Parametric measures of variability induced by risk measures

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ABSTRACT

We present a general framework for a comparative theory of variability measures, with a particular focus on the recently introduced one-parameter families of inter-Expected Shortfall differences and inter-expectile differences, that are explored in detail and compared with the widely known and applied inter-quantile differences.

From the mathematical point of view, our main result is a characterization of symmetric and comonotonic variability measures as mixtures of inter-Expected Shortfall differences, under a few additional technical conditions. Further, we study the stochastic orders induced by the pointwise comparison of inter-Expected Shortfall and inter-expectile differences, and discuss their relationship with the dilation order. From the statistical point of view, we establish asymptotic consistency and normality of the natural estimators and provide a rule of the thumb for cross-comparisons.

Finally, we study the empirical behavior of the considered classes of variability measures on the S&P 500 Index under various economic regimes, and explore the comparability of different time series according to the introduced stochastic orders.

1. Introduction

Several measures of distributional variability are widely used in statistics, probability, economics, finance, physical sciences, and other disciplines. In this paper, we study a general theory of variability measures with an emphasis on three symmetric one-parameter families generated by popular parametric risk measures: Value-at-Risk (VaR), Expected Shortfall (ES), and expectiles. The corresponding induced variability measures are the *inter-quantile difference*, the *inter-ES difference*, and the *inter-expectile difference*. While the first one is a classical measure of statistical dispersion widely used e.g. in box plots, the other two are, to the best of our knowledge, relatively new: the inter-ES difference appears in Example 4 of Wang et al. (2020b) as a signed Choquet integral, and the inter-expectile difference has been studied in Bellini et al. (2020) via a connection to option prices. The present paper is a first unifying study, focused on their comparative qualitative and quantitative properties.

The mathematical theory of risk measures is extensive, and a standard reference is Föllmer and Schied (2016). As it is well-known, VaR is simply a quantile and ES is a coherent risk measure in the sense of Artzner et al. (1999). Both VaR and ES are implemented in current banking and insurance regulation frameworks; we refer to McNeil et al. (2015) for a comprehensive background and Wang and Zitikis (2021) for a more recent account. Expectiles, originally introduced in the statistical literature by Newey and Powell (1987), have received an increasing attention in risk management, as it has been shown that they are the only elicitable coherent risk measures (Ziegel (2016)). We refer e.g. to Bellini et al. (2014) and Bellini and Di Bernardino (2015) for more on the theory and financial applications of expectiles. For a comparison of the above risk measures in the context of regulatory capital calculation, see Embrechts et al. (2014) and Emmer et al. (2015).

The theory of variability measures has been studied from different angles; see David (1998) for a review in the context of the measurement of statistical dispersion. A mathematical formulation closer to our setting is the notion of deviation measure introduced in Rockafellar et al. (2006), and further developed by Grechuk et al. (2009, 2010). A similar notion of variability measure was proposed by Furman et al. (2017) with an emphasis on the Gini deviation. We will explain in Section 2 the differences between our general definition and the ones given in the literature; in particular, the inter-quantile difference does not satisfy the definition of deviation measure of Rockafellar et al. (2006) due to its lack of convexity.

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Our main contribution is a collection of results towards a general theory of variability measures, with particular emphasis on the three parametric classes mentioned above. Various novel properties are studied to underline the special role these measures play among other variability measures. Since statistical inference for VaR, ES, and expectiles is well developed (see e.g. Shorack and Wellner (2009) for VaR and Krätschmer and Zähle (2017) for the expectiles), the estimation of the corresponding variability measures is quite straightforward.

The rest of the paper is organized as follows. In the remainder of this section, we introduce some notation. The definitions of the three classes of variability measures induced by VaR, ES, and the expectiles are presented in Section 2, with some basic properties. In Section 3, we summarize many properties of some common variability measures which are arguably desirable in practice. A characterization result of these measures is established. The stochastic ordering of the three classes of variability measures based on pointwise comparison is discussed in 4. In Section 5, we discuss non-parametric estimation of the three classes of variability measures. We obtain the asymptotic normality and the asymptotic variances explicitly for the empirical estimators. It may be undesirable and financial unjustifiable to choose the same probability level for the three classes of variability measures induced by VaR, ES, and the expectiles; see Li and Wang (2022) for a detailed analysis on plausible equivalent probability levels when ES is to replace VaR. A simple analysis of a cross-comparison of an equivalent probability level for the variability measures using different distributions is carried out in Section 6. A small empirical analysis using the variability measures on the S&P 500 index is conducted in Section 7, where we observe the differences between these variability measures during different economic regimes. Further, we explore the symmetric variability orders between log-returns of Facebook and Berkshire Hathaway in 2020. In Section 8, we conclude the paper with some discussions on the suitability of the three classes in different situations. Appendix A contains a list of classic variability measures, and proofs of all results are put in Appendix B.

Notation. Throughout the paper, L^q is the set of all random variables in an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with finite q-th moment, $q \in (0, \infty)$, and L^∞ is the set of essentially bounded random variables. $\mathcal{X} = L^0$ is the set of all random variables, and $\mathcal{M}$ is the set of all distributions on $\mathbb{R}$. For any $X \in L^0$, F_X represents the distribution function of X, F_X^{-1} its left-quantile function, and U_X is a uniform random variable such that $F_X^{-1}(U_X) = X$ almost surely. The existence of such a U_X for any X is given, for example, in Lemma A.32 of Föllmer and Schied (2016). Two random variables X and Y are said to be comonotonic if there exist two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $X = f(X + Y)$ and $Y = g(X + Y)$. We write $X \stackrel{d}{=} Y$ if X and Y have the same distribution. In this paper, the terms "increasing" and "decreasing" are meant in the non-strict sense.

2. Definitions

2.1. Basic requirements for variability measures

Generally speaking, a variability measure is a functional $\nu : \mathcal{X} \to [0, \infty]$ that quantifies the magnitude of variability of random variables. In order for our definition to be as general as possible, we only require three natural properties.

Definition 1. A *variability measure* is a functional $\nu : \mathcal{X} \to [0, \infty]$ satisfying the following properties.

(A1) *Law invariance*: if $X, Y \in \mathcal{X}$ and $X \stackrel{d}{=} Y$, then $\nu(X) = \nu(Y)$.
(A2) *Standardization*: $\nu(m) = 0$ for all $m \in \mathbb{R}$.

(A3) *Positive homogeneity*: there exists $\alpha \in [0, \infty)$ such that $\nu(\lambda X) = \lambda^\alpha \nu(X)$ for any $\lambda > 0$ and $X \in \mathcal{X}$. The number α is called the *homogeneity index* of ν.

The three properties in Definition 1 are the most basic, and they are satisfied by virtually all examples in the literature; a useful variability measure typically satisfies other desirable properties (see Section 3).

Some examples of classic variability measures are given in Appendix A. Notice that in the literature there are some relative measures of variability that are only defined for positive random variables, such as the Gini coefficient or the relative deviation (see Appendix A). In this paper, we do not deal with these cases, although our definition can be easily amended to include them by replacing $\mathcal{X}$ with a positive convex cone. We call the set $\mathcal{X}_\nu = \{X \in \mathcal{X} : \nu(X) < \infty\}$ the effective domain of ν.

Remark 1. A deviation measure in the sense of Rockafellar et al. (2006) satisfies, in addition to (A2) and (A3) with homogeneity index 1, also subadditivity and strict positivity for non-constant random variables. As we will see in Section 3, the latter two properties are not satisfied by the inter-quantile difference. For this reason, our more general definition is more suitable here than the one of Rockafellar et al. (2006). Alternatively, Furman et al. (2017) required location-invariance instead of positive homogeneity, but this property is not satisfied by relative variability measures. Thus, we identify (A1), (A2), (A3) as the defining properties of a variability measure, and all other properties, such as location invariance and subadditivity, will be additional properties that may or may not be satisfied, as we will discuss in see Section 3.

Remark 2. In applications, we may choose the domain $\mathcal{X}$ of a variability measure as a convex cone contained in L^0. For risk measures, the domain plays an essential role, which is often chosen as a general convex cone containing L^∞, because many risk measures cannot be naturally extended to L^0; see e.g., Filipović and Svindland (2012). For variability measures defined on a convex cone $\mathcal{X} \subset L^0$, since it takes non-negative values (thus, no issues with $\infty - \infty$ which occur for some risk measures), we could always extend the domain by mapping $L^0 \setminus \mathcal{X}$ to $\{\infty\}$ without affecting the properties studied in this paper.

2.2. Three one-parameter families of risk measures

Value at Risk (VaR), Expected Shortfall (ES) and expectiles are very popular financial risk measures (see e.g. Embrechts et al. (2014) and Emmer et al. (2015)). We recall the basic definitions below.

(i) The right-VaR (right-quantile): for $p \in (0, 1)$,

$$Q_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) > p\}, \quad X \in \mathcal{X}.$$

The left-VaR (left-quantile): for $p \in (0, 1)$,

$$Q_p^-(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant p\}, \quad X \in \mathcal{X}.$$

(ii) The ES: for $p \in (0, 1)$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p} \int_p^1 Q_r(X) \mathrm{d}r, \quad X \in \mathcal{X}.$$

The left-ES: for $p \in (0, 1)$,

$$\mathrm{ES}_p^-(X) = \frac{1}{p} \int_0^p Q_r(X) \mathrm{d}r, \quad X \in \mathcal{X}.$$

(iii) The expectile: for $p \in (0, 1)$,

$$\mathrm{ex}_p(X) = \min\{x \in \mathbb{R} : p\mathbb{E}[(X - x)_+] \leqslant (1 - p)\mathbb{E}[(X - x)_-]\},$$
$$X \in L^1.$$

In the above, Q_p and Q_p^- are finite on L^0, while ES_p, ES_p^- and ex_p are finite on L^1. We only define expectiles on L^1 since generalizing them beyond L^1 is not natural; on the other hand, ES can be naturally defined on a set larger than L^1 by taking possibly infinite values.

2.3. Three one-parameter families of variability measures

We now introduce the variability measures induced by the aforementioned risk measures, that are the main object of the paper.

(i) The inter-quantile difference: for $p \in [1/2, 1)$,

$$\Delta_p^Q(X) = Q_p(X) - Q_{1-p}^-(X), \quad X \in \mathcal{X}.$$

It is obvious that Δ_p^Q is finite on $\mathcal{X} = L^0$.

(ii) The inter-ES difference: for $p \in (0, 1)$,

$$\Delta_p^{\mathrm{ES}}(X) = \mathrm{ES}_p(X) - \mathrm{ES}_{1-p}^-(X), \quad X \in \mathcal{X}.$$

Here, ES_p takes values in $(-\infty, \infty]$, and ES_{1-p}^- takes values in $[-\infty, \infty)$, and hence the above Δ_p^{ES} is well defined on $\mathcal{X}$.

(iii) The inter-expectile difference: for $p \in (1/2, 1)$,

$$\Delta_p^{\mathrm{ex}}(X) = \mathrm{ex}_p(X) - \mathrm{ex}_{1-p}(X), \quad X \in L^1,$$

and we set by definition $\Delta_p^{\mathrm{ex}}(X) = \infty$ for $X \in \mathcal{X} \setminus L^1$.

We consider also the limiting cases

$$\Delta_1^Q(X) = \Delta_1^{\mathrm{ES}}(X) = \Delta_1^{\mathrm{ex}}(X) = \mathrm{ess\text{-}sup}(X) - \mathrm{ess\text{-}inf}(X), \quad X \in \mathcal{X},$$

which is the range functional, and it is simply denoted by Δ_1. Both Δ_p^Q and Δ_p^{ES} belong to the class of distortion riskmetrics (Wang et al. (2020a,b)), with many convenient theoretical properties. On the other hand, Δ_p^{ex} does not belong to this class, but it also has several nice properties, inherited from those of expectiles.

In Theorems 1-2 and Table 1 below, the range of p is $p \in [1/2, 1)$ for Δ_p^Q, $p \in (1/2, 1)$ for Δ_p^{ex}, and $p \in (0, 1)$ for Δ_p^{ES}.

Theorem 1. *For each p, the following statements hold.*

(i) Δ_p^Q, Δ_p^{ES}, Δ_p^{ex} *and* Δ_1 *are variability measures.*

(ii) *The effective domains of* Δ_p^Q, Δ_p^{ES}, Δ_p^{ex} *and* Δ_1 *are* L^0, L^1, L^1, *and* L^∞, *respectively.*

(iii) *Each of* Δ_p^Q, Δ_p^{ES} *and* Δ_p^{ex} *is increasing in p.*

(iv) *For each* $X \in \mathcal{X}$, *the following alternative formulations hold:*

$$\Delta_p^Q(X) = Q_p(X) + Q_p(-X),$$
$$\Delta_p^{\mathrm{ES}}(X) = \mathrm{ES}_p(X) + \mathrm{ES}_p(-X),$$
$$\Delta_p^{\mathrm{ex}}(X) = \mathrm{ex}_p(X) + \mathrm{ex}_p(-X).$$

It is straightforward to check that for $p = 1/2$, Δ_p^{ES} is equal to two times the mean median deviation (see Appendix A, item (v)). The next proposition shows that it suffices to consider $p \in [1/2, 1)$, as we will tacitly assume in most results of the next sections.

Proposition 1. *For each* $p \in (0, 1)$, $(1 - p)\Delta_p^{\mathrm{ES}} = p\Delta_{1-p}^{\mathrm{ES}}$, *and* $\Delta_p^{\mathrm{ES}} = \frac{1}{1-p}\int_p^1 \Delta_q^Q \, \mathrm{d}q.$

3. Comparative properties and characterization

In this section, we study comparative advantages of Δ_p^Q, Δ_p^{ES} and Δ_p^{ex}, among with several other measures of variability, namely the standard deviation (STD), the variance, the mean absolute deviation (MAD), the Gini deviation (Gini-D), and the range; see Appendix A for the definition of these classic variability measures.

We consider the following additional properties of a variability measure ν, which are all arguably desirable in some situations. In what follows, $\leqslant_{\mathrm{cx}}$ is the convex order, defined by $X \leqslant_{\mathrm{cx}} Y$ if $\mathbb{E}[\phi(X)] \leqslant \mathbb{E}[\phi(Y)]$ for all convex $\phi : \mathbb{R} \to \mathbb{R}$ such that the above two expectations exist.

(B1) *Relevance*: $\nu(X) > 0$ if X is not a constant, and there exists $\beta \in (0, \infty)$ such that $\nu(X) \leqslant \beta$ for all $X \in \mathcal{X}$ with $|X| \leqslant 1$.

(B2) *Continuity*: $\nu((X \wedge M) \vee (-M)) \to \nu(X)$ as $M \to \infty$ for all $X \in \mathcal{X}$.

(B3) *Symmetry*: $\nu(X) = \nu(-X)$ for all $X \in \mathcal{X}$.

(B4) *Comonotonic additivity (C-additivity)*: $\nu(X + Y) = \nu(X) + \nu(Y)$ for all comonotonic $X, Y \in \mathcal{X}$.

(B5) *Convex order consistency (Cx-consistency)*: $\nu(X) \leqslant \nu(Y)$ if $X \leqslant_{\mathrm{cx}} Y$.

(B6) *Convexity*: $\nu(\lambda X + (1 - \lambda)Y) \leqslant \lambda \nu(X) + (1 - \lambda)\nu(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$.

(B7) *Mixture concavity (M-concavity)*: $\hat{\nu}$ is concave, where $\hat{\nu} : \mathcal{M} \to [0, \infty]$ is defined by $\hat{\nu}(F) = \nu(X)$ for $X \sim F$.

(B8) *Location invariance (L-invariance)*: $\nu(X + c) = \nu(X)$ for all $X \in \mathcal{X}$ and $c \in \mathbb{R}$.

Relevance (B1) requires ν to report a positive value for all non degenerate distributions, and the value of $\nu(X)$ should not explode if $|X| \leqslant 1$. Continuity (B2) is very weak and unspecific to the effective domain of ν. If ν is finite on L^q for some $q \geqslant 1$, then (B2) is implied by L^q continuity. Symmetry (B3) means that ν is indifferent to the positive and the negative sides of the distribution, and this property is in sharp contrast to classic risk measures for which positive and negative values are interpreted very differently (deficit/surplus or loss/profit). The symmetry property of the measures of variability motivates and simplifies the application of the measures. C-additivity (B4) is a convenient functional property which allows for a characterization result below. The properties (B5)-(B7) describe natural requirements for ν to increase when the underlying distribution is more spread out in some sense; see Wang et al. (2020a) for further motivation and explanations of these properties. Finally, (B8) requires that variability is measured independently of the location of the distribution and is indeed imposed as an axiom for measures of variability by Furman et al. (2017).

In Table 1 below, α represents the homogeneity index. Table 1 shows properties of different variability measures including the inter-quantile, inter-ES, and inter-expectile differences, as well as the aforementioned classic variability measures.

Theorem 2. *The statements in Table 1 hold true.*

The proof of Theorem 2, thus checking the properties in Table 1, relies on several existing results on properties of risk measures and distortion riskmetrics from Newey and Powell (1987), Bellini et al. (2014, 2018a), Liu et al. (2020) and Wang et al. (2020a).

Notably, the inter-ES difference satisfies all properties (B1)-(B8), along with the Gini deviation and the range. Next, we establish that any variability measure satisfying (B1)-(B8) admits a representation as a mixture of Δ_p^{ES} for $p \in (0, 1]$.

Theorem 3. *The following statements are equivalent for a variability measure* $\nu : \mathcal{X} \to [0, \infty]$:

Table 1
Properties of variability measures.

	Δ_p^Q	Δ_p^{ES}	Δ_p^{ex}	variance	STD	MAD	Gini-D	range
relevance	NO	YES	YES	YES	YES	YES	YES	YES
continuity	YES	YES	YES	YES	YES	YES	YES	YES
symmetry	YES	YES	YES	YES	YES	YES	YES	YES
C-additivity	YES	YES	NO	NO	NO	NO	YES	YES
Cx-consistency	NO	YES	YES	YES	YES	YES	YES	YES
convexity	NO	YES	YES	YES	YES	YES	YES	YES
M-concavity	NO	YES	NO	YES	YES	NO	YES	YES
L-invariance	YES	YES	YES	YES	YES	YES	YES	YES
homogeneity (α)	1	1	1	2	1	1	1	1
effective domain	L^0	L^1	L^1	L^2	L^2	L^1	L^1	L^∞

(i) ν satisfies (B1)-(B8).
(ii) ν satisfies (B1)-(B4) and one of (B5)-(B6).
(iii) ν is a mixture of Δ_p^{ES}, that is, there exists a finite Borel measure $\mu \neq 0$ on $(0, 1]$ such that

$$\nu(X) = \int_0^1 \Delta_p^{ES}(X) \mathrm{d}\mu(p), \quad X \in \mathcal{X}. \tag{1}$$

The measure μ in (1) for a given ν is generally not unique. Using Proposition 1, we can require μ in (1) to be supported on $[1/2, 1]$ instead of $(0, 1]$.

Example 1. There are three variability measures in Table 1 that satisfy all of (B1)-(B8), and each admit a representation as in Theorem 3. We give below a corresponding measure μ for each of them.

1. Δ_p^{ES} for $p \in (0, 1)$: $\mu = \delta_p$.
2. The Gini deviation: $\mu(\mathrm{d}x) = (1 - x)\mathrm{d}x$ on $[0, 1]$.
3. The range Δ_1: $\mu = \delta_1$.

As we have seen from Theorem 2, all of Δ_p^Q, Δ_p^{ES}, Δ_p^{ex} are invariant under location shifts. In the next result, we show that each of the one-parameter families Δ_p^Q, Δ_p^{ES}, Δ_p^{ex} characterize a symmetric distribution up to location shifts.

Proposition 2. *Suppose that X has a symmetric distribution, i.e., $X \overset{\mathrm{d}}{=} -X$. Each of the curves $p \mapsto \Delta_p^Q(X)$, $p \mapsto \Delta_p^{ES}(X)$ and $p \mapsto \Delta_p^{ex}(X)$ for $p \in (1/2, 1)$, if it is finite, uniquely determines the distribution of X.*

Remark 3. If the distribution of X is not symmetric, none of $p \mapsto \Delta_p^Q(X)$, $p \mapsto \Delta_p^{ES}(X)$ and $p \mapsto \Delta_p^{ex}(X)$ for $p \in (1/2, 1)$ determines its distribution up to location shifts. This is because the inter-quantile difference curve $p \mapsto Q_p - Q_{1-p}^-$ does not determine the quantile curve $p \mapsto Q_p$. For instance, given a quantile curve $p \mapsto Q_p(X)$, we can define another quantile curve $p \mapsto Q_p(Y)$ by

$$Q_p(Y) = Q_p(X) + f(p), \quad p \in (0, 1),$$

where $f(p)$ is any continuous function satisfying $f(p) = f(1 - p)$ for $p \in (0, 1/2)$, such that $Q_p(Y)$ is increasing in p. The inter-quantile difference curves of X and Y are the same, but the distributions of X and Y are not the same up to a location shift unless f is a constant.

Remark 4. From Kusuoka (2001) it is well known that any coherent risk measure admits a representation as a supremum of mixtures of ES; see Bellini et al. (2014) for the case of expectiles. One naturally wonders whether an inter-expectile difference can be represented as the supremum of mixtures of inter-ES differences, i.e.,

the supremum over functions of form (1). Rather surprisingly, it turns out that such a relationship does not hold in general, as illustrated by Example 3 in Section 4.

4. Symmetric variability orders

Since the variability measures can be easily estimated from real data (see Section 5 below), one may conclude some ordering relations between two data sets with ordered measures of variability. For this purpose, we consider stochastic orders induced by pointwise comparison of inter-quantile, inter-ES, and inter-expectile differences. The first case has been studied in Townsend and Colonius (2005) under the name of quantile spread order, defined as follows:

$$X \leqslant_{QS} Y \text{ if } \Delta_p^Q(X) \leqslant \Delta_p^Q(Y), \text{ for each } p \in (1/2, 1).$$

Note that the order $\leqslant_{QS}$ is weaker than the well-known dispersive order, defined by

$$X \leqslant_{\mathrm{disp}} Y \text{ if } Q_\beta(X) - Q_\alpha(X) \leqslant Q_\beta(Y) - Q_\alpha(Y),$$

$$\text{for each } 0 < \alpha < \beta < 1,$$

for which we refer e.g. to Müller and Stoyan (2009) and Shaked and Shantikumar (2009). We define two stochastic orders based on inter-ES and inter-expectile differences as follows:

$$X \leqslant_{\Delta\text{-ES}} Y \text{ if } \Delta_p^{ES}(X) \leqslant \Delta_p^{ES}(Y), \text{ for each } p \in (1/2, 1),$$

$$X \leqslant_{\Delta\text{-ex}} Y \text{ if } \Delta_p^{ex}(X) \leqslant \Delta_p^{ex}(Y), \text{ for each } p \in (1/2, 1).$$

It turns out that for symmetric random variables, these orders are equivalent to the dilation order $\leqslant_{\mathrm{dil}}$, defined by

$$X \leqslant_{\mathrm{dil}} Y \text{ if } X - \mathbb{E}[X] \leqslant_{\mathrm{cx}} Y - \mathbb{E}[Y],$$

as shown in (v) and (vi) below; the other properties are summarized in the following.

Proposition 3. *Let $X, Y \in L^1$. The following statements hold:*

(i) *For any $c \in \mathbb{R}$, $X \leqslant_{\Delta\text{-ES}} Y \iff X + c \leqslant_{\Delta\text{-ES}} Y$; $X \leqslant_{\Delta\text{-ex}} Y \iff X + c \leqslant_{\Delta\text{-ex}} Y$.*
(ii) *If $|a| \geqslant 1$, then $X \leqslant_{\Delta\text{-ES}} aX$ and $X \leqslant_{\Delta\text{-ex}} aX$;*
(iii) *$X \leqslant_{\Delta\text{-ES}} Y \iff X \leqslant_{\Delta\text{-ES}} -Y$; $X \leqslant_{\Delta\text{-ex}} Y \iff X \leqslant_{\Delta\text{-ex}} -Y$.*
(iv) *$X \leqslant_{QS} Y \implies X \leqslant_{\Delta\text{-ES}} Y$.*
(v) *$X \leqslant_{\mathrm{dil}} Y \implies X \leqslant_{\Delta\text{-ES}} Y$ and $X \leqslant_{\Delta\text{-ex}} Y$.*
(vi) *If X and Y are symmetric with respect to their means, then $X \leqslant_{\mathrm{dil}} Y \iff X \leqslant_{\Delta\text{-ES}} Y \iff X \leqslant_{\Delta\text{-ex}} Y$.*

In case X or Y is not symmetric, then the equivalence relations in (vi) may fail, as the following simple example shows. Therefore, the two new orders $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ are generally weaker than the dilation order. This provides more flexibility for these new orders in real-data applications, as we will illustrate in Section 7.

Example 2. Let

$$X = \begin{cases} -3/4 & \text{with prob. } 1/3 \\ -1/2 & \text{with prob. } 1/3 \\ 5/4 & \text{with prob. } 1/3 \end{cases} \quad \text{and} \quad Y = \begin{cases} -1 & \text{with prob. } 1/2 \\ 1 & \text{with prob. } 1/2. \end{cases}$$

Then $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and

$$\Delta_p^{ES}(X) = \begin{cases} 2 & 2/3 \leqslant p \leqslant 1 \\ \frac{2}{3(1-p)} & 1/2 \leqslant p \leqslant 2/3, \end{cases} \quad \text{and}$$

$$\Delta_p^{ES}(Y) = 2, \quad 1/2 \leqslant p \leqslant 1.$$

Hence, $\Delta_p^{ES}(X) \leqslant \Delta_p^{ES}(Y)$ for each $p \in [1/2, 1]$. Also, by a straightforward computation,

$$\text{ex}_p(X) = \begin{cases} \frac{6p-3}{4+4p} & \text{if } 0 \leqslant p \leqslant 1/8 \\ \frac{10p-5}{8-4p} & \text{if } 1/8 \leqslant p \leqslant 1, \end{cases} \quad \text{and}$$

$$\text{ex}_p(Y) = 2p - 1 \text{ for } 0 \leqslant p \leqslant 1;$$

$$\Delta_p^{ex}(X) = \begin{cases} \frac{30p-15}{4(2-p)(1+p)} & \text{if } 1/2 \leqslant p \leqslant 7/8 \\ \frac{4p-2}{2-p} & \text{if } 7/8 \leqslant p \leqslant 1, \end{cases} \quad \text{and}$$

$$\Delta_p^{ex}(Y) = 4p - 2 \text{ for } 1/2 \leqslant p \leqslant 1.$$

It follows that $\Delta_p^{ex}(X) \leqslant \Delta_p^{ex}(Y)$ for each $p \in [1/2, 1]$. However, $X \not\leqslant_{dil} Y$ because X and Y have the same mean, and the support of X is not contained in that of Y. This shows that $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ do not imply $\leqslant_{dil}$.

Finally, in the asymmetric case the $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ orders are not related. In the next example we have that $X \leqslant_{\Delta\text{-ES}} Y$ but $X \not\leqslant_{\Delta\text{-ex}} Y$, and a (real-data) example in which the opposite situation occurs can be found in Section 7.

Example 3. Let

$$X = \begin{cases} -1 & \text{with prob. } 1/4 \\ 1 & \text{with prob. } 3/4 \end{cases} \quad \text{and} \quad Y = \begin{cases} -1 & \text{with prob. } 1/4 \\ 0 & \text{with prob. } 1/4 \\ 1 & \text{with prob. } 1/2. \end{cases}$$

Then

$$\Delta_p^{ES}(X) = \begin{cases} 2 & 3/4 \leqslant p \leqslant 1 \\ \frac{1}{2(1-p)} & 1/2 \leqslant p \leqslant 3/4, \end{cases} \quad \text{and}$$

$$\Delta_p^{ES}(Y) = \begin{cases} 2 & 3/4 \leqslant p \leqslant 1 \\ 1 + \frac{1}{4(1-p)} & 1/2 \leqslant p \leqslant 3/4. \end{cases}$$

Hence $\Delta_p^{ES}(X) \leqslant \Delta_p^{ES}(Y)$ for each $p \in [1/2, 1]$ and $X \leqslant_{\Delta\text{-ES}} Y$. Also,

$$\text{ex}_p(X) = \frac{4p-1}{1+2p} \text{ for } 0 \leqslant p \leqslant 1 \quad \text{and}$$

$$\text{ex}_p(Y) = \begin{cases} \frac{3p-1}{2p+1} & \text{if } 0 \leqslant p \leqslant 1/3 \\ \frac{3p-1}{2} & \text{if } 1/3 \leqslant p \leqslant 1, \end{cases}$$

$$\Delta_p^{ex}(X) = \frac{12p-6}{(1+2p)(3-2p)} \text{ for } 1/2 \leqslant p \leqslant 1 \quad \text{and}$$

$$\Delta_p^{ex}(Y) = \begin{cases} \frac{-6p^2+17p-7}{6-4p} & \text{if } 2/3 \leqslant p \leqslant 1 \\ \frac{6p-3}{2} & \text{if } 1/2 \leqslant p \leqslant 2/3. \end{cases}$$

Since $\Delta_{2/3}^{ex}(X) = \frac{18}{35} > \frac{1}{2} = \Delta_{2/3}^{ex}(Y)$, it follows that $X \not\leqslant_{\Delta\text{-ex}} Y$.

5. Non-parametric estimators

The properties of non-parametric estimators of $\Delta_p^Q(X)$, $\Delta_p^{ES}(X)$ and $\Delta_p^{ex}(X)$ can be derived from those of VaR, ES and expectiles, as we will explain in this section.

Suppose $X_1, X_2, \ldots, X_n \in L^1$ is an iid sample from a random variable X. Recall that the empirical distribution $\widehat{F}_n$ of $X_1, \ldots, X_n$ is given by

$$\widehat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\{X_j \leqslant x\}}, \quad x \in \mathbb{R}.$$

Let $\widehat{\Delta}_p^Q(n)$ be the empirical estimator of $\Delta_p^Q(X)$, obtained by applying Δ_p^Q to the empirical distribution of $X_1, \ldots, X_n$. Similarly, let $\widehat{\Delta}_p^{ES}(n)$ and $\widehat{\Delta}_p^{ex}(n)$ be the empirical estimators of $\Delta_p^{ES}(X)$ and $\Delta_p^{ex}(X)$. We will establish consistency and asymptotic normality of the empirical estimators, based on corresponding results on empirical estimators of VaR, ES and expectiles in the literature, e.g., Chen and Tang (2005), Chen (2008), and Krätschmer and Zähle (2017). We make the following standard regularity assumption on the distribution of the random variable X.

(R) The distribution F of $X \in L^1$ is supported on a convex set and has a positive density function f on the support.

Denote by $g = f \circ F^{-1}$ and let $p \in (1/2, 1)$. We will show in the next theorem that the asymptotic variances of the empirical estimators for Δ_p^Q and Δ_p^{ES} are given by, respectively,

$$\sigma_Q^2 = \frac{p(1-p)}{(g(p))^2} + \frac{p(1-p)}{(g(1-p))^2} - 2\frac{(1-p)^2}{g(p)g(1-p)}, \tag{2}$$

$$\sigma_{ES}^2 = \frac{1}{(1-p)^2} \left(\int_{[p,1]^2 \cup [0,1-p]^2} - 2 \int_{[p,1] \times [0,1-p]} \right) \frac{s \wedge t - st}{g(t)g(s)} dt ds, \tag{3}$$

and that for Δ_p^{ex} is given by

$$\sigma_{ex}^2 = s_p^{ex} + s_{1-p}^{ex} - 2c_p^{ex}, \tag{4}$$

where for $r \in \{p, 1-p\}$,

$$f_{r,F}^{ex}(t) = \frac{(1-r)\mathbb{1}_{\{t \leqslant \text{ex}_r(X)\}} + r\mathbb{1}_{\{t > \text{ex}_r(X)\}}}{(1-2r)F(\text{ex}_r(X)) + r}, \quad t \in \mathbb{R},$$

$$s_r^{ex} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{r,F}^{ex}(t) f_{r,F}^{ex}(s) F(t \wedge s)(1 - F(t \vee s)) dt ds,$$

$$c_r^{ex} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{r,F}^{ex}(t) f_{1-r,F}^{ex}(s) F(t \wedge s)(1 - F(t \vee s)) dt ds.$$

Theorem 4. *Suppose that $p \in (1/2, 1)$ and Assumption (R) holds.*

(i) $\widehat{\Delta}_p^Q(n) \xrightarrow{p} \Delta_p^Q(X)$, $\widehat{\Delta}_p^{ES}(n) \xrightarrow{p} \Delta_p^{ES}(X)$ *and* $\widehat{\Delta}_p^{ex}(n) \xrightarrow{p} \Delta_p^{ex}(X)$ *as* $n \to \infty$.

(ii) If $X \in L^{2+\delta}$ for some $\delta > 0$, then

$$\sqrt{n}(\widehat{\Delta}_p^Q(n) - \Delta_p^Q(X)) \xrightarrow{d} N(0, \sigma_Q^2),$$

$$\sqrt{n}(\widehat{\Delta}_p^{ES}(n) - \Delta_p^{ES}(X)) \xrightarrow{d} N(0, \sigma_{ES}^2),$$

$$\sqrt{n}(\widehat{\Delta}_p^{ex}(n) - \Delta_p^{ex}(X)) \xrightarrow{d} N(0, \sigma_{ex}^2),$$

where σ_Q^2, σ_{ES}^2 and σ_{ex}^2 are given in (2), (3) and (4), respectively.

(a) Histogram of $\widehat{\Delta}_p^Q(n)$ for $N(0,1)$.

(b) Histogram of $\widehat{\Delta}_p^Q(n)$ for Pareto(4).

(c) Histogram of $\widehat{\Delta}_p^{ES}(n)$ for $N(0,1)$.

(d) Histogram of $\widehat{\Delta}_p^{ES}(n)$ for Pareto(4).

(e) Histogram of $\widehat{\Delta}_p^{ex}(n)$ for $N(0,1)$.

(f) Histogram of $\widehat{\Delta}_p^{ex}(n)$ for Pareto(4).
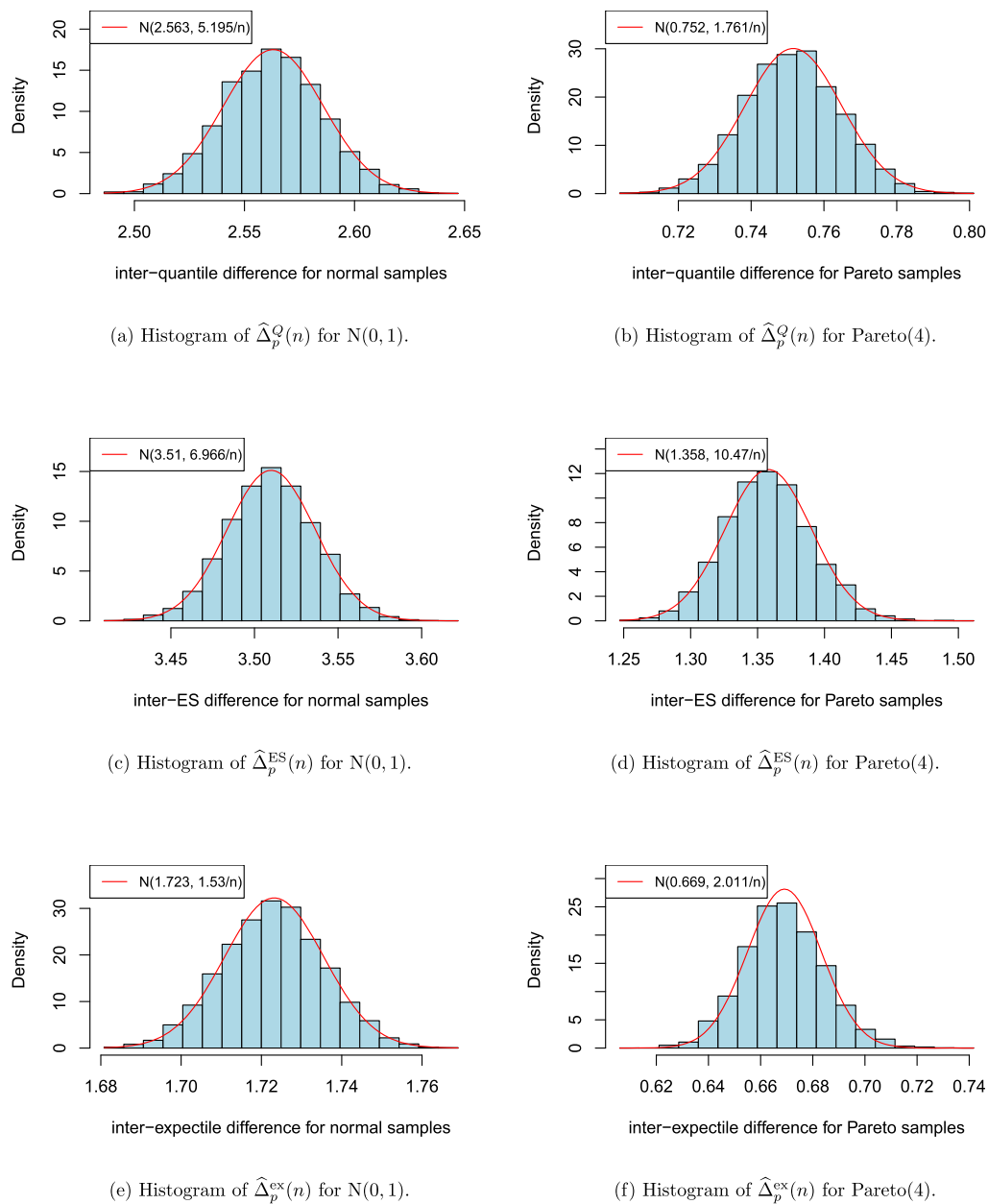
Fig. 1. Histograms of empirical estimators for simulated normal and Pareto risks, plotted against the density of their asymptotic normal distributions in Theorem 4 (with variance normalized by the sample size n). Each histogram is computed from 5,000 replications with sample size $n = 10,000$. The parameter p is set to 0.9 in all simulation experiments.

Simulation results are presented in Fig. 1 for $p = 0.9$ in the case of standard normal and Pareto risks with tail index 4, that confirm the asymptotic normality of the empirical estimators in Theorem 4. More general asymptotic results for α-mixing processes could be similarly established using results in Chen (2008) and Krätschmer and Zähle (2017). For the sake of space we do not discuss here the case of dependent observations.

Remark 5. For part (i) of Theorem 4, the assumption (R) is used to guarantee that the empirical quantiles converge to the true quantile (more precisely, we only need the quantile function to be continuous at p and $1 - p$); this is not needed for the consistency statements on Δ_p^{ES} and Δ_p^{ex} in part (i).

Remark 6. For a convex risk measure $\rho : H^\Psi \to \mathbb{R}$ on an Orlicz heart H^Ψ, Krätschmer et al. (2014, Theorem 2.6) showed that the

empirical estimator $\widehat{\rho}(n)$ is strongly consistent for a stationary and ergodic data sequence $(X_n)_{n \in \mathbb{N}}$; that is, $\widehat{\rho}(n) \to \rho(X_1)$ almost surely. For a general variability measure ν, a similar result holds if we further assume that ν is norm-continuous on H^Ψ, following the same arguments as in the proof of Theorem 2.6 of Krätschmer et al. (2014), where the only used property of ρ is norm-continuity. This statement includes consistency of $\widehat{\Delta}_p^{ES}(n)$ and $\widehat{\Delta}_p^{ex}(n)$ on L^1 in part (i) of Theorem 4 as special cases. For real-valued convex risk measures, norm-continuity is implied by monotonicity and convexity. To establish such continuity, monotonicity is essential and it plays the role of positivity in the Namioka-Klee theorem for linear functions; see Biagini and Frittelli (2009). For convex variability measures, since monotonicity is not satisfied, we have to assume norm-continuity for the analog of Theorem 2.6 of Krätschmer et al. (2014).
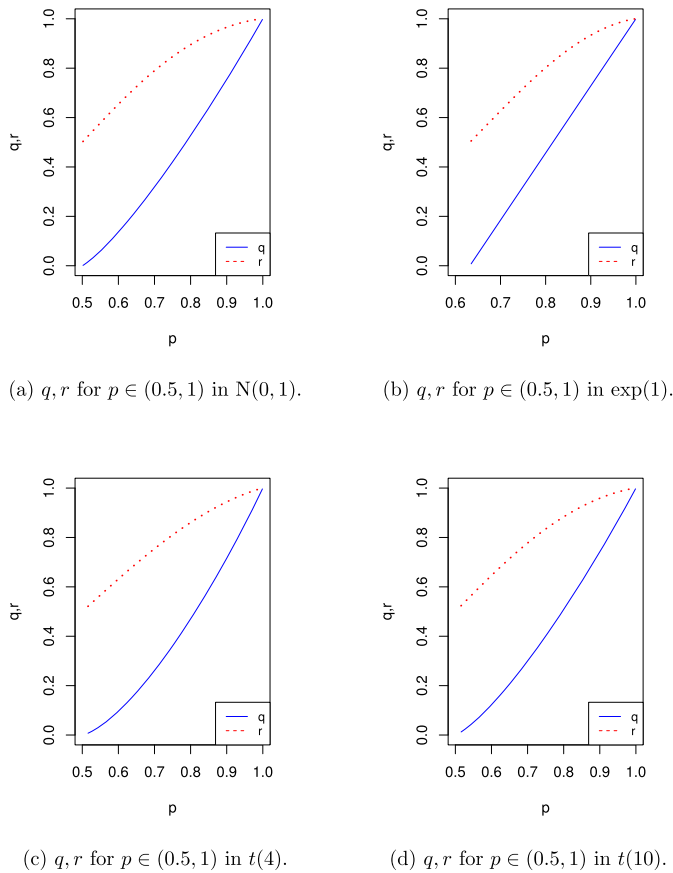
(a) q, r for $p \in (0.5, 1)$ in $N(0, 1)$.

(b) q, r for $p \in (0.5, 1)$ in $\exp(1)$.

(c) q, r for $p \in (0.5, 1)$ in $t(4)$.

(d) q, r for $p \in (0.5, 1)$ in $t(10)$.

Fig. 2. q, r such that $\Delta_p^Q = \Delta_q^{ES} = \Delta_r^{ex}$ for $p \in (0.5, 1)$.

6. A rule of thumb for cross comparison

As mentioned in the introduction, we are interested in comparing the inter-quantile, the inter-ES and the inter-expectile differences. Due to the different meanings of the parameter p in VaR_p, ES_p and ex_p, there is no reason to directly compare Δ_p^Q, Δ_p^{ES} and Δ_p^{ex} using the same probability level p. For a fair cross comparison, we may calibrate p, q, r such that the variability measures have the same value, that is,

$$\Delta_p^Q = \Delta_q^{ES} = \Delta_r^{ex},$$

for some common choices of distributions. In particular, we will consider normal (N), t- and exponential distributions as benchmarks, and the curves of q and r in terms of p for these distributions are plotted in Fig. 2. We observe that the values of r are typically much closer to 1 than the corresponding p or q. The matching value of q is smaller than the corresponding p but the relationship between q and p is close to linear; a corresponding observation on comparing VaR and ES is noted by Li and Wang (2022), where they obtained the ratios $(1 - q)/(1 - p) \approx 2.5$ for normal risks and $(1 - q)/(1 - p) = e \approx 2.72$ for exponential risks (this corresponds to the straight line in Fig. 2b).

In empirical studies, it has been costumary in the literature to use the matching values for normal distribution as a rule of thumb for general comparisons; note that the location and scale parameters are irrelevant for such a comparison due to location-invariance and positive homogeneity. Roughly, we obtain

$$\Delta_p^Q \approx \Delta_q^{ES} \approx \Delta_r^{ex}$$

for $(p, q, r) \in \{(0.9, 0.75, 0.97), (0.95, 0.875, 0.99), (0.99, 0.97, 0.999)\}$. For the particular choice of $p = 0.95$, it means that

$\Delta_{0.95}^Q \approx \Delta_{0.875}^{ES} \approx \Delta_{0.99}^{ex}$ for normal risks. We will compare these variability measures on real data in the next section, confirming the well-known departure from normality of financial returns.

7. Empirical analysis

In this section, we illustrate the three classes of variability measures studied in this paper by means of a few empirical studies on financial data.

We first analyze the difference between the performances of these variability measures during different periods of time (different economic regimes). Our data are the historical price movements spanned from 01/04/1999 to 06/30/2020 of the S&P 500 index.[1] We use its daily log-loss data[2] over the observation period with moving window of 253 days for daily estimation of the variability measures. To compare the relative performance of the three measures, we report the ratios $\Delta_q^{ES}/\Delta_r^{ex}$ and Δ_q^{ES}/Δ_p^Q for the S&P 500 daily log-losses in Figs. 3 and 4 using the rule of thumb for (p, q, r) obtained in Section 6 induced by the normal distribution. In Fig. 3, spikes in the ratio of Δ_q^{ES}/Δ_p^Q are located around the 2008 subprime crisis and the COVID-19 period. On the other hand, the ratio $\Delta_q^{ES}/\Delta_r^{ex}$ in Fig. 4 experiences a down-slide around the subprime crisis and the COVID-19 period. These results suggest that Δ_q^{ES} is more sensitive to extremely large losses than Δ_p^Q, but Δ_r^{ex} is even more sensitive than Δ_q^{ES}. Recall that these ratios should be 1 if the underlying losses are normally distributed, whereas we observe $\Delta_q^{ES}/\Delta_p^Q > 1$ and $\Delta_q^{ES}/\Delta_r^{ex} < 1$ for most dates during the period of 2000 - 2020 ($\Delta_q^{ES}/\Delta_r^{ex}$ is almost always smaller than 1). Hence, Figs. 3 and 4 confirm that the log-losses of S&P 500 are not normally distributed, and in fact, they typically show paretian tails, as is well studied in the literature (see, e.g., McNeil et al. (2015)).

As a second empirical illustration, we compare the distributions of the log-returns of Facebook and Berkshire Hathaway Inc. during the year 2020, displayed in Fig. 5. In this very peculiar year Facebook made +33.09% with annualized volatility 46.16%, and Berkshire Hathaway's made only +2.37% with annualized volatility 35.02%. In order to check if the two distributions are comparable in one of the symmetric variability orders considered in Section 4, we recall that an equivalent condition for the dilation order is given by

$$X \leqslant_{\text{dil}} Y \iff ES_p(X) - \mathbb{E}[X] \leqslant ES_p(Y) - \mathbb{E}[Y],$$

for each $p \in (0, 1)$.

We see in the left panel of Fig. 6 that there is an intersection point in the $ES_p - \mathbb{E}$ curves, so Facebook's log-returns do not dominate Berkshire Hathaway's according to the dilation order (and vice versa). In this specific example, this is due to the presence of two large values in the distribution of Berkshire Hathaway's daily log-returns. On the contrary, looking at the center and left panels of Fig. 6 we see that there are no intersection points, so Facebook's log-returns dominate Berkshire Hathaway's according to both the $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ orders. Hence, both $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ are able to model an ordering relation in the variability between two distributions, when the classic dilation order fails to hold, and this shows the additional flexibility of the new orders over the classic notion.

As a third example, we compare the distributions of log-returns of the S&P500 Index in 2008 and in 2020, displayed in Fig. 7. As in the previous example, we plot the relevant curves in Fig. 8. Here there is an intersection point both in the left and in the center
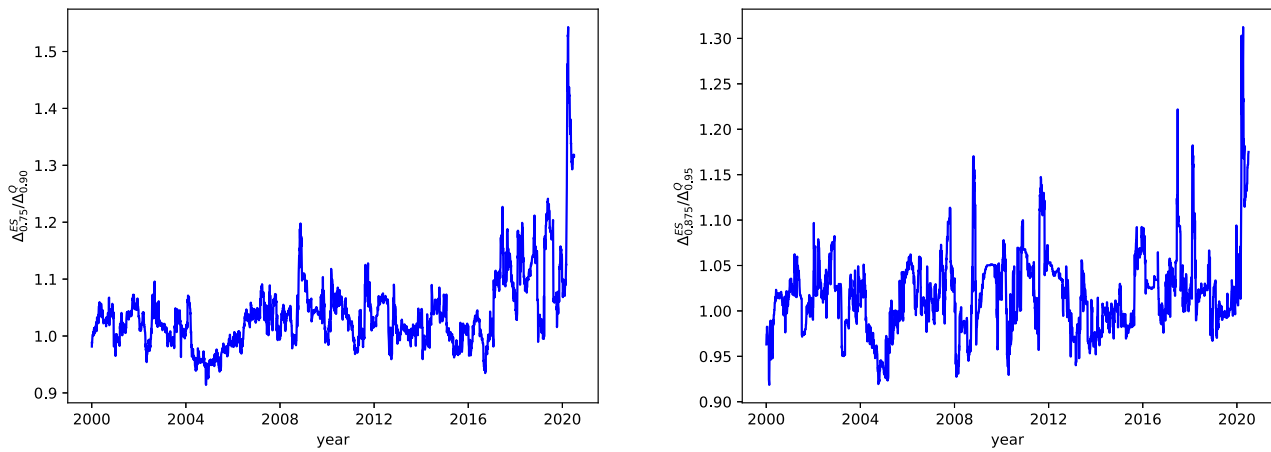

Fig. 3. The ratio of Δ_q^{ES} to Δ_p^Q using S&P 500 daily log-loss data (Jan 2000 - Jun 2020). Left: $(p, q) = (0.9, 0.75)$. Right: $(p, q) = (0.95, 0.875)$.
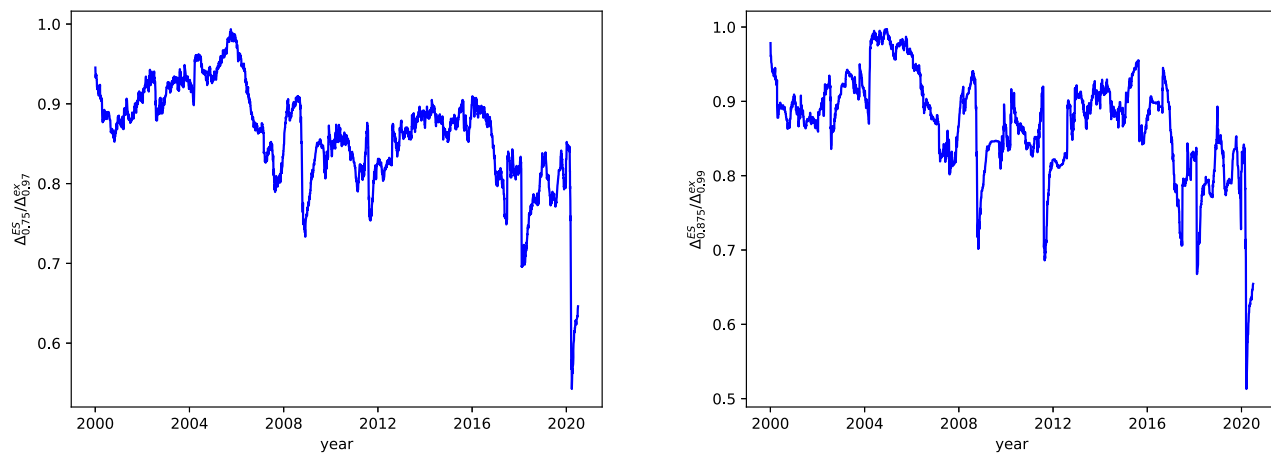

Fig. 4. The ratio of Δ_q^{ES} to Δ_r^{ex} using S&P 500 daily log-loss data (Jan 2000 - Jun 2020). Left: $(q, r) = (0.75, 0.97)$. Right: $(q, r) = (0.875, 0.99)$.

Table 2
Number of occurrences of the symmetric variability orders $\leqslant_{dil}$, $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ in the $78 = 13 \times 12/2$ pairs of years of daily log-returns of the S&P500 Index, ranging from 2008 to 2020, and corresponding pairs. For brevity we report only years' last two digits.

N	$\leqslant_{dil}$	$\leqslant_{\Delta\text{-ES}}$	$\leqslant_{\Delta\text{-ex}}$	Pairs of years (20XX)
66	✓	✓	✓	all the others
6	✗	✓	✓	(09, 11), (10, 15), (11, 18), (12, 14), (12, 13), (15, 18)
0	✗	✓	✗	–
2	✗	✗	✓	(08, 20), (10, 18)
4	✗	✗	✗	(12, 16), (12, 19), (13, 14), (16, 19)

panel, and no intersections in the right panel, so only the $\leqslant_{\Delta\text{-ex}}$ order applies.

In order to give a first exploratory assessment of how often the various symmetric variability orders do apply, we checked the comparability of daily log-returns of the S&P500 Index for each pair of years ranging from 2008 to 2020, for a total of $78 = 13 \times 12/2$ pairs. The results are reported in Table 2. It turns out that in 66 cases the $\leqslant_{dil}$ order applies, and so as a consequence also the other two weaker orders apply. In the remaining 12 cases, one or both of the $\leqslant_{\Delta\text{-ES}}$ and $\leqslant_{\Delta\text{-ex}}$ orders apply in 8 cases, so when the $\leqslant_{dil}$ order does not apply, we have a fraction of $8/12 \simeq 67\%$ of cases in which the data can still be compared. Notice also that the $\leqslant_{\Delta\text{-ES}}$ order without the $\leqslant_{\Delta\text{-ex}}$ order never occurred for this dataset; however, Example 3 in Section 4 shows that also this situation is theoretically possible.

8. Conclusion

In this paper, we introduce variability measures induced by three very popular parametric families of risk measures, that is, the inter-quantile, the inter-ES, and the inter-expectile differences. The three classes of variability measures enjoy many nice theoretical properties (Theorem 1); in particular, each of them characterizes symmetric distributions up to a location shift (Proposition 2). We study several desirable functional properties of general variability measures including the above three classes and many other classic ones; a grand summary is obtained in Theorem 2 and Table 1. The family of variability measures that satisfy a set of desirable properties is characterized as mixtures of inter-ES differences (Theorem 3). It is important to note that the three classes of variability measures introduced in this paper are well defined on L^1 and that
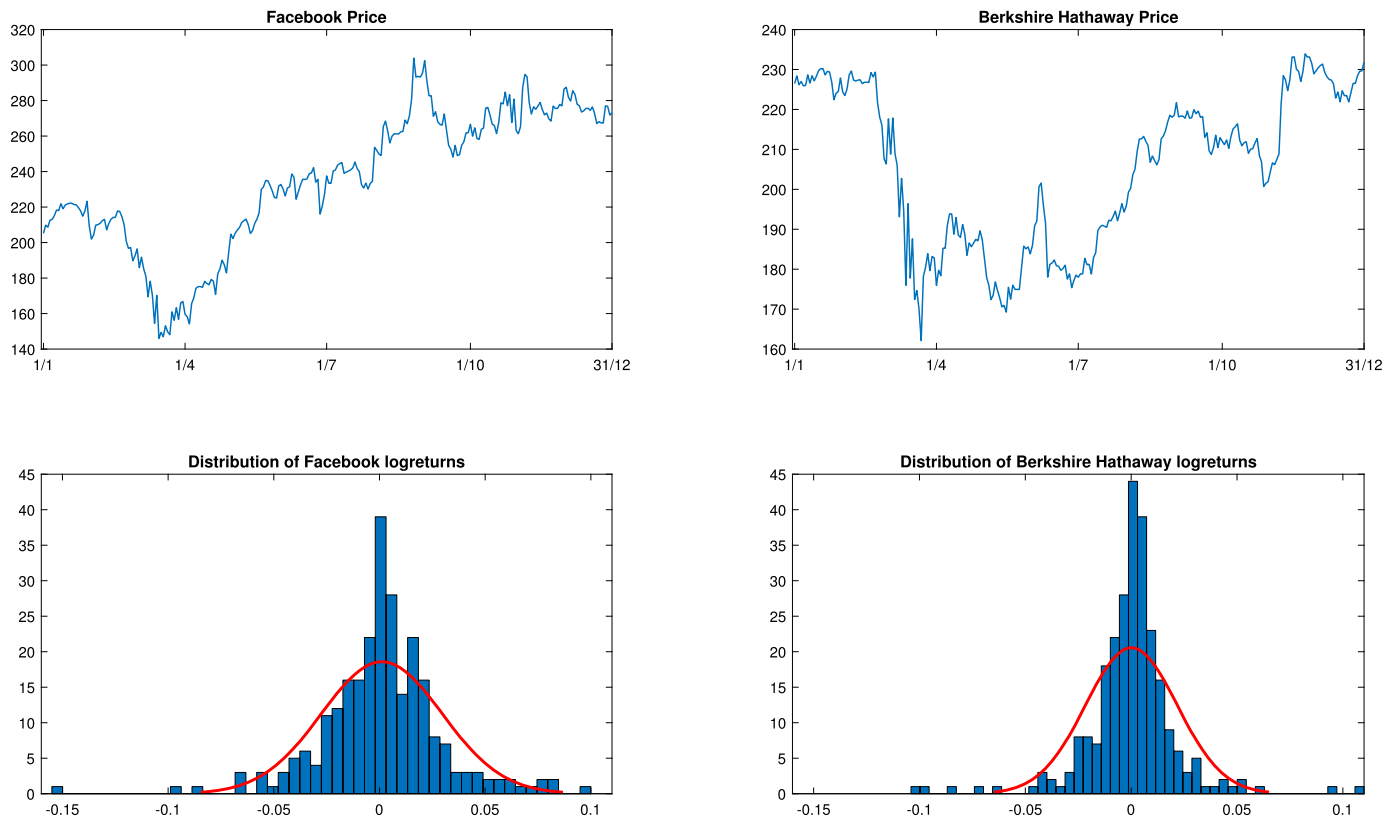
Fig. 5. Prices and log-return distributions of Facebook and Berkshire Hathaway in 2020.
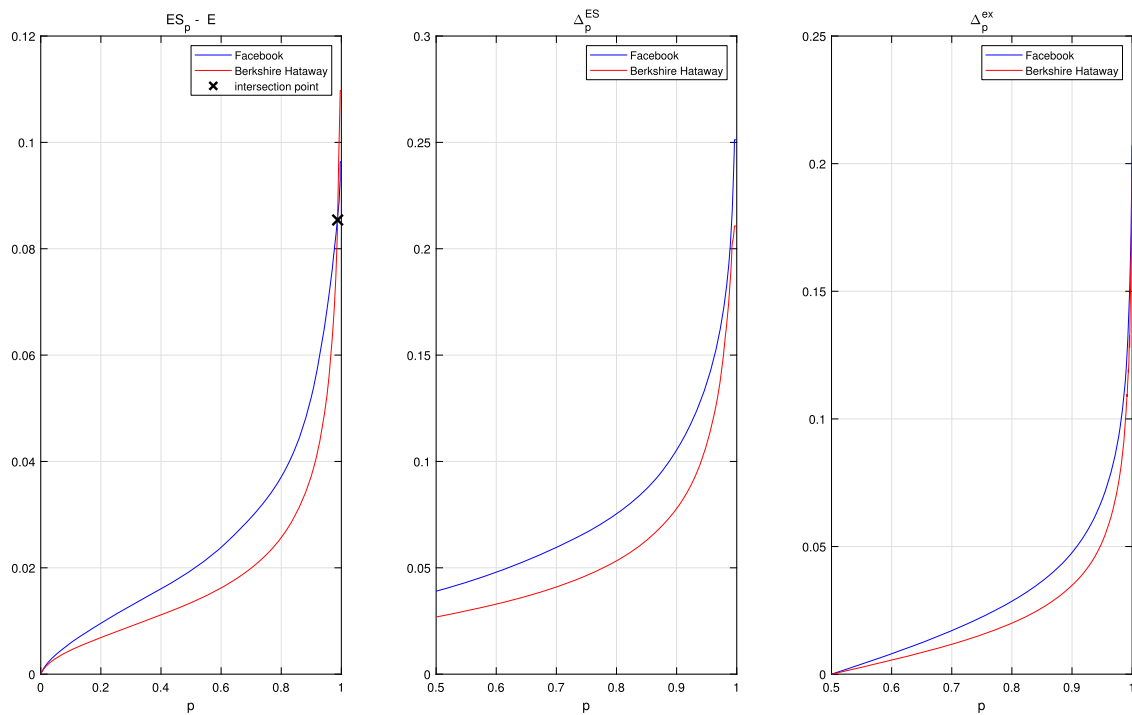

Fig. 6. Symmetric variability orderings between log-returns of Facebook and Berkshire Hathaway in 2020. To check comparability, we plot $ES_p - \mathbb{E}$ (left panel), Δ_p^{ES} (center panel) and Δ_p^{ex} (right panel) as a function of p. Facebook's log-returns dominate Berkshire Hathaway's in the $\leqslant_{\Delta\text{-}ES}$ and $\leqslant_{\Delta\text{-}ex}$ orders, but not in the $\leqslant_{dil}$ order.

each depends on a single parameter which allows for flexible applications. This distinguishes them from other deviation measures (e.g., Rockafellar et al. (2006)) where no parametric family is given. The empirical estimators of the inter-quantile, the inter-ES, and

the inter-expectile differences can be formulated based on those of VaR, ES and the expectile, and the asymptotic normality of the estimators is established (Theorem 4). In the financial application, we observe that the behavior of these variability measures is similar to
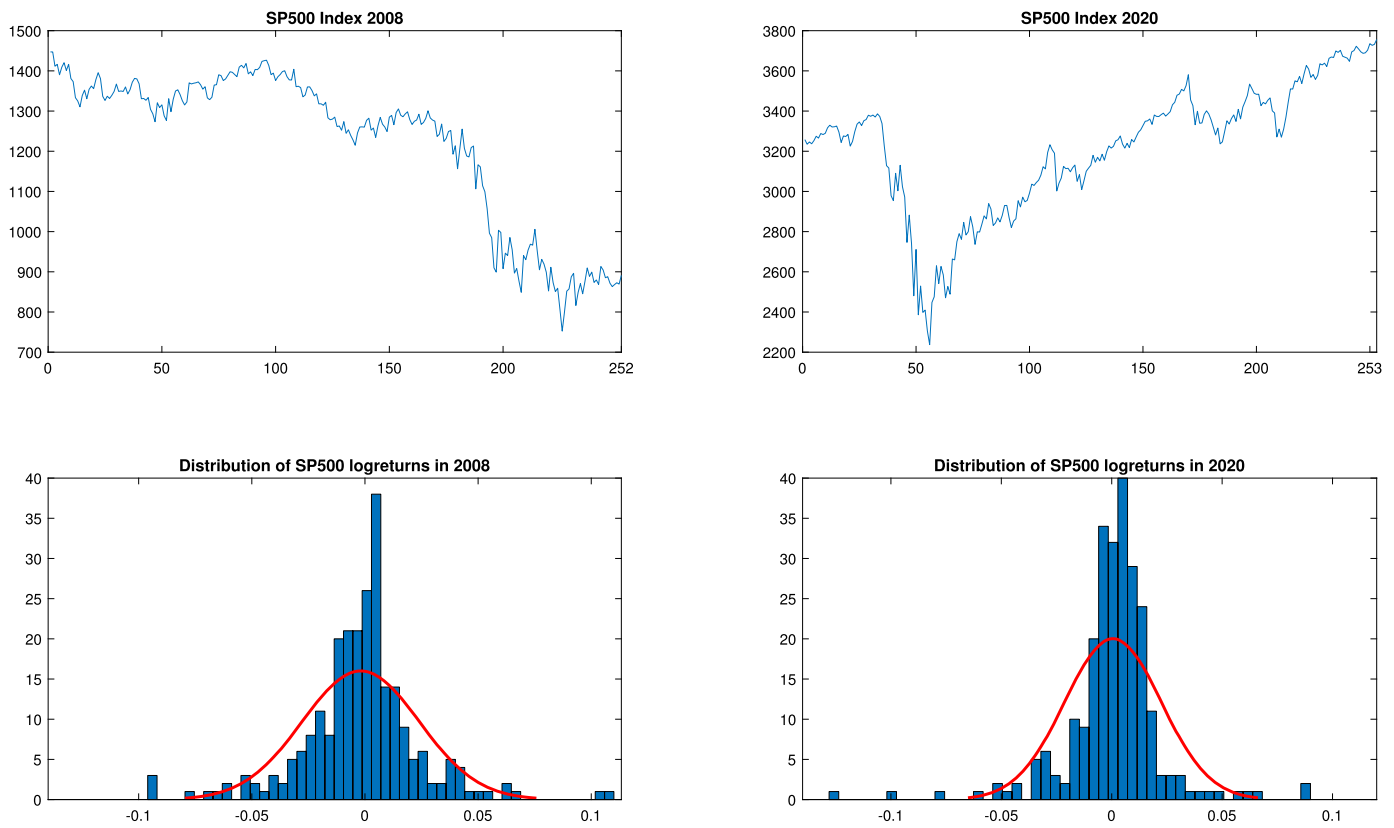
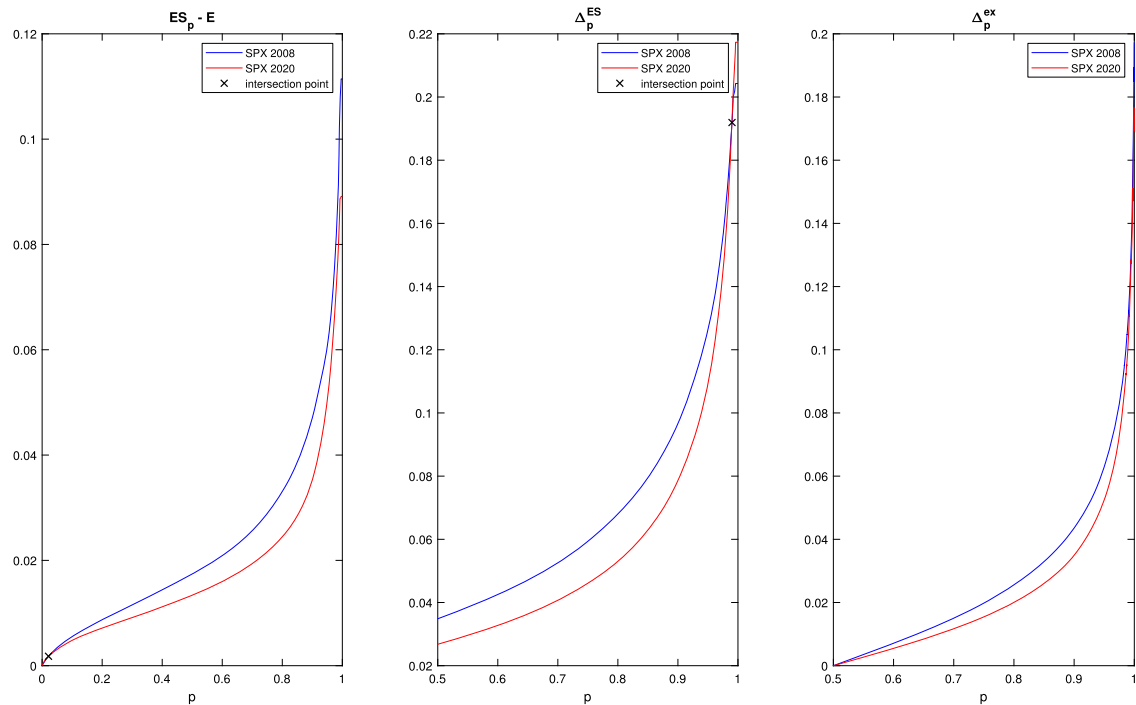**Fig. 7.** Values and log-return distributions of the SPX Index in 2008 and 2020.

Fig. 8. Symmetric variability orderings between log-returns of the SPX Index in 2008 and 2020. To check comparability, we plot $\mathrm{ES}_p - \mathbb{E}$ (left panel), Δ_p^{ES} (center panel) and Δ_p^{ex} (right panel) as a function of p. The SPX 2008 log-returns dominate the 2020 log-returns in the $\leqslant_{\Delta\text{-ex}}$ order, but not in the $\leqslant_{\mathrm{dil}}$ and in the $\leqslant_{\Delta\text{-ES}}$ orders.

the corresponding parametric families of risk measures. However, a comparison of different ratio of the variability measures reveals that Δ^{ex} is the most sensitive to extreme losses, and Δ^Q is the least sensitive.

For the end-user, if tail risk is of particular concern, then Δ^{ex} may be a better variability measure to use, as it captures tail-heaviness quite effectively. However, Δ^{ex} is usually cumbersome in computation and optimization because of the lack of explicit

formulas in terms of quantile or distribution functions; another technical disadvantage is that Δ^{ex} is not concave with respect to mixtures. On the other hand, if robustness is more important and tail risk is not relevant, then Δ^Q is a good choice, because quantiles are easy to compute and they are generally more robust than coherent risk measures including ES and expectiles (see Cont et al. (2010)). Moreover, Δ^Q is well defined on risks without a finite mean; nevertheless we should keep in mind that Δ^Q ignores tail risk just like a quantile. Finally, Δ^{ES} lies somewhere in between Δ^Q and Δ^{ex} regarding the above considerations, which giving rise to a good compromise; further, it is the only one among the three classes that is concave with respect to mixtures (see Table 1), and it is the building block for many other measures of variability (see Theorem 3).

In the literature, risk measures are commonly defined on a space of both positive and negative random variables. For this reason, our variability measures are also defined on such spaces, and we omit a detailed study of relative variability measures which are defined only for positive random variables. Relative variability measures include important examples such as the relative deviation and the Gini coefficient; see Appendix A. By replacing classic risk measures with relative risk measures (e.g., Peng et al. (2012)), one could define new classes of relative risk measures. On the other hand, other parametric families of risk measures, such as entropic risk measures (e.g., Föllmer and Schied (2016)) and RVaR (e.g., Embrechts et al. (2018)), can also be used to design flexible variability measures.

Declaration of competing interest

Acknowledgements

Appendix A. Classic variability measures

Below we list some classic variability measures, which are formulated on their respective effective domains.

(i) The variance (Var)

$$\mathbb{E}[(X - \mathbb{E}[X])^2], \quad X \in L^2.$$

(ii) The standard deviation (STD):

$$\sqrt{\mathrm{Var}(X)}, \quad X \in L^2.$$

(iii) The range (Δ_1):

$$\mathrm{ess\text{-}sup}(X) - \mathrm{ess\text{-}inf}(X), \quad X \in L^\infty.$$

(iv) The mean absolute deviation (MAD):

$$\mathbb{E}[|X - \mathbb{E}[X]|], \quad X \in L^1.$$

(v) The mean median deviation (MMD):

$$\min_{x \in \mathbb{R}} \mathbb{E}[|X - x|] = \mathbb{E}[|X - Q_{1/2}(X)|], \quad X \in L^1.$$

(vi) The Gini deviation (Gini-D):

$$\frac{1}{2} \mathbb{E}[|X_1 - X_2|], \quad X \in L^1, \ X_1, X_2, X \text{ are iid}.$$

(vii) The relative deviation:

$$\frac{\mathrm{SD}(X)}{\mathbb{E}[X]}, \quad X \in L_+^2.$$

(viii) The Gini coefficient:

$$\frac{\mathbb{E}[|X_1 - X_2|]}{2\mathbb{E}[X]} = \frac{\mathrm{Gini}(X)}{\mathbb{E}[X]}, \quad X \in L_+^1, \ X_1, X_2, X \text{ are iid}.$$

Here, L_+^q, $q \in [0, \infty]$ is the set of all non-negative random variables X in L^q with $\mathbb{P}(X > 0) > 0$.

Appendix B. Proofs of main results

Proof of Theorem 1. (i) Law invariance (A1) is obvious. For standardization (A2), note that the risk measures $\rho \in \{Q_p, Q_p^-, \mathrm{ES}_p, \mathrm{ES}_p^-, \mathrm{ex}_p\}$ are all monetary (Föllmer and Schied (2016)) and satisfies $\rho(m) = m$ for any constant m. Hence, for a constant m, $\Delta_p^Q(m) = \Delta_p^{\mathrm{ES}}(m) = \Delta_p^{\mathrm{ex}}(m) = 0$. Positive homogeneity follows from that of Q_p, Q_p^-, ES_p, ES_p^- and ex_p.

(ii) The effective domains of these variability measures can be easily checked from the effective domain of the corresponding risk measures.

(iii) Since Q_p is increasing in p and Q_{1-p}^- is decreasing in p, Δ_p^Q is increasing in p. The same applies to Δ_p^{ES} and Δ_p^{ex}.

(iv) It is well known that, for $X \in L^0$, $Q_p(-X) = -Q_{1-p}^-(X)$; see e.g., Föllmer and Schied (2016, (4.44)). Hence, $\Delta_p^Q(X) = Q_p(X) + Q_p(-X)$.

The formula for Δ_p^{ES}, $\mathrm{ES}_p(X) - \mathrm{ES}_{1-p}^-(X) = \mathrm{ES}_p(X) + \mathrm{ES}_p(-X)$, follows directly from definition.

Next we show the formula for Δ_p^{ex}. From Newey and Powell (1987), the expectile $\mathrm{ex}_p(X)$, for $p \in (1/2, 1)$ is the unique solution x to

$$p\mathbb{E}[(X - x)_+] = (1 - p)\mathbb{E}[(X - x)_-]. \tag{5}$$

Hence, the expectile of $-X$ satisfies

$$(1 - p)\mathbb{E}[(-X - \mathrm{ex}_{1-p}(-X))_+] = p\mathbb{E}[(-X - \mathrm{ex}_{1-p}(-X))_-].$$

This is equivalent to

$$p\mathbb{E}[(X + \mathrm{ex}_{1-p}(-X))_+] = (1 - p)\mathbb{E}[(X + \mathrm{ex}_{1-p}(-X))_-].$$

The uniqueness of solution x to (5) implies $-\mathrm{ex}_{1-p}(X) = \mathrm{ex}_p(-X)$. Hence,

$$\Delta_p^{\mathrm{ex}}(X) = \mathrm{ex}_p(X) - \mathrm{ex}_{1-p}(X) = \mathrm{ex}_p(X) + \mathrm{ex}_p(-X),$$

thus the desired formula. $\square$

Proof of Proposition 1. By definition, for $X \in L^1$,

$$(1 - p)\Delta_p^{\mathrm{ES}}(X) = (1 - p)\frac{1}{1 - p} \int_p^1 (Q_r(X) - Q_{1-r}(X)) \, \mathrm{d}r$$

$$= \int_{1-p}^1 (Q_r(X) - Q_{1-r}(X)) \, \mathrm{d}r$$

$$- \int_{1-p}^p (Q_r(X) - Q_{1-r}(X)) \, \mathrm{d}r$$

$$= p\Delta_{1-p}^{\mathrm{ES}}(X) - \int_{1-p}^p Q_r(X)\mathrm{d}r + \int_{1-p}^p Q_r(X)\mathrm{d}r$$

$$= p\Delta_{1-p}^{\mathrm{ES}}(X).$$

By Theorem 1,

$$\Delta_p^{\mathrm{ES}}(X) = \frac{1}{1-p}\int_p^1 Q_q(X)\mathrm{d}q + \frac{1}{1-p}\int_p^1 Q_q(-X)\mathrm{d}q$$

$$= \frac{1}{1-p}\int_p^1 \Delta_q^Q(X)\mathrm{d}q.$$

Hence, the desired statements hold. □

Proof of Theorem 2. We first explain some general observations on all variability measures in Table 1. The effective domains and the homogeneity indices follow directly from definition. Continuity (B2) is implied by L^q continuity since all variability measures are finite and thus continuous on their effective domains. Symmetry (B3) and location invariance (B8) are straightforward to check, and they hold for all variability measures in Table 1.

The conditions (B5)-(B7) are connected. In particular, Theorem 3 of Wang et al. (2020a) states that (B5)-(B7) are equivalent for *distortion riskmetrics*, which are functionals satisfying (A1), (B4) and some continuity assumptions. It is well known that the inter-quantile differences and the inter-ES differences are distortion riskmetrics.

Next, we explain that convexity (B6) implies Cx-consistency (B5) for all variability measures we consider. By Theorem 2.2 of Liu et al. (2020), all law-invariant convex risk functionals, i.e., functionals satisfying (A1), (B6) and (B8), can be written as the supremum of a family of convex distortion riskmetrics. Each distortion riskmetric is Cx-consistent as stated in Theorem 3 of Wang et al. (2020a), and hence (B5) is implied by (B6). The only negative statement for (B5) is made for the inter-quantile difference, which is a non-convex distortion riskmetric; this is shown in Table 1 of Wang et al. (2020a), which contains also a list of other examples of distortion riskmetrics with their corresponding properties. Hence, the inter-quantile difference does not satisfy any of (B5)-(B7).

It remains to verify (B1), (B4), (B6), (B7) for each variability measure.

(i) The following example shows that Δ_p^Q does not satisfy (B1). Take $\varepsilon > 0$ such that $p + \varepsilon < 1$ and $X \sim \mathrm{Bernoulli}(1 - p - \varepsilon)$. Notice that X is not a constant but $\Delta_p^Q(X) = Q_p(X) - Q_{1-p}^-(X) = 0 - 0 = 0$. C-additivity (B4) is satisfied since Δ_p^Q is a distortion riskmetric. (B6)-(B7) are explained above.

(ii) Δ_p^{ES}, Gini-D and range are all convex distortion riskmetrics; see Table 1 of Wang et al. (2020a). Hence, they all satisfy (B4)-(B7). Relevance (B1) can be easily verified.

(iii) If X is not a constant, by Newey and Powell (1987, Theorem 1), ex_p is strictly increasing in $p \in (0, 1)$, which means that $\Delta_p^{\mathrm{ex}}(X) = \mathrm{ex}_p(X) - \mathrm{ex}_{1-p}(X) > 0$ for $p \in (1/2, 1)$. By Proposition 7 of Bellini et al. (2014), ex_p is increasing in X, so for $|X| \leqslant 1$, $-1 \leqslant \mathrm{ex}_p(X) \leqslant 1$ for $p \in (0, 1)$. Thus $\Delta_p^{\mathrm{ex}}(X) \leqslant 2$ and Relevance (B1) is satisfied. Convexity (B6) is satisfied by Theorem 1 (iv) and convexity of expectiles.

We show that M-concavity (B7) is not satisfied by $\Delta_p^{\mathrm{ex}}(X)$ via the following example from Bellini et al. (2018a). Take $p = 1/10$. Define X by $\mathbb{P}(X = -1) = 1/2$, and $\mathbb{P}(X = 1) = 1/2$; Y by $\mathbb{P}(Y = 0) = 2/3$, $\mathbb{P}(Y = 5) = 1/3$. Then $\Delta_{1/10}^{\mathrm{ex}}(X) = -\frac{8}{5}$ and $\Delta_{1/10}^{\mathrm{ex}}(Y) = -\frac{800}{209}$.

Let $F = \frac{9}{10}F_X + \frac{1}{10}F_Y$ and $Z \sim F$. Then

$$\Delta_{1/10}^{\mathrm{ex}}(Z) = -\frac{2531}{1311} < \frac{9}{10}\Delta_{1/10}^{\mathrm{ex}}(X) + \frac{1}{10}\Delta_{1/10}^{\mathrm{ex}}(Y) = -\frac{9524}{5225},$$

and hence Δ_p^{ex} is not mixture concave.

C-additivity (B4) is not satisfied since by Theorem 3, a variability measure satisfying (B1)-(B5) must satisfy (B7).

(iv) For the variance, Relevance (B1) can be easily verified. Variance does not satisfy (B4) since (B4) requires the homogeneity index to be 1. For (B6), the variance is well known to be convex (Deprez and Gerber (1985)); see also Example 2.2 of Liu et al. (2020). The variance satisfies M-concavity (B7) because of the well known equality

$$\sigma^2(X) = \min_{x \in \mathbb{R}} \mathbb{E}[(X - x)^2], \quad X \in L^2.$$

Since σ^2 is the minimum of mixture-linear functionals, we know that it is mixture concave.

(v) For STD, Relevance (B1) can be easily verified. C-additivity (B4) is not satisfied by STD since STD is not additive for comonotonic random variables X and Y with correlation less than 1. STD is convex (B6); see Example 2.1 of Liu et al. (2020). To show that STD satisfies M-concavity (B7), take $X, Y \in L^1$ and let $Z \sim \lambda F_X + (1 - \lambda)F_Y$ for $\lambda \in [0, 1]$. By definition,

$$\sigma^2(Z) - (\lambda\sigma(X) + (1-\lambda)\sigma(Y))^2$$

$$= \lambda(1-\lambda)\left(\mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[X]\mathbb{E}[Y] - 2\sigma(X)\sigma(Y)\right)$$

$$= \lambda(1-\lambda)\left(\mathbb{E}^2[X] + \sigma^2(X) + \mathbb{E}^2[Y]\right.$$

$$\left. + \sigma^2(Y) - 2\mathbb{E}[X]\mathbb{E}[Y] - 2\sigma(X)\sigma(Y)\right)$$

$$= \lambda(1-\lambda)\left((\mathbb{E}[X] - \mathbb{E}[Y])^2 + (\sigma(X) - \sigma(Y))^2\right) \geqslant 0,$$

which is equivalent to $\sigma(Z) \geqslant \lambda\sigma(X) + (1-\lambda)\sigma(Y)$.

(vi) For the mean absolute deviation (MAD), Relevance (B1) can be easily verified. MAD satisfies convexity (B6), since, for $\lambda \in [0, 1]$ and $X, Y \in L^1$,

$$\mathbb{E}[|\lambda X + (1-\lambda)Y - \lambda\mathbb{E}[X] - (1-\lambda)\mathbb{E}[Y]|]$$

$$\leqslant \mathbb{E}[|\lambda X - \lambda\mathbb{E}[X]|] + \mathbb{E}[|(1-\lambda)(Y - \mathbb{E}[Y])|]$$

$$= \lambda\mathbb{E}[|X - \mathbb{E}[X]|] + (1-\lambda)\mathbb{E}[|Y - \mathbb{E}[Y]|].$$

We give an example showing that MAD does not satisfy M-concavity (B7). Take $X \sim \mathrm{Bernoulli}(1/3)$, and $Y \stackrel{\mathrm{d}}{=} -X$. Let $F = \frac{1}{2}F_X + \frac{1}{2}F_Y$ and $Z \sim F$. It is easy to calculate that $\mathbb{E}[X] = 1/3$, $\mathbb{E}[Y] = -1/3$, $\mathbb{E}[Z] = 0$, and $\mathbb{E}[|X - \mathbb{E}[X]|] = \mathbb{E}[|Y - \mathbb{E}[Y]|] = 4/9$. On the other hand,

$$\mathbb{E}[|Z - \mathbb{E}[Z]|] = \frac{1}{2}\mathbb{E}[|X|] + \frac{1}{2}\mathbb{E}[|Y|] = \frac{1}{3}.$$

Therefore,

$$\mathbb{E}[|Z - \mathbb{E}[Z]|] < \frac{1}{2}\mathbb{E}[|X - \mathbb{E}[X]|] + \frac{1}{2}\mathbb{E}[|Y - \mathbb{E}[Y]|],$$

and hence MAD is not mixture concave.

C-additivity (B4) is not satisfied by MAD since by Theorem 3, a variability measure satisfies (B1)-(B5) must satisfy (B7). □

Proof of Theorem 3. Write the functional $\nu_\mu = \int_0^1 \Delta_p^{\mathrm{ES}}\mathrm{d}\mu(p)$, which is the right-hand side of (1). First, obviously (i) implies (ii). It is also straightforward to check that (iii) implies (i), since Δ_p^{ES} for $p \in (0, 1]$ satisfies (B1)-(B8) by Theorem 2, and so is ν_μ; the only non-trivial statement is (B2) of ν_μ which is guaranteed by

Theorem 5 of Wang et al. (2020a), which shows that the representation ν_μ belongs to a class of convex distortion riskmetrics with continuity (B2). Below, we show (ii)⟹(iii).

Let $\mathcal{X}_\nu$ be the effective domain of ν. Take $X \in \mathcal{X}_\nu$ such that $\nu(X) > 0$. By (B4), $\nu(2X) = \nu(X) + \nu(X) = 2\nu(X)$. Hence, the homogeneity index of ν is 1.

Suppose that Cx-consistency (B5) holds. Take any $X, Y \in \mathcal{X}_\nu$ and let $X' \overset{d}{=} X$ and $Y' \overset{d}{=} X$ such that X' and Y' are comonotonic. It is well known that $X + Y \leqslant_{cx} X' + Y'$; see e.g., Theorem 3.5 of Rüschendorf (2013). Using (B4) and (B5), we have

$$\nu(X + Y) \leqslant \nu(X' + Y') = \nu(X') + \nu(Y') = \nu(X) + \nu(Y).$$

Therefore, ν is subadditive, that is,

$$\nu(X + Y) \leqslant \nu(X) + \nu(Y) \text{ for all } X, Y \in \mathcal{X}. \tag{6}$$

Note that convexity (B6) and homogeneity (A3) with $\alpha = 1$ together also imply subadditivity. Hence, either assuming (B5) or (B6), we get (6). It follows from (6) and (B1) that there exists $\beta > 0$ such that $\nu(Y) - \nu(X) \leqslant \nu(Y - X) \leqslant \beta \|Y - X\|_\infty$ where $\|Y - X\|_\infty$ is the essential supremum of $|Y - X|$. Hence, ν is uniformly continuous with respect to the supremum norm. Moreover, as a consequence of (B1), (A3) and (6), $\mathcal{X}_\nu$ is a convex cone that contains L^∞.

Theorem 1 of Wang et al. (2020a) states that a real functional on a convex cone that is uniformly continuous with respect to the supremum norm, law-invariant, and satisfying (B2) and (B4) is a distortion riskmetric in the sense of that paper; see (7) below. Further, Theorem 3 of Wang et al. (2020a) says that each of (B5)-(B7) is equivalent to the convexity of a distortion riskmetric. Hence, ν is a convex distortion riskmetric on $\mathcal{X}_\nu \cap L^1$. Theorem 5 of Wang et al. (2020a) gives a representation of convex distortion riskmetrics; that is, ν has a representation, for some finite measures μ_1 and μ_2,

$$\nu(X) = \int_0^1 \mathrm{ES}_p(X)\mathrm{d}\mu_1(p) + \int_0^1 \mathrm{ES}_p(-X)\mathrm{d}\mu_2(p), \quad X \in \mathcal{X}_\nu \cap L^1. \tag{7}$$

By symmetry (B3), we know

$$\nu(X) = \nu(-X)$$
$$= \int_0^1 \mathrm{ES}_p(X)\mathrm{d}\mu_2(p) + \int_0^1 \mathrm{ES}_p(-X)\mathrm{d}\mu_1(p), \quad X \in \mathcal{X}_\nu \cap L^1.$$

Hence, we can take $\mu = (\mu_1 + \mu_2)/2$, and get

$$\nu(X) = \int_0^1 \Delta_p^{\mathrm{ES}}(X)\mathrm{d}\mu(p), \quad X \in \mathcal{X}_\nu \cap L^1.$$

Relevance (B1) implies $\mu \neq 0$, which in turn implies $\mathcal{X}_\nu \subset L^1$, as the effective domain of Δ_p^{ES} is L^1 for $p \in (0, 1)$. Hence, the two functionals ν and ν_μ coincide on $\mathcal{X}_\nu$ which contains L^∞. Also note that both ν and ν_μ satisfy continuity (B2), and hence one can approximate any random variable outside $\mathcal{X}_\nu$ with truncated random variables, and obtain that ν and ν_μ also coincide on $\mathcal{X}$. □

Proof of Proposition 2. (i) If X has a symmetric distribution, then by Theorem 1 (iv), we have

$$\Delta_p^Q(X) = Q_p(X) - Q_{1-p}^-(X)$$
$$= -Q_{1-p}^-(-X) - Q_{1-p}^-(X) = -2Q_{1-p}^-(X).$$

Assume X_1 and X_2 are symmetric distributions with finite $\Delta_p^Q(X_1) = \Delta_p^Q(X_2)$ for $p \in (\frac{1}{2}, 1)$. It follows that $Q_p^-(X_1) = Q_p^-(X_2)$ for $p \in (0, \frac{1}{2})$. By the left-continuity of the left-quantile, $Q_{1/2}^-(X_1) = Q_{1/2}^-(X_2)$. By symmetry of the distribution of X, we have $Q_p^-(X_1) = Q_p^-(X_2)$ almost every p, and thus X_1 and X_2 have the same distribution.

(ii) If X has a symmetric distribution, then similarly to (i), we have $\Delta_p^{\mathrm{ES}}(X) = -2\mathrm{ES}_{1-p}^-(X)$. Assume that X_1 and X_2 have symmetric distributions with finite $\Delta_p^{\mathrm{ES}}(X_1) = \Delta_p^{\mathrm{ES}}(X_2)$ for $p \in (\frac{1}{2}, 1)$. It follows that for $p \in (0, \frac{1}{2})$, $\mathrm{ES}_p^-(X_1) = \mathrm{ES}_p^-(X_2)$ holds, which means

$$\int_0^p Q_r(X_1)\mathrm{d}r = \int_0^p Q_r(X_2)\mathrm{d}r. \tag{8}$$

By taking a derivative of both sides of (8) with respect to p, we get

$$Q_p(X_1) = Q_p(X_2)$$

at all common continuity points p of $p \mapsto Q_p(X_1)$ and $p \mapsto Q_p(X_1)$. Since both functions are right-continuous, we know that the two functions are identical. This argument can be applied to any $p \in (0, \frac{1}{2})$. Similarly to part (i), we conclude that X_1 and X_2 have the same distribution.

(iii) If X has a symmetric distribution, then similarly to (i), we have

$$\Delta_p^{\mathrm{ex}}(X) = 2\mathrm{ex}_p(X) = -2\mathrm{ex}_{1-p}(X).$$

Suppose X_1 and X_2 have symmetric distributions with finite $\Delta_p^{\mathrm{ex}}(X_1) = \Delta_p^{\mathrm{ex}}(X_2)$ for $p \in (\frac{1}{2}, 1)$. Then $\mathrm{ex}_p(X_1) = \mathrm{ex}_p(X_2)$ for $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. By symmetry, we observe that $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$, which means $\mathrm{ex}_{\frac{1}{2}}(X_1) = \mathrm{ex}_{\frac{1}{2}}(X_2) = 0$, so $\mathrm{ex}_p(X_1) = \mathrm{ex}_p(X_2)$ for $p \in (0, 1)$.

The expectile has alternative definitions from Newey and Powell (1987),

$$\mathrm{ex}_p(X) = \mathbb{E}[X] + \frac{2p - 1}{1 - p}\mathbb{E}[(X - \mathrm{ex}_p(X))_+],$$

which leads to

$$\mathbb{E}[(X_1 - \mathrm{ex}_p(X_1))_+] = \mathbb{E}[(X_2 - \mathrm{ex}_p(X_2))_+].$$

Since $\mathrm{ex}_p(X)$ is continuous in p and takes all values in the range of X, we know

$$\mathbb{E}[(X_1 - x)_+] = \mathbb{E}[(X_2 - x)_+]$$

for all $x \in \mathbb{R}$, implying that the distributions of X_1 and X_2 are identical. □

Proof of Proposition 3. (i), (ii), (iii) follow immediately, respectively from location invariance, positive homogeneity of order 1 and symmetry of Δ_p^{ES} and Δ_p^{ex}, while (iv) follows immediately from the second part of the thesis of Proposition 1.

(v) By passing if necessary to the random variables $\tilde{X} = X - \mathbb{E}[X]$ and $\tilde{Y} = Y - \mathbb{E}[Y]$, from (i) we can assume without loss of generality that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Then $X \leqslant_{dil} Y \Rightarrow X \leqslant_{cx} Y$, and the thesis follows from Cx-consistency of Δ_p^{ES} and Δ_p^{ex}, for each $p \in (1/2, 1)$.

(vi) As in (v), we can assume without loss of generality that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Then $\Delta_p^{\mathrm{ES}}(X) = 2\mathrm{ES}_p(X)$, so $X \leqslant_{\Delta\text{-ES}} Y \Rightarrow \mathrm{ES}_p(X) \leqslant \mathrm{ES}_p(Y) \Rightarrow \int_p^1 Q_r(X)\mathrm{d}r \leqslant \int_p^1 Q_r(Y)\mathrm{d}r$, for each $p \in$

(1/2, 1). From symmetry and the assumption $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ it follows that the same inequality holds also for each $p \in (0, 1/2)$, that implies $X \leqslant_{\mathrm{cx}} Y$ by Theorem 3.A.5 in Shaked and Shantikumar (2009). Similarly, under symmetry $\Delta_p^{\mathrm{ex}}(X) = 2\mathrm{ex}_p(X)$, so $X \leqslant_{\Delta\text{-ex}} Y \Rightarrow \mathrm{ex}_p(X) \leqslant \mathrm{ex}_p(Y)$ for each $p \in (1/2, 1)$, and since $\mathrm{ex}_p(X) = \mathrm{ex}_p(-X) = -\mathrm{ex}_{1-p}(X)$, the opposite inequality holds for $p \in (0, 1/2)$. By reasoning as in the proof of Theorem 12 of Bellini et al. (2018b), it follows that $\pi_X(x) \leqslant \pi_Y(x)$ for each $x \in \mathbb{R}$, where $\pi_X(x) := \mathbb{E}[(X - x)_+]$ and $\pi_Y(x) := \mathbb{E}[(Y - x)_+]$ are the usual stop-loss transforms of X and Y; the thesis then follows from Theorem 3.A.1 of Shaked and Shantikumar (2009). □

Proof of Theorem 4. (i) Let $\widehat{Q}_p(n)$, $\widehat{\mathrm{ES}}_p(n)$, and $\widehat{\mathrm{ex}}_p(n)$ be the empirical estimators of $Q_p(X)$, $\mathrm{ES}_p(X)$, and $\mathrm{ex}_p(X)$ based on n sample data points. It is well known (e.g., Bahadur (1966)) that $\widehat{Q}_r(n) \xrightarrow{\mathrm{p}} Q_r(X)$ at each r of continuous point of $Q_r(X)$, which implies $\widehat{\Delta}_p^Q(n) \xrightarrow{\mathrm{p}} \Delta_p^Q(X)$ under assumption (R). Since ES_p and ex_p are law-invariant convex risk measures, by Theorem 2.6 of Krätschmer et al. (2014), $\widehat{\mathrm{ES}}_r(n) \xrightarrow{\mathrm{p}} \mathrm{ES}_r(X)$ and $\widehat{\mathrm{ex}}_r(n) \xrightarrow{\mathrm{p}} \mathrm{ex}_r(X)$ for each r. Hence we have $\widehat{\Delta}_p^{\mathrm{ES}}(n) \xrightarrow{\mathrm{p}} \Delta_p^{\mathrm{ES}}(X)$ and $\widehat{\Delta}_p^{\mathrm{ex}}(n) \xrightarrow{\mathrm{p}} \Delta_p^{\mathrm{ex}}(X)$.

(ii) By Proposition 1 of Shorack and Wellner (2009, p.640), if assumption (R) is satisfied, then we have

$$\sqrt{n}\left(\widehat{Q}_p(n) - Q_p(X)\right) \xrightarrow{\mathrm{d}} \frac{B_p}{g(p)}. \tag{9}$$

where B_p is a standard Brownian bridge. With assumption (R), $Q_p(X) = Q_p^-(X)$. Hence,

$$\sqrt{n}\left(\widehat{\Delta}_p^Q(n) - \Delta_p^Q(X)\right) \xrightarrow{\mathrm{d}} \frac{B_p}{g(p)} - \frac{B_{1-p}}{g(1-p)},$$

which has a Gaussian distribution. Using the covariance property of the Brownian bridge, that is, $\mathrm{Cov}[B_t, B_s] = s - st$ for $s < t$, we have

$$\mathrm{Cov}\left[\frac{B_p}{g(p)}, \frac{B_{1-p}}{g(1-p)}\right] = \frac{(1-p)^2}{g(p)g(1-p)}.$$

Therefore, $\sqrt{n}(\widehat{\Delta}_p^Q(n) - \Delta_p^Q(X)) \xrightarrow{\mathrm{d}} N(0, \sigma_Q^2)$, where σ_Q^2 is in (2), namely,

$$\sigma_Q^2 = \frac{p(1-p)}{g^2(p)} + \frac{p(1-p)}{g^2(1-p)} - 2\frac{(1-p)^2}{g(p)g(1-p)}.$$

Next, we address the inter-ES difference. Applying the convergence in (9) to ES_p, we obtain

$$\sqrt{n}\left(\widehat{\mathrm{ES}}_p(n) - \mathrm{ES}_p(X)\right) \xrightarrow{\mathrm{d}} \frac{1}{1-p}\int_p^1 \frac{B_s}{g(s)}\mathrm{d}s,$$

and thus

$$\sqrt{n}(\widehat{\Delta}_p^{\mathrm{ES}}(n) - \Delta_p^{\mathrm{ES}}(X)) \xrightarrow{\mathrm{d}} \frac{1}{1-p}\int_p^1 \frac{B_s}{g(s)}\mathrm{d}s - \frac{1}{1-p}\int_0^{1-p} \frac{B_s}{g(s)}\mathrm{d}s.$$

Note that

$$\mathrm{Var}\left[\frac{1}{1-p}\int_p^1 \frac{B_s}{g(s)}\mathrm{d}s\right] = \mathbb{E}\left[\frac{1}{(1-p)^2}\int_p^1\int_p^1 \frac{B_s B_t}{g(s)g(t)}\mathrm{d}t\mathrm{d}s\right]$$

$$= \frac{1}{(1-p)^2}\int_p^1\int_p^1 \frac{s \wedge t - st}{g(s)g(t)}\mathrm{d}t\mathrm{d}s,$$

and

$$\frac{1}{(1-p)^2}\mathrm{Cov}\left[\int_p^1 \frac{1}{g(t)}B_t\mathrm{d}t, \int_0^{1-p} \frac{1}{g(t)}B_t\mathrm{d}t\right]$$

$$= \frac{1}{(1-p)^2}\int_p^1\int_0^{1-p} \frac{s \wedge t - st}{g(t)g(s)}\mathrm{d}t\mathrm{d}s.$$

Hence, $\sqrt{n}(\widehat{\Delta}_p^{\mathrm{ES}}(n) - \Delta_p^{\mathrm{ES}}(X)) \xrightarrow{\mathrm{d}} N(0, \sigma_{\mathrm{ES}}^2)$, with σ_{ES}^2 given in (3), namely,

$$\sigma_{\mathrm{ES}}^2 = \frac{1}{(1-p)^2}\left(\int_{[p,1]^2 \cup [0,1-p]^2} -2\int_{[p,1]\times[0,1-p]}\right)\frac{s \wedge t - st}{g(t)g(s)}\mathrm{d}t\mathrm{d}s.$$

For the inter-expectile difference, we use Theorem 3.2 of Krätschmer and Zähle (2017). The conditions for this theorem are satisfied in our setting noting that $X \in L^{2+\delta}$; see Remark 3.4 of Krätschmer and Zähle (2017). We obtain, for $p \in (1/2, 1)$,

$$\sqrt{n}(\widehat{\mathrm{ex}}_p(n) - \mathrm{ex}_p(X)) \to N(0, s_p^{\mathrm{ex}})$$

where for $r \in \{1 - p, p\}$,

$$s_r^{\mathrm{ex}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{r,F}^{\mathrm{ex}}(t)f_{r,F}^{\mathrm{ex}}(s)F(t \wedge s)(1 - F(t \vee s))\mathrm{d}t\mathrm{d}s,$$

and

$$f_{r,F}^{\mathrm{ex}}(t) = \frac{(1-r)\mathbb{1}_{\{t \leqslant \mathrm{ex}_r(X)\}} + r\mathbb{1}_{\{t > \mathrm{ex}_r(X)\}}}{(1-2r)F(\mathrm{ex}_r(X)) + r}.$$

Similar arguments as above lead to

$$\sqrt{n}(\widehat{\Delta}_p^{\mathrm{ex}}(n) - \Delta_p^{\mathrm{ex}}(X)) \xrightarrow{\mathrm{d}} N(0, s_p^{\mathrm{ex}} + s_{1-p}^{\mathrm{ex}} - 2c_p^{\mathrm{ex}}),$$

where

$$c_p^{\mathrm{ex}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{p,F}^{\mathrm{ex}}(t)f_{1-p,F}^{\mathrm{ex}}(s)F(t \wedge s)(1 - F(t \vee s))\mathrm{d}t\mathrm{d}s.$$

This completes the proof. □

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