, p}_{t})$ we find that this subgraph is a cycle of order 4, which we means $K\underbrace{p, p, \ldots, p}_{t} \bigtriangledown \overline{K\underbrace{q, q, \ldots, q}_{t}}$ is not a threshold graph by Theorem 1.2.9.

We present a conjecture for the sandpile group of $K_{p,p} \bigtriangledown \overline{K_{q,q}}$ that has verified for $p, q \leqslant 30$.

Conjecture 4.2.3. *Let $p \geqslant q+1 \geqslant 3$. The sandpile group of $K_{p,p} \bigtriangledown \overline{K_{q,q}}$ is equal to $\mathbb{Z} \oplus \mathbb{Z}_{p+2q}^{2(p-q)} \oplus \mathbb{Z}_{(p+2q)(2p+q)}^{2(q-1)} \oplus \mathbb{Z}_{p(p+q)} \oplus \mathbb{Z}_4$.*

A small change to $\overline{K_{n-q}} \bigtriangledown K_q$ can result in a graph that falls under Conjecture 4.1.2, but not Theorem 4.1.3. The graph $((\overline{K_q} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_p$ has the degree sequence $\underbrace{p+q+1, p+q+1, \ldots, p+q+1}_{p}, p+q, p, \underbrace{p+1, p+1, \ldots, p+1}_{q}$. The column lengths of the Ferrers diagram is as follows

$\underbrace{p+q+2, p+q+2, \ldots, p+q+2}_{p}, p+q+1, p, \underbrace{p+1, p+1, \ldots, p+1}_{q-1}, 0$. By reordering them to create the sequence b, we get $\underbrace{p+q+2, p+q+2, \ldots, p+q+2}_{p}, p, p+q+1, \underbrace{p+1, p+1, \ldots, p+1}_{q-1}, 0$. Which makes it a graph that does not fit Theorem 4.1.3.

Theorem J. *Let $p, q \geqslant 2$ be integers such that $(p+1) \mid (q+1)$. Then the sandpile group of $((\overline{K_q} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_p$ is isomorphic to*

$$\mathbb{Z}_{p+q+2}{}^{p-2} \oplus \mathbb{Z}_{p+1}{}^{q-1} \oplus \mathbb{Z}_{p(p+q+1)(p+q+2)}.$$

Proof. The group presentation with relation matrix $L(((\overline{K_q} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_p)$ is

$$\begin{aligned} P = \langle x_0, x_1, \ldots, x_{p+q+1} \mid\ & x_i^{p+q+2}(x_0x_1\ldots x_{p+q+1})^{-1}\ (i \in \{0, 1, \ldots p-1\}), \\ & x_{p+j}^{p+1}(x_0x_1\ldots x_{p-1}x_{p+q})^{-1}\ (j \in \{0, 1, \ldots q-1\}), \\ & x_{p+q}^{p+q}(x_0x_1\ldots x_{p+q-1})^{-1}, \\ & x_{p+q+1}^{p}(x_0x_1\ldots x_{p-1})^{-1}\rangle^{ab}. \end{aligned}$$

Let $y_0 = x_0x_1\ldots x_{p-1}, y_1 = x_px_{p+1}\ldots x_{p+q-1}$,

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{p+q+1}, y_0, y_1}} \mid x_i^{p+q+2}(x_0x_1\ldots x_{p+q+1})^{-1}\ (i \in \{0, 1, \ldots p-1\}),$$

$$x_{p+j}^{p+1}(x_0x_1\dots x_{p-1}x_{p+q})^{-1}\ (j\in\{0,1,\dots q-1\}),$$

$$x_{p+q}^{p+q}(x_0x_1\dots x_{p+q-1})^{-1},$$

$$x_{p+q+1}^{p}(x_0x_1\dots x_{p-1})^{-1}, \underline{\underline{y_0=x_0x_1\dots x_{p-1}}}, \underline{\underline{y_1=x_px_{p+1}\dots x_{p+q-1}}}\rangle^{ab}$$

$$=\langle x_0,x_1,\dots,x_{p+q+1},y_0,y_1 \mid \underline{\underline{x_i^{p+q+2}(y_0y_1x_{p+q}x_{p+q+1})^{-1}}}\ (i\in\{0,1,\dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}(y_0x_{p+q})^{-1}}}\ (j\in\{0,1,\dots q-1\}), \underline{\underline{x_{p+q}^{p+q}(y_0y_1)^{-1}}},$$

$$\underline{\underline{x_{p+q+1}^{p}(y_0)^{-1}}}, y_0=x_0x_1\dots x_{p-1}, y_1=x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}.$$

Eliminate $y_0=x_{p+q+1}^{p}$

$$P=\langle \underline{\underline{x_0,x_1,\dots,x_{p+q+1},y_1}} \mid \underline{\underline{x_i^{p+q+2}((x_{p+q+1}^{p})y_1x_{p+q}x_{p+q+1})^{-1}}}\ (i\in\{0,1,\dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}((x_{p+q+1}^{p})x_{p+q})^{-1}}}\ (j\in\{0,1,\dots q-1\}), \underline{\underline{x_{p+q}^{p+q}((x_{p+q+1}^{p})y_1)^{-1}}},$$

$$\underline{\underline{(x_{p+q+1}^{p})=x_0x_1\dots x_{p-1}}}, y_1=x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}$$

$$=\langle x_0,x_1,\dots,x_{p+q+1},y_1 \mid \underline{\underline{x_i^{p+q+2}(x_{p+q+1}^{p+1}y_1x_{p+q})^{-1}}}\ (i\in\{0,1,\dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}(x_{p+q+1}^{p}x_{p+q})^{-1}}}\ (j\in\{0,1,\dots q-1\}), \underline{\underline{x_{p+q}^{p+q}(x_{p+q+1}^{p}y_1)^{-1}}},$$

$$\underline{\underline{x_{p+q+1}^{p}=x_0x_1\dots x_{p-1}}}, y_1=x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}.$$

Eliminate $y_1=x_{p+q}^{p+q}x_{p+q+1}^{-p}$

$$P=\langle \underline{\underline{x_0,x_1,\dots,x_{p+q+1}}} \mid \underline{\underline{x_i^{p+q+2}(x_{p+q+1}^{p+1}(x_{p+q}^{p+q}x_{p+q+1}^{-p})x_{p+q})^{-1}}}\ (i\in\{0,1,\dots p-1\}),$$

$$x_{p+j}^{p+1}(x_{p+q+1}^{p}x_{p+q})^{-1}\ (j\in\{0,1,\dots q-1\}),$$

$$x_{p+q+1}^{p}=x_0x_1\dots x_{p-1}, \underline{\underline{x_{p+q}^{p+q}x_{p+q+1}^{-p}=x_px_{p+1}\dots x_{p+q-1}}}\rangle^{ab}$$

$$=\langle x_0,x_1,\dots,x_{p+q+1} \mid \underline{\underline{x_i^{p+q+2}(x_{p+q+1}^{1}x_{p+q}^{p+q+1})^{-1}}}\ (i\in\{0,1,\dots p-1\}),$$

$$x_{p+j}^{p+1}(x_{p+q+1}^{p}x_{p+q})^{-1}\ (j\in\{0,1,\dots q-1\}),$$

$$x_{p+q+1}^{p}=x_0x_1\dots x_{p-1}, x_{p+q}^{p+q}x_{p+q+1}^{-p}=x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}.$$

Eliminate $x_{p+q+1}=x_0^{p+q+2}x_{p+q}^{-(p+q+1)}$

$$P=\langle \underline{\underline{x_0,x_1,\dots,x_{p+q}}} \mid \underline{\underline{x_i^{p+q+2}((x_0^{p+q+2}x_{p+q}^{-(p+q+1)})x_{p+q}^{p+q+1})^{-1}}}\ (i\in\{\underline{1},2,\dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}((x_0^{p+q+2}x_{p+q}^{-(p+q+1)})^{p}x_{p+q})^{-1}}}\ (j\in\{0,1,\dots q-1\}),$$

$$\underline{\underline{(x_0^{p+q+2}x_{p+q}^{-(p+q+1)})^{p}=x_0x_1\dots x_{p-1}}}, \underline{\underline{x_{p+q}^{p+q}(x_0^{p+q+2}x_{p+q}^{-(p+q+1)})^{-p}=x_px_{p+1}\dots x_{p+q-1}}}\rangle^{ab}$$

$$=\langle x_0,x_1,\dots,x_{p+q} \mid \underline{\underline{(x_ix_0^{-1})^{p+q+2}}}\ (i\in\{1,2,\dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}x_0^{-p(p+q+2)}x_{p+q}^{p(p+q+1)-1}}}\ (j\in\{0,1,\dots q-1\}),$$

$$\underline{\underline{x_0^{p(p+q+2)}x_{p+q}^{-p(p+q+1)} = x_0x_1\dots x_{p-1}}}, x_{p+q}^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}.$$

Let $w_i = x_ix_0^{-1}$ $(i \in \{1, 2, \dots, p-1\})$,

$$P = \langle \underline{\underline{x_0, x_1, \dots, x_{p+q}, w_1, w_2, \dots, w_{p-1}}} \mid (x_ix_0^{-1})^{p+q+2} \ (i \in \{1, 2, \dots p-1\}),$$

$$x_{p+j}^{p+1}x_0^{-p(p+q+2)}x_{p+q}^{p(p+q+1)-1} \ (j \in \{0, 1, \dots q-1\}),$$

$$x_0^{p(p+q+2)}x_{p+q}^{-p(p+q+1)} = x_0x_1\dots x_{p-1},$$

$$x_{p+q}^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}, \underline{\underline{w_i = x_ix_0^{-1}}}\rangle^{ab}.$$

Eliminate $x_i = w_ix_0$ $(i \in \{1, 2, \dots, p-1\})$,

$$P = \langle \underline{\underline{x_0, x_p, x_{p+1}, \dots, x_{p+q}, w_1, w_2, \dots, w_{p-1}}} \mid \underline{\underline{(w_ix_0x_0^{-1})^{p+q+2}}} \ (i \in \{1, 2, \dots p-1\}),$$

$$x_{p+j}^{p+1}x_0^{-p(p+q+2)}x_{p+q}^{p(p+q+1)-1} \ (j \in \{0, 1, \dots q-1\}),$$

$$\underline{\underline{x_0^{p(p+q+2)}x_{p+q}^{-p(p+q+1)} = x_0(w_1x_0)(w_2x_0)\dots(w_{p-1}x_0)}},$$

$$x_{p+q}^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}$$

$$= \langle x_0, x_1, \dots, x_{p+q}, w_1, w_2, \dots, w_{p-1} \mid \underline{\underline{w_i^{p+q+2}}} \ (i \in \{1, 2, \dots p-1\}),$$

$$x_{p+j}^{p+1}x_0^{-p(p+q+2)}x_{p+q}^{p(p+q+1)-1} \ (j \in \{0, 1, \dots q-1\}),$$

$$\underline{\underline{x_0^{p(p+q+1)}x_{p+q}^{-p(p+q+1)} = w_1w_2\dots w_{p-1}}}, x_{p+q}^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}\rangle^{ab}.$$

Let $a = x_0x_{p+q}^{-1}$,

$$P = \langle \underline{\underline{x_0, x_p, x_{p+1}, \dots, x_{p+q}, w_1, w_2, \dots, w_{p-1}, a}} \mid w_i^{p+q+2} \ (i \in \{1, 2, \dots p-1\}),$$

$$x_{p+j}^{p+1}x_0^{-p(p+q+2)}x_{p+q}^{p(p+q+1)-1} \ (j \in \{0, 1, \dots q-1\}),$$

$$x_0^{p(p+q+1)}x_{p+q}^{-p(p+q+1)} = w_1w_2\dots w_{p-1},$$

$$x_{p+q}^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}, \underline{\underline{a = x_0x_{p+q}^{-1}}}\rangle^{ab}.$$

Eliminate $x_{p+q} = x_0a^{-1}$,

$$P = \langle \underline{\underline{x_0, x_p, x_{p+1}, \dots, x_{p+q-1}, w_1, w_2, \dots, w_{p-1}, a}} \mid w_i^{p+q+2} \ (i \in \{1, 2, \dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}x_0^{-p(p+q+2)}(x_0a^{-1})^{p(p+q+1)-1}}} \ (j \in \{0, 1, \dots q-1\}),$$

$$\underline{\underline{x_0^{p(p+q+1)}(x_0a^{-1})^{-p(p+q+1)} = w_1w_2\dots w_{p-1}}},$$

$$\underline{\underline{(x_0a^{-1})^{(p+1)(p+q)+p}x_0^{-p(p+q+2)} = x_px_{p+1}\dots x_{p+q-1}}}\rangle^{ab}$$

$$= \langle x_0, x_p, x_{p+1}, \dots, x_{p+q-1}, w_1, w_2, \dots, w_{p-1}, a \mid w_i^{p+q+2} \ (i \in \{1, 2, \dots p-1\}),$$

$$\underline{\underline{x_{p+j}^{p+1}x_0^{-p-1}a^{-p(p+q+1)+1}}} \ (j \in \{0, 1, \dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, \underline{\underline{a^{-(p+1)(p+q)-p}x_0^q = x_px_{p+1}\dots x_{p+q-1}}}\rangle^{ab}.$$

Let $z_{p+j} = x_{p+j}x_0^{-1}a^{-1}$ $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, x_p, x_{p+1}, \dots, x_{p+q-1}, w_1, w_2, \dots, w_{p-1}, a, z_p, z_{p+1}, \dots, z_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$x_{p+j}^{p+1}x_0^{-p-1}a^{-p(p+q+1)+1} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, a^{-(p+1)(p+q)-p}x_0^q = x_px_{p+1}\dots x_{p+q-1}, \underline{\underline{z_{p+j} = x_{p+j}x_0^{-1}a^{-1}}}\rangle^{ab}.$$

Eliminate $x_{p+j} = z_{p+j}x_0a$ $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \dots, w_{p-1}, a, z_p, z_{p+1}, \dots, z_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$\underline{\underline{(z_{p+j}x_0a)^{p+1}x_0^{-p-1}a^{-p(p+q+1)+1}}} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, \underline{\underline{a^{-(p+1)(p+q)-p}x_0^q = (z_px_0a)(z_{p+1}x_0a)\dots(z_{p+q-1}x_0a)}}\rangle^{ab}$$

$$= \langle x_0, w_1, w_2, \dots, w_{p-1}, a, z_p, z_{p+1}, \dots, z_{p+q-1} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$\underline{\underline{z_{p+j}^{p+1}a^{-p(p+q)+2}}} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, \underline{\underline{a^{-(p+2)(p+q)} = z_pz_{p+1}\dots z_{p+q-1}}}\rangle^{ab}.$$

Let $u_{p+j} = z_{p+j}a^{-p-q}$ $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \dots, w_{p-1}, a, z_p, z_{p+1}, \dots, z_{p+q-1}, u_p, u_{p+1}, \dots, u_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$z_{p+j}^{p+1}a^{-p(p+q)+2} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, a^{-(p+2)(p+q)} = z_pz_{p+1}\dots z_{p+q-1}, \underline{\underline{u_{p+j} = z_{p+j}a^{-p-q}}}\rangle^{ab}.$$

Eliminate $z_{p+j} = u_{p+j}a^{p+q}$ $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \dots, w_{p-1}, a, u_p, u_{p+1}, \dots, u_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$\underline{\underline{(u_{p+j}a^{p+q})^{p+1}a^{-p(p+q)+2}}} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, \underline{\underline{a^{-(p+2)(p+q)} = (u_pa^{p+q})(u_{p+1}a^{p+q})\dots(u_{p+q-1}a^{p+q})}}\rangle^{ab}$$

$$= \langle x_0, w_1, w_2, \dots, w_{p-1}, a, u_p, u_{p+1}, \dots, u_{p+q-1} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$\underline{\underline{u_{p+j}^{p+1}a^{p+q+2}}} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, \underline{\underline{a^{-(p+q+2)(p+q)} = u_pu_{p+1}\dots u_{p+q-1}}}\rangle^{ab}.$$

Let $v_{p+j} = u_{p+j}a^{-m}$, $m = \dfrac{p+q+2}{p+1}$, $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \dots, w_{p-1}, a, u_p, u_{p+1}, \dots, u_{p+q-1}, v_p, v_{p+1}, \dots, v_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1,2,\dots p-1\}),$$

$$u_{p+j}^{p+1}a^{p+q+2} \ (j \in \{0,1,\dots q-1\}),$$

$$a^{p(p+q+1)} = w_1w_2\dots w_{p-1}, a^{-(p+q+2)(p+q)} = u_pu_{p+1}\dots u_{p+q-1}, \underline{\underline{v_{p+j} = u_{p+j}a^{-m}}}\rangle^{ab}.$$

Eliminate $u_{p+j} = z_{p+j}a^m$ $(j \in \{0,1,2,\dots,q-1\})$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \ldots, w_{p-1}, a, v_p, v_{p+1}, \ldots, v_{p+q-1}}} \mid w_i^{p+q+2} \ (i \in \{1, 2, \ldots p-1\}),$$

$$\underline{\underline{v_{p+j}^{p+1}}} \ (j \in \{0, 1, \ldots q-1\}), a^{p(p+q+1)} = w_1 w_2 \ldots w_{p-1},$$

$$\underline{\underline{a^{-(p+q+2)(p+q)} = a^{-qm}(v_p v_{p+1} \ldots v_{p+q-1})}}\rangle^{ab}$$

$$= \langle x_0, w_1, w_2, \ldots, w_{p-1}, a, v_p, v_{p+1}, \ldots, v_{p+q-1} \mid w_i^{p+q+2} \ (i \in \{1, 2, \ldots p-1\}),$$

$$v_{p+j}^{p+1} \ (j \in \{0, 1, \ldots q-1\}), a^{p(p+q+1)} = w_1 w_2 \ldots w_{p-1},$$

$$\underline{\underline{a^{-(p+q+2)(p+q)+mq} = v_p v_{p+1} \ldots v_{p+q-1}}}\rangle^{ab}.$$

Eliminate $w_{p-1} = a^{p(p+q+1)} w_1 w_2 \ldots w_{p-2})^{-1}, v_{p+q-1} = a^{-(p+q+2)(p+q)+mq}(v_p v_{p+1} \ldots v_{p+q-2})^{-1}$,

$$P = \langle \underline{\underline{x_0, w_1, w_2, \ldots, w_{p-2}, a, v_p, v_{p+1}, \ldots, v_{p+q-2}}} \mid w_i^{p+q+2} \ (i \in \{1, 2, \ldots \underline{\underline{p-2}}\}),$$

$$\underline{\underline{(a^{p(p+q+1)}(w_1 w_2 \ldots w_{p-2})^{-1})^{p+q+2}}},$$

$$v_{p+j}^{p+1} \ (j \in \{0, 1, \ldots \underline{\underline{q-2}}\}), \underline{\underline{(a^{-(p+q+2)(p+q)+mq}(v_p v_{p+1} \ldots v_{p+q-2})^{-1})^{p+1}}}\rangle^{ab}$$

$$= \langle x_0, w_1, w_2, \ldots, w_{p-2}, a, v_p, v_{p+1}, \ldots, v_{p+q-2} \mid w_i^{p+q+2} \ (i \in \{1, 2, \ldots p-2\}),$$

$$\underline{\underline{a^{p(p+q+1)(p+q+2)}}},$$

$$v_{p+j}^{p+1} \ (j \in \{0, 1, \ldots q-2\}), \underline{\underline{a^{(p+1)((p+q+2)(p+q)-mq)}}}\rangle^{ab}.$$

As $(p+1)((p+q+2)(p+q) - mq) = (p+1)((p+q+2)(p+q) - \dfrac{p+q+2}{q+1}q)$

$= \dfrac{p+1}{q+1}(p+q+2)(pq+q^2+p)$, and $p(p+q+1)(p+q+2) = (p+q+2)(p^2+pq+p)$,

we can see that $\dfrac{p+1}{q+1}(p+q+2)(pq+q^2+p) > (p+q+2)(p^2+pq+p)$,

as $(p+1) \mid (q+1) \implies p \leqslant q$. Therefore

$$P = \langle x_0, w_1, w_2, \ldots, w_{p-2}, a, v_p, v_{p+1}, \ldots, v_{p+q-2} \mid w_i^{p+q+2} \ (i \in \{1, 2, \ldots p-2\}),$$

$$\underline{\underline{a^{p(p+q+1)(p+q+2)}}}, v_{p+j}^{p+1} \ (j \in \{0, 1, \ldots q-2\})\rangle^{ab}$$

$$P \cong \mathbb{Z} \oplus \mathbb{Z}_{p(p+q+1)(p+q+2)} \oplus \underbrace{\mathbb{Z}_{p+q+2} \oplus \mathbb{Z}_{p+q+2} \oplus \cdots \oplus \mathbb{Z}_{p+q+2}}_{p-2} \oplus \underbrace{\mathbb{Z}_{p+1} \oplus \mathbb{Z}_{p+1} \oplus \cdots \oplus \mathbb{Z}_{p+1}}_{q-1}.$$

□

Example 4.2.4. *The sandpile group of* $((\overline{K_4} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_4$

The degree sequence of $((\overline{K_4} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_4$ is $9, 9, 9, 9, 8, 5, 5, 5, 5, 4$, which gives us the Ferrers diagram in Figure 4.3.

Using Theorem 4.1.1, we can see that the eigenvalues are

$$a_0, a_1, \ldots, a_9 = 10, 10, 10, 10, 9, 5, 5, 5, 4, 0.$$

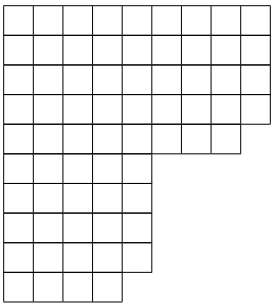

Figure 4.3: Ferrers diagram of degree sequence of $((\overline{K_4} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_4$

Now, by taking the columns $a_1, a_2, \ldots, a_{s=9}$ and reordering with a new label we get

$$b_1, b_2, \ldots, b_8 = 10, 10, 10, 4, 9, 5, 5, 5.$$

We use b to construct the graph H,

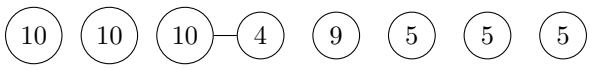

Figure 4.4: The graph H for $((\overline{K_4} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_4$

Using Conjecture 4.1.2 to calculate the sandpile group of $((\overline{K_4} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_4$, we get

$$\mathbb{Z}_{40} \oplus {\mathbb{Z}_{10}}^2 \oplus {\mathbb{Z}_5}^3 \oplus \mathbb{Z}_9.$$

Which is isomorphic to the result obtained using Theorem J as we get

$$p(p+q+1)(p+q+2) = 4 \times 9 \times 10 = 360,$$

$$\mathbb{Z}_{360} \oplus {\mathbb{Z}_{10}}^2 \oplus {\mathbb{Z}_5}^3.$$

4.3 Construction using block matrix

In Section 3.3.2, we discussed circulant graphs with 4 distinct eigenvalues where we show that their adjacency matrix is a block matrix. Expanding on this idea we explore two block matrix constructions that are not regular. A *star graph* S_q is the complete bipartite graph $K_{1,q-1}$, we assume that $v_0 \in V(S_q)$ is the vertex with degree $q-1$. We use S_q in a block matrix construction with an antidiagonal matrix and the other with an identity matrix. This construction can be thought of as adding a matching to two isolated star graphs in two different ways.

Let Γ_1 be a graph with an adjacency matrix A

$$A = \begin{pmatrix} A(S_q) & I_{anti} \\ I_{anti} & A(S_q) \end{pmatrix},$$

where S_q is the star tree graph of order q.

Theorem K. *The sandpile group of* Γ_1 *is* $\mathbb{Z}_{12(q+1)} \oplus {\mathbb{Z}_3}^{q-4}$.

Proof.

$$P = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid x_0^q (x_1 x_2 \ldots x_{q-1})^{-1} y_{q-1}^{-1} = 1,$$
$$y_0^q (y_1 y_2 \ldots y_{q-1})^{-1} x_{q-1}^{-1} = 1, x_0^{-1} x_i^2 y_{q-1-i}^{-1}, y_0^{-1} y_i^2 x_{q-1-i}^{-1} \ (i = 1, 2, \ldots, q-1) \rangle^{ab}.$$

Notice that x, y indices are modulo q.

To simplify the relation $(x_0^q (x_1 x_2 \ldots x_{q-1})^{-1} y_{q-1}^{-1})$, we take the relations $x_0^{-1} x_i^2 y_{q-1-i}^{-1}$ $(i = 1, 2, \ldots, q-1)$ and add them all as a multiplier to the relation above. We also simplify the relation $(y_0^q (y_1 y_2 \ldots y_{q-1})^{-1} x_{q-1}^{-1})$ by taking the relations $y_0^{-1} y_i^2 x_{q-1-i}^{-1}$ $(i = 1, 2, \ldots, q-1)$ and add them all as a multiplier in the same way.

$$P = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid \underline{\underline{(x_0 x_1 \ldots x_{q-1})(y_0 y_1 \ldots y_{q-1})^{-1}}}, \underline{\underline{(y_0 y_1 \ldots y_{q-1})(x_0 x_1 \ldots x_{q-1})^{-1}}},$$
$$x_0^{-1} x_i^2 y_{q-1-i}^{-1}, y_0^{-1} y_i^2 x_{q-1-i}^{-1} \ (i = 1, 2, \ldots, q-1) \rangle^{ab}$$
$$= \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid \underline{\underline{(x_0 x_1 \ldots x_{q-1})(y_0 y_1 \ldots y_{q-1})^{-1}}},$$
$$x_0^{-1} x_i^2 y_{q-1-i}^{-1}, y_0^{-1} y_i^2 x_{q-1-i}^{-1} \ (i = 1, 2, \ldots, q-1) \rangle^{ab}.$$

Eliminate $y_i = x_0^{-1} x_{q-1-i}^2$ $(i = 1, 2, \ldots, q-2)$

$$P = \langle \underline{\underline{x_0, x_1, \ldots, x_{q-1}, y_0, y_{q-1}}} \mid \underline{\underline{(x_0 x_1 \ldots x_{q-1})(y_0 y_{q-1})^{-1} (x_0^{q-2} (x_1 \ldots x_{q-2})^{-2})}},$$
$$\underline{\underline{y_0^{-1} (x_0^{-1} x_{q-1-i}^2)^2 x_{q-1-i}^{-1}}} \ (i = 1, 2, \ldots, q-2), x_0^{-1} x_{q-1}^2 y_0^{-1}, y_0^{-1} y_{q-1}^2 x_0^{-1} \rangle^{ab}$$
$$= \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_{q-1} \mid \underline{\underline{x_0^{q-1} x_{q-1} (x_1 \ldots x_{q-2} y_0 y_{q-1})^{-1}}},$$
$$\underline{\underline{y_0^{-1} x_0^{-2} x_i^3}} \ (i = 1, 2, \ldots, q-2), x_0^{-1} x_{q-1}^2 y_0^{-1}, y_0^{-1} y_{q-1}^2 x_0^{-1} \rangle^{ab}.$$

Eliminate $x_0 = x_{q-1}^2 y_0^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_0, y_{q-1}}} \mid \underline{\underline{(x_{q-1}^2 y_0^{-1})^{q-1} x_{q-1} (x_1 \ldots x_{q-2} y_0 y_{q-1})^{-1}}},$$
$$y_0^{-1} (x_{q-1}^2 y_0^{-1})^{-2} x_i^3 \ (i = 1, 2, \ldots, q-2), y_0^{-1} y_{q-1}^2 (x_{q-1}^2 y_0^{-1})^{-1} \rangle^{ab}$$
$$= \langle x_1, x_2, \ldots, x_{q-1}, y_0, y_{q-1} \mid \underline{\underline{x_{q-1}^{2q-1} y_0^{-q} (x_1 \ldots x_{q-2} y_{q-1})^{-1}}},$$
$$y_0 x_{q-1}^{-4} x_i^3 \ (i = 1, 2, \ldots, q-2), y_{q-1}^2 x_{q-1}^{-2} \rangle^{ab}.$$

Let $a = y_{q-1} x_{q-1}^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_0, y_{q-1}, a}} \mid x_{q-1}^{2q-1} y_0^{-q} (x_1 \ldots x_{q-2} y_{q-1})^{-1},$$

$$y_0 x_{q-1}^{-4} x_i^3 \ (i = 1, 2, \ldots, q-2), y_{q-1}^2 x_{q-1}^{-2}, \underline{\underline{a = y_{q-1} x_{q-1}^{-1}}} \rangle^{ab}.$$

Eliminate $y_{q-1} = a x_{q-1}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_0, a}} \mid \underline{\underline{x_{q-1}^{2q-1} y_0^{-q} (x_1 \ldots x_{q-2} a x_{q-1})^{-1}}}, y_0 x_{q-1}^{-4} x_i^3 \ (i = 1, 2, \ldots, q-2), \underline{\underline{a^2}} \rangle^{ab}.$$

Eliminate $y_0 = x_{q-1}^4 x_{q-2}^{-3}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, a}} \mid \underline{\underline{x_{q-1}^{2q-2} (x_{q-1}^4 x_{q-2}^{-3})^{-q} (x_1 \ldots x_{q-2} a)^{-1}}}, \underline{\underline{(x_{q-1}^4 x_{q-2}^{-3}) x_{q-1}^{-4} x_i^3}} \ (i = 1, 2, \ldots, \underline{\underline{q-3}}), a^2 \rangle^{ab}$$

$$= \langle x_1, x_2, \ldots, x_{q-1}, a \mid \underline{\underline{x_{q-1}^{-2(q-1)} x_{q-2}^{3q-1} (x_1 \ldots x_{q-3} a)^{-1}}}, \underline{\underline{(x_{q-2}^{-1} x_i)^3}} \ (i = 1, 2, \ldots, q-3), a^2 \rangle^{ab}.$$

Let $b_i = x_{q-2}^{-1} x_i \ (i = 1, 2, \ldots, q-3)$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, a, b_1, b_2, \ldots, b_{q-3}}} \mid \underline{\underline{x_{q-1}^{-2(q-1)} x_{q-2}^{3q-1} (x_1 \ldots x_{q-3} a)^{-1}}}, \underline{\underline{(x_{q-2}^{-1} x_i)^3}} \ (i = 1, 2, \ldots, q-3), a^2,$$

$$\underline{\underline{b_i = x_{q-2}^{-1} x_i}} \rangle^{ab}.$$

Eliminate $x_i = x_{q-2} b_i \ (i = 1, 2, \ldots, q-3)$

$$P = \langle \underline{\underline{x_{q-2}, x_{q-1}, a, b_1, b_2, \ldots, b_{q-3}}} \mid \underline{\underline{x_{q-1}^{-2(q-1)} x_{q-2}^{2(q+1)} (b_1 \ldots b_{q-3} a)^{-1}}}, \underline{\underline{b_i^3}} \ (i = 1, 2, \ldots, q-3), a^2 \rangle^{ab}.$$

Eliminate $a = x_{q-1}^{-2(q-1)} x_{q-2}^{2q-4} (b_1 \ldots b_{q-3})^{-1}$

$$P = \langle \underline{\underline{x_{q-2}, x_{q-1}, b_1, b_2, \ldots, b_{q-3}}} \mid b_i^3 \ (i = 1, 2, \ldots, q-3), (x_{q-1}^{-2(q-1)} x_{q-2}^{2(q+1)} (b_1 \ldots b_{q-3})^{-1})^2 \rangle^{ab}.$$

Let $c = b_1 \ldots b_{q-3}, d = x_{q-1}^{-1} x_{q-2}$

$$P = \langle \underline{\underline{x_{q-2}, x_{q-1}, b_1, b_2, \ldots, b_{q-3}, c, d}} \mid b_i^3 \ (i = 1, 2, \ldots, q-3),$$

$$(x_{q-1}^{-2(q-1)} x_{q-2}^{2(q+1)} (b_1 \ldots b_{q-3})^{-1})^2, \underline{\underline{c = b_1 \ldots b_{q-3}}}, \underline{\underline{d = x_{q-1}^{-1} x_{q-2}}} \rangle^{ab}.$$

Eliminate $x_{q-2} = x_{q-1} d, b_{q-3} = (b_1 \ldots b_{q-4})^{-1} c$

$$P = \langle \underline{\underline{x_{q-1}, b_1, b_2, \ldots, b_{q-4}, c, d}} \mid b_i^3 \ (i = 1, 2, \ldots, \underline{\underline{q-4}}), \underline{\underline{((b_1 \ldots b_{q-4})^{-1} c)^3}}, \underline{\underline{(d^{2(q+1)} c^{-1})^2}} \rangle^{ab}$$

$$= \langle x_{q-1}, b_1, b_2, \ldots, b_{q-4}, c, d \mid b_i^3 \ (i = 1, 2, \ldots, q-4), \underline{\underline{c^3}}, \underline{\underline{(d^{2(q+1)} c^{-1})^2}} \rangle^{ab}$$

$$= \langle x_{q-1}, b_1, b_2, \ldots, b_{q-4} \mid b_i^3 \ (i = 1, 2, \ldots, q-4) \rangle^{ab} \oplus \langle c, d \mid c^3, (d^{2(q+1)} c^{-1})^2 \rangle^{ab}.$$

By converting $\langle c, d \mid c^3, (d^{2(q+1)} c^{-1})^2 \rangle^{ab}$ into a matrix we get,

$$M = \begin{pmatrix} -2 & 4(q+1) \\ 3 & 0 \end{pmatrix}$$

$$\mathrm{SNF}(M) = \mathrm{diag}(l, \det(M)/l),$$

where $l = \gcd(4(q+1), -2, 0, 3)$. Therefore, $\mathrm{SNF}(M) = \mathrm{diag}(1, 12(q+1))$. □

Let Γ_2 be a graph with an adjacency matrix A

$$A = \begin{pmatrix} A(S_q) & I \\ I & A(S_q) \end{pmatrix},$$

where S_q is the star tree graph of order q.

Theorem L. *The sandpile group of* Γ_2 *is* $\mathbb{Z}_{3(q+2)} \oplus {\mathbb{Z}_3}^{q-3}$.

Proof.

$$\begin{aligned} P = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid\ & x_0^q (x_1 x_2 \ldots x_{q-1})^{-1} y_0^{-1} = 1, \\ & y_0^q (y_1 y_2 \ldots y_{q-1})^{-1} x_0^{-1} = 1, \\ & x_0^{-1} x_i^2 y_i^{-1}, y_0^{-1} y_i^2 x_i^{-1}\ (i = 1, 2, \ldots, q-1) \rangle^{ab}. \end{aligned}$$

Notice that x, y indices are modulo q.

To simplify the relation $x_0^q (x_1 x_2 \ldots x_{q-1})^{-1} y_0^{-1})$ we take the relations $x_0^{-1} x_i^2 y_i^{-1}$ $(i = 1, 2, \ldots, q-1)$ and add them all as a multiplier to the relation above. We also simplify the relation $(y_0^q (y_1 y_2 \ldots y_{q-1})^{-1} x_0^{-1})$ by taking the relations $y_0^{-1} y_i^2 x_i^{-1}$ $(i = 1, 2, \ldots, q-1)$ and add them all as a multiplier in the same way.

$$\begin{aligned} P = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid\ & \underline{\underline{(x_0 x_1 \ldots x_{q-1})(y_0 y_1 \ldots y_{q-1})^{-1}}}, \\ & \underline{\underline{(y_0 y_1 \ldots y_{q-1})(x_0 x_1 \ldots x_{q-1})^{-1}}}, x_0^{-1} x_i^2 y_i^{-1}, y_0^{-1} y_i^2 x_i^{-1}\ (i = 1, 2, \ldots, q-1) \rangle^{ab} \\ = \langle x_0, x_1, \ldots, x_{q-1}, y_0, y_1, \ldots, y_{q-1} \mid\ & \underline{\underline{(x_0 x_1 \ldots x_{q-1})(y_0 y_1 \ldots y_{q-1})^{-1}}}, \\ & x_0^{-1} x_i^2 y_i^{-1}, y_0^{-1} y_i^2 x_i^{-1}\ (i = 1, 2, \ldots, q-1) \rangle^{ab}. \end{aligned}$$

Eliminate $y_i = x_0^{-1} x_i^2$ $(i = 1, 2, \ldots, q-2)$

$$\begin{aligned} P = \langle & \underline{\underline{x_0, x_1, \ldots, x_{q-1}, y_0, y_{q-1}}} \mid \underline{\underline{x_0^{q-1} (x_1 x_2 \ldots x_{q-1})(y_0 y_{q-1})^{-1} (x_1 \ldots x_{q-2})^{-2}}}, \\ & \underline{\underline{y_0^{-1} (x_0^{-1} x_i^2)^2 x_i^{-1}}}\ (i = 1, 2, \ldots, \underline{\underline{q-2}}), \underline{\underline{x_0^{-1} x_{q-1}^2 y_{q-1}^{-1}}}, \underline{\underline{y_0^{-1} y_{q-1}^2 x_{q-1}^{-1}}} \rangle^{ab} \\ = \langle & x_0, x_1, \ldots, x_{q-1}, y_0, y_{q-1} \mid \underline{\underline{x_0^{q-1} x_{q-1} (x_1 x_2 \ldots x_{q-2})(y_0 y_{q-1})^{-1}}}, \\ & \underline{\underline{y_0^{-1} x_0^{-2} x_i^3}}\ (i = 1, 2, \ldots, q-2), x_0^{-1} x_{q-1}^2 y_{q-1}^{-1}, y_0^{-1} y_{q-1}^2 x_{q-1}^{-1} \rangle^{ab}. \end{aligned}$$

Eliminate $x_0 = x_{q-1}^2 y_{q-1}^{-1}$

$$\begin{aligned} P = \langle & \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_0, y_{q-1}}} \mid \underline{\underline{(x_{q-1}^2 y_{q-1}^{-1})^{q-1} x_{q-1} (x_1 x_2 \ldots x_{q-2})^{-1} (y_0 y_{q-1})^{-1}}}, \\ & \underline{\underline{y_0^{-1} (x_{q-1}^2 y_{q-1}^{-1})^{-2} x_i^3}}\ (i = 1, 2, \ldots, q-2), y_0^{-1} y_{q-1}^2 x_{q-1}^{-1} \rangle^{ab} \\ = \langle & x_1, x_2, \ldots, x_{q-1}, y_0, y_{q-1} \mid \underline{\underline{x_{q-1}^{2q-1} y_{q-1}^{-q} (y_0 x_1 x_2 \ldots x_{q-2})^{-1}}}, \end{aligned}$$

$$\underline{\underline{y_{q-1}^2 x_{q-1}^{-4} y_0^{-1} x_i^3}}\ (i = 1, 2, \ldots, q-2), y_0^{-1} y_{q-1}^2 x_{q-1}^{-1} \rangle^{ab}.$$

Eliminate $y_0 = y_{q-1}^2 x_{q-1}^{-1}$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_{q-1}}} \mid \underline{\underline{x_{q-1}^{2q-1} y_{q-1}^{-q} ((y_{q-1}^2 x_{q-1}^{-1}) x_1 x_2 \ldots x_{q-2})^{-1}}},$$

$$\underline{\underline{y_{q-1}^2 x_{q-1}^{-4} (y_{q-1}^2 x_{q-1}^{-1})^{-1} x_i^3}}\ (i = 1, 2, \ldots, q-2) \rangle^{ab}$$

$$= \langle x_1, x_2, \ldots, x_{q-1}, y_{q-1} \mid \underline{\underline{x_{q-1}^{2q} y_{q-1}^{-(q+2)}}} (x_1 x_2 \ldots x_{q-2})^{-1}, \underline{\underline{(x_{q-1}^{-1} x_i)^3}}\ (i = 1, 2, \ldots, q-2) \rangle^{ab}.$$

Let $a_i = x_{q-1}^{-1} x_i\ (i = 1, 2, \ldots, q-2)$

$$P = \langle \underline{\underline{x_1, x_2, \ldots, x_{q-1}, y_{q-1}, a_1, a_2, \ldots, a_{q-2}}} \mid x_{q-1}^{2q} y_{q-1}^{-(q+2)} (x_1 x_2 \ldots x_{q-2})^{-1}, (x_{q-1}^{-1} x_i)^3,$$

$$\underline{\underline{a_i = x_{q-1}^{-1} x_i}}\ (i = 1, 2, \ldots, q-2) \rangle^{ab}.$$

Eliminate $x_i = x_{q-1} a_i\ (i = 1, 2, \ldots, q-2)$

$$P = \langle \underline{\underline{x_{q-1}, y_{q-1}, a_1, a_2, \ldots, a_{q-2}}} \mid \underline{\underline{x_{q-1}^{q+2} y_{q-1}^{-(q+2)} (a_1 a_2 \ldots a_{q-2})^{-1}}}, \underline{\underline{a_i^3}}\ (i = 1, 2, \ldots, q-2) \rangle^{ab}.$$

Eliminate $a_{q-2} = x_{q-1}^{q+2} y_{q-1}^{-(q+2)} (a_1 a_2 \ldots a_{q-3})^{-1}$

$$P = \langle \underline{\underline{x_{q-1}, y_{q-1}, a_1, a_2, \ldots, a_{q-3}}} \mid a_i^3\ (i = 1, 2, \ldots, \underline{\underline{q-3}}), \underline{\underline{(x_{q-1}^{q+2} y_{q-1}^{-(q+2)} (a_1 a_2 \ldots a_{q-2})^{-1})^3}} \rangle^{ab}$$

$$= \langle x_{q-1}, y_{q-1}, a_1, a_2, \ldots, a_{q-3} \mid a_i^3\ (i = 1, 2, \ldots, q-3), \underline{\underline{(x_{q-1} y_{q-1}^{-1})^{3(q+2)}}} \rangle^{ab}.$$

Let $c = x_{q-1} y_{q-1}^{-1}$

$$P = \langle \underline{\underline{x_{q-1}, y_{q-1}, a_1, a_2, \ldots, a_{q-3}, c}} \mid a_i^3\ (i = 1, 2, \ldots, q-3), (x_{q-1} y_{q-1}^{-1})^{3(q+2)}, \underline{\underline{c = x_{q-1} y_{q-1}^{-1}}} \rangle^{ab}.$$

Eliminate $x_{q-1} = c y_{q-1}$

$$P = \langle \underline{\underline{y_{q-1}, a_1, a_2, \ldots, a_{q-3}, c}} \mid a_i^3\ (i = 1, 2, \ldots, q-3), \underline{\underline{c^{3(q+2)}}} \rangle^{ab}.$$

□

We observe that even though Γ_1 and Γ_2 share the same degree sequence and are similar in their construction, their sandpile groups are slightly different.

4.4 Edge tiling construction

The chain cyclic graph is described in Section 1.3, it can be constructed by tiling the edges of a list of cycle graphs. Here, rather than tiling cycle graphs as a chain, we start with a cycle C_r and a list of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ and tile each edge $e_i \in E(C_r)$ with an edge in Γ_i, we call this construction a *necklace of graphs*, for which we present the following new theorem for a necklace of cycles.

Theorem 4.4.1. *Suppose we have an edge tiling construction of a list of cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_r}$ and a cycle*

C_r denoted by Γ_1, where $r \geqslant 3$, $k_i \geqslant 3$, $i \in \{1, 2, \ldots, r\}$. Define another edge tiling construction of a list of cycles $C_{t_1}, C_{t_2}, \ldots, C_{t_r}$ and a cycle C_r denoted by Γ_2 where $t_1, t_2, \ldots, t_r$ is a permutation of $k_1, k_2, \ldots, k_r$. The sandpile group of Γ_1 is isomorphic to the sandpile group of Γ_2.

Proof. As we are working with a planar graph, Theorem 1.11.3 tells us that a graph sandpile group is isomorphic to its dual graph sandpile group. It is not hard to verify that the dual graph of both Γ_1, Γ_2 has Laplacian matrix row/column equivalent to

$$\begin{pmatrix} k_1 & 0 & \cdots & 0 & -1 & 1-k_1 \\ 0 & k_2 & \cdots & 0 & -1 & 1-k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k_r & -1 & 1-k_r \\ -1 & -1 & \cdots & -1 & r & 0 \\ 1-k_1 & 1-k_2 & \cdots & 1-k_r & 0 & t \end{pmatrix},$$

where $t = k_1 + k_2 + \cdots + k_r - r$. □

Note that Theorem 4.4.1 on the necklace of cycles is similar to Theorem 1.11.6 on the chain cyclic graph which was introduced in Section 1.11.

Using r copies of K_n, and C_r in an edge tiling construction with C_r we have the following two conjectures, both verified for $3 \leqslant r \leqslant 6$, $3 \leqslant n \leqslant 6$.

Conjecture 4.4.2. *Suppose we have an edge tiling construction of r K_n graphs and a cycle C_r where $r \geqslant 3$, $n \geqslant 3$, then the resulting sandpile group is isomorphic to*

$$\mathbb{Z}_n^{r(n-2)-2} \oplus \mathbb{Z}_{2rn}$$

Conjecture 4.4.3. *Suppose we have an edge tiling construction of r C_n graphs and a cycle C_r where $r \geqslant 3$, $n \geqslant 3$, then the resulting sandpile group is isomorphic to*

$$\mathbb{Z}_n^{r-2} \oplus \mathbb{Z}_{rn(n-1)}$$

4.5 Recursive construction

The Sierpiński Sieve graph is based on the well known Sierpiński triangle fractal, which can be thought of as a recursive edge subdivision process. Start with a triangle graph (the cycle graph C_3)

$$C_3 = G(\{a, b, c\}, \{(a, b), (a, c), (b, c)\}),$$

which is the base Sierpiński Sieve graph S_1. Now, to get the next Sierpiński Sieve graph S_2, we start by subdividing the edges $(a, b), (a, c), (b, c)$. The resulting graph

$$T = G(\{a, b, c, d, e, f\}, \{(a, d), (b, d), (a, e), (c, e), (b, f), (c, f)\}).$$

We then create an inner triangle in T by adding the edges $(d, e), (d, f), (e, f)$, which gives us the Sierpiński Sieve graph S_2. In general, by repeating this process for each triangle in S_i we get the next Sierpiński Sieve graph S_{i+1}.

We have a conjecture for the sandpile group of the Sierpiński Sieve graph, which has been verified for $3 \leqslant i \leqslant 5$.

Conjecture 4.5.1. *Let S_k be the Sierpiński Sieve Graph, where $k \geqslant 3$. Let $f(n) = 3^{\lceil n/2 \rceil} \cdot 5^{\lfloor n/2 \rfloor}$. The sandpile group of S_k is isomorphic to*

$$\mathbb{Z}_3 \oplus \mathbb{Z}_6^{3^{k-2}-1} \oplus \mathbb{Z}_{6 \cdot f(1)} \oplus \mathbb{Z}_{6 \cdot f(2)}^{3^{k-3}-1} \oplus \cdots \oplus \mathbb{Z}_{6 \cdot f(2k-5)} \oplus \mathbb{Z}_{90 \cdot f(2k-5)}$$

4.6 Future work

One area to explore further is the sandpile group of the complete multipartite graph $K_{n_1, n_2, \ldots, n_k}$ when some $n_i = 1$ as we believe it is not covered by Theorem 1.11.1. The sandpile group of threshold graphs that are not covered by Theorem 4.1.3 would be a good area to investigate further as we noticed that the difficulty increased to some extent once we altered $\overline{K_q} \bigtriangledown K_p$, to $((\overline{K_q} \bigtriangledown K_1) \cup K_1) \bigtriangledown K_p$, but it is not clear yet why this is the case. We also think that better understanding of how the sandpile group changes under certain edge/vertex tiling operations may resolve the conjectures provided in this chapter more generally.

Chapter 5

The Rank of the Sandpile Group

In this chapter, we investigate the rank of the sandpile group and present graphs with cyclic sandpile group and other graphs of a sandpile group rank at least 2. We investigate the sandpile group of $K_2 \bigtriangledown P_m$ to determine when it is cyclic and when it is not cyclic (in which case we show it has rank 2). We have a conjecture for when the sandpile group of this graph is cyclic (see Conjecture 5.2.18) and we prove a result for when it is not (see Theorem M).

5.1 Introduction

Let Γ be a connected graph of order n and size m. Suppose that the Smith normal form of the Laplacian matrix of Γ is $\mathrm{SNF}(L(\Gamma)) = \mathrm{diag}(a_0, a_1, \ldots, a_{n-2}, 0)$. The minimum number of generators of the sandpile group of Γ can be defined as $\psi(\Gamma) = \Big|\{a_i | a_i > 1, i \in \{0, 1, \ldots, n-2\}\}\Big|$. This leads us to the next result by Lorenzini (see [43, page 277]),

$$\psi(\Gamma) \leqslant m - (n-1). \tag{5.1}$$

An immediate observation from Equation (5.1), is that in order to make sure that the sandpile group of a connected graph Γ is cyclic ($\psi(\Gamma) = 1$), we must have $m = n$. In this case we get a tree graph with an extra (unique) edge added. This graph will have a cycle subgraph.

In [44] Lorenzini suggested that it is common for graphs to have a cyclic sandpile groups, and presented a family of graphs with cyclic sandpile groups. In [62, Conjecture 4.2], Wagner conjectured that the majority of connected graphs have a cyclic sandpile group. Clancy, Leake and Payne presented in [17] the following conjecture

$$\lim_{n\to\infty} \frac{T(n)}{2^{\binom{n}{2}}} = \prod_{i=1}^{\infty} \zeta(2i+1)^{-1} \approx 0.79353, \tag{5.2}$$

where $T(n)$ is the number of connected graphs of order n that have a cyclic sandpile group, and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

is the Riemann zeta function. Later on Wood proved that the right-hand side of this equation is an upper bound for the probability on the left-hand side (see [66]).

Our motivation in this chapter is to explore families of graphs that are cyclic and try to explore when they are non cyclic.

5.1.1 Cayley graphs with cyclic sandpile groups

Let $(m_{ii})_{k\times k}$ be a matrix such that, all entries are integers which are $m_{ij} \leqslant 0, i \neq j$. We will call this matrix class M-*Matrix* (they are also known as *avalanche-finite matrices*, see [28, Page 4]).

Let G be a group, $\chi_0, \chi_1, \dots \chi_k$ be its irreducible characters, where χ_0 is the trivial character ($\chi(g) = 1$ for all g). And let γ be an n-dimensional faithful representation of G with character χ_γ. Let $A = (a_{ij})$ be an $(k+1) \times (k+1)$ integer matrix such that

$$\chi_\gamma \cdot \chi_i = \sum_{j=0}^{k} a_{ij}\chi_j.$$

Note that a_{ij} are non-negative as the product of two characters is a character by the following well known proposition (see for example [35, Theorem 19.18, page 206]).

Proposition 5.1.1. *Let χ and ψ be characters of the groups G, and H respectively. Then $\chi \cdot \psi$ is a character defined as $(\chi \cdot \psi)(g, h) = \chi(g)\psi(h)$.*

The extended McKay-Cartan matrix is defined as $C' = nI - A$. If we remove the row and column corresponding to the trivial character χ_0 we get the McKay-Cartan matrix C.

Theorem 5.1.2. [5, Theorem 1.2] *The McKay-Cartan matrix C is an M matrix.*

Theorem 5.1.3. [5, Theorem 6.9] *Let G be a group, and let $\gamma : G \to \mathrm{SL_n}(\mathbb{C})$ be a faithful representation of G, where $\mathrm{SL_n}(\mathbb{C})$ is the special linear group of $n \times n$ complex matrices of determinant 1. Then there is a surjection*

$$\mathrm{K}(\gamma) \to \widehat{G}.$$

Where $\mathrm{K}(\gamma)$ is the critical group ($\mathrm{coker}(C)$), and $\widehat{G}$ is the character group of G.

In the case when G is abelian, $S = \{s_1, s_2, \cdots, s_n\}$ is a multiset of elements of G such that $s_i \neq 0, \sum_{i=1}^{n} s_i = 0$. We can see that there is an isomorphism between G and its character group $\widehat{G} \cong G$. Given that the generating set of G satisfy the condition $s_i \neq 0, \sum_{i=1}^{n} s_i = 0$, we find in $\widehat{G}$ a corresponding set of irreducible characters $\chi = \chi_1, \chi_1, \dots \chi_n$ that sums to the element corresponding to 0 in G (χ_0 in $\widehat{G}$). Let $\phi(s_1), \phi(s_2), \dots, \phi(s_n)$ be the mapping of these characters from S, we have $\sum_{i=1}^{n} s_i = 0 \Rightarrow \phi(\sum_{i=1}^{n} s_i) = \prod_{i=1}^{n} \chi_i = \chi_0$, where χ_0 is the trivial character. If G (similarly $\widehat{G}$) is the base group for a Cayley graph $\Gamma(G, S)$, where S is its generating set, G will be acting on $V(\Gamma(G, S))$ by multiplication $g(x) = gx$. This will give rise to a faithful representation γ of G,

where $V(\Gamma(G,S))$ is the basis of $\mathbb{C}^{|G|}$. To see that γ is in $\mathrm{SL_n}(\mathbb{C})$, we have $\prod_{i=1}^{n}\chi_i$ corresponds to the modules $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ which means γ is in $\mathrm{GL_n}(C)$. The determinant of $\prod_{i=1}^{n}\chi_i$ is 1 as $\det(\chi_0) = 1$. Therefore γ is in $\mathrm{SL_n}(\mathbb{C})$.

Corollary 5.1.4. [54, Problem 1.9] *Let $\Gamma = \mathrm{CayD}(G,S)$ be a Cayley digraph, where G is a finite abelian group, and $S = \{s_1, s_2, \cdots, s_n\}$ is a multiset such that $s_i \neq 0, \sum_{i=1}^{n} s_i = 0$. Then we have the following homomorphism which is also a surjection*

$$\mathrm{Sp}(\Gamma) \twoheadrightarrow G.$$

Based on this we have the following corollary.

Corollary 5.1.5. *Let G be a finite abelian group with S as its generating set, and let $\Gamma = \mathrm{Cay}(G,S)$, be a Cayley graph. If the sandpile group of Γ is cyclic then G is cyclic.*

This tells us that we can get a non cyclic Sandpile group by constructing a Cayley graph with a non cyclic group.

5.1.2 Graphs with non-cyclic sandpile group

It is easy to construct a graph that has a sandpile group of rank 3 or more, for example the following proposition is an example of such constructions.

Proposition 5.1.6. *Let Γ be a connected graph of order n, and let $r \geqslant 3$. The sandpile group of $K_r \bigtriangledown \Gamma$ is not cyclic.*

Proof. By Theorem 1.2.2, the Laplacian eigenvalues of $K_r \bigtriangledown \Gamma$ will contain the eigenvalue $r+n$ with multiplicity at least equal to r. Then

$$\begin{aligned} P_{K_r \bigtriangledown \Gamma}(x) &= \frac{x-n-r}{(x-n)(x-r)}((x-n)(x-n-r)^{r-1})P_\Gamma(x-r) \\ &= \frac{x-n-r}{(x-n)(x-r)}((x-n)(x-n-r)^{r-1})(x-r)Q(x-r) \\ &= (x-n-r)^r Q_\Gamma(x-r). \end{aligned}$$

Now Theorem 1.6.5 tells us that $n-r$ divides at least r invariant factors in $\mathrm{SNF}(K_r \bigtriangledown \Gamma)$. Hence the sandpile group of $K_r \bigtriangledown \Gamma$ is not cyclic if $r \geqslant 3$ (note that $n-r$ divides 0 the connectivity eigenvalue). □

Proposition 5.1.6 can be also derived from the following lemma presented in [60].

Lemma 5.1.7. [60, Lemma 2.1] *Let G be a graph of order n. The Laplacian matrix of $K_m \bigtriangledown G$ is equivalent to*

$$I_1 \oplus (m+n)I_{m-2} \oplus ((m+n)I_n - L(\overline{G})) \oplus 0_1.$$

Let Γ be a graph, a *bridge* is an edge that is not contained in any cycle in Γ. If such an edge exists in Γ, then we call Γ a *bridge graph*. Theorem 1.11.4 is about the sandpile group of graphs with an articulation point, and as the bridge graph can be treated as two graphs and K_2 in the middle connecting them by two articulation points. This leads us to the next corollary.

Corollary 5.1.8. *Let Γ be a bridge graph then for any bridge e, the sandpile group of Γ is equal to the sandpile group of Γ/e.*

Proof. Γ can be represented by two separate components Γ_1, Γ_2 and an edge connecting them K_2 (the bridge). Select the vertices $x \in V(\Gamma_1)$, $v_1, v_2 \in V(K_2)$ and $y \in V(\Gamma_2)$, then we have

$$S(((\Gamma_1 \cup K_2)_{\phi(x,v_1)} \cup \Gamma_2)_{\phi(v_2,y)}) \cong S(\Gamma_1) \oplus \mathbb{Z}_1 \oplus S(\Gamma_2) \cong S(\Gamma).$$

□

Let Γ_1, Γ_2 be the graphs defined in Corollary 5.1.8, with sandpile groups $\mathrm{Sp}(\Gamma_1)$, $\mathrm{Sp}(\Gamma_2)$. The sandpile group of Γ is obviously not cyclic if either $\mathrm{Sp}(\Gamma_1)$, $\mathrm{Sp}(\Gamma_2)$ is a non-cyclic group. Otherwise if both are $\mathrm{Sp}(\Gamma_1)$, $\mathrm{Sp}(\Gamma_2)$ are cyclic groups, then by Theorem 1.8.1, $\mathrm{Sp}(\Gamma)$ is cyclic if and only if $\gcd(|\mathrm{Sp}(\Gamma_1)|, |\mathrm{Sp}(\Gamma_2)|) = 1$.

As we discussed earlier in Section 1.11, it is conjectured that the majority of graphs have a cyclic sandpile group. The sandpile group of some classes of planar graphs are known to be cyclic (see for example [41] and [13]).

5.2 A graph family that has a sandpile group of rank between 1 and 2

The graph $K_2 \bigtriangledown P_n$ is an example of a graph family that does not always have a cyclic sandpile group. We will be examining the sandpile group of this graph in this section. This graph has been determined more generally for $K_m \bigtriangledown P_n$ in [60], from which cite the result below.

Theorem 5.2.1. [60, Theorem 1.1] *The sandpile group of $K_m \bigtriangledown P_n$ is isomorphic to*

$$\mathbb{Z}_{\gcd(m+n,\alpha_n,\beta_n)} \oplus \mathbb{Z}_{m+n}^{m-2} \oplus \mathbb{Z}_{((m+n)\alpha_n)/\gcd(m+n,\alpha_n,\beta_n)},$$

where $\alpha_l = (m+2)\alpha_{l-1} - \alpha_{l-2}$, $\alpha_1 = 1$, $\alpha_2 = m+2$ and $\beta_l = (m+2)\beta_{l-1} - \beta_{l-2} + (n-l+1)$, $\beta_1 = 0$, $\beta_2 = n-2$.

The main result we have in this section is Theorem M which is based on Theorem 5.2.1. In Theorem M we present a condition for which the sandpile group of $K_m \bigtriangledown P_n$ is always non-cyclic. The sandpile group of $K_2 \bigtriangledown P_n$ is $\mathbb{Z}_{\gcd(2+n,\alpha_n,\beta_n)} \oplus \mathbb{Z}_{((2+n)\alpha_n)/\gcd(2+n,\alpha_n,\beta_n)}$. Using Lemma 5.2.3, we show that the sandpile group of $K_2 \bigtriangledown P_n$ can be expressed as $\mathbb{Z}_{\gcd(2+n,a_n,a_n)} \oplus \mathbb{Z}_{((2+n)a_n)/\gcd(2+n,a_n,b_n)}$, where $a_l = (m+2)a_{l-1} - a_{l-2}$, $a_0 = 0$, $a_1 = 1$ and $b_l = (m+2)b_{l-1} - b_{l-2} + (l+m-1)$, $b_0 = -1$, $b_1 = 0$. By exploring the properties a_n, b_n, we show that

for $n = m2^t + 2$, $m \geqslant 1$, $t \geqslant 1$, we always get $\gcd(a_n, b_n, n+2) = 4x$, $x \geqslant 1$, which implies that the rank of the sandpile group of $K_2 \bigtriangledown P_n$ is 2.

Theorem M. *Let $n = m2^t + 2$, where $t \geqslant 2$ and m is an odd number, then the rank of sandpile group of $K_2 \bigtriangledown P_n$ is 2.*

Lemma 5.1.7 simplifies the problem of calculating the Smith normal form of $L(K_m \bigtriangledown G)$ to just calculating the Smith normal form of $(m+n)I_n - L(\overline{G})$. And as a special case we have,

Lemma 5.2.2. [60, Lemma 2.2] *The Smith Normal Form of $(m+n)I_n - L(\overline{P_n})$ is equal to*

$$\mathrm{SNF}((m+n)I_n - L(\overline{P_n})) = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{n-2}, \gcd(m+n, \alpha_n, \beta_n), ((m+n)\alpha_n)/\gcd(m+n, \alpha_n, \beta_n)).$$

The proof of Theorem 5.2.1 uses Lemma 5.2.3, which can be obtained using Lemma 5.1.7. However, we provide a more direct and simpler proof below. This proof results in two numbers a_l, b_l that are slightly different from α_l, β_l introduced in Theorem 5.2.1.

Let

$$a_l = (m+2)a_{l-1} - a_{l-2}, a_0 = 0, a_1 = 1,$$

and

$$b_l = (m+2)b_{l-1} - b_{l-2} + (l+m-1), b_0 = -1, b_1 = 0.$$

We can rewrite Lemma 5.2.2 using a_l, b_l instead of α_a, β_l.

Lemma 5.2.3. *The Smith Normal Form of $(m+n)I_n - L(\overline{P_n})$ is equal to*

$$\mathrm{SNF}((m+n)I_n - L(\overline{P_n})) = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{n-2}, \gcd(m+n, a_n, b_n), ((m+n)a_n)/\gcd(m+n, a_n, b_n)).$$

Proof. Let R_i, C_j denote the i-th row, and the j-th column of $(m+n)I_n - L(\overline{G})$ respectively. We apply the following row/column operations.

Eliminate the ones in the upper and lower triangle except next to the diagonal in Rows $R_i, i = 1, 2, \ldots, n-1$.

$R_i \leftarrow R_i + R_{i+1}$, for $i = 1, 2, \ldots, n-1$,

$R_n \leftarrow R_n + R_{n-i}$, for $i = 1, 2, \ldots, n-1$,

Eliminate the values left in the upper triangle and the diagonal.

$C_2 \leftarrow C_2 + (m+3)C_1$,

$C_3 \leftarrow C_3 + (m+3)C_2 - (m+3)C_1$,

$C_i \leftarrow C_i + (m+3)C_{i-1} - (m+3)C_{i-2} + C_{i-3}$, for $i = 4, 5, \ldots, n-1$,

$C_n \leftarrow C_n + C_{n-1} - (m+2)C_{n-2} + C_{n-3}$.

Now we have,

$$\begin{pmatrix} a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ b_2 & b_3 & b_4 & \cdots & b_{n-1} & b_n & n+m \end{pmatrix}.$$

Now, it is easy to clear the values $a_2, a_3, \ldots a_{n-1}, b_2, b_3, \ldots, b_{n-1}$ using the following operations

$$R_1 \leftarrow R_1 + a_i R_i, \text{ for } i = 2, 3, \ldots, n-1,$$

$$R_n \leftarrow R_n + b_i R_i, \text{ for } i = 2, 3, \ldots, n-1.$$

Finally, the SNF of the submatrix $\begin{pmatrix} a_n & 0 \\ b_n & n+m \end{pmatrix}$ is $\operatorname{diag}(\gcd(m+n, a_n, b_n), ((m+n)a_n)/\gcd(m+n, a_n, b_n)))$.

□

Following this, the sandpile group of $K_2 \bigtriangledown P_n$ can be written as $\mathbb{Z}_{\gcd(2+n,a_n,b_n)} \oplus \mathbb{Z}_{((2+n)a_n)/\gcd(2+n,a_n,b_n)}$. By examining the values of a_n and b_n, we can prove a few facts about these numbers by induction.

We notice that a_n is sequence (A001353) of the On-Line Encyclopedia of Integer Sequences (see [51]), and we also cite Propositions 5.2.4, 5.2.5 from the same source, then we add more propositions that serve as a starting point in the process of proving Theorem M.

Proposition 5.2.4. [51, Sequence (A001353)] *For each $n \geqslant 1$, we have*

$$a_n = 3a_{n-1} + 2\sum_{i=1}^{n-2} a_i + 1$$

Proposition 5.2.5. [51, Sequence (A001353)] *For each $n \geqslant 1$, we have*

$$3a_n^2 + 1 = (2a_n - a_{n-1})^2$$

Proposition 5.2.6. *For each $n \geqslant 1$, we have*

$$3a_n - 2b_n = n + 2.$$

Proof. The base case $n = 0$ holds as $3(0) - 2(-1) = 2$.

Suppose that $3a_i - 2b_i = i + 2$, for $i = 1, 2, \ldots, n-1$, then

$$\begin{aligned}
3a_n - 2b_n &= 3(4a_{n-1} - a_{n-2}) - 2(4b_{n-1} - b_{n-2} + n + 1) \\
&= 4(3a_{n-1} - 2b_{n-1}) - (3a_{n-2} - 2b_{n-2}) - 2(n+1) \\
&= 4(n+1) - n - 2(n+1) \\
&= n + 2.
\end{aligned}$$

□

Proposition 5.2.7.

$$b_n = b_{n-1} + a_n + \sum_{i=1}^{n-2} a_i, \ n \geqslant 1$$

Proof. The base case $n = 1$ holds as $b_1 = -1 + 1 = 0$.

Suppose that $b_j = b_{j-1} + a_j + \sum_{i=1}^{j-2} a_i$, for $j = 2, 3, \ldots, n-1$, then

$$\begin{aligned}
b_n &= 4b_{n-1} - b_{n-2} + n + 1 \\
&= 4(b_{n-2} + a_{n-1} + \sum_{i=1}^{n-3} a_i) - (b_{n-3} + a_{n-2} + \sum_{i=1}^{n-4} a_i) + n + 1 \\
&= (4b_{n-2} - b_{n-3} + n) + (4a_{n-1} - a_{n-2}) + \sum_{i=2}^{n-3} (4a_i - a_{i-1}) + 4a_1 + 1 \\
&= b_{n-1} + a_n + \sum_{i=2}^{n-3} a_{i+1} + 4 + 1 \\
&= b_{n-1} + a_n + \sum_{i=3}^{n-2} a_i + a_2 + a_1 \\
&= b_{n-1} + a_n + \sum_{i=1}^{n-2} a_i.
\end{aligned}$$

□

Proposition 5.2.8. *For each $n \geqslant 1$, we have*

$$\gcd(a_n, b_n, n+2) = \gcd(b_n - a_n, n+2).$$

Proof.

$$\begin{aligned}
\gcd(a_n, b_n, n+2) &= \gcd(\gcd(a_n, b_n), n+2) \\
&= \gcd(\gcd(a_n, b_n - a_n), n+2) \\
&= \gcd(\gcd(a_n - 2(b_n - a_n), b_n - a_n), n+2) \\
&\quad \text{(As } 3a_n - 2b_n = n+2 \implies a_n - 2(b_n - a_n) = n+2.\text{)}
\end{aligned}$$

$$= \gcd(\gcd(n+2, b_n - a_n), n+2)$$
$$= \gcd(b_n - a_n, n+2).$$

□

Proposition 5.2.9. *For each $j, k \geqslant 1$,*

$$a_k \mid a_{jk}.$$

Proof. The base case when $j = 1$, $k \geqslant 1$ holds as $a_k \mid a_k$.

Suppose that $a_k \mid a_{(j-1)k}$ so that $a_{(j-1)k} = a_k d$, for some $d \geqslant 1$, then we have

$$\begin{aligned}
a_{jk} &= 4a_{jk-1} - a_{jk-2} \\
&= a_2 a_{jk-1} - a_1 a_{jk-2} \\
&= a_2(4a_{jk-2} - a_{jk-3}) - a_1 a_{jk-2} \\
&= (4a_2 - a_1)a_{jk-2} - a_2 a_{jk-3} \\
&= a_3 a_{jk-2} - a_2 a_{jk-3} \\
&\vdots \\
&= a_{k+1} a_{jk-k} - a_k a_{jk-k-1} \\
&= a_k(a_{k+1} d - a_{jk-k-1}).
\end{aligned}$$

So a_k divides a_{jk} as required. □

The divisibility of $b_n - a_n$ is very important to explore as it is desirable to understand when $K_2 \bigtriangledown P_n$ is cyclic. Now define the sequence $c_l = 4c_{l-1} - c_{l-2} + n + 1$, $c_0 = -1$, $c_1 = -1$, $n \geqslant 1$, which is equivalent to $b_l - a_l$. And let $d_n = 4d_{n-1} - d_{n-2} + 2, d_2 = 1, d_3 = 6$ which we can see easily that is equivalent to $d_n = c_n - c_{n-2}$.

Proposition 5.2.10. *For each $k \geqslant 1$, we have*

$$a_{2k} = a_{k+1} a_k - a_k a_{k-1}.$$

Proof.

$$\begin{aligned}
a_{2k} &= 4a_{2k-1} - a_{2k-2} \\
&= (4)a_{2k-1} - (1)a_{2k-2} \\
&= a_2 a_{2k-1} - a_1 a_{2k-2} \\
&= a_2(4a_{2k-2} - a_{2k-3}) - a_1 a_{2k-2} \\
&= (4a_2 - a_1)a_{2k-2} - a_2 a_{2k-3}
\end{aligned}$$

$$= a_3 a_{2k-2} - a_2 a_{2k-3}$$

$$\vdots$$

$$= a_{k+1} a_k - a_k a_{k-l}.$$

□

Proposition 5.2.11. *For each $k \geqslant 1$, we have*

$$d_k = a_k - 2a_{k-1} - 1.$$

Proof.

$$\begin{aligned}
d_k &= 4d_{k-1} - d_{k-2} + 2 \\
&= a_2 d_{k-1} - a_1 d_{k-2} + 2a_1 \\
&= a_3 d_{k-2} - a_2 d_{k-3} + 2\sum_{i=1}^{2} a_i \\
&\vdots \\
&= a_l d_{k-l+1} - a_{l-1} d_{k-l} + 2\sum_{i=1}^{l-1} a_i \\
&= a_{k-2} d_3 - a_{k-3} d_2 + 2\sum_{i=1}^{k-3} a_i \quad (\text{as } l = k-2) \\
&= 6a_{k-2} - a_{k-3} + 2\sum_{i=1}^{k-3} a_i \\
&= 4a_{k-2} - a_{k-3} + 2a_{k-2} + 2\sum_{i=1}^{k-3} a_i \\
&= a_{k-1} + 2\sum_{i=1}^{k-2} a_i \\
&= a_k - 2a_{k-1} - 1 \quad (2\sum_{i=1}^{k-2} a_i = a_k - 3a_{k-1} - 1, \text{ by Proposition 5.2.4})
\end{aligned}$$

□

Proposition 5.2.12. *For each $k \geqslant 1$, we have*

$$d_{k+1} - d_k = a_k + a_{k-1}.$$

Proof. By Proposition 5.2.11

$$d_{k+1} - d_k = (a_{k+1} - 2a_k - 1) - (a_k - 2a_{k-1} - 1) = a_{k+1} - 3a_k + 2a_{k-1} = a_k + a_{k-1}.$$

□

Proposition 5.2.13. *For each $k \geqslant 1$, we have*

$$d_{2k} = (d_{k+1} - d_k)^2, \text{ and } d_{k+1} - d_k \text{ is an odd number.}$$

Proof.

$$\begin{aligned}
d_{2k} &= a_{2k} - 2a_{2k-1} - 1 \text{ (Proposition 5.2.11)} \\
&= a_{k+1}a_k - a_k a_{k-1} - 2(a_k a_k - a_{k-1}a_{k-1}) - 1 \text{ (Proposition 5.2.10)} \\
&= a_{k+1}a_k - 2a_k a_k + 2a_{k-1}a_{k-1} - a_k a_{k-1} - 1 \\
&= a_k(a_{k+1} - 2a_k) + a_{k-1}(2a_{k-1} - a_k) - 1 \\
&= a_k(2a_k - a_{k-1}) + a_{k-1}(2a_{k-1} - a_k) - 1 \\
&= 2a_k a_k - a_k a_{k-1} + 2a_{k-1}a_{k-1} - a_{k-1}a_k - 1 \\
&= 2{a_k}^2 - 2a_k a_{k-1} + 2{a_{k-1}}^2 - 1 \\
&= 2{a_k}^2 - 2a_k a_{k-1} + 2{a_{k-1}}^2 - ({a_k}^2 - 4a_k a_{k-1} + {a_{k-1}}^2) \text{ (Proposition 5.2.5)} \\
&= {a_k}^2 + 2a_k a_{k-1} + {a_{k-1}}^2 \\
&= (a_k + a_{k-1})^2 \\
&= (d_{k+1} - d_k)^2 \text{ (Proposition 5.2.12).}
\end{aligned}$$

For the second part, $d_{k+1} - d_k = a_k + a_{k-1}$ (Proposition 5.2.12). As two consecutive numbers has to be an odd (or even) number followed by an even (or odd) number. Suppose without loss of generality that $k = 2t$. Then $a_2 = 4 \mid a_{2t}$ which proves that a_{2t} is an even number. a_{2t-1} can be shown to be odd by induction as the base case $(t = 1)$ is $a_1 = 1$. Assume that $a_{2(t-1)-1}$ is odd, then $a_{2t-1} = 4a_{2(t-1)} - a_{2(t-1)-1}$, which is an odd number. This implies that $a_k + a_{k-1} = d_{k+1} - d_k$ is an odd number. □

Proposition 5.2.14.

$$2^5 \mid d_{4k+1}, \text{ for each } k \geqslant 1.$$

Proof. The base case when $k = 1$ holds as $d_5 = 96$ and $2^5 \mid 96$.

Suppose that $2^5 \mid d_{4(k-1)+1}$ such that $d_{4(k-1)+1} = 2^5 s$, then we have

$$\begin{aligned}
d_{4k+1} &= 4d_{4k} - d_{4k-1} + 2 \\
&= 15d_{4k-1} - 4d_{4k-2} + 10 \\
&= 56d_{4k-2} - 15d_{4k-3} + 40 \\
&= 209d_{4k-3} - 56d_{4k-4} + 152
\end{aligned}$$

$$= 209d_{4(k-1)+1} - 56d_{4(k-1)} + 152$$

$$= 209 \cdot 2^5 s - 2^3(7d_{4(k-1)} + 19)$$

(Proposition 5.2.13 makes it possible to write $d_{4(k-1)}$ as a square of an odd number)

$$= 209 \cdot 2^5 s - 2^3(7(2t+1)^2 - 19)$$

$$= 209 \cdot 2^5 s - 2^3(7(4t^2 + 4t + 1) - 19)$$

$$= 209 \cdot 2^5 s - 2^3(28t^2 + 28t - 12)$$

$$= 209 \cdot 2^5(s - (7t^2 + 7t - 3))$$

□

The following lemma will be useful on the divisibility of c_l by 2 when $l = n$.

Lemma 5.2.15. *Let $p = 2^i(2s+1)$, $q = 2^j(2t+1)$ then, if $i = j$, $2^{i+1} \mid (p+q)$. Otherwise $2^{\min(i,j)} \mid (p+q)$. Note that $\min(i,j)$ is the 2-adic order that divides $p+q$.*

Proof. If $i = j$, then $p + q = 2^i(2s+1) + 2^i(2t+1) = 2^{i+1}(s+t+1)$. Otherwise, suppose without loss of generality that $i < j$ then, $p + q = 2^i(2s+1) + 2^j(2t+1) = 2^i(2s + 1 + 2^{j-i+1}t + 2^{j-i})$. It is clear that $(2s + 1 + 2^{j-i+1}t + 2^{j-i})$ is an odd number. □

Notice that as $d_5 = 96 = 2^5 \cdot 3$ and $d_9 = 18816 = 2^7 \cdot 147$, this implies that subsequent values of d_{8k-3} which is equal to $d_{8k-3} = 209d_{8(k-1)+1} - 56d_{8(k-1)} + 152$ will be maximally divisible by 2^5 (Lemma 5.2.15) as 2^5 is the highest power of 2 that divides $56d_{8(k-1)} + 152$. On the other hand $d_{8k+1} = 209d_{8(k-1)-3} - 56d_{8(k-1)-4} + 152$ will be at least divisible by 2^6 (Lemma 5.2.15) as both $209d_{8(k-1)-3}$ and $56d_{8(k-1)-4} + 152$ are maximally divisible by 2^5.

Proposition 5.2.16. *Let $n = 2^t + 2$, where $t \geqslant 2$, then $2^{t-1} \mid c_n$.*

Proof.

$$c_{2^t+2} = 4c_{2^t+1} - c_{2^t} + 2^t + 3$$

$$= 4(c_{2^t+1} - c_{2^t-1}) + c_{2^t-2} + 2$$

$$= 4(c_{2^t+1} - c_{2^t-1} + c_{2^t-3}) - c_{2^t-4} + 2^t + 1$$

$$= 4((c_{2^t+1} - c_{2^t-1}) + (c_{2^t-3} - c_{2^t-5})) + c_{2^t-6} + 2^2$$

$\vdots$ (after 2^{t-1} steps similar to above)

$$= 4((c_{2^t+1} - c_{2^t-1}) + (c_{2^t-3} - c_{2^t-5}) + \cdots + (c_{2^t-2^{t-1}+3} - c_{2^t-2^{t-1}+1})) + c_{2^t-2^t+2} + 2^{t-1}$$

$$= 4((c_{2^t+1} - c_{2^t-1}) + (c_{2^t-3} - c_{2^t-5}) + \cdots + (c_5 - c_3)) + c_2 + 2^{t-1}$$

$$= 4(d_{2^t+1} + d_{2^t-3} + \cdots + d_5) + c_2 + 2^{t-1}$$

$$= 4(d_{2^t+1} + d_{2^t-3} + \cdots + d_5) + 0 + 2^{t-1}$$

$$= 4(d_{2^t+1} + d_{2^t-3} + \cdots + d_5) + 2^{t-1}$$

$$= 4(2^6 s_1 + 2^5 s_2 + 2^6 s_3 + \cdots + 2^6 s_{2^{t-1}-1} + 2^5 s_{2^{t-1}}) + 2^{t-1} \text{ (Proposition 5.2.14)}$$

$$= 4 \cdot 2^{t+3}(s_1 + s_2 + s_3 + \cdots + s_{2^{t-1}-1} + s_{2^{t-1}}) + 2^{t-1} \text{ (applying recursively Lemma 5.2.15)}$$

$$= 2^{t-1}(64(s_1 + s_2 + s_3 + \cdots + s_{2^{t-1}-1} + s_{2^{t-1}}) + 1)$$

□

Corollary 5.2.17. *Let $n = m2^t + 2$, where $t \geqslant 2$ and m is an odd number, then $2^{t-1} \mid c_n$*

Proof. The base case when $m = 1$, $n = 2^t + 2$ holds by Proposition 5.2.16.

Suppose that $2^{t-1} \mid c_{(m-2)2^t+2}$, then we have

$$c_{m2^t+2} = 4c_{m2^t+1} - c_{m2^t} + m2^t + 3$$

$$= 4(c_{m2^t+1} - c_{m2^t-1}) + c_{m2^t-2} + 2$$

$$= 4(c_{m2^t+1} - c_{m2^t-1} + c_{m2^t-3}) - c_{m2^t-4} + m2^t + 1$$

$$= 4((c_{m2^t+1} - c_{m2^t-1}) + (c_{m2^t-3} - c_{m2^t-5})) + c_{m2^t-6} + 2^2$$

$\vdots$ (after 2^{t-1} steps similar to above)

$$= 4((c_{m2^t+1} - c_{m2^t-1}) + (c_{m2^t-3} - c_{m2^t-5}) + \cdots + (c_{m2^t-2^{t-1}+3} - c_{m2^t-2^{t-1}+1})) + c_{m2^t+2-2^{t+1}} + 2^{t-1}$$

$$= 4(d_{m2^t+1} + d_{m2^t-3} + \cdots + d_{m2^t-2^{t-1}+3}) + c_{m2^t+2-2^{t+1}} + 2^{t-1}$$

$$= 4(d_{m2^t+1} + d_{m2^t-3} + \cdots + d_{m2^t-2^{t-1}+3}) + c_{2^t(m-2)+2} + 2^{t-1}$$

$$= 4(2^6 s_1 + 2^5 s_2 + 2^6 s_3 + \cdots + 2^6 s_{2^{t-1}-1} + 2^5 s_{2^{t-1}}) + 2^{t-1}u + 2^{t-1} \text{ (Proposition 5.2.14)}$$

$$= 4 \cdot 2^{t+3}(s_1 + s_2 + s_3 + \cdots + s_{2^{t-1}-1} + s_{2^{t-1}}) + 2^{t-1}u + 2^{t-1} \text{ (applying recursively Lemma 5.2.15)}$$

$$= 2^{t-1}(64(s_1 + s_2 + s_3 + \cdots + s_{2^{t-1}-1} + s_{2^{t-1}}) + u + 1)$$

□

Now we get to prove Theorem M.

Proof of Theorem M. We can clearly see that the sandpile group of $K_2 \bigtriangledown P_n$ is non cyclic when $n = m2^t + 2$ as $\gcd(a_n, b_n, n+2) = \gcd(b_n - a_n, n+2) = 4x$. This follows from Theorem 5.2.1 and Corollary 5.2.17 as it implies that $2^{t-1} \mid \gcd(2+n, a_n, b_n)$. □

There are other cases of the sandpile group of $K_2 \bigtriangledown P_n$ being non cyclic other than what we have in Theorem M. Our computations show that the sandpile group of $K_2 \bigtriangledown P_n$ is non-cyclic group when $n = 15r + 3$, $1 \leqslant r \leqslant 6$.

If $\gcd(2+n, a_n, b_n) = 1$, then the sandpile group $K_2 \bigtriangledown P_n$ is $\mathbb{Z}_{\gcd(2+n,a_n,a_n)} \oplus \mathbb{Z}_{((2+n)a_n)/\gcd(2+n,a_n,b_n)}$ which is equal to $\mathbb{Z}_{(2+n)a_n}$, a cyclic group. Determining when the sandpile group of $K_2 \bigtriangledown P_n$ is cyclic has proven

to be a lot harder. Our computations show, $K_2 \bigtriangledown P_n$ is cyclic when $n = 6k+1,\ n < 55$, and $n = 6k-1,\ n < 185$. Also, when $n = 2^i - 2$, the sandpile group of $K_2 \bigtriangledown P_n$ is cyclic up to $i < 24$.

The only remaining possibility which seems to be hard to prove is the following,

Conjecture 5.2.18. *If $n = p$ or $n = q - 2$, where p and q are prime numbers, then the sandpile group of $K_2 \bigtriangledown P_n$ is cyclic.*

It is conjectured that many recurrence sequences of second order including the Fibonacci and Lucas sequences, should have infinitely many prime numbers (See for example [31, Page 17]).

5.3 Future work

There are a few graphs that we could explore. For example the graph in Figure 5.1 is a wheel graph with fewer edges connecting the centre compared to $K_1 \bigtriangledown C_{17}$, this graph has a cyclic sandpile group unlike the wheel graph $K_1 \bigtriangledown C_{17}$ which has a sandpile group of rank 2. This can be generalised. Another example which seems interesting is the graph in Figure 5.2. However, we do not have a way yet of generalising this graph while maintaining the cyclic property of its sandpile group.

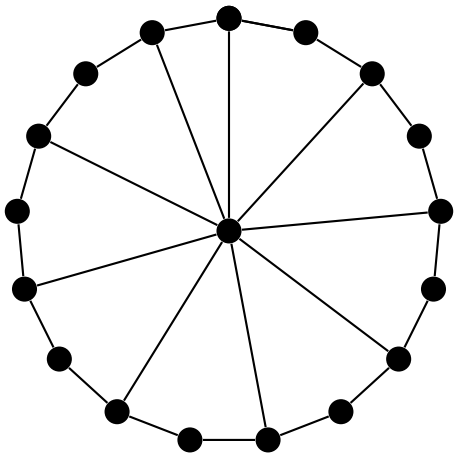

Figure 5.1: A graph with cyclic sandpile group of order 99905.

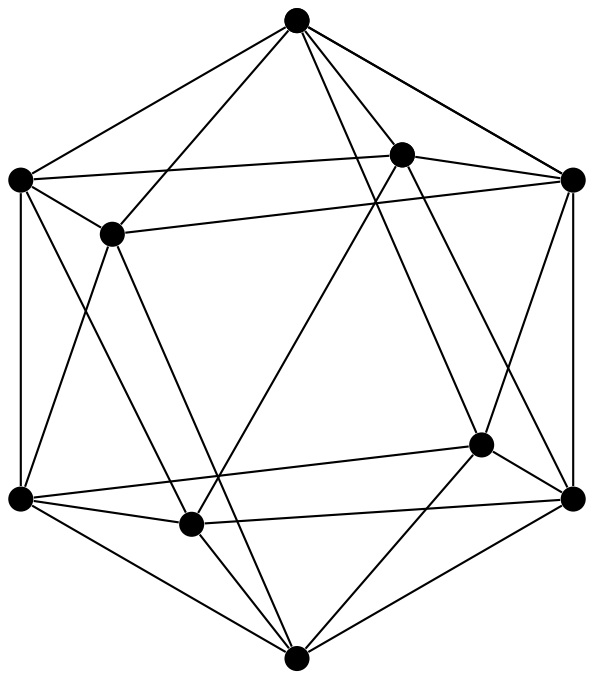

Figure 5.2: A 5-regular graph with cyclic sandpile group of order 336072.

Chapter 6

Computational Methods applied to the Smith and Sandpile Groups

This chapter introduces Maple code and examples that we have used to conjecture and prove many of the ideas presented in this thesis.

6.1 Introduction

Maple is a software that allows mathematicians to write programs to compute a variety of mathematical objects. For our research, we mainly use `GraphTheory` and `LinearAlgebra` packages.

Throughout this chapter, we explain how we have used Maple to investigate the Smith and sandpile groups and search for interesting graph classes. In order to find these graphs and calculate their Smith & sandpile groups, we mainly perform the following steps

1. Enumerate a subset of graphs to identify and parameterise a class of graphs to study.
2. Select candidate classes and see how far we can generalise them, then write a procedure to construct these graphs and prepare their adjacency and Laplacian matrices.
3. Calculate the Smith normal form, eigenvalues and other properties we are interested in of these matrices.
4. Present the results alongside a drawing of the graph selected when it is useful.

6.2 Graph construction in Maple

A graph can be constructed in Maple using a set of edges, where each edge is a set of two elements representing the two vertices. We can also use a square binary matrix (the adjacency matrix) to create a graph object, which is what we tend to use in this chapter as it is easier to work with matrices.

For example, to define a cycle power graph (Definition 1.2.10) we can define the following procedure,

Cycle power adjacency matrix procedure

```
CyclePowerGraphAdjacencyMatrix := proc (m, p)
    local i, u, A;
    description "returns the adjacency matrix for cycle power graph.";
    u := ZeroVector[row](m);
    for i from 2 to p+1 do
        u[i] := 1
    end do;
    for i from m-p+1 to m do
        u[i] := 1
    end do;
    A := Matrix(m, shape = Circulant[u]);
    return A
end proc
```

Then we create a graph as follows

Cycle power graph of order 6

```
G := Graph(CyclePowerGraphAdjacencyMatrix(6, 2));
```

Of course Maple provides a list of graph procedures that allows us to use them to use off the shelf without needing to define a procedure like the example above. The list of graphs are available in `SpecialGraphs` package. For example, the following code calculates and prints the sandpile group of Kneser graphs as we did in Chapter 3, and we can see in Figure 6.1 the output of the case when $t = 5$. Note that $K(5, 2)$ is the Petersen graph.

Kneser graphs $K(n, 2)$ sandpile group calculation

```
for i from 5 to t do
    G:= KneserGraph(i, 2):
    PrintSandpileGroup(sprintf("Graph #%d (order = %d)",
                                i-4, i*(i-1)/2), G)
end do;
```

$$\textit{Graph \#1 (order = 10):}$$

$$\textit{Order} = 10\,,\ \textit{Degrees} = \begin{bmatrix} 3 \\ 10 \end{bmatrix},$$

$$\textit{Laplacian Eigenvalues} = \begin{bmatrix} 0 & 5 & 2 \\ 1 & 4 & 5 \end{bmatrix},\ \textit{integral} = \textit{true},$$

$$\textit{Sandpile Group} = \begin{bmatrix} 1 & 2 & 10 \\ 5 & 1 & 3 \end{bmatrix},\ \textit{Divisibility} = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$$

Figure 6.1: `PrintSandpileGroup` output for the Petersen graph.

6.3 Calculating the Smith normal form and other graph properties

In order to present the Smith normal form and other graph properties in a simple way, we developed a set of procedures that allows that to happen. The first procedure is `ValueMultiplicity`, which takes a vector and returns a matrix of each entry on the first row and the number of occurrences of that entry beneath it on the second row.

ValueMultiplicity procedure

```
ValueMultiplicity := proc (V)
    local VUnique, VMultiplicity, VSpec, t, v;
    description "View a vector as unique value multiplicity matrix";
    VUnique := MakeUnique(convert(V, list));
    VMultiplicity := convert(Vector(1 .. NumElems(VUnique)), list);
    t := 1;
    for v in VUnique do
        VMultiplicity[t] := Occurrences(v, convert(V, list));
        t := t+1
    end do;
    VSpec := convert([VUnique, VMultiplicity], Matrix);
    return VSpec
end proc
```

Now we can calculate the Smith normal form of a matrix and represent the diagonal as a value multiplicity pair. `GroupOnMatrixInvariants` procedure uses `SmithForm` procedure to calculate the Smith normal form as a matrix, it then returns the diagonal as a vector.

GroupOnMatrixInvariants procedure

```
GroupOnMatrixInvariants := proc (L)
    local m;
    description "The group on matrix invariants";
    m := RowDimension(L);
    return LinearAlgebra[Transpose](Diagonal(SmithForm(L)))
end proc;
```

We can now create another procedure `SmithGroup` to calculate the Smith normal form from the adjacency matrix of its input graph and represent it as a value multiplicity matrix.

SmithGroup procedure

```
SmithGroup := proc (G)
    description "The Smith group of a graph G";
    return ValueMultiplicity(GroupOnMatrixInvariants(AdjacencyMatrix(G)))
end proc
```

Similarly, for the sandpile group, `SandpileGroupInvariants` uses `GroupOnMatrixInvariants` procedure to calculate the Smith normal form as a diagonal vector, it then removes the last element as it is 0 given that we are working with a Laplacian matrix.

SandpileGroupInvariants procedure

```
SandpileGroupInvariants := proc (L)
    local m;
    description "The sandpile group invariants";
    m := RowDimension(L);
    return SubVector(GroupOnMatrixInvariants(L), [1 .. m-1])
end proc;
```

Similar to `SmithGroup`, we create `SandpileGroup` procedure to work against the Laplacian matrix of its input graph.

SandpileGroup procedure

```
SandpileGroup := proc (G)
    description "the sandpile group of a graph G";
    return ValueMultiplicity(SandpileGroupInvariants(LaplacianMatrix(G)))
end proc;
```

Now we can calculate the Smith group and various properties of a graph using the following procedure

PrintSmithGroup procedure

```
PrintSmithGroup := proc (title, G)
    local D, n, nST, v, iv, e, EV, SP, SPu, SPd, m, t;
    description "print the Smith group";
    D := ValueMultiplicity(DegreeSequence(G));
    n := NumberOfVertices(G);
    v := Eigenvalues(AdjacencyMatrix(G));
    iv := IsIntegralVector(v);
    if not iv then
        v := evalf(v)
    end if;
    EV := ValueMultiplicity(v);
    SP := SmithGroup(G);
    SPd := DivisibilityView(SP[1, 1 .. ()]);
    t := LineSeparatorString("_", 80);
    FormatPrint(
        cat("%s:\nOrder = %d, ",
            "Degrees = %a,\nAdjacency Eigenvalues = %a, ",
            "integral = %a,\nSmith Group = %a, Divisibility=%a\n%s"
        ),
        title, n, D, EV, iv, SP, SPd, t
    )
end proc
```

Here we use use `ValueMultiplicity` to display the degrees and adjacency matrix eigenvalues as value multiplicity matrices. `PrintSandpileGroup` is an analog procedure to `PrintSmithGroup` but for the Laplacian matrix.

PrintSandpileGroup procedure

```
PrintSandpileGroup := proc (title, G)
    local D, n, v, iv, EV, SP, SPu, SPd, m, t;
    description "print the sandpile group";
    D := ValueMultiplicity(DegreeSequence(G));
    n := NumberOfVertices(G);
    v := Eigenvalues(LaplacianMatrix(G));
    iv := IsIntegralVector(v);
    if not iv then
        v := evalf(v)
    end if;
    EV := ValueMultiplicity(v);
    SP := SandpileGroup(G);
    SPd := DivisibilityView(SP[1, 1 .. ()]);
    t := LineSeparatorString("_", 80);
    FormatPrint(
        cat("%s:\nOrder = %d, ",
            "Degrees = %a,\nLaplacian Eigenvalues = %a, ",
            "integral = %a,\nSandpile Group = %a, Divisibility=%a\n%s"
        ),
        title, n, D, EV, iv, SP, SPd, t
    )
end proc;
```

Notice that we call a few other procedures `DivisibilityView`, `IsIntegralVector`, `LineSeparatorString`, and `FormatPrint` which we explain now. `DivisibilityView` is mainly used to evaluate a_i/a_{i+1}, for a vector $(a_1, a_2, \ldots, a_n)$, $i \in \{1, 2, \ldots, n-1\}$. We mainly use this for the unique elements of the Smith normal form diagonal.

DivisibilityView procedure

```
DivisibilityView := proc (v)
    local n;
    description "As v elements satisfy v_i = d_i v_{i+1}, we return d_i";
    n := Dimension(v);
    if n < 1 then
        return Vector([])
    else
        return Concatenate(
                2,
                Vector([v[1]]),
                ElementDivide(
                    convert(SubVector(Vector(v), [2 .. n]), Array),
                    convert(v[1 .. n-1], Array)
                )
            )
    end if
end proc;
```

`IsIntegralVector` enables us to test if a given vector is integral or not, which we mainly use to test if a graph is integral or not. Note that the integrality of a vector is based on Maple recognising each entry as of type integer. Of course, in a mathematical context, we would only consider a vector integral if we can prove it to be the case.

IsIntegralVector procedure

```
IsIntegralVector := proc (v)
    local t, i;
    t := Dimension(v);
    for i to t do
        if not type(v[i], integer) then
            return false
        end if
    end do;
    return true
end proc
```

FormatPrint makes it possible to print out objects similar to printf procedure in Maple while maintaining the view we get when using print procedure. It uses FormatString which is mostly the code in pprintf procedure that is available on MaplePrime website (see [46]).

FormatPrint procedure

```
FormatPrint := proc (S::string)
    return print(FormatString(S, args[2 .. -1]))
end proc;
```

FormatString procedure

```
FormatString := proc (S::string)
    local X, L, R, n, i, j, k;
    L := [StringTools:-RegSplit("%a", S)];
    X := [args[2 .. -1]];
    R := NULL;
    j := 1;
    for i to nops(L) do
        n := StringTools:-CountCharacterOccurrences(L[i], "%");
        R := R,Typesetting:-mi(sprintf(L[i], seq(X[k], k = j .. j+n-1)));
        j := j+n;
        if j <= nops(X) then
            R := R, op(map(Typesetting:-Typeset, [X[j]]));
            j := j+1
        end if
    end do;
    Typesetting:-mrow(R)
end proc;
```

LineSeparatorString is a simple procedure which produces a line separator given a character and a length n by calling the string procedure cat n times.

6.4 Graph enumeration

Maple provides the NonIsomorphicGraphs procedure to enumerate graphs, this can be used practically for generic graph search of small order. For example, the following code prints out the Laplacian matrix, the sandpile group and a drawing of graphs that are connected with no leaves and have non-cyclic sandpile groups. It makes

sense to eliminate graphs with leaves as they have the same sandpile group as the graph we have when we remove all leaves (this is implied by Theorem 1.11.4). However, we can also use the same example code below with `IsCyclic(SP)` as a filter rather than `not IsCyclic(SP)` to explore cyclic graphs as well.

Enumeration of connected graphs with no leaves and having non-cyclic sandpile group

```
for i from 3 to t do
    Lp := ([NonIsomorphicGraphs])(i,
    output = graphs,
    outputform = adjacency,
    restrictto = connected);
    s := ColumnDimension(Lp);
    satisfy_property_count := 0;
    for j to s do
        G := Graph(Lp[j]);
        deg := DegreeSequence(G);
        SP := SandpileGroup(G);
        if and(not(has(deg, 1)), not IsCyclic(SP)) then
            print(DrawGraph(G, style = spring));
            print(LaplacianMatrix(G));
            PrintSandpileGroup(sprintf("Graph #%d (n = %d)", j, i), G);
            satisfy_property_count := satisfy_property_count+1;
        end if
    end do;
    FormatPrint(
        cat("The number of graphs of order",
            " %d with the conditions above is %d/%d"),
        i,
        satisfy_property_count,
        s
    )
end do
```

Note that we use `IsCyclic` procedure to filter out graphs with cyclic sandpile group.

IsCyclic procedure

```
IsCyclic := proc (VM)
    local i, n;
    description "Is a group cyclic given its SNF muliplicity value?";
    n := Dimension(VM[1, 1 .. ()]);
    if 3 <= n then
        return false
    elif n = 2 then
        return has(VM[1, 1 .. ()], 1) and VM[2, 2] = 1
    else
        return VM[1, 1] = 1 or VM[2, 1] = 1
    end if
end proc;
```

We can define other filters such as `IsFinite`, and `IsTrivial` to test groups in general. These are more applicable to the Smith group as it is well understood that the sandpile group of a graph Γ is finite if and only if Γ is connected, and trivial if and only if Γ is a tree.

IsFinite procedure

```
IsFinite := proc (VM)
    description "Is a group finite given its SNF muliplicity value?";
    return not has(VM[1, 1 .. ()], 0)
end proc;
```

IsTrivial procedure

```
IsTrivial := proc (VM)
    description "Is a group trivial given SNF its muliplicity value?";
    return VM[1, 1] = 1 and Dimension(VM[1, 1 .. ()]) = 1
end proc;
```

`NonIsomorphicGraphs` allows enumeration of graphs of a certain order with a few restriction options such as being connected, regular and the number of edges in these graphs to name a few options as a filtering mechanism. To make sure we are not limited by the list of available filters available in Maple nor with having to wait for the list of the objects we need to explore to be ready before we could enumerate a subset of these graphs, our solution is to develop an alternative set of enumeration procedures. We can define a graph Γ using a binary vector that represents the entries of the upper triangle entries of $A(\Gamma)$ excluding the diagonal. More precisely,

suppose the entries of $A(\Gamma)$ are $a_{i,j}$, $i, j \in \{0, 1, \ldots, n-1\}, n = |V(\Gamma)|$, then we can use a vector v to assign the graph values to $a_{i,j}$ as follows

$$a_{i,j} = a_{j,i} = v_{ni+j+1}, \quad \text{for all } i \leqslant j.$$

In order to simplify creating the vector v, we can convert from a number to a v and vice versa using the procedures `BinaryVectorFromNumber`, and `NumberFromBinaryVector`, and `NextBinaryVector`. `BinaryVectorFromNumber` takes a number u and optionally and integer d that specifies the minimum vector dimension required and converts u to a binary vector v. Note that the smallest digit will be stored at $v[1]$. `NumberFromBinaryVector` does the converse operation to `BinaryVectorFromNumber`. `NextBinaryVector` procedure helps us to increment a binary vector to save computation time.

MultipartiteGraphFromBinaryVector procedure

```
MultipartiteGraphFromBinaryVector := proc (p, v)
    local t, s, b, e, n, i, j, k, c, A;
    description "Defines a multipartite graph";
    t := Dimension(p); n := 0;
    for i to t do
        n := n+p[i]
    end do;
    A := Matrix(n); s := Dimension(v);
    b := 1; e := 0; i := 1;
    for j to t do e := e+p[j];
        for k from b to e do
            for c from e+1 to n do
                if i <= s then
                    A(k, c) := v[i]; A(c, k) := v[i]; i := i+1
                end if
            end do
        end do;
        b := e+1
    end do;
    return Graph(A)
end proc;
```

In the procedure above we construct a multipartite graph using a vector p that defines the each partition size in some order, and another vector v that contains the entries $a_{ij}, j > i$ that can be set to 1 without violating the

conditions for multipartite graph. In order to be able to enumerate all multipartite graphs of a particular partitions structure we need the following procedure.

MultipartiteGraphBinaryVectorSize procedure

```
MultipartiteGraphBinaryVectorSize := proc (p)
    local x, n, t, i;
    t := Dimension(p);
    x := 0;
    n := 0;
    for i to t do
        x := x+p[i]^2;
        n := n+p[i]
    end do;
    return floor((1/2)*n^2-(1/2)*x)
end proc
```

Now as an example we use the following code to enumerate all bipartite graphs of order 2×2. This can also produce graphs that are isomorphic to other graphs of the same order.

Enumeration of some bipartite graphs

```
n := 2;
m := 2;
v := Vector([n, m]);
t := MultipartiteGraphBinaryVectorSize(v);
b := Vector(t);
for i from 0 to 2^t-1 do
    G := MultipartiteGraphFromBinaryVector(v, b);
    FormatPrint("%a", AdjacencyMatrix(G));
    PrintSmithGroup(sprintf("Graph #%d", i), G);
    b := NextBinaryVector(b)
end do
```

We can define some procedures to construct graphs using a binary vector for other graph classes as we will see next.

6.4.1 Enumeration of circulant graphs

In Chapters 2 and 3 we discussed various constructions of circulant graphs. This was made possible using another set of enumeration procedures. We can create a circulant graph from a binary vector but with a different interpretation to how we construct multipartite graphs. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_n, S), S = \{s_0, s_1, \ldots, s_{n-1}\}$, then we can use a vector v of order t assign the elements of S as follows,

$$s_i = s_{n-i} = v_i,\ i \in \{0, 1, \ldots, t\}.$$

Note that n, and t are not independent as they need to satisfy the condition $n \geqslant 2t$.

We can use `CirculantGraphFromBinaryVector` procedure for this construction.

CirculantGraphFromBinaryVector procedure

```
CirculantGraphFromBinaryVector := proc (m, v)
    local t, i, u, A;
    description "returns a circulant graph defined by v.";
    t := min(Dimension(v), (1/2)*m);
    u := ZeroVector[row](m);
    for i to t do
        u[i+1] := v[i];
        u[m-i+1] := v[i]
    end do;
    A := Matrix(m, shape = Circulant[u]);
    return Graph(A)
end proc;
```

To enumerate circulant graphs we use `CirculantGraphFromBinaryVector` for even order and odd order graphs separately. This makes sense as each binary vector v can be used to define both circulant graphs of order $2|v|$ and $2|v| + 1$. This enumeration is likely to include graphs that are isomorphic to other graphs of the same order.

Enumeration of circulant graphs of even order

```
B := BinaryVectorFromNumber(0, n);
for i from 0 to 2^n - 1 do
    G := CirculantGraphFromBinaryVector(2 * n, B);
    PrintSandpileGroup(sprintf("Graph #%d", i), G);
    B := NextBinaryVector(B)
end do
```

Enumeration of circulant graphs of odd order

```
B := BinaryVectorFromNumber(0, n);
for i from 0 to 2^n - 1 do
    G := CirculantGraphFromBinaryVector(2 * n + 1, B);
    PrintSandpileGroup(sprintf("Graph #%d", i), G);
    B := NextBinaryVector(B)
end do
```

We can of course calculate the eigenvalues of each circulant graph that we enumerate and use `IsIntegralVector` procedure to determine if it is integral or not. But, Theorem 2.1.5 in Chapter 2 gives us a more efficient approach to achieve this. We start by creating an analog procedure to `CirculantGraphFromBinaryVector` to work with the generating set of a Cayley graph.

CirculantGraphFromGeneratingSet procedure

```
CirculantGraphFromGeneratingSet := proc (n, s)
    local A, u, t, i;
    u := ZeroVector(n);
    t := nops(s);
    for i to t do
        u[s[i]+1] := 1;
        u[n-s[i]+1] := 1
    end do;
    A := Matrix(n, shape = Circulant[u]);
    return Graph(A)
end proc;
```

Given a number n that we would like to explore all integral circulant graphs of that order, we can calculate the divisors $D(n)$ and use the power set of $D(n)\backslash\{n\}$ to enumerate each graph (this can include graphs that

are isomorphic to other graphs of order n if So's conjecture is wrong). And as we are interested in the number of components an integral circulant has we calculate the greatest common divisor of a set using a `GCDofSet` procedure that is easy to define. Now we can enumerate, connected integral circulant graphs of order n. To make calculations more efficient, we calculate the $S(d)$ for each divisor d in advance before starting enumeration of graphs.

Enumeration of connected integral circulant graphs of order n

```
d := divisors(n);
k := nops(d);
ps := powerset(k-1);
psn := nops(ps);
MS := [];
for i to k-1 do
    MS := [op(MS), MultipliersSet(n, d[i])]
end do;
for i to psn do u := {}; s := {};
    for j to nops(ps[i]) do
        u := union(u, MS[ps[i][j]]);
        s := union(s, {d[ps[i][j]]})
    end do;
    if GCDofSet(s) = 1 then
        G := CirculantGraphFromGeneratingSet(n, u);
        PrintSandpileGroup(sprintf("ICG(%d, %a)", n, s), G)
    end if
end do
```

`MultipliersSet` calculates the set $S(d) = \{k \ : \ 1 \leqslant k \leqslant n, \gcd(k,n) = d\}$ for each $d \in D(n)\backslash\{n\}$ which we use to construct the generating set to create an integral circulant graph. Note that we exclude the divisor n as we are not interested in graphs having loops.

MultipliersSet procedure

```
MultipliersSet := proc (n, d)
    local u, i;
    u := {};
    for i from d to n-1 do
        if gcd(n, i) = d then
            u := union(u, {i})
        end if
    end do;
    return u
end proc
```

If we are more concerned with enumerating a set of graphs with specific divisor set for higher n values we can use define a simple `IntegralCirculantGraph` procedure and enumerate through them as follows.

IntegralCirculantGraph procedure

```
IntegralCirculantGraph := proc (n, T)
    local u, i;
    u := {};
    for i to nops(T) do
        u := union(u, MultipliersSet(n, T[i]))
    end do;
    return CirculantGraphFromGeneratingSet(n, u)
end proc
```

`IntegralCirculantGraph` is used next to evaluate the sandpile group of integral circulant graphs of the form $\mathrm{ICG}(p^i, \{1, p^k\})$. See Conjectures 2.3.6, 2.3.7 for more details.

Using IntegralCirculantGraph to test Conjectures 2.3.6, 2.3.7

```
p := 2;
k := 2:
r := 4;
u := {1, p^k};
for i from k + 2 to r do
    n := p^i;
    G := IntegralCirculantGraph(n, u);
    PrintSandpileGroup(sprintf("ICG(%d, %a)", n, u), G)
end do
```

It is worth noting that all the enumeration methods presented in this chapter except `NonIsomorphicGraphs` procedure can produce graphs that are isomorphic to other graphs of the same order. As for integral circulant graphs, if So's conjecture is true (see Conjecture 2.1.8), then we should only get non isomorphic integral circulant graphs when we enumerate them.

In Chapter 3, working with some circulant graphs, it turned out to be useful to define these circulant graphs using a 0/1 string pattern. An example of this is the graph

$$C(r,t) = \mathrm{Cay}(\mathbb{Z}_{rt}, \{1, rt-1\} \cup \{k, k+1, k+2 \mid k \in \{r-1, 2r-1, \ldots, (t-1)r-1\}\}).$$

We define the procedure `StringCirculantGraph` to create such graphs generically.

StringCirculantGraph procedure

```
StringCirculantGraph:=proc(prefix, prefixCount, mid, midCount,
                           suffix, suffixCount)
    local A, n, u, i, t, j;
    description "returns the sandpile for string circulants.";
    n := Length(prefix)*prefixCount
        +Length(mid)*midCount
        +Length(suffix)*suffixCount;
    u := Vector(n);
    i := 1;
    for j to prefixCount do
        for t to Length(prefix) do
            u[i] := parse(prefix[t]);
            i := i+1
        end do
    end do;
    for j to midCount do
        for t to Length(mid) do
            u[i] := parse(mid[t]);
            i := i+1
        end do
    end do;
    for j to suffixCount do
        for t to Length(suffix)
            do u[i] := parse(suffix[t]);
            i := i+1
            end do
        end do;
    A := Matrix(n, shape = Circulant[u]);
    return Graph(A)
end proc;
```

Now we can calculate the sandpile group for some instances of $C(6, n)$.

Example of some C(6, n) graphs

```
prefix := "01"; mid := "000111"; suffix := "0001";
for k from 1 to n do
    G := StringCirculantGraph(prefix, 1, mid, k, suffix, 1);
    PrintSandpileGroup(sprintf("Graph #%d", k), G)
end do
```

Theorem 2.4.2 in Chapter 2 on the number of connected integral circulant graphs, was formulated based on calculations of the number of lunar partition sums, which is based on lunar arithmetic. Here we present the procedures we used to understand this result.

NumberOfLunarPartitionSums procedure

```
NumberOfLunarPartitionSums := proc (v)
    local vdim, pps, c2, z, sgn, x, y;
    vdim := Dimension(v);
    pps := powerset(vdim);
    c2 := 0;
    for x to nops(pps) do
        sgn := (-1)^(mod(nops(pps[x]), 2));
        z := 1;
        for y to vdim do
            if y in pps[x] then
                z := z*v[y]
            else
                z := z*(v[y]+1)
            end if
        end do;
        c2 := c2+sgn*2^z
    end do;
    c2 := (1/2)*c2;
    return c2
end proc
```

Here, we do not need to construct an integral circulant graph but only work with the divisors of n and count the generating divisor subsets that are coprime.

NumberOfLunarPartitionSums on the number of connected integral circulant graphs (Theorem 2.4.2)

```
m := Matrix([[1, 2], [5, 3]]);
mdim := RowDimension(m);
n := 1;
for i to mdim do
    n := n*ithprime(m(i, 1))^m(i, 2)
end do;
d := divisors(n);
k := nops(d);
ps := powerset(k-1);
psn := nops(ps);
c := 0;
for i to psn do
    u := {};
    s := {};
    for j to nops(ps[i]) do
        s := union(s, {d[ps[i][j]]})
    end do;
    if GCDofSet(s) = 1 then
        c := c+1
    end if
end do;
FormatPrint(
    "For integral graphs of order %d with prime factorization %a.",
    n,
    ifactor(n)
);
FormatPrint(
    "The number of connected integral circulant graphs are %d.",
    c
);
FormatPrint(
    "The number of lunar partition sums for prime powers of n is %d.",
    NumberOfLunarPartitionSums(m(() .. (), 2))
)
```

In case we need to check a lunar sum we have the following procedure.

LunarSum procedure

```
LunarSum := proc (v, u)
    local i, k, m;
    k := min(Dimension(v), Dimension(u));
    if Dimension(u) < Dimension(v) then
        m := LinearAlgebra[Copy](v)
    else
        m := LinearAlgebra[Copy](u)
    end if;
    for i to k do
        m[i] := max(v[i], u[i])
    end do;
    return m
end proc;
```

6.5 Strongly walk regular graph test

In Chapter 3, we briefly discussed strongly walk regular graphs (Definition 1.4.7). In this section, a procedure that tries to calculate the constants a_l, c_l, and k_l of a potentially strongly walk regular graph is presented. We use Theorem 6.5.1 to develop a test procedure for strongly walk regular graphs.

Theorem 6.5.1. [23, Lemma 2.1] *Suppose we have a graph* Γ*, and let* $n = |V(\Gamma)|$*, and* A *be the adjacency matrix of* Γ*, then* Γ *is a strongly* l*-walk regular graph* $\mathrm{swrg}_l(a_l, c_l, k_l)$*,* $l > 0$ *if and only if,*

$$A^l + (c_l - a_l)A + (c_l - k_l)I = c_l J,$$

where I*,* J *are the* $n \times n$ *identity and all ones matrices respectively.*

`StronglyWalkRegularParameters` accepts a graph and a number t that specifies the maximum walk length we would like to run the test for. The test works by testing if the powers of A satisfies the equation in-Theorem 6.5.1. Once it is satisfied, `StronglyWalkRegularParameters` returns the vector (a_l, c_l, k_l, l), otherwise, as we are only testing t times, the vector returned will be $(0, 0, 0, t)$.

StronglyWalkRegularParameters procedure

```
StronglyWalkRegularParameters := proc (G, t)
    local A, Ac, Ie, L, U, T, R, P, n, i, j, k, r, B, X;
    description "test for up to power t if a graph is walk regular";
    A := AdjacencyMatrix(G); Ac := AdjacencyMatrix(GraphComplement(G));
    n := RowDimension(A); Ie := IdentityMatrix(n);
    T := Vector(3); U := Vector(3); R := Vector(4); P := permute(3, 3);
    B := LinearAlgebra[Copy](A); R[4] := 2;
    for i from 2 to t do
        r := 1; R[4] := i;
        B := Multiply(B, A); L := MakeUnique(convert(B, list));
        if nops(L) < 3 then
            r := 3;
            for j to nops(L) do
                U[j+1] := L[j]
            end do
        end if;
        if nops(L) <= 3 then
            for k to r do
                for j to nops(L) do
                    T[j] := L[j]
                end do;
                if nops(L) < 3 then T[3] := U[k] end if;
                for j to nops(P) do
                    X := T[P[j, 1]]*A+T[P[j, 2]]*Ie+T[P[j, 3]]*Ac;
                    if Equal(B, X) then
                        R[1] := T[P[j, 1]]; R[2] := T[P[j, 2]];
                        R[3] := T[P[j, 3]];
                        return LinearAlgebra:-Transpose(R)
                    end if
                end do
            end do
        end if
    end do;
    return LinearAlgebra:-Transpose(R)
end proc;
```

`StronglyWalkRegularParameters` is a test for strongly walk regular graphs. We use it on the graphs discussed in Chapter 3, and we managed to determine the strongly walk regular graph parameters for the following graphs $\overline{\overline{C_6} \otimes \mathcal{J}_3}$ and a graph that has the adjacency matrix $\begin{pmatrix} A(K_4) & I \\ I & A(K_4) \end{pmatrix}$.

$\mathrm{swrg}_3(39, 38, 18)$

```
k := 3;
t := 6;
C := AdjacencyMatrix(GraphComplement(CycleGraph(t)));
Jn := Matrix(1 .. k, 1 .. k, 1);
G := GraphComplement(Graph(KroneckerProduct(Jn, C)));
print(WalkRegularParameters(G, 15))
```

$\mathrm{swrg}_3(10, 6, 6)$

```
t := 4;
A := AdjacencyMatrix(CompleteGraph(t));
B := LinearAlgebra[Transpose](IdentityMatrix(t));
G := Graph(Matrix([[A, B], [B, A]]));
print(WalkRegularParameters(G, 15))
```

6.6 Testing conjectures on some graph constructions

In Chapter 4, we explore different types of graph constructions, including graph product, graphs with block adjacency matrices as well as edge and vertex tiling constructions.

The sandpile group of the graph product $K_{p,p} \bigtriangledown \overline{K_{q,q}}$, for which we have Conjecture 4.2.3, can be verified using the following code.

Testing Conjecture 4.2.3 on the sandpile group of $K_{p,p} \bigtriangledown \overline{K_{q,q}}$

```
for i from 2 to q do
    H := GraphComplement(CompleteGraph(i, i));
    for j from i+1 to p do
        T := CompleteGraph(j, j);
        G := GraphJoin(H, T);
        PrintSandpileGroup(sprintf("Graph (%d, %d)", i, j), G)
    end do
end do
```

Using a similar verification code, we can test another graph product $C_p \bigtriangledown \overline{K_2}$. We presented Conjecture 1.12.1 in Chapter 1 as an example of a sandpile group with a non trivial recurrence relation.

Testing Conjecture 1.12.1 on the sandpile group of $C_p \bigtriangledown \overline{K_2}$

```
H := GraphComplement(CompleteGraph(2));
for i from 3 to r do
    G := GraphJoin(CycleGraph(i), H);
    PrintSandpileGroup(sprintf("Graph #%d", i), G)
end do
```

The Sierpiński Sieve graph is a graph that is constructed by subdividing edges of triangles and connecting the resulting vertices to create an inner triangle (see Section 4.5 for more details).

We start with SubdivisionMatrix procedure that returns the adjacency matrix after subdivision.

SubdivisionMatrix procedure

```
SubdivisionMatrix := proc (B, v, u)
    local n, A;
    n := RowDimension(B);
    A := Matrix(n+1);
    A(1 .. n, 1 .. n) := LinearAlgebra[Copy](B);
    A(v, u) := 0;
    A(u, v) := 0;
    A(n+1, v) := 1;
    A(v, n+1) := 1;
    A(n+1, u) := 1;
    A(u, n+1) := 1;
    return A
end proc;
```

Now we can define a procedure to recursively construct the Sierpiński Sieve graph adjacency matrix given an adjacency matrix and the vertices defining the target triangle for subdivision operation.

SierpinskiSieveSubdivisionContructionMatrix procedure

```
SierpinskiSieveSubdivisionContructionMatrix := proc (A, u, v, w, k)
    local C, p, q, r;
    C := A;
    if k = 1 then
        return C
    else
        C := SubdivisionMatrix(C, u, v);
        p := RowDimension(C);
        C := SubdivisionMatrix(C, u, w);
        q := RowDimension(C);
        C := SubdivisionMatrix(C, v, w);
        r := RowDimension(C);
        C(p, q) := 1;
        C(q, p) := 1;
        C(p, r) := 1;
        C(r, p) := 1;
        C(q, r) := 1;
        C(r, q) := 1;
        C := SierpinskiSieveSubdivisionContructionMatrix(C,u,p,q,k-1);
        C := SierpinskiSieveSubdivisionContructionMatrix(C,v,p,r,k-1);
        C := SierpinskiSieveSubdivisionContructionMatrix(C,w,q,r,k-1);
        return C
    end if
end proc;
```

Finally, we define `SierpinskiSieveGraph` to create the Sierpiński Sieve graph without needing to pass extra parameters as we do in SierpinskiSieveSubdivisionContructionMatrix procedure.

SierpinskiSieveGraph procedure

```
SierpinskiSieveGraph := proc (c)
    return Graph(SierpinskiSieveSubdivisionContructionMatrix(
                    AdjacencyMatrix(CycleGraph(3)), 1, 2, 3, c))
end proc;
```

Conjecture 4.5.1 is on the sandpile group of the Sierpiński sieve graph, the following code can be used to verify

this conjecture.

Testing Conjecture 4.5.1 on the sandpile group of Sierpiński sieve graph

```
for i to t do
    G := SierpinskiSieveGraph(i);
    print(DrawGraph(G, style = spring));
    PrintSandpileGroup(sprintf("Graph #%d", i), G)
end do
```

Theorem 1.11.4 gives us the ability to calculate the sandpile graph of a graph with an articulation point, this can be thought of as a vertex tiling of two graphs. The resulting sandpile group after a vertex tiling is the product of the sandpile groups of the two graphs we started with before vertex tiling.

Inspired by this, we wanted to understand what happens to the sandpile when we perform vertex tiling to a list of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_i$, then we tile a vertex from Γ_1, to another in Γ_i to form a necklace of graphs (see Section 4.4 for more details). A simple form of this construction is to think of it as starting with a cycle C_i, then edge tile the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_i$ to it. Even though we used vertex tiling procedure to achieve this in the code below, we still get the same necklace of graphs construction.

To start, we need to be able to define a graph as a union of disconnected graphs with duplicates as Maple's `GraphUnion` doesn't allow that. To do this we define `MatrixUnion` procedure to create the adjacency matrix of two adjacency matrices, then we wrap this in `GraphUnionWithDuplicates` to work directly with graph structures.

MatrixUnion procedure

```
MatrixUnion := proc (A, B)
    local n, m, J, C;
    n := RowDimension(A);
    m := RowDimension(B);
    J := Matrix(n, m, fill = 0);
    C := Matrix(n+m);
    C[1 .. n, 1 .. n] := A;
    C[1 .. n, n+1 .. n+m] := J;
    C[n+1 .. n+m, 1 .. n] := LinearAlgebra[Transpose](J);
    C[n+1 .. n+m, n+1 .. n+m] := B;
    return C
end proc;
```

GraphUnionWithDuplicates procedure

```
GraphUnionWithDuplicates := proc (G, H)
    return Graph(MatrixUnion(AdjacencyMatrix(G), AdjacencyMatrix(H)))
end proc;
```

Now we can define the procedure GraphsListJoiningCircle to make a necklace of graphs as desired.

GraphsListJoiningCircle procedure

```
GraphsListJoiningCircle := proc (L)
    local H, n, m, k, i;
    description "returns the graph of graphs Joining as a circle.";
    k := nops(L);
    n := 0;
    for i to k do
        n := n+GraphOrder(L[i])
    end do;
    H := TileTwoVertices(GraphUnionWithDuplicates(L[1], L[2]),
                         1, 1+GraphOrder(L[1])
         );
    m := GraphOrder(H);
    for i from 2 to k-1 do
        H := TileTwoVertices(GraphUnionWithDuplicates(H, L[i+1]),
                             m-i+2, 1+GraphOrder(H)
             );
        m := m+GraphOrder(L[i+1])
    end do;
    H := TileTwoVertices(H, GraphOrder(L[1]), m-k+2);
    return H
end proc;
```

The operations of tiling vertices and calculating the order of a graph are defined below,

TileTwoVertices procedure

```
TileTwoVertices := proc (H, v, u)
    local A;
    A := LinearAlgebra[Copy](AdjacencyMatrix(H));
    A(() .. (), v) := A(() .. (), v)+A(() .. (), u);
    A(v, () .. ()) := A(v, () .. ())+A(u, () .. ());
    A := DeleteRow(A, u);
    A := DeleteColumn(A, u);
    return Graph(AdjacencyMatrix(Graph(A)))
end proc;
```

GraphOrder procedure

```
GraphOrder := proc (G)
    return nops(Vertices(G))
end proc;
```

Now we have a way to calculate the sandpile group of a list of graphs joined as a necklace as in the following example

GraphsListJoiningCircle Example

```
G := GraphsListJoiningCircle(
                    [CycleGraph(3), CycleGraph(4), CycleGraph(5)]);
PrintSandpileGroup(sprintf("Graph #%d", 1), G)
```

In order to simplify creating a necklace of the same graph repeated k times we define

`GraphsJoiningCircle` procedure,

GraphsJoiningCircle Procedure

```
GraphsJoiningCircle := proc (G, k)
    local L, i;
    description "returns the graph of Graphs Joining as a circle graph.";
    L := [];
    for i to k do
        L := [op(L), G]
    end do;
    return GraphsListJoiningCircle(L)
end proc
```

This simplified construction allows us to make two conjecture about the sandpile groups of a necklace of cycles and another of complete graphs (see Conjectures 4.4.3, 4.4.2).

Testing Conjecture 4.4.3 on the sandpile group of a cycle with a cycle graph tiled at each edge

```
C := CycleGraph(5);
G := GraphsJoiningCircle(C, 5);
PrintSandpileGroup(sprintf("Graph #%d", 1), G)
```

Testing Conjecture 4.4.2 on the sandpile group of a cycle with a complete graph tiled at each edge

```
C := CompleteGraph(5);
G := GraphsJoiningCircle(C, 5);
PrintSandpileGroup(sprintf("Graph #%d", 1), G)
```

6.7 Future work

It is desirable to expand our study of the Smith and sandpile groups with more classes of graphs. For example the following code can be used to enumerate more classes of graphs such as regular graphs that are non-planar and have cyclic sandpile groups. This gives us interesting examples such as the graph in Figure 5.2. We hope to find a generalised family of graphs that shares similar properties to study further.

Exploring small graphs that has cyclic sandpile groups

```
t := 10;
for i from 7 to t do
    Lp := ([NonIsomorphicGraphs])(i, output = graphs,
            outputform = adjacency, restrictto = regular);
    s := ColumnDimension(Lp);
    satisfy_property_count := 0;
    for j to s do
        G := Graph(Lp[j]);
        deg := DegreeSequence(G);
        SP := SandpileGroup(G);
        if and(not(has(deg, 1)), IsCyclic(SP)) then
            H := GraphComplement(G);
            print(DrawGraph(G, style = spring));
            print(LaplacianMatrix(G));
            satisfy_property_count := satisfy_property_count+1
        end if
    end do;
    FormatPrint(cat("The number of graphs of order %d",
                    "that satisfy the conditions above is %d/%d"),
                i,
                satisfy_property_count,
                s
    )
end do
```

A set of constructions we would like to explore is cycle chains that has a path of length l shared between each pair. This is motivated by the chain cyclic graphs that we mentioned in Chapter 5.

`TileTwoPaths` procedure, takes two paths from two separate graphs and edge tiles them into one path, which is defined below,

TileTwoPaths procedure

```
TileTwoPaths := proc (G, H, e, f)
    local ng, nh, Ag, Ah, A, B, e_len, i;
    description "returns the graph after tiling path e in G and f in H.";
    Ag := AdjacencyMatrix(G);
    Ah := AdjacencyMatrix(H);
    ng := RowDimension(Ag);
    nh := RowDimension(Ah);
    e_len := min(nops(e), nops(f));
    B := Matrix(ng+nh);
    A := MatrixUnion(Ag, Ah);
    for i to e_len do
        B(() .. (), e[i]) := A(() .. (), e[i])+A(() .. (), ng+f[i]);
        B(e[i], () .. ()) := A(e[i], () .. ())+A(ng+f[i], () .. ())
    end do;
    for i to e_len do
        A(() .. (), e[i]) := B(() .. (), e[i]);
        A(e[i], () .. ()) := B(e[i], () .. ())
    end do;
    for i to e_len do
        A := DeleteRow(A, ng+f[i]+1-i);
        A := DeleteColumn(A, ng+f[i]+1-i)
    end do;
    return Graph(AdjacencyMatrix(Graph(A)))
end proc;
```

This enables us to explore as a first step 3 cycles of different sizes with a path of length 2 shared between each pair as in the code below.

Exploring cycle chain graphs with a path of length 2 shared between each pair

```
m := 5; t := 6; s := 6; r := 6;
for k from m to t do
    for i from m to s do
        for j from m to r do
            T := CycleGraph(k); G1 := CycleGraph(i); G2 := CycleGraph(j);
            t1 := [k-2, k-1, k+0]; e1 := [1, 2, 3];
            U := TileTwoPaths(T, G1, t1, e1);
            q := GraphOrder(U);
            t2 := [k-2, q, q-1]; e2 := [1, 2, 3];
            U := TileTwoPaths(U, G2, t2, e2);
            print(DrawGraph(U, style = spring));
            PrintSandpileGroup(sprintf("Graph #k=%d,i=%d,j = %d",
                                    k, i, j), U)
        end do
    end do
end do
```

And finally, here we have a planar graph, a modified wheel graph that seems to have cyclic sandpile structure, which we would like to understand better.

Exploring Wheel graph with some edges removed

```
for i to 10 do
    n := 2*i+1;
    A := Copy(AdjacencyMatrix(GraphJoin(CycleGraph(n),
                                        CompleteGraph(1))));
    for j to n-2 do
        if mod(j, 2) = 1 then
            A(n+1, j) := 0; A(j, n+1) := 0
        end if
    end do;
    G := Graph(A);
    PrintSandpileGroup(sprintf("Graph #%d", i), G)
end do
```

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