

# Central limit theorem and self-normalized Cramér-type moderate deviation for Euler-Maruyama scheme

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We consider a stochastic differential equation and its Euler-Maruyama (EM) scheme, under some appropriate conditions, they both admit a unique invariant measure, denoted by  $\pi$  and  $\pi_\eta$  respectively ( $\eta$  is the step size of the EM scheme). We construct an empirical measure  $\Pi_\eta$  of the EM scheme as a statistic of  $\pi_\eta$ , and use Stein's method developed in Fang, Shao and Xu (*Probab. Theory Related Fields* **174** (2019) 945–979) to prove a central limit theorem of  $\Pi_\eta$ . The proof of the self-normalized Cramér-type moderate deviation (SNCMD) is based on a standard decomposition on Markov chain, splitting  $\eta^{-1/2}(\Pi_\eta(\cdot) - \pi(\cdot))$  into a martingale difference series sum  $\mathcal{H}_\eta$  and a negligible remainder  $\mathcal{R}_\eta$ . We handle  $\mathcal{H}_\eta$  by the time-change technique for martingale, while prove that  $\mathcal{R}_\eta$  is exponentially negligible by concentration inequalities, which have their independent interest. Moreover, we show that SNCMD holds for  $x = o(\eta^{-1/6})$ , which has the same order as that of the classical result in Shao (*J. Theoret. Probab.* **12** (1999) 385–398), Jing, Shao and Wang (*Ann. Probab.* **31** (2003) 2167–2215).

**Keywords:** Stochastic differential equation; Euler-Maruyama scheme; central limit theorem; self-normalized Cramér-type moderate deviation; Stein's method

## 1. Introduction

We consider the following stochastic differential equation (SDE) on  $\mathbb{R}^d$ :

$$dX_t = g(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x, \tag{1.1}$$

where  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  and  $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfy Assumption 2.1 below, and  $B_t$  is a  $d$ -dimensional standard Brownian motion. Given a step size  $\eta$ , the Euler-Maruyama (EM) scheme of (1.1) reads as

$$\theta_{k+1} = \theta_k + \eta g(\theta_k) + \sqrt{\eta} \sigma(\theta_k) \xi_{k+1}, \quad k \geq 0, \tag{1.2}$$

where  $(\xi_k)_{k \geq 1}$  are i.i.d. standard  $d$ -dimensional normal random vectors. When  $g$  and  $\sigma$  are both Lipschitz, (1.1) admits a unique strong solution and the following strong approximation error bound holds, see Mao [26]: for any  $T > 0$ ,

$$\mathbb{E}|X_T - \theta_{\lfloor T/\eta \rfloor}|^2 \leq C_T \eta, \tag{1.3}$$

the constant  $C_T$  usually tends to  $\infty$  as  $T \rightarrow \infty$  and  $\lfloor x \rfloor$  denotes the integer part of  $x$  for a  $x > 0$ . When  $g$  or  $\sigma$  is irregular, there have recently been some works, see Bao, Huang and Yuan [2] for the convergence rate of degenerate SDEs. We refer the reader to Bao and Shao [3], Shao [35] for the

EM scheme of path-dependent SDEs and to Bao and Yuan [4] for that of stochastic differential delay equations.

Let us first discuss a special case of (1.1) in which  $\sigma(x) \equiv I_{d \times d}$ ,  $d \times d$  identity matrix, and  $g(x) = -\nabla U(x)$  with  $U$  being a potential, it is well known that (1.1) is a gradient system and admits a unique ergodic measure  $\pi$  proportional to  $e^{-U(x)}$  from Roberts and Tweedie [34]. (1.2) is called unadjusted Langevin algorithm (ULA) with constant step size, Roberts and Tweedie [34] mainly established some criteria for the ergodicity of  $\theta_k$ , while Dalalyan [9], Theorem 2, gave an explicit error in total variation distance between  $\theta_k$  and  $\pi$  in terms of  $d, k, \eta$  when  $\nabla U$  is Lipschitz and strong convex. Replacing the strong convexity assumption in Dalalyan [9] with a strong convexity at infinity condition, Majka, Mijatović and Szpruch [25] used a coupling method to show the Wasserstein-2 distance between  $\theta_k$  and  $\pi$  were bounded by  $C[(1 - \eta)^{k/2} + \eta^{1/4}]$ . When  $\nabla U$  is third order differentiable with a appropriate growth condition but not necessarily Lipschitz, Fang, Shao and Xu [19] showed that as long as the above (1.2) admits a unique ergodic measure  $\pi_\eta$ , then the Wasserstein-1 distance between  $\pi_\eta$  and  $\pi$  is bounded by  $\sqrt{\eta}$  up to a logarithmic correction. For more research about Langevin algorithm, we refer the reader to Durmus and Moulines [13,14], Chatterji et al. [6] and the references therein.

The motivations of studying the central limit theorem (CLT) and the self-normalized Cramér type moderate deviation (SNCMD) of ULA are two folds. One is that there have been many central limit theorems and moderate deviation results for Markov chain Monte Carlo (MCMC) algorithm, see Dupuis and Johnson [12], Meyn and Tweedie [28], Del Moral, Hu and Wu [11], Nyquist [30], Tierney [40], whereas there are very few these type of fluctuation theorems for Langevin algorithm. The other is that our result provides a new example for SNCMD for dependent time series, and also a new example that applies Stein’s method to prove SNCMD, see Chen, Fang and Shao [7], Shao, Zhang and Zhang [37]. Note that there are not many results for SNCMD for dependent time series, see Chen et al. [8], Fan [15], Fan et al. [16,17], Fang, Luo and Shao [18], Jing, Wang and Zhou [21], Shao and Zhou [38] and the references therein.

Let us briefly describe our main results and methods as follows. We construct an empirical measure  $\Pi_\eta$  as a statistic of the ergodic measure  $\pi_\eta$  of (1.2), for any function  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$  (see the definition of  $C_b^2(\mathbb{R}^d, \mathbb{R})$  below), we study the CLT and SNCMD of  $\Pi_\eta(h)$ . In order to prove the CLT, we apply Stein’s method developed in Fang, Shao and Xu [19]. Assumption 2.1 guarantees that (1.2) admits a unique invariant measure  $\pi_\eta$ , while the restriction of  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$  ensures that the solution  $\varphi$  of Stein’s equation (2.9) has bounded 4th order derivatives. Note that the ergodicity of (1.1) does not imply that of (1.2), see Roberts and Tweedie [34]. The proof of SNCMD is based on a standard decomposition on Markov chain, splitting  $\eta^{-1/2}(\Pi_\eta(h) - \pi(h))$  into a martingale difference series sum  $\mathcal{H}_\eta$  and a negligible remainder  $\mathcal{R}_\eta$ . We handle  $\mathcal{H}_\eta$  by the time-change technique for martingale, while prove that  $\mathcal{R}_\eta$  is exponentially negligible by concentration inequalities, which have their independent interest. Moreover, we show that SNCMD holds for  $x = o(\eta^{-1/6})$ , which has the same order as that of the classical result in Shao [36], Jing, Shao and Wang [20]. Indeed, the limit  $\lim_{\eta \rightarrow 0} (\Pi_\eta(h) - \pi(h)) = 0$  can be understood as a law of large number (LLN), after zooming in on it by a scale  $\eta^{-1/2}$ ,  $\eta^{-1/2}(\Pi_\eta(h) - \pi(h))$  has a normal distributed fluctuation. Our result showed that this fluctuation is uniformly comparable with normal distribution for all  $x \in (c\eta^{1/6}, o(\eta^{-1/6}))$ . In contrast, Shao et al.’s result means that by zooming in on  $\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}X$  with a scale  $n^{1/2}$ ,  $n^{1/2}(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}X)$  has a normal distributed fluctuation uniformly comparable with normal distribution for all  $x \in [0, o(n^{1/6})]$ .

The paper is organized as the following. Our main results are stated and discussed in Section 2. In Section 3, we provide some preliminary lemmas. The proof of the CLT is given in Section 4. In Section 5, we give the proof of SNCMD. The details of the proof of preliminary lemmas are deferred to Appendix.

We finish this section by introducing some notations which will be frequently used in sequel. For  $x \in \mathbb{R}^d$ ,  $x_i$  denotes the  $i$ -th element of  $x$ . For function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , denote  $\nabla_{i,j,k}^3 f(x) = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$  with

$i, j, k = 1, 2, \dots, d$ .  $C_b^k(\mathbb{R}^d, \mathbb{R})$  with  $k \geq 1$  denotes the collection of all bounded  $k$ -th order continuously differentiable functions. The symbols  $C$  and  $c$  denote positive numbers depending on  $g$  and  $\sigma$ ,  $C_p$  and  $c_p$  denote positive numbers depending on  $g, \sigma$  and the parameter  $p$ . Their values may vary from line to line. We denote the Euclidean norm of  $\mathbb{R}^d$  by  $|\cdot|$  and for higher rank tensors by  $\|\cdot\|$ . For function  $f$ , we denote  $\|f\| = \sup_{x \in \mathbb{R}^d} \|f(x)\|$ . If a random variable  $\xi$  has a probability distribution  $\mu$ , we write  $\xi \sim \mu$ . Let  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  be two nonnegative real number sequences, if there exist some  $C > 0$  such that  $|a_n| \leq C b_n$ , we write  $a_n = O(b_n)$ . If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ , we write  $a_n = o(b_n)$ .

## 2. Main results

**Assumption 2.1.**  $\sigma(x) \equiv \sigma$  with  $\sigma$  being an invertible  $d \times d$  matrix.  $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is second order differentiable. There exist  $L, K_1 > 0$  and  $K_2 \geq 0$  such that for every  $x, y \in \mathbb{R}^d$

$$|g(x) - g(y)| \leq L|x - y|, \tag{2.1}$$

$$\langle g(x) - g(y), x - y \rangle \leq -K_1|x - y|^2 + K_2. \tag{2.2}$$

Moreover, the second order derivative of  $g$  is bounded.

**Remark 2.2.** It is easy to see that the assumption (2.1) implies

$$|g(x)|^2 \leq 2L^2|x|^2 + 2|g(0)|^2, \quad \|\nabla g\| \leq L, \tag{2.3}$$

and that the assumption (2.2) and Young’s inequality imply

$$\begin{aligned} \langle x, g(x) \rangle &= \langle x - 0, g(x) - g(0) \rangle + \langle x, g(0) \rangle \\ &\leq -K_1|x|^2 + K_2 + \frac{K_1}{2}|x|^2 + \frac{1}{2K_1}|g(0)|^2 = -\frac{K_1}{2}|x|^2 + C. \end{aligned} \tag{2.4}$$

The condition that  $g$  has bounded second order derivative is only needed for proving the regularity to the solution of Stein’s equation.

Under Assumption 2.1, the Euler-Maruyama scheme reads as

$$\theta_{k+1} = \theta_k + \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1}, \quad k \geq 0, \tag{2.5}$$

where  $\theta_0 = x$  and  $(\xi_k)_{k \geq 1}$  are i.i.d. standard  $d$ -dimensional normal random vectors.

**Lemma 2.3.** Under Assumption 2.1, SDE (1.1) and  $(\theta_k)_{k \geq 0}$  are both ergodic with invariant measures  $\pi$  and  $\pi_\eta$  respectively.

**Proof.** The proof will be given in Appendix A. □

The generator  $\mathcal{A}$  of (1.1) is given by

$$\mathcal{A}f(x) = \langle g(x), \nabla f(x) \rangle + \frac{1}{2} \langle \sigma \sigma^T, \nabla^2 f(x) \rangle_{\text{HS}}, \tag{2.6}$$

where  $T$  is the transpose operator and  $\langle A, B \rangle_{HS} := \sum_{i,j=1}^d A_{ij} B_{ij}$  for  $A, B \in \mathbb{R}^{d \times d}$ , and  $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$ . To approximate the behavior of  $(X_t)_{t \geq 0}$ , we can use the Euler-Maruyama scheme to discrete (1.1).

For a small  $\eta \in (0, 1)$ , define

$$\Pi_\eta(\cdot) = \frac{1}{[\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}]-1} \delta_{\theta_k}(\cdot), \tag{2.7}$$

where  $\delta_y(\cdot)$  is a delta measure of  $y$ , i.e., for any  $A \subset \mathbb{R}^d$ ,  $\delta_y(A) = 1$  if  $y \in A$  and  $\delta_y(A) = 0$  if  $y \notin A$ . We shall see that  $\Pi_\eta$  is an asymptotically consistent statistic of  $\pi$  as  $\eta \rightarrow 0$ .

Parallel to the CLT and tail probability estimates of MCMC algorithms, see Roberts and Rosenthal [33], it is natural to consider those for  $\Pi_\eta$ . For a test function  $h : \mathbb{R}^d \rightarrow \mathbb{R}$ , we consider the limit of  $\frac{\Pi_\eta(h) - \pi(h)}{\sqrt{\eta}}$  with  $\pi(h) = \int_{\mathbb{R}^d} h(x)\pi(dx)$ . Our first main result is

**Theorem 2.4.** *Suppose that Assumption 2.1 holds. Let  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$ , then we have*

$$\frac{1}{\sqrt{\eta}}(\Pi_\eta(h) - \pi(h)) \Rightarrow N(0, \pi(|\sigma^T \nabla \varphi|^2)), \quad \text{as } \eta \rightarrow 0, \tag{2.8}$$

where  $\varphi$  is the solution to the following Stein's equation:

$$h - \pi(h) = \mathcal{A}\varphi, \tag{2.9}$$

and  $\mathcal{A}$  is the generator (2.6) of the SDE (1.1).

Let  $\mathbb{E}_k[\cdot]$  and  $\mathbb{P}_k(\cdot)$  be respectively the conditional expectation  $\mathbb{E}[\cdot|\theta_k]$  and conditional probability  $\mathbb{P}(\cdot|\theta_k)$ . Let  $\Phi(x)$  be the standard normal distribution function. Denote <sup>1</sup>

$$\mathcal{Y}_\eta = \frac{1}{[\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}]-1} |\sigma^T \nabla \varphi(\theta_k)|^2, \quad \mathcal{W}_\eta = \frac{\eta^{-\frac{1}{2}}(\Pi_\eta(h) - \pi(h))}{\sqrt{\mathcal{Y}_\eta}}.$$

Our second main result is the SNCMD of  $\mathcal{W}_\eta$  as follows.

**Theorem 2.5.** *Suppose that Assumption 2.1 holds. Let  $\theta_0 \sim \pi_\eta$  and  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$ , we have*

$$\frac{\mathbb{P}(\mathcal{W}_\eta \geq x)}{1 - \Phi(x)} = 1 + O(x\eta^{\frac{1}{6}} + \eta^{\frac{1}{6}}) \tag{2.10}$$

uniformly for  $c\eta^{\frac{1}{6}} \leq x = o(\eta^{-\frac{1}{6}})$  as  $\eta$  vanishes, where  $c, O$  and  $o$  depend on  $L, K_1, K_2, |g(0)|^2, \sigma$ .

For the simplicity of notations below, without loss of generality, we assume from now on that  $\eta \in (0, 1)$  is a small number such that  $\eta^{-1}$  is an integer. We also denote

$$m = \eta^{-2}$$

<sup>1</sup>Prof. Fuqing Gao suggested that we replace the self-normalized factor  $\frac{1}{[\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}]-1} (\sigma^T \nabla \varphi(\theta_k), \xi_{k+1})^2$  in the previous version by  $\frac{1}{[\eta^{-2}]} \sum_{k=0}^{[\eta^{-2}]-1} |\sigma^T \nabla \varphi(\theta_k)|^2$ . Since  $(\theta_k)_{k \geq 0}$  is observable whereas  $(\xi_k)_{k \geq 1}$  is not known, the new self-normalized factor is more natural.

and often write  $\eta^{-1}$  as  $m\eta$  for notational simplicity. Denote

$$\Delta\theta_k = \theta_{k+1} - \theta_k, \quad k \geq 0.$$

### 3. Auxiliary lemmas for Theorem 2.4 and Theorem 2.5

#### 3.1. The strategy of proving Theorem 2.4 and Theorem 2.5

The strategy of proving Theorem 2.4 and Theorem 2.5 is to decompose  $\eta^{-\frac{1}{2}}(\Pi_\eta(h) - \pi(h))$  into a martingale and a remainder as in (3.1) below, showing that the remainder is negligible, while the martingale converges weakly to a normal distribution and satisfies the SNCMD. This type of decomposition is typical for proving CLT for semi-martingales, see e.g., Teh, Thiery and Vollmer [39], (28).

**Lemma 3.1.** *Let  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$ , a solution to Stein’s equation (2.9) is given by*

$$\varphi(x) = - \int_0^\infty \mathbb{E}[h(X_t(x)) - \pi(h)]dt,$$

where  $X_t(x)$  is the solution of equation (1.1) with initial value  $x$ . Moreover,

$$\|\nabla^k \varphi\| \leq C, \quad k = 0, 1, 2, 3, 4.$$

**Proof.** Denote  $\hat{h} = h - \pi(h)$  and  $P_t h(x) = \mathbb{E}[h(X_t(x))]$ . Following the exponential ergodicity of  $\{X_t\}_{t \geq 0}$ , i.e. (A.1), one has

$$\left| \int_0^\infty P_s \hat{h}(x) ds \right| \leq \int_0^\infty |P_s \hat{h}(x)| ds \leq CV(x) \int_0^\infty e^{-cs} ds < \infty.$$

Thus  $\int_0^\infty P_s \hat{h}(x) ds$  is well defined. For any  $\varepsilon > 0$ , it is known that  $\varepsilon - \mathcal{A}$  is invertible (cf. Applebaum [1], pp. 158-159), and

$$(\varepsilon - \mathcal{A})^{-1} \hat{h} = \int_0^\infty e^{-\varepsilon t} P_t \hat{h} dt,$$

i.e.,

$$\varepsilon \int_0^\infty e^{-\varepsilon t} P_t \hat{h} dt - \hat{h} = \mathcal{A} \left( \int_0^\infty e^{-\varepsilon t} P_t \hat{h} dt \right).$$

Let  $\varepsilon \rightarrow 0+$ ,

$$\varepsilon \int_0^\infty e^{-\varepsilon t} P_t \hat{h} dt - \hat{h} \rightarrow -\hat{h}, \quad \int_0^\infty e^{-\varepsilon t} P_t \hat{h} dt \rightarrow \int_0^\infty P_t \hat{h} dt.$$

Since  $\mathcal{A}$  is a closed operator (cf. Partington [31], Theorem 2.2.6),  $\int_0^\infty P_t \hat{h} dt$  is in the domain of  $\mathcal{A}$  and

$$\hat{h} = \mathcal{A} \left( - \int_0^\infty P_t \hat{h} dt \right).$$

By Krylov and Priola [22], Theorem 2.6, we know that  $\varphi \in C_b^3(\mathbb{R}^d, \mathbb{R})$ . Denoting  $\varphi_i = \partial_{x_i} \varphi$  for  $i = 1, \dots, d$ , it satisfies

$$\mathcal{A}\varphi_i = \partial_{x_i} h - \partial_{x_i} g\varphi,$$

it is easy to check that the right hand side of this equation belongs to  $C_b^1(\mathbb{R}^d)$  by Assumption 2.1, we know that  $\varphi_i \in C_b^3(\mathbb{R}^d, \mathbb{R})$  by Krylov and Priola [22], Theorem 2.6. Hence,  $\varphi \in C_b^4(\mathbb{R}^d, \mathbb{R})$ .  $\square$

By Stein’s equation (2.9), we have,

$$\begin{aligned} \Pi_\eta(h) - \pi(h) &= \frac{1}{m} \sum_{k=0}^{m-1} (h(\theta_k) - \pi(h)) = \frac{1}{m} \sum_{k=0}^{m-1} \mathcal{A}\varphi(\theta_k) \\ &= \eta \sum_{k=0}^{m-1} [\mathcal{A}\varphi(\theta_k)\eta - (\varphi(\theta_{k+1}) - \varphi(\theta_k))] + \eta \sum_{k=0}^{m-1} (\varphi(\theta_{k+1}) - \varphi(\theta_k)) \\ &= \eta[\varphi(\theta_m) - \varphi(\theta_0)] + \eta \sum_{k=0}^{m-1} [\mathcal{A}\varphi(\theta_k)\eta - (\varphi(\theta_{k+1}) - \varphi(\theta_k))]. \end{aligned}$$

(2.5), (2.6) and the Taylor expansion yield that

$$\begin{aligned} &\mathcal{A}\varphi(\theta_k)\eta - (\varphi(\theta_{k+1}) - \varphi(\theta_k)) \\ &= \frac{\eta}{2} \langle \nabla^2 \varphi(\theta_k), \sigma \sigma^T \rangle_{\text{HS}} - \sqrt{\eta} \langle \nabla \varphi(\theta_k), \sigma \xi_{k+1} \rangle - \frac{1}{2} \langle \nabla^2 \varphi(\theta_k), (\Delta \theta_k)(\Delta \theta_k)^T \rangle_{\text{HS}} \\ &\quad - \frac{1}{6} \int_0^1 \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 \varphi(\theta_k + t \Delta \theta_k) (\Delta \theta_k)_{i_1} (\Delta \theta_k)_{i_2} (\Delta \theta_k)_{i_3} dt. \end{aligned}$$

This, together with the previous two relations and  $\Delta \theta_k = \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1}$ , implies

$$\eta^{-\frac{1}{2}} (\Pi_\eta(h) - \pi(h)) = \mathcal{H}_\eta + \mathcal{R}_\eta, \tag{3.1}$$

where, as we shall see below,  $\mathcal{H}_\eta$  is a martingale and  $\mathcal{R}_\eta$  is a remainder, given by

$$\mathcal{H}_\eta = -\eta \sum_{k=0}^{m-1} \langle \nabla \varphi(\theta_k), \sigma \xi_{k+1} \rangle, \quad \mathcal{R}_\eta = -\sum_{i=1}^6 \mathcal{R}_{\eta, i},$$

with

$$\begin{aligned} \mathcal{R}_{\eta,1} &= \sqrt{\eta} (\varphi(\theta_0) - \varphi(\theta_m)), \quad \mathcal{R}_{\eta,2} = \frac{\eta^{\frac{3}{2}}}{2} \sum_{k=0}^{m-1} \langle \nabla^2 \varphi(\theta_k), (\sigma \xi_{k+1})(\sigma \xi_{k+1})^T - \sigma \sigma^T \rangle_{\text{HS}}, \\ \mathcal{R}_{\eta,3} &= \frac{\eta^2}{2} \sum_{k=0}^{m-1} \left[ \langle \nabla^2 \varphi(\theta_k), g(\theta_k)(\sigma \xi_{k+1})^T \rangle_{\text{HS}} + \langle \nabla^2 \varphi(\theta_k), \sigma \xi_{k+1}(g(\theta_k))^T \rangle_{\text{HS}} \right], \\ \mathcal{R}_{\eta,4} &= \frac{\eta^2}{6} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 \varphi(\theta_k + t \Delta \theta_k) (\sigma \xi_{k+1})_{i_1} (\sigma \xi_{k+1})_{i_2} (\sigma \xi_{k+1})_{i_3} dt, \\ \mathcal{R}_{\eta,5} &= \frac{\eta^{\frac{5}{2}}}{2} \sum_{k=0}^{m-1} \langle \nabla^2 \varphi(\theta_k), g(\theta_k)g(\theta_k)^T \rangle_{\text{HS}} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\eta^{\frac{7}{2}}}{6} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 \varphi(\theta_k + t \Delta \theta_k)(g(\theta_k))_{i_1}(g(\theta_k))_{i_2}(g(\theta_k))_{i_3} dt, \\
 \mathcal{R}_{\eta, 6} & = \frac{\eta^{\frac{5}{2}}}{2} \sum_{k=0}^{m-1} \int_0^1 \sum_{i_1, i_2, i_3=1}^d \left[ \nabla_{i_1, i_2, i_3}^3 \varphi(\theta_k + t \Delta \theta_k)(g(\theta_k))_{i_1}(\sigma \xi_{k+1})_{i_2}(\sigma \xi_{k+1})_{i_3} \right. \\
 & \quad \left. + \sqrt{\eta} \nabla_{i_1, i_2, i_3}^3 \varphi(\theta_k + t \Delta \theta_k)(g(\theta_k))_{i_1}(g(\theta_k))_{i_2}(\sigma \xi_{k+1})_{i_3} \right] dt.
 \end{aligned}$$

A crucial lemma for estimating the remainder  $\mathcal{R}_\eta$  is Lemma 3.3, which has a long proof as below. To better understand the proof’s strategy, we give a continuous version as the following. For the solution  $X_t$  of SDE (1.1) and a constant  $\gamma > 0$  which will be chosen later, Itô’s formula implies

$$\begin{aligned}
 |X_t|^2 - |x|^2 & = \int_0^t 2\langle X_s, g(X_s) \rangle ds + \int_0^t 2\langle X_s, \sigma dB_s \rangle + t \|\sigma\|^2 \\
 & \leq - \int_0^t K_1 |X_s|^2 ds + \int_0^t 2\langle X_s, \sigma dB_s \rangle + (C + \|\sigma\|^2)t,
 \end{aligned}$$

where the second line follows (2.4). Then we have

$$\mathbb{E} \exp \left\{ \gamma |X_t|^2 + \int_0^t \gamma K_1 |X_s|^2 ds \right\} \leq e^{\gamma |x|^2} e^{\gamma(C + \|\sigma\|^2)t} \mathbb{E} \exp \left\{ \int_0^t 2\gamma \langle X_s, \sigma dB_s \rangle \right\}.$$

Hölder’s inequality and the exponential martingale property yield

$$\begin{aligned}
 & \mathbb{E} \exp \left\{ \int_0^t 2\gamma \langle X_s, \sigma dB_s \rangle \right\} \\
 & \leq \left( \mathbb{E} \exp \left\{ \int_0^t 4\gamma \langle X_s, \sigma dB_s \rangle - \int_0^t 8\gamma^2 |X_s^T \sigma|^2 ds \right\} \right)^{\frac{1}{2}} \left( \mathbb{E} \exp \left\{ \int_0^t 8\gamma^2 |X_s^T \sigma|^2 ds \right\} \right)^{\frac{1}{2}} \\
 & = \left( \mathbb{E} \exp \left\{ \int_0^t 8\gamma^2 |X_s^T \sigma|^2 ds \right\} \right)^{\frac{1}{2}} \leq \left( \mathbb{E} \exp \left\{ \int_0^t \gamma K_1 |X_s|^2 ds \right\} \right)^{\frac{1}{2}},
 \end{aligned}$$

where we choose  $\gamma$  small enough such that  $8\gamma \|\sigma\|^2 \leq K_1$  in the last inequality. That is

$$\begin{aligned}
 \mathbb{E} \exp \left\{ \gamma |X_t|^2 + \int_0^t \gamma K_1 |X_s|^2 ds \right\} & \leq e^{\gamma |x|^2} e^{\gamma(C + \|\sigma\|^2)t} \left( \mathbb{E} \exp \left\{ \int_0^t \gamma K_1 |X_s|^2 ds \right\} \right)^{\frac{1}{2}} \\
 & \leq e^{\gamma |x|^2} e^{\gamma(C + \|\sigma\|^2)t} \left( \mathbb{E} \exp \left\{ \gamma |X_t|^2 + \int_0^t \gamma K_1 |X_s|^2 ds \right\} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Hence by (2.3), we have

$$\begin{aligned}
 \mathbb{E} \exp \left\{ \gamma |X_t|^2 + \int_0^t \frac{\gamma K_1}{2L^2} |g(X_s)|^2 ds \right\} & \leq \mathbb{E} \exp \left\{ \gamma |X_t|^2 + \int_0^t \gamma K_1 |X_s|^2 ds \right\} e^{\frac{\gamma K_1}{L^2} |g(0)|^2 t} \\
 & \leq C e^{ct}.
 \end{aligned}$$

Replacing  $\frac{\gamma K_1}{2L^2}$  by  $\tilde{\gamma}$ , we can get

$$\mathbb{E} \exp \left\{ \gamma |X_t|^2 \right\} \leq C e^{ct}, \quad \mathbb{E} \exp \left\{ \int_0^t \tilde{\gamma} |g(X_s)|^2 ds \right\} \leq C e^{ct}.$$

### 3.2. Auxiliary lemmas for $\mathcal{R}_\eta$

We will give in this subsection several lemmas of  $\mathcal{R}_\eta$  which play a crucial role in proving main results. Their proofs will be given in Appendix B and the supplemental article Lu, Tan and Xu [24]. In order to estimate the tail probability of  $\mathcal{R}_\eta$ , we need the following four lemmas, the first three lemmas paving a way for proving the last.

**Lemma 3.2.** *Let  $\Psi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\Psi_2 : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  both be measurable functions. We have*

$$\mathbb{E}_k \left[ \exp \{ \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1}) \} \right] \leq \left( \mathbb{E}_k \left[ \exp \left\{ 2|\Psi_1(\theta_k)|^2 \|\sigma\|^2 + 2\Psi_2(\theta_k, \xi_{k+1}) \right\} \right] \right)^{\frac{1}{2}}$$

for  $k = 0, \dots, m - 1$ . Moreover, we have

$$\mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} (\langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1})) \right\} \leq \left( \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} 2 \left( |\Psi_1(\theta_k)|^2 \|\sigma\|^2 + \Psi_2(\theta_k, \xi_{k+1}) \right) \right\} \right)^{\frac{1}{2}},$$

and

$$\begin{aligned} & \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} (\langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1})) \right\} \\ & \leq \left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} 2 \left( |\Psi_1(\theta_k)|^2 \|\sigma\|^2 + \Psi_2(\theta_k, \xi_{k+1}) \right) \right\} \right)^{\frac{1}{2}}. \end{aligned}$$

**Lemma 3.3.** *Under Assumption 2.1, there exist  $\eta_0 > 0$  and  $\gamma_0 > 0$ , both depending on  $L, K_1, K_2, |g(0)|^2$  and  $\sigma$ , such that as  $\eta < \eta_0$  and  $\gamma < \gamma_0$ ,*

$$\mathbb{E}_0 \exp \left\{ \gamma \eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 \right\} \leq C e^{c(\eta^{-1} + |\theta_0|^2)}, \tag{3.2}$$

where  $C$  and  $c$  depend on  $L, K_1, K_2, |g(0)|^2, \sigma$  and  $\gamma$ . Moreover, if  $\theta_0 \sim \pi_\eta$ ,

$$\mathbb{E} \exp \left\{ \gamma \eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 \right\} \leq C e^{c\eta^{-1}}, \tag{3.3}$$

where  $C$  and  $c$  depend on  $L, K_1, K_2, |g(0)|^2, \sigma$  and  $\gamma$ . This particular implies that for all  $x > 0$ ,

$$\mathbb{P}_0 \left( \eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 > x \right) \leq C e^{c_1(\eta^{-1} + |\theta_0|^2)} e^{-c_2 x}, \tag{3.4}$$



where  $C, c_1, c_2$  depends on  $L, K_1, K_2, |g(0)|^2, \sigma$  and  $\gamma_0$ . Moreover, if  $\theta_0 \sim \pi_\eta$ ,

$$\mathbb{P}\left(\eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 > x\right) \leq C e^{c_1 \eta^{-1}} e^{-c_2 x}, \tag{3.5}$$

where  $C, c_1, c_2$  depends on  $L, K_1, K_2, |g(0)|^2, \sigma$  and  $\gamma_0$ .

**Lemma 3.4.** Let  $\Psi : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$  be measurable function satisfying the conditions

$$\mathbb{E}_k[\Psi(\theta_k, \xi_{k+1})] = 0 \quad \text{and} \quad \mathbb{P}_k(|\Psi(\theta_k, \xi_{k+1})| \leq K(1 + |\xi_{k+1}|^2)) = 1$$

for  $k = 0, \dots, m - 1$ , where  $K \in (0, \infty)$  is an arbitrary constant. Then we have

$$\mathbb{E}\left[\exp\left\{\frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \Psi(\theta_k, \xi_{k+1})\right\}\right] \leq C,$$

where  $C$  depends on  $K$ .

**Lemma 3.5.** Suppose that Assumption 2.1 holds. Let  $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$  and  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  be the solution of (2.9). We have

$$\mathbb{P}_0(|\mathcal{R}_\eta| > x) \leq C e^{c_1 \|\theta_0\|^2} \left( e^{-c_2 \eta^{-\frac{1}{2}} x^{\frac{1}{2}}} \mathbb{1}_{\{x < \eta^{-1}\}} + e^{-c_2 \eta^{-\frac{3}{5}} x^{\frac{2}{5}}} \mathbb{1}_{\{x \geq \eta^{-1}\}} + e^{-c_2 \eta^{-2\bar{\gamma}} x^{\frac{2}{3}}} \right),$$

where  $0 < \bar{\gamma} < \frac{1}{4}$  and  $x \geq c \max\{\eta^{\frac{3}{2}-6\bar{\gamma}}, \eta^{\frac{3}{2}\bar{\gamma}}, \eta^{\frac{1}{2}}\}$ . Here  $C, c, c_1, c_2$  depends on  $L, K_1, K_2, |g(0)|^2, \sigma$ . Moreover, for  $\theta_0 \sim \pi_\eta$ , we have

$$\mathbb{P}(|\mathcal{R}_\eta| > x) \leq C \left( e^{-c_1 \eta^{-\frac{1}{2}} x^{\frac{1}{2}}} \mathbb{1}_{\{x < \eta^{-1}\}} + e^{-c_1 \eta^{-\frac{3}{5}} x^{\frac{2}{5}}} \mathbb{1}_{\{x \geq \eta^{-1}\}} + e^{-c_1 \eta^{-2\bar{\gamma}} x^{\frac{2}{3}}} \right),$$

where  $0 < \bar{\gamma} < \frac{1}{4}$  and  $x \geq c \max\{\eta^{\frac{3}{2}-6\bar{\gamma}}, \eta^{\frac{3}{2}\bar{\gamma}}, \eta^{\frac{1}{2}}\}$ . Here  $C, c, c_1$  depends on  $L, K_1, K_2, |g(0)|^2, \sigma$ .

The proof of Lemma 3.5 is much more complicated and we give details in the supplemental article Lu, Tan and Xu [24].

### 4. Proof of Theorem 2.4

We first introduce following lemma which paves a way to proving the convergence of martingale  $\mathcal{H}_\eta$ . Its proof borrows the idea of the Stein’s method in Fang, Shao and Xu [19], Theorem 2.5.

**Lemma 4.1.** Let  $\pi$  and  $\pi_\eta$  be the same as those in Lemma 2.3,  $\varphi$  be the solution of Stein’s equation (2.9). We have

$$|\pi_\eta(|\sigma^T \nabla \varphi|^2) - \pi(|\sigma^T \nabla \varphi|^2)| \leq C \eta^{\frac{1}{2}}.$$

Here,  $C$  depends on  $g$  and  $\sigma$ .

**Proof.** We shall use the stationary Markov chain trick in Fang, Shao and Xu [19], Theorem 2.5. Let  $\{\theta_k\}_{k \geq 0}$  be the Markov chain with initial value  $\theta_0 \sim \pi_\eta$ . (2.5) implies that

$$\mathbb{E}_0[\Delta\theta_0] = \eta g(\theta_0), \quad \mathbb{E}_0[(\Delta\theta_0)(\Delta\theta_0)^T] = \eta^2 g(\theta_0)g^T(\theta_0) + \eta\sigma\sigma^T. \tag{4.1}$$

The Taylor expansion and the stationarity of  $(\theta_k)_{k \geq 0}$  yield

$$\begin{aligned} 0 &= \mathbb{E}[|\sigma^T \nabla\varphi(\theta_1)|^2 - |\sigma^T \nabla\varphi(\theta_0)|^2] \\ &= \mathbb{E}[|\nabla|\sigma^T \nabla\varphi(\theta_0)|^2, \Delta\theta_0] + \frac{1}{2} \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \Delta\theta_0(\Delta\theta_0)^T \rangle_{\text{HS}}] \\ &\quad + \frac{1}{6} \int_0^1 \mathbb{E} \left[ \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 |\sigma^T \nabla\varphi(\theta_0 + t\Delta\theta_0)|^2 (\Delta\theta_0)_{i_1} (\Delta\theta_0)_{i_2} (\Delta\theta_0)_{i_3} \right] dt. \end{aligned} \tag{4.2}$$

For the first and the second terms, by (4.1), we have

$$\begin{aligned} \mathbb{E}[\langle \nabla|\sigma^T \nabla\varphi(\theta_0)|^2, \Delta\theta_0 \rangle] &= \mathbb{E}[\langle \nabla|\sigma^T \nabla\varphi(\theta_0)|^2, \mathbb{E}_0[\Delta\theta_0] \rangle] = \mathbb{E}[\langle \nabla|\sigma^T \nabla\varphi(\theta_0)|^2, \eta g(\theta_0) \rangle], \\ \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \Delta\theta_0(\Delta\theta_0)^T \rangle_{\text{HS}}] &= \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \mathbb{E}_0[\Delta\theta_0(\Delta\theta_0)^T] \rangle_{\text{HS}}] \\ &= \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \eta^2 g(\theta_0)g^T(\theta_0) + \eta\sigma\sigma^T \rangle_{\text{HS}}]. \end{aligned}$$

Combining equalities above with (2.6) and (4.2), we have

$$\begin{aligned} \mathbb{E}[\mathcal{A}(|\sigma^T \nabla\varphi(\theta_0)|^2)] &= -\frac{1}{2} \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \eta g(\theta_0)g^T(\theta_0) \rangle_{\text{HS}}] \\ &\quad - \frac{1}{6\eta} \int_0^1 \mathbb{E} \left[ \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 |\sigma^T \nabla\varphi(\theta_0 + t\Delta\theta_0)|^2 (\Delta\theta_0)_{i_1} (\Delta\theta_0)_{i_2} (\Delta\theta_0)_{i_3} \right] dt. \end{aligned}$$

For the first term, the boundedness of  $\|\nabla^3\varphi\|$  and (A.5) imply

$$\left| \frac{1}{2} \mathbb{E}[\langle \nabla^2|\sigma^T \nabla\varphi(\theta_0)|^2, \eta g(\theta_0)g^T(\theta_0) \rangle_{\text{HS}}] \right| \leq C\pi_\eta(|g|^4)^{1/2} \eta \leq C\eta.$$

For the second term, by (A.5), we can get

$$\mathbb{E}[|g(\theta_0)|^3] \leq (\mathbb{E}[|g(\theta_0)|^4])^{3/4} = \pi_\eta(|g|^4)^{3/4} < \infty.$$

Cauchy's inequality and the boundedness of  $\|\nabla^4\varphi\|$  imply

$$\begin{aligned} &\left| \frac{1}{6\eta} \int_0^1 \mathbb{E} \left[ \sum_{i_1, i_2, i_3=1}^d \nabla_{i_1, i_2, i_3}^3 |\sigma^T \nabla\varphi(\theta_0 + t\Delta\theta_0)|^2 (\Delta\theta_0)_{i_1} (\Delta\theta_0)_{i_2} (\Delta\theta_0)_{i_3} \right] dt \right| \\ &\leq \frac{C}{\eta} \int_0^1 \mathbb{E}[\|\nabla^3|\sigma^T \nabla\varphi(\theta_0 + t\Delta\theta_0)|^2\| |\Delta(\theta_0)|^3] dt \leq C(\eta^2 \mathbb{E}[|g(\theta_0)|^3] + \eta^{\frac{1}{2}} \mathbb{E}[|\xi_1|^3]) \leq C\eta^{\frac{1}{2}}. \end{aligned}$$

Here, the constant  $C$  depends on  $\sigma$  and  $g$ . Hence we have

$$|\mathbb{E}[\mathcal{A}(|\sigma^T \nabla \varphi(\theta_0)|^2)]| \leq C\eta^{\frac{1}{2}}.$$

From Stein's equation (2.9), we deduce

$$|\pi_\eta(|\sigma^T \nabla \varphi|^2) - \pi(|\sigma^T \nabla \varphi|^2)| = |\mathbb{E}[|\sigma^T \nabla \varphi(\theta_0)|^2] - \pi(|\sigma^T \nabla \varphi|^2)| = |\mathbb{E}[\mathcal{A}(|\sigma^T \nabla \varphi(\theta_0)|^2)]| \leq C\eta^{\frac{1}{2}}.$$

□

**Lemma 4.2.** *Under the condition of Theorem 2.4, we have*

$$\mathcal{H}_\eta \Rightarrow N(0, \pi(|\sigma^T \nabla \varphi|^2)).$$

**Proof.** Recall  $\mathcal{H}_\eta = -\eta \sum_{i=0}^{m-1} \langle \nabla \varphi(\theta_i), \sigma \xi_{i+1} \rangle$ . We denote

$$Z_i = \langle \nabla \varphi(\theta_i), \sigma \xi_{i+1} \rangle, \quad i = 0, \dots, m-1.$$

McLeish [27], Theorem 2.3, will imply the result if we can verify the conditions

$$\mathbb{E} \max_{0 \leq i \leq m-1} \{\eta |Z_i|\} \rightarrow 0, \tag{4.3}$$

$$\eta^2 \sum_{i=0}^{m-1} Z_i^2 \rightarrow \pi(|\sigma^T \nabla \varphi|^2) \text{ in probability.} \tag{4.4}$$

Denoting  $\hat{Z}_i = Z_i 1_{\{|Z_i|^2 \leq \eta^{-1}\}}$  and  $\check{Z}_i = Z_i 1_{\{|Z_i|^2 > \eta^{-1}\}}$ , we have

$$\eta^2 \left( \max_{0 \leq i \leq m-1} \{|Z_i|\} \right)^2 = \eta^2 \max_{0 \leq i \leq m-1} \{|Z_i|^2\} \leq \eta^2 \max_{0 \leq i \leq m-1} \{|\hat{Z}_i|^2\} + \eta^2 \max_{0 \leq i \leq m-1} \{|\check{Z}_i|^2\}.$$

It is easily to see that the first term converges to 0 in probability. For the second term, we have

$$\eta^2 \mathbb{E} \max_{0 \leq i \leq m-1} \{|\check{Z}_i|^2\} \leq \eta^2 \sum_{0 \leq i \leq m-1} \mathbb{E} |\check{Z}_i|^2.$$

Since  $\mathbb{E}[Z_i^2]$  is finite,  $\mathbb{E}|\check{Z}_i|^2$  converges to 0 as  $\eta \rightarrow 0$  for each  $i$ , this implies that  $\eta^2 \times \mathbb{E}[\max_{0 \leq i \leq m-1} \{|\check{Z}_i|^2\}]$  converges to 0. Hölder's inequality yields (4.3).

For (4.4), we can finish the proof if we verify

$$\begin{aligned} & \mathbb{E}[\eta^2 \sum_{i=0}^{m-1} (Z_i^2 - \pi(|\sigma^T \nabla \varphi|^2))]^2 \tag{4.5} \\ & \leq 2\mathbb{E}[\eta^2 \sum_{i=0}^{m-1} (Z_i^2 - \pi_\eta(|\sigma^T \nabla \varphi|^2))]^2 + 2\left(\eta^2 \sum_{i=0}^{m-1} (\pi_\eta(|\sigma^T \nabla \varphi|^2) - \pi(|\sigma^T \nabla \varphi|^2))\right)^2 \rightarrow 0. \end{aligned}$$

By Lemma 4.1, the second term converges to 0. For the first term, a straight calculation gives that

$$\mathbb{E}[\eta^2 \sum_{i=0}^{m-1} (Z_i^2 - \pi_\eta(|\sigma^T \nabla \varphi|^2))]^2 = \eta^4 \sum_{i=0}^{m-1} \mathbb{E}[Z_i^2 - \pi_\eta(|\sigma^T \nabla \varphi|^2)]^2$$

$$+2\eta^4 \sum_{i,j=0,i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))(Z_j^2 - \pi_\eta(|\sigma^T \varphi|^2)) \right].$$

For the first term, the boundedness of  $\|\nabla\varphi\|$  implies

$$\begin{aligned} \mathbb{E}[Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2)]^2 &\leq 2\mathbb{E}[Z_i^4] + 2\pi_\eta(|\sigma^T \varphi|^2)^2 \\ &\leq 2\mathbb{E}[|\nabla\varphi(\theta_i)|^4 \|\sigma\|^4 |\xi_{i+1}|^4] + 2\pi_\eta(|\sigma^T \varphi|^2)^2 \\ &\leq C + 2\pi_\eta(|\sigma^T \varphi|^2)^2. \end{aligned} \tag{4.6}$$

Then we have

$$\eta^4 \sum_{i=0}^{m-1} \mathbb{E}[Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2)]^2 \leq C\eta^4 m \rightarrow 0.$$

For the second term, we can calculate that

$$\begin{aligned} &\sum_{i,j=0,i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))(Z_j^2 - \pi_\eta(|\sigma^T \varphi|^2)) \right] \\ &= \sum_{i,j=0,i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))\mathbb{E}_{i+1}[Z_j^2 - \pi_\eta(|\sigma^T \varphi|^2)] \right] \\ &= \sum_{i,j=0,i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))\mathbb{E}_{i+1}[|\sigma^T \nabla\varphi(\theta_j)|^2 - \pi_\eta(|\sigma^T \varphi|^2)] \right] \\ &\leq \sum_{i,j=0,i < j}^{\lfloor \nu \rfloor} \mathbb{E} \left[ |Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2)|(1 + |\theta_{i+1}|^2)e^{-c(j-i-1)} \right], \end{aligned}$$

where the last inequality follows from (A.3). By Hölder’s inequality, we have

$$\begin{aligned} &\sum_{i,j=0,i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))(Z_j^2 - \pi_\eta(|\sigma^T \varphi|^2)) \right] \\ &\leq \sum_{i,j=0,i < j}^{m-1} \left[ \left( \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ (1 + |\theta_{i+1}|^2)e^{-c(j-i-1)} \right]^2 \right)^{\frac{1}{2}} \right] \\ &\leq C \sum_{i,j=0,i < j}^{m-1} e^{-c(j-i)} \left( 1 + \mathbb{E}|\theta_{i+1}|^4 \right)^{\frac{1}{2}}, \end{aligned}$$

where the boundedness of  $\mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))^2 \right]$  follows from (4.6). Now we estimate  $\mathbb{E}|\theta_{i+1}|^4$ . A similar calculation with (A.4) yields

$$\mathbb{E}[|\theta_{i+1}|^4] = \mathbb{E}[\mathbb{E}_i|\theta_{i+1}|^4] \leq (1 - K_1\eta + c\eta^2)\mathbb{E}[|\theta_i|^4] + C\eta.$$

By iteration with initial data  $\theta_0 = x$ , we obtain

$$\mathbb{E}[|\theta_{i+1}|^4] \leq C\eta \sum_{k=0}^i (1 - K_1\eta + c\eta^2)^k + |x|^4(1 - K_1\eta + c\eta^2)^{i+1}.$$

Choosing  $\eta$  small enough such that  $1 - K_1\eta + c\eta^2 < 1$  gives

$$\mathbb{E}[|\theta_k|^4] \leq |x|^4 + \frac{C}{K_1 + c\eta}, \quad k = 0, 1, \dots$$

Combining the relationships above, we have

$$\begin{aligned} & 2\eta^4 \sum_{i,j=0, i < j}^{m-1} \mathbb{E} \left[ (Z_i^2 - \pi_\eta(|\sigma^T \varphi|^2))(Z_j^2 - \pi_\eta(|\sigma^T \varphi|^2)) \right] \leq C\eta^4 \sum_{\substack{i,j=0, \\ i < j}}^{m-1} e^{-c(j-i)} \\ &= C\eta^4 \sum_{\substack{i,j=0, \\ 0 < j-i \leq \ln m}}^{m-1} e^{-c(j-i)} + C\eta^4 \sum_{\substack{i,j=0, \\ \ln m < j-i}}^{m-1} e^{-c(j-i)} \\ &\leq C\eta^4 m \ln m + C\eta^4 e^{-c \ln m} (m - \ln m)^2 \rightarrow 0. \end{aligned}$$

Hence we prove the first term of (4.5) converges to 0 and finish the proof. □

**Proof of Theorem 2.4.** We have shown in (3.1) that

$$\eta^{-\frac{1}{2}}(\Pi_\eta(h) - \pi(h)) = \mathcal{H}_\eta + \mathcal{R}_\eta.$$

Here  $\mathcal{H}_\eta$  weakly converges to  $N(0, \pi(|\sigma^T \nabla \varphi|^2))$  by Lemma 4.2. Lemma 3.5 implies  $\mathcal{R}_\eta$  converges to 0 in probability with fixed initial value  $\theta_0$ . Thus  $\eta^{-\frac{1}{2}}(\Pi_\eta(h) - \pi(h)) \Rightarrow N(0, \pi(|\sigma^T \nabla \varphi|^2))$ . □

## 5. Proof of Theorem 2.5

### 5.1. Self-normalized Cramér-type moderate deviation of $\mathcal{H}_\eta$

In order to prove the Cramér-type moderate deviation result for  $\mathcal{H}_\eta$ , we introduce following concentration inequality for stationary process.

**Lemma 5.1.** *Suppose that the conditions of Theorem 2.5 hold. Then, for any  $y > 0$*

$$\mathbb{P} \left( \left| \sum_{i=0}^{k-1} |\sigma^T \varphi(\theta_i)|^2 - k\pi_\eta(|\sigma^T \varphi|^2) \right| > y \right) \leq 2e^{-Cy^2k^{-1}}, \quad k \in \mathbb{N}.$$

Here,  $C$  depends on  $g$  and  $\sigma$ .

**Proof.** Since  $\theta_0 \sim \pi_\eta$ ,  $(\theta_k)_{k \geq 0}$  is stationary. Following Dedecker and Gouëzel [10], (6), with  $||\sigma^T \varphi(\theta_k)|^2 - \pi_\eta(|\sigma^T \varphi|^2)| \leq C$ , we can get the result immediately. □

**Lemma 5.2.** *Under the conditions of Theorem 2.5, one has*

$$\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) = 1 + O(x\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}}),$$

uniformly for  $\eta^{\frac{1}{3}} \leq x = o(\eta^{-\frac{1}{3}})$  as  $\eta$  tends to zero. Here,  $O$  and  $o$  depend on  $g, \sigma$ .

**Proof.** We first prove the upper bound of  $\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x))$ . Notice that  $\mathbb{E}\mathcal{Y}_\eta = \pi_\eta(|\sigma^T \nabla \varphi|^2)$  by the fact  $\theta_0 \sim \pi_\eta$ , without loss of generality, we may assume  $\mathbb{E}\mathcal{Y}_\eta = \pi_\eta(|\sigma^T \nabla \varphi|^2) = 1$ . For  $y$  such that  $0 < y\eta^2 < 1$  which will be chosen later, Lemma 5.1 implies

$$\begin{aligned} \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{\mathcal{Y}_\eta} \geq x\right) &= \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{\mathcal{Y}_\eta} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| > y\right) + \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{\mathcal{Y}_\eta} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| \leq y\right) \\ &\leq \mathbb{P}\left(\eta^{-2} |1 - \mathcal{Y}_\eta| > y\right) + \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{1 - y\eta^2} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| \leq y\right) \\ &\leq 2e^{-Cy^2\eta^2} + \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{1 - y\eta^2} \geq x\right). \end{aligned} \tag{5.1}$$

Define

$$\tilde{\mathcal{H}}_t = \eta \int_0^t \nabla \varphi(\theta_{[s]})^T \sigma dB_s$$

for any  $t \in \mathbb{R}^+$  which is a continuous martingale. Denote its sharp bracket process by  $\langle \tilde{\mathcal{H}} \rangle(s, t) = \eta^2 \int_s^t |\sigma^T \nabla \varphi(\theta_{[r]})|^2 dr$  and  $\langle \tilde{\mathcal{H}} \rangle(t) = \langle \tilde{\mathcal{H}} \rangle(0, t)$  for simplicity. It is easy to see

$$\begin{aligned} \tilde{\mathcal{H}}_m &\stackrel{d}{=} \mathcal{H}_\eta, \\ \langle \tilde{\mathcal{H}} \rangle(t) &= \eta^2 \int_0^t |\sigma^T \nabla \varphi(\theta_{[s]})|^2 ds = \sum_{i=0}^{[t]-1} \eta^2 |\sigma^T \nabla \varphi(\theta_i)|^2 + \eta^2 \int_{[t]}^t |\sigma^T \nabla \varphi(\theta_{[s]})|^2 ds. \end{aligned}$$

Denoting the stopping time  $T_1 = \inf\{s : \langle \tilde{\mathcal{H}} \rangle(s) > 1\}$ , Dambis-Dubins-Schwarz Theorem (cf. Revuz and Yor [32], Theorem 5.1.6) yields that  $\tilde{\mathcal{H}}_{T_1}$  is a  $\mathcal{F}_{T_1}$ -Brownian motion and  $\tilde{\mathcal{H}}_{T_1} \sim N(0, 1)$ . Then we have

$$\begin{aligned} \mathbb{P}\left(\mathcal{H}_\eta / \sqrt{1 - y\eta^2} \geq x\right) &= \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1} + \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq x\right) \\ &\leq \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0\right) + \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq x - c_0\right) \end{aligned} \tag{5.2}$$

with small  $c_0$  satisfying  $0 < c_0 \leq x$  which will be chosen later. For the second term on the right hand side, since  $\tilde{\mathcal{H}}_{T_1} \sim N(0, 1)$ ,

$$\mathbb{P}\left(\frac{\tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq x - c_0\right) = 1 - \Phi(\sqrt{1 - y\eta^2}(x - c_0)). \tag{5.3}$$

For the first term and  $\alpha \in (0, 1)$ , we have

$$\begin{aligned} \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0\right) &= \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0, T_1 \notin [m - m^\alpha, m + m^\alpha]\right) \\ &\quad + \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0, T_1 \in [m - m^\alpha, m + m^\alpha]\right). \end{aligned} \tag{5.4}$$

Without loss of generality, we may assume that  $m^\alpha$  is an integer. The definition of  $T_1$  implies  $\{T_1 < m - m^\alpha\} = \{\langle \tilde{\mathcal{H}} \rangle(m - m^\alpha) > 1\}$  and  $\{T_1 > m + m^\alpha\} = \{\langle \tilde{\mathcal{H}} \rangle(m + m^\alpha) < 1\}$ . Then we can obtain

$$\begin{aligned} \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0, T_1 \notin [m - m^\alpha, m + m^\alpha]\right) &\leq \mathbb{P}(T_1 < m - m^\alpha) + \mathbb{P}(T_1 > m + m^\alpha) \\ &= \mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m - m^\alpha) > 1\right) + \mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m + m^\alpha) < 1\right). \end{aligned}$$

Following Lemma 5.1, one has

$$\begin{aligned} \mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m - m^\alpha) > 1\right) &= \mathbb{P}\left(\sum_{i=0}^{m - m^\alpha - 1} |\sigma^\top \nabla \varphi(\theta_i)|^2 - (m - m^\alpha) > m - (m - m^\alpha)\right) \\ &\leq e^{-Cm^{2\alpha}(m - m^\alpha)^{-1}} \leq e^{-Cm^{2\alpha-1}}. \end{aligned}$$

Similarly, we can get  $\mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m + m^\alpha) < 1\right) \leq e^{-Cm^{2\alpha-1}}$ . That is

$$\mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0, T_1 \notin [m - m^\alpha, m + m^\alpha]\right) \leq 2e^{-Cm^{2\alpha-1}}. \tag{5.5}$$

For the second term of (5.4), we have

$$\begin{aligned} &\mathbb{P}\left(\frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1 - y\eta^2}} \geq c_0, T_1 \in [m - m^\alpha, m + m^\alpha]\right) \\ &\leq \mathbb{P}\left(\sup_{s \in [m - m^\alpha, m + m^\alpha]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) \\ &\leq \mathbb{P}\left(\sup_{s \in [m, m + m^\alpha]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) + \mathbb{P}\left(\sup_{s \in [m - m^\alpha, m]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right). \end{aligned} \tag{5.6}$$

For the first term and positive number  $y'$  which will be chosen later, one has

$$\begin{aligned} &\mathbb{P}\left(\sup_{s \in [m, m + m^\alpha]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) \\ &\leq \mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m, m + m^\alpha) > \eta^2 y'\right) \\ &\quad + \mathbb{P}\left(\sup_{s \in [m, m + m^\alpha]} (\tilde{\mathcal{H}}_s - \tilde{\mathcal{H}}_m) \geq c_0 \sqrt{1 - y\eta^2}, \langle \tilde{\mathcal{H}} \rangle(m, m + m^\alpha) \leq \eta^2 y'\right). \end{aligned}$$

The first probability can be estimate by Lemma 5.1, that is,

$$\mathbb{P}\left(\langle \tilde{\mathcal{H}} \rangle(m, m + m^\alpha) > \eta^2 y'\right) \leq e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}}.$$

For the second probability, the Bernstein inequality (cf. Barlow, Jacka and Yor [5], Proposition 4.2.3(1)) implies

$$\begin{aligned} & \mathbb{P}\left(\sup_{s \in [m, m+m^\alpha]} (\tilde{\mathcal{H}}_s - \tilde{\mathcal{H}}_m) \geq c_0 \sqrt{1 - y\eta^2}, \langle \tilde{\mathcal{H}} \rangle(m, m + m^\alpha) \leq \eta^2 y'\right) \\ & \leq \exp\left\{-\frac{c_0^2(1 - y\eta^2)}{2\eta^2 y'}\right\} = e^{-\frac{c_0^2(m-y)}{2y'}}. \end{aligned}$$

Thus we have

$$\mathbb{P}\left(\sup_{s \in [m, m+m^\alpha]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) \leq e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}} + e^{-\frac{c_0^2(m-y)}{2y'}}. \tag{5.7}$$

For the second term of (5.6),

$$\begin{aligned} & \mathbb{P}\left(\sup_{s \in [m-m^\alpha, m]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) \\ & \leq \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\sup_{s \in [m-k-1, m-k]} (\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_s) \geq c_0 \sqrt{1 - y\eta^2}\right) \\ & \leq \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{m-k} \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) + \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\sup_{s \in [m-k-1, m-k]} (\tilde{\mathcal{H}}_{m-k} - \tilde{\mathcal{H}}_s) \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right). \end{aligned}$$

For the first probability, the stability of  $\theta_k$  and (5.7) yield

$$\begin{aligned} \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{m-k} \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) &= \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\tilde{\mathcal{H}}_{m+k} - \tilde{\mathcal{H}}_m \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) \\ &\leq m^\alpha \mathbb{P}\left(\sup_{s \in [m, m+m^\alpha]} (\tilde{\mathcal{H}}_s - \tilde{\mathcal{H}}_m) \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) \\ &\leq m^\alpha \left( e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}} + e^{-\frac{c_0^2(m-y)}{8y'}} \right). \end{aligned}$$

For the second probability, by the boundedness of  $\nabla\varphi$ , we have

$$\begin{aligned} & \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\sup_{s \in [m-k-1, m-k]} (\tilde{\mathcal{H}}_{m-k} - \tilde{\mathcal{H}}_s) \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) \\ &= \sum_{k=0}^{m^\alpha-1} \mathbb{P}\left(\sup_{s \in [m-k-1, m-k]} \int_s^{m-k} \eta(\nabla\varphi(\theta_{[r_1]}))^T \sigma dB_r \geq \frac{c_0}{2} \sqrt{1 - y\eta^2}\right) \end{aligned}$$



$$\begin{aligned} &\leq \sum_{k=0}^{m^\alpha-1} \mathbb{P} \left( \sup_{m-k-1 \leq s \leq t \leq m-k} |B_t - B_s| \geq \frac{Cc_0}{\eta} \sqrt{1-y\eta^2} \right) \\ &\leq m^\alpha \mathbb{P} \left( \sup_{0 \leq s \leq 1, 0 \leq t-s \leq 1} |B_{s+(t-s)} - B_s| \geq \frac{Cc_0}{\eta} \sqrt{1-y\eta^2} \right). \end{aligned}$$

Following [23], Theorem 12.1.c, we can get

$$\sum_{k=0}^{m^\alpha-1} \mathbb{P} \left( \sup_{s \in [m-k-1, m-k]} (\tilde{\mathcal{H}}_{m-k} - \tilde{\mathcal{H}}_s) \geq \frac{c_0}{2} \sqrt{1-y\eta^2} \right) \leq cm^\alpha e^{-\frac{c_0^2}{\eta^2}(1-y\eta^2)}.$$

Hence we have

$$\begin{aligned} &\mathbb{P} \left( \frac{\tilde{\mathcal{H}}_m - \tilde{\mathcal{H}}_{T_1}}{\sqrt{1-y\eta^2}} \geq c_0, T_1 \in [m-m^\alpha, m+m^\alpha] \right) \\ &\leq (1+m^\alpha) \left( e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}} + e^{-\frac{c_0^2(m-y)}{8y'}} \right) + cm^\alpha e^{-C\eta^{-2}c_0^2(1-y\eta^2)}. \end{aligned} \tag{5.8}$$

Combining (5.1-5.8), we obtain

$$\begin{aligned} \mathbb{P} \left( \mathcal{H}_\eta / \sqrt{\mathcal{Y}_\eta} \geq x \right) &\leq 1 - \Phi(\sqrt{1-y\eta^2}(x-c_0)) + 2e^{-Cy^2\eta^2} + 2e^{-Cm^{2\alpha-1}} \\ &\quad + (1+m^\alpha) \left( e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}} + e^{-\frac{c_0^2(m-y)}{8y'}} \right) + cm^\alpha e^{-C\eta^{-2}c_0^2(1-y\eta^2)}. \end{aligned} \tag{5.9}$$

By the following well known estimate of normal distribution (cf. Fan et al. [16], (4.1))

$$\frac{1}{\sqrt{2\pi}(1+x)} e^{-\frac{x^2}{2}} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{\pi}(1+x)} e^{-\frac{x^2}{2}}, \quad x \geq 0, \tag{5.10}$$

we have

$$\begin{aligned} &\mathbb{P} \left( \mathcal{H}_\eta / \sqrt{\mathcal{Y}_\eta} \geq x \right) / (1 - \Phi(x)) \\ &\leq \frac{1 - \Phi(\sqrt{1-y\eta^2}(x-c_0))}{1 - \Phi(x)} + \sqrt{2\pi}(1+x) \left[ 2e^{-Cy^2\eta^2 + \frac{x^2}{2}} + 2e^{-Cm^{2\alpha-1} + \frac{x^2}{2}} \right. \\ &\quad \left. + (1+m^\alpha) \left( e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha} + \frac{x^2}{2}} + e^{-\frac{c_0^2(m-y)}{8y'} + \frac{x^2}{2}} \right) + cm^\alpha e^{-C\eta^{-2}c_0^2(1-y\eta^2) + \frac{x^2}{2}} \right]. \end{aligned}$$

For the normal distribution part, by (5.10) again, we have

$$\begin{aligned} &\frac{1 - \Phi(\sqrt{1-y\eta^2}(x-c_0))}{1 - \Phi(x)} \\ &= \frac{1 - \Phi(\sqrt{1-y\eta^2}(x-c_0))}{1 - \Phi(x)} 1_{\{x \geq 1\}} + \frac{1 - \Phi(\sqrt{1-y\eta^2}(x-c_0))}{1 - \Phi(x)} 1_{\{0 \leq x < 1\}} \end{aligned} \tag{5.11}$$

$$\begin{aligned}
 &\leq \frac{\sqrt{2}(1+x)}{1+\sqrt{1-y\eta^2}(x-c_0)} e^{\frac{1}{2}x^2-\frac{1}{2}(1-y\eta^2)(x-c_0)^2} \mathbf{1}_{\{x \geq 1\}} \\
 &\quad + \left(1 + \int_x^x \frac{e^{-\frac{t^2}{2}} dt}{\sqrt{1-y\eta^2}(x-c_0)} / \int_x^\infty e^{-\frac{t^2}{2}} dt\right) \mathbf{1}_{\{0 \leq x < 1\}} \\
 &\leq \frac{\sqrt{2}(1+x)}{1+\sqrt{1-y\eta^2}(x-c_0)} e^{-\frac{1}{2}c_0^2+xc_0+\frac{1}{2}(x-c_0)^2y\eta^2} \mathbf{1}_{\{x \geq 1\}} \\
 &\quad + \left[1 + (1+x)(x - \sqrt{1-y\eta^2}(x-c_0)) e^{-\frac{1}{2}c_0^2+xc_0+\frac{1}{2}(x-c_0)^2y\eta^2}\right] \mathbf{1}_{\{0 \leq x < 1\}} \\
 &\leq e^{-\frac{1}{2}c_0^2+xc_0+C(x-c_0)^2y\eta^2} \mathbf{1}_{\{x \geq 1\}} + \left[1 + C(x - \sqrt{1-y\eta^2}(x-c_0))\right] \mathbf{1}_{\{0 \leq x < 1\}}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) \\
 &\leq e^{-\frac{1}{2}c_0^2+xc_0+C(x-c_0)^2y\eta^2} \mathbf{1}_{\{x \geq 1\}} + \left[1 + C(x - \sqrt{1-y\eta^2}(x-c_0))\right] \mathbf{1}_{\{0 \leq x < 1\}} \\
 &\quad + \sqrt{2\pi}(1+x) \left[2e^{-Cy^2\eta^2+\frac{x^2}{2}} + 2e^{-Cm^{2\alpha-1}+\frac{x^2}{2}}\right. \\
 &\quad \left.+ (1+m^\alpha)\left(e^{-\frac{C(y'-m^\alpha)^2}{m^\alpha}+\frac{x^2}{2}} + e^{-\frac{c_0^2(m-y)}{8y'}+\frac{x^2}{2}}\right) + cm^\alpha e^{-C\eta^{-2}c_0^2(1-y\eta^2)+\frac{x^2}{2}}\right].
 \end{aligned}$$

To guarantee the limit of the first two terms is 1 and the last term is 0 as  $\eta \rightarrow 0$ , i.e.  $m \rightarrow \infty$ . We need  $y^2\eta^2 \rightarrow \infty$ ,  $2Cy^2\eta^2 > x^2$ ,  $\eta^2x^2y \rightarrow 0$  and  $x = o(c_0^{-1})$ . Choosing  $y = \eta^{-\frac{4}{3}}$ ,  $c_0 = \eta^{\frac{1}{3}}$ ,  $y' = \eta^{-\frac{2}{3}}$  and  $\alpha = 2/3$ , one has

$$\begin{aligned}
 &\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) \\
 &\leq e^{-c(\eta^{\frac{2}{3}}-x\eta^{\frac{1}{3}}-x^2\eta^{\frac{2}{3}})} \mathbf{1}_{\{x \geq 1\}} + \left[1 + C(x - (1-\eta^{\frac{2}{3}})^{-\frac{1}{2}}(x-\eta^{\frac{1}{3}}))\right] \mathbf{1}_{\{0 < x < 1\}} + e^{C(x^2-\eta^{-\frac{2}{3}})} \\
 &\leq 1 + C(x\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}})
 \end{aligned}$$

converges to 1 uniformly for  $\eta^{\frac{1}{3}} \leq x = o(\eta^{-1/3})$  as  $\eta$  tends to 0.

For the lower bound of  $\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x))$ , we have

$$\begin{aligned}
 \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) &\geq \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| \leq y\right) \\
 &\geq \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{1+\eta^2y}} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| \leq y\right) \\
 &= \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{1+\eta^2y}} \geq x\right) - \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{1+\eta^2y}} \geq x, \eta^{-2} |1 - \mathcal{Y}_\eta| > y\right)
 \end{aligned}$$

$$\geq \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_{T_1}}{\sqrt{1+\eta^2y}} \geq x + c_0\right) - \mathbb{P}\left(\frac{\tilde{\mathcal{H}}_{T_1} - \tilde{\mathcal{H}}_m}{\sqrt{1+\eta^2y}} \geq c_0\right) - \mathbb{P}\left(\eta^{-2}|1 - \mathcal{Y}_\eta| > y\right).$$

Similar with the estimate of the upper bound, (5.10), (5.4) and Lemma 5.1 imply

$$\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) \geq 1 - C(x\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}})$$

converges to 1 uniformly for  $\eta^{\frac{1}{3}} \leq x = o(\eta^{-1/3})$  as  $\eta$  tends to 0. Hence, we have

$$\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) = 1 + O(x\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}})$$

uniformly for  $\eta^{\frac{1}{3}} \leq x = o(\eta^{-1/3})$  as  $\eta$  vanishes. □

### 5.2. Proof of Theorem 2.5

**Proof of Theorem 2.5.** We have proved the following decomposition,

$$\eta^{-\frac{1}{2}}(\Pi_\eta(h) - \pi(h)) = \mathcal{R}_\eta + \mathcal{H}_\eta.$$

Noting that, for any  $x > 0$  and  $0 < y < x$ , we have

$$\mathbb{P}(\mathcal{W}_\eta \geq x) = \mathbb{P}\left(\frac{\mathcal{R}_\eta + \mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) \leq \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x - y\right) + \mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right). \tag{5.12}$$

For the first term, Lemma 5.2 yields that

$$\mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x - y\right) / (1 - \Phi(x - y)) = 1 + O((x - y)\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}})$$

uniformly for  $\eta^{\frac{1}{3}} \leq x - y = o(\eta^{-\frac{1}{3}})$  as  $\eta$  tends to zero. We take  $\eta^{\frac{1}{3}} < x = o(\eta^{-\alpha})$  and  $y = o(1)$  such that  $\alpha \leq 1/3$ ,  $xy \rightarrow 0$  and  $x - y \geq \eta^{\frac{1}{3}}$ , here  $y$  will be chosen later. Similar with the calculation of (5.11), (5.10) yields

$$\frac{1 - \Phi(x - y)}{1 - \Phi(x)} = 1 + O(xy + y).$$

Hence,

$$\begin{aligned} \frac{\mathbb{P}(\mathcal{H}_\eta/\sqrt{\mathcal{Y}_\eta} \geq x - y)}{1 - \Phi(x)} &= \frac{\mathbb{P}(\mathcal{H}_\eta/\sqrt{\mathcal{Y}_\eta} \geq x - y)}{1 - \Phi(x - y)} \frac{1 - \Phi(x - y)}{1 - \Phi(x)} \\ &= 1 + O(x\eta^{\frac{1}{3}} + \eta^{\frac{1}{3}} + xy + y) \end{aligned} \tag{5.13}$$

as  $\eta$  vanishes.

For the second term of (5.12), we have

$$\begin{aligned} \mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right) &\leq \mathbb{P}(\mathcal{Y}_\eta < \mathbb{E}\mathcal{Y}_\eta - y) + \mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y, \mathcal{Y}_\eta \geq \mathbb{E}\mathcal{Y}_\eta - y\right) \\ &\leq \mathbb{P}(\mathbb{E}\mathcal{Y}_\eta - \mathcal{Y}_\eta > y) + \mathbb{P}\left(\mathcal{R}_\eta \geq y\sqrt{\mathbb{E}\mathcal{Y}_\eta - y}\right). \end{aligned}$$

For the first probability, Lemma 5.1 yields that

$$\mathbb{P}(\mathbb{E}\mathcal{Y}_\eta - \mathcal{Y}_\eta > y) \leq e^{-Cy^2\eta^{-2}}.$$

For the second probability, following the stationary of  $\theta_k$  and Lemma 3.5, one has

$$\mathbb{P}\left(\mathcal{R}_\eta \geq y\sqrt{\mathbb{E}\mathcal{Y}_\eta - y}\right) = \mathbb{P}\left(\mathcal{R}_\eta \geq y\sqrt{\pi(|\sigma^T \nabla \varphi|^2) - y}\right) \leq Ce^{-c\eta^{-2\bar{\gamma}}y^{\frac{2}{3}}},$$

as  $y \geq c \max\{\eta^{\frac{3}{2}-6\bar{\gamma}}, \eta^{\frac{3}{2}\bar{\gamma}}, \eta^{\frac{1}{2}}\} = c\eta^{\frac{3}{2}-6\bar{\gamma}}$  where  $\frac{1}{5} \leq \bar{\gamma} < \frac{1}{4}$ . Hence, we have

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right) \leq C\left(e^{-c\eta^{-2}y^2} + e^{-c\eta^{-2\bar{\gamma}}y^{\frac{2}{3}}}\right).$$

This, together with (5.10), implies

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right) / (1 - \Phi(x)) \leq C(1+x)e^{\frac{1}{2}x^2} \mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right).$$

It converges to 0 as  $\eta \rightarrow 0$  uniformly for

$$\eta^{\frac{1}{3}} \leq x = o\left(\min\{\eta^{-1}y, \eta^{-\bar{\gamma}}y^{\frac{1}{3}}\}\right).$$

Since Lemma 5.2 holds uniformly as  $\eta^{\frac{1}{3}} + y \leq x = o(\eta^{-\alpha})$ , we need to choose  $\alpha$ ,  $y$  and  $\bar{\gamma}$  such that

$$\min\{\eta^{-1}y, \eta^{-\bar{\gamma}}y^{\frac{1}{3}}\} \geq \eta^{-\alpha}.$$

By taking  $\alpha = 1/6$ ,  $y = c\eta^{\frac{1}{6}}$  and  $\bar{\gamma} = 2/9$ , we can get

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right) / (1 - \Phi(x)) \leq C(1+x) \exp\{c(x^2 - \eta^{-\frac{1}{3}})\} \rightarrow 0 \tag{5.14}$$

uniformly for  $c\eta^{\frac{1}{6}} \leq x = o(\eta^{-\frac{1}{6}})$  as  $\eta$  vanishes.

Following (5.12), (5.13) and (5.14), we have

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta + \mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) \leq 1 + C(x\eta^{1/6} + \eta^{1/6}). \tag{5.15}$$

uniformly for  $c\eta^{\frac{1}{6}} \leq x = o(\eta^{-\frac{1}{6}})$  as  $\eta$  tends to zero.

On the other hand,

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta + \mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) \geq \mathbb{P}\left(\frac{\mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x + y\right) - \mathbb{P}\left(\frac{-\mathcal{R}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq y\right).$$

Similar as the proof of (5.15), Lemmas 3.5 and 5.2 yield that

$$\mathbb{P}\left(\frac{\mathcal{R}_\eta + \mathcal{H}_\eta}{\sqrt{\mathcal{Y}_\eta}} \geq x\right) / (1 - \Phi(x)) \geq 1 - C(x\eta^{\frac{1}{6}} + \eta^{\frac{1}{6}}),$$

uniformly for  $c\eta^{\frac{1}{6}} \leq x = o(\eta^{-\frac{1}{6}})$  as  $\eta$  tends to zero. Combining the last inequality with (5.15), we deduce that

$$\mathbb{P}(\mathcal{W}_\eta \geq x) / (1 - \Phi(x)) = 1 + O\left(x\eta^{1/6} + \eta^{1/6}\right),$$

uniformly for  $c\eta^{\frac{1}{6}} \leq x = o(\eta^{-\frac{1}{6}})$  as  $\eta$  tends to zero. □

## Appendix A: Proofs of lemmas in Section 2

**Proof of Lemma 2.3.** We first give the proof of the ergodicity of  $(X_t)_{t \geq 0}$ . Following Roberts and Tweedie [34], Theorem 2.1, it is easy to verify the irreducibility of  $(X_t)_{t \geq 0}$ . For the Lyapunov function  $V(x) = |x|^2 + 1$ , following (2.4) and (2.6), we have

$$\mathcal{A}V(x) = \langle g(x), 2x \rangle + \|\sigma\|^2 \leq -K_1|x|^2 + C \leq -\frac{K_1}{2}V(x) + (C + \frac{K_1}{2})1_{\{|x| \leq \sqrt{2C/K_1+1}\}}.$$

By Meyn and Tweedie [29], Theorem 6.1,  $(X_t)_{t \geq 0}$  is exponential ergodic with invariant measure  $\pi$  satisfying

$$|\mathbb{E}[h(X_t^x) - \pi(h)]| \leq CV(x)e^{-ct}. \tag{A.1}$$

Then we consider the ergodicity of  $(\theta_k)_{k \geq 0}$ . Denote its transition probability by  $P(x, dy)$  for  $x, y \in \mathbb{R}^d$ . For any open set  $A \in \mathbb{R}^d$  and initial value  $x$ , since  $\xi_1$  is a normal random vector, we have

$$P(x, A) = \mathbb{P}(x + \eta g(x) + \sqrt{\eta}\sigma\xi_1 \in A) > 0.$$

Suppose  $P^k(x, A) > 0$  for some integer  $k > 1$ , then we have

$$P^{k+1}(x, A) = \int_{\mathbb{R}^d} P(x, y)P^k(y, A)dy > 0.$$

The induction yields that  $(\theta_k)_{k \geq 0}$  is irreducible. Following (2.3),(2.4) and (2.5), one has

$$\begin{aligned} \mathbb{E}_k[V(\theta_{k+1})] &= \mathbb{E}_k[|\theta_k + \eta g(\theta_k) + \sqrt{\eta}\sigma\xi_{k+1}|^2] + 1 \\ &= |\theta_k|^2 + |\eta g(\theta_k)|^2 + \eta\|\sigma\|^2 + 2\langle \theta_k, \eta g(\theta_k) \rangle + 1 \\ &\leq (1 - K_1\eta + 2L^2\eta^2)|\theta_k|^2 + 2|g(0)|^2\eta^2 + \eta\|\sigma\|^2 + 2C\eta + 1 \\ &\leq (1 - \frac{1}{2}K_1\eta + 2L^2\eta^2)V(\theta_k) + b1_D(\theta_k). \end{aligned} \tag{A.2}$$

Here  $b = \frac{1}{2}K_1\eta - 2L^2\eta^2 + 2|g(0)|^2\eta^2 + \eta\|\sigma\|^2 + 2C\eta$  and set  $D = \{|x| \leq \frac{2b}{\eta K_1}\}$ . There exists  $\eta_0$  such that for  $\eta \leq \eta_0$ ,  $1 - \frac{1}{2}K_1\eta + 2L^2\eta^2 < 1$ . By Roberts and Tweedie [34], (29), we deduce that  $\theta_k$  is ergodic when  $\eta \leq \eta_0$ , that is

$$|\mathbb{E}[h(\theta_k) - \pi_\eta(h)]| \leq CV(\theta_0)e^{-ck}. \tag{A.3}$$

□

Moreover, for the function  $\tilde{V}(x) = |x|^4 + 1$ , similarly with the calculation of (A.2), we can get

$$\begin{aligned} \mathbb{E}_k[\tilde{V}(\theta_{k+1})] &= \mathbb{E}_k[|\theta_{k+1}|^4 + 1] \\ &\leq (1 - 2K_1\eta + C_1\eta^2)|\theta_k|^4 + C_2\eta|\theta_k|^2 + C_3\eta^2 + 1, \end{aligned}$$

where  $C_1, C_2, C_3$  depend on  $\sigma, K_1, L$  and  $C$  in (2.3), (2.4). Then we have

$$\mathbb{E}_k[\tilde{V}(\theta_{k+1})] \leq (1 - K_1\eta + C_1\eta^2)\tilde{V}(\theta_k) + \tilde{b}1_{\tilde{D}}(\theta_k), \tag{A.4}$$

where  $\tilde{b} = \frac{C_2^2}{4K_1}\eta + K_1\eta + C_3\eta^2$ ,  $\tilde{D} = \{|x|^2 \leq (\frac{C_3-C_1}{K_1}\eta + 1 + (\frac{C_2}{2K_1})^2)^{\frac{1}{2}} + \frac{C_2}{2K_1}\}$ . For small enough  $\eta$  such that  $1 - K_1\eta + C_1\eta^2 < 1$ , let  $\theta_0$  take the ergodic measure  $\pi_\eta$ , then  $(\theta_k)_{k \geq 0}$  is stationary and (A.4) implies

$$\pi_\eta(\tilde{V}) \leq (1 - K_1\eta + C_1\eta^2)\pi_\eta(\tilde{V}) + \tilde{b},$$

i.e.

$$\pi_\eta(\tilde{V}) \leq \frac{\tilde{b}}{K_1\eta - C_1\eta^2}. \tag{A.5}$$

Notice that for any  $k = 0, \dots, m$  and positive number  $\gamma$ , we have

$$\begin{aligned} &\mathbb{E}_k \left[ e^{\gamma|\theta_{k+1}|^2} \right] \\ &= \mathbb{E}_k \left[ \exp\{\gamma|\theta_k|^2 + \gamma|\eta g(\theta_k)|^2 + \gamma\eta|\sigma\xi_{k+1}|^2 + 2\gamma\langle\theta_k, \eta g(\theta_k)\rangle + 2\sqrt{\eta}\gamma\langle\sigma^T(\theta_k + \eta g(\theta_k)), \xi_{k+1}\rangle\} \right] \\ &= e^{\gamma|\theta_k|^2 + \gamma|\eta g(\theta_k)|^2 + 2\gamma\langle\theta_k, \eta g(\theta_k)\rangle} \mathbb{E}_k \left[ \exp\{\gamma\eta|\sigma\xi_{k+1}|^2 + 2\sqrt{\eta}\gamma\langle\sigma^T(\theta_k + \eta g(\theta_k)), \xi_{k+1}\rangle\} \right]. \end{aligned}$$

A straight calculation to the conditional expectation with respect to the Gaussian random variable  $\xi_{k+1}$  yields

$$\mathbb{E}_k \left[ \exp\{\gamma\eta|\sigma\xi_{k+1}|^2 + 2\sqrt{\eta}\gamma\langle\sigma^T(\theta_k + \eta g(\theta_k)), \xi_{k+1}\rangle\} \right] \leq 2 \exp \left\{ 4\eta\gamma^2\|\sigma\|^2|\theta_k + \eta g(\theta_k)|^2 \right\},$$

here  $\gamma$  is chosen small enough such that  $\gamma\|\sigma\|^2 \leq 1/4$ . This estimate, together with (2.3) and (2.4), implies

$$\begin{aligned} \mathbb{E}_k \left[ e^{\gamma|\theta_{k+1}|^2} \right] &\leq 2 \exp \left\{ (1 - K_1\eta + 4\eta\gamma + C_1\eta^2)\gamma|\theta_k|^2 + 3C\gamma\eta \right\} \\ &\leq (1 - K_1\eta + 4\eta\gamma + C_1\eta^2)e^{\gamma|\theta_k|^2} + \tilde{b}, \end{aligned}$$

with  $\eta$  and  $\gamma$  are small enough such that  $1 - K_1\eta + 4\eta\gamma + C_1\eta^2 < 1$  and  $\bar{b}$  is big enough such that the second inequality holds. Let  $\theta_0$  take the ergodic measure  $\pi_\eta$ , then we have

$$\pi_\eta(e^{\gamma|\cdot|^2}) \leq \frac{\bar{b}}{K_1\eta - 4\eta\gamma - C_1\eta^2}. \tag{A.6}$$

## Appendix B: The proof of lemmas in Section 4

**Proof of Lemma 3.2.** For the first inequality, by using Hölder’s inequality, we can get

$$\begin{aligned} & \mathbb{E}_k \exp \{ \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1}) \} \\ & \leq \left( \mathbb{E}_k \exp \{ 2 \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - 2 |\sigma^T \Psi_1(\theta_k)|^2 \} \right)^{\frac{1}{2}} \left( \mathbb{E}_k \exp \{ 2 \Psi_2(\theta_k, \xi_{k+1}) + 2 |\sigma^T \Psi_1(\theta_k)|^2 \} \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $\xi_{k+1}$  is gaussian distributed and independent of  $\theta_k$ , a straightforward calculation gives

$$\left( \mathbb{E}_k \exp \{ 2 \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - 2 |\sigma^T \Psi_1(\theta_k)|^2 \} \right)^{\frac{1}{2}} = 1.$$

Hence, we have

$$\mathbb{E}_k \exp \{ \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1}) \} \leq \left( \mathbb{E}_k \exp \left\{ 2 |\Psi_1(\theta_k)|^2 \|\sigma\|^2 + 2 \Psi_2(\theta_k, \xi_{k+1}) \right\} \right)^{\frac{1}{2}}.$$

For the second inequality of Lemma 3.2, by the same way we have

$$\begin{aligned} & \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} \left( \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle + \Psi_2(\theta_k, \xi_{k+1}) \right) \right\} \\ & \leq \left( \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} 2 \left( \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - |\sigma^T \Psi_1(\theta_k)|^2 \right) \right\} \right)^{\frac{1}{2}} \\ & \quad \times \left( \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} 2 \left( |\sigma^T \Psi_1(\theta_k)|^2 + \Psi_2(\theta_k, \xi_{k+1}) \right) \right\} \right)^{\frac{1}{2}} \\ & = \left( \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} 2 \left( |\Psi_1(\theta_k)|^2 \|\sigma\|^2 + \Psi_2(\theta_k, \xi_{k+1}) \right) \right\} \right)^{\frac{1}{2}}, \end{aligned}$$

where the following relation is obtained by a standard conditional argument:

$$\begin{aligned} & \mathbb{E} \exp \left\{ \sum_{k=0}^{m-1} 2 \left( \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - |\sigma^T \Psi_1(\theta_k)|^2 \right) \right\} \\ & = \mathbb{E} \left[ \exp \left\{ \sum_{k=0}^{m-2} 2 \left( \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - |\sigma^T \Psi_1(\theta_k)|^2 \right) \right\} \mathbb{E}_{m-1} \left[ e^{2 \langle \Psi_1(\theta_{m-1}), \sigma \xi_m \rangle - 2 |\sigma^T \Psi_1(\theta_{m-1})|^2} \right] \right] \end{aligned}$$

$$= \mathbb{E} \exp \left\{ \sum_{k=0}^{m-2} 2 \left( \langle \Psi_1(\theta_k), \sigma \xi_{k+1} \rangle - |\sigma^T \Psi_1(\theta_k)|^2 \right) \right\} = \dots = 1.$$

A similar calculation gives the third inequality. □

**Proof of Lemma 3.3.** Since  $\theta_{k+1} = \theta_k + \eta g(\theta_k) + \sqrt{\eta} \sigma \xi_{k+1}$ , it is easy to calculate that

$$|\theta_{k+1}|^2 - |\theta_k|^2 = \eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2 + 2 \langle \eta \theta_k, g(\theta_k) \rangle + 2 \langle \sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k), \sigma \xi_{k+1} \rangle.$$

Summing these equalities from  $k = 0$  to  $k = m - 1$ , we obtain

$$|\theta_m|^2 - |\theta_0|^2 = \sum_{k=0}^{m-1} \left[ \eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2 + 2 \langle \eta \theta_k, g(\theta_k) \rangle + 2 \langle \sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k), \sigma \xi_{k+1} \rangle \right] \tag{B.1}$$

For  $\gamma > 0$ , (2.4) and (B.1) imply

$$\begin{aligned} & \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2} \gamma \eta |\theta_k|^2 \right\} \leq \mathbb{E}_0 \exp \left\{ - \sum_{k=0}^{m-1} \gamma \langle \eta \theta_k, g(\theta_k) \rangle \right\} e^{C\gamma\eta^{-1}} \\ & \leq \mathbb{E}_0 \exp \left\{ \frac{\gamma |\theta_0|^2}{2} + \frac{\gamma}{2} \sum_{k=0}^{m-1} \left[ \eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2 + 2 \langle \sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k), \sigma \xi_{k+1} \rangle \right] \right\} e^{C\gamma\eta^{-1}} \\ & = \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \left[ \frac{\gamma}{2} \left( \eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2 \right) + \gamma \langle \sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k), \sigma \xi_{k+1} \rangle \right] \right\} e^{\frac{\gamma |\theta_0|^2}{2} + C\gamma\eta^{-1}}. \end{aligned}$$

By Lemma 3.2 with  $\Psi_1(\theta_k) = \gamma(\sqrt{\eta}\theta_k + \eta^{\frac{3}{2}}g(\theta_k))$  and  $\Psi_2(\theta_k, \xi_{k+1}) = \frac{\gamma}{2}(\eta^2|g(\theta_k)|^2 + \eta|\sigma\xi_{k+1}|^2)$  therein, we have

$$\begin{aligned} & \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \left[ \frac{\gamma}{2} \left( \eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2 \right) + \gamma \langle \sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k), \sigma \xi_{k+1} \rangle \right] \right\} \\ & \leq \left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \left( \gamma (\eta^2 |g(\theta_k)|^2 + \eta |\sigma \xi_{k+1}|^2) + 2\gamma^2 |\sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k)|^2 \|\sigma\|^2 \right) \right\} \right)^{\frac{1}{2}} \\ & \leq \left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} 2\gamma \eta |\sigma \xi_{k+1}|^2 \right\} \right)^{\frac{1}{4}} \\ & \quad \times \left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} (2\gamma \eta^2 |g(\theta_k)|^2 + 4\gamma^2 |\sqrt{\eta} \theta_k + \eta^{\frac{3}{2}} g(\theta_k)|^2 \|\sigma\|^2) \right\} \right)^{\frac{1}{4}}. \end{aligned}$$

For the first expectation, we take some  $\gamma'_0$  and  $\eta'_0$  such that  $1 - 4\gamma'_0\eta'_0\|\sigma\|^2 > 0$ . Then for any  $\gamma < \gamma'_0$  and  $\eta < \eta'_0$ , we have

$$\mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} 2\gamma \eta |\sigma \xi_{k+1}|^2 \right\} \leq \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} 2\gamma \eta \|\sigma\|^2 |\xi_{k+1}|^2 \right\}$$



$$\begin{aligned} &= \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{2\gamma\eta\|\sigma\|^2x^2 - \frac{1}{2}x^2\}dx \right)^{md} \\ &= (1 - 4\gamma\eta\|\sigma\|^2)^{-\frac{md}{2}}. \end{aligned}$$

For the second expectation, by (2.3), we can choose some  $\gamma_0''$  and  $\eta_0''$  such that as  $\gamma < \gamma_0''$  and  $\eta < \eta_0''$

$$2\gamma\eta^2|g(\theta_k)|^2 + 4\gamma^2|\sqrt{\eta}\theta_k + \eta^{\frac{3}{2}}g(\theta_k)|^2\|\sigma\|^2 \leq \frac{K_1}{2}\gamma\eta|\theta_k|^2 + C\eta\gamma, \tag{B.2}$$

which leads to

$$\mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} (2\gamma\eta^2|g(\theta_k)|^2 + 4\gamma^2|\sqrt{\eta}\theta_k + \eta^{\frac{3}{2}}g(\theta_k)|^2\|\sigma\|^2) \right\} \leq e^{C\gamma\eta^{-1}} \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\}.$$

Hence, for  $\gamma < \gamma_0 = \gamma_0' \wedge \gamma_0''$  and  $\eta < \eta_0 = \eta_0' \wedge \eta_0''$ , we have

$$\mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\} \leq e^{\frac{\gamma|\theta_0|^2}{2} + C\gamma\eta^{-1}} (1 - 4\gamma\eta\|\sigma\|^2)^{-\frac{md}{8}} \left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\} \right)^{\frac{1}{4}},$$

i.e.,

$$\left( \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\} \right)^{\frac{3}{4}} \leq e^{\frac{\gamma|\theta_0|^2}{2} + C\gamma\eta^{-1}} (1 - 4\gamma\eta\|\sigma\|^2)^{-\frac{md}{8}}.$$

Then we have

$$\begin{aligned} \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\} &\leq e^{\frac{2\gamma|\theta_0|^2}{3} + C\gamma\eta^{-1}} \left( (1 - 4\gamma\eta\|\sigma\|^2)^{-\frac{1}{4\gamma\eta\|\sigma\|^2}} \right)^{\frac{2\gamma\|\sigma\|^2d}{3\eta}} \\ &\leq e^{\frac{2\gamma|\theta_0|^2}{3} + C\gamma\eta^{-1}}. \end{aligned} \tag{B.3}$$

This, together with (2.3), implies

$$\begin{aligned} \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{4L^2}\gamma\eta|g(\theta_k)|^2 \right\} &\leq \mathbb{E}_0 \exp \left\{ \sum_{k=0}^{m-1} \frac{K_1}{2}\gamma\eta|\theta_k|^2 \right\} e^{\frac{K_1|g(0)|^2}{2L^2}\gamma\eta^{-1}} \\ &\leq C e^{c(\eta^{-1} + |\theta_0|^2)}. \end{aligned}$$

Writing  $\tilde{\gamma} = \frac{K_1}{4L^2}\gamma$  and replacing the  $\gamma$  in (3.2) by  $\tilde{\gamma}$ , we immediately finish the proof of (3.2).

For (3.3) with  $\theta_0 \sim \pi_\eta$ , (3.2) and (A.6) yield

$$\mathbb{E} \exp \left\{ \gamma\eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 \right\} = \mathbb{E} \left[ \mathbb{E}_0 \exp \left\{ \gamma\eta \sum_{k=0}^{m-1} |g(\theta_k)|^2 \right\} \right] \leq C e^{c\eta^{-1}}.$$

The inequalities (3.4) and (3.5) immediately follow by Chebyshev’s inequality. □

**Proof of Lemma 3.4.** It is easy to see that

$$\mathbb{E} \exp \left\{ \frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \Psi(\theta_k, \xi_{k+1}) \right\} = \mathbb{E} \left[ \exp \left\{ \frac{1}{\sqrt{m}} \sum_{n=0}^{m-2} \Psi(\theta_k, \xi_{k+1}) \right\} \mathbb{E}_{m-1} \left[ e^{\frac{1}{\sqrt{m}} \Psi(\theta_{m-1}, \xi_m)} \right] \right].$$

By Taylor expansion, we deduce that

$$\begin{aligned} \mathbb{E}_{m-1} \left[ e^{\frac{1}{\sqrt{m}} \Psi(\theta_{m-1}, \xi_m)} \right] &= \mathbb{E}_{m-1} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{\sqrt{m}} \Psi(\theta_{m-1}, \xi_m) \right)^n \right] \\ &= 1 + \sum_{n=2}^{\infty} \mathbb{E}_{m-1} \left[ \frac{1}{n!} \left( \frac{1}{\sqrt{m}} \Psi(\theta_{m-1}, \xi_m) \right)^n \right] \\ &\leq 1 + \sum_{n=2}^{\infty} \mathbb{E}_{m-1} \left[ \frac{1}{n!} \left( \frac{K}{\sqrt{m}} (1 + |\xi_m|^2) \right)^n \right]. \end{aligned}$$

For each element, we have

$$\begin{aligned} \mathbb{E}_{m-1} \left[ \frac{1}{n!} \left( \frac{K}{\sqrt{m}} (1 + |\xi_m|^2) \right)^n \right] &\leq \mathbb{E}_{m-1} \left[ \frac{2^{n-1}}{n!} \left( \frac{K^n}{m^{\frac{n}{2}}} (1 + |\xi_m|^{2n}) \right) \right] \\ &\leq \frac{2^{n-1} K^n}{n! m^{\frac{n}{2}}} (1 + d^{n-1} (2n - 1)!) \leq \frac{(4Kd)^n}{m^{\frac{n}{2}}}. \end{aligned}$$

For small enough  $\eta$  such that  $\frac{4Kd}{\sqrt{m}} = 4Kd\eta < 1$ , we have

$$\mathbb{E}_{m-1} \left[ e^{\frac{1}{\sqrt{m}} \Psi(\theta_{m-1}, \xi_m)} \right] \leq 1 + \frac{\frac{(4Kd)^2}{m}}{1 - \frac{4Kd}{\sqrt{m}}} = 1 + \frac{(4Kd)^2}{m - 4Kd\sqrt{m}}.$$

Inductively, we can get

$$\mathbb{E} \exp \left\{ \frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \Psi(\theta_k, \xi_{k+1}) \right\} \leq \left( 1 + \frac{(4Kd)^2}{m - 4Kd\sqrt{m}} \right)^m \leq C. \quad \square$$

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## Supplementary Material

Supplement to “Central limit theorem and self-normalized Cramér-type moderate deviation for Euler-Maruyama scheme” (DOI: [10.3150/21-BEJ1372SUPP](https://doi.org/10.3150/21-BEJ1372SUPP); .pdf). The supplement gives the detailed proof of Lemma 3.5.

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