

Some observations on the dimension of Fano K-moduli

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Abstract

In this short note we show the unboundedness of the dimension of the K-moduli space of n -dimensional Fano varieties, and that the *dimension of the stack* can also be unbounded while, simultaneously, the dimension of the corresponding coarse space remains bounded.

1 Main statement

Moduli spaces of K-stable Fano varieties have been intensively investigated in the last decade, both from a general theory point of view as well as via the study of explicit examples. There are two objects of interest, the moduli stack of K-semistable Fano varieties \mathcal{M}_K and its good moduli space, in the sense of Alper, M_K which parametrises K-polystable varieties. We refer the reader to [16] for a survey in the construction of these objects in the case of smoothable varieties (cf. [17]) and to [11] for the most recent construction in the general case. In this note, we observe the following:

Theorem 1.1. *For each $n > 1$ the dimension (as a variety) of the K-moduli spaces M_K of n -dimensional Fano varieties is unbounded. Moreover, the dimension of the K-moduli stack \mathcal{M}_K can be arbitrarily big, while the dimension of its coarse variety M_K remains bounded.*

Here, for *dimension of the K-moduli stack \mathcal{M}_K* at a given K-polystable point $[X]$ we mean the difference between the dimension of the versal space of (K-semistable) \mathbb{Q} -Gorenstein deformations of the variety X minus the dimension of its reductive automorphism group, cf. [15, Section 0AFL].

It is well-known that smooth Fano manifolds, and more generally ε -log terminal Fano varieties (where $\varepsilon > 0$ is fixed), form a bounded family in a fixed dimension [4]. Thus to construct such examples we need to consider non-smoothable varieties whose Kawamata log terminal (klt) singularities get worse and worse.

Our main theorem is a quick consequence of these two dimensional easy examples.

Proposition 1.2. *Consider the following two families of K-polystable normal surfaces:*

1. $X_l := (\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_l$, for $l \geq 2$, where the generator 1 of the cyclic group \mathbb{Z}_l acts by

$$([z_0 : z_1], [w_0 : w_1]) \mapsto ([\zeta z_0 : z_1], [\zeta^{-1} w_0 : w_1]),$$

where ζ is a primitive l -root of unity.

2. $Y_l := \mathbb{P}^2/\mathbb{Z}_l$ for $l \geq 3$, l odd, where the generator 1 of the cyclic group \mathbb{Z}_l acts by

$$[z_0 : z_1 : z_2] \mapsto [\zeta z_0 : \zeta^{-1} z_1 : z_2],$$

where ζ is a primitive l -root of unity.

Then

1. the dimension as a variety of the K -moduli space M_K at $[X_l]$ is equal to $2l - 3$ if $l \neq 2, 4$, and equal to 2 (resp. 6) for $l = 2$ (resp. $l = 4$).
2. the dimension of the K -moduli stack \mathcal{M}_K at $[Y_l]$ is equal to $l - 3$ for $l \neq 3, 9$ and equal to 4 (resp. 8) for $l = 3$ (resp. 9). However, $[Y_l]$ is an isolated K -polystable point for $l \neq 3, 9$.

The surfaces X_l with $l = 2, 4$ and Y_l with $l = 3, 9$ are actually \mathbb{Q} -Gorenstein smoothable and they appear in the boundary of K -moduli of smooth del Pezzo surfaces of degree 4, 2, 3 and 1 respectively [13, (Ex. 5.7, (4.3), Ex. 3.10)].

Note that if we had considered Y_l for l even, we would have a \mathbb{Z}_2 subgroup fixing the line $z_2 = 0$ (thus the more natural way to think about the quotient is as a pair (X, D) , considering the line at infinity with weight $\frac{2}{l}$). This pair will be only \log - K -polystable, not X .

Proof of Theorem 1.1. It simply follows by taking $\tilde{X}_l = X_l \times \mathbb{P}^{n-2}$. Of course, being the product of two K -polystable varieties, such n -dimensional varieties are still K -polystable [18] and hence the dimension of the K -moduli spaces at $[\tilde{X}_l]$ tends to infinity with l . Similarly, one may take $\tilde{Y}_l = Y_l \times \mathbb{P}^{n-2}$ and have that the moduli stack has arbitrary dimension while $[\tilde{Y}_l]$ is still an isolated K -polystable point. Note, moreover, that Y_l (or \tilde{Y}_l) for $l \neq 3, 9$ actually give examples where the K -moduli reduces to a point while there are many non isomorphic strictly K -semistable Fano varieties around Y_l (we are unaware if a similar phenomenon can occur for smooth Fano manifolds too; note that Y_l has klt singularities). \square

These are toric examples, thus they suggest the following problem:

Problem 1.3. *Study in detail the local theory of K -moduli of toric del Pezzo surfaces.*

We expect that such investigations are interesting and important when studying moduli spaces of non-necessarily \mathbb{Q} -Gorenstein smoothable del Pezzo surfaces.

The proof of the above Proposition is based on the local study of K -stability for \mathbb{Q} -Gorenstein deformations of the surfaces, which is possible even in this non-smoothable setting thanks to the the recent works [5, 6]. This type of computations have been performed for the \mathbb{Q} -Gorenstein smoothable cases of the above examples in [13]. A similar strategy to show interesting behaviour of K -moduli spaces near toric varieties has also been considered in [8] to show that the moduli can be reducible and non-reduced.

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After writing up a first draft of this problem in late December 2020, we found out that the first example in Proposition 1.2 was considered a few weeks before in [10], when studying the K -stability of hypersurfaces in $\mathbb{P}(1, 1, a, a)$. We would like to thank A. Petracci for having a look at an early draft of our manuscript and giving us very useful comments which improved our manuscript.

2 Proof of Proposition 1.2

Proposition 1.2 is a consequence of the next few lemmas.

Lemma 2.1. *For the surface X_l above we have that $\text{Sing}(X_l) = \{2A_{l-1}, 2\frac{1}{l}(1, 1)\}$, and the connected component to the identity is $\text{Aut}_0(X_l) = (\mathbb{C}^*)^2$. Similarly for Y_l we have that $\text{Sing}(Y_l) = \{A_{l-1}, 2\frac{1}{l}(1, 2)\}$ and $\text{Aut}_0(Y_l) = (\mathbb{C}^*)^2$.*

Proof. Let's consider the Y_l case (X_l is completely analogous and we omit it). The singularities of Y_l correspond to points on \mathbb{P}^2 where \mathbb{Z}_l acts with non-trivial stabilizer. Near $[0 : 0 : 1]$ the action has weight $(1, -1)$ resulting in a A_{l-1} canonical singularity. Similarly near the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ the action as weight $(1, 2)$ resulting in $\frac{1}{l}(1, 2)$ quotient singularities. The statement about the automorphism follows by noting that $\text{Aut}_0(Y_l) \cong \text{Aut}_0(\mathbb{P}^2; S) \cong (\mathbb{C}^*)^2$, where $S = \{[0 : 0 : 1], [1 : 0 : 0], [0 : 1 : 0]\}$ and $\text{Aut}_0(\mathbb{P}^2; S)$ is the connected component of the automorphism group containing the identity and fixing the subset S . For a similar computation see [12, Lemma 3.1]. \square

Remark 2.2. *Note that there is no $\epsilon > 0$ such that the non-Du Val singularities of the set of varieties $\{X_l\}_{l \geq 2}$ and $\{Y_l\}_{l \geq 3, l \text{ odd}}$, are ϵ -log terminal. Indeed, each of the two singular points $\frac{1}{l}(1, 1)$ in X_l is locally analytically isomorphic to the affine cone over the rational normal curve $C_l \subset \mathbb{P}^l$ and its resolution has exceptional locus $E \cong \mathbb{P}^1$ with $E^2 = -l$. It follows that their log discrepancies equal to $\frac{2}{l} - 1 \rightarrow -1$ as $l \rightarrow \infty$, and moreover $-K_{X_l}$ is \mathbb{Q} -Cartier (with Cartier index going to infinity) with $(-K_{X_l})^2 = \frac{8}{l} \rightarrow 0$ as $l \rightarrow \infty$, and similarly for Y_l .*

Lemma 2.3. *X_l and Y_l are K -polystable Fano variety whose space of \mathbb{Q} -Gorenstein deformations are given by*

1. $\text{qDef}(X_l) \cong \text{qDef}(A_{l-1}) \oplus \text{qDef}(A_{l-1}) \cong \mathbb{C}^{2(l-1)}$ for $l \neq 2, 4$.
2. $\text{qDef}(Y_l) \cong \text{qDef}(A_{l-1}) \cong \mathbb{C}^{(l-1)}$ for $l \neq 3, 9$.

Proof of Lemma 2.3. Y_l (and X_l) is a K -polystable del Pezzo surface as \mathbb{Z}_l acts by isometries with respect to the Fubini-Study metric in \mathbb{P}^2 (with respect to the product in $\mathbb{P}^1 \times \mathbb{P}^1$ of the product of the Fubini-Study metrics in \mathbb{P}^1 , respectively). Hence, both X_l and Y_l inherit an (orbifold) Kähler-Einstein metric and consequently they are K -polystable by [3].

By [1, Lemma 6], it follows that there are no local-to-global obstructions to \mathbb{Q} -Gorenstein deformations on del Pezzo surfaces. Since X_l is toric it does not admit *equisingular deformations* (i.e. non-trivial deformations to a non-isomorphic projective variety with the same singularities), e.g., [14, Lemma 4.4]. Hence all \mathbb{Q} -Gorenstein deformations must come from local \mathbb{Q} -Gorenstein deformations of the singularities. Thus

$$\text{qDef}(Y_l) = \prod_{p \in \text{Sing}(Y_l)} \text{qDef}(p) \tag{2.1}$$

and similar for X_l .

Note that any deformation of A_{l-1} is \mathbb{Q} -Gorenstein and given by the versal family $xy = z^l + a_{l-2}z^{l-2} + \dots + a_0$. Hence the vector $(a_0, a_1, \dots, a_{l-2})$ defines a point in $\text{qDef}(A_{l-1})$ and $\text{qDef}(A_{l-1}) \cong \mathbb{C}^{l-1}$. The proof follows from Lemma 2.1, once we show that $\text{qDef}(p) = \{0\}$ for p non-Du Val. We will do this for Y_l , since the case of X_l is very similar.

We claim the two $\frac{1}{l}(1, 2)$ singularities of Y_l are \mathbb{Q} -Gorenstein rigid (i.e. they do not admit \mathbb{Q} -Gorenstein deformations) if $l \neq 3, 9$, and \mathbb{Q} -Gorenstein smoothable otherwise. Let $w = \text{hcf}(l, 3)$, $r > 0$ such that $l = wr$, $m \geq 0$ and $0 \leq w_0 < r$ such that $w = mr + w_0$. It is well known (see e.g. [1]) that a quotient singularity $\frac{1}{l}(1, 2)$ is \mathbb{Q} -Gorenstein rigid if and only if $m = 0$, or equivalently if $w = w_0$. Moreover, $\frac{1}{l}(1, 2)$ is \mathbb{Q} -Gorenstein smoothable (often known as a *T-singularity*) if and only if $w_0 = 0$ and a *primitive T-singularity* if in addition $m = 1$.

The number $w = \text{hcf}(l, 3)$ can only be 1 or 3. If $w = 1$, then $l = wr = r$ and $1 = w = mr + w_0$ implies that $m = 0$ so $\frac{1}{l}(1, 2)$ is \mathbb{Q} -Gorenstein rigid. If $w = 3$ then $l = 3k$ for some $k \in \mathbb{N}$ but in fact, that means that $l = 3k = wr = 3r$, so $l = 3r$. If $r = 1$ then $m = 1$ and $w_0 = 0$ so $\frac{1}{3}(1, 2)$ is a primitive T-singularity. The case $r = 2$ is excluded, otherwise l would be even. If $r = 3$, then $m = 1$ and $w_0 = 0$ and $\frac{1}{9}(1, 2)$ is \mathbb{Q} -Gorenstein smoothable. If $r \geq 4$, (r odd) then $m = 0$ and $\frac{1}{3r}(1, 2)$ is \mathbb{Q} -Gorenstein rigid. Hence, whenever $l \neq 3, 9$ we have $\text{qDef}(\frac{1}{l}(1, 2)) = \{0\}$.

For X_l similar computations show that the singularities $\frac{1}{l}(1, 1)$ are \mathbb{Q} -Gorenstein rigid for $l \neq 2, 4$. \square

Remark 2.4. For $l = 2$, X_l has four A_1 singularities giving a four dimensional versal space of deformation. For $l = 4$, the deformation space has (beside the deformations coming from the two A_3 singularities) \mathbb{Q} -Gorenstein deformations coming from the one dimensional family of \mathbb{Q} -Gorenstein smoothings of the $\frac{1}{4}(1, 1)$ singularities. For $l = 3$, Y_l is just the unique cubic surface with $3A_2$ -singularities, given by $xyz = t^3$ (and the only strictly K -polystable surface in the K -moduli of del Pezzo surfaces of degree 3). The case $l = 9$ was studied in [13, Example 3.10] and it appears in the boundary of the K -moduli compactification of smooth del Pezzo surfaces of degree 1.

Lemma 2.5. The natural action of $G = \text{Aut}_0(X_l) \cong (\mathbb{C}^*)^2$ ($G' = \text{Aut}_0(Y_l) \cong (\mathbb{C}^*)^2$) on $\text{qDef}(X_l)$ (respectively $\text{qDef}(Y_l)$) for $l \neq 2, 4$ (resp. $l \neq 3, 9$) is not effective. Moreover:

1. The action on $\text{qDef}(X_l) \cong \mathbb{C}^{2(l-1)}$ of $G / \cap_x (G_x) \cong \mathbb{C}^*$ with $t = \lambda_1 \lambda_2 \in G / \cap_x (G_x)$, is given by

$$(a_0, a_1, \dots, a_{l-2}, a'_0, \dots, a'_{l-2}) \mapsto (t^l a_0, t^{l-1} a_1, \dots, t^2 a_{l-2}, t^{-l} a'_0, \dots, t^{-2} a'_{l-2});$$

2. The action on $\text{qDef}(Y_l) \cong \mathbb{C}^{l-1}$ of $G' / \cap_x (G'_x) \cong \mathbb{C}^*$ with $t = \lambda_1 \lambda_2 \in G' / \cap_x (G'_x)$, is given by

$$(a_0, a_1, \dots, a_{l-2}) \mapsto (t^l a_0, t^{l-1} a_1, \dots, t^2 a_{l-2}).$$

Proof. Let us start with Y_l . In local coordinates near the A_{l-1} -point $[0 : 0 : 1]$ we can take coordinates on $\text{Aut}_0(Y_l) \cong (\mathbb{C}^*)^2$ -action such that the action is just given by

$$(u, v) \mapsto (\lambda_1^{-1} u, \lambda_2^{-1} v).$$

Taking invariants for the \mathbb{Z}_l -action $x = u^l$, $y = v^l$ and $z = uv$, we get the induced action on the A_{l-1} -quotient singularity $xy = z^l$ given by $(\lambda_1^{-l} x, \lambda_2^{-l} y, (\lambda_1 \lambda_2)^{-1} z)$. Considering then the natural action induced on the versal deformation family of the singularity $xy = z^l + a_{l-2} z^{l-2} + \dots + a_0$, we get that

$$(a_0, a_1, \dots, a_{l-2}) \mapsto ((\lambda_1 \lambda_2)^l a_0, (\lambda_1 \lambda_2)^{l-1} a_1, \dots, (\lambda_1 \lambda_2)^2 a_{l-2}).$$

In particular note that the action is non effective since the action of the subtorus $(s, s^{-1}) \subseteq (\mathbb{C}^*)^2$ is clearly trivial. Finally, putting $t = \lambda_1 \lambda_2$ we obtain our statement for Y_l .

The statement for X_l is completely analogous, but (crucially) noticing that if we take coordinates on $\text{Aut}_0(X_l)$ to be such that near the point $([0 : 1], [0 : 1])$ the action is again given by

$(u, v) \mapsto (\lambda_1^{-1}u, \lambda_2^{-1}v)$, then near the point $([1 : 0], [1 : 0])$ one get an action with *opposite* weights. From there the statements follows immediately. \square

Descriptions of the local actions for the smoothable cases of X_l and Y_l can be found in [13]. Also note that since the above action is not effective (with a \mathbb{C}^* as stabilizer) all the small deformations will have a residual \mathbb{C}^* -action on them.

Lemma 2.6. *When $l \neq 2, 4$ the K -moduli space near $[X_l]$ is (étale locally) described by the affine GIT quotient $\mathbb{C}^{2(l-1)}/\mathbb{C}^*$, where the \mathbb{C}^* -action is given as in Lemma 2.5. Similarly for Y_l when $l \geq 4$ $l \neq 9$, the K -moduli space near $[Y_l]$ is (étale locally) described by the affine GIT quotient $\mathbb{C}^{l-1}/\mathbb{C}^*$,*

Proof. Any \mathbb{Q} -Gorenstein deformation of X_l and Y_l is still a Fano variety since the canonical $K_{\mathcal{X}}$ of the total space of a deformation \mathcal{X} is \mathbb{Q} -Cartier and ampleness is an open condition. Moreover, the deformation is singular, since it is flat and $K_{\mathcal{X}}^2 \notin \mathbb{Z}$ (alternatively, as pointed out by a referee, it is singular because there are rigid singularities). Then the characterization of those varieties in the deformation which are K -polystable follows by the local GIT description of non-necessarily smoothable Fano varieties in [5, Proof of Theorem 4.5], cf.[2, Remark 2.11], where it is shown that K -semistability is an open condition and that K -polystability can be checked locally by considering the action of the automorphisms. \square

We are now ready to conclude the proof of our Proposition 1.2:

Proof of Proposition 1.2. . For Y_l it is clear that that all points near zero in $\text{qDef}(Y_l)$ are K -semistable by openness. However, note that all such points are destabilized to zero since

$$\lim_{t \rightarrow 0} (t^l a_0, \dots, t^2 a_{l-2}) = 0.$$

Hence only 0 is GIT polystable, and Y_l is an isolated K -polystable variety. However, by [15, Lemma 98.12.1], the dimension of the stack at the point Y_l is equal to

$$\dim_{Y_l}(\mathcal{M}_K) = \dim \text{qDef}(Y_l) - \dim \text{Aut}(Y_l) = (l - 1) - 2 = l - 3.$$

For X_l it is now sufficient to compute the dimension (as a variety) of the GIT quotient $\mathbb{C}^{2(l-1)}/\mathbb{C}^*$ above. But it is clear that the generic orbit is closed (with no further stabilizer). Indeed, if coordinates a_j and a'_j in Lemma 2.5 are all non-zero, then the orbits are given by the closed set $a_j a'_j = c_j \neq 0$, with $j = 0, \dots, l - 2$. Hence $\dim_{\mathbb{C}} M_K$ near $[X_l]$ is simply given by $2(l - 1) - 1 = 2l - 3$ as claimed. \square

Observe that if we consider a deformation of X_l which smooths only one of the two A_{l-1} singularities, the resulting variety is strictly K -semistable and never K -polystable, since in order to obtain K -polystable varieties we need to deform the two A_{l-1} singularities *simultaneously* by the same computation as for the Y_l case in the last paragraph of the proof of Proposition 1.2.

Note also that for X_l , $l \neq 2, 4$, since the action is not effective, we also have a discrepancy between the dimension of the stack and the dimension of the coarse space (which is then one dimension *bigger* than expected).

3 Some final comments

The general small deformation X_t of X_l is then a K-polystable variety which is also Kähler-Einstein by [9]. Moreover the second Betti number gets bigger and bigger as l goes to infinity: indeed, smoothing out an A_{l-1} -singularity introduces a chain of S^2 of length $l - 1$, giving distinct homological classes. Hence:

Corollary 3.1. *There are K-polystable/Kähler-Einstein del Pezzo surfaces with arbitrarily big second Betti number.*

We should also observe that this moduli space corresponds to the moduli of Kähler-Einstein orbifolds with positive cosmological constant, hence giving also examples of moduli spaces of positive Einstein orbifolds of unbounded dimension. Thus, from a more differential geometric perspective, it would be interesting to know if a bound on the second Betti number would instead force the dimension of the moduli spaces of such metrics to stay bounded.

Finally, note that the unboundedness of the dimension can be avoided by bounding below either the volume or the singularities. Indeed, that is what [7] proves, where the measure of boundedness used for the singularities is the alpha-invariant. This does not contradict our example, as we had that $K_{X_l}^2 \rightarrow 0$ as l grows and the log discrepancies were monotonously decreasing with l towards -1 . What is remarkable of this example is not that a bound below on the volume or the singularities are required to achieve boundedness of families, there were plenty of examples of this behaviour in [7]. What is remarkable is that removing such bounds not only gives an infinite number of families (whose dimension, one may think could, in principle, be uniformly bounded), but it also gives infinite dimension of the moduli.

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