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Sojourn times of Gaussian and related random fields

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Abstract. This paper is concerned with the asymptotic analysis of sojourn times of random fields with continuous sample paths. Under a very general framework we show that there is an interesting relationship between tail asymptotics of sojourn times and that of supremum. Moreover, we establish the uniform double-sum method to derive the tail asymptotics of sojourn times. In the literature, based on the pioneering research of S. Berman the sojourn times have been utilised to derive the tail asymptotics of supremum of Gaussian processes. In this paper we show that the opposite direction is even more fruitful, namely knowing the asymptotics of supremum of random processes and fields (in particular Gaussian) it is possible to establish the asymptotics of their sojourn times. We illustrate our findings considering i) two dimensional Gaussian random fields, ii) chi-process generated by stationary Gaussian processes and iii) stationary Gaussian queueing processes.

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1. Introduction & First Result

Let $X(t), t \in E$ be a random field with compact parameter set $E \subset \mathbb{R}^d, d \geq 1$ and almost surely continuous sample paths. For a given level $u \in \mathbb{R}$ define the excursion set of X above the level u by

$$A_u(X) \coloneqq \{t \in E : X(t) > u\}.$$

The probability that A_u is not empty

$$\mathbb{P}\{A_u(X) \neq \emptyset\} = \mathbb{P}\{\exists t \in E : X(t) > u\} = \mathbb{P}\left\{\sup_{t \in E} X(t) > u\right\} =: p_u$$

is widely studied in the literature under the asymptotic regime $u \to \infty$, and the assumption that X has marginals with infinite upper endpoint; see, e.g., Piterbarg (1996); Adler and Taylor (2007) for X being Gaussian processes and related random fields.
Define the Lebesgue volume of $A_u(X)$ by

$$Vol(A_u(X)) = \int_E \mathbb{I}(X(t) > u)dt.$$

For specific cases, commonly $d = 1$ and X is stationary, asymptotic results as $u \to \infty$ are also known for the probability that the volume of the excursion set (occupation time or sojourn time) exceeds $v(u)x, x \geq 0$, i.e., approximations of

$$r_u(x) \coloneqq \mathbb{P}\{Vol(A_u(X)) > v(u)x\}, \quad u \to \infty$$

for some specific positive scale function v and $x \geq 0$ are available, see the seminal contribution Berman (1982).

The non-stationary case has been considered in Berman (1985a,b). See also Berman (1992) for the comprehensive introduction of extremes of sojourns for Gaussian processes.
In this contribution we are mainly interested in the formalisation of the uniform double-sum method for sojourns of random processes and fields focusing on the multidimensional case $d \geq 2$, for which no asymptotic results for $r_u(z)$ are available in the literature.
The first question of our study is whether we can determine a positive scaling functions $v(u), u > 0$ and some survival function $\bar{F}$ such that

$$\begin{aligned}
&\lim_{u\to\infty} \mathbb{P}\left\{Vol(A_u(X)) > v(u)x \middle| Vol(A_u(X)) > 0\right\} \\
&= \lim_{u\to\infty} \mathbb{P}\left\{Vol(A_u(X)) > v(u)x \middle| \sup_{t\in E} X(t) > u\right\} = \bar{F}(x)
\end{aligned} \tag{1.1}$$

is valid for all $x \geq 0$. If (1.1) holds for some x positive such that $\bar{F}(x) > 0$ the asymptotics of $r_u(x)$ is proportional to that of p_u, i.e.,

$$r_u(x) \sim \bar{F}(x)p_u, \quad u \to \infty.$$

Here $a(t) \sim b(t)$ means $\lim_{t\to\infty} a(t)/b(t) = 1$.

The following theorem states tractable conditions that imply (1.1) for X as above and $E = E_u$. In order to avoid repetition, all Gaussian processes hereafter are assumed to have almost surely continuous sample paths.

Theorem 1.1. *Let $E_u, u > 0$ be compact sets of $\mathbb{R}^d$ such that $\lim_{u\to\infty} \mathbb{P}\left\{\sup_{t\in E_u} X(t) > u\right\} = 0$. Suppose that there exist collections of Lebesgue measurable disjoint compact sets $I_k(u,n), k \in K_{u,n}$ with $K_{u,n}$ non-empty countable index sets such that*

$$E(u,n) := \bigcup_{k\in K_{u,n}} I_k(u,n) \subset E_u,$$

then (1.1) holds with $E = E_u$ if the following three conditions are satisfied:

A1) (Reduction to relevant sets)

$$\lim_{n\to\infty} \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in E_u\setminus E(u,n)} X(t) > u\right\}}{\mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\}} = 0.$$

A2) (Uniform single-sum approximation) There exists $v(u) > 0$ and $\bar{F}_n, n \geq 1$ such that

$$\lim_{u\to\infty} \sup_{k\in K_{u,n}} \left| \frac{\mathbb{P}\left\{Vol(\{t \in I_k(u,n) : X(t) > u\}) > v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in I_k(u,n)} X(t) > u\right\}} - \bar{F}_n(x) \right| = 0, \quad x \geq 0, \ n \geq 1 \tag{1.2}$$

and for all $x \geq 0$

$$\bar{F}(x) := \lim_{n\to\infty} \bar{F}_n(x) \in (0,1]. \tag{1.3}$$

A3) (Double-sum negligibility) For all large n and large u the set $K_{u,n}$ has at least two elements and

$$\lim_{n\to\infty} \limsup_{u\to\infty} \frac{\sum_{i\neq j, i,j\in K_{u,n}} \mathbb{P}\left\{\sup_{t\in I_i(u,n)} X(t) > u, \sup_{t\in I_j(u,n)} X(t) > u\right\}}{\sum_{k\in K_{u,n}} \mathbb{P}\left\{\sup_{t\in I_k(u,n)} X(t) > u\right\}} = 0.$$

For $X(t), t \in \mathbb{R}$ being a Gaussian process, Dębicki et al. (2017c) shows that conditions A1)-A3) are satisfied under very general assumptions on X. In this case the choice of the family of sets $E(u,n)$ is strongly governed by the set of maximizers of the variance function of X and local behaviour of the variance around its neighbourhood, while sets $I_k(u,n)$ are chosen according to the local structure of the correlation function of X, in a similar fashion as used in the method of *the double sum* for suprema of Gaussian processes; see e.g., Piterbarg (1996, Chapter 2). From Dębicki et al. (2017c), we can formulate some general conditions on X that imply

$$\lim_{u\to\infty} \sup_{k\in K_{u,n}} \left| \frac{\mathbb{P}\left\{\sup_{t\in I_k(u,n)} X(t) > u\right\}}{\Xi_k(u)} - C_n \right| = 0 \tag{1.4}$$

for some known deterministic functions $\Xi_k(u)$, $k \in K_{u,n}$ and C_n positive constants such that $\lim_{n\to\infty} C_n = C \in (0,\infty)$. In order to prove (1.2) if (1.4) holds, we shall prove that

$$\lim_{u\to\infty} \sup_{k\in K_{u,n}} \left| \frac{\mathbb{P}\left\{Vol(\{t \in I_k(u,n) : X(t) > u\}) > v(u)x\right\}}{\Xi_k(u)} - D_n(x) \right| = 0, \tag{1.5}$$

where $D_n,\ n \geq 1$ are deterministic functions such that $\lim_{n\to\infty} D_n(x) = D(x) > 0,\ x \geq 0$. This then in turn implies that (1.3) holds with

$$\bar{F}(x) = \frac{D(x)}{C}.$$

Note that in case that D is continuous at $x = 0$ we also expect that $C = D(0)$.

In the literature various results are known for supremum of functions of Gaussian vector processes, for instance for chi-square processes, chaos of Gaussian processes, order statistics of Gaussian processes, (see, e.g., Piterbarg, 1994, 1996; Hashorva and Ji, 2015; Bai, 2019) or reflected Gaussian processes modelling a queueing process with Gaussian input (see, e.g., Norros, 1994; Hüsler and Piterbarg, 1999; Dębicki, 2002; Piterbarg, 2001; Dębicki and Mandjes, 2003; Hüsler and Piterbarg, 2004; Dieker, 2005; Mandjes, 2007; Dębicki and Liu, 2016, 2018). In Section 3 we illustrate the applicability of Theorem 1.1 by the analysis of three diverse families of stochastic processes: 1) Gaussian random fields (GRF's), 2) chi-processes and 3) reflected fractional Brownian motions. For all this families of stochastic processes the available results in the literature show that both A1) and A3) hold under quite general conditions; see Section 2. Hence, in view of Theorem 1.1, in order to get (1.1) it suffices to determine $\bar{F}$ in A2).

In insurance applications, investigation of sojourn ruin is of particular importance. Therein, cumulative Parisian ruin is used instead of sojourn ruin, see e.g., Ji (2020); Kriukov (2022); Krystecki (2022); Jasnovidov (2021) and the references therein. Besides the above examples, our findings can also be applied to many other GRF's. For instance, multi-dimensional GRF's with $d \geq 3$, non-stationary chi-process or chi-square process, Gaussian chaos process, non-stationary Gaussian fluid queues and so on. However, we shall not analyse these random processes or fields in this paper.

Brief organisation of the rest of the paper. In Section 2 we introduce some notation and Berman-type constants that play the core role in the description of $\bar{F}$. In Section 3, we provide examples that illustrate the derived in Theorem 1.1 technique for getting (1.1). Some technical lemmas are given in Section 4; their proofs are deferred to Section 6. The proofs of the main contributions of this paper are presented in Section 5.

2. Berman-type constants

We begin with the introduction of the Berman-type constants for given independent fBm's $B_{\alpha_i}(s), s \in \mathbb{R}$ with Hurst index $\alpha_i/2 \in (0,1],\ i = 1,2$. For given continuous functions h_1, h_2 set

$$W_{\alpha_1,\alpha_2,h_1,h_2}(t) := \sum_{i=1}^{2}(W_{\alpha_i}(t_i) - h_i(t_i)), \quad t = (t_1,t_2) \in \mathbb{R}^2, \quad W_{\alpha_i}(t_i) = \sqrt{2}B_{\alpha_i}(t_i) - |t_i|^{\alpha_i}\ .$$

Set next $B_0(s) \equiv 0, s \in \mathbb{R}$. For $\alpha_i \in [0,2], i = 1,2,\ x \geq 0$ and $E \subset \mathbb{R}^2$ a compact set, let

$$\mathcal{B}^{h_1,h_2}_{\alpha_1,\alpha_2}(x,E) = \int_{\mathbb{R}} \mathbb{P}\left\{\int_E \mathbb{I}(W_{\alpha_1,\alpha_2,h_1,h_2}(t) > z)dt > x\right\} e^z dz$$

and if the limit exists, define

$$\mathcal{B}_{\alpha_1,\alpha_2}^{b_1|t_1|^{\beta_1},b_2|t_2|^{\beta_2}}(x) := \lim_{S\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{b_1|t_1|^{\beta_1},b_2|t_2|^{\beta_2}}(x, G(S,\alpha_1,\beta_1,\alpha_2,\beta_2))}{S^{\mathbb{I}(\alpha_1<\beta_1)+\mathbb{I}(\alpha_2<\beta_2)}},$$

where

$$G(S,\alpha_1,\beta_1,\alpha_2,\beta_2) = \begin{cases} [0,S]^2, & \alpha_1<\beta_1, \alpha_2<\beta_2, \\ [-S,S]\times[0,S], & \alpha_1\ge\beta_1, \alpha_2<\beta_2, \\ [0,S]\times[-S,S], & \alpha_1<\beta_1, \alpha_2\ge\beta_2, \\ [-S,S]^2 & \alpha_1\ge\beta_1, \alpha_2\ge\beta_2. \end{cases}$$

We omit superscripts h_i's if $h_1(s)=h_2(s)=0, s\in\mathbb{R}$ and then we put in our notation $\beta_1=\beta_2=\infty$ (this implies that $\alpha_1<\beta_1$ and $\alpha_2<\beta_2$). Notice that for $x=0$, $\mathcal{B}_{\alpha_1,\alpha_2}^{h_1,h_2}(x)$ reduces to the classical Pickands or Piterbarg constants, see e.g., Piterbarg (1996). The one-dimensional Berman type constant is defined by

$$\mathcal{B}_\alpha(x,[a,b]) = \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[a,b]} \mathbb{I}(W_\alpha(s)>t)ds > x\right\} e^t dt$$

for $\alpha\in(0,2], a<b, a,b\in\mathbb{R}$, and

$$\mathcal{B}_\alpha(x) = \lim_{S\to\infty}\frac{\mathcal{B}_\alpha(x,[0,S])}{S}.$$

One can refer to Dębicki et al. (2019) and Dębicki et al. (2020b) for the existence and properties of one-dimensional Berman constants. For $x=0$, $\mathcal{H}_\alpha := \mathcal{B}_\alpha(0)$ reduces to the classical Pickands constant; see, e.g., Piterbarg (1996).

The next lemma deals with properties of

$$\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}\left(x, \prod_{i=1}^m [0,n_i]\right) := \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[0,n_1]} \mathbb{I}\left\{\sup_{t_i\in[0,n_i],i=2,\dots,m} \sum_{i=1}^m W_{\alpha_i}(t_i) > s\right\} dt_1 > x\right\} e^s ds$$

for $\alpha_i\in(0,2], i=1,\dots,m$ and $m\ge 1$.

Lemma 2.1. *For any $x\ge 0$, and $n_1>0$*

$$\begin{aligned}\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}(x,n_1) &:= \lim_{n_i\to\infty, i=2,\dots,m} \frac{\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}(x, \prod_{i=1}^m[0,n_i])}{\prod_{i=2}^m n_i} \\ &= \prod_{i=2}^m \mathcal{H}_{\alpha_i} \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[0,n_1]} \mathbb{I}\left\{W_{\alpha_1}(t)>s\right\} dt > x\right\} e^s ds \in (0,\infty) \qquad (2.1)\end{aligned}$$

and

$$\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}(x) := \lim_{n\to\infty}\frac{\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}(x,n)}{n} = \mathcal{B}_{\alpha_1}(x)\prod_{i=2}^m \mathcal{H}_{\alpha_i} \in (0,\infty). \qquad (2.2)$$

Remark 2.2. The limits in (2.1) are finite and positive and $\widehat{\mathcal{B}}_{\alpha_1,\dots,\alpha_m}(x,n_1)$ is a continuous function of $x\in[0,n_1)$ which follows from the combination of Lemma 2.1 and Dębicki et al. (2020b, Lem 4.1). The claim of Lemma 2.1 still holds if we replace B_{α_i} by X_i being independent centered Gaussian processes with stationary increments and variance function satisfying some regular conditions as e.g., in Dębicki (2002).

3. Illustrating examples

In this section we shall apply Theorem 1.1 to three classes of processes:

i) GRF's;
ii) chi-process generated by a stationary Gaussian process;
iii) stationary reflected fractional Brownian motions with drift.

3.1. *Sojourns of GRF's.* Although numerous results for the tail asymptotics of supremum of GRF's are available for both stationary and non-stationary cases (see e.g., Piterbarg, 1996, 2015), asymptotic behaviour of sojourns times for random fields have not been analysed so far in the literature. It follows from the available results in the literature, that A1) holds under quite general conditions, for instance when the variance function has a unique point of maximum and X satisfies a global Hölder continuity condition, see e.g., Piterbarg (1996). The main tool for proving A1) is the so-called Piterbarg inequality, see Piterbarg (1996, Thm 8.1) and the recent contribution Dębicki et al. (2017a). In the following we set

$$\sigma(t) = \sqrt{Var(X(t))}.$$

Under some further weak assumptions on σ and the covariance function of X, also A3) has been shown to hold for a wide collection of cases of interest, see Piterbarg (1996); Dębicki et al. (2016). Thus, in light of Theorem 1.1, in order to prove (1.1) for GRF's the main task is the explicit calculation of $\bar{F}$.

3.1.1. *Stationary GRF's.* First we consider X being a centered stationary GRF with $\sigma(t) = 1, t \in E \subset \mathbb{R}^2$ and the correlation function $r(t,s) = \rho(t-s)$, $t, s \in \mathbb{R}^2$ satisfying

$$1 - r(t_1, t_2, s_1, s_2) \sim a_1|t_1 - s_1|^{\alpha_1} + a_2|t_2 - s_2|^{\alpha_2}, \quad (t_1,t_2),(s_1,s_2) \in E, |t_i - s_i| \to 0, i = 1, 2, \tag{3.1}$$

with $a_i > 0$ and $\alpha_i \in (0, 2]$, $i = 1, 2$. Moreover, suppose that

$$r(t_1, t_2, s_1, s_2) < 1, \quad (t_1,t_2),(s_1,s_2) \in E, (t_1,t_2) \neq (s_1,s_2). \tag{3.2}$$

For notational simplicity we shall consider $E = [0, T_1] \times [0, T_2]$, with T_1, T_2 positive constants. Results for general hypercubes or even for bounded Jordan measurable sets $E \subset \mathbb{R}^d$ follow with similar calculations.

Proposition 3.1. *Let $X(t), t \in E = [0, T_1] \times [0, T_2]$ be a centred stationary GRF which satisfies (3.1) and (3.2) and assume that $v(u) = a_1^{-1/\alpha_1} a_2^{-1/\alpha_2} u^{-2/\alpha_1 - 2/\alpha_2}$. Then for all $x \geq 0$*

$$\lim_{u\to\infty} \mathbb{P}\left\{ \int_E \mathbb{I}(X(t) > u)dt > v(u)x \Big| \sup_{t\in E} X(t) > u \right\} = \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}_{\alpha_1,\alpha_2}(0)}.$$

Remark 3.2. The case that $T_i = T_{i,u}, i = 1, 2$ depend on u needs some extra care. $T_{i,u}$'s should not be too small, i.e.,

$$\lim_{u\to\infty} T_{i,u} u^{2/\alpha_i} = \infty, \quad i = 1, 2.$$

On the other side $T_{i,u}$'s cannot be too large. For some $\beta \in (0, 1)$ we shall require

$$\lim_{u\to\infty} T_{1,u} T_{2,u} e^{-\beta u^2/2} = 0.$$

3.1.2. *GRF's with non-constant variance.* Suppose that $t^* = (t_1^*, t_2^*) \in E = [-T_1, T_1] \times [-T_2, T_2]$ is a unique inner point of E such that $\sigma(t^*) = \sup_{t\in E} \sigma(t) = 1$ and further for some positive constants b_i, β_i, $i = 1, 2$

$$1 - \sigma(t) \sim b_1 |t_1 - t_1^*|^{\beta_1} + b_2 |t_2 - t_2^*|^{\beta_2}, \quad t = (t_1, t_2) \in E, \ \|t - t^*\| \to 0. \tag{3.3}$$

Here $\|\cdot\|$ denotes the Euclidean norm. Moreover, for the correlation function r of X we shall assume that

$$1 - r(t, s) \sim a_1 |t_1 - s_1|^{\alpha_1} + a_2 |t_2 - s_2|^{\alpha_2} \tag{3.4}$$

as $t, s \in E, \|t - t^*\|, \|s - t^*\| \to 0$ with $a_i > 0$ and $\alpha_i \in (0, 2]$, $i = 1, 2$, $s = (s_1, s_2)$. Below we interpret $\infty \cdot 0$ as 0.

Proposition 3.3. *If $X(t), t \in E$ is a centered GRF which satisfies (3.3) and (3.4) and $v(u) = \prod_{i=1}^{2} \left(a_i^{-1/\alpha_i^*} u^{-2/\min(\alpha_i, \beta_i)} \right)$ with $\alpha_i^* = \alpha_i \mathbb{I}(\alpha_i \le \beta_i) + \infty \mathbb{I}(\alpha_i > \beta_i)$, then for all $x \ge 0$*

$$\lim_{u\to\infty} \mathbb{P}\left\{ \int_E \mathbb{I}(X(t) > u) dt > v(u) x \Big| \sup_{t\in E} X(t) > u \right\} = \frac{\mathcal{B}_{\hat{\alpha}_1, \hat{\alpha}_2}^{\bar{a}_1 b_1 |t_1|^{\beta_1}, \bar{a}_2 b_2 |t_2|^{\beta_2}}(x)}{\mathcal{B}_{\hat{\alpha}_1, \hat{\alpha}_2}^{\bar{a}_1 b_1 |t_1|^{\beta_1}, \bar{a}_2 b_2 |t_2|^{\beta_2}}(0)},$$

where

$$\bar{a}_i = \begin{cases} 0 & \alpha_i < \beta_i \\ \frac{1}{a_i} & \alpha_i = \beta_i \\ 1 & \alpha_i > \beta_i \end{cases}, \quad \hat{\alpha}_i = \begin{cases} \alpha_i & \alpha_i \le \beta_i \\ 0 & \alpha_i > \beta_i \end{cases}, i = 1, 2.$$

3.2. *Sojourns of chi-processes.* Let $X(t), t \in [0, T]$ be a centered stationary Gaussian process with unit variance and correlation function satisfying

$$1 - r(s, t) \sim a |t - s|^{\alpha}, \quad |s - t| \to 0, \quad \alpha \in (0, 2]$$

and for all $s \ne t$, $s, t \in [0, T]$

$$r(s, t) < 1.$$

Define the chi-process χ with $m \ge 1$ degrees by

$$\chi(t) := \sqrt{\sum_{i=1}^{m} X_i^2(t)}, \quad t \in \mathbb{R}, \tag{3.5}$$

where $X_i, 1 \le i \le m$ are iid copies of X. The exact asymptotics of $\mathbb{P}\left\{\sup_{t\in[0,T]} \chi(t) > u\right\}$ has been investigated in Piterbarg (1994, 1996); Hashorva and Ji (2015). In the following theorem we consider the sojourn time of χ.

Proposition 3.4. *Let χ be defined as in (3.5). If $v(u) = a^{-1/\alpha}u^{-2/\alpha}$, then for all $x \ge 0$*

$$\lim_{u\to\infty} \mathbb{P}\left\{\int_{[0,T]} \mathbb{I}(\chi(t) > u)dt > v(u)x \Big| \sup_{t\in[0,T]} \chi(t) > u\right\} = \frac{\mathcal{B}_\alpha(x)}{\mathcal{B}_\alpha(0)}.$$

3.3. *Sojourns of stationary reflected fractional Brownian motion with drift.* Consider a stationary reflected fractional Brownian motion with drift $Q(t), t \ge 0$, i.e.,

$$Q(t) := \sup_{s\ge t} \left(B_\alpha(s) - B_\alpha(t) - c(s-t)\right),$$

where B_α is an fBm with Hurst parameter $\alpha/2 \in (0,1)$ and $c \in (0,\infty)$. Motivated by some applications to queueing theory, the seminal paper Hüsler and Piterbarg (1999) studied the tail asymptotics of $Q(0)$. Later on, Piterbarg (2001) considered the tail asymptotics of the supremum of $Q(t)$ over a finite time horizon.

Recently, the findings of Piterbarg have been extended to Gaussian processes with stationary increments Dębicki and Liu (2016). We consider next the case of fBm and note that a more general case of Gaussian processes with stationary increments can be also dealt with using results from Dębicki and Liu (2016). In the following we consider $E_u = [0, T_u]$, where T_u is a non-negative function of $u > 0$.

Proposition 3.5. *Let $v(u) = u^{\frac{2(\alpha-1)}{\alpha}} \left(\frac{\sqrt{2}(\tau^*)^\alpha}{1+c\tau^*}\right)^{2/\alpha}$ with $\tau^* = \frac{\alpha}{c(2-\alpha)}$ and $\alpha \in (0,2)$.*
i) If $\lim_{u\to\infty} \frac{T_u}{v(u)} = T \in (0,\infty)$, then for $T > x \ge 0$

$$\lim_{u\to\infty} \mathbb{P}\left\{\int_{[0,T_u]} \mathbb{I}(Q(t) > u)dt > v(u)x \Big| \sup_{t\in[0,T_u]} Q(t) > u\right\} = \frac{\mathcal{B}_\alpha(x,[0,T])}{\mathcal{B}_\alpha(0,[0,T])}.$$

ii) If $\lim_{u\to\infty} \frac{T_u}{v(u)} = \infty$ and $T_u < e^{\beta u^{2-\alpha}}$ with $\beta \in \left(0, \left(\frac{1+c\tau^*}{\sqrt{2}(\tau^*)^{\alpha/2}}\right)^2\right)$, then for all $x \ge 0$

$$\lim_{u\to\infty} \mathbb{P}\left\{\int_{[0,T_u]} \mathbb{I}(Q(t) > u)dt > v(u)x \Big| \sup_{t\in[0,T_u]} Q(t) > u\right\} = \frac{\mathcal{B}_\alpha(x)}{\mathcal{B}_\alpha(0)}.$$

Remark 3.6. i) Note that $\lim_{u\to\infty} v(u) = \infty$ for $\alpha > 1$, and $\lim_{u\to\infty} v(u) = 0$ for $\alpha < 1$.

ii) Conclusion in i) of Proposition 3.5 still holds for $x > T$ since both sides in the equality of i) are 0. However, it becomes tricky for the case $T = x$. We consider two special cases for $T = x$. If $T = x$ and $T_u \le xv(u)$ for u sufficiently large, then both sides in the equality of i) are 0. If $T = x$ and $T_u > xv(u)$ for sufficiently large u, as $u \to \infty$

$$\mathbb{P}\left\{\int_{[0,T_u]} \mathbb{I}(Q(t) > u)dt > v(u)x \Big| \sup_{t\in[0,T_u]} Q(t) > u\right\} \sim \frac{\mathbb{P}\left\{\inf_{t\in[0,T_u]} Q(t) > u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T_u]} Q(t) > u\right\}}.$$

Combining the above two cases for $T = x$, we conclude that the limit for $T = x$ generally does not exist.

4. Auxiliary lemmas

In this section we collect some lemmas that play important, although mostly technical role in the proofs of results given in Sections 1-3. Their proofs are deferred to Section 6. We begin with a lemma which is an extension of Theorem 2.1 from Dębicki et al. (2017c). Suppose that for a compact $d-$dimensional hyperrectangle $K \subset \mathbb{R}^d$ we have

$$I_k(u,n) = \{t_{u,n,k} + (v_1(u)t_1, \dots v_d(u)t_d) : t \in K\},$$

where $v_i(u) > 0$, $i = 1, \dots, d$ and $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. Then, by transforming time, we have

$$\begin{aligned}
&\mathbb{P}\{Vol(\{t \in I_k(u,n) : X(t) > u\}) > v(u)z\} \\
&= \mathbb{P}\left\{\int_{I_k(u,n)} \mathbb{I}(X(t) > u)dt > v(u)z\right\} \\
&= \mathbb{P}\left\{\int_K \mathbb{I}(X(t_{u,n,k} + (v_1(u)t_1, ..., v_d(u)t_d)) > u)dt > z\right\},
\end{aligned}$$

where $v(u) = \prod_{i=1}^d v_i(u)$.
Motivated by these calculations, we consider next $\xi_{u,j}(t), t \in E_1,\ j \in S_u,\ u \geq 0$ a family of centered GRF's with continuous sample paths and variance function $\sigma_{u,j}^2$.
Suppose in the following that S_u is a countable set for all u large.
For simplicity in the following we assume that $0 \in E_1$. For a random variable Z, we set $\overline{Z} = \frac{Z}{\sqrt{Var(Z)}}$ if $Var(Z) > 0$.

We introduce next three assumptions:

C0: $\{g_{u,j}, j \in S_u\}$ is a sequence of deterministic functions of u satisfying

$$\lim_{u\to\infty} \inf_{j \in S_u} g_{u,j} = \infty.$$

C1: $Var(\xi_{u,j}(0)) = 1$ for all large u and any $j \in S_u$ and there exists some bounded continuous function h on E_1 such that

$$\lim_{u\to\infty} \sup_{s\in E_1, j\in S_u} \left|g_{u,j}^2 \left(1 - \sigma_{u,j}(s)\right) - h(s)\right| = 0.$$

C2: There exists a centered GRF $\zeta(s), s \in \mathbb{R}^d$ with a.s. continuous sample paths such that

$$\lim_{u\to\infty} \sup_{s,s'\in E_1, j\in S_u} \left|g_{u,j}^2 \left(Var(\overline{\xi}_{u,j}(s) - \overline{\xi}_{u,j}(s'))\right) - 2Var(\zeta(s) - \zeta(s'))\right| = 0. \tag{4.1}$$

C3: There exist positive constants C, ν, u_0 such that

$$\sup_{j\in S_u} g_{u,j}^2 Var(\overline{\xi}_{u,j}(s) - \overline{\xi}_{u,j}(s')) \leq C\|s - s'\|^\nu$$

holds for all $s, s' \in E_1, u \geq u_0$.

Denote by $C(E_i), i=1,2$ the Banach space of all continuous functions $f: E_i \mapsto \mathbb{R}$, with $E_i \subset \mathbb{R}^{d_i}, d_i \geq 1, i=1,2$ being compact rectangles equipped with the sup-norm.

Let $\Gamma: C(E_1) \to C(E_2)$ be a continuous functional satisfying

F1: For any $f \in C(E_1)$, and $a>0, b \in \mathbb{R}$, $\Gamma(af+b) = a\Gamma(f)+b$;

F2: There exists $c>0$ such that

$$\sup_{t\in E_2} \Gamma(f)(t) \leq c \sup_{s \in E_1} f(s), \quad \forall f \in C(E_1).$$

Hereafter, $Q_i, i \in \mathbb{N}$ are some positive constants which might be different from line to line and $f(u,n) \sim g(u), u \to \infty, n \to \infty$ means that

$$\lim_{n\to\infty} \lim_{u\to\infty} \frac{f(u,n)}{g(u)} = 1.$$

Lemma 4.1. *Let $\{\xi_{u,j}(s), s \in E_1, j \in S_u, u \geq 0\}$ be a family of centered GRF's defined as above satisfying* **C0-C3** *and let Γ satisfy* **F1-F2**. *Let η be a positive σ-finite measure on E_2 being equivalent with the Lebesgues measure on E_2. If for all large u and all $j \in S_u$*

$$\mathbb{P}\left\{\sup_{t\in E_2} \Gamma(\xi_{u,j})(t) > g_{u,j}\right\} > 0,$$

then for all $x \in [0, \eta(E_2))$

$$\lim_{u\to\infty} \sup_{j \in S_u} \left| \frac{\mathbb{P}\left\{\int_{E_2} \mathbb{I}\left(\Gamma(\xi_{u,j})(t) > g_{u,j}\right) \eta(dt) > x\right\}}{\Psi(g_{u,j})} - \mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)\right| = 0, \tag{4.2}$$

where Ψ is the tail of the standard normal distribution and

$$\mathcal{B}_\zeta^{\Gamma,h,\eta}(x,E_2) := \int_{\mathbb{R}} \mathbb{P}\left\{\int_{E_2} \mathbb{I}\big(\Gamma(\sqrt{2}\zeta - Var(\zeta) - h)(t) + y > 0\big)\eta(dt) > x\right\} e^{-y} dy$$

and the function $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x,E_2)$ is continuous at $x \in (0, \eta(E_2))$.

Lemma 4.2. *Let $x \geq 0$. Then*

(i) $\mathcal{B}_{\alpha_1,\alpha_2}(x) = \lim_{n\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} \in (0,\infty)$,

(ii) $\lim_{n\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} \in (0,\infty)$,

(iii) $\lim_{n\to\infty} \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x,[-n,n]^2) \in (0,\infty)$.

5. Proofs

5.1. ***Proof of Theorem 1.1.*** Let next $A_u(X) := \{t \in E_u : X(t) > u\}$. For all $x \geq 0$ and all u positive, since $v(u)$ is non-negative, assuming that $\mathbb{P}\left\{\sup_{t\in E_u} X(t) > u\right\} > 0$ we have

$$\begin{aligned}\pi(u) &:= \mathbb{P}\left\{Vol(A_u(X)) > v(u)x \middle| Vol(A_u(X)) > 0\right\}\\ &= \mathbb{P}\left\{Vol(A_u(X)) > v(u)x \middle| \sup_{t\in E_u} X(t) > u\right\}\\ &= \frac{\mathbb{P}\left\{\int_E \mathbb{I}(X(t) > u)dt > v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in E_u} X(t) > u\right\}}\end{aligned}$$

and further for all $n \geq 1$

$$\begin{aligned}\pi(u) &\geq \frac{\mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\} + \mathbb{P}\left\{\sup_{t\in E_u\setminus E(u,n)} X(t) > u\right\}},\\ \pi(u) &\leq \frac{\mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\}} + \frac{\mathbb{P}\left\{\sup_{t\in E_u\setminus E(u,n)} X(t) > u\right\}}{\mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\}}.\end{aligned}$$

Applying A1, it follows that

$$\pi(u) \sim \frac{\mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\}} =: \pi(u,n), \quad u \to \infty, n \to \infty.$$

If $K_{u,n}$ has only one element for all u, n large, the claim follows straightforwardly by A2. We suppose next that $K_{u,n}$ has at least two elements for all u, n large. In order to proceed we shall apply the standard scheme utilising Bonferroni inequality. Set therefore

$$\Sigma_{u,n} := \sum_{k\in K_{u,n}} \mathbb{P}\left\{\sup_{t\in I_k(u,n)} X(t) > u\right\}, \quad \Sigma\Sigma_{u,n} := \sum_{i\neq j, i,j\in K_{u,n}} \mathbb{P}\left\{\sup_{t\in I_i(u,n)} X(t) > u, \sup_{t\in I_j(u,n)} X(t) > u\right\}.$$

By the Bonferroni inequality

$$\Sigma_{u,n} - \Sigma\Sigma_{u,n} \leq \mathbb{P}\left\{\sup_{t\in E(u,n)} X(t) > u\right\} \leq \Sigma_{u,n}.$$

The asymptotic behaviour of the probability of interest in the above inequality can be derived if the following two-step procedure is successful (which will work in our settings here). First we determine the exact asymptotics of the upper bound and then in a second step we show that the correction in the lower bound is asymptotically negligible.

Now we want to apply the same idea for the sojourn functional, here the analysis is however more involved. Observe first that for any $u > 0$

$$
\begin{aligned}
&\mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}\\
&\le \mathbb{P}\left\{\sum_{k\in K_{u,n}} \int_{I_k(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}\\
&\le \mathbb{P}\left\{\exists k \in K_{u,n}, \int_{I_k(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}\\
&\quad +\mathbb{P}\left\{\exists i,j \in K_{u,n}, i \neq j, \int_{I_i(u,n)} \mathbb{I}(X(t) > u)dt > 0, \int_{I_j(u,n)} \mathbb{I}(X(t) > u)dt > 0\right\}\\
&\le \hat{\pi}(u,n) + \Sigma\Sigma_{u,n},
\end{aligned}
$$

where

$$
\hat{\pi}(u,n) = \sum_{k\in K_{u,n}} \mathbb{P}\left\{\int_{I_k(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}.
$$

Using the Bonferroni inequality again we have

$$
\begin{aligned}
\mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\} &\ge \mathbb{P}\left\{\exists k \in K_{u,n}, \int_{I_k(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}\\
&\ge \hat{\pi}(u,n) - \Sigma\Sigma_{u,n}.
\end{aligned}
$$

The sojourn integral can then be approximated by $\hat{\pi}(u,n)$ if we show the correction in the lower bound is negligible. We have

$$
\begin{aligned}
\limsup_{u\to\infty} \pi(u,n) &\le \limsup_{u\to\infty} \frac{\hat{\pi}(u,n) + \Sigma\Sigma_{u,n}}{\Sigma_{u,n} - \Sigma\Sigma_{u,n}} = \limsup_{u\to\infty} \frac{\hat{\pi}(u,n)}{\Sigma_{u,n}} \times \frac{1 + \limsup_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\hat{\pi}(u,n)}}{1 - \limsup_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\Sigma_{u,n}}},\\
\liminf_{u\to\infty} \pi(u,n) &\ge \liminf_{u\to\infty} \frac{\hat{\pi}(u,n) - \Sigma\Sigma_{u,n}}{\Sigma_{u,n}} = \liminf_{u\to\infty} \frac{\hat{\pi}(u,n)}{\Sigma_{u,n}} - \limsup_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\Sigma_{u,n}}.
\end{aligned}
$$

By (1.2) in A2 for any $n \ge 1$ and $x \ge 0$

$$
\limsup_{u\to\infty} \frac{\hat{\pi}(u,n)}{\Sigma_{u,n}} = \liminf_{u\to\infty} \frac{\hat{\pi}(u,n)}{\Sigma_{u,n}} = \bar{F}_n(x)
$$

implying

$$
\begin{aligned}
\bar{F}_n(x) - \limsup_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\Sigma_{u,n}} &\le \liminf_{u\to\infty} \pi(u,n)\\
&\le \limsup_{u\to\infty} \pi(u,n) \le \bar{F}_n(x) \frac{1 + \limsup\limits_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\bar{F}_n(x)\Sigma_{u,n}}}{1 - \limsup\limits_{u\to\infty} \frac{\Sigma\Sigma_{u,n}}{\Sigma_{u,n}}}.
\end{aligned}
\tag{5.1}
$$

In view of A3, letting $n \to \infty$ in the above inequalities we have that for $x \ge 0$

$$
\lim_{n\to\infty} \lim_{u\to\infty} \pi(u,n) = \bar{F}(x) \in (0,1].
$$

This completes the proof. □

5.2. ***Proof of Lemma 2.1.*** By the independence of W_{α_i}'s for any positive $n_1, \ldots, n_m$

$$
\begin{aligned}
\widehat{\mathcal{B}}_{\alpha_1,\ldots,\alpha_m}\left(x, \prod_{i=1}^m [0,n_i]\right) &= \mathbb{E}\left\{\int_{\mathbb{R}} \mathbb{I}\left(\int_{[0,n_1]} \mathbb{I}\left\{\sup_{t_i\in[0,n_i], i=2,\ldots,m} \sum_{i=1}^m W_{\alpha_i}(t_i) > s\right\} dt_1 > x\right) e^s ds\right\} \\
&= \mathbb{E}\left\{e^{\sum_{i=2}^m \sup_{t_i\in[0,n_i]} W_{\alpha_i}(t_i)} \int_{\mathbb{R}} \mathbb{I}\left(\int_{[0,n_1]} \mathbb{I}\left\{W_{\alpha_1}(t_1) > s\right\} dt_1 > x\right) e^s ds\right\} \\
&= \prod_{i=2}^m \mathbb{E}\left\{\sup_{t_i\in[0,n_i]} e^{W_{\alpha_i}(t_i)}\right\} \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[0,n_1]} \mathbb{I}\left\{W_{\alpha_1}(t_1) > s\right\} dt_1 > x\right\} e^s ds.
\end{aligned}
$$

Hence the claim follows by the definition of Pickands and Berman constants. □

5.3. *Proof of Proposition 3.1.* The proof will be established by checking that A1-A3 in Theorem 1.1 are satisfied. We begin with the introduction of partition

$$
I_{k_1,k_2}(u,n) = \prod_{i=1}^2 [a_i^{-1/\alpha_i} u^{-2/\alpha_i} k_i n, a_i^{-1/\alpha_i} u^{-2/\alpha_i}(k_i+1)n],
$$

for

$$
0 \le k_i \le [T_i a_i^{1/\alpha_i} u^{2/\alpha_i} n^{-1}] - 1 =: N_i(u,n), \ \ i = 1, 2.
$$

Let next

$$
K_{u,n} = \{(k_1,k_2) : 0 \le k_1 \le N_1(u,n), 0 \le k_2 \le N_2(u,n)\}
$$

and $E(u,n) = \bigcup_{(k_1,k_2)\in K_{u,n}} I_{k_1,k_2}(u,n)$, hence $E(u,n) \subset E$.

Condition A1. It follows straightforwardly from Lemma 7.1 in Piterbarg (1996) that

$$
\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\} \sim \sum_{0\le k_i\le N_i(u,n), i=1,2} \mathbb{P}\left\{\sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u\right\}, \quad u\to\infty, n\to\infty, \tag{5.2}
$$

which implies the validity of condition A1.

Condition A2. Let for $t = (t_1, t_2)$

$$
\begin{aligned}
\xi_{u,n,k_1,k_2}(t) &= X(a_1^{-1/\alpha_1} u^{-2/\alpha_1}(k_1 n + t_1), a_2^{-1/\alpha_2} u^{-2/\alpha_2}(k_2 n + t_2)), \\
v(u) &= a_1^{-1/\alpha_1} a_2^{-1/\alpha_2} u^{-2/\alpha_1 - 2/\alpha_2}.
\end{aligned}
$$

We derive the uniform asymptotics, as $u \to \infty$, of

$$
\mathbb{P}\left\{Vol(\{t \in I_{k_1,k_2}(u,n) : X(t) > u\}) > v(u)x\right\} = \mathbb{P}\left\{\int_{[0,n]^2} \mathbb{I}(\xi_{u,n,k_1,k_2}(t) > u) dt > x\right\},
$$

with $x \ge 0$. For this, we check conditions **C0-C3** of Lemma 4.1 with $\Gamma(f) = f, f \in C([0,n]^2)$. First note that **C0-C1** follow trivially with $h = 0$ and $g_{u,j} = u$. Moreover, by (3.1), we have

$$
\lim_{u\to\infty} \sup_{\substack{0\le k_i\le N_i(u,n),\\ i=1,2}} \sup_{s,t\in[0,n]^2} \left| u^2 Var(\xi_{u,n,k_1,k_2}(t) - \xi_{u,n,k_1,k_2}(s)) - 2Var\left(\sum_{i=1}^2 B_{\alpha_i}(t_i) - \sum_{i=1}^2 B_{\alpha_i}(s_i)\right)\right| = 0,
$$

with $B_{\alpha_i}, i=1,2$ being two independent fBms' with indices $\alpha_i/2$, respectively. This implies that **C2** is satisfied with $\zeta(t)=\sum_{i=1}^2 B_{\alpha_i}(t_i)$. Additionally, in light of (3.1) and the stationarity, we have that

$$\sup_{0\le k_i\le N_i(u,n)+1, i=1,2} u^2 Var(\xi_{u,n,k_1,k_2}(t)-\xi_{u,n,k_1,k_2}(s))\le C\|t-s\|^{\min(\alpha_1,\alpha_2)}, \quad s,t\in[0,n]^2$$

implying **C3**. Consequetnly, by Lemma 4.1

$$\lim_{u\to\infty}\sup_{0\le k_i\le N_i(u,n), i=1,2}\left|\frac{\mathbb{P}\left\{Vol(\{t\in I_{k_1,k_2}(u,n): X(t)>u\})>v(u)x\right\}}{\Psi(u)}-\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)\right|=0. \quad (5.3)$$

Hence Piterbarg (1996, Lem 6.1) yields

$$\lim_{u\to\infty}\sup_{0\le k_i\le N_i(u,n), i=1,2}\left|\frac{\mathbb{P}\left\{Vol(\{t\in I_{k_1,k_2}(u,n): X(t)>u\})>v(u)x\right\}}{\mathbb{P}\left\{\sup_{t\in I_{k_1,k_2}(u,n)} X(t)>u\right\}}-\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{\mathcal{B}_{\alpha_1,\alpha_2}(0,[0,n]^2)}\right|=0.$$

Since, by (i) of Lemma 4.2, for all $x\ge 0$ we have

$$\mathcal{B}_{\alpha_1,\alpha_2}(x)=\lim_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2}\in(0,\infty), \quad (5.4)$$

then

$$\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}_{\alpha_1,\alpha_2}(0)}=\lim_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{\mathcal{B}_{\alpha_1,\alpha_2}(0,[0,n]^2)}\in(0,1], \quad x\ge 0, \quad (5.5)$$

which confirms that A2 holds with $\bar F(x)=\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}_{\alpha_1,\alpha_2}(0)}$.

Condition A3. By (7.4) in the proof of Piterbarg (1996, Lem 7.1), for all large u and n

$$\sum_{0\le k_i,k_i'\le N_i(u,n), i=1,2,(k_1,k_2)\ne(k_1',k_2')}\mathbb{P}\left\{\sup_{t\in I_{k_1,k_2}(u,n)} X(t)>u, \sup_{t\in I_{k_1',k_2'}(u,n)} X(t)>u\right\}$$
$$\le\left(\frac{\mathbb{C}_2}{\sqrt{n}}+e^{-\mathbb{C}_1 n^{\mathbb{C}}}\right)\mathbb{P}\left\{\sup_{t\in E} X(t)>u\right\},$$

where $\mathbb{C},\mathbb{C}_1$ and $\mathbb{C}_2$ are some positive constants, which gives that A3 is satisfied. This completes the proof. □

5.4. *Proof of Proposition 3.3.* Without loss of generality, we assume that $t^*=(0,0)$. The proof relies on verification that A1-A3 in Theorem 1.1 are satisfied. We begin by introducing some notation. Let

$$I_{k_1,k_2}(u,n)=\prod_{i=1}^2[k_i v_i(u)n,(k_i+1)v_i(u)n], \quad v_i(u)=a_i^{-1/\alpha_i^*}u^{-2/\min(\alpha_i,\beta_i)}, i=1,2 \quad (5.6)$$

and $v(u)=v_1(u)v_2(u)$, where $\alpha_i^*=\alpha_i\mathbb{I}(\alpha_i\le\beta_i)+\infty\mathbb{I}(\alpha_i>\beta_i)$. Additionally, let

$$e(t)=\frac{1-\sigma(t)}{\sum_{i=1}^2 b_i|t_i|^{\beta_i}}-1, |t|\ne 0, \ e_u=\sup_{0<|t_i|<\left(\frac{\ln u}{u}\right)^{2/\beta_i}}|e(t)|,$$

and set

$$N'_i(u,n) = \left[\frac{(e_u^{-1/4} \wedge \ln u)^{2/\beta_i}}{u^{2/\beta_i} v_i(u) n} \right], \ i = 1, 2.$$

We distinguish between different scenarios according to the values of $\alpha_i, \beta_i, i = 1, 2$.

Case $\alpha_i < \beta_i, i = 1, 2$. In this scenario

$$v_i(u) = a_i^{-1/\alpha_i} u^{-2/\alpha_i}, i = 1, 2, \quad K_{u,n} = \{(k_1, k_2) : 0 \leq |k_i| \leq N'_i(u,n), i = 1, 2\}$$

and $E(u,n) = \bigcup_{(k_1,k_2)\in K_{u,n}} I_{k_1,k_2}(u,n)$.

Conditions A1 and A3. Following the same reasoning as in the proof of Proposition 3.1, the validity of conditions A1 and A3 follows straightforwardly from (34), (40) and (41) in Dębicki et al. (2017b).

Condition A2. Let

$$\xi_{u,n,k_1,k_2}(t) = \overline{X}(v_1(u)(k_1 n + t_1), v_2(u)(k_2 n + t_2)),$$

$$u^-_{n,k_1,k_2} = u \inf_{t \in I_{k_1,k_2}(u,n)} \frac{1}{\sigma(t)}, \quad u^+_{n,k_1,k_2} = u \sup_{t \in I_{k_1,k_2}(u,n)} \frac{1}{\sigma(t)}. \tag{5.7}$$

Then we have the following bounds

$$\mathbb{P}\{Vol(\{t \in I_{k_1,k_2}(u,n) : X(t) > u\}) \geq v(u)x\} \leq \mathbb{P}\left\{ \int_{[0,n]^2} \mathbb{I}(\xi_{u,n,k_1,k_2}(t) > u^-_{n,k_1,k_2}) dt > x \right\},$$

$$\mathbb{P}\{Vol(\{t \in I_{k_1,k_2}(u,n) : X(t) > u\}) \geq v(u)x\} \geq \mathbb{P}\left\{ \int_{[0,n]^2} \mathbb{I}(\xi_{u,n,k_1,k_2}(t) > u^+_{n,k_1,k_2}) dt > x \right\}.$$

In order to derive the uniform asymptotics of the above terms we check conditions **C0-C3** of Lemma 4.1 with $\Gamma(f) = f,\ f \in C([0,n]^2)$ for $\xi_{u,n,k_1,k_2}(t),\ (k_1,k_2) \in K_{u,n}$.

Note that **C0-C1** holds with $h = 0$ and $g_{u,j} = u^{\pm}_{n,k_1,k_2}$. By (3.3) and (3.4), we have

$$\lim_{u\to\infty} \sup_{\substack{s,t\in[0,n]^2,\\ (k_1,k_2)\in K_{u,n}}} \left| (u^{\pm}_{n,k_1,k_2})^2 Var(\xi_{u,n,k_1,k_2}(t) - \xi_{u,n,k_1,k_2}(s)) - 2Var\left(\sum_{i=1}^2 B_{\alpha_i}(t_i) - \sum_{i=1}^2 B_{\alpha_i}(s_i) \right) \right| = 0,$$

where $B_{\alpha_i}, i = 1, 2$ are two independent fBm's with indices $\alpha_i, i = 1, 2$ respectively. This confirms that **C2** holds with $\zeta(t_1, t_2) = B_{\alpha_1}(t_1) + B_{\alpha_2}(t_2)$. By (3.4), we have

$$\sup_{(k_1,k_2)\in K_{u,n}} (u^{\pm}_{n,k_1,k_2})^2 (Var(\xi_{u,n,k_1,k_2}(t) - \xi_{u,n,k_1,k_2}(s))) \leq Q||s-t||^{\min(\alpha_1,\alpha_2)}, \quad s,t \in [0,n]^2.$$

Thus **C3** is satisfied.

Therefore, by Lemma 4.1, we have that for $0 \leq x < n^2$

$$\lim_{u\to\infty} \sup_{(k_1,k_2)\in K_{u,n}} \left| \frac{\mathbb{P}\left\{ \int_{[0,n]^2} \mathbb{I}(\xi_{u,n,k_1,k_2}(t) > u^{\pm}_{n,k_1,k_2}) dt > x \right\}}{\Psi(u^{\pm}_{n,k_1,k_2})} - \mathcal{B}_{\alpha_1,\alpha_2}(x, [0,n]^2) \right| = 0. \tag{5.8}$$

Since

$$\lim_{u\to\infty} \sup_{(k_1,k_2)\in K_{u,n}} \left| \frac{\Psi(u^-_{n,k_1,k_2})}{\Psi(u^+_{n,k_1,k_2})} - 1 \right| = 0 \tag{5.9}$$

(see Section 6 for the validation of (5.9)), by (5.8) we obtain for $0 \le x < n^2$

$$\lim_{u\to\infty} \sup_{(k_1,k_2)\in K_{u,n}} \left| \frac{\mathbb{P}\left\{Vol(\{t \in I_{k_1,k_2}(u,n) : X(t) > u\}) \ge v(u)x\right\}}{\Psi(u^-_{n,k_1,k_2})} - \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2) \right| = 0.$$

Therefore, (1.2) holds with

$$\bar{F}_n(x) = \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{\mathcal{B}_{\alpha_1,\alpha_2}(0,[0,n]^2)}, \quad x \ge 0.$$

Finally, by (5.5), we have that A2 holds. Thus the claim is established with

$$\bar{F}(x) = \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}_{\alpha_1,\alpha_2}(0)}.$$

Case $\alpha_1 = \beta_1, \alpha_2 < \beta_2$. In this case $v_i(u) = a_i^{-1/\alpha_i} u^{-2/\alpha_i}, i = 1,2$. Let

$$\hat{I}_{k_2}(u,n) = I_{-1,k_2}(u,n) \cup I_{0,k_2}(u,n), \quad E_1(u,n) = \bigcup_{k_2\in K_{u,n}} \hat{I}_{k_2}(u,n), \tag{5.10}$$

where $K_{u,n} := \{k_2 \in \mathbb{Z} : |k_2| \le N'_2(u,n)\}$.

Conditions A1 and A3. Analogously to the previous case, conditions A1 and A3 hold with $E(u,n) := E_1(u,n)$ and $I_k(u,n) := \hat{I}_{k_2}(u,n)$, by (34), (46), (48) and (49) of Dębicki et al. (2017b).

Condition A2. Rewrite (3.3) as

$$\frac{1}{\sigma(t)} = \left(1 + (1+e_1(t_1))b_1|t_1|^{\beta_1}\right)\left(1 + (1+e_2(t_2))b_2|t_2|^{\beta_2}\right)$$

for some functions $e_1(t_1)$ and $e_2(t_2)$ which satisfy

$$\lim_{u\to\infty} \sup_{t\in E_1(u,n)} |e_i(t_i)| = 0, \quad i = 1,2.$$

Let

$$\xi_{u,n,k_2}(t) = \frac{\overline{X}(v_1(u)t_1, v_2(u)(k_2 n + t_2))}{1 + b_1|v_1(u)t_1|^{\beta_1}(1+e_1(v_1(u)t_1))}, \quad v(u) = a_1^{-1/\alpha_1} a_2^{-1/\alpha_2} u^{-2/\alpha_1 - 2/\alpha_2},$$

$$u^-_{k_2,n} = u \inf_{t\in \hat{I}_{k_2}(u,n)} (1 + b_2|t_2|^{\beta_2}(1+e_2(t_2))), \quad u^+_{k_2,n} = u \sup_{t\in \hat{I}_{k_2}(u,n)} (1 + b_2|t_2|^{\beta_2}(1+e_2(t_2))).$$

Then it follows that

$$\mathbb{P}\left\{Vol(\{t \in \hat{I}_{k_2}(u,n) : X(t) > u\}) > v(u)x\right\} \le \mathbb{P}\left\{\int_{[-n,n]\times[0,n]} \mathbb{I}(\xi_{u,n,k_2}(t) > u^-_{k_2,n})dt > x\right\},$$

$$\mathbb{P}\left\{Vol(\{t \in \hat{I}_{k_2}(u,n) : X(t) > u\}) > v(u)x\right\} \ge \mathbb{P}\left\{\int_{[-n,n]\times[0,n]} \mathbb{I}(\xi_{u,n,k_2}(t) > u^+_{k_2,n})dt > x\right\}.$$

Straightforward application of Lemma 4.1 with $\Gamma(f) = f$, $f \in C([-n,n]\times[0,n])$ and $h(t) = a_1^{-1}b_1|t_1|^{\alpha_1}$ in **C1**, gives that for $0 \le x < 2n^2$

$$\lim_{u\to\infty}\sup_{k_2\in K_{u,n}}\left|\frac{\mathbb{P}\left\{\int_{[-n,n]\times[0,n]}\mathbb{I}(\xi_{u,n,k_2}(t) > u^{\pm}_{k_2,n})dt > x\right\}}{\Psi(u^{\pm}_{k_2,n})} - \mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(x,[-n,n]\times[0,n])\right| = 0.$$

Similarly to (5.9), we have

$$\lim_{u\to\infty}\sup_{k_2\in K_{u,n}}\left|\frac{\Psi(u^-_{k_2,n})}{\Psi(u^+_{k_2,n})} - 1\right| = 0. \tag{5.11}$$

Consequently, for $0 \le x < 2n^2$

$$\lim_{u\to\infty}\sup_{k_2\in K_{u,n}}\left|\frac{\mathbb{P}\left\{Vol(\{t\in\hat{I}_{k_2}(u,n) : X(t) > u\}) > v(u)x\right\}}{\Psi(u^-_{k_2,n})} - \mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(x,[-n,n]\times[0,n])\right| = 0. \tag{5.12}$$

Thus (1.2) holds with

$$\bar{F}_n(x) = \frac{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(x,[-n,n]\times[0,n])}{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(0,[-n,n]\times[0,n])}.$$

By (ii) of Lemma 4.2 it follows that

$$\lim_{n\to\infty}\bar{F}_n(x) = \frac{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(0)} \in (0,1], \tag{5.13}$$

which confirms that A2 holds. Thus, applying Theorem 1.1, we establish the claim with

$$\bar{F}(x) = \frac{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(x)}{\mathcal{B}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}_{\alpha_1,\alpha_2}(0)}.$$

<u>**Case** $\alpha_1 = \beta_1, \alpha_2 = \beta_2$.</u> In this case we have $v_i(u) = a_i^{-1/\alpha_i}u^{-2/\alpha_i}, i = 1,2$. Let

$$E(u,n) := \hat{I}(u,n) := \bigcup_{i,j=-1,0} I_{i,j}(u,n). \tag{5.14}$$

<u>*Conditions A1 and A3.*</u> It follows from (34) and (52) in the proof of theorem 3.1 of Dębicki et al. (2017b) that A1 holds. Since we take only one interval $I_1(u,n)$, condition A3 is not applicable to this case.

<u>*Condition A2.*</u> Let

$$\xi_{u,n}(t) = X(v_1(u)t_1, v_2(u)t_2)), \quad v(u) = a_1^{-1/\alpha_1}a_2^{-1/\alpha_2}u^{-2/\alpha_1}u^{-2/\alpha_2}.$$

Then

$$\mathbb{P}\left\{Vol(\{t\in\hat{I}(u,n) : X(t) > u\}) \ge v(u)x\right\} = \mathbb{P}\left\{\int_{[-n,n]^2}\mathbb{I}(\xi_{u,n}(t) > u)dt > x\right\}.$$

In order to derive the asymptotics of the above term, similarly to the previous cases, we observe that **C1** in Lemma 4.1 holds with $h(t) = a_1^{-1}b_1|t_1|^{\alpha_1} + a_2^{-1}b_2|t_2|^{\alpha_2}$ while **C2** and **C3** have been checked in the case of $\alpha_i < \beta_i, i = 1, 2$.

Hence we have

$$\lim_{u\to\infty} \left| \frac{\mathbb{P}\left\{\int_{[-n,n]^2} \mathbb{I}(\xi_{u,n}(t) > u)dt > x\right\}}{\Psi(u)} - \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x, [-n, n]^2) \right| = 0.$$

Combining the above with the fact that, by (iii) of Lemma 4.2,

$$\lim_{n\to\infty} \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x, [-n, n]^2) \in (0, \infty)$$

we conclude that A2 holds with

$$\bar{F}(x) = \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x)}{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(0)} \in (0, 1].$$

Hence we establish the claim.

For the cases $\alpha_1 > \beta_1, \alpha_2 = \beta_2$, and $\alpha_1 > \beta_1, \alpha_2 > \beta_2$, we can establish the claim similarly to the case of $\alpha_1 = \beta_1, \alpha_2 = \beta_2$. For the case $\alpha_1 > \beta_1, \alpha_2 < \beta_2$, the proof is similar to the case of $\alpha_1 = \beta_1$, $\alpha_2 < \beta_2$. This completes the proof. □

5.5. *Proof of Proposition 3.4*. In order to apply Theorem 1.1, we introduce some useful notation. Let

$$I_k(u, n) = [kv(u)n, (k + 1)v(u)n], \quad N(u, n) = \left[\frac{T}{v(u)n}\right] - 1,$$

and $E(u, n) = \bigcup_{k\in K_{u,n}} I_k(u, n)$, with $K_{u,n} = \{k \in \mathbb{N} : 0 \le k \le N(u, n)\}$ and $v(u) = a^{-1/\alpha}u^{-2/\alpha}$. We denote by

$$Z(t, \theta) = \sum_{i=1}^{m} X_i(t)v_i(\theta), \quad A = [0, \pi]^{m-2} \times [0, 2\pi),$$

where $\theta = (\theta_1, \dots, \theta_{m-1})$ and $v_1(\theta) = \cos\theta_1$, $v_2(\theta) = \sin\theta_1 \cos\theta_2$, $v_3(\theta) = \sin\theta_1 \sin\theta_2 \cos\theta_3$, $\dots, v_{m-1}(\theta) = (\prod_{i=1}^{m-2} \sin\theta_i) \cos\theta_{m-1}$, $v_m(\theta) = \prod_{i=1}^{m-1} \sin\theta_i$. In this proof, we will use that

$$\chi(t) = \sup_{\theta\in A} Z(t, \theta).$$

We split the set A into (setting $k = (k_1, \dots, k_{m-1})$)

$$A = \bigcup_{k\in\Lambda} A_k, \quad \Lambda = \{(k_1, \dots, k_{m-1}) : 1 \le k_i \le L, 1 \le i \le m - 2, 1 \le k_{m-1} \le 2L\},$$

where

$$A_k = \prod_{i=1}^{m-1} \left[\frac{(k_i - 1)\pi}{L}, \frac{k_i\pi}{L}\right], \quad k_{m-1} \le 2L - 1,$$

$$A_{k_1,\dots,k_{m-2},2L} = \left(\prod_{i=1}^{m-2} \left[\frac{(k_i - 1)\pi}{L}, \frac{k_i\pi}{L}\right]\right) \times \left[2\pi - \frac{\pi}{L}, 2\pi\right),$$

and L is a positive integer. Moreover, let

$$\begin{aligned} \pi_1(u) &:= \sum_{k\neq k',k,k'\in\Lambda} \mathbb{P}\left\{ \sup_{t\in[0,v(u)n],\theta\in A_k} Z(t,\theta)>u, \sup_{t\in[0,v(u)n],\theta\in A_{k'}} Z(t,\theta)>u \right\}, \qquad (5.15)\\ \Sigma\Sigma_{u,n} &:= \sum_{0\le k_1<k_2\le N(u,n)} \mathbb{P}\left\{ \sup_{t\in I_{k_1}(u,n)} \chi(t)>u, \sup_{t\in I_{k_2}(u,n)} \chi(t)>u \right\}\\ &= \sum_{0\le k_1<k_2\le N(u,n)} \mathbb{P}\left\{ \sup_{(t,\theta)\in I_{k_1}(u,n)\times A} Z(t,\theta)>u, \sup_{(t,\theta)\in I_{k_2}(u,n)\times A} Z(t,\theta)>u \right\}\\ &\le \sum_{0\le k_1<k_2\le N(u,n),i,j\in\Lambda} \mathbb{P}\left\{ \sup_{(t,\theta)\in I_{k_1}(u,n)\times A_i} Z(t,\theta)>u, \sup_{(t,\theta)\in I_{k_2}(u,n)\times A_j} Z(t,\theta)>u \right\}. \end{aligned}$$

Denote by (with $k=(k_1,\ldots,k_{m-1}), l=(l_1,\ldots,l_{m-1})$)

$$\begin{aligned} J_{k,l}(u) &= \prod_{i=1}^{m-1}\left[\frac{(k_i-1)\pi}{L}+l_iu^{-1}n_1, \frac{(k_i-1)\pi}{L}+(l_i+1)u^{-1}n_1\right],\\ \Lambda_1(u) &= \left\{l: 0\le l_i\le \left[\frac{\pi u}{Ln_1}\right], 1\le i\le m-1\right\} \end{aligned}$$

and let

$$p_k^*(u)=\sum_{l,l'\in\Lambda_1(u),l\neq l'} \mathbb{P}\left\{ \sup_{t\in[0,v(u)n],\theta\in J_{k,l}(u)} Z(t,\theta)>u, \sup_{t\in[0,v(u)n],\theta\in J_{k,l'}(u)} Z(t,\theta)>u \right\}. \qquad (5.16)$$

Conditions A1 and A3. Condition A1 follows from Piterbarg (1996, Cor 7.3) while A3 can be deduced from equations (7.4), (7.6) and (7.18) in the proofs of Piterbarg (1996, Lem 7.1, Thm 7.1).

Condition A2. Let us put

$$\pi(n,u):=\mathbb{P}\left\{\int_{[0,v(u)n]}\mathbb{I}(\chi(t)>u)dt>v(u)x\right\}=\mathbb{P}\left\{\int_{[0,v(u)n]}\mathbb{I}\left(\sup_{\theta\in A}Z(t,\theta)>u\right)dt>v(u)x\right\}.$$

To verify A2, by stationarity we have to find the asymptotics of $\pi(n,u)$ as $u\to\infty$, which is given in the following lemma.

Lemma 5.1. *For $n>x$*

$$\pi(n,u)\sim\frac{\widehat{\mathcal{B}}_{\alpha,2,\ldots,2}(x,n)}{\widehat{\mathcal{B}}_{\alpha,2,\ldots,2}(0,n)}\mathbb{P}\left\{\sup_{[0,v(u)n]}\chi(t)>u\right\},\quad u\to\infty.$$

Proof of Lemma 5.1. Let $D_k=\{t\in[0,v(u)n]:\sup_{\theta\in A_k}Z(t,\theta)>u\}$. Then we have

$$\begin{aligned} \int_{[0,v(u)n]}\mathbb{I}\left(\sup_{\theta\in A}Z(t,\theta)>u\right)dt &= \int_{[0,v(u)n]}\mathbb{I}_{\bigcup_{k\in\Lambda}D_k}(t)dt\\ &\le \sum_{k\in\Lambda}\int_{[0,v(u)n]}\mathbb{I}_{D_k}(t)dt \end{aligned}$$

and

$$\int_{[0,v(u)n]} \mathbb{I}\left(\sup_{\theta\in A} Z(t,\theta) > u\right) dt \quad \geq \quad \sum_{k\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k}(t)dt - \sum_{k\neq k',k,k'\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k \bigcap D_{k'}}(t)dt.$$

Note that

$$\begin{aligned}
\pi(n,u) &\geq \mathbb{P}\left\{ \sum_{k\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k}(t)dt - \sum_{k\neq k',k,k'\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k \bigcap D_{k'}}(t)dt > v(u)x \right\} \\
&\geq \mathbb{P}\left\{ \sum_{k\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k}(t)dt > v(u)(x+\epsilon), \sum_{k\neq k',k,k'\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k \bigcap D_{k'}}(t)dt \leq v(u)\epsilon \right\} \\
&\geq \mathbb{P}\left\{ \sum_{k\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k}(t)dt > v(u)(x+\epsilon) \right\} - \mathbb{P}\left\{ \sum_{k\neq k',k,k'\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k \bigcap D_{k'}}(t)dt > v(u)\epsilon \right\} \\
&\geq \mathbb{P}\left\{ \sum_{k\in\Lambda} \int_{[0,v(u)n]} \mathbb{I}_{D_k}(t)dt > v(u)(x+\epsilon) \right\} - \pi_1(u) \\
&\geq \sum_{k\in\Lambda^*} p_k(x+\epsilon,u) - 2\pi_1(u),
\end{aligned}$$

where $\epsilon > 0$ and $\pi_1(u)$ is given in (5.15) and

$$\begin{aligned}
p_k(x,u) &= \mathbb{P}\left\{ \int_{[0,v(u)n]} \mathbb{I}\left(\sup_{\theta\in A_k} Z(t,\theta) > u\right) dt > v(u)x \right\}, \\
\Lambda^* &= \{k\in\Lambda, 1<k_i<L, 1\leq i\leq m-2, k_{m-1}\neq 1, L, 2L\}.
\end{aligned}$$

Similarly we get

$$\pi(n,u) \leq \sum_{k\in\Lambda} p_k(x,u) + \pi_1(u).$$

Hence

$$\sum_{k\in\Lambda^*} p_k(x+\epsilon,u) - 2\pi_1(u) \leq \pi(n,u) \leq \sum_{k\in\Lambda} p_k(x,u) + \pi_1(u). \tag{5.17}$$

◇ *Upper bound for $p_k(x,u)$*. A direct calculations show

$$\begin{aligned}
Var(Z(t,\theta)) &= 1, \\
Corr(Z(t,\theta), Z(t',\theta')) &= Corr(X(t),X(t'))\left(\cos(\theta_1-\theta_1') - \sin\theta_1\sin\theta_1'(1-\cos(\theta_2-\theta_2'))\right. \\
&\quad \left. - \cdots - \left(\prod_{i=1}^{m-2}\sin\theta_i\sin\theta_i'\right)\left(1-\cos(\theta_{m-1}-\theta_{m-1}')\right)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
1 - Corr(Z(t,\theta), Z(t',\theta')) &\sim a|t-t'|^\alpha + \frac{1}{2}(\theta_1-\theta_1')^2 + \frac{\sin^2\theta_1}{2}(\theta_2-\theta_2')^2 \\
&\quad + \frac{1}{2}\left(\prod_{i=1}^{m-2}\sin^2\theta_i\right)(\theta_{m-1}-\theta_{m-1}')^2, \quad |t-t'|\to 0, ||\theta-\theta'||\to 0. \quad (5.18)
\end{aligned}$$

We have

$$p_k(x,u) \le \sum_{l\in\Lambda_1(u)} \mathbb{P}\left\{\int_{[0,v(u)n]} \mathbb{I}\left(\sup_{\theta\in J_{k,l}(u)} Z(t,\theta)>u\right)dt > v(u)x\right\} + p_k^*(u), \tag{5.19}$$

where $p_k^*(u)$ is given in (5.16). Let

$$\begin{aligned} Z_{u,k,l}(t,\theta) &= Z\left(v(u)t, \frac{(k_1-1)\pi}{L} + l_1 u^{-1} n_1 + u^{-1} c_1^{-1}(\theta_{k,l}(u))\theta_1, ..., \right. \\ &\left. \frac{(k_{m-1}-1)\pi}{L} + l_{m-1} u^{-1} n_1 + u^{-1} c_{m-1}^{-1}(\theta_{k,l}(u))\theta_{m-1}\right), \end{aligned}$$

and $G_l = \prod_{i=1}^{m-1}[0, c_i(\theta_{k,l}(u))n_1]$, where $c_k(\theta) = 2^{-1/2}\prod_{i=1}^{k-1}|\sin\theta_i|$, $2\le k\le m-1$, $c_1(\theta)=2^{-1/2}$, $\theta_{k,l}(u) = \left(\frac{(k_1-1)\pi}{L} + l_1 u^{-1} n_1, \ldots, \frac{(k_{m-1}-1)\pi}{L} + l_{m-1}u^{-1}n_1\right)$. Noting that

$$G_l = \prod_{i=1}^{m-1}[0, c_i(\theta_{k,l}(u))n_1] \subset \prod_{i=1}^{m-1}[0, c_{k,i}^+ n_1] =: G_k^+, \quad c_{k,i}^+ = \sup_{\theta\in A_k} c_i(\theta),$$

we have

$$\begin{aligned} \mathbb{P}\left\{\int_{[0,v(u)n]} \mathbb{I}\left(\sup_{\theta\in J_{k,l}(u)} Z(t,\theta)>u\right)dt > v(u)x\right\} &= \mathbb{P}\left\{\int_{[0,n]} \mathbb{I}\left(\sup_{\theta\in G_l} Z_{u,k,l}(t,\theta)>u\right)dt > v(u)x\right\} \\ &\le \mathbb{P}\left\{\int_{[0,n]} \mathbb{I}\left(\sup_{\theta\in G_k^+} Z_{u,k,l}(t,\theta)>u\right)dt > v(u)x\right\}. \end{aligned}$$

A straightforward application of Lemma 4.1 for $\Gamma: C([0,n]\times G_k^+) \to C([0,n])$ defined by $\Gamma(f) = \sup_{\theta\in G_k^+} f(t,\theta)$, $f\in C([0,n]\times G_k^+)$, where $h=0$ in **C1** and $\zeta(t,\theta) = B_\alpha(t) + \sum_{i=1}^{m-1} N_i\theta_i$, with $N_i, i=1,\ldots,m-1$ being independent standard normal random variables independent of B_α, implies that for all $x\ge 0$ we have

$$\lim_{u\to\infty}\sup_{l\in\Lambda_1(u)}\left|\frac{\mathbb{P}\left\{\int_{[0,n]}\mathbb{I}\left(\sup_{\theta\in G_k^+} Z_{u,k,l}(t,\theta)>u\right)dt > v(u)x\right\}}{\Psi(u)} - \widehat{\mathcal{B}}_{\alpha,2,\ldots,2}(x,[0,n]\times G_k^+)\right| = 0. \tag{5.20}$$

By (7.18) in the proof of Piterbarg (1996, Thm 7.1), we have

$$p_k^*(u) = o\left(u^{m-1}\Psi(u)\right), \quad u\to\infty, n_1\to\infty. \tag{5.21}$$

Hence, by (5.19)-(5.21) and using Lemma 2.1 we have

$$\begin{aligned} p_k(x,u) &\le \limsup_{n_1\to\infty} \frac{\widehat{\mathcal{B}}_{\alpha,2,\ldots,2}(x,[0,n]\times G_k^+)}{(n_1)^{m-1}}\left(\frac{\pi}{L}\right)^{m-1} u^{m-1}\Psi(u) \\ &\le \widehat{\mathcal{B}}_{\alpha,2,\ldots,2}(x,n)\prod_{i=1}^{m-1} c_{k,i}^+\left(\frac{\pi}{L}\right)^{m-1} u^{m-1}\Psi(u), \quad u\to\infty, n\to\infty. \end{aligned}$$

◇ *Lower bound for $p_k(x,u)$*. By (5.17), we have that for $\epsilon > 0$

$$\begin{aligned} p_k(x,u) &\geq \sum_{l\in\Lambda_2(u)} \mathbb{P}\left\{\int_{[0,v(u)n]} \mathbb{I}\left(\sup_{\theta\in J_{k,l}(u)} Z(t,\theta) > u\right) dt > v(u)(x+\epsilon)\right\} - 2p_k^*(u) \\ &\geq \sum_{l\in\Lambda_2(u)} \mathbb{P}\left\{\int_{[0,n]} \mathbb{I}\left(\sup_{\theta\in G_k^-} Z_{u,k,l}(t,\theta) > u\right) dt > v(u)(x+\epsilon)\right\} - 2p_k^*(u), \end{aligned}$$

where

$$\Lambda_2(u) = \left\{l : 0 \leq l_i \leq \left[\frac{\pi u}{L n_1}\right] - 1, 1 \leq i \leq m-1\right\},$$

$$G_l = \prod_{i=1}^{m-1} [0, c_i(\theta_{k,l}(u))n_1] \supset \prod_{i=1}^{m-1} [0, c_{k,i}^- n_1] =: G_k^-, \quad c_{k,i}^- = \min_{\theta\in A_k} c_i(\theta).$$

By (5.20), (5.21), Lemma 2.1 and Remark 2.2 we have for $n > x$

$$\begin{aligned} p_k(x,u) &\geq \liminf_{n_1\to\infty} \frac{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x+\epsilon, [0,n]\times G_k^-)}{(n_1)^{m-1}} \left(\frac{\pi}{L}\right)^{m-1} u^{m-1}\Psi(u) \\ &\geq \widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x+\epsilon, n) \prod_{i=1}^{m-1} c_{k,i}^- \left(\frac{\pi}{L}\right)^{m-1} u^{m-1}\Psi(u) \\ &\geq \widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x, n) \prod_{i=1}^{m-1} c_{k,i}^- \left(\frac{\pi}{L}\right)^{m-1} u^{m-1}\Psi(u), \quad u\to\infty, \epsilon\to 0. \end{aligned}$$

◇ *Asymptotics of $\pi(n,u)$*. By (7.6) in Piterbarg (1996)

$$\pi_1(u) = o\left(u^{m-1}\Psi(u)\right), \quad u\to\infty, L\to\infty.$$

Therefore, in view of (5.17)

$$\begin{aligned} \limsup_{u\to\infty} \frac{\pi(n,u)}{u^{m-1}\Psi(u)} &\leq \limsup_{L\to\infty} \sum_{k\in\Lambda} \left(\prod_{i=1}^{m-1} c_{k,i}^+\right) \left(\frac{\pi}{L}\right)^{m-1} \widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x,n), \\ \liminf_{u\to\infty} \frac{\pi(n,u)}{u^{m-1}\Psi(u)} &\geq \liminf_{L\to\infty} \sum_{k\in\Lambda^*} \left(\prod_{i=1}^{m-1} c_{k,i}^-\right) \left(\frac{\pi}{L}\right)^{m-1} \widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x,n). \end{aligned}$$

Using the fact that

$$\limsup_{L\to\infty} \sum_{k\in\Lambda} \left(\prod_{i=1}^{m-1} c_{k,i}^+\right) \left(\frac{\pi}{L}\right)^{m-1} = \liminf_{L\to\infty} \sum_{k\in\Lambda^*} \left(\prod_{i=1}^{m-1} c_{k,i}^-\right) \left(\frac{\pi}{L}\right)^{m-1} = Vol(S_{m-1})$$

it follows that

$$\pi(n,u) \sim \frac{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x,n)}{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(0,n)} \mathbb{P}\left\{\sup_{[0,v(u)n]} \chi(t) > u\right\}, \quad u\to\infty.$$

This completes the proof of Lemma 5.1. □

Condition A2 continued. Lemma 2.1 yields that for $x \geq 0$

$$\frac{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x)}{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(0)} = \lim_{n\to\infty} \frac{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x,[0,n])}{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(0,[0,n])} \in (0,1].$$

Hence A2 holds with

$$\bar{F}(x) = \frac{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(x)}{\widehat{\mathcal{B}}_{\alpha,2,\dots,2}(0)}, \quad x \ge 0.$$

Thus we establish the claim and hence the proof is complete. □

5.6. *Proof of Proposition 3.5.* We first apply Theorem 1.1 to derive the asymptotics for case ii) of Proposition 3.5. Let

$$E(u,n) = \bigcup_{i=0}^{N(u,n)} I_i(u,n), \quad I_i(u,n) = [iv(u)n, (i+1)v(u)n], \ N(u,n) = \left[\frac{T_u}{nv(u)}\right] - 2$$

and

$$v(u) = u^{\frac{2(\alpha-1)}{\alpha}} \left(\frac{(\tau^*)^{\alpha/2}}{1+c\tau^*}\right)^{2/\alpha}, \quad \tau^* = \frac{\alpha}{c(2-\alpha)}.$$

Let

$$Z(s,t) = \frac{B_\alpha(s) - B_\alpha(t)}{1+c(s-t)}, \ I_i'(u,n) = [iq(u)n, (i+1)q(u)n], \ q(u) = u^{-1}v(u),$$

and

$$\begin{aligned}
&\Sigma\Sigma(u,n)\\
&:= \sum_{i\neq j, 0\le i,j\le N(u,n)} \mathbb{P}\left\{\sup_{t\in I_i(n,u)} Q(t) > u, \sup_{t\in I_j(n,u)} Q(t) > u\right\}\\
&= \sum_{i\neq j, 0\le i,j\le N(u,n)} \mathbb{P}\left\{\sup_{t\in I_i(n,u), s\ge t} (B_\alpha(s) - B_\alpha(t) - c(s-t)) > u\right\}.\\
&\qquad\qquad \left.\sup_{t\in I_j(n,u), s\ge t} (B_\alpha(s) - B_\alpha(t) - c(s-t)) > u\right\}\\
&= \sum_{i\neq j, 0\le i,j\le N(u,n)} \mathbb{P}\left\{\sup_{t\in I_i'(n,u), s\ge t} Z(s,t) > u^{1-\alpha/2}, \sup_{t\in I_j'(n,u), s\ge t} Z(s,t) > u^{1-\alpha/2}\right\},
\end{aligned}$$

where in the last equality we use the self-similarity of fBm. Moreover, let

$$L_i(u) = [\tau^* + iq(u)n_1, \tau^* + (i+1)q(u)n_1], \quad M(u) = \left[\frac{u^{\alpha/2}\ln u}{v(u)n_1}\right], \tag{5.22}$$

$$G(u) = \{s : |s-\tau^*| < u^{\alpha/2-1}\ln u\}, \quad G^c(u) = [0,\infty) \setminus G(u) \tag{5.23}$$

and

$$\pi_2(u) = \sum_{-M(u)-1\le i<j\le M(u)+1} \mathbb{P}\left\{\sup_{t\in[0,q(u)n], s\in L_i(u)} Z(s,t) > u^{1-\alpha/2}, \sup_{t\in[0,q(u)n], s\in L_j(u)} Z(s,t) > u^{1-\alpha/2}\right\}. \tag{5.24}$$

Conditions A1 and A3. Condition A1 follows from Theorems 3.1-3.3 of Dębicki and Liu (2016) while A3 is due to Lemma 5.6 of Dębicki and Liu (2016) and the upper bounds of $\Sigma_i(u), i = 1,2,3,4$ in the proof of Dębicki and Liu (2016, Thm 3.1).

Condition A2. Due to stationarity of the process Q, in order to show (1.2) it suffices to find the exact asymptotics of $\mathbb{P}\left\{\int_{[0,v(u)n]}\mathbb{I}(Q(t)>u)dt>v(u)x\right\}$ as $u\to\infty$. By the self-similarity of B_α, we have

$$\begin{aligned}\mathbb{P}\left\{\int_{[0,v(u)n]}\mathbb{I}(Q(t)>u)dt>v(u)x\right\} &= \mathbb{P}\left\{\int_{[0,v(u)n]}\mathbb{I}\left(\sup_{s\ge t}(B_\alpha(s)-B_\alpha(t)-c(s-t))>u\right)dt>v(u)x\right\}\\ &= \mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u^{1-\alpha/2}\right)dt>q(u)x\right\}.\end{aligned}$$

We set next

$$m(u)=\frac{1+c\tau^*}{(\tau^*)^{\alpha/2}}u^{1-\alpha/2},\quad \tau^*=\frac{\alpha}{c(2-\alpha)},$$
$$A=\left(\frac{\alpha}{c(2-\alpha)}\right)^{-\alpha/2}\frac{2}{2-\alpha},\quad B=\left(\frac{\alpha}{c(2-\alpha)}\right)^{-\alpha/2-1}\frac{\alpha}{2}.$$

Lemma 5.2. *For $n>x$*

$$\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u^{1-\alpha/2}\right)dt>q(u)x\right\}\sim\widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)\sqrt{\frac{2A}{B}}\frac{u}{m(u)v(u)}\Psi(m(u)),\quad u\to\infty. \tag{5.25}$$

Proof. Upper bound. Using the fact that (set $u_\alpha=u^{1-\alpha/2}$

$$\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u_\alpha\right)\le\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)+\mathbb{I}\left(\sup_{s\in G^c(u)}Z(s,t)>u_\alpha\right)$$

we obtain

$$\begin{aligned}&\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\\ &\le\mathbb{P}\left\{\int_{[0,q(u)n]}\left(\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)+\mathbb{I}\left(\sup_{s\in G^c(u)}Z(s,t)>u_\alpha\right)\right)dt>q(u)x\right\}\\ &\le\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\\ &\quad+\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G^c(u)}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\\ &\quad+\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)dt>0,\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G^c(u)}Z(s,t)>u_\alpha\right)dt>0\right\}\\ &\le\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}+2\mathbb{P}\left\{\sup_{t\in[0,q(u)n],s\in G^c(u)}Z(s,t)>u_\alpha\right\}.\end{aligned}$$

Using that (recall (5.22), (5.23))

$$\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)\le\sum_{|i|\le M(u)+1}\mathbb{I}\left(\sup_{s\in L_i(u)}Z(s,t)>u_\alpha\right)$$

we have

$$\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\in G(u)}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\le\pi_1(u)+\pi_2(u),$$

where $\pi_2(u)$ is given in (5.24) and

$$\pi_1(u) = \sum_{|i|\le M(u)+1} \mathbb{P}\left\{\int_{t\in[0,q(u)n]} \mathbb{I}\left(\sup_{s\in L_i(u)} Z(s,t) > u_\alpha\right) dt > q(u)x\right\}. \tag{5.26}$$

By Dębicki and Liu (2016, Lem 5.6) we obtain

$$\mathbb{P}\left\{\sup_{t\in[0,q(u)n],s\in G^c(u)} Z(s,t) > u_\alpha\right\} = o\left(\mathbb{P}\left\{\sup_{t\in[0,v(u)n]} Q(t) > u\right\}\right), \quad u\to\infty$$

and in light of the upper bounds of $\Lambda_i(u), i = 1,2,3,4$ in the proof of Theorem 3.1 of Dębicki and Liu (2016)

$$\pi_2(u) = o\left(\mathbb{P}\left\{\sup_{t\in[0,v(u)n]} Q(t) > u\right\}\right), \quad u\to\infty, n_1\to\infty. \tag{5.27}$$

Next we focus on $\pi_1(u)$. Rewrite

$$\mathbb{P}\left\{\int_{t\in[0,q(u)n]} \mathbb{I}\Big(\sup_{s\in L_i(u)} Z(s,t) > u_\alpha\Big) dt > q(u)x\right\} = \mathbb{P}\left\{\int_{t\in[0,n]} \mathbb{I}\Big(\sup_{s\in[0,n_1]} Z_{u,i}(s,t) > m(u)\Big) dt > x\right\},$$

where

$$Z_{u,i}(s,t) = \frac{B_\alpha(\tau^* + q(u)(in_1+s)) - B_\alpha(q(u)t)}{1 + c(\tau^* + q(u)(in_1+s-t))} \cdot \frac{1+c\tau^*}{(\tau^*)^{\alpha/2}}.$$

Let for $0<\epsilon<1$

$$m_i^{\pm}(u) = m(u)\left(1 + \left(\frac{B}{2A} \pm \epsilon\right) q(u)(in_1 \pm n)^2\right).$$

A direct calculation shows (see also Lemmas 5.3-5.4 in Dębicki and Liu, 2016) that

$$m_i^-(u) \le m(u)(Var(Z_{u,i}(s,t)))^{-1/2} \le m_i^+(u), \quad |i| \le M(u)+1 \tag{5.28}$$

and

$$\lim_{u\to\infty} \sup_{|i|\le M(u)+1,(s,t)\ne(s',t'),(s,t),(s',t')\in[0,n_1]\times[0,n]} \left|(m_i^{\pm}(u))^2 \frac{1 - Corr(Z_{u,i}(s,t), Z_{u,i}(s',t'))}{|t-t'|^\alpha + |s-s'|^\alpha} - 1\right| = 0. \tag{5.29}$$

Hence

$$\mathbb{P}\left\{\int_{t\in[0,n]} \mathbb{I}\left(\sup_{s\in[0,n_1]} Z_{u,i}(s,t) > m(u)\right) dt > x\right\} \le \mathbb{P}\left\{\int_{t\in[0,n]} \mathbb{I}\left(\sup_{s\in[0,n_1]} \overline{Z}_{u,i}(s,t) > m_i^-(u)\right) dt > x\right\}.$$

Next, by Lemma 4.1 applied to $\Gamma : C([0,n]\times[0,n_1]) \to C([0,n])$ defined by $\Gamma(f) = \sup_{t\in[0,n_1]} f(s,t)$, $f \in C([0,n]\times[0,n_1])$, with $h = 0$ in **C0-C1** and **C2** satisfied with $\zeta(s,t) = B_\alpha(s) + B'_\alpha(t)$, we have

$$\lim_{u\to\infty} \sup_{|i|\le M(u)+1} \left|\frac{\mathbb{P}\left\{\int_{t\in[0,n]} \mathbb{I}\left(\sup_{s\in[0,n_1]} \overline{Z}_{u,i}(s,t) > m_i^-(u)\right) dt > x\right\}}{\Psi(m_i^-(u))} - \widehat{\mathcal{B}}_{\alpha,\alpha}(x,[0,n]\times[0,n_1])\right| = 0 \tag{5.30}$$

and in light of Lemma 2.1, we have

$$\begin{aligned}\pi_1(u) &\le \widehat{\mathcal{B}}_{\alpha,\alpha}(x,[0,n]\times[0,n_1]) \sum_{|i|\le M(u)+1} \Psi(m_i^-(u))\\ &\le \widehat{\mathcal{B}}_{\alpha,\alpha}(x,[0,n]\times[0,n_1])\Psi(m(u)) \sum_{|i|\le M(u)+1} e^{-m^2(u)\left(\frac{B}{2A}-\epsilon\right)\left(u^{-1}v(u)(in_1)\right)^2}\\ &\le \frac{\widehat{\mathcal{B}}_{\alpha,\alpha}(x,[0,n]\times[0,n_1])}{n_1}\sqrt{\frac{2A\pi}{B}}\frac{u}{m(u)v(u)}\Psi(m(u))\\ &\sim \widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)\sqrt{\frac{2A\pi}{B}}\frac{u}{m(u)v(u)}\Psi(m(u)) \qquad (5.31)\end{aligned}$$

as $u\to\infty, n_1\to\infty, \epsilon\to 0$. Therefore, we conclude that

$$\mathbb{P}\left\{\int_{[0,q(u)n]} \mathbb{I}\left(\sup_{s\ge t} Z(s,t)>u_\alpha\right)dt>q(u)x\right\} \le \widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)\sqrt{\frac{2A\pi}{B}}\frac{u}{m(u)v(u)}\Psi(m(u)), \quad u\to\infty.$$

<u>*Lower bound*</u>. Observe that for u sufficiently large, $s>t$ holds for all $s\in G(u), t\in[0,q(u)n]$. Therefore,

$$\mathbb{P}\left\{\int_{[0,q(u)n]} \mathbb{I}\left(\sup_{s\ge t} Z(s,t)>u_\alpha\right)dt>q(u)x\right\} \ge \mathbb{P}\left\{\int_{[0,q(u)n]} \mathbb{I}\left(\sup_{s\in G(u)} Z(s,t)>u_\alpha\right)dt>q(u)x\right\}.$$

By the fact that

$$\begin{aligned}\mathbb{I}\left(\sup_{s\in G(u)} Z(s,t)>u_\alpha\right) &\ge \sum_{|i|\le M(u)} \mathbb{I}\left(\sup_{s\in L_i(u)} Z(s,t)>u_\alpha\right)\\ &\quad - \sum_{-M(u)\le i<j\le M(u)} \mathbb{I}\left(\sup_{s\in L_i(u)} Z(s,t)>u_\alpha, \sup_{s\in L_j(u)} Z(s,t)>u_\alpha\right)\\ &=: A_1(u,t)-A_2(u,t)\end{aligned}$$

it follows that for $\epsilon>0$ (recall $q(u)=u^{-1}v(u)$)

$$\begin{aligned}&\mathbb{P}\left\{\int_{[0,q(u)n]} \mathbb{I}\left(\sup_{s\ge t} Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\\ &\ge \mathbb{P}\left\{\int_{[0,q(u)n]} \left(A_1(u,t)-A_2(u,t)\right)dt>q(u)x\right\}\\ &\ge \mathbb{P}\left\{\int_{[0,q(u)n]} A_1(u,t)dt>q(u)(x+\epsilon), \int_{[0,q(u)n]} A_2(u,t)dt<q(u)\epsilon\right\}\\ &\ge \mathbb{P}\left\{\int_{[0,q(u)n]} A_1(u,t)dt>q(u)(x+\epsilon)\right\} - \mathbb{P}\left\{\int_{[0,q(u)n]} A_2(u,t)dt\ge q(u)\epsilon\right\}\\ &\ge \mathbb{P}\left\{\exists i:|i|\le M(u), \int_{[0,q(u)n]} \mathbb{I}\left(\sup_{s\in L_i(u)} Z(s,t)>u_\alpha\right)dt>q(u)(x+\epsilon)\right\}-\pi_2(u)\\ &\ge \sum_{|i|\le M(u)} \mathbb{P}\left\{\int_{t\in[0,q(u)n]} \mathbb{I}\left(\sup_{s\in L_i(u)} Z(s,t)>u_\alpha\right)dt>q(u)(x+\epsilon)\right\}-2\pi_2(u), \qquad (5.32)\end{aligned}$$

where $\pi_2(u)$ is defined in (5.24). Similarly as in (5.31) and in light of (5.27), we have

$$\begin{aligned}\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u_\alpha\right)dt>q(u)x\right\} &\ge \widehat{\mathcal{B}}_{\alpha,\alpha}(x+\epsilon,n)\sqrt{\frac{2A}{B}}\frac{u}{m(u)v(u)}\Psi(m(u))\\ &\ge \widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)\sqrt{\frac{2A}{B}}\frac{u}{m(u)v(u)}\Psi(m(u))\end{aligned}$$

as $u\to\infty$, $\epsilon\to 0$. Consequently, for $n>x$

$$\mathbb{P}\left\{\int_{[0,q(u)n]}\mathbb{I}\left(\sup_{s\ge t}Z(s,t)>u_\alpha\right)dt>q(u)x\right\}\sim \widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)\sqrt{\frac{2A}{B}}\frac{u}{m(u)v(u)}\Psi(m(u)),\quad u\to\infty. \tag{5.33}$$

□

Moreover, by Lemma 2.1

$$\frac{\mathcal{B}_\alpha(x)}{\mathcal{B}_\alpha(0)}=\frac{\widehat{\mathcal{B}}_{\alpha,\alpha}(x)}{\widehat{\mathcal{B}}_{\alpha,\alpha}(0)}=\lim_{n\to\infty}\frac{\widehat{\mathcal{B}}_{\alpha,\alpha}(x,n)}{\widehat{\mathcal{B}}_{\alpha,\alpha}(0,n)}\in(0,1].$$

Thus A2 holds with

$$\bar{F}(x)=\frac{\mathcal{B}_\alpha(x)}{\mathcal{B}_\alpha(0)},\quad x\ge 0.$$

This completes the proof of case ii).

For case i), note that if $x=0$, the claim clearly holds. Next we suppose that $0<x<T$. By (5.25) for any $0<\epsilon<\min(x/2,(T-x)/2)$,

$$\begin{aligned}\mathbb{P}\left\{\int_{[0,T_u]}\mathbb{I}(Q(t)>u)dt>v(u)x\right\} &\le \mathbb{P}\left\{\int_{[0,v(u)(T+\epsilon)]}\mathbb{I}(Q(t)>u)dt>v(u)x\right\}\\ &\le \mathbb{P}\left\{\int_{[0,v(u)T]}\mathbb{I}(Q(t)>u)dt>v(u)(x-\epsilon)\right\}\\ &\sim \frac{\widehat{\mathcal{B}}_{\alpha,\alpha}(x-\epsilon,T)}{\widehat{\mathcal{B}}_{\alpha,\alpha}(0,T)}\mathbb{P}\left\{\sup_{t\in[0,v(u)T]}Q(t)>u\right\},\quad u\to\infty.\end{aligned}$$

Analogously, we have

$$\mathbb{P}\left\{\int_{[0,T_u]}\mathbb{I}(Q(t)>u)dt>v(u)x\right\}\ge\frac{\widehat{\mathcal{B}}_{\alpha,\alpha}(x+\epsilon,T)}{\widehat{\mathcal{B}}_{\alpha,\alpha}(0,T)}\mathbb{P}\left\{\sup_{t\in[0,v(u)T]}Q(t)>u\right\},\quad u\to\infty.$$

In light of Remark 2.2 we establish the claim by letting $\epsilon\to 0$ in the above inequalities. This completes the proof. □

6. Appendix

Proof of Lemma 4.1 For notational simplicity denote by $\rho_{u,j}$ the correlation function of the random field $\xi_{u,j}$. Further set

$$\chi_{u,j}(s):=g_{u,j}(\overline{\xi}_{u,j}(s)-\rho_{u,j}(s,0)\overline{\xi}_{u,j}(0)),\quad s\in E_1$$

and

$$f_{u,j}(s,y) := y\rho_{u,j}(s,0) - g_{u,j}^2\left(1-\rho_{u,j}(s,0)\right) - g_{u,j}^2\frac{1-\sigma_{u,j}(s)}{\sigma_{u,j}(s)},\ s\in E_1, y\in\mathbb{R}.$$

Conditioning on $\xi_{u,j}(0)$, by **F1** and using that $\overline{\xi}_{u,j}(0)$ and $\overline{\xi}_{u,j}(s)-\rho_{u,j}(s,0)\overline{\xi}_{u,j}(0)$ are mutually independent we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\int_{E_2}\mathbb{I}\left\{\Gamma\left(g_{u,j}(\xi_{u,j}(s)-g_{u,j})\right)(t)>0\right\}\eta(dt)>x\right\}\\
&\quad=\frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi}g_{u,j}}\int_{\mathbb{R}}\exp\left(-y-\frac{y^2}{2g_{u,j}^2}\right)\times\\
&\qquad\times\mathbb{P}\left\{\int_{E_2}\mathbb{I}\left\{\Gamma\left(g_{u,j}(\xi_{u,j}(s)-g_{u,j})\right)(t)>0\right\}\eta(dt)>x|\xi_{u,j}(0)=g_{u,j}+yg_{u,j}^{-1}\right\}dy\\
&\quad=\frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi}g_{u,j}}\int_{\mathbb{R}}\exp\left(-y-\frac{y^2}{2g_{u,j}^2}\right)\mathbb{P}\left\{\int_{E_2}\mathbb{I}\left\{\Gamma\left(\sigma_{u,j}(s)\left(\chi_{u,j}(s)+f_{u,j}(s,y)\right)\right)(t)>0\right\}\eta(dt)>x\right\}dy\\
&\quad=\frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi}g_{u,j}}\int_{\mathbb{R}}\exp\left(-y-\frac{y^2}{2g_{u,j}^2}\right)\mathcal{I}_{u,j}(y;x)dy,
\end{aligned}$$

where

$$\mathcal{I}_{u,j}(y;x) := \mathbb{P}\left\{\int_{E_2}\mathbb{I}\left\{\Gamma\left(\sigma_{u,j}(s)\left(\chi_{u,j}(s)+f_{u,j}(s,y)\right)\right)(t)>0\right\}\eta(dt)>x\right\}.$$

Noting that

$$\lim_{u\to\infty}\sup_{j\in S_u}\left|\frac{\frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi}g_{u,j}}}{\Psi(g_{u,j})}-1\right|=0$$

in order to show the claim it suffices to prove that

$$\lim_{u\to\infty}\sup_{j\in S_u}\left|\int_{\mathbb{R}}\exp\left(-y-\frac{y^2}{2g_{u,j}^2}\right)\mathcal{I}_{u,j}(y;x)dy-\mathcal{B}_\zeta^{\Gamma,h,\eta}(x,E_2)\right|=0 \tag{6.1}$$

for all $x\ge 0$. In view of **C3** it follows that that for $u>u_0$

$$Var(\chi_{u,j}(s)-\chi_{u,j}(s'))\le g_{u,j}^2\mathbb{E}\left\{\overline{\xi}_{u,j}(s)-\overline{\xi}_{u,j}(s')\right\}^2\le Q_1\|s-s'\|^\nu,\quad s,s'\in E_1,$$

with $\nu>0$. Further, by **C0-C2** for each $y\in\mathbb{R}$

$$\lim_{u\to\infty}\sup_{j\in S_u,s\in E_1}\left|f_{u,j}(s,y)-y+\sigma_\zeta^2(s)+h(s)\right|=0. \tag{6.2}$$

Hence, by **F2**

$$\begin{aligned}
\sup_{j\in S_u}e^{-y}\mathcal{I}_{u,j}(y;x) &\le e^{-y}\sup_{j\in S_u}\mathbb{P}\left\{\sup_{t\in E_2}\Gamma\left(\chi_{u,j}(s)+f_{u,j}(s,y)\right)(t)>0\right\}\\
&\le e^{-y}\sup_{j\in S_u}\mathbb{P}\left\{\sup_{s\in E_1}\left\{\chi_{u,j}(s)+f_{u,j}(s,y)\right\}>0\right\}\\
&\le e^{-y}\sup_{j\in S_u}\mathbb{P}\left\{\sup_{s\in E_1}\chi_{u,j}(s)>Q_2\left|y\right|-Q_3\right\}\\
&\le Q_4\left|y\right|^{2n/\nu-1}e^{-Q_5y^2-y},\quad y<-M,
\end{aligned} \tag{6.3}$$

where in the last inequality we used Piterbarg inequality and $M > 0$. Moreover, it follows trivially that for all $x \geq 0$

$$\sup_{j \in S_u} e^{-y} \mathcal{I}_{u,j}(y;x) \leq e^{-y}, \quad y \in \mathbb{R}. \tag{6.4}$$

Therefore by the dominated convergence theorem and assumption **C0**

$$\sup_{j \in S_u} \left| \int_{\mathbb{R}} \exp\left(-y - \frac{y^2}{2g_{u,j}^2}\right) \mathcal{I}_{u,j}(y;x)dy - \int_{\mathbb{R}} e^{-y} \mathcal{I}_{u,j}(y;x)dy \right|$$
$$\leq \int_{\mathbb{R}} \sup_{j \in S_u} \left(e^{-y} \mathcal{I}_{u,j}(y;x)(1 - e^{-y^2/(2g_{u,j}^2)})\right) dy \to 0, \quad u \to \infty.$$

Hence in order to prove the convergence in (6.1) it suffices to show that

$$\lim_{u \to \infty} \sup_{j \in S_u} \left| \int_{\mathbb{R}} e^{-y} \mathcal{I}_{u,j}(y;x)dy - \mathcal{B}_{\zeta}^{\Gamma,h,\eta}(x, E_2) \right| = 0 \tag{6.5}$$

for all $x \in [0, \eta(E_2))$.

Weak convergence. The claim follows from the same arguments as in Dębicki et al. (2020a, Lem 4.3, 4.7), where the precise meaning of uniform weak convergence is also given. Thus let $C(E_1)$ denote the Banach space of all continuous functions on the compact set E_1 equipped with supremum norm. For any $s, s' \in E_1$, by **C2** we have

$$Var(\chi_{u,j}(s) - \chi_{u,j}(s')) = g_{u,j}^2 \left(\mathbb{E}\left\{\bar{\xi}_{u,j}(s) - \bar{\xi}_{u,j}(s')\right\}^2 - (\rho_{u,j}(s,0) - \rho_{u,j}(s',0))^2\right) \to 2Var(\zeta(s) - \zeta(s'))$$

uniformly with respect to $j \in S_u$ as $u \to \infty$. Hence, the finite-dimensional distributions of $\chi_{u,j}(s), s \in E_1$ weakly converge to those of $\sqrt{2}\zeta(s), s \in E_1$ uniformly with respect to $j \in S_u$. In view of **C3**, we know that the measures on $C(E_1)$ induced by $\{\chi_{u,j}(s), s \in E_1, j \in S_u\}$ are uniformly tight for large u, and by **C1**, $\sigma_{u,j}(s)$ converges to 1 uniformly for $s \in E_1$ and $j \in S_u$ as $u \to \infty$. Therefore, $\{\sigma_{u,j}(s)\chi_{u,j}(s), s \in E_1\}$ converge weakly to $\{\sqrt{2}\zeta(s), s \in E_1\}$ as $u \to \infty$ uniformly with respect to $j \in S_u$, which together with (6.2) implies that for each $y \in \mathbb{R}$, the probability measures on $C(E_1)$ induced by $\{\chi_{u,j}^f(s,y), s \in E_1\}$ converges weakly as $u \to \infty$ to those induced by $\{\zeta_h(s) + y, t \in E_1\}$ uniformly with respect to $j \in S_u$, where

$$\chi_{u,j}^f(s,y) := \sigma_{u,j}(s)\left(\chi_{u,j}(s) + f_{u,j}(s,y)\right) \quad \text{and} \quad \zeta_h(s) := \sqrt{2}\zeta(s) - \sigma_{\zeta}^2(t) - h(s).$$

Continuous mapping theorem implies that for each $y \in \mathbb{R}$, the push-forward probability measures $P_{u,y}$ on $C(E_2)$ induced by $\{\Gamma\left(\chi_{u,j}^f(\cdot,y)\right)(t), t \in E_2\}$ converges weakly to the push-forward probability measure P_y induced by $\{\Gamma\left(\zeta_h\right)(t) + y, t \in E_2\}$ as $u \to \infty$ uniformly with respect to $j \in S_u$.

The issues regarding the non-continuity of the sojourn functional are discussed in Berman (1973, Lem 4.2). A sequence of functions $f_n \in C(E_2)$ converges to $f \in C(E_2)$ as $n \to \infty$ with respect to uniform topology if $f_n \to f$ uniformly as $n \to \infty$. Since η is absolutely continuous with respect to Lebesgue measure on E_2 we can define the set

$$A_* = \left\{f \in C(E_2) : \int_{E_2} \mathbb{I}(f(t) = 0)\eta(dt) > 0\right\},$$

which is measurable in the completion $\mathcal{C}^\mu$ of $\mathcal{C}$ with respect to μ, where $\mathcal{C}$ is the Borel σ-field of $C_2(E)$. Its complement belongs to $\mathcal{C}^\mu$, i.e.,

$$A_*^c = C(E_2) \setminus A_* \in \mathcal{C}^\mu.$$

Any function $f \in A_*^c$ is a continuity point of the sojourn functional $J : C(E_2) \mapsto [0, \eta(E_2)]$, where

$$J(f) = \int_{E_2} \mathbb{I}(f(t) > 0)\eta(dt), f \in C(E_2).$$

This functional is measurable $\mathcal{C}/\mathcal{B}(\mathbb{R})$ by the assumption on η. We shall show that it is continuous at any $f \in A_*^c$. Let such f be given. By the definition of the integral such f is not equal to zero on any compact interval of $\mathbb{R}$. Let $f_n \to f$ uniformly as $n \to \infty$. Then $\mathbb{I}(f_n(t) > 0) \to \mathbb{I}(f(t) > 0)$ as $n \to \infty$ for almost all $t \in \mathbb{R}$ (with respect to Lebesgue measure). Hence by dominated convergence theorem we have $J(f_n) \to J(f)$ as $n \to \infty$, which means that the functional is continuous for all $f \in A_*^c$. Recall that P_y is the push-forward (image measure) on $C(E_2)$ with respect to $\Gamma(\xi_h) + y$. We claim that

$$P_y(A_*) > 0$$

is possible only for y in a countable set of $\mathbb{R}$. Indeed, any $f \in A_*$ is such that it is constant equal to zero on a compact interval. Consequently, $P_y(A_*) > 0$ means that the functions $f \in A_*$ are constant equal to $-y$ on some interval of $\mathbb{R}$. If this is true for two different y's, then the intervals where f is constant equal $-y$ must be disjoint, therefore this can be true only for countable y's.

Alternatively, using the fact that $\mathbb{P}\{\Gamma(\zeta_h)(t) + y = 0\} = 0$ a.e., $y \in \mathbb{R}$, by the σ-finiteness of η, Fubini-Tonelli theorem yields

$$\int_{\mathbb{R}} \mathbb{E}\left\{\int_{E_2} \mathbb{I}(\Gamma(\zeta_h)(t) + y = 0)\eta(dt)\right\} dy = \int_{E_2}\int_{\mathbb{R}} \mathbb{P}\{\Gamma(\zeta_h)(t) + y = 0\} dy\eta(dt) = 0.$$

Hence for almost all $y \in \mathbb{R}$

$$\mathbb{E}\left\{\int_{E_2} \mathbb{I}(\Gamma(\zeta_h)(t) + y = 0)\eta(dt)\right\} = 0,$$

which means that, for almost all $y \in \mathbb{R}$

$$P_y(A_*) = \mathbb{P}\left\{\int_{E_2} \mathbb{I}(\Gamma(\zeta_h)(t) + y = 0)\eta(dt) > 0\right\} = 0.$$

Consequently, since $J(f)$ is continuous for $f \in A_*^c$, by the continuous mapping theorem, as $u \to \infty$

$$\int_{E_2} \mathbb{I}\left(\Gamma\left(\chi_{u,j}^f(\cdot, y)\right)(t) > 0\right)\eta(dt) \tag{6.6}$$

weakly converges to

$$\int_{E_2} \mathbb{I}\left(\Gamma\left(\zeta_h\right)(t) + y > 0\right)\eta(dt)$$

uniformly with respect to $j \in S_u$ for almost all $y \in \mathbb{R}$.

Convergence on continuity points. Define

$$\mathcal{I}(y; x) := \mathbb{P}\left\{\int_{E_2} \mathbb{I}\left(\Gamma(\zeta_h)(t) + y > 0\right)\eta(dt) > x\right\}.$$

We draw a similar argument as in Theorem 1.3.1 of Berman (1992) to verify (6.5) for all continuity points $x \in (0, \eta(E_2))$ of $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$. Let $x_0 \in (0, \eta(E_2))$ be such a continuity point, that is

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(\mathcal{I}(y; x_0 + \varepsilon) - \mathcal{I}(y; x_0 - \varepsilon)\right) e^{-y} dy = 0.$$

Since for large M and all $x \geq 0$ by **F2** as in the derivation of (6.3) we have

$$e^{-y}\mathcal{I}(y; x) \leq Q_4' \left|y\right|^{2n/\nu - 1} e^{-Q_5 y^2 - y}, \quad y < -M \tag{6.7}$$

it follows from the dominated convergence theorem that

$$\int_{\mathbb{R}} \left(\mathcal{I}(y; x_0+) - \mathcal{I}(y; x_0-)\right) e^{-y} dy = 0$$

and thus by the monotonicity of $\mathcal{I}(y; x)$ in x for each fixed y, x_0 is a continuous point of $\mathcal{I}(y; x)$ for a.e. $y \in \mathbb{R}$. Thus by (6.6) for a.e. $y \in \mathbb{R}$

$$\lim_{u \to \infty} \sup_{j \in S_u} \left|\mathcal{I}_{u,j}(y; x_0) - \mathcal{I}(y; x_0)\right| = 0. \tag{6.8}$$

As shown in (6.3), (6.4) and (6.7) it follows from the dominated convergence theorem that

$$\begin{aligned} &\sup_{j \in S_u} \left| \int_{\mathbb{R}} e^{-y} \mathcal{I}_{u,j}(y; x_0) dy - \int_{\mathbb{R}} e^{-y} \mathcal{I}(y; x_0) dy \right| \\ &\leq \int_{\mathbb{R}} \sup_{j \in S_u} \left|\mathcal{I}_{u,j}(y; x_0) - \mathcal{I}(y; x_0)\right| e^{-y} dy \to 0, \quad u \to \infty \end{aligned} \tag{6.9}$$

establishing the proof for all continuity points $x \in (0, \eta(E_2))$. Moreover, for the case that $x = 0$, (6.9) also holds by replacing sojourn with supremum. This can be shown directly without any continuity requirement for $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$ at $x = 0$.

Continuity of $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$. Next we show that $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$ is continuous at any $x \in (0, \eta(E_2))$ using that η is equivalent with Lebesgue measure on E_2. Note that $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$ is clearly right continuous at 0. Next we show the continuity at $x \in (0, E_2)$. The claimed continuity at x follows if we show

$$\int_{\mathbb{R}} \mathbb{P}\left\{A_y\right\} e^{-y} dy = 0, \quad A_y = \left\{ \int_{E_2} \mathbb{I}\big(\Gamma(\zeta_h)(t) + y > 0\big) \eta(dt) = x \right\}, \quad y \in \mathbb{R}.$$

Note that A_y is an event for any y which is consequence of Fubini-Tonelli theorem. If for $0 < x < \eta(E_2)$ we have

$$\int_{E_2} \mathbb{I}\big(\Gamma(\zeta_h)(t) + y > 0\big) \eta(dt) = x,$$

then using the fact that $\Gamma(\zeta_h)(t)$ is continuous over E_2 and the Lebesgue measure is absolutely continuous with respect to η, we have that for any $y' > y$

$$\int_{E_2} \mathbb{I}\big(\Gamma(\zeta_h)(t) + y' > 0\big) \eta(dt) > x.$$

This implies that $A_y \cap A_{y'} = \emptyset, y \neq y', y, y' \in \mathbb{R}$. Noting that the continuity of $\Gamma(\zeta_h)$ guarantees the measurability of A_y, and $\{y : y \in \mathbb{R} \quad \text{such that} \quad \mathbb{P}\left\{A_y\right\} > 0\}$ is a countable set because if it were not we would find countably many (disjoint) A_y such that $\sum \mathbb{P}\left\{A_y\right\} = \infty$.

Thus we get $\int_{\mathbb{R}} \mathbb{P}\{A_y\} e^{-y} dy = 0$, hence $\mathcal{B}_\zeta^{\Gamma,h,\eta}(x, E_2)$ is continuous on $(0, \eta(E_2))$, establishing the claim. □

Before proceeding to the proof of Lemma 4.2, under the notation introduced in the proof of Proposition 3.1, we denote and analyze

$$\Sigma\Sigma_1(u,n) := \sum_{0\le k_i,k_i'\le N_i(u,n), i=1,2, (k_1,k_2)\neq(k_1',k_2')} \mathbb{P}\left\{ \sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u, \sup_{t\in I_{k_1',k_2'}(u,n)} X(t) > u \right\}, \quad (6.10)$$

$$\Sigma\Sigma_2(u,n) := \sum_{0\le 2k_i,2k_i'\le N_i(u,n), i=1,2, (k_1,k_2)\neq(k_1',k_2')} \mathbb{P}\left\{ \sup_{t\in I_{2k_1,2k_2}(u,n)} X(t) > u, \sup_{t\in I_{2k_1',2k_2'}(u,n)} X(t) > u \right\}, \quad (6.11)$$

$$\Theta(u) := T_1 T_2 a_1^{1/\alpha_1} a_2^{1/\alpha_2} u^{2/\alpha_1 + 2/\alpha_2} \Psi(u). \quad (6.12)$$

Moreover, following notation introduced in the proof of Proposition 3.3, let

$$\Sigma\Sigma_3''(u,n) := \sum_{|k_i|,|k_i'|\le N_i'(u,n), i=1,2, (k_1,k_2)\neq(k_1',k_2')} \mathbb{P}\left\{ \sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u, \sup_{t\in I_{k_1',k_2'}(u,n)} X(t) > u \right\}.$$

$$\hat{I}_{k_2}(u,n) := I_{-1,k_2}(u,n) \cup I_{0,k_2}(u,n), \quad E_1(u,n) := \bigcup_{|k_2|\le N_2'(u,n)} \hat{I}_{k_2}(u,n), \quad (6.13)$$

and

$$\Sigma_3'(u,n) := \sum_{|k_i|\le N_i'(u,n)+1, i=1,2,\ k_1\neq -1,0} \mathbb{P}\left\{ \sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u \right\}, \quad (6.14)$$

$$\Sigma\Sigma_3(u,n) := \sum_{|k_2|,|k_2'|\le N_2'(u,n), k_2\neq k_2'} \mathbb{P}\left\{ \sup_{t\in \hat{I}_{k_2}(u,n)} X(t) > u, \sup_{t\in \hat{I}_{k_2'}(u,n)} X(t) > u \right\}, \quad (6.15)$$

$$\Sigma\Sigma_4(u,n) := \sum_{|2k_2|,|2k_2'|\le N_2'(u,n)-1, k_2\neq k_2'} \mathbb{P}\left\{ \sup_{t\in \hat{I}_{2k_2}(u,n)} X(t) > u, \sup_{t\in \hat{I}_{2k_2'}(u,n)} X(t) > u \right\}. \quad (6.16)$$

Lemma 6.1. *Under the assumptions of Proposition 3.1*

$$\mathbb{P}\left\{ \sup_{t\in E} X(t) > u \right\} \sim \sum_{0\le k_i\le N_i(u,n), i=1,2} \mathbb{P}\left\{ \sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u \right\} \sim \mathbb{C}_0 \Theta(u) \quad (6.17)$$

as $u\to\infty$, $n\to\infty$, where $\mathbb{C}_0 > 0$. Moreover, for all large u and n

$$\Sigma\Sigma_1(u,n) \le \left(\frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}} \right) \Theta(u), \quad \Sigma\Sigma_2(u,n) \le e^{-\mathbb{C}_1 n^{\mathbb{C}}} \Theta(u),$$

where $\mathbb{C}, \mathbb{C}_1$ and $\mathbb{C}_2$ are some positive constants.

Proof of Lemma 6.1 Asymptotics (6.17) follow from Piterbarg (1996, Lem 7.1), while the bounds can be deduced from equations (7.4) and (7.6) in the proof of the aforementioned lemma. □

Lemma 6.2. *Under the assumptions of Proposition 3.3, for* $\alpha_i < \beta_i, i = 1, 2,$

$$\mathbb{P}\left\{\sup_{t\in E\setminus E(u,n)} X(t) > u\right\} = o\left(\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\}\right)$$

as $u \to \infty, n \to \infty,$ *and*

$$\Sigma\Sigma_3''(u,n) = o\left(\sum_{0\le k_i\le N_i'(u,n), i=1,2} \mathbb{P}\left\{\sup_{t\in I_{k_1,k_2}(u,n)} X(t) > u\right\}\right),$$

as $u \to \infty, n \to \infty$. *For* $\alpha_1 = \beta_1, \alpha_2 < \beta_2$

$$\mathbb{P}\left\{\sup_{t\in E\setminus E_1(u,n)} X(t) > u\right\} = o\left(\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\}\right),$$

as $u \to \infty$, $n \to \infty$, *and for* u *and* n *sufficiently large*

$$\Sigma\Sigma_3(u,n) \le \left(\frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}}\right)\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\},$$

$$\Sigma_3'(u,n) \le e^{-\mathbb{C}_1 n^{\mathbb{C}}}\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\},$$

$$\Sigma\Sigma_4(u,n) \le e^{-\mathbb{C}_1 n^{\mathbb{C}}}\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\}.$$

For $\alpha_1 = \beta_1$ *and* $\alpha_2 = \beta_2$

$$\mathbb{P}\left\{\sup_{t\in E\setminus \bigcup_{i,j\in\{-1,0\}} I_{i,j}(u,n)} X(t) > u\right\} = o\left(\mathbb{P}\left\{\sup_{t\in E} X(t) > u\right\}\right),$$

as $u \to \infty, n \to \infty$.

Proof of Lemma 6.2 The proof of Lemma 6.2 follows from Dębicki et al. (2017b). Specifically, the first one follows from (34), the second one from (40) and (41), the third one from (34) and (46), the fourth one from (48) and (49), the fifth one from (46), the six one from (48), and the last one from (34) and (52) in the proof of Dębicki et al. (2017b, Thm 3.1). □

Now we are in the position to prove Lemma 4.2.

Proof of Lemma 4.2 *Ad (i)*. We follow notation introduced in the proof of Proposition 3.1. For any $n, n_1 > \sqrt{x}$, we have

$$\Sigma_1^-(u,n_1) - \Sigma\Sigma_1(u,n_1) \le \mathbb{P}\left\{\int_{E(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\} \le \Sigma_1^+(u,n) + \Sigma\Sigma_1(u,n), \quad (6.18)$$

where $\Sigma\Sigma_1(u,n)$ is given in (6.10) and

$$\Sigma_1^{\pm}(u,n) = \sum_{0\le k_i\le N_i(u,n)\pm 1, i=1,2} \mathbb{P}\left\{\int_{I_{k_1,k_2}(u,n)} \mathbb{I}(X(t) > u)dt > v(u)x\right\}.$$

By (5.3), it follows that

$$\begin{aligned}\Sigma_1^+(u,n) &\le \sum_{0\le k_i\le N_i(u,n), i=1,2} \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)\Psi(u)\\ &\le \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2}\Theta(u), \quad u\to\infty,\end{aligned}$$

where $\Theta(u)$ is defined in (6.12). Analogously, we obtain the lower bound

$$\Sigma_1^-(u,n) \;\ge\; \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2}\Theta(u), \quad u\to\infty.$$

Lemma 6.1 shows that for u and n sufficiently large

$$\Sigma\Sigma_1(u,n) \le \left(\frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}}\right)\Theta(u).$$

Dividing both sides of (6.18) by $\Theta(u)$ and letting $u\to\infty$, we have

$$\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2)}{n_1^2} - \frac{\mathbb{C}_2}{\sqrt{n_1}} - e^{-\mathbb{C}_1 n_1^{\mathbb{C}}} \le \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} + \frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}}.$$

The above implies that

$$\limsup_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} \;=\; \liminf_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} < \infty.$$

Next we show that

$$\liminf_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} > 0.$$

Observe that

$$\mathbb{P}\left\{\int_E \mathbb{I}(X(t)>u)dt > v(u)x\right\} \ge \Sigma_2(u,n) - \Sigma\Sigma_2(u,n), \tag{6.19}$$

where $\Sigma\Sigma_2(u)$ is given in (6.11) and

$$\Sigma_2(u,n) = \sum_{0\le 2k_i\le N_i'(u,n), i=1,2} \mathbb{P}\left\{\int_{I_{2k_1,2k_2}(u,n)} \mathbb{I}(X(t)>u)dt > v(u)x\right\}.$$

In light of (5.3), we have

$$\begin{aligned}\Sigma_2(u,n) &\ge \sum_{0\le 2k_i\le N_i'(u,n), i=1,2} \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)\Psi(u)\\ &\ge \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{4n^2}\Theta(u), \quad u\to\infty.\end{aligned}$$

Moreover, by Lemma 6.1 we have, for u and n large enough

$$\Sigma\Sigma_2(u,n) \le e^{-\mathbb{C}_1 n^{\mathbb{C}}}\Theta(u).$$

Combination of upper bound in (6.18) and lower bound in (6.19) leads to

$$\liminf_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2} \ge \frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2)}{4n_1^2} - e^{-\mathbb{C}_1 n_1^{\mathbb{C}}}. \tag{6.20}$$

For $n_1 > \sqrt{x}$

$$\begin{aligned}\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2) &= \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[0,n_1]^2} \mathbb{I}\left(\sum_{i=1}^{2}(\sqrt{2}B_{\alpha_i}(t_i)-|t_i|^{\alpha_i})>s\right)dt>x\right\}e^s ds\\ &\geq \int_{\mathbb{R}} \mathbb{P}\left\{\inf_{t\in[0,n_1]^2}\sum_{i=1}^{2}\left(\sqrt{2}B_{\alpha_i}(t_i)-|t_i|^{\alpha_i}\right)>s\right\}e^s ds>0,\end{aligned}$$

which combined with the monotonicity of $\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2)$ in n_1 and (6.20) implies that for sufficiently large n_1

$$\liminf_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)}{n^2}\geq\frac{\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2)-4n_1^2e^{-\mathbb{C}_1 n_1^{\mathbb{C}}}}{4n_1^2}>0,$$

establishing the proof of (i).

Ad (ii). We follow notation introduced in the proof of Proposition 3.3 for the case $\alpha_1=\beta_1$ and $\alpha_2<\beta_2$. Let next for $u>0$

$$E_2(u):=\left[-\left(\frac{e_u^{-1/4}\wedge\ln u}{u}\right)^{2/\beta_1},\left(\frac{e_u^{-1/4}\wedge\ln u}{u}\right)^{2/\beta_1}\right]\times\left[-\left(\frac{e_u^{-1/4}\wedge\ln u}{u}\right)^{2/\beta_2},\left(\frac{e_u^{-1/4}\wedge\ln u}{u}\right)^{2/\beta_2}\right],$$

$$\begin{gathered}I_{k_1,k_2}(u,n):=[k_1v_1(u)n,(k_1+1)v_1(u)n]\times[k_2v_2(u)n,(k_2+1)v_2(u)n],\\ \Theta_1(u):=2\hat{\Gamma}(1/\beta_2+1)a_2^{1/\alpha_2}b_2^{-1/\beta_2}u^{2/\alpha_2-2/\beta_2}\Psi(u),\end{gathered}$$

where $\hat{\Gamma}(\cdot)$ is the gamma function and

$$e_u=\sup_{0<|t_i|<\left(\frac{\ln u}{u}\right)^{2/\beta_i},i=1,2}|e(t)|,\quad e(t)=\frac{1-\sigma(t)}{\sum_{i=1}^2 b_i|t_i|^{\beta_i}}-1,|t|\neq 0.$$

Observe that

$$\begin{aligned}\mathbb{P}\left\{\int_{E_2(u)}\mathbb{I}(X(t)>u)dt>v(u)x\right\} &\geq \mathbb{P}\left\{\int_{E_1(u,n)}\mathbb{I}(X(t)>u)dt>v(u)x\right\},\\ \mathbb{P}\left\{\int_{E_2(u)}\mathbb{I}(X(t)>u)dt>v(u)x\right\} &\leq \mathbb{P}\left\{\int_{\bigcup_{|k_2|\leq N_2'(u,n)+1}\hat{I}_{k_2}(u,n)}\mathbb{I}(X(t)>u)dt>v(u)x\right\}\\ &\quad+\mathbb{P}\left\{\sup_{E_2(u)\setminus(\bigcup_{|k_2|\leq N_2'(u,n)+1}\hat{I}_{k_2}(u,n))}X(t)>u\right\}.\end{aligned}$$

Hence it follows that

$$\Sigma_3^-(u,n_1)-\Sigma\Sigma_3(u,n_1)\leq\mathbb{P}\left\{\int_{E_2(u)}\mathbb{I}(X(t)>u)dt>v(u)x\right\}\leq\Sigma_3^+(u,n)+\Sigma_3'(u,n),\tag{6.21}$$

with

$$\Sigma_3^{\pm}(u,n)=\sum_{|k_2|\leq N_2'(u,n)\pm1}\mathbb{P}\left\{\int_{\hat{I}_{k_2}(u,n)}\mathbb{I}(X(t)>u)dt>v(u)x\right\},$$

where $I_{k_1,k_2}(u,n)$ is defined in (5.6) and Σ_3' and $\Sigma\Sigma_3$ are given in (6.14) and (6.15) respectively. Noting that (5.12) also holds for $|k_2| \le N_2'(u,n)+1$, we have for $x \ge 0$

$$\begin{aligned}\Sigma_3^{\pm}(u,n) &\sim \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n]) \sum_{|k_2|\le N_2'(u,n)+1} \Psi(u_{k_2,n}^{\pm}) \\ &\sim \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])\Psi(u) \sum_{|k_2|\le N_2'(u,n)+1} e^{-u^2 b_2(|k_2|v_2(u)n)^{\beta_2}} \\ &\sim \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n}\Theta_1(u), \quad u\to\infty.\end{aligned}$$

In light of Lemma 6.2, we have that for u and n sufficiently large

$$\Sigma\Sigma_3(u,n)+\Sigma_3'(u,n) \le \left(\frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}}\right)\Theta_1(u).$$

Dividing both sides of (6.21) by $\Theta_1(u)$ respectively and letting $u\to\infty$, we have that

$$\frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n_1,n_1]\times[0,n_1])}{n_1} - \frac{\mathbb{C}_2}{\sqrt{n_1}} - e^{-\mathbb{C}_1 n_1^{\mathbb{C}}} \le \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} + \frac{\mathbb{C}_2}{\sqrt{n}} + e^{-\mathbb{C}_1 n^{\mathbb{C}}},$$

which gives that

$$\liminf_{n\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} = \limsup_{n\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} < \infty.$$

Moreover, we have

$$\mathbb{P}\left\{\int_{E_2(u)} \mathbb{I}(X(t)>u)dt > v(u)x\right\} \ge \Sigma_4(u,n) - \Sigma\Sigma_4(u,n),$$

where $\Sigma\Sigma_4(u,n)$ is defined in (6.16) and

$$\Sigma_4(u,n) = \sum_{|2k_2|\le N_2'(u,n)-1} \mathbb{P}\left\{\int_{\hat{I}_{2k_2}(u,n)} \mathbb{I}(X(t)>u)dt > v(u)x\right\}.$$

By (5.12), for $x\ge 0$ we have

$$\begin{aligned}\Sigma_4(u,n) &\sim \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n]) \sum_{|2k_2|\le N_2'(u,n)-1} \Psi(u_{k_2,n}^{-}) \\ &\sim \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{2n}\Theta_1(u), \quad u\to\infty.\end{aligned}$$

By Lemma 6.2, for u and n sufficiently large, we have

$$\Sigma\Sigma_4(u,n) \le e^{-\mathbb{C}_1 n^{\mathbb{C}}}\Theta_1(u).$$

In view of (6.21) for the upper bound, we have

$$\liminf_{n\to\infty} \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} \ge \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n_1,n_1]\times[0,n_1])}{n_1} - e^{-\mathbb{C}_1 n_1^{\mathbb{C}}}.$$

Noting that for $n > \sqrt{x}$

$$\begin{aligned}
&\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])\\
&= \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[-n,n]\times[0,n]} \mathbb{I}\left(\sum_{i=1}^{2}(B_{\alpha_i}(t_i)-|t_i|^{\alpha_i})-a_1^{-1}b_1|t_1|^{\alpha_1}>s\right)dt>x\right\}e^s ds\\
&\geq \int_{\mathbb{R}} \mathbb{P}\left\{\inf_{t\in[-n,n]\times[0,n]}\left(\sum_{i=1}^{2}(B_{\alpha_i}(t_i)-|t_i|^{\alpha_i})-a_1^{-1}b_1|t_1|^{\alpha_1}\right)>s\right\}e^s ds>0,
\end{aligned}$$

and by the monotonicity of $\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])$ with respect to n, we have, for n_1 sufficiently large,

$$\liminf_{n\to\infty}\frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n,n]\times[0,n])}{n} \geq \frac{\mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},0}(x,[-n_1,n_1]\times[0,n_1])}{n_1}-e^{-\mathbb{C}_1 n_1^{\mathbb{C}}}>0.$$

This completes the proof of (ii).

Ad (iii). We follow notation introduced in the proof of Proposition 3.3 for the case $\alpha_i=\beta_i$, $i=1,2$ Observe that

$$\Sigma_5(u,n)\leq \mathbb{P}\left\{\int_{E'(u,n)}\mathbb{I}(X(t)>u)dt>v(u)x\right\}\leq \Sigma_5(u,n)+\Sigma\Sigma_5(u,n), \tag{6.22}$$

where $E'(u,n)=\bigcup_{(k_1,k_2)\in K_{u,n}} I_{k_1,k_2}(u,n)$ and

$$\begin{aligned}
\Sigma_5(u,n) &= \mathbb{P}\left\{\int_{\hat{I}(u,n)}\mathbb{I}(X(t)>u)dt>v(u)x\right\},\\
\Sigma\Sigma_5(u,n) &= \sum_{|k_i|\leq N_i'(u,n),k_i\neq -1,0,i=1,2}\mathbb{P}\left\{\sup_{t\in I_{k_1,k_2}(u,n)}\overline{X}(t)>u_{n,k_1,k_2}^{-}\right\},
\end{aligned}$$

with u_{n,k_1,k_2}^{-} defined in (5.7) and $\hat{I}(u,n)$ in (5.14). In light of (5.8) and (3.3), we have that for u sufficiently large

$$\begin{aligned}
\Sigma\Sigma_5(u,n) &\leq \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)\sum_{|k_i|\leq N_i'(u,n),k_i\neq -1,0,i=1,2}\Psi(u_{n,k_1,k_2}^{-})\\
&\leq \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)\Psi(u)\sum_{|k_i|\leq N_i'(u,n),k_i\neq -1,0,i=1,2}e^{-a_1^{-1}b_1|k_1^*n|^{\beta_1}-a_2^{-1}b_2|k_2^*n|^{\beta_2}}\\
&\leq \mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n]^2)e^{-Q_1(n^{\beta_1}+n^{\beta_2})}\Psi(u),
\end{aligned}$$

where $k_i^*=k_iI_{\{k_i>0\}}+(|k_i|-1)I_{\{k_i<0\}}, i=1,2.$

Hence dividing (6.22) by $\Psi(u)$ and letting $u\to\infty$, we have for any $n,n_1>\sqrt{x}$

$$\begin{aligned}
0 &< \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x,[-n,n]^2)\\
&\leq \mathcal{B}_{\alpha_1,\alpha_2}^{a_1^{-1}b_1|t_1|^{\alpha_1},a_2^{-1}b_2|t_2|^{\alpha_2}}(x,[-n_1,n_1]^2)+\mathcal{B}_{\alpha_1,\alpha_2}(x,[0,n_1]^2)e^{-Q_1(n_1^{\beta_1}+n_1^{\beta_2})}.
\end{aligned}$$

Letting $n\to\infty$ with n_1 fixed in the above inequality, we complete the proof. □

Proof of (5.9): Observe that

$$\frac{\Psi(u^-_{n,k_1,k_2})}{\Psi(u^+_{n,k_1,k_2})} \sim e^{\frac{\left(u^+_{n,k_1,k_2}\right)^2-\left(u^-_{n,k_1,k_2}\right)^2}{2}}, \quad u\to\infty$$

uniformly with respect to $0 \le |k_i| \le N_i'(u,n), i=1,2$. Furthermore, by (3.3), for u sufficiently large

$$\begin{aligned}
\left(u^+_{n,k_1,k_2}\right)^2-\left(u^-_{n,k_1,k_2}\right)^2 &= u^2\left(\sup_{t\in I_{k_1,k_2}(u,n)}\frac{1}{\sigma^2(t)}-\inf_{t\in I_{k_1,k_2}(u,n)}\frac{1}{\sigma^2(t)}\right)\\
&= u^2\sup_{s,t\in I_{k_1,k_2}(u,n)}\frac{\sigma^2(t)-\sigma^2(s)}{\sigma^2(t)\sigma^2(s)}\\
&\le 4u^2\sup_{s,t\in I_{k_1,k_2}(u,n)}|\sigma(t)-\sigma(s)|\\
&= 4u^2\sup_{s,t\in I_{k_1,k_2}(u,n)}\left|(1+e(t))\sum_{i=1}^2 b_i|t_i|^{\beta_i}-(1+e(s))\sum_{i=1}^2 b_i|s_i|^{\beta_i}\right|\\
&\le 4u^2\sup_{s,t\in I_{k_1,k_2}(u,n)}\left|\sum_{i=1}^2 b_i|t_i|^{\beta_i}-\sum_{i=1}^2 b_i|s_i|^{\beta_i}\right|+8u^2\sup_{t\in I_{k_1,k_2}(u,n)}|e(t)|\sum_{i=1}^2 b_i|t_i|^{\beta_i}\\
&\le 4u^2\sum_{i=1}^2 b_i\beta_i|\theta_i|^{\beta_i-1}v_i(u)n+8u^2\sup_{t\in I_{k_1,k_2}(u,n)}|e(t)|\sum_{i=1}^2 b_i|t_i|^{\beta_i},
\end{aligned}$$

where $e(t)=\frac{1-\sigma(t)}{\sum_{i=1}^2 b_i|t_i|^{\beta_i}}-1, |t|\neq 0$ and $\theta_i\in(k_iv_i(u)n,(k_i+1)v_i(u)n)$. Using the fact that

$$N_i'(u,n)=\left[\frac{(e_u^{-1/4}\wedge\ln u)^{2/\beta_i}}{u^{2/\beta_i}v_i(u)n}\right] \text{ and } \lim_{u\to\infty}e_u=0,$$

we have that

$$u^2\sup_{t\in I_{k_1,k_2}(u,n)}|e(t)|\sum_{i=1}^2 b_i|t_i|^{\beta_i}\le 2e_u\sum_{i=1}^2 b_i(e_u^{-1/4}\wedge\ln u)^2\to 0,$$

as $u\to\infty$ uniformly with respect to $0\le|k_i|\le N_i'(u,n), i=1,2$. For $\beta_i\ge 1, i=1,2$,

$$\begin{aligned}
u^2\sum_{i=1}^2 b_i\beta_i|\theta_i|^{\beta_i-1}v_i(u)n &\le u^2\sum_{i=1}^2 b_i\beta_i\left(\frac{\ln u}{u}\right)^{\frac{2(\beta_i-1)}{\beta_i}}v_i(u)n\\
&\le \sum_{i=1}^2 2a_i^{-1/\alpha_i}b_i\beta_iu^{2/\beta_i-2/\alpha_i}(\ln u)^{\frac{2(\beta_i-1)}{\beta_i}}n\to 0, \quad u\to\infty
\end{aligned}$$

uniformly with respect to $0\le|k_i|\le N_i'(u,n), i=1,2$, where $(\theta_1,\theta_2)\in I_{k_1,k_2}(u,n)$. For $0<\beta_i<1, i=1,2$,

$$\begin{aligned}
u^2\sup_{s,t\in I_{k_1,k_2}(u,n)}\left|\sum_{i=1}^2 b_i|t_i|^{\beta_i}-\sum_{i=1}^2 b_i|s_i|^{\beta_i}\right| &\le u^2\sum_{i=1}^2 b_i\beta_i|\theta_i|^{\beta_i-1}v_i(u)n\\
&\le u^2\sum_{i=1}^2 b_i\beta_i|v_i(u)n|^{\beta_i}\to 0, \quad u\to\infty,
\end{aligned}$$

holds uniformly for $0 \le |k_i| \le N_i{}'(u,n), k_i \neq -1, 0, i = 1, 2$. For $0 < \beta_i < 1, k_i = -1, 0, i = 1, 2$

$$\begin{aligned} u^2 \sup_{s,t \in I_{k_1,k_2}(u,n)} \left| \sum_{i=1}^2 b_i |t_i|^{\beta_i} - \sum_{i=1}^2 b_i |s_i|^{\beta_i} \right| &\le u^2 \sup_{s,t \in I_{k_1,k_2}(u,n)} \left(\sum_{i=1}^2 b_i |t_i|^{\beta_i} + \sum_{i=1}^2 b_i |s_i|^{\beta_i} \right) \\ &\le 2u^2 \sum_{i=1}^2 b_i |v_i(u) n|^{\beta_i} \\ &= 2 \sum_{i=1}^2 a_i^{-\beta_i/\alpha_i} b_i n^{\beta_i} u^{2-2\beta_i/\alpha_i} \to 0, \quad u \to \infty. \end{aligned}$$

Therefore, we can conclude that

$$\left(u^+_{n,k_1,k_2} \right)^2 - \left(u^-_{n,k_1,k_2} \right)^2 \to 0$$

as $u \to \infty$ uniformly with respect to $0 \le |k_i| \le N'_i(u,n), i = 1, 2$ establishing the proof. $\square$

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