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RELATIONAL CONTRACTS: PUBLIC VERSUS PRIVATE SAVINGS

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Work on relational employment agreements often predicts low payments or termination for poor performance. The possibility of saving can, however, limit the effectiveness of monetary incentives in motivating an employee with diminishing marginal utility for consumption. We study the role of savings and their observability in optimal relational contracts. We focus on the case where players are not too patient, and hence the constant first-best effort cannot be implemented. If savings are hidden, the relationship eventually deteriorates over time. In particular, both payments and effort decline. On the other hand, if savings are public, consumption is initially high, so the agent's savings fall over time, and effort and payments to the agent increase. The findings thus suggest how tacit agreements on consumption can forestall the deterioration of dynamic relationships in which the agent can save.

KEYWORDS: Relational contracts, consumption smoothing preferences, private savings.

1. INTRODUCTION

WORK ON RELATIONAL CONTRACTS has examined the role of commitment in employment relationships. Yet, the role of employee savings in this context is still not well understood. From the perspective of incentive provision, the possibility that the employee accumulates wealth creates a potential difficulty. If the employee can become wealthy, he will be less reliant on the employer's pay, as his options outside the relationship are then more attractive. We study how this difficulty is managed in optimal relational contracts, focusing on the role of the observability of savings and consumption.

In our model, an employer (the principal) is in a repeated relationship with a worker (the agent). The agent provides effort determining output and also chooses how much to consume and to save. The principal has linear preferences over money, and the agent has concave utility for consumption implying a preference for smooth consumption streams. The agent cannot commit to future effort or consumption and the principal cannot commit to future pay. Effort and pay are publicly observable, and we contrast the situation

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where savings/consumption are private to the agent with that in which these decisions are public information. We characterize relational contracts that maximize the principal's discounted payoff.

Private Savings. We begin by considering the case where the agent can privately consume and save. The "private savings problem" has by now a long tradition in work on dynamic contracting where the principal commits to the contract. Particularly relevant to our paper is work on private savings in dynamic moral hazard, where there is imperfect monitoring of effort (see a detailed discussion of the literature in Section 6).[1] There is little work, however, on agreements with private savings but without commitment. To our knowledge, the main exception is Ábrahám and Laczó (2018) who study constrained-efficient agreements between two risk-averse agents in a setting of mutual insurance (without effort) when there is both private and public savings, and which we discuss below.

We contribute to the understanding of relationships with private savings and without commitment by characterizing optimal relational contracts in the employment setting described above. It is without loss to consider punishments for public deviations that involve autarky; this means the cessation of effort and pay. Two types of incentive compatibility conditions are then relevant. The first is standard in work on relational contracts: the principal's payment cannot be larger than the continuation value the relationship has to her. This sets a limit on how much the principal can credibly pay to the agent. The second type of compatibility condition is new and concerns the incentive of the agent to follow the consumption and effort specified in the agreement. While there are many ways in which the agent can deviate jointly in effort and consumption, we identify the critical deviations to be those where the agent (a) follows the agreed effort up to any given date and then shirks by supplying zero effort forever, and (b) reduces consumption from the beginning so that he perfectly smooths over his lifetime the income derived up to the date at which he begins to shirk. These deviations thus involve the agent secretly saving more in anticipation of a public defection on the agreement. Such deviations represent the optimal ones for the agent: provided they are not profitable, agent incentive compatibility is assured.

Now consider the principal's optimal contract and in particular the role of the constraints preventing the agent's critical deviations. If the principal was able to commit to payments, she would ask for a constant (first-best) level of effort and pay to the agent would rise over time. The reason the agent's pay must be increasing is the following. As the agent obediently works more periods, he values additional payments less. This is because the agent can smooth his earnings over his lifetime and because of diminishing marginal utility for consumption. It follows that the agent must be paid more at later dates to induce the same level of effort.

Returning to the case without commitment, when the players are sufficiently patient, the first-best effort can be sustained. Otherwise, we find that effort is initially constant, and then eventually declines over time. The reason is related to the need to compensate the agent more at later dates for any given level of effort, as identified above. As time passes, the agent's higher pay relative to effort reduces the profits of the principal. Because future profits are lower, the principal can then only credibly promise lower levels

[1]Contributions include Werning (2002), Kocherlakota (2004), Ábrahám and Pavoni (2005), Mitchell and Zhang (2010), Ábrahám, Koehne, and Pavoni (2011), Edmans, Gabaix, Sadzik, and Sannikov (2012), He (2012), and Di Tella and Sannikov (2021).

of pay. This in turn depresses the sustainable effort and profits, which creates a feedback loop.

The effects described above are new. The dynamics of the optimal contract are driven by the constraints ensuring the agent does not engage in the "double deviations" of secretly increasing savings and later publicly defecting. This is not the case in Ábrahám and Laczó (2018). They solve a relaxed problem that omits constraints related to double deviations where agents secretly increase savings and then later quit the agreement for autarky. They then check numerically that this is justified when the return on savings is not too high, concluding that private savings in the constrained-efficient agreements are zero in this case. They therefore argue that the "characteristics of the constrained-efficient allocations… are the same" (p. 17) whether or not the agent can privately save. When the return on savings is higher, however, agents' double deviations cannot be ignored, and so no results are provided.

Due to the forces explained above, pay to the agent is eventually strictly declining with time whenever the principal cannot achieve the first-best payoff. Because equilibrium consumption is constant, the agent's savings are eventually increasing. These predictions contrast with what occurs in the dynamic moral hazard literature with private savings where the players fully commit. In particular, in situations where the agent can borrow as well as save, the principal is unconstrained in the timing of payments and the optimal timing of pay is indeterminate. For instance, the principal can always delay payments, effectively "saving for the agent." Given the indeterminacy in pay, the convention has been to consider pay that requires the agent to have zero savings in equilibrium. Papers where pay is indeterminate and this convention is invoked include Ábrahám, Koehne, and Pavoni (2011), Edmans et al. (2012), and Williams (2015) (an early reference is Cole and Kocherlakota (2001), although this studies an insurance setting).

Public Savings. We compare our results with the case where consumption and savings by the agent are observable by the principal. This case is also new to the principal-agent literature. For the public-consumption case, first-best effort and consumption is again sustainable when the players are sufficiently patient. Otherwise, the dynamics of the relationship stand in sharp contrast to what occurs for private savings. First, the agent's consumption is distorted: it is high in the initial periods and lower in later ones. Also, the relational contract induces the agent to dissave, worsening his outside option from exiting the contract. As the agent becomes poorer, he is more willing to trade high effort for pay and the relationship becomes more profitable for the principal. The level of pay and effort that can be sustained increases with time. The advantage of a relationship in which the agent becomes impoverished with time is that the principal's higher profitability at later dates relaxes her credibility constraint in early periods. This increases the pay that is credible early on and increases the sustainable level of effort. Impoverishment is shown to continue indefinitely, with the balance on the agent's account approaching a level at which the first best is sustainable. That is, we obtain convergence to efficiency in the long run.

The fact that the agent becomes poorer over time is reminiscent of immiseration results such as Thomas and Worrall (1990) where the agent's utility declines without bound with probability one. However, note that the classical immiseration results are driven by the provision of incentives for information revelation, rather than the absence of commitment, which is responsible for the agent's impoverishment in our paper.

Broader Implications. The contrast between private and public savings may have broader implications for settings with limited commitment where agents can invest in their outside options. Savings is one possible investment, but other possibilities include physical capital accumulation, for example, Kehoe and Perri (2002), or investment in human capital, for example, Voena (2015). In such settings, lower outside options tend to enhance the efficiency of the relationship, and optimal investments must be determined in light of such effects. As noted, the role of private investment in outside options has been explored little to date. With private investments, optimal relationships can be shaped by agents' abilities to gradually and secretly invest from the beginning. In this sense, what can matter is agents' *potential* outside options, that is, the ones they can access if investing more from the beginning than they do on path. We demonstrate this in our setting where the possibility of private investment in outside options hampers efficiency and causes the relationship to deteriorate over time. To our knowledge, we are the first to establish such effects.

The organization of the rest of the paper is as follows. Section 2 introduces the setting. Section 3 solves the case where both principal and agent can commit. Section 4 addresses the case with limited commitment where savings are private, and Section 5 the case with limited commitment where savings are public. Section 6 provides a review of the literature. Appendix contains proofs of key results. The Appendix in the Online Supplementary Material (Dilmé and Garrett (2023)) discusses anecdotal evidence suggesting the importance of high consumption for the efficiency of principal-agent relationships.

2. SETTING

Environment and Preferences. A principal and agent meet in discrete time at dates $t = 1, 2, \ldots$. Letting $r > 0$ be the interest rate that will apply to the balance on the agent's savings account, we suppose the players have a common discount factor $\delta = \frac{1}{1+r}$. In every period t, first the agent exerts an effort e_t and consumes an amount c_t. Then the principal makes a discretionary payment w_t to the agent. These variables are all restricted to be nonnegative.

The agent has initial savings balance $b_1 > 0$ as well as access to a savings technology (with the interest rate r as specified above). Throughout, the initial balance is common knowledge between the principal and agent. The agent's balance at time $t+1 > 1$ then satisfies

$$b_{t+1} = \frac{b_t + w_t - c_t}{\delta} = b_1 \delta^{-t} + \sum_{s=1}^{t} \delta^{s-t-1}(w_s - c_s). \tag{1}$$

Balances can, in principle, be negative (i.e., the agent can borrow). We say that the agent's intertemporal budget constraint is satisfied in case

$$\sum_{t=1}^{\infty} \delta^{t-1} c_t \leq b_1 + \sum_{t=1}^{\infty} \delta^{t-1} w_t. \tag{2}$$

The agent's felicity from consumption c_t in any period t is denoted $v(c_t)$, where $v : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. We assume that $v(c)$ is real-valued for $c > 0$, and takes value $-\infty$ at $c = 0$. We further assume that v, when evaluated on positive consumption values, is twice continuously differentiable, strictly increasing, and strictly concave. In addition, v is onto all of $\mathbb{R}$, implying $\lim_{c \searrow 0} v(c) = -\infty$.

The agent's disutility of effort e_t is $\psi(e_t)$. We assume that ψ is continuously differentiable, strictly increasing, strictly convex, and such that $\psi(0) = \psi'(0) = 0$, and that $\lim_{e\to\infty} \psi'(e) = \infty$.

The agent's period-t payoff is $v(c_t) - \psi(e_t)$, while the principal's is $e_t - w_t$; hence, we interpret effort as equal to the output enjoyed by the principal.

Relational Contracts. We focus for tractability on *deterministic* relational contracts. We identify relational contracts with their outcomes; denote them $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. We restrict attention to contracts that satisfy the following feasibility constraints.

DEFINITION 2.1: A *feasible relational contract* is a sequence $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the following conditions:

1. **Positivity:** $\tilde{e}_t, \tilde{w}_t, \tilde{c}_t \geq 0$ for all t.
2. **Balance dynamics and constraint:** Conditions (1) and (2) hold.
3. **Boundedness:** The sequences of consumption, pay and effort $((\tilde{c}_t)_{t\geq 1}$, $(\tilde{w}_t)_{t\geq 1}$, and $(\tilde{e}_t)_{t\geq 1})$ are bounded.

While the first and second conditions reflect features of the environment introduced above, the third condition guarantees that the players' payoffs are well-defined in a feasible contract.

3. FIRST BEST AND FULL COMMITMENT TO THE CONTRACT

Consider first the problem of maximizing the principal's payoff by choice of a feasible relational contract subject only to the constraint that the agent is initially willing to participate. If the agent does not participate, a possibility we describe as "autarky," we stipulate that he consumes $(1 - \delta)b_1$ per period. This is the optimal consumption for the agent among consumption streams satisfying the intertemporal budget constraint in equation (2) given that all payments are set to zero. Therefore, we consider maximizing the principal's payoff over feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ such that the payoff of the agent

$$\sum_{t=1}^{\infty} \delta^{t-1}\big(v(\tilde{c}_t) - \psi(\tilde{e}_t)\big) \tag{3}$$

is no lower than his autarky value, $\frac{1}{1-\delta}v((1 - \delta)b_1)$.

PROPOSITION 3.1: *Consider maximizing the principal's discounted payoff by choice of feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, subject to ensuring the agent a payoff at least his autarky value $\frac{1}{1-\delta}v((1 - \delta)b_1)$. In any optimal feasible contract, effort and consumption are constant at $e^{\mathrm{FB}}(b_1) > 0$ and $c^{\mathrm{FB}}(b_1) > (1 - \delta)b_1$, respectively, being the unique solutions to*:

1. ***First-order condition***: $\psi'(e^{\mathrm{FB}}(b_1)) = v'(c^{\mathrm{FB}}(b_1))$, *and*
2. ***Agent's indifference condition***: $v(c^{\mathrm{FB}}(b_1)) - \psi(e^{\mathrm{FB}}(b_1)) = v((1 - \delta)b_1)$.

Furthermore, the payoff of the principal is $V^{\mathrm{FB}}(b_1) \equiv \frac{1}{1-\delta}(e^{\mathrm{FB}}(b_1) - (c^{\mathrm{FB}}(b_1) - (1 - \delta)b_1))$, which is a strictly decreasing function of b_1.

Note that the first-best policies depend on both b_1 and δ, since they depend on the value of autarky consumption $(1 - \delta)b_1$ (see Condition 2). However, we reduce the notational burden by making dependence only on b_1 explicit. Note also that the proposition does not specify the timing of payments. The only requirement on payments is that they are feasible and satisfy the agent's budget constraint (2) with equality. Payments may be constant, in which case they equal $c^{\mathrm{FB}}(b_1) - (1-\delta)b_1$ in each period. Sections 4.1 and 5.1 discuss how, when the principal fully commits but the agent cannot, sufficient backloading of payments is enough to ensure the agent's continued obedience to a first-best contract.

4. UNOBSERVABLE CONSUMPTION

We now suppose the principal can observe the agent's effort, but not the consumption choices nor the agent's balance. Given the absence of commitment, we are interested to determine feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, which coincide with outcomes of a perfect Bayesian equilibrium (PBE) of a dynamic game. These represent the outcomes that are sustainable by a relational contract, and among which we can consider optimizing the principal's payoff.

We begin by defining the histories in our game. For $t \geq 0$, a t-history for the agent is $h_t^A = (e_s, w_s, c_s)_{1\leq s<t}$, which gives the observed effort, payments and consumption up until time $t-1$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^A = \mathbb{R}_+^{3(t-1)}$ (with the convention that $\mathbb{R}_+^0 = \emptyset$). Note that, given h_t^A and the agent's initial balance b_1, we can completely determine the evolution of the balance up to date t using equation (1). We denote the date-t balance by $b(h_t^A)$. A t-history for the principal is $h_t^P = (e_s, w_s)_{1\leq s<t}$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^P = \mathbb{R}_+^{2(t-1)}$.

A strategy for the agent is then a collection of functions

$$\alpha_t : \mathcal{H}_t^A \to \mathbb{R}_+^2, \quad t \geq 1,$$

and a strategy for the principal is a collection of functions

$$\sigma_t : \mathcal{H}_t^P \times \mathbb{R}_+ \to \mathbb{R}_+, \quad t \geq 1.$$

Here, α_t maps the t-history of the agent to a pair (e_t, c_t) of effort and consumption. Also, σ_t maps the t-history of the principal, together with the agent's effort choice e_t, to a payment w_t.

As noted above, we will restrict attention to equilibria whose outcomes coincide with a feasible relational contract. However, we do not restrict the strategies that are available to the players. Certain strategies imply, for instance, the violation of the agent's intertemporal budget constraint in equation (2). To ensure that the agent finds it optimal to satisfy this constraint, we make the following assumption on payoffs. While the principal's payoff is as specified above (and so given by $\sum_{t=1}^{\infty} \delta^{t-1}(e_t - w_t)$), the agent obtains the payoff $\sum_{t=1}^{\infty} \delta^{t-1}(v(c_t) - \psi(e_t))$ if the constraint in equation (2) is satisfied, and obtains payoff $-\infty$ otherwise.

To obtain the set of feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that are PBE outcomes, it suffices to consider PBE where publicly observed deviations from the agreed outcomes are punished by "autarky." This means that, if the agent deviates from the agreed effort $\tilde{e}_t$, or if the principal deviates from the agreed payment $\tilde{w}_t$, the principal makes no payments and the agent exerts no effort from then on; the agent perfectly smoothing the balance of his account over the infinite future.[2] If the agent's balance is

[2]The reason we can consider autarky punishments is that they deliver the lowest possible individually rational payoffs for the players.

negative when autarky begins, the intertemporal budget constraint in equation (2) is necessarily violated (as the agent receives no further payments), the agent must earn payoff $-\infty$, and so we can specify for instance that the agent consumes zero in every period. Note that deviations by the agent from the specified consumption, provided they are not accompanied by any deviation in effort, go unpunished (i.e., the principal continues to adhere to the payments specified by the agreement).

If the agent plans to always choose effort in accordance with the contract, he optimally consumes

$$\bar{c}_\infty \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1} \tilde{w}_s\right)$$

in every period. Clearly, any contract to which the agent is willing to adhere must then specify $\tilde{c}_t = \bar{c}_\infty$ for all t. To conclude that the agent does not want to deviate from the contract, it is then enough to show that he does not gain by planning to shirk on effort for the first time at any given date t, while making all other choices optimally. Suppose then that the agent plans to shirk for the first time at some date t, and so puts effort equal to $\tilde{e}_s$ for all $s < t$, and then optimally sets it equal to zero at all later dates. Then the agent optimally sets consumption equal to

$$\bar{c}_{t-1} \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{t-1} \delta^{s-1} \tilde{w}_s\right) \tag{4}$$

at all dates, so as to completely smooth consumption and exhaust lifetime earnings. Note that this corresponds to the double deviation mentioned in the Introduction.

Given the above, the maximum payoff the agent achieves when deviating in choice of effort for the first time at date t is

$$\frac{1}{1-\delta} v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1} \delta^{s-1} \psi(\tilde{e}_s).$$

Hence, the agent does not want to deviate from the agreement if and only if, for all $t \geq 1$,

$$\frac{1}{1-\delta} v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1} \delta^{s-1} \psi(\tilde{e}_s) \leq \frac{1}{1-\delta} v(\bar{c}_\infty) - \sum_{s=1}^{\infty} \delta^{s-1} \psi(\tilde{e}_s). \tag{$\mathrm{AC}_t^{\mathrm{un}}$}$$

The principal remains willing to continue abiding by the agreement if and only if, at each time t, the payment $\tilde{w}_t$ that is due is less than her continuation payoff in the agreement. The exact requirement is that, for all $t \geq 1$,

$$\tilde{w}_t \leq \sum_{s=t+1}^{\infty} \delta^{s-t} (\tilde{e}_s - \tilde{w}_s). \tag{PC_t}$$

The following result states that the above constraints determine whether a feasible relational contract is the outcome of a PBE.

PROPOSITION 4.1: *Fix a feasible contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. *It is the outcome of a PBE if and only if, for all* $t \geq 1$, *Conditions* ($\mathrm{AC}_t^{\mathrm{un}}$) *and* ($\mathrm{PC}_t$) *are satisfied, and* $\tilde{c}_t = \bar{c}_\infty$.

Necessity of the conditions in the proposition follow for the reasons described above. To obtain sufficiency, we completely specify PBE strategies and beliefs in the proof.

From now on, we refer to a contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that satisfies the conditions of Proposition 4.1 as "self-enforceable." Our task reduces to characterizing feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that maximize the principal's payoff subject to the requirement of being self-enforceable. We term such contracts "optimal."

To determine the properties of optimal contracts, we first show that we can restrict attention to contracts with a particular pattern of payments over time. This pattern involves paying the agent as early as possible, subject to satisfying the agent's incentive constraints. This requires that the agent's obedience constraints in Condition ($\mathrm{AC}_t^{\mathrm{un}}$) hold with equality for all $t \geq 1$. Inspired by the terminology of Board (2011), we refer to this condition as "fastest payments."

LEMMA 4.1: *For any optimal contract, there is another optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ with the same sequence of efforts and consumption such that, for all $t \geq 1$,*[3]

$$\frac{v(\bar{c}_{t-1})}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) = \frac{v\big((1-\delta)b_1\big)}{1-\delta}. \qquad (\mathrm{FP}_t^{\mathrm{un}})$$

An explanation for the result is as follows. First, note that it is optimal to hold the agent to his outside option, and hence

$$\frac{v(\bar{c}_{\infty})}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1}\psi(\tilde{e}_t) = \frac{v\big((1-\delta)b_1\big)}{1-\delta}. \qquad (5)$$

If Condition (5) does not hold, $\tilde{e}_1$ can be slightly increased while keeping the rest of the contract the same so that the constraints ($\mathrm{AC}_t^{\mathrm{un}}$) and ($\mathrm{PC}_t$) continue to hold for all t. Second, when ($\mathrm{FP}_t^{\mathrm{un}}$) holds for all t, the agent is paid as early as possible while preserving the constraints ($\mathrm{AC}_t^{\mathrm{un}}$). The agent cannot be paid earlier, otherwise he will prefer to work obediently for a certain number of periods, save his income at a higher rate than specified in the agreement, and then quit by exerting no effort. It is easily seen that moving payments earlier in time only relaxes the "principal's constraints" (PC_t).

Concerning "fastest payments," we have the following result.

LEMMA 4.2: *Consider a feasible relational contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that satisfies Condition ($\mathrm{FP}_t^{\mathrm{un}}$) at all dates. For any t, if $\tilde{e}_t > 0$, then*

$$\tilde{w}_t \in \left(\frac{\psi(\tilde{e}_t)}{v'(\bar{c}_{t-1})}, \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_t)}\right). \qquad (6)$$

Since $\bar{c}_t$ is increasing in t, the lemma implies that the ratio $\frac{\tilde{w}_t}{\psi(\tilde{e}_t)}$ increases with t. This result translates the agent's incentive constraints ($\mathrm{AC}_t^{\mathrm{un}}$) into the conclusion that the agent becomes more expensive to compensate with time. One explanation is as follows. The

[3]The conclusion of Lemma 4.1 also follows if we permit the agent to make payments to the principal, alongside his choice of effort. The same argument as for Lemma 4.1 then establishes that, in an optimal contract, the agent never makes any payment to the principal, so results on optimal contracts are unaffected.

longer the agent obediently works, the more he is paid in total. Since he can smooth his consumption of these payments over his entire lifetime, and since he has concave utility of consumption, he values additional payments less. Therefore, the payments needed to keep the agent obediently in the relationship, relative to the disutility of effort incurred, increase with time.

Apart from the observation in Lemma 4.2, the usefulness of Lemma 4.1 is that it permits the design of the relational contract to be reduced to the choice of an effort sequence $(\tilde{e}_t)_{t\geq 1}$. From $(\tilde{e}_t)_{t\geq 1}$, we can obtain $(\bar{c}_t)_{t\geq 1}$ using (FP_t^{un}) (so the corresponding consumption $\tilde{c}_t = \bar{c}_\infty$ is also pinned down). Then $(\tilde{w}_t)_{t\geq 1}$ is obtained from equation (4), and $(\tilde{b}_t)_{t\geq 1}$ from equation (1). We next discuss the implementation of first-best contracts (Section 4.1), before moving to optimal contracts when there is no first-best contract that is self-enforceable (Section 4.2).

4.1. *Implementing the First-Best Outcome*

Lemma 4.1 is also useful for understanding the conditions under which the principal obtains the first-best payoff. For instance, we can observe that the first-best effort and consumption, which are constant over time and equal to $e^{\text{FB}}(b_1)$ and $c^{\text{FB}}(b_1)$, can be implemented when the principal can commit to the agreement, but the agent cannot commit. For this, we simply suppose the principal agrees to payments satisfying the conditions in equation (FP_t^{un}), provided the agent chooses effort obediently. Any deviation by the agent from the required effort is met with zero payments from then on. Because first-best effort is constant, and by Lemma 4.2, the payments determined by equation (FP_t^{un}) are increasing over time. Since these represent the *earliest* payments that satisfy the agent's incentive constraints, the result makes clear that backloading of pay is essential to achieving first-best outcomes when the principal can fully commit.[4]

Now consider whether the principal can attain the first-best payoff when neither the principal nor agent can commit; that is, whether there is a first-best contract that is self-enforceable. We can restrict attention to payments that satisfy the conditions in equation (FP_t^{un}). As mentioned, Lemma 4.2 then implies that these payments increase over time. In the long run, payments approach $\psi(e^{\text{FB}}(b_1))/v'(c^{\text{FB}}(b_1))$. Because the principal's constraints (PC_t) tighten over time, verifying they are always satisfied amounts to verifying that

$$\frac{\psi\big(e^{\text{FB}}(b_1)\big)}{v'\big(c^{\text{FB}}(b_1)\big)} \leq \frac{\delta}{1-\delta}\left(e^{\text{FB}}(b_1) - \frac{\psi\big(e^{\text{FB}}(b_1)\big)}{v'\big(c^{\text{FB}}(b_1)\big)}\right). \tag{7}$$

The right-hand side is the limiting value of the principal's future discounted profits in the agreement, while the left-hand side is the limiting value of the payment to the agent. We have the following result.

PROPOSITION 4.2: *Suppose that neither the principal nor agent can commit to the terms of the agreement and that consumption is unobservable. Then the principal attains the first-best payoff in an optimal contract if and only if Condition* (7) *is satisfied.*

[4] Delaying payments relative to the ones determined by equation (FP_t^{un}), while holding their NPV constant, only relaxes the agent's constraints (AC_t^{un}). Given the principal is assumed able to commit to these payments, such delayed payments also constitute an optimal implementation of first-best outcomes.

While understanding the parameter range for which Condition (7) holds is clearly important for understanding optimal contracts, this is complicated by the dependence of the first-best policy on both b_1 and δ. Nonetheless, if we vary δ while allowing b_1 to adjust, holding $b_1(1-\delta)$ constant, then the first-best consumption and effort remain constant. There is then a threshold value of δ above which Condition (7) is satisfied, and below which it fails.

4.2. *Main Characterization for Unobservable Consumption*

We now state our main result for the unobservable consumption case, which is a characterization of optimal effort when the first-best effort cannot be sustained.

PROPOSITION 4.3: *An optimal relational contract exists. Suppose the principal cannot attain the first-best payoff in a self-enforceable contract* (*i.e.*, *Condition* (7) *is not satisfied*). *Then, for any optimal contract, there is a date $\bar{t} \geq 1$ such that effort is constant up to this date, and is subsequently strictly decreasing.*[5] *Effort converges to a value $\tilde{e}_\infty > 0$ in the long run.*

The dynamics of optimal effort when the principal cannot attain the first-best payoff can be explained as follows. There may be some initial periods when the effort is constant. This occurs if the principal's constraint (PC_t) is initially slack. Given that we consider "fastest payments," the payments rise over these periods for the reasons discussed in relation to Lemma 4.2. Given the principal cannot achieve the first-best payoff, it turns out that the principal's constraint eventually binds, and so payments must be reduced. This is only possible by reducing the level of effort. Note that how much effort can be asked without violating the principal's constraint depends on the future profitability of the relationship. Profitability declines over time, both because higher payments must be made relative to the agent's disutility of effort (see Lemma 4.2), and because the effort that can be requested is less. The fact that profitability declines contributes to the decline in effort, which creates a feedback loop.

Our approach to proving Proposition 4.3 relies on variational arguments. For contracts that fail to exhibit the dynamics described in the proposition, we construct more profitable contracts satisfying all the constraints in Proposition 4.1. We demonstrate some of these arguments below.

One useful result toward establishing Proposition 4.3 links the dynamics of effort to the dates at which the principal's constraint (PC_t) is slack (rather than holding with equality).

LEMMA 4.3: *Suppose that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is an optimal relational contract. Suppose that the principal's constraint is slack at t^*, that is, $\tilde{w}_{t^*} < \sum_{s=t^*+1}^{\infty} \delta^{s-t^*}(\tilde{e}_s - \tilde{w}_s)$. Then $\tilde{e}_{t^*+1} \leq \tilde{e}_{t^*}$; also, if $t^* > 1$, then $\tilde{e}_{t^*-1} \leq \tilde{e}_{t^*}$.*

The proof (in the Appendix) proceeds by showing that, if the conclusion of the lemma fails, then effort can be smoothed raising the principal's profits. Such smoothing is profitable given that the disutility of effort is strictly convex (so that differences in effort across periods are inefficient). An immediate implication of the lemma is that effort is constant over any sequence of periods for which the principal's constraint is slack, explaining why effort may be constant in the initial periods.

[5]There are indeed cases where the value $\bar{t}$ is strictly greater than one. See Lemma A.9 in the Appendix.

A further key part of the proof of Proposition 4.3 is to show that effort strictly decreases from a finite date $\bar{t}$ onwards. The main steps of this argument can be explained as follows. Building on Lemma 4.3, we are able to show (in Lemma A.5 in the Appendix) that effort is weakly decreasing with time. Lemma A.6 then establishes that, if the principal's constraint (PC_t) holds with equality at some date $\hat{t}$, then $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$ and the constraint holds with equality also at $\hat{t}+1$. Hence effort strictly decreases from $\hat{t}$ onwards.

The argument for Lemma A.6 can be summarized as follows. By assumption, the principal's constraint (PC_t) at date $\hat{t}$ holds as an equality, that is,

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}}(\tilde{e}_s - \tilde{w}_s).$$

We are able to show that $\tilde{e}_{\hat{t}+1} - \tilde{w}_{\hat{t}+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > \hat{t}+1$. This follows because $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ for all t (as established in Lemma A.1), because effort is weakly decreasing over time (as noted above), and making use of Lemma 4.2 (which recall implies that the ratio of payments to disutility of effort increases with time). Therefore,

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}}(\tilde{e}_s - \tilde{w}_s) > \sum_{s=\hat{t}+2}^{\infty} \delta^{s-\hat{t}-1}(\tilde{e}_s - \tilde{w}_s) \geq \tilde{w}_{\hat{t}+1},$$

where the second inequality is the principal's constraint (PC_t) at date $\hat{t}+1$. Hence, (again using Lemma 4.2) effort is *strictly* lower in period $\hat{t}+1$ (i.e., $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$). In turn, using Lemma 4.3, the principal's constraint must hold again with equality at $\hat{t}+1$. So, we have shown that, if the principal's constraint holds with equality at a given date, it must hold with equality from then on, and so effort strictly decreases with time.

The above argument assumes that the principal's constraint (PC_t) holds with equality at some date. To show this must in fact be the case when the principal cannot attain the first-best payoff, assume to the contrary that these constraints are always slack. Then Lemma 4.3 implies that optimal effort is constant at all dates, say at a value $\tilde{e}_\infty$ (using the notation of the proposition). Letting the payments and the equilibrium consumption $\bar{c}_\infty$ be determined by equation (FP_t^{un}), payments increase over time, and converge to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$. The principal's constraint (PC_t) is then satisfied at all dates if and only if

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right),$$

where the left-hand side can be read as the limiting payment to the agent, while the right-hand side is the limiting NPV of future profits to the principal. For the most profitable choice of a constant effort $\tilde{e}^*_\infty$, this inequality holds as equality. The principal's constraints (PC_t) tighten over time, but never hold with equality.

Because effort is below the first-best level, we have $\psi'(\tilde{e}^*_\infty) < v'(\bar{c}^*_\infty)$, with $\bar{c}^*_\infty$ the level of agent consumption that corresponds to a contract with constant effort $\tilde{e}^*_\infty$. It follows that any sufficiently small adjustment to the effort policy that raises the NPV of effort, together with a change in payments and consumption that leaves the agent's payoff in the contract unchanged, raises profits. We therefore suggest a perturbation to the constant-effort contract (see Lemma A.7 in the Appendix) that increases the NPV of effort, but (assuming that payments continue to satisfy (FP_t^{un})) leaves the principal's constraints (PC_t) intact. To be more precise, we consider increasing effort by a little at date one and lowering it by a constant amount in future periods. If we only raise effort at date one, leaving other effort values unchanged and assuming that payments are adjusted to satisfy (FP_t^{un}) at all dates, the principal's constraint (PC_t) is eventually violated (since v is strictly concave and total pay increases, it becomes more costly to compensate the agent for his effort; in particular, payments must increase in all periods). Therefore, the reduction in effort at future dates is a "correction" intended to relax the principal's constraint (PC_t) when it is tightest.

We have established then that, when the first-best payoff is not attainable, the principal's constraint (PC_t) holds with equality from some date onwards. At these dates, the principal is indifferent between paying the agent and reneging. This feature is the same as in the optimal contracts of Ray (2002) (although his model is quite general, it does not include the possibility of savings or investments).

It remains to translate the findings of Proposition 4.3 into predictions for payments and the agent's balance. Note, however, that while Lemma 4.1 tells us it is optimal for Condition (FP_t^{un}) to hold at all dates, other contracts with a different timing for payments may be optimal. We therefore provide a partial converse for Lemma 4.1.

PROPOSITION 4.4: *Suppose the principal cannot attain the first-best payoff in a self-enforceable contract. Fix any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *and let* $\bar{t}$ *be the date from which effort is strictly decreasing (see Proposition 4.3). Then Condition* (FP_t^{un}) *holds for all* $t > \bar{t}$. *Payments to the agent strictly decrease from date* $\bar{t}+1$ *onwards, while the agent's balances strictly increase.*

The reason payments satisfying Condition (FP_t^{un}) are strictly decreasing from date $\bar{t}+1$ is explained above. The fact that the agent's balance increases over time then follows straightforwardly from equation (1) and from equation (2) taken to hold with equality. In particular, note that

$$\tilde{b}_t = \frac{\bar{c}_\infty}{1-\delta} - \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau,$$

which strictly increases with t when payments fall over time.

Note that, when $\bar{t} > 1$, the principal's constraint (PC_t) is initially slack. In this case, Condition (FP_t^{un}) need not hold at $t < \bar{t}$, and so payments before date $\bar{t}$ are not uniquely determined. When this "fastest payments" condition is nonetheless taken to hold, payments in fact increase over time up to date $\bar{t}$ (as was mentioned above).

5. OBSERVED CONSUMPTION

We now study the case where, at each time t, before making the payment w_t, the principal can observe the agent's past and present-period effort choices $(e_s)_{s=1}^t$ as well as past and present-period consumption choices $(c_s)_{s=1}^t$. Since payments and consumption are commonly observed, the balance b_t at the beginning of each period t is also commonly known (as deduced from equation (1)).

The game is now one of complete information, and we consider subgame perfect Nash equilibrium (SPNE). Both players observe at date t the history $h_t = (e_s, w_s, c_s)_{1\leq s<t}$. The set of such histories at each date t is $\mathcal{H}_t = \mathbb{R}_+^{3(t-1)}$. Reusing notation from Section 4 introduces no confusion, so we describe a strategy for the agent as a collection of functions

$$\alpha_t : \mathcal{H}_t \to \mathbb{R}_+^2, \quad t \geq 1,$$

and a strategy for the principal as a collection of functions

$$\sigma_t : \mathcal{H}_t \times \mathbb{R}_+^2 \to \mathbb{R}_+, \quad t \geq 1.$$

Here, α_t maps the public t-history to a pair (e_t, c_t) of effort and consumption. Also, σ_t maps the public t-history, together with the observed effort and consumption choices (e_t, c_t) of the agent, to a payment w_t. We assume that payoffs are as specified in Section 4 (i.e., the agent earns a payoff $-\infty$ in case his intertemporal budget constraint (2) is violated).

Again we identify a relational contract with the equilibrium outcomes, and we want to characterize contracts that maximize the principal's payoff. A first step is then to determine equilibrium outcomes $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ that are feasible relational contracts. Analogous to the arguments made in the previous section, we begin supposing deviations from the agreed outcomes are punished by "autarky." That is, when either player deviates from the contract, all future effort and payments cease, and the agent perfectly smooths his balance over time. In autarky, the agent consumes $b_t(1-\delta)$ when his balance is $b_t > 0$, and we specify zero consumption in case the balance is $b_t \leq 0$ (in the latter case, the agent can only obtain a payoff of $-\infty$ since violating the intertemporal budget constraint in equation (2) implies this payoff; hence we might as well set consumption to zero). Now, autarky follows not only deviations in effort and payments, but also in consumption.

Suppose that the agreed contract is $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, and deviations are punished by autarky. The agent's payoff, if complying until date $t-1$ and optimally failing to comply from t onwards, is now

$$\sum_{s=1}^{t-1} \delta^{s-1}\big(v(\tilde{c}_s) - \psi(\tilde{e}_s)\big) + \delta^{t-1} \frac{v\big(\max\{0, (1-\delta)\tilde{b}_t\}\big)}{1-\delta}.$$

This takes into account that the agent who deviates at date t optimally exerts no effort from then on, and consumes $\max\{0, (1-\delta)\tilde{b}_t\}$ per period as explained above. Thus, the agent is willing to follow the prescription of the contract if and only if, at all dates t,

$$\frac{v\big(\max\{0, (1-\delta)\tilde{b}_t\}\big)}{1-\delta} \leq \sum_{s=t}^{\infty} \delta^{s-t}\big(v(\tilde{c}_s) - \psi(\tilde{e}_s)\big). \tag{$\mathrm{AC}_t^{\mathrm{ob}}$}$$

The reason for the difference to Condition ($\mathrm{AC}_t^{\mathrm{un}}$) is that publicly honoring the agreement up to date $t-1$ ensures that the agent begins period t with the specified balance $\tilde{b}_t$, which in turn determines the wealth he has available to spend in autarky. Condition ($\mathrm{AC}_t^{\mathrm{un}}$), on the other hand, takes into account that the agent who plans to publicly deviate at date t (by shirking on effort) can save in advance for this event, because consumption is not observed.

We can characterize equilibrium outcomes as follows.

PROPOSITION 5.1: *Fix a feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. It is the outcome of an SPNE in the environment where consumption is observed if and only if, for all $t \geq 1$, Conditions ($\mathrm{AC}_t^{\mathrm{ob}}$) and ($\mathrm{PC}_t$) are satisfied.*

Notice here that the principal's constraint (PC_t) is the one in Section 4. A feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the conditions in the proposition is again termed "self-enforceable" and a self-enforceable contract that maximizes the principal's payoff is "optimal." We can now state a result similar to Lemma 4.1.

LEMMA 5.1: *For any optimal contract, there exists another optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *with the same effort and consumption, with* $\tilde{b}_t > 0$ *for all* t, *and where the timing of payments ensures that agent constraints hold with equality in all periods; that is, for all* $t \geq 1$,

$$\frac{v\big(\tilde{b}_t(1-\delta)\big)}{1-\delta} = \sum_{s=t}^{\infty} \delta^{s-t}\big(v(\tilde{c}_s) - \psi(\tilde{e}_s)\big). \tag{8}$$

Lemma 5.1 implies that we can focus on relational contracts where, for all $t \geq 1$,

$$\frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_t\big) = v(\tilde{c}_t) - \psi(\tilde{e}_t) + \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{t+1}\big). \tag{$\mathrm{FP}_t^{\mathrm{ob}}$}$$

This says that the agent is indifferent between quitting at date t (i.e., ceasing to exert effort) and smoothing the balance $\tilde{b}_t$ optimally over the infinite future, and instead working one more period, exerting effort $\tilde{e}_t$ and consuming $\tilde{c}_t$, before quitting at date $t+1$ and smoothing the balance $\tilde{b}_{t+1}$ over the infinite future.

5.1. *Implementing the First-Best Outcome*

Let us turn now to the question of when the principal can attain the first-best payoff in a self-enforceable relational contract. As for the case with private savings, we can begin by asking how the principal implements the first-best outcomes if she can fully commit to payments (but the agent cannot commit). We can again answer this question by focusing on the earliest payments, where equation (8) is satisfied at all dates, noting that delayed payments (with the same NPV) will also do the job. Any deviation in effort or consumption leads to a cessation of pay. Given effort and consumption constant at the first-best levels $e^{\mathrm{FB}}(b_1)$ and $c^{\mathrm{FB}}(b_1)$, the agent's balance under the specified payments is constant and equal to b_1. Therefore, the payment is constant over time and equal to $w^{\mathrm{FB}}(b_1) \equiv c^{\mathrm{FB}}(b_1) - (1-\delta)b_1$. This shows an important difference between the solutions to the principal's full-commitment problem when the agent's consumption is observed rather than unobserved. With observed consumption, payments can be made earlier without the agent quitting the agreement; in particular, they are constant rather than rising over time. This is because agent deviations of secretly saving and then quitting are not available as any deviation in consumption is observed and so punished by a cessation of pay.

Now turn to the question of when the principal is able to obtain the first-best payoff when she cannot commit. Note that the principal's continuation payoff in a first-best contract with the earliest payments is $V^{\mathrm{FB}}(b_1) = (e^{\mathrm{FB}}(b_1) - w^{\mathrm{FB}}(b_1))/(1-\delta)$. Using the above observations, we have the following result.

PROPOSITION 5.2: *Suppose that consumption is observable. Then the principal attains the first-best payoff in an optimal relational contract if and only if*

$$w^{\mathrm{FB}}(b_1) \leq \frac{\delta}{1-\delta}\big(e^{\mathrm{FB}}(b_1) - w^{\mathrm{FB}}(b_1)\big). \tag{9}$$

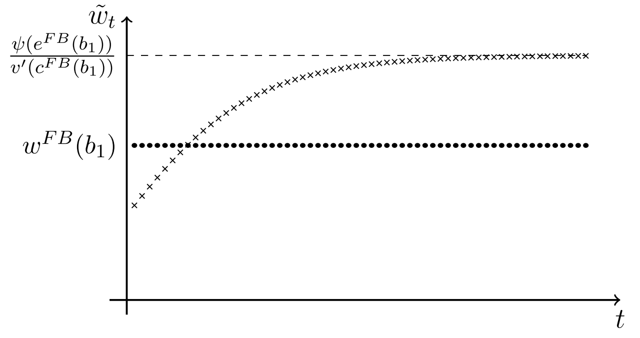

FIGURE 1.—Payments for optimal relational contracts satisfying fastest payments when equation (7) holds (and so the principal obtains her first-best payoff), in the unobservable case (crosses) and observable case (circles).

Condition (9) is more easily satisfied than Condition (7) (the condition for the unobservable consumption case). This follows immediately from showing that

$$w^{\mathrm{FB}}(b_1) < \frac{\psi\big(e^{\mathrm{FB}}(b_1)\big)}{v'\big(c^{\mathrm{FB}}(b_1)\big)}. \tag{10}$$

Here, $w^{\mathrm{FB}}(b_1)$ is the constant payment to the agent in the observed-consumption case, as specified above. On the other hand, $\frac{\psi(e^{\mathrm{FB}}(b_1))}{v'(c^{\mathrm{FB}}(b_1))}$ is the limiting payment for the unobserved-consumption case (assuming that payments satisfy the "fastest payments" condition in equation ($\mathrm{FP}_t^{\mathrm{un}}$)).

The key insight is that, in the observed-consumption case, the principal's constraints (PC_t) are identical in every period, since payments remain constant. In contrast, in the unobserved-consumption case, we saw that they tighten over time. After enough time, the payments in the unobserved-consumption case exceed the constant payments in the observed-consumption case (note that the NPV of payments in both cases is the same), which makes the principal's constraints more difficult to satisfy. Figure 1 illustrates the payments in optimal contracts achieving the first-best payoff for the principal in the unobserved and observed consumption cases.

To derive the inequality (10) formally, observe that by concavity of v, and because $c^{\mathrm{FB}}(b_1) > (1-\delta)b_1$, we have

$$v\big(c^{\mathrm{FB}}(b_1)\big) - v\big((1-\delta)b_1\big) > v'\big(c^{\mathrm{FB}}(b_1)\big)\big(c^{\mathrm{FB}}(b_1) - (1-\delta)b_1\big) = v'\big(c^{\mathrm{FB}}(b_1)\big)w^{\mathrm{FB}}(b_1).$$

The result then follows because the first-best effort and consumption satisfy $v(c^{\mathrm{FB}}(b_1)) - v((1-\delta)b_1) = \psi(e^{\mathrm{FB}}(b_1))$ by Condition 2 of Proposition 3.1.

5.2. *Optimal Contract With Observed Consumption*

Now consider the problem of characterizing an optimal contract when the principal's first-best payoff is *not* attainable. We restrict attention to "fastest payments" as given in Lemma 5.1 and show that this timing of payments is necessary for optimality in Proposition 5.4 below.

Under fastest payments, we can write the problem of finding the principal's optimal contract recursively, with the balance $\tilde{b}_t$ as the state variable for the relationship. Since an optimal contract maximizes the principal's continuation profits, an optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ must solve a sequence of subproblems with value $V(\tilde{b}_t)$ given by

$$V(\tilde{b}_t) = \max_{e_t, b_{t+1}, c_t \in \mathbb{R}_+} \big(e_t - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V(b_{t+1})\big) \tag{11}$$

subject to the agent's indifference condition (8), which can be written as

$$v(c_t) - \psi(e_t) + \frac{\delta}{1-\delta} v\big((1-\delta) b_{t+1}\big) = \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_t\big), \tag{12}$$

and to the principal's constraint (PC_t), which can be written as

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V(b_{t+1}). \tag{13}$$

The left-hand side of (13) can be understood as the date-t payment w_t, which is divided into date-t consumption $c_t \in \mathbb{R}_+$ and savings $\delta b_{t+1} - \tilde{b}_t \in \mathbb{R}$. Nonnegativity of the payment $\delta b_{t+1} - \tilde{b}_t + c_t$ is assured by the equality (12) and the concavity of v. The same equality ensures that, given $\tilde{b}_t$ is strictly positive, optimal c_t and b_{t+1} must be strictly positive also.

We show that any optimal policy for the principal can be characterized as follows.

PROPOSITION 5.3: *An optimal contract exists. Suppose that an optimal contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ *fails to obtain the first-best payoff* $V^{\text{FB}}(b_1)$ (*i.e.*, *equation* (9) *does not hold*). *Then the agent's balance* $\tilde{b}_t$ *and consumption* $\tilde{c}_t$ *decline strictly over time, with* $\tilde{b}_t \to \tilde{b}_\infty$ *for some* $\tilde{b}_\infty > 0$. *Effort* $\tilde{e}_t$ *and the payments* $\tilde{w}_t$ *increase strictly over time. We have* $V(\tilde{b}_t) \to V^{\text{FB}}(\tilde{b}_\infty)$ *as* $t \to \infty$, *and effort and consumption converge to first-best levels for the balance* $\tilde{b}_\infty$.

A heuristic account of the forces behind this result is as follows. When the principal's constraint ((PC_t) or equivalently (13)) binds, effort is suppressed. That is, if the principal could increase effort and credibly increase payments to keep the agent as well off, she would gain by doing so. Also, the principal's value function $V(\cdot)$ is strictly decreasing; intuitively, because a lower balance makes the agent cheaper to compensate to keep him in the agreement. Therefore, for any date t, reducing the balance b_{t+1} increases the principal's continuation payoff $V(b_{t+1})$ and relaxes the principal's date-t constraint (13). It follows that the principal asks the agent to consume earlier than he would like, driving the balance down over time. This continues to a point where, given the revised balance, the contract is close to first best, and so the value of continuing to distort consumption vanishes.

It is worth pointing out here that the dynamics of $V(\tilde{b}_t)$ are determinative of both the dynamics of effort and payments. When there is no self-enforceable first-best contract, $V(\tilde{b}_t)$ strictly increases with t, and moreover, the principal's constraint (13) binds. The latter implies that, for all t, both $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ and $V(\tilde{b}_t) = \tilde{e}_t - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) = \tilde{e}_t$.

A further part of our analysis worth highlighting is a Euler equation

$$1 - \frac{v'\big((1-\delta)\tilde{b}_{t+1}\big)}{v'(\tilde{c}_t)} = \frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})}\left(1 - \frac{v'\big((1-\delta)\tilde{b}_{t+1}\big)}{v'(\tilde{c}_{t+1})}\right), \tag{14}$$

which must hold for an optimal contract at all dates t, and which we use to derive several key properties. This condition is derived (in Lemma A.11) by fixing the contract at and before $t-1$, and from date $t+2$ onwards, and then requiring the contractual variables at t and $t+1$ to be chosen optimally. The equation captures the relationship between a static distortion in effort and a dynamic distortion in consumption. In particular, when the principal's first-best payoff cannot be attained, we are able to show that $\psi'(\tilde{e}_{t+1}) < v'(\tilde{c}_{t+1})$ for all t (reflecting a static (downward) distortion in effort), and correspondingly $(1-\delta)\tilde{b}_{t+1} < \tilde{c}_{t+1} < \tilde{c}_t$ (i.e., consumption strictly decreases over time, which is a dynamic distortion). A trade-off between the static and dynamic distortions should be anticipated, since asking the agent to consume excessively early in the relationship increases the agent's marginal utility of consumption later on, which makes him easier to motivate and permits higher effort and profits at later dates. In turn, this relaxes the principal's credibility constraint (PC_t), permitting higher payments and, therefore, effort also early in the relationship. As $\tilde{b}_t \to \tilde{b}_\infty$, consumption falls to its lower bound, becoming almost constant, so $\frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} \to 1$, which accords with convergence of effort and consumption to first-best levels.

Finally, analogous to Proposition 4.4, we provide a result on the uniqueness of the timing of payments.

PROPOSITION 5.4: *Suppose the principal cannot attain the first-best payoff in a self-enforceable relational contract. Then, in any contract that is optimal for the principal, Condition* (FP_t^{ob}) *holds at all dates.*

The logic of this result is that, if the Condition (FP_t^{ob}) fails, then payments can be made earlier in time, while maintaining the agent constraints (AC_t^{ob}). This induces slack in the principal's constraint (PC_t), which can then be exploited by increasing payments, consumption and effort, increasing profits. As noted for the case of unobservable consumption, such an observation is related to arguments in Ray (2002).

6. LITERATURE REVIEW

There is by now a large literature on relational contracts (see MacLeod (2007), and Malcomson (2015), for a review). Much of it involves settings with exogenous uncertainty, while our environment is deterministic. Our model is similar to the principal-agent model of Bull (1987), where overlapping generations of short-lived agents have concave utility for money. A key difference is that we examine inefficient equilibria, whereas Bull provides only conditions for efficiency.

Though not principal-agent models, other set-ups where agents face a consumption/storage decision, and where there is limited commitment, are studied by Kehoe and Perri (2002), Ligon, Thomas, and Worrall (2000), Wahhaj (2010), and Voena (2015).[6] Storage is public in all these settings, while an important part of our focus has been on the case of hidden savings. Also, these papers feature noise in the environment. While this introduces mutual insurance among agents, it restricts tractability and limits the availability of analytic results. Ábrahám and Laczó (2018) is closely related to these works, but like our paper includes an analysis of hidden savings (the Introduction explained how our analysis differs). In all these papers, when there are public consumption/storage decisions,

[6]All these papers have the feature that agents' outside options are determined through accumulated storage or savings. Other papers with this feature include Thomas and Worrall (1994, Section 4) and Garicano and Rayo (2017).

optimal agreements resolve trade-offs on the level of storage or savings. Notably, Wahhaj provides conditions under which savings are distorted downwards and links these effects to social pressure toward low savings in tribal societies. Our findings for the public savings case are related to this point as they show how reduced savings can enhance the efficiency of the relationship over time. It is worth noting that similar ideas have been suggested in the law literature. Henderson and Spindler (2004) have argued through a range of examples that social norms limiting employee savings tend to make them more dependent on employment relationships, and in turn make them easier to incentivize (see the Appendix in the Online Supplementary Material for further discussion).

Savings in our model can be viewed as a kind of investment in the agent's outside option. Our paper therefore connects to work on outside options in relational contracts more generally. We can compare to settings where outside options are either private but exogenous or endogenous but public. With respect to the former, Halac (2012) considers a model where the principal's outside option is her private information and is exogenous and persistent. Examples of the latter include Englmaier and Fahn (2019) and Malcomson (2021), where initial one-time public investments affect payoffs both inside and outside the agreement (see also Halac (2015), for a related model). Similarly, Fahn, Merlo, and Wamser (2019) consider a setting where the up-front one-shot decision is the capital structure of the firm and this decision influences payoffs when the relational agreement breaks down.

Our work on private savings is related to the literature on moral hazard with private savings where the principal has commitment power; see footnote 1 for references. A major difference is that monitoring of effort is perfect in our model. This contrasts with work on imperfect monitoring where there is a trade-off between incentives for more efficient effort and the additional riskiness of pay. Another crucial difference is that dynamics in our model are driven by limited commitment on both sides of the relationship.

A key goal of the literature on moral hazard with private savings has been to obtain analytic characterizations of optimal contracts. This has been challenging, however, because of the complexity of potential agent deviations involving saving more and shirking. Most of the papers that characterize contracts analytically do so via a first-order approach where complex deviations can at first instance be ignored in the study of a "relaxed optimization program" that addresses only local deviations in effort and consumption (examples include Ábrahám, Koehne, and Pavoni (2011), Edmans et al. (2012), Williams (2015), and Di Tella and Sannikov (2021)). An exception is Mitchell and Zhang (2010) where, similar to our paper, the binding constraints relate to global deviations of shirking and saving. A key simplification in our optimization program is that relevant agent deviations involve pairing public deviations in effort with a choice of consumption that is optimal in light of the agent's reduced pay. This simplification is not available when effort is imperfectly observed. The tractability of our model permits predictions on the optimal contract without strong restrictions on model primitives such as functional form assumptions on preferences, which have been common in the literature.

Similar to our paper, an objective of the moral hazard literature with private savings has been comparison to the case where savings are public. Given full commitment, if the principal observes consumption/savings, then there is no loss in taking pay to equal consumption and having the principal in this sense "save for the agent." It has then been understood since Rogerson (1985) that optimal contracts in repeated moral hazard force the agent to consume more up front than privately optimal; if the agent could save privately and consume later, he would do so. Early consumption is driven here by the optimal provision of incentives for imperfectly monitored effort, rather than the absence of commitment as in our paper.

APPENDIX: PROOFS

PROOF OF PROPOSITION 3.1: We maximize the principal's payoff over feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying

$$\sum_{t\geq 1} \delta^{t-1}\big(v(\tilde{c}_t) - \psi(\tilde{e}_t)\big) \geq \frac{1}{1-\delta} v\big(b_1(1-\delta)\big). \tag{15}$$

Considering this maximization, we may assume that equation (2) holds with equality.

For any feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, let

$$\tilde{c} \equiv (1-\delta)\sum_{t\geq 1}\delta^{t-1}\tilde{c}_t, \qquad \tilde{e} \equiv \psi^{-1}\left((1-\delta)\sum_{t\geq 1}\delta^{t-1}\psi(\tilde{e}_t)\right), \quad \text{and} \quad \tilde{w} \equiv (1-\delta)\sum_{t\geq 1}\delta^{t-1}\tilde{w}_t.$$

It is optimal for the principal to use "stationary" contracts with constant effort, pay, and consumption. Indeed, any feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the inequality in equation (15) can be weakly improved (while satisfying all constraints) by the contract $(\tilde{e}'_t, \tilde{w}'_t, \tilde{c}'_t, \tilde{b}'_t)_{t\geq 1}$ with, for all t, $\tilde{e}'_t = \tilde{e}$, $\tilde{w}'_t = \tilde{w}$, $\tilde{c}'_t = \tilde{c}$ and $\tilde{b}'_t = b_1$. Moreover, using the strict convexity of ψ and strict concavity of v, effort and consumption must be constant in *any* optimal contract.

Now consider the optimal specification of the stationary contract. Because equation (2) holds with equality, $\tilde{w} = \tilde{c} - (1-\delta)b_1$. Also, optimality implies $v(\tilde{c}) - \psi(\tilde{e}) = v(b_1(1-\delta))$. Using this, we may write

$$\tilde{c} = v^{-1}\big(\psi(\tilde{e}) + v\big(b_1(1-\delta)\big)\big). \tag{16}$$

The principal's payoff can then be written as

$$\tilde{e} - \big(\tilde{c} - (1-\delta)b_1\big) = \tilde{e} - v^{-1}\big(\psi(\tilde{e}) + v\big(b_1(1-\delta)\big)\big) + (1-\delta)b_1.$$

This is strictly concave in $\tilde{e}$ and an optimum corresponds to the first-order condition $1 - \frac{\psi'(\tilde{e})}{v'(\tilde{c})} = 0$ (with $\tilde{c}$ given by equation (16)), giving Condition 1. By the assumption that $\psi'(0) = 0$, we have $\tilde{e} > 0$, and hence $\tilde{c} > b_1(1-\delta)$.

To see that the principal's payoff is strictly decreasing in b_1, consider the optimal (stationary) contract specified above, and reduce the initial balance b_1 by $\varepsilon \in (0, b_1)$. Reducing per-period consumption by $\varepsilon(1-\delta)$, leaving payments unchanged, yields a new feasible contract. The agent's balance remains constant at $b_1 - \varepsilon$ and the inequality in equation (15) holds strictly. A strict improvement to the contract then comprises increasing date-1 effort by a small amount (without a violation of equation (15)). *Q.E.D.*

A.1. *Proofs of the Results in Section 4*

PROOF OF PROPOSITION 4.1: **Necessity.** Follows from arguments in the main text.

Sufficiency. We fix some feasible contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the conditions of the proposition and we provide strategies and beliefs that constitute a PBE with outcome $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$.

We specify a PBE as follows. On the principal's side, put $\sigma_t(h_t^P, e_t) = \tilde{w}_t$ if $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$ for all $s \leq t-1$ and $e_t = \tilde{e}_t$, and put $\sigma_t(h_t^P, \tilde{e}_t) = 0$ otherwise. For the agent's strategy, put $\alpha_t(h_t^A) = (\tilde{e}_t, \tilde{c}_t)$ if $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$. Put $\alpha_t(h_t^A) =$

$(0, \max\{0, (1-\delta)b(h_t^A)\}\})$ whenever $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s \leq t-1$. This specification of the agent's strategy will guarantee its sequential optimality at date-t histories where $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s < t$.[7]

Now consider the agent's equilibrium strategy for the remaining possible histories h_t^A, where $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$ for all $s \leq t-1$, and yet $c_s \neq \tilde{c}_s$ for some values $s \leq t-1$. First, in case

$$b(h_t^A) + \sum_{\tau \geq t} \delta^{\tau-t}\tilde{w}_\tau \tag{17}$$

is nonpositive, the agent's payoff is necessarily $-\infty$ and we might as well put $e_t = \tilde{e}_t$ and $c_t = 0$. If (17) is instead strictly positive, then an optimal continuation strategy for the agent, given the principal's strategy, should induce a continuation outcome of the following form. There should be some $t' \geq t$ so that effort is $e_s = \tilde{e}_s$ for all $s \in \{t, t+1, \dots, t'-1\}$, and so that effort is $e_s = 0$ for $s \geq t'$ (we allow $t' = +\infty$, in which case $e_s = \tilde{e}_s$ for all $s \geq t$). Consumption should be specified optimally. In the absence of deviation by the principal, this means that, for all $s \geq t$, $c_s = \max\{0, (1-\delta)(b(h_t^A) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t}\tilde{w}_\tau)\}$.

Note that the existence of an optimal "public deviation" time t' (possibly $+\infty$) follows because, for all t, all histories h_t^A for which there is, as yet, no public deviation,

$$\frac{v\left(\max\left\{0, (1-\delta)\left(b(h_t^A) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t}\tilde{w}_\tau\right)\right\}\right)}{1-\delta} - \sum_{\tau=t}^{t'-1} \delta^{\tau-t}\psi(\tilde{e}_\tau)$$

$$\longrightarrow \frac{v\left((1-\delta)\left(b(h_t^A) + \sum_{\tau=t}^{\infty} \delta^{\tau-t}\tilde{w}_\tau\right)\right)}{1-\delta} - \sum_{\tau=t}^{\infty} \delta^{\tau-t}\psi(\tilde{e}_\tau)$$

as $t' \to \infty$. In determining the agent's continuation strategy at date t and private history h_t^A, we take t' to be the largest value that attains the optimal payoff for the agent (it could be $+\infty$). Hence, the strategy specifies that, at private history h_t^A for the agent, effort is $e_t = \tilde{e}_t$ if $t' > t$ and $e_t = 0$ if $t' = t$, and consumption is $c_t = (1-\delta)(b(h_t^A) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t}\tilde{w}_\tau)$.

Finally, let us specify (degenerate) principal beliefs on the agent's previous consumption choices. Denote believed consumption up to date t by $(\hat{c}_s)_{s=1}^t$. Absent public deviation by date t, $\hat{c}_s = \tilde{c}_s$ at each date $s \leq t$. If instead $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s \leq t-1$, or if $e_t \neq \tilde{e}_t$, let t' be the first date of such a public deviation. If $e_{t'} \neq \tilde{e}_{t'}$ (so the agent is first to publicly deviate),

$$\hat{c}_s = (1-\delta)\left(b_1 + \sum_{\tau=1}^{t'-1} \delta^{\tau-1}\tilde{w}_\tau\right)$$

for all $s \in \{1, \dots, t'-1\}$, while, for all $s \in \{t', \dots, t\}$, $\hat{c}_s = (1-\delta)\hat{b}_s$, where $\hat{b}_s$ are beliefs on the agent's balance determined recursively from the principal's payments and the agent's

[7]This is because, at time t after a public deviation, there are two possibilities. If $b(h_t^A) < 0$, then the agent's payoff is $-\infty$ independently of the strategy he uses, because the feasibility constraint (2) is violated (as he no receives any further payment by the principal). If, instead, $b(h_t^A) \geq 0$, the agent smooths consumption by consuming $(1-\delta)b(h_t^A)$ in all subsequent periods.

believed consumption (i.e., $\hat{b}_s = (\hat{b}_{s-1} + w_{s-1} - \hat{c}_{s-1})/\delta$, with $\hat{b}_1 = b_1$). If $e_{t'} = \tilde{e}_{t'}$ (so the principal is first to publicly deviate), then $\hat{c}_s = \tilde{c}_s$ for all $s \le t'$ and $\hat{c}_s = \max\{0, (1-\delta)\hat{b}_s\}$ for all $s = t'+1, \dots, t$ (again, the values of $\hat{b}_s$ are determined recursively by $\hat{b}_s = (\hat{b}_{s-1} + w_{s-1} - \hat{c}_{s-1})/\delta$, with $\hat{b}_1 = b_1$). These beliefs are consistent with updating of the principal's prior beliefs according to the specified strategy of the agent whenever there is no public evidence the agent's strategy has not been followed.

Now let us verify the sequential optimality of the above strategies, given beliefs. First, at any information set at which the principal has not yet observed a deviation, satisfaction of Condition (PC_t) implies the principal optimally sets $w_t = \tilde{w}_t$. If instead the principal has observed a deviation, she obtains at most zero, since the agent exerts no effort, and hence paying $w_t = 0$ is optimal. Finally, observe that the agent's strategy is constructed to be sequentially optimal. *Q.E.D.*

PROOF OF LEMMA 4.1: Fix a contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\ge1}$, which is optimal, and hence respects the conditions of Proposition 4.1. Then Condition (5) holds, as explained in the main text. We cannot have

$$\frac{v(\bar{c}_{t-1})}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) \tag{18}$$

exceed $\frac{v(b_1(1-\delta))}{1-\delta}$ at any date t, as otherwise the inequality ($\mathrm{AC}_t^{\mathrm{un}}$) fails.

Finally, suppose that (18) is strictly less than $\frac{v(b_1(1-\delta))}{1-\delta}$ at some increasing sequence of dates $(t_n)_{n=1}^N$, where N may be finite or infinite. For each n, there is $\varepsilon_n > 0$ such that

$$\frac{1}{1-\delta} v\big(\bar{c}_{t_n-1} + \delta^{t_n-2}\varepsilon_n(1-\delta)\big) - \sum_{s=1}^{t_n-1} \delta^{s-1}\psi(\tilde{e}_s) = \frac{v\big(b_1(1-\delta)\big)}{1-\delta}.$$

Increase $\tilde{w}_{t_n-1}$ by ε_n, and reduce $\tilde{w}_{t_n}$ by $\frac{\varepsilon_n}{\delta}$; note that this leads to a change in $\bar{c}_{t_n-1}$, but does not affect $\bar{c}_t$ for $t \ne t_n$. Making the adjustment for each n yields a contract for which Condition ($\mathrm{FP}_t^{\mathrm{un}}$) holds at all t. Because ψ is nonnegative, $\bar{c}_t$ is a nondecreasing sequence, and hence all payments in the new contract are nonnegative. Hence, the new contract is feasible, and the agent's constraints ($\mathrm{AC}_t^{\mathrm{un}}$) are satisfied. Also, the principal's constraints (PC_t) are satisfied (moving payments forward only relaxes the principal's constraints). *Q.E.D.*

PROOF OF LEMMA 4.2: Observe from Condition ($\mathrm{FP}_t^{\mathrm{un}}$) evaluated at consecutive dates, we have

$$v\big(\bar{c}_{t-1} + (1-\delta)\delta^{t-1}\tilde{w}_t\big) - v(\bar{c}_{t-1}) = (1-\delta)\delta^{t-1}\psi(\tilde{e}_t).$$

By the fundamental theorem of calculus, we have

$$\int_0^{\tilde{w}_t} v'\big(\bar{c}_{t-1} + (1-\delta)\delta^{t-1}x\big)\,\mathrm{d}x = \psi(\tilde{e}_t),$$

and hence $k\tilde{w}_t = \psi(\tilde{e}_t)$ for some $k \in (v'(\bar{c}_t), v'(\bar{c}_{t-1}))$, which proves the result. *Q.E.D.*

PROOF OF LEMMA 4.3: **Proof that** $\tilde{e}_{t^*+1} \le \tilde{e}_{t^*}$. Suppose, for the sake of contradiction, that $\tilde{e}_{t^*+1} > \tilde{e}_{t^*}$. We can choose a new contract with efforts $(\tilde{e}'_t)_{t\ge1}$, and payments $(\tilde{w}'_t)_{t\ge1}$

chosen to satisfy equation ($\mathrm{FP}_t^{\mathrm{un}}$), such that they coincide with the original policy except in periods t^* and t^*+1. In these periods, $\tilde{e}'_{t^*}$ and $\tilde{e}'_{t^*+1}$ are such that $\tilde{e}_{t^*} < \tilde{e}'_{t^*} \leq \tilde{e}'_{t^*+1} < \tilde{e}_{t^*+1}$ and

$$\psi(\tilde{e}'_{t^*}) + \delta\psi(\tilde{e}'_{t^*+1}) = \psi(\tilde{e}_{t^*}) + \delta\psi(\tilde{e}_{t^*+1}),$$

which implies (by convexity of ψ) that $\tilde{e}'_{t^*} + \delta\tilde{e}'_{t^*+1} > \tilde{e}_{t^*} + \delta\tilde{e}_{t^*+1}$. We then have also that $\tilde{w}_{t^*} < \tilde{w}'_{t^*}$ and $\tilde{w}'_{t^*} + \delta\tilde{w}'_{t^*+1} = \tilde{w}_{t^*} + \delta\tilde{w}_{t^*+1}$ (since the NPV of payments does not change, equilibrium consumption does not change in any period t; so the balance at date t^*+1 is larger under the new contract). Provided the changes are small, the principal's constraint (PC_t) at t^* remains satisfied. The above observations imply $\tilde{w}'_{t^*+1} < \tilde{w}_{t^*+1}$, so the principal's constraint is relaxed at date t^*+1. Since the NPV of output goes up, the principal's constraint is relaxed at all periods before t^*.[8] The contract after date t^*+1 is unaffected. The modified contract is thus self-enforceable, and it is strictly more profitable than the original, establishing a contradiction.

Proof that $\tilde{e}_{t^*-1} \leq \tilde{e}_{t^*}$. Analogous and omitted. *Q.E.D.*

PROOF OF PROPOSITIONS 4.2 AND 4.3: The proof of Proposition 4.3 is divided into nine lemmas. The proof of Proposition 4.2 is provided in the process, in Lemma A.7. Throughout, we restrict attention to payments determined under the restriction to "fastest payments," that is, satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$).

The following result provides a bound on effort in an optimal contract.

LEMMA A.1: *In an optimal contract,* $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ *for all* t.

PROOF: Take a contract satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$) for all t, and let t^* be the first date at which $\psi'(\tilde{e}_{t^*}) > v'(\bar{c}_\infty)$. We can adjust such a contract by reducing date t^* effort by some $\eta \in (0, \tilde{e}_{t^*})$ (holding effort at other dates fixed). This determines a new contract, with adjusted consumption and payments, again satisfying ($\mathrm{FP}_t^{\mathrm{un}}$) for all t. Let us index the revised effort policy by the date-t^* adjustment η, writing $\tilde{e}_t(\eta)$ for all t. Correspondingly, write

$$\bar{c}_\infty(\eta) \equiv (1-\delta)\left(b_1 + \sum_{s\geq 1}\delta^{s-1}\tilde{w}_s(\eta)\right),$$

where $(\tilde{w}_s(\eta))_{s\geq 1}$ are the payments determined from the adjusted effort policy. Then

$$\frac{v(\bar{c}_\infty(0)) - v(\bar{c}_\infty(\eta))}{1-\delta} = \delta^{t^*-1}\big(\psi(\tilde{e}_{t^*}(0)) - \psi(\tilde{e}_{t^*}(\eta))\big).$$

Differentiating with respect to η,

$$\frac{\bar{c}'_\infty(\eta)}{1-\delta} = \frac{-\delta^{t^*-1}\psi'(\tilde{e}_{t^*}(\eta))}{v'(\bar{c}_\infty(\eta))}.$$

[8]Note that, for the new contract, the principal's constraint at any date $\hat{t}$ may be written as $\sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}'_t \leq \sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}'_t$. For $\hat{t} < t^*$, this inequality is satisfied strictly since $\sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}'_t = \sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}_t$, while $\sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}'_t > \sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}_t$.

This expression coincides with the derivative of the NPV of payments to the agent with respect to η. The derivative of the principal's profits is therefore

$$-\delta^{t^*-1} + \frac{\delta^{t^*-1}\psi'(\tilde{e}_{t^*}(\eta))}{v'(\bar{c}_\infty(\eta))},$$

which is strictly positive for $\eta \in [0, \bar{\eta})$, with $\bar{\eta}$ satisfying $\psi'(\tilde{e}_{t^*}(\bar{\eta})) = v'(\bar{c}_\infty(\bar{\eta}))$. The effect on profit from reducing date t^* effort by $\bar{\eta}$ is therefore to strictly increase it. Note (from ($\mathrm{FP}_t^{\mathrm{un}}$)) that payments $\tilde{w}_t(\bar{\eta})$ are reduced for all $t \geq t^*$, with the implication that the principal's constraints (PC_t) are relaxed for all t. Hence, the new contract is self-enforceable. Note then that $\psi'(\tilde{e}_t(\bar{\eta})) < v'(\bar{c}_\infty(\bar{\eta}))$ for all $t < t^*$. We can therefore continue iteratively, by proceeding to the next date $t > t^*$ at which $v'(\bar{c}_\infty(\bar{\eta})) < \psi'(\tilde{e}_t(\bar{\eta}))$, if any, and reducing effort precisely as for at t^*. Proceeding sequentially, we obtain a self-enforceable contract for which $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ at all dates t, and which is strictly more profitable than the original. *Q.E.D.*

This observation is used to prove existence of an optimal contract.

LEMMA A.2: *An optimal relational contract exists.*

PROOF: As we already observed, assuming "fastest payments," the relational contract is determined solely by the effort policy $(\tilde{e}_t)_{t\geq 1}$. Hence, the payoff obtained by the principal can be written

$$W((\tilde{e}_t)_{t=1}^\infty) = \sum_{t=1}^\infty \delta^{t-1}\tilde{e}_t - \sum_{t=1}^\infty \delta^{t-1}\tilde{w}_t,$$

where each $\tilde{w}_t$ is recursively obtained from ($\mathrm{FP}_t^{\mathrm{un}}$). Note that, from Lemma A.1, we can restrict attention to effort policies in $[0, z(v'(b_1(1-\delta)))]^\infty$, where z denotes the inverse of ψ'.

Now, let $W^{\sup}$ be the supremum of $W(\cdot)$ over effort policies $(\tilde{e}_t)_{t\geq 1}$ in $[0, z(v'(b_1(1-\delta)))]^\infty$ for which the implied contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ (i.e., the one implied by ($\mathrm{FP}_t^{\mathrm{un}}$)) satisfies the principal's constraints (PC_t) (such contracts are feasible and satisfy all the conditions of Proposition 4.1). Note the set is nonempty; for instance, because effort constant at zero is in the set.

Consider then a sequence of policies $((\tilde{e}_t^n)_{t=1}^\infty)_{n=1}^\infty$ in $[0, z(v'(b_1(1-\delta)))]^\infty$ and with $W((\tilde{e}_t^n)_{t=1}^\infty) > W^{\sup} - 1/n$ for all n, and for which the contract defined by each effort policy (using ($\mathrm{FP}_t^{\mathrm{un}}$)) satisfies the principal's constraints (PC_t). There then exists a sequence $(\tilde{e}_t^\infty)_{t\geq 1} \in [0, z(v'(b_1(1-\delta)))]^\infty$ and a subsequence $((\tilde{e}_t^{n_k})_{t\geq 1})_{k\geq 1}$ convergent pointwise to $(\tilde{e}_t^\infty)_{t\geq 1}$. Let $(\tilde{w}_t^\infty)_{t\geq 1}$ be the payments corresponding to $(\tilde{e}_t^\infty)_{t\geq 1}$ determined using ($\mathrm{FP}_t^{\mathrm{un}}$).

Note that, for any policy $(\tilde{e}_t)_{t\geq 1}$ in $[0, z(v'(b_1(1-\delta)))]^\infty$, using that the payments $(\tilde{w}_t)_{t\geq 1}$ determined by ($\mathrm{FP}_t^{\mathrm{un}}$) are bounded, as well as continuity of v,

$$\frac{v\left((1-\delta)b_1 + (1-\delta)\sum_{s=1}^\infty \delta^{s-1}\tilde{w}_s\right)}{1-\delta} - \sum_{t=1}^\infty \delta^{t-1}\psi(\tilde{e}_t) = \frac{v(b_1(1-\delta))}{1-\delta}. \tag{19}$$

Notice also that $\sum_{t=1}^\infty \delta^{t-1}\psi(\tilde{e}_t^{n_k}) \to \sum_{t=1}^\infty \delta^{t-1}\psi(\tilde{e}_t^\infty)$ as $k \to \infty$ (by continuity of ψ and discounting). Therefore, we have (by equation (19), using that v is strictly increasing) that

$\sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t^{n_k} \to \sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t^{\infty}$. Since, also, $\sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t^{n_k} \to \sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t^{\infty}$, we can conclude that $W((\tilde{e}_t^{\infty})_{t\geq 1}) = W^{\sup}$.

Our result will then follow if we can show that the contract defined by $(\tilde{e}_t^{\infty})_{t\geq 1}$ satisfies the principal's constraints (PC_t). Suppose with a view to contradiction that there is some t^* at which the principal's constraint does not hold, and so $\tilde{w}_{t^*}^{\infty} > \sum_{s=t^*+1}^{\infty} \delta^{s-t^*} (\tilde{e}_s^{\infty} - \tilde{w}_s^{\infty})$. It is easily verified, from ($\mathrm{FP}_t^{\mathrm{un}}$) and the pointwise convergence of $(\tilde{e}_t^{n_k})_{t\geq 1}$ to $(\tilde{e}_t^{\infty})_{t\geq 1}$ and $(\tilde{w}_t^{n_k})_{t\geq 1}$ to $(\tilde{w}_t^{\infty})_{t\geq 1}$, that for large enough k, $\tilde{w}_{t^*}^{n_k} > \sum_{s=t^*+1}^{\infty} \delta^{s-t^{n_k}} (\tilde{e}_s^{n_k} - \tilde{w}_s^{n_k})$, contradicting that the contract determined by $(\tilde{e}_t^{n_k})_{t\geq 1}$ satisfies the principal's constraints (PC_t). *Q.E.D.*

We next establish the following regarding the nondegeneracy of optimal contracts.

LEMMA A.3: *The principal obtains a strictly positive payoff in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Moreover, $\tilde{e}_t$ and $\tilde{w}_t$ are strictly positive at all dates t.*

PROOF: Consider effort set constant to some level $\tilde{e} > 0$. Let $g(\tilde{e}) = \sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t$ be the NPV of payments that must be made to the agent when satisfying the indifference conditions ($\mathrm{FP}_t^{\mathrm{un}}$), given that effort is constant at $\tilde{e}$. This satisfies

$$v\big((1-\delta)b_1 + (1-\delta)g(\tilde{e})\big) = \psi(\tilde{e}) + v\big((1-\delta)b_1\big).$$

Differentiating with respect to $\tilde{e}$ yields

$$g'(\tilde{e}) = \frac{\psi'(\tilde{e})}{(1-\delta)v'\big((1-\delta)b_1 + (1-\delta)g(\tilde{e})\big)}.$$

Since the principal's payoff is $\frac{1}{1-\delta}\tilde{e} - g(\tilde{e})$, and since $\psi'(0) = 0$, it follows that the principal's payoff is strictly positive for small positive $\tilde{e}$. Moreover, by Lemma 4.2, payments determined by the conditions ($\mathrm{FP}_t^{\mathrm{un}}$) rise over time approaching

$$\frac{\psi(\tilde{e})}{v'\big((1-\delta)b_1 + (1-\delta)g(\tilde{e})\big)}$$

which is $o(\tilde{e})$ as $\tilde{e} \to 0$ (i.e., vanishes much faster than $\tilde{e}$). Hence, when $\tilde{e}$ is small enough, all the principal constraints (PC_t) are satisfied. Hence, the contract determined from specifying constant effort $\tilde{e}$, for small $\tilde{e}$, is self-enforceable and generates a strictly positive payoff.

Now we show that, in an optimal contract, effort is strictly positive in every period. Suppose that payments $\tilde{w}_t$ are determined from effort using ($\mathrm{FP}_t^{\mathrm{un}}$). First, note that the principal's continuation profits $\sum_{s=t}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s)$ must be strictly positive at all dates. Otherwise, this expression is zero at some date t, and so $\tilde{w}_{t-1} = 0$. Condition ($\mathrm{FP}_t^{\mathrm{un}}$) then implies that $\tilde{e}_{t-1} = 0$. Iterating backwards, the optimal profit is zero in contradiction with the previous claim. Suppose then that effort is zero at some date, and consider a date t such that effort is zero at this date but strictly positive at the subsequent date. Then $\tilde{w}_t = 0$ and so the principal's constraint (PC_t) is slack at date t. However, this contradicts Lemma 4.3. The restriction to "fastest payments" then implies that all payments are also strictly positive, as stated in the lemma. *Q.E.D.*

We now establish an important property of relational contracts: they become (approximately) stationary in the long run.

LEMMA A.4: *Suppose that* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *is an optimal relational contract satisfying* (FP_t^{un}). *Then there exists an effort/payment pair* $(\tilde{e}_\infty, \tilde{w}_\infty)$ *such that* $\lim_{t\to\infty}(\tilde{e}_t, \tilde{w}_t) = (\tilde{e}_\infty, \tilde{w}_\infty)$.

PROOF: **Step 0.** We observe first that, for an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfying (FP_t^{un}),

$$\lim_{t\to\infty}\left(\tilde{w}_t - \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_\infty)}\right) = 0.$$

This follows from Lemma 4.2, recalling that $(\tilde{e}_t)_{t\geq 1}$ is bounded.

Step 1. Define $\bar{e} \equiv \limsup_{t\to\infty} \tilde{e}_t$, which we know from Lemma A.1 is no greater than $z(v'(\bar{c}_\infty))$, where z is the inverse of ψ'. We now show that, for any $e \in [0, \bar{e}]$,

$$\frac{\psi(e)}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(e - \frac{\psi(e)}{v'(\bar{c}_\infty)}\right). \tag{20}$$

Note, by convexity of ψ, if the inequality (20) is satisfied at $\bar{e}$, then it is satisfied for all $e \in [0, \bar{e}]$.

Assume now for the sake of contradiction that the inequality (20) is not satisfied for some $e \in [0, \bar{e}]$. Then we must have

$$\frac{\psi(\bar{e})}{v'(\bar{c}_\infty)} > \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}\right). \tag{21}$$

Observe then that there is a sequence $(\varepsilon_t)_{t=1}^\infty$ convergent to zero such that, for all $t \geq 1$,

$$\tilde{e}_t - \tilde{w}_t \leq \bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)} + \varepsilon_t.$$

This follows because $\tilde{w}_t - \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_\infty)} \to 0$ as $t \to \infty$ (by Step 0), because $e - \frac{\psi(e)}{v'(\bar{c}_\infty)}$ increases over effort levels e in $[0, \bar{e}]$ (since $\psi'(\bar{e}) \leq v'(\bar{c}_\infty)$ by Lemma A.1), and by definition of $\bar{e}$ as $\limsup_{t\to\infty} \tilde{e}_t$.

We therefore have that

$$\limsup_{t\to\infty} \sum_{s\geq t+1} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s) \leq \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}\right) < \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)},$$

where the last inequality holds by equation (21). However, Step 0 implies that the superior limit of payments to the agent must be $\frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}$, which implies that the principal's constraint (PC_t) is not satisfied at some time t. This is a contradiction.

Step 2. We complete the proof by showing that $\liminf_{t\to\infty} \tilde{e}_t = \bar{e}$. This is immediate if $\bar{e} = 0$, so assume $\bar{e} > 0$. Assume, for the sake of contradiction, that $\liminf_{t\to\infty} \tilde{e}_t < \bar{e}$. In this case, there exists some $t' > 1$ such that $\tilde{e}_{t'} < \min\{\bar{e}, \tilde{e}_{t'+1}\}$.

Step 2a. We have

$$\tilde{w}_{t'} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}\right). \tag{22}$$

This follows because (i) $\tilde{w}_{t'} \leq \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)}$ by Lemma 4.2 and the assumption that payments satisfy condition (FP_t^{un}); (ii) $\frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}(\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)})$, by assumption that $\tilde{e}_{t'} < \bar{e}$ and by

Step 1; and (iii) $\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \le e_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}$ because $z(v'(\bar{c}_\infty)) \ge \tilde{e}_{t'+1} > \tilde{e}_{t'}$ (recall that the inequality $z(v'(\bar{c}_\infty)) \ge \tilde{e}_{t'+1}$ is established in Lemma A.1).

Step 2b. We now obtain a contradiction to Lemma 4.3 by showing that the principal's constraint (PC_t) is slack at t'. Note first that, for any $t \ge 1$, we have

$$\tilde{w}_{t+1} - \tilde{w}_t = \frac{\bar{c}_{t+1} - \bar{c}_t}{\delta^t(1-\delta)} - \frac{\bar{c}_t - \bar{c}_{t-1}}{\delta^{t-1}(1-\delta)} \ge \frac{v(\bar{c}_{t+1}) - v(\bar{c}_t)}{\delta^t(1-\delta)v'(\bar{c}_t)} - \frac{v(\bar{c}_t) - v(\bar{c}_{t-1})}{\delta^{t-1}(1-\delta)v'(\bar{c}_t)}$$
$$= \frac{\psi(\tilde{e}_{t+1}) - \psi(\tilde{e}_t)}{v'(\bar{c}_t)},$$

where we used that v is concave and equation ($\mathrm{FP}_t^{\mathrm{un}}$). Hence, $\tilde{e}_{t+1} > \tilde{e}_t$ implies $\tilde{w}_{t+1} > \tilde{w}_t$.

Since t' was chosen so that $\tilde{e}_{t'+1} > \tilde{e}_{t'}$, we have $\tilde{w}_{t'+1} > \tilde{w}_{t'}$. Hence,

$$\tilde{w}_{t'} < (1-\delta)\tilde{w}_{t'} + \delta\tilde{w}_{t'+1} \le \delta\left(\tilde{e}_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}\right) + \delta \sum_{s\ge t'+2} \delta^{s-t'-1}(\tilde{e}_s - \tilde{w}_s)$$
$$\le \sum_{s\ge t'+1} \delta^{s-t'}(\tilde{e}_s - \tilde{w}_s),$$

where the second inequality uses (i) equation (22) from Step 2a, and (ii) the principal's constraint (PC_t) in period $t'+1$. The third inequality uses that $\tilde{w}_{t'+1} \le \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}$, which follows from Lemma 4.2. *Q.E.D.*

The following lemma determines that, in an optimal contract, effort is weakly decreasing.

LEMMA A.5: *In an optimal contract, the effort policy $(\tilde{e}_t)_{t\ge 1}$ is a weakly decreasing sequence. Therefore, for all t, $\tilde{e}_t \ge \tilde{e}_\infty \equiv \lim_{s\to\infty} \tilde{e}_s$.*

PROOF: By Lemma A.4, $(\tilde{e}_t)_{t=1}^\infty$ is a convergent sequence, so using the notation in its proof, we have $\tilde{e}_\infty = \bar{e}$. Step 2 in the proof of Lemma A.4 proves that there is no time t' such that $\tilde{e}_{t'} < \min\{\bar{e}, \tilde{e}_{t'+1}\}$. Hence, there is no t' such that $\tilde{e}_{t'} < \tilde{e}_\infty$.

Now suppose, for the sake of contradiction, that $(\tilde{e}_t)_{t=1}^\infty$ is not a weakly decreasing sequence. Thus, there exists a date t' where $\max_{t>t'} \tilde{e}_t > \tilde{e}_{t'}$ (the maximum exists by the first part of this proof, and because $\lim_{t\to\infty} \tilde{e}_t = \tilde{e}_\infty$ by Lemma A.4). Let $t^*(t')$ be the smallest value $t > t'$ where the maximum is attained, that is, $\tilde{e}_{t^*(t')} = \max_{t>t'} \tilde{e}_t$.

For any $s > t^*(t')$,

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{t^*(t')})} \ge \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{s-1})} \ge \tilde{e}_s - \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})}$$
$$> \tilde{e}_s - \tilde{w}_s. \tag{23}$$

The first inequality follows from Lemma 4.2; the second inequality follows because $\bar{c}_{s-1} \ge \bar{c}_{t^*(t')}$. The third inequality follows because $e - \frac{\psi(e)}{v'(\bar{c}_{s-1})}$ is increasing in e over $[0, z(v'(\bar{c}_\infty))]$, and because $\tilde{e}_s \le \tilde{e}_{t^*(t')}$ for $s > t^*(t')$ by definition of $t^*(t')$. The fourth inequality follows because $\tilde{w}_s > \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})}$ by Lemma 4.2.

Equation (23) implies that

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > (1-\delta) \sum_{s \geq t^*(t')+1} \delta^{s-t^*(t')-1}(\tilde{e}_s - \tilde{w}_s),$$

so that

$$\begin{aligned}
\sum_{s \geq t^*(t')} \delta^{s-t^*(t')}(\tilde{e}_s - \tilde{w}_s) &= \tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} + \delta \sum_{s \geq t^*(t')+1} \delta^{s-t^*(t')-1}(\tilde{e}_s - \tilde{w}_s) \\
&> (1-\delta) \sum_{s \geq t^*(t')+1} \delta^{s-t^*(t')-1}(\tilde{e}_s - \tilde{w}_s) + \delta \sum_{s \geq t^*(t')+1} \delta^{s-t^*(t')-1}(\tilde{e}_s - \tilde{w}_s) \\
&= \sum_{s \geq t^*(t')+1} \delta^{s-t^*(t')-1}(\tilde{e}_s - \tilde{w}_s). \qquad (24)
\end{aligned}$$

Recall from Lemma 4.3 that the principal's constraint must hold with equality at $t^*(t') - 1$ (since $\tilde{e}_{t^*(t')} > \tilde{e}_{t^*(t')-1}$ by the definition of $t^*(t')$). The inequality (24) then implies (given satisfaction of the principal's constraint (PC$_t$)) that $\tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')}$. But then, recalling Lemma 4.2, we have

$$\frac{\psi(\tilde{e}_{t^*(t')-1})}{v'(\bar{c}_{t^*(t')-1})} > \tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')} > \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{t^*(t')-1})}.$$

Hence, $\tilde{e}_{t^*(t')-1} > \tilde{e}_{t^*(t')}$, contradicting the definition of $t^*(t')$. *Q.E.D.*

Having shown that the effort is weakly decreasing in an optimal relational contract (Lemma A.5), we now show that it is strictly decreasing when the principal's constraint holds with equality.

LEMMA A.6: *If the principal's constraint* (PC$_t$) *holds with equality at some date* t^*, *then* $\tilde{e}_{t^*} > \tilde{e}_{t^*+1}$. *Hence, by Lemma* 4.3, *the principal's constraint also holds with equality at* $t^* + 1$.

PROOF: The same arguments we used in Lemma A.5 to establish the inequalities in (23) imply that $\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > t^* + 1$. In turn, this means that, if the principal's constraint (PC$_t$) holds with equality at t^*, then $\tilde{w}_{t^*} > \tilde{w}_{t^*+1}$. Indeed, because the principal's constraint holds with equality at t^*,

$$\begin{aligned}
\tilde{w}_{t^*} &= \delta\left(\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} + \delta \sum_{s \geq t^*+2} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\right) \\
&> \delta\left((1-\delta) \sum_{s \geq t^*+2} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s) + \delta \sum_{s \geq t^*+2} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\right) \\
&= \sum_{s \geq t^*+2} \delta^{s-t^*-1}(\tilde{e}_s - \tilde{w}_s) \\
&\geq \tilde{w}_{t^*+1}.
\end{aligned}$$

The final inequality follows from the principal's constraint (PC$_t$) at date $t^* + 1$. Using Lemma 4.2, we have $\tilde{e}_{t^*+1} < \tilde{e}_{t^*}$. *Q.E.D.*

Lemma A.6 implies that, given payments satisfy condition ($\mathrm{FP}_t^{\mathrm{un}}$), if the principal's constraint (PC_t) holds with equality at some date, then effort is strictly decreasing forever after (and the principal's constraints (PC_t) hold with equality forever after). Our next goal is therefore to establish when the principal attains the first-best payoff. Also, when the first-best payoff is unattainable, establish that there is necessarily a date at which the principal's constraint is satisfied with equality.

LEMMA A.7: *The principal attains her first-best payoff if and only if Condition* (7) *holds. If this is not satisfied, there is a time $t^* \in \mathbb{N}$ such that the principal's constraint is slack if and only if $t < t^*$. Hence, effort is constant up to date $t^* - 1$ and strictly decreases from date t^*.*

PROOF: Consider payments satisfying ($\mathrm{FP}_t^{\mathrm{un}}$), for all t, determined given first-best effort. By Lemma 4.2, payments rise over time to $\frac{\psi(e^{\mathrm{FB}}(b_1))}{v'(c^{\mathrm{FB}}(b_1))}$. Per-period profits fall over time to $e^{\mathrm{FB}}(b_1) - \frac{\psi(e^{\mathrm{FB}}(b_1))}{v'(c^{\mathrm{FB}}(b_1))}$. So, Condition (7) is necessary and sufficient for implementation of the first-best.

Assume now that Condition (7) fails, and fix an optimal contract that is not first-best. We want to show that the principal's constraint (PC_t) holds with equality at some t^*. This requires ruling out that it holds as a strict inequality at all dates, so suppose for a contradiction that it does. Effort is then constant over all periods (by Lemma 4.3), but not first-best. Letting $\tilde{e}_\infty$ be the constant effort level and $\bar{c}_\infty$ equilibrium consumption, Proposition 3.1 then implies that $v'(\bar{c}_\infty) \neq \psi'(\tilde{e}_\infty)$. By Lemma A.1, we have $v'(\bar{c}_\infty) > \psi'(\tilde{e}_\infty)$. By Lemma A.3, we have $\tilde{e}_\infty > 0$.

Note that $\tilde{w}_t$ increases over time to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$ (from Lemma 4.2). We claim then that

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right). \tag{25}$$

If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} > \frac{\delta}{1-\delta}(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)})$, the principal's constraint is violated for large enough t. If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} < \frac{\delta}{1-\delta}(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)})$, we have $\tilde{w}_t$ remains bounded below $\sum_{s=t+1}^{\infty}\delta^{s-t}(\tilde{e}_\infty - \tilde{w}_t)$. Without violating ($\mathrm{PC}_t$), effort can be increased by a small constant amount across all periods (with payments adjusted to satisfy ($\mathrm{FP}_t^{\mathrm{un}}$)). This increases profits. Note then that Condition (25) can be written as $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \delta\tilde{e}_\infty$. Because ψ is strictly convex, we have $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} > \delta$.

Now consider a contract with effort increased at date 1 by $\varepsilon > 0$ and reduced from date 2 onwards by $\kappa(\varepsilon) > 0$ to be determined. Payments are adjusted to satisfy Condition ($\mathrm{FP}_t^{\mathrm{un}}$).

We let $\bar{c}_\infty(\varepsilon, \kappa(\varepsilon))$ denote equilibrium consumption under the new plan (naturally, $\bar{c}_\infty(0,0)$ is consumption under the original plan). The new consumption satisfies

$$\begin{aligned}&\frac{1}{1-\delta}v\big(\bar{c}_\infty\big(\varepsilon, \kappa(\varepsilon)\big)\big) - \frac{1}{1-\delta}v\big(\bar{c}_\infty(0,0)\big)\\ &\quad = \psi(\tilde{e}_\infty + \varepsilon) - \psi(\tilde{e}_\infty) - \frac{\delta}{1-\delta}\big(\psi(\tilde{e}_\infty) - \psi\big(\tilde{e}_\infty - \kappa(\varepsilon)\big)\big).\end{aligned}$$

We aim to choose $\kappa(\varepsilon)$ so that the principal's constraint (PC_t) is just satisfied. Define

$$f(\varepsilon, k) \equiv \frac{\psi(\tilde{e}_\infty - k)}{v'\big(\bar{c}_\infty(\varepsilon, k)\big)} - \delta(\tilde{e}_\infty - k). \tag{26}$$

We then define $\kappa(\varepsilon)$ by $f(\varepsilon, \kappa(\varepsilon)) = 0$ for positive ε in a neighborhood of 0. We will use the implicit function theorem to show that such a local solution $\kappa(\varepsilon)$ exists.

To apply the implicit function theorem, note that $f(\varepsilon, k)$ is continuously differentiable in a neighborhood of $(\varepsilon, k) = (0, 0)$. The derivative of $f(\varepsilon, k)$ with respect to k, evaluated at $(\varepsilon, k) = (0, 0)$, is

$$f_2(0,0) = \delta - \frac{\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)} + v''\big(\bar{c}_\infty(0,0)\big)\left(\frac{\delta\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)^3}\right)\psi(\tilde{e}_\infty).$$

This is strictly negative, using that $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$. The derivative $f(\varepsilon, k)$ instead with respect to ε, evaluated at $(\varepsilon, k) = (0, 0)$, is

$$f_1(0,0) = -v''\big(\bar{c}_\infty(0,0)\big)\left(\frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)^3}\right)\psi(\tilde{e}_\infty).$$

The implicit function theorem then gives us that κ is locally well-defined by $f(\varepsilon, \kappa(\varepsilon)) = 0$ on some interval around 0, unique, and continuously differentiable, with derivative approaching

$$\kappa'(0) = -\frac{f_1(0,0)}{f_2(0,0)} = \frac{v''\big(\bar{c}_\infty(0,0)\big)\left(\frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)^3}\right)\psi(\tilde{e}_\infty)}{\delta - \frac{\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)} + v''\big(\bar{c}_\infty(0,0)\big)\left(\frac{\delta\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)^3}\right)\psi(\tilde{e}_\infty)} < \frac{1-\delta}{\delta} \tag{27}$$

as $\varepsilon \to 0$ (the strict inequality follows because $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$).

For small enough ε, the adjusted contract satisfies the principal's constraints (PC_t). Indeed, when ε is small, the constraint (PC_t) remains slack at date $t = 1$. For all other dates, the constraint (PC_t) holds because $f(\varepsilon, \kappa(\varepsilon)) = 0$, and by Lemma 4.2.

For small enough ε, the principal's profits strictly increase. The NPV of effort increases by

$$\varepsilon - \frac{\delta}{1-\delta}\kappa(\varepsilon) = \left(1 - \frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon + o(\varepsilon)$$

($o(\varepsilon)$ represents terms that vanish faster than ε as $\varepsilon \to 0$). From inequality (27), we have $1 - \frac{\delta}{1-\delta}\kappa'(0) > 0$, and so the increase in effort is strictly positive for ε small enough. Using that payments continue to satisfy Condition ($\mathrm{FP}_t^{\mathrm{un}}$), a marginal increase in the NPV of effort is compensated by an increase in the NPV of payments to the agent by $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}$. Therefore, the principal's payoff under the new contract increases by

$$\left(1 - \frac{\psi'(\tilde{e}_\infty)}{v'\big(\bar{c}_\infty(0,0)\big)}\right)\left(1 - \frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon + o(\varepsilon),$$

which is strictly positive for small enough ε, recalling that $v'(\bar{c}_\infty(0,0)) > \psi'(\tilde{e}_\infty)$. *Q.E.D.*

We have established that, when the principal cannot attain the first-best payoff, effort is eventually strictly decreasing to a value $\tilde{e}_\infty$ (identified in the proposition). We now show that $\tilde{e}_\infty > 0$, which requires only ruling out $\tilde{e}_\infty = 0$.

LEMMA A.8: *Suppose the principal cannot attain the first-best payoff. In any optimal contract, the limiting value of effort $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ is strictly positive.*

PROOF: Consider any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ that does not achieve the principal's first-best payoff, and suppose for a contradiction that $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ is equal to zero.

Because $\psi'(0) = 0$, there exists a value $\hat{e} \in (0, \tilde{e}_1)$ satisfying $\frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} \leq \delta \hat{e}$. Note that $\lim_{t\to\infty}(\tilde{e}_t - \tilde{w}_t) = 0$. Hence, there is T satisfying that (i) $\frac{\delta^{T-1}}{1-\delta}\psi(\hat{e}) < \psi(\tilde{e}_1) - \psi(\hat{e})$, (ii) $\tilde{e}_t < \hat{e}$ for all $t \geq T$, and (iii) $\tilde{e}_t - \tilde{w}_t < \hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}$ for all $t \geq T$.

Now, let $m \equiv \sum_{t=T}^\infty \delta^{t-1}(\psi(\hat{e}) - \psi(\tilde{e}_t)) > 0$, and define a new contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$ by specifying $\tilde{e}'_1$ to satisfy $\psi(\tilde{e}'_1) = \psi(\tilde{e}_1) - m$, $\tilde{e}'_t = \tilde{e}_t$ for all $t \in \{2, \dots, T-1\}$, as well as $\tilde{e}'_t = \hat{e}$ for all $t \geq T$. Let effort determine the other variables, by satisfaction of Condition (FP_t^{un}).

Note we have $\sum_{t=1}^\infty \delta^{t-1}\psi(\tilde{e}'_t) = \sum_{t=1}^\infty \delta^{t-1}\psi(\tilde{e}_t)$, and hence $\bar{c}_\infty$ and the NPV of the payments are the same in the new contract. In addition, $\psi(\tilde{e}'_1) = \psi(\tilde{e}_1) - m > \psi(\tilde{e}_1) - \frac{\delta^{T-1}}{1-\delta}\psi(\hat{e}) > \psi(\hat{e})$. Hence, $\tilde{e}'_1 > \hat{e}$. From strict convexity of ψ, the principal's payoff is now strictly higher. It remains to show self-enforceability, which will follow from satisfaction of constraints (PC_t).

Note that $\tilde{w}'_t < \tilde{w}_t$ for all $t < T$ by concavity of v. This shows that per-period profits satisfy $\tilde{e}'_t - \tilde{w}'_t > \tilde{e}_t - \tilde{w}_t$ for $t \in \{2, \dots, T-1\}$. In addition, profits $\tilde{e}'_t - \tilde{w}'_t$ at $t \geq T$ exceed $\hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} > \tilde{e}_t - \tilde{w}_t$, since $\tilde{w}'_t < \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}$ for $t \geq T$ (this follows by Lemma 4.2). This shows constraint (PC_t) is satisfied at all $t < T$. To see this for $t \geq T$, note that

$$\tilde{w}'_t < \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(\hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}\right) < \sum_{s=t+1}^\infty \delta^{s-t}(\tilde{e}'_s - \tilde{w}'_s). \qquad \textit{Q.E.D.}$$

We now show the claim in footnote 5: $\bar{t}$ (defined in the proposition) may be greater than 1.

LEMMA A.9: *For any v and ψ admitted in the model set-up, there exists a discount factor δ and initial balance b_1 such that (i) the principal obtains less than the first-best payoff, and (ii) for any optimal contract, the principal's constraint (PC_t) is satisfied strictly for $t = 1, 2$.*

PROOF: Fix v and ψ satisfying the properties in the model set-up, and fix a scalar $\gamma > 0$. Define the function $b_1(\delta) = \frac{\gamma}{1-\delta}$. As explained in Section 4.1, there is then a threshold value $\delta^* \in (0, 1)$ such that $\delta \geq \delta^*$ and $b_1 = b_1(\delta)$ implies the principal can attain the first-best payoff in a self-enforceable contract, while $\delta < \delta^*$ and $b_1 = b_1(\delta)$ implies this is not the case. We aim to show that the principal's constraint (PC_t) is slack over some initial periods when δ is below, but close enough to, δ^*, and when $b_1 = b_1(\delta)$. We do so in three steps. In these steps, we let δ parameterize the environment, leaving $b_1 = b_1(\delta)$ implicit.

Step 1. First, by considering constant effort policies, it is easily seen that the principal's payoff in an optimal contract approaches that for parameter δ^* as $\delta \to \delta^*$ from below.

Step 2. Next, let e^* be the first-best effort for parameter δ^*. We show that, for any $\varepsilon > 0$ and period T, there exists $\hat{\delta}(T, \varepsilon)$ such that, for $\delta \in (\hat{\delta}(T, \varepsilon), \delta^*)$, $\max_{t\leq T} |\tilde{e}_t - e^*| < \varepsilon$, where $(\tilde{e}_t)_{t\geq 1}$ is any optimal effort policy for parameter δ.

By Lemma A.1, any optimal effort policy is contained in $[0, z(v'(\gamma))]^\infty$. The principal's payoff for a self-enforceable contract with efforts $(\tilde{e}_t)_{t=1}^\infty$ (and satisfying condition (FP_t^{un}))

is

$$\sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t - \frac{v^{-1}\left((1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} \psi(\tilde{e}_t) + v(\gamma)\right)}{1-\delta} + b_1(\delta),$$

which varies continuously in δ, with the continuity uniform over $\delta \leq \delta^*$ and effort policies contained in $[0, z(v'(\gamma))]^\infty$.

Fix $\delta = \delta^*$, and fix any $\varepsilon > 0$ and $T \in \mathbb{N}$. There is then $\nu > 0$ such that the following is true. For any effort policy $(\tilde{e}_t)_{t\geq 1}$ contained in $[0, z(v'(\gamma))]^\infty$ and satisfying $\max_{t\leq T} |\tilde{e}_t - e^{\mathrm{FB}}(b_1)| \geq \varepsilon$, and for payments satisfying condition ($\mathrm{FP}_t^{\mathrm{un}}$), the principal's payoff is less than that sustained by the first-best contract by at least ν. This follows from uniqueness of the first-best policy and continuity of the principal's objective in the effort policy $(\tilde{e}_t)_{t\geq 1}$. However, the aforementioned continuity of the principal's payoff in δ, together with Step 1, implies that, when δ is close enough to (but below) δ^*, any effort policy satisfying $\max_{t\leq T} |\tilde{e}_t - e^{\mathrm{FB}}(b_1)| \geq \varepsilon$ cannot be optimal.

Step 3. For $\delta = \delta^*$, under the first-best policy, the principal's constraint (PC_t) is satisfied strictly at all dates. Then, provided ε is small enough, and T large enough, the constraint is satisfied strictly under an optimal contract for at least the first two dates when $\delta \in (\hat{\delta}(T, \varepsilon), \delta^*)$. *Q.E.D.*

(End of the proof of Propositions 4.2 and 4.3.) *Q.E.D.*

PROOF OF PROPOSITION 4.4: Fix an optimal relational contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. We want to show that condition ($\mathrm{FP}_t^{\mathrm{un}}$) holds at all dates $t > \bar{t}$ (with $\bar{t}$ identified in the proposition). Suppose for a contradiction this is not the case, and so the condition fails at some $t' > \bar{t}$. Since the contract is optimal, the condition in equation (5) holds (by the arguments in the main text). We therefore have

$$\begin{aligned} &\frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{t'-1} \delta^{s-1} \tilde{w}_s\right)\right)}{1-\delta} - \sum_{s=1}^{t'-1} \delta^{s-1} \psi(\tilde{e}_s) \\ &\quad < \frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1} \tilde{w}_s\right)\right)}{1-\delta} - \sum_{t\geq 1} \delta^{t-1} \psi(\tilde{e}_t) \\ &\quad = \frac{v\big((1-\delta) b_1\big)}{1-\delta}. \end{aligned} \tag{28}$$

Now consider the relational contract with the same effort and consumption but where payments ensure the satisfaction of condition ($\mathrm{FP}_t^{\mathrm{un}}$) at all dates. Denote the payments by $\tilde{w}_t'$ for all t, and note that equation (28) holds with equality when each $\tilde{w}_s$ is replaced by $\tilde{w}_s'$. Therefore, $\sum_{s=t'}^{\infty} \delta^{s-t'} \tilde{w}_s' < \sum_{s=t'}^{\infty} \delta^{s-t'} \tilde{w}_s$, and hence the principal's constraint (PC_t) is slack at t' under the payments $\tilde{w}_t'$. Because $t' > \bar{t}$, we have $\tilde{e}_{t'} < \tilde{e}_{t'-1}$. This contradicts Lemma 4.3.

The final part of the proposition concerns the observation that payments are strictly decreasing from $\bar{t} + 1$ onwards. Considering the principal's constraint (PC_t) holds with equality at these dates, it is enough to observe that profits $\tilde{e}_t - \tilde{w}_t$ are strictly decreasing in t after date $\bar{t}+1$. This follows by the same arguments that establish the inequalities in equation (23). That the agent's balances are strictly increasing from date $\bar{t}+1$ is established in the main text. *Q.E.D.*

A.2. *Proofs of the Results in Section 5*

PROOF OF PROPOSITION 5.1: **Necessity.** Immediate from the arguments in the main text.

Sufficiency. Let $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ be a feasible contract satisfying conditions ($\mathrm{AC}_t^{\mathrm{ob}}$) and ($\mathrm{PC}_t$). Specify strategies $(\alpha_t)_{t\geq 1}$ and $(\sigma_t)_{t\geq 1}$ for the agent and principal as follows. Provided that $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$, the agent consumes $\tilde{c}_t$ and chooses effort $\tilde{e}_t$. Otherwise, the agent consumes $\max\{0, (1-\delta)b(h_t)\}$, with $b(h_t)$ the balance determined recursively from b_1, given the history $h_t = (e_s, c_s, w_s)_{s=1}^{t-1}$ and chooses effort zero (if $b(h_t) < 0$, the agent violates his intertemporal budget constraint and earns payoff $-\infty$). Provided that $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$, and $(e_t, c_t) = (\tilde{e}_t, \tilde{c}_t)$, the principal pays $\tilde{w}_t$. Otherwise, she pays zero.

Now check the agent does not want to deviate at any history. Suppose there has been no deviation by t. Given the principal's strategy, the agent's continuation payoff is $\sum_{s=t}^{\infty} \delta^{s-t}(v(\tilde{c}_s) - \psi(\tilde{e}_s))$ by not deviating. By deviating at t, he gets paid zero from t onwards and optimally consumes $\max\{0, (1-\delta)\tilde{b}_t\}$ per period with continuation payoff at most $\frac{v(\max\{0,(1-\delta)\tilde{b}_t\})}{1-\delta}$. So, the inequality ($\mathrm{AC}_t^{\mathrm{ob}}$) implies no deviation is profitable. At any other history, the agent anticipates no further payments, so consuming $\max\{0, (1-\delta)b(h_t)\}$ and ceasing effort is optimal.

Now check that the principal does not want to deviate at any history. This follows when there has been no past deviation by condition (PC_t) because the $t+1$ continuation payoff from not deviating, $\sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s)$, is larger than the date-t payment $\tilde{w}_t$ that could be avoided. Also, when there has been a deviation, the principal clearly finds paying zero optimal. *Q.E.D.*

PROOF OF LEMMA 5.1: Fix an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ and suppose that Condition (8) is not satisfied for some t. Since the contract is self-enforceable, the inequality in equation ($\mathrm{AC}_t^{\mathrm{ob}}$) holds for all t. First, note that because the contract is optimal, the inequality in equation ($\mathrm{AC}_t^{\mathrm{ob}}$) cannot hold as a strict inequality at date $t = 1$. Otherwise, both $\tilde{c}_1$ and $\tilde{w}_1$ can be reduced by the same small amount, leaving all constraints intact but increasing the principal's payoff.

If the inequality in equation ($\mathrm{AC}_t^{\mathrm{ob}}$) is strict at any $t > 1$, we can consider a new contract with payment reduced at date t by $\varepsilon > 0$, and with payment increased at date $t-1$ by $\delta\varepsilon$, keeping consumption and effort unchanged. The change increases $\tilde{b}_t$ by ε and, for appropriately chosen ε, the constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) holds with equality. All the other agent constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are unaffected. The principal's constraint (PC_t) is slackened at date t, and unaffected at all other dates. The new contract attains the principal's optimal payoff.

Now note that the above adjustments can be applied sequentially at the dates for which ($\mathrm{AC}_t^{\mathrm{ob}}$) holds as a strict inequality starting with the earliest one, yielding a contract for which equation (8) and the conditions (PC_t) hold at all dates.

Finally, when equation (8) is satisfied for all t, all payments are nonnegative given that disutility of effort is nonnegative (this can be seen from equation ($\mathrm{FP}_t^{\mathrm{ob}}$)). Hence, the implied contract can be shown to be feasible. Also, balances $\tilde{b}_t$ remain strictly positive at all t, as the agent's constraint (AC_t^{ob}) for date 1 guarantees the agent earns a finite equilibrium payoff. *Q.E.D.*

PROOF OF PROPOSITION 5.2: Follows from the arguments in the main text. *Q.E.D.*

PROOF OF PROPOSITION 5.3: It will be useful to write the recursive problem in the main text by substituting out agent effort. To this end, define a function $\hat{e}$ by

$$\hat{e}(c_t, b_t, b_{t+1}) \equiv \psi^{-1}\left(v(c_t) + \frac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big) - \frac{1}{1-\delta}v\big((1-\delta)b_t\big)\right) \tag{29}$$

for $c_t, b_t, b_{t+1} > 0$, and $v(c_t) + \frac{\delta}{1-\delta}v((1-\delta)b_{t+1}) - \frac{1}{1-\delta}v((1-\delta)b_t) \geq 0$. Given our assumption of "fastest payments" (condition (FP_t^{ob})), and given values $\tilde{c}_t$, $\tilde{b}_t$, and $\tilde{b}_{t+1}$, the date-t effort must be given by $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$.

We can then write the principal's optimal payoff given balance $\tilde{b}_t > 0$ (which we establish in Lemma A.13 can be attained by a self-enforceable contract) as follows:

$$V(\tilde{b}_t) = \max_{c_t, b_{t+1} > 0}\big(\hat{e}(c_t, \tilde{b}_t, b_{t+1}) - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V(b_{t+1})\big) \tag{30}$$

subject to the principal's constraint

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V(b_{t+1}) \tag{31}$$

and to the requirement that the implied effort is nonnegative, that is,

$$v(c_t) + \frac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big) - \frac{1}{1-\delta}v\big((1-\delta)\tilde{b}_t\big) \geq 0. \tag{32}$$

The proof of Proposition 5.3 will now consist of eight lemmas. These lemmas mainly are concerned with the dynamics of an optimal contract. Lemma A.13 establishes the existence of an optimal contract.

LEMMA A.10: *In any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, $\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t > 0$ *for all* t, *and* $V(\tilde{b}_t) \in (0, V^{\text{FB}}(\tilde{b}_t)]$ *for all* t. *Also*, $\psi'(\tilde{e}_t) \leq v'(\tilde{c}_t)$ *for all* t, *and* $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ *only if* $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$.

PROOF: **Proof that** $\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t > 0$ **for all** t **and that** $V(\tilde{b}_t) \in (0, V^{\text{FB}}(\tilde{b}_t)]$ **for all** t. First, note that $\tilde{c}_t, \tilde{b}_t > 0$ for all t follows immediately from our assumption that the conditions (FP_t^{ob}) hold at all dates t, and because $b_1 > 0$.

Now consider the condition on $V(\tilde{b}_t)$. The principal can never do better than offering the first-best contract; that is, $V(\tilde{b}) \leq V^{\text{FB}}(\tilde{b})$ for all $\tilde{b} > 0$. That $V(\tilde{b}) > 0$ for all $\tilde{b} > 0$ follows by considering a stationary (i.e., constant) contract with small but positive effort.

We now prove that $\tilde{w}_t > 0$ for all t. Assume for a contradiction that $\tilde{w}_t = 0$ for some t, so that $\delta\tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t = 0$. Then $\tilde{c}_t = (1-\delta)\tilde{b}_t$ and $\tilde{b}_{t+1} = \tilde{b}_t$ (the only possibility for condition (32) to be satisfied), and so $\tilde{e}_t = \hat{e}((1-\delta)\tilde{b}_t, \tilde{b}_t, \tilde{b}_t) = 0$. Hence, $V(\tilde{b}_t) = \delta V(\tilde{b}_t)$; that is, $V(\tilde{b}_t) = 0$. But this contradicts the claim that $V(\tilde{b}_t) > 0$.

To prove that $\tilde{e}_t > 0$ for all t, suppose to the contrary that $\tilde{e}_t = 0$ for some t. If $\tilde{w}_t < \delta V(\tilde{b}_{t+1})$, we can raise effort to $\check{e}_t = \varepsilon$ at date t for $\varepsilon > 0$; raise date-t consumption to

$$\check{c}_t = v^{-1}\left(\psi(\varepsilon) - \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{t+1}\big) + \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_t\big)\right);$$

and raise the date-t payment to $\check{w}_t = \tilde{w}_t + \check{c}_t - \tilde{c}_t$. Thus, the agent's balance at $t+1$ remains unchanged, and the only adjustments to the contract are at date t. For ε sufficiently small, we have $\check{w}_t < \delta V(\tilde{b}_{t+1})$, so the principal's constraints (PC_t) are satisfied. By construction, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied. The principal's payoff strictly increases, so the original contract with effort $\tilde{e}_t = 0$ was not optimal, a contradiction. The remaining case is where $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$. In this case $V(\tilde{b}_t) = 0$, but this again contradicts the claim that $V(\tilde{b}_t) > 0$.

Proof that $\psi'(\tilde{e}_t) \le v'(\tilde{c}_t)$ for all t, and $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ only if $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$. Define

$$\underline{c}(\tilde{b}_t, \tilde{b}_{t+1}) \equiv v^{-1}\left(\frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_t\big) - \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{t+1}\big)\right),$$

interpreted as the lowest consumption level that permits the constraint (32) to be satisfied, for fixed values of $\tilde{b}_t$ and $\tilde{b}_{t+1}$. Consider the problem of maximizing

$$\hat{e}(c_t, \tilde{b}_t, \tilde{b}_{t+1}) - (\delta\tilde{b}_{t+1} - \tilde{b}_t + c_t) + \delta V(\tilde{b}_{t+1}) \tag{33}$$

with respect to c_t on $[\underline{c}(\tilde{b}_t, \tilde{b}_{t+1}), +\infty)$. Given that $\hat{e}(\cdot, \tilde{b}_t, \tilde{b}_{t+1})$ is a continuously differentiable and strictly concave function, and that $\lim_{c\to+\infty} \hat{e}_1(c, \tilde{b}_t, \tilde{b}_{t+1}) = 0$,[9] there is a unique solution of the maximization problem, denoted $c^*(\tilde{b}_t, \tilde{b}_{t+1})$. Furthermore, since $\psi'(0) = 0$, we have that $c^*(\tilde{b}_t, \tilde{b}_{t+1}) > \underline{c}(\tilde{b}_t, \tilde{b}_{t+1})$, and the first-order condition establishes

$$\psi'\big(\hat{e}\big(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1}\big)\big) = v'\big(c^*(\tilde{b}_t, \tilde{b}_{t+1})\big).$$

If we have $\delta\tilde{b}_{t+1} - \tilde{b}_t + c^*(\tilde{b}_t, \tilde{b}_{t+1}) \le \delta V(\tilde{b}_{t+1})$, then it is clear that optimality requires $\tilde{c}_t = c^*(\tilde{b}_t, \tilde{b}_{t+1})$, and so $\tilde{e}_t = \hat{e}(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1})$. If instead $\delta\tilde{b}_{t+1} - \tilde{b}_t + c^*(\tilde{b}_t, \tilde{b}_{t+1}) > \delta V(\tilde{b}_{t+1})$, given the concavity of (33) in c_t, we must have

$$\tilde{c}_t = \delta V(\tilde{b}_{t+1}) - \delta\tilde{b}_{t+1} + \tilde{b}_t < c^*(\tilde{b}_t, \tilde{b}_{t+1}).$$

Note that in this case, $\tilde{w}_t = \delta\tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t = \delta V(\tilde{b}_{t+1})$. Moreover,

$$\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) < \hat{e}\big(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1}\big),$$

and so we have $\psi'(\tilde{e}_t) < v'(c^*(\tilde{b}_t, \tilde{b}_{t+1})) < v'(\tilde{c}_t)$. *Q.E.D.*

We now establish the Euler equation in the main text and monotonicity of consumption.

LEMMA A.11: *Any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\ge1}$ satisfies the Euler equation* (14) *in all periods. Furthermore, $\tilde{c}_t \ge \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$ for all t.*

[9] Here, $\hat{e}_1$ denotes the derivative of $\hat{e}$ with respect to the first argument.

PROOF: We divide the proof in three steps:

Step 1. Fix an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Consider a contract $(\check{e}_t, \check{c}_t, \check{w}_t, \check{b}_t)_{t\geq 1}$, coinciding with the original contract in all periods except for periods t and $t+1$ (so, also, $\check{b}_t = \tilde{b}_t$). We specify the new contract to also satisfy equation ($\mathrm{FP}_t^{\mathrm{ob}}$) at all dates. Hence,

$$v(\check{c}_t) - \psi(\check{e}_t) + \frac{\delta}{1-\delta} v\left(\frac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right) = \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_t\big), \tag{34}$$

$$\begin{aligned} & v\left(\frac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta \tilde{b}_{t+2}\right) - \psi(\check{e}_{t+1}) + \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{t+2}\big) \\ & \quad = \frac{1}{1-\delta} v\left(\frac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right), \end{aligned} \tag{35}$$

which uses that consumption in period $t+1$ under the new contract is $\check{c}_{t+1} = \frac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta \tilde{b}_{t+2}$ (guaranteeing the agent has savings $\tilde{b}_{t+2}$ at date $t+2$).

Fix $\check{e}_t = \tilde{e}_t$ and $\check{w}_{t+1} = \tilde{w}_{t+1}$. Equations (34) and (35) implicitly define $\check{e}_{t+1}$ and $\check{w}_t$ as functions of $\check{c}_t$. Let these functions be denoted $\hat{e}_{t+1}(\cdot)$ and $\hat{w}_t(\cdot)$, respectively. We can use the implicit function theorem to compute the derivatives at $\check{c}_t = \tilde{c}_t$:

$$\hat{e}'_{t+1}(\tilde{c}_t) = \frac{v'(\tilde{c}_t)\big(v'\big((1-\delta)\tilde{b}_{t+1}\big) - v'(\tilde{c}_{t+1})\big)}{\delta \psi'(\tilde{e}_{t+1}) v'\big((1-\delta)\tilde{b}_{t+1}\big)} \quad \text{and} \quad \hat{w}'_t(\tilde{c}_t) = 1 - \frac{v'(\tilde{c}_t)}{v'\big((1-\delta)\tilde{b}_{t+1}\big)}.$$

Note that the original contract is obtained by setting $\check{c}_t = \tilde{c}_t$. If $\check{c}_t$ is changed from $\tilde{c}_t$ to $\tilde{c}_t + \varepsilon$, for some (positive or negative) ε small, the total effect on the continuation payoff of the principal at time t is $(-\hat{w}'_t(\tilde{c}_t) + \delta \hat{e}'_{t+1}(\tilde{c}_t))\varepsilon + o(\varepsilon)$. Hence, a necessary condition for optimality is that $-\hat{w}'_t(\tilde{c}_t) + \delta \hat{e}'_{t+1}(\tilde{c}_t) = 0$, which is equivalent to the Euler equation (14).

The Euler equation implies that if $v'(\tilde{c}_{t+1}) = \psi'(\tilde{e}_{t+1})$, then $\tilde{c}_t = \tilde{c}_{t+1}$. From Lemma A.10, if instead $v'(\tilde{c}_{t+1}) \neq \psi'(\tilde{e}_{t+1})$, then $v'(\tilde{c}_{t+1}) > \psi'(\tilde{e}_{t+1})$. There are then three possibilities:

1. If both sides of the Euler equation are strictly positive, then $\tilde{c}_t < \tilde{c}_{t+1} < (1-\delta)\tilde{b}_{t+1}$.
2. If both sides of the Euler equation are zero, then $\tilde{c}_t = \tilde{c}_{t+1} = (1-\delta)\tilde{b}_{t+1}$.
3. If both sides of the Euler equation are strictly negative, then $\tilde{c}_t > \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$.

Step 2. We now prove that if $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$ then $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t$. Assume first that there is a period t such that $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$. Hence, since $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) > 0$ (recall Lemma A.10) we have $\tilde{b}_{t+1} > \tilde{b}_t$. This shows that each side of the Euler equation is strictly positive. Since $v'(\tilde{c}_{t+1})/\psi'(\tilde{e}_{t+1}) \geq 1$ (from Lemma A.10), $(1-\delta)\tilde{b}_{t+1} > \tilde{c}_{t+1} \geq \tilde{c}_t$. The result then follows by induction.

Step 3. We prove that $\tilde{c}_t > (1-\delta)\tilde{b}_t$ for all $t > 1$; it then follows immediately from Step 1 that consumption is (weakly) decreasing in t. Assume then, for the sake of contradiction, that there is a $t' > 1$ such that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We will construct a self-enforceable contract that is strictly more profitable than the original, contradicting the optimality of the original.

We first make some preliminary observations. From Step 2, we have that $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t'$. Also, since effort is always strictly positive (by Lemma A.10),

$$\sum_{s\geq t'} \delta^{s-t'} v(\tilde{c}_s) > \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_{t'}\big).$$

Hence, there must be a period $s \geq t'$ where $\tilde{c}_{s+1} > \tilde{c}_{t'}$. Let t'' be the earliest such period, and note that it satisfies $\tilde{c}_{t''+1} > \tilde{c}_{t''}$. Additionally, we can observe that, for all t,

$$\tilde{b}_t + \sum_{\tau \geq t} \delta^{\tau - t} \tilde{w}_\tau = \sum_{\tau \geq t} \delta^{\tau - t} \tilde{c}_\tau. \tag{36}$$

If this is not the case (the right-hand side is strictly smaller), then applying equation (1) repeatedly, we have $\tilde{b}_t \to \infty$ and so the agent's constraint (AC_t^{ob}) must be violated for large t.

Now let us construct the more profitable contract for the principal, given our assumption that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We first construct a self-enforceable contract $(\tilde{e}_t^{\text{new}}, \tilde{c}_t^{\text{new}}, \tilde{w}_t^{\text{new}}, \tilde{b}_t^{\text{new}})_{t \geq 1}$ in which the agent obtains a strictly higher payoff than in the original, while the principal obtains the same payoff. We then show how that contract can be further adjusted to obtain one, which is strictly better for the principal. In the new contract that is better for the agent, we maintain $\tilde{w}_t^{\text{new}} = \tilde{w}_t$ and $\tilde{e}_t^{\text{new}} = \tilde{e}_t$ for all t, but specify different consumption ($\tilde{c}_t^{\text{new}}$) and balances ($\tilde{b}_t^{\text{new}}$).

The change in agent consumption is to specify *constant* consumption $\bar{c}$ in each period from t'' onwards, where

$$\bar{c} \equiv (1-\delta) \sum_{\tau \geq t''} \delta^{\tau - t''} \tilde{c}_\tau. \tag{37}$$

That is, $\tilde{c}_t^{\text{new}} = \bar{c}$ for all $t \geq t''$, while $\tilde{c}_t^{\text{new}} = \tilde{c}_t$ for $t < t''$. Notice that, $\bar{c} < (1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t}\tilde{c}_\tau$ for all $t > t''$.

Balances are determined by equation (1), so they are $\tilde{b}_t^{\text{new}} = \tilde{b}_t$ for $t \leq t''$, and

$$\tilde{b}_t^{\text{new}} = \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau=t''}^{t-1} \delta^{\tau-t}(\tilde{w}_\tau - \bar{c})$$

for all $t > t''$. Observe then that, for all $t > t''$,

$$\tilde{b}_t^{\text{new}} + \sum_{\tau \geq t} \delta^{\tau-t}\tilde{w}_\tau = \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau \geq t''} \delta^{\tau-t}\tilde{w}_\tau - \sum_{\tau=t''}^{t-1} \delta^{\tau-t}\bar{c} = \sum_{\tau \geq t''} \delta^{\tau-t}\tilde{c}_\tau - \sum_{\tau=t''}^{t-1} \delta^{\tau-t}\bar{c} = \frac{\bar{c}}{1-\delta},$$

where the second equality uses equation (36) and the third equation (37). Hence, for all $t > t''$,

$$\tilde{b}_t^{\text{new}} + \sum_{\tau \geq t} \delta^{\tau-t}\tilde{w}_\tau = \frac{\bar{c}}{1-\delta} < \sum_{\tau \geq t} \delta^{\tau-t}\tilde{c}_\tau = \tilde{b}_t + \sum_{\tau \geq t} \delta^{\tau-t}\tilde{w}_\tau,$$

where the second equality follows from equation (36). This implies that $\tilde{b}_t^{\text{new}} < \tilde{b}_t$ for all $t > t''$.

Now, we want to show that the contract $(\tilde{e}_t^{\text{new}}, \tilde{c}_t^{\text{new}}, \tilde{w}_t^{\text{new}}, \tilde{b}_t^{\text{new}})_{t \geq 1}$ is self-enforceable. Because effort and payments are unchanged relative to the original contract, the principal's constraints (PC_t) remain intact. For all $t \leq t''$, the agent obtains a strictly higher continuation payoff in the new contract. This follows from concavity of v, and because consumption from date t'' onwards is constant in the new contract, while its NPV is the same as in the original. Since agent balances are also unchanged, constraints (AC_t^{ob}) are then satisfied at $t \leq t''$ as strict inequalities.

To see how the agent's constraints change at each $t > t''$, define $\bar{c}^{(t)} \equiv (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{c}_\tau$. Consider the original contract, and suppose that the agent's consumption is changed from date t onwards, being set equal to $\bar{c}^{(t)}$ in all such periods. The agent's payoff increases from the smoothing of consumption, and so

$$\sum_{\tau\geq t}\delta^{\tau-t}\big(v(\bar{c}^{(t)})-\psi(\tilde{e}_\tau)\big)\geq\sum_{\tau\geq t}\delta^{\tau-t}\big(v(\tilde{c}_\tau)-\psi(\tilde{e}_\tau)\big)\geq\frac{1}{1-\delta}v\big(\tilde{b}_t(1-\delta)\big), \tag{38}$$

where the second inequality follows because the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied in the original contract.

Equation (38) implies $\bar{c}^{(t)} \geq \tilde{b}_t(1-\delta)$. Therefore, since v is concave, we have

$$v(\bar{c}^{(t)})-v\big(\bar{c}^{(t)}-(1-\delta)(\tilde{b}_t-\tilde{b}_t^{\mathrm{new}})\big)\leq v\big(\tilde{b}_t(1-\delta)\big)-v\big(\tilde{b}_t(1-\delta)-(1-\delta)(\tilde{b}_t-\tilde{b}_t^{\mathrm{new}})\big). \tag{39}$$

Note that $\bar{c} = \bar{c}^{(t)} - (1-\delta)(\tilde{b}_t - \tilde{b}_t^{\mathrm{new}})$. Combining equations (38) and (39), we therefore have that, for all $t > t''$,

$$\sum_{\tau\geq t}\delta^{\tau-t}\big(v(\bar{c})-\psi(\tilde{e}_\tau)\big)\geq\frac{1}{1-\delta}v\big(\tilde{b}_t^{\mathrm{new}}(1-\delta)\big).$$

This shows that, for the contract $(\tilde{e}_t^{\mathrm{new}}, \tilde{c}_t^{\mathrm{new}}, \tilde{w}_t^{\mathrm{new}}, \tilde{b}_t^{\mathrm{new}})_{t\geq1}$, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied also at dates $t > t''$.

We have thus shown that $(\tilde{e}_t^{\mathrm{new}}, \tilde{c}_t^{\mathrm{new}}, \tilde{w}_t^{\mathrm{new}}, \tilde{b}_t^{\mathrm{new}})_{t\geq1}$ is a self-enforceable contract. Moreover, we saw that the constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied strictly at all $t \leq t''$. We can therefore further adjust the contract by raising effort at t'' by a small amount $\varepsilon > 0$ such that, without other changes to the contract, all constraints remain intact. The principal earns a strictly higher payoff than in the original contract, contradicting the optimality of the original. *Q.E.D.*

We can then provide the key result that balances decrease over time toward $\tilde{b}_\infty$.

LEMMA A.12: *In any optimal contract, $(\tilde{b}_t)_{t\geq1}$ is weakly decreasing: it is constant if the contract attains the first-best payoff, and strictly decreasing toward some $\tilde{b}_\infty > 0$ otherwise.*

PROOF: **Step 0.** If the first-best payoff is achievable at b_1, then equilibrium consumption and effort is uniquely determined by the conditions in Proposition 3.1. Because we assume payments satisfy equation (8), balances $(\tilde{b}_t)_{t\geq1}$ are constant. Suppose from now on that $V(b_1) < V^{\mathrm{FB}}(b_1)$.

Step 1. Proof that $(\tilde{b}_t)_{t\geq1}$ is weakly decreasing. Consider an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq1}$. To show that the balance $\tilde{b}_t$ is weakly decreasing, we suppose for a contradiction that $\tilde{b}_{\hat{t}+1} > \tilde{b}_{\hat{t}}$ for some date $\hat{t}$. We construct a self-enforceable contract that achieves strictly higher profits for the principal.

Step 1a. First, denote a new contract by $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq1}$, which is taken to coincide with the original contract until $\hat{t}-1$, with $\tilde{e}'_{\hat{t}} = \tilde{e}_{\hat{t}}$. For dates $t \geq \hat{t}$, let

$$\tilde{c}'_t = \bar{c} \equiv (1-\delta)\sum_{\tau\geq\hat{t}}\delta^{\tau-\hat{t}}\tilde{w}_\tau + (1-\delta)\tilde{b}_{\hat{t}} = (1-\delta)\sum_{\tau\geq\hat{t}}\delta^{\tau-\hat{t}}\tilde{c}_\tau,$$

where the last equality is for the same reason as equation (36). For dates $t \geq \hat{t}+1$, let $\tilde{e}'_t = \bar{e}$, where $\bar{e}$ is defined by

$$\psi(\bar{e}) = (1-\delta)\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1}\psi(\tilde{e}_\tau).$$

Let also, for all $t \geq \hat{t}$, $\tilde{w}'_t = \bar{w}$, where $\bar{w} = (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau$. Thus, we must have $\tilde{b}'_t = \bar{b} \equiv \tilde{b}_{\hat{t}}$ for all $t \geq \hat{t}$.

Step 1b. We now want to show that, for the new contract, the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) is satisfied at all dates. Begin with dates $\hat{t}+1$ onwards when the contract is stationary. Note first that, by the previous lemma, we must have $\tilde{c}_{\hat{t}} \geq \bar{c}$. Therefore,

$$\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\bar{c} \geq \sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\tilde{c}_\tau.$$

Also, the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same for both the original contract and the new contract. The fact that the original contract satisfies the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) at date $\hat{t}+1$, plus the observation that $\bar{b} < \tilde{b}_{\hat{t}+1}$, then implies

$$\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}v(\bar{c}) - \sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\psi(\bar{e}) > \frac{1}{1-\delta}v\big((1-\delta)\bar{b}\big), \tag{40}$$

which means that the agent's constraint is satisfied as a *strict inequality* from $\hat{t}+1$ onwards.

Note then that

$$\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}v(\bar{c}) \geq \sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}v(\tilde{c}_\tau).$$

Also, the NPV of the disutility of effort is the same from $\hat{t}$ onwards under both policies. Therefore, constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) continues to be satisfied at $\hat{t}$, and by the same logic all earlier periods.

Step 1c. Now we show that the principal's constraint (PC_t) is satisfied in all periods. Because the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same under both contracts, and because ψ is convex, we have $\bar{e} \geq (1-\delta)\sum_{\tau\geq\hat{t}+1}\delta^{\tau-\hat{t}-1}\tilde{e}_\tau$. Therefore,

$$\begin{aligned}\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}}\tilde{e}'_\tau - \sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}'_\tau = \frac{\delta\bar{e}}{1-\delta} - \sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau \geq \sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}}\tilde{e}_\tau - \sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau \\ \geq 0,\end{aligned} \tag{41}$$

where the second inequality is by (PC_t) in the original contract. Hence, the principal's constraint is satisfied under the new contract at date $\hat{t}$. Because $\tilde{e}'_t$ is constant for $t \geq \hat{t}+1$, and because $\tilde{w}'_t$ is constant for $t \geq \hat{t}$, the principal's constraint holds also from $\hat{t}+1$ onwards. It is then readily checked that the principal's constraint is satisfied also for dates before $\hat{t}$.

Step 1d. Finally, the contract can be further (slightly) adjusted to a self-enforceable contract with a strictly higher payoff for the principal. The original contract was taken to satisfy

$$v(\tilde{c}_{\hat{t}}) - \psi(\tilde{e}_{\hat{t}}) = \frac{1}{1-\delta}\Big(v\big((1-\delta)\bar{b}\big) - \delta v\big((1-\delta)\tilde{b}_{\hat{t}+1}\big)\Big) < v\big((1-\delta)\bar{b}\big).$$

Hence,

$$\psi(\tilde{e}_{\hat{t}}) > v(\tilde{c}_{\hat{t}}) - v\big((1-\delta)\bar{b}\big) \geq v(\bar{c}) - v\big((1-\delta)\bar{b}\big) > \psi(\bar{e}),$$

where the final inequality follows from (40). Hence, $\tilde{e}_{\hat{t}} > \bar{e}$. Recall that $\tilde{e}'_{\hat{t}} = \tilde{e}_{\hat{t}}$, and $\tilde{e}'_{\tau} = \bar{e}$ for $\tau > \hat{t}$; so we have $\tilde{e}'_{\hat{t}} > \tilde{e}'_{\tau}$ for all $\tau > \hat{t}$.

Now, pick $\tilde{e}''_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$, with $\tilde{e}'_{\hat{t}+1} < \tilde{e}''_{\hat{t}+1} < \tilde{e}''_{\hat{t}} < \tilde{e}'_{\hat{t}}$ and such that

$$\psi\big(\tilde{e}''_{\hat{t}}\big) + \frac{\delta}{1-\delta}\psi\big(\tilde{e}''_{\hat{t}+1}\big) = \psi\big(\tilde{e}'_{\hat{t}}\big) + \frac{\delta}{1-\delta}\psi\big(\tilde{e}'_{\hat{t}+1}\big).$$

Substitute $\tilde{e}''_{\hat{t}}$ for $\tilde{e}'_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$ for $\tilde{e}'_{\tau}$, for all $\tau \geq \hat{t}+1$, in the contract defined in Step 1a. The agent's value from remaining in the contract from $\hat{t}$ onwards remains unchanged, so the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) remains satisfied at $\hat{t}$, and at all earlier dates. Note that, due to (40), the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) at dates $\hat{t}+1$ onwards are slack under the contract defined in Step 1a, and hence continue to be satisfied under the contract with the further modification, provided the adjustment in effort is small. Moreover, because ψ is strictly convex, the NPV of effort from date $\hat{t}$ onwards increases; so the principal's payoff strictly increases. Also, the principal's constraints (PC_t) clearly continue to be satisfied. Thus, we have constructed a self-enforceable contract that is strictly more profitable for the principal than the original, completing Step 1.

Step 2. Proof that if $V(\tilde{b}_1) < V^{\mathrm{FB}}(\tilde{b}_1)$ then $(\tilde{b}_t)_{t\geq 1}$ is a strictly decreasing sequence.

Step 2a. Consider an optimal contract. We first prove that if $\tilde{b}_{\hat{t}} = \tilde{b}_{\hat{t}+1}$ for some $\hat{t} \geq 1$, then $V(\tilde{b}_{\hat{t}}) = V^{\mathrm{FB}}(\tilde{b}_{\hat{t}})$. To do this, note that if $\tilde{b}_{\hat{t}} = \tilde{b}_{\hat{t}+1}$ for some $\hat{t}$, then it is optimal to specify $\tilde{c}_\tau = \tilde{c}_{\hat{t}}$, $\tilde{w}_\tau = \tilde{w}_{\hat{t}}$, $\tilde{b}_\tau = \tilde{b}_{\hat{t}}$, and $\tilde{e}_\tau = \tilde{e}_{\hat{t}}$ for all $\tau > \hat{t}$. The Euler equation (14) then requires that $\psi'(\tilde{e}_\tau) = v'(\tilde{c}_\tau)$ for all $\tau \geq \hat{t}+1$,[10] and so also $\psi'(\tilde{e}_{\hat{t}}) = v'(\tilde{c}_{\hat{t}})$. Then $\tilde{e}_\tau$ and $\tilde{c}_\tau$ satisfy, for all $\tau \geq \hat{t}$, the conditions in Proposition 3.1, given initial balance $\tilde{b}_{\hat{t}}$. Therefore, they are the first-best effort and consumption given this balance. This shows that $V(\tilde{b}_{\hat{t}}) = V^{\mathrm{FB}}(\tilde{b}_{\hat{t}})$, as desired.

Step 2b. Now we consider any optimal contract, and show the following. If $V(\tilde{b}_1) < V^{\mathrm{FB}}(\tilde{b}_1)$, then $V(\tilde{b}_t) < V^{\mathrm{FB}}(\tilde{b}_t)$ for all $t \geq 1$, and in addition, $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing.

Suppose that, for some $\hat{t}$, $V(\tilde{b}_{\hat{t}}) < V^{\mathrm{FB}}(\tilde{b}_{\hat{t}})$, which by Step 1 and Step 2a implies $\tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$. The result will follow by induction if we can show that $V(\tilde{b}_{\hat{t}+1}) < V^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$. Hence, suppose for a contradiction that the contract achieves the first-best continuation payoff for the principal at date $\hat{t}+1$, given the balance is $\tilde{b}_{\hat{t}+1}$ (i.e., suppose $V(\tilde{b}_{\hat{t}+1}) = V^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$). This implies that $\tilde{e}_\tau = e^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$ and $\tilde{c}_\tau = c^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$ for all $\tau > \hat{t}$. By assumption that equation (8) holds in all periods, we then have $\tilde{b}_\tau = \tilde{b}_{\hat{t}+1}$ for all $\tau > \hat{t}+1$. Hence, the contract is stationary from $\hat{t}+1$ onwards; in particular, the payment is constant at $\tilde{w}_\tau = \bar{w}$ for $\tau \geq \hat{t}+1$, for some value $\bar{w}$.

[10]To see this, note Lemma A.11 implies that $\tilde{c}_\tau > \tilde{b}_\tau$ for all $\tau \geq \hat{t}+1$.

From the Euler equation (14) and the fact that $v'(\tilde{c}_{\hat{t}+1}) = \psi'(\tilde{e}_{\hat{t}+1})$, we have $\tilde{c}_{\hat{t}} = \tilde{c}_{\hat{t}+1}$. Hence, using $\tilde{b}_{\hat{t}+2} = \tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$, we have (using ($\mathrm{FP}^{\mathrm{ob}}_t$))

$$\begin{aligned}\psi(\tilde{e}_{\hat{t}}) &= v(\tilde{c}_{\hat{t}}) + \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{\hat{t}+1}\big) - \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_{\hat{t}}\big) \\ &< v(\tilde{c}_{\hat{t}+1}) + \frac{\delta}{1-\delta} v\big((1-\delta)\tilde{b}_{\hat{t}+2}\big) - \frac{1}{1-\delta} v\big((1-\delta)\tilde{b}_{\hat{t}+1}\big) = \psi(\tilde{e}_{\hat{t}+1}).\end{aligned}$$

Consequently, $\tilde{e}_{\hat{t}} < \tilde{e}_{\hat{t}+1}$, and so $\frac{\psi'(\tilde{e}_{\hat{t}})}{v'(\tilde{c}_{\hat{t}})} < \frac{\psi'(\tilde{e}_{\hat{t}+1})}{v'(\tilde{c}_{\hat{t}+1})} = 1$. We then know (from Lemma A.10) that the principal's constraint (PC_t) binds at $\hat{t}$, and so

$$\sum_{s \geq \hat{t}+1} \delta^{s-\hat{t}} \tilde{e}_s = \sum_{s \geq \hat{t}} \delta^{s-\hat{t}} \tilde{w}_s = \sum_{s \geq \hat{t}} \delta^{s-\hat{t}} \tilde{c}_s - \tilde{b}_{\hat{t}},$$

where the second equality follows for the same reason as for equation (36). Using that $\tilde{e}_\tau = e^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}+1$, and $\tilde{c}_\tau = c^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}$, we have

$$\delta e^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1}) = c^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}} < c^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}+1} = \bar{w} = \tilde{w}_{\hat{t}+1}.$$

That $\delta e^{\mathrm{FB}}(\tilde{b}_{\hat{t}+1}) < \tilde{w}_{\hat{t}+1}$ means the principal's constraint (PC_t) in period $\hat{t}+1$ (as well as at future dates) is violated, so we reach our contradiction. This completes Step 2.

Step 3. Proof that $\tilde{b}_\infty > 0$. Consider an optimal contract, and suppose that $V(\tilde{b}_1) < V^{\mathrm{FB}}(\tilde{b}_1)$. We saw that $(\tilde{b}_t)_{t\geq 1}$ is a strictly decreasing sequence. By Lemma 5.1, $\tilde{b}_t > 0$ for all t, so the limit $\lim_{t\to\infty} \tilde{b}_t$ exists and is nonnegative. We want to show this limit, call it $\tilde{b}_\infty$, is strictly positive.

We first show $\lim_{b\searrow 0} \frac{c^{\mathrm{FB}}(b)-(1-\delta)b}{e^{\mathrm{FB}}(b)} = 0$ and so, by Proposition 5.2, there exists some $\bar{b} > 0$ such that an optimal contract achieves the first-best payoff of the principal for all $b \leq \bar{b}$. This follows after noting that $v(c^{\mathrm{FB}}(b)) - v((1-\delta)b) = \psi(e^{\mathrm{FB}}(b)) > 0$, so we have that $\lim_{b\searrow 0} c^{\mathrm{FB}}(b) = 0$ or $\lim_{b\searrow 0} e^{\mathrm{FB}}(b) = +\infty$. Since $\psi'(e^{\mathrm{FB}}(b)) = v'(c^{\mathrm{FB}}(b))$ we have, in fact, that both $\lim_{b\searrow 0} c^{\mathrm{FB}}(b) = 0$ and $\lim_{b\searrow 0} e^{\mathrm{FB}}(b) = +\infty$, which establishes the result. Next, recall from Step 2 that, given $V(b_1) < V^{\mathrm{FB}}(b_1)$, the sequence $(\tilde{b}_t)_{t\geq 1}$ of balances in the optimal contract is strictly decreasing and such that $V(\tilde{b}_t) < V^{\mathrm{FB}}(\tilde{b}_t)$ for all t. That is, $\tilde{b}_t$ remains above $\bar{b}$, and so converges to some value $\tilde{b}_\infty \geq \bar{b}$. *Q.E.D.*

It is now convenient to prove the existence of an optimal contract.

LEMMA A.13: *An optimal contract exists.*

PROOF: The argument is related to Lemma 1 of Thomas and Worrall (1994), which obtains the principal's value function as a fixed point of an appropriate functional mapping.

If $\delta \geq \frac{c^{\mathrm{FB}}(b_1)-(1-\delta)b_1}{e^{\mathrm{FB}}(b_1)} \in (0,1)$, then there is a self-enforceable efficient contract (by Proposition 5.2), and so existence is established. The remainder of the proof is needed for the values b_1 such that there is no self-enforceable first-best contract.

Given any sequence $(c_s, b_{s+1})_{s=t}^\infty$, and assuming "fastest payments," we can completely define the continuation contract from date t, with effort given at each date $s \geq t$ by $\hat{e}(c_s, b_s, b_{s+1})$ (recall equation (29)), and the payment given by $\delta b_{s+1} - b_s + c_s$. We then denote by $\Pi(b_t)$ the sequences $(c_s, b_{s+1})_{s=t}^{\infty}$, which are part of feasible and self-enforceable contracts beginning with balance b_t. Note that these sequences satisfy, for all $s \geq t$,

$$\delta b_{s+1} - b_s + c_s \leq \sum_{\tau=s+1}^{\infty} \delta^{\tau-s}\big(\hat{e}(c_\tau, b_\tau, b_{\tau+1}) - (\delta b_{\tau+1} - b_\tau + c_\tau)\big)$$

as well as

$$v(c_s) + \frac{\delta}{1-\delta} v\big((1-\delta)b_{s+1}\big) - \frac{1}{1-\delta} v\big((1-\delta)b_s\big) \geq 0.$$

Note also that $\Pi(b_t)$ is not empty: for instance, it contains the "autarky" continuation contract, where $c_s = (1-\delta)b_t$ and $b_s = b_t$ for all $s \geq t$ (recall that $\hat{e}((1-\delta)b_t, b_t, b_t) = 0$).

Given any $b_t > 0$, let the value of the principal's problem of determining a feasible and self-enforceable contract be given by

$$V(b_t) \equiv \sup_{(c_s, b_{s+1})_{s=t}^{\infty} \in \Pi(b_t)} \sum_{s=t}^{\infty} \delta^{s-t}\big(\hat{e}(c_s, b_s, b_{s+1}) - (\delta b_{s+1} - b_s + c_s)\big).$$

Note that $V(b_t)$ is no greater than the first-best value $V^{\mathrm{FB}}(b_t)$. Usual arguments imply that the continuation payoff of the principal in an optimal contract (if it exists) is a fixed point of an operator defined by

$$TW(b_t) \equiv \sup_{c_t > 0, b_{t+1} > 0} \big(\hat{e}(c_t, b_t, b_{t+1}) - (\delta b_{t+1} - b_t + c_t) + \delta W(b_{t+1})\big) \tag{42}$$

subject to the principal's constraint

$$\delta b_{t+1} - b_t + c_t \leq \delta W(b_{t+1}) \tag{43}$$

and to

$$v(c_t) + \frac{\delta}{1-\delta} v\big((1-\delta)b_{t+1}\big) - \frac{1}{1-\delta} v\big((1-\delta)b_t\big) \geq 0. \tag{44}$$

Note that the operator T is monotone: if $W_1 \geq W_2$, then $TW_1 \geq TW_2$. Also, we have $TV^{\mathrm{FB}} \leq V^{\mathrm{FB}}$. Applying T to both sides, we have that $(T^n V^{\mathrm{FB}}(b_t))_{n \geq 1}$ is a decreasing sequence for all $b_t > 0$. Therefore, there is some pointwise limit of $T^n V^{\mathrm{FB}}$, call it $\bar{V}$. Straightforward continuity arguments show that $\bar{V}$ is a fixed point of T.

We now make four observations to be used in the completion of the proof.

Observation 1. First-best value is strictly decreasing. We now show that V^{FB} is a strictly decreasing function. First, notice that, for any $b_t > 0$,

$$V^{\mathrm{FB}}(b_t) = \frac{1}{1-\delta} \max_{w} \big\{\psi^{-1}\big(v\big(b_t(1-\delta) + w\big) - v\big(b_t(1-\delta)\big)\big) - w\big\}. \tag{45}$$

At the optimal choice of w (which is strictly positive), we have $c^{\mathrm{FB}}(b_t) = b_t(1-\delta) + w$, and

$$e^{\mathrm{FB}}(b_t) = \psi^{-1}\big(v\big(b_t(1-\delta) + w\big) - v\big(b_t(1-\delta)\big)\big).$$

Therefore, by the envelope theorem,

$$\frac{d}{db_t} V^{\mathrm{FB}}(b_t) = \frac{v'\big(c^{\mathrm{FB}}(b_t)\big) - v'\big(b_t(1-\delta)\big)}{\psi'\big(e^{\mathrm{FB}}(b_t)\big)} = 1 - \frac{v'\big(b_t(1-\delta)\big)}{\psi'\big(e^{\mathrm{FB}}(b_t)\big)}.$$

Using the theorem of the maximum and the fact that the objective in equation (45) is strictly concave, e^{FB} is a continuous function, so V^{FB} is continuously differentiable.

Now note that, for any $b_t > 0$, $\psi'(e^{\mathrm{FB}}(b_t)) = v'(c^{\mathrm{FB}}(b_t)) < v'(b_t(1-\delta))$, and so $\frac{\mathrm{d}}{\mathrm{d}b_t}V^{\mathrm{FB}}(b_t) < 0$, which establishes the result.

Observation 2. Bounded choice variables. Consider the program (42) to (44) given $b_t > 0$ and $W = \bar{V}$. We show that (a) the values of b_{t+1} that satisfy the constraints of this program are contained in a bounded interval $[l^b(b_t), u^b(b_t)]$ with $l^b(b_t) > 0$, and (b) consumption choices c_t are contained in a bounded interval $[l^c(b_t), u^c(b_t)]$ with $l^c(b_t) > 0$.

First, we show that b_{t+1} must be bounded above by some $u^b(b_t)$. Observe that

$$\lim_{b_{t+1}\to\infty} \delta\big(V^{\mathrm{FB}}(b_{t+1}) - b_{t+1}\big) = -\infty,$$

because V^{FB} is decreasing. Therefore, using $\bar{V} \leq V^{\mathrm{FB}}$, we have

$$\lim_{b_{t+1}\to\infty} \delta\big(\bar{V}(b_{t+1}) - b_{t+1}\big) = -\infty. \tag{46}$$

Satisfaction of (43) then implies that the choice of b_{t+1} must be bounded above.

We now show that, given b_t, satisfaction of the constraints in equations (43) and (44) implies that b_{t+1} must be no less than some $l^b(b_t) > 0$. In particular, given any $b_t > 0$, we show that at least one of these constraints is violated whenever b_{t+1} is taken sufficiently close to zero.

The constraints in equations (43) and (44) are satisfied only if

$$v(c_t) \geq \frac{1}{1-\delta}v\big((1-\delta)b_t\big) - \frac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big) \quad \text{and} \quad c_t \leq b_t + \delta\big(V^{\mathrm{FB}}(b_{t+1}) - b_{t+1}\big).$$

Combining these two equations, we have

$$\begin{aligned} V^{\mathrm{FB}}(b_{t+1}) &\geq \tilde{V}(b_{t+1}) \\ &\equiv \frac{v^{-1}\left(\dfrac{1}{1-\delta}v\big((1-\delta)b_t\big) - \dfrac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big)\right) - b_t}{\delta} + b_{t+1}. \end{aligned} \tag{47}$$

Now, notice that the right-hand side of equation (47) tends to $+\infty$ as $b_{t+1} \to 0$. Hence, joint satisfaction of the constraints requires $\lim_{b_{t+1}\to 0} V^{\mathrm{FB}}(b_{t+1}) = +\infty$ and

$$\lim_{b_{t+1}\to 0} \frac{\tilde{V}(b_{t+1})}{V^{\mathrm{FB}}(b_{t+1})} \leq 1.$$

We will instead show that this limit is $+\infty$.

From l'Hôpital's rule, we have that

$$\lim_{b_{t+1}\to 0} \frac{\tilde{V}(b_{t+1})}{V^{\mathrm{FB}}(b_{t+1})} = \lim_{b_{t+1}\to 0} \frac{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}\tilde{V}(b_{t+1})}{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{\mathrm{FB}}(b_{t+1})}.$$

Let us first calculate the following limit:

$$\lim_{b_{t+1}\to 0}\frac{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}\tilde{V}(b_{t+1})-1}{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{\mathrm{FB}}(b_{t+1})-1}=\lim_{b_{t+1}\to 0}\frac{-\dfrac{v'\big((1-\delta)b_{t+1}\big)}{v'\Big(v^{-1}\Big(\dfrac{1}{1-\delta}v\big((1-\delta)b_t\big)-\dfrac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big)\Big)\Big)}}{-\dfrac{v'\big((1-\delta)b_{t+1}\big)}{\psi'\big(e^{\mathrm{FB}}(b_{t+1})\big)}}$$

$$=\lim_{b_{t+1}\to 0}\frac{\psi'\big(e^{\mathrm{FB}}(b_{t+1})\big)}{v'\Big(v^{-1}\Big(\dfrac{1}{1-\delta}v\big((1-\delta)b_t\big)-\dfrac{\delta}{1-\delta}v\big((1-\delta)b_{t+1}\big)\Big)\Big)}$$

$$=+\infty.$$

To see the last equality, note that $\lim_{b_{t+1}\to 0}V^{\mathrm{FB}}(b_{t+1})=+\infty$ implies $\lim_{b_{t+1}\to 0}e^{\mathrm{FB}}(b_{t+1})=+\infty$. Also, the denominator takes positive values and has a finite limit as $b_{t+1}\to 0$. Using that $\frac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{\mathrm{FB}}(b_{t+1})<0$, it is then readily seen that in fact

$$\lim_{b_{t+1}\to 0}\frac{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}\tilde{V}(b_{t+1})}{\dfrac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{\mathrm{FB}}(b_{t+1})}=+\infty.$$

We have therefore shown that, given a date-t balance $b_t>0$, the choices of b_{t+1} that are available in the program (42) to (44) come from some bounded set $[l^b(b_t),u^b(b_t)]$ with $l^b(b_t)>0$. It is then immediate (using equations (43) and (44)) that consumption c_t must be chosen from some bounded interval $[l^c(b_t),u^c(b_t)]$ with $l^c(b_t)>0$ as well.

Observation 3. The function $\bar{V}$ is continuous. We now show that $\bar{V}$ is continuous. We will use that there is a decreasing and strictly positive function $\kappa(b)$ such that $\bar{V}(b)\geq\kappa(b)$ for all $b>0$. This can be seen by recalling that, for all $b>0$, $\bar{V}(b)$ is the limit of $T^nV^{\mathrm{FB}}(b)$, and by verifying that the latter is, for all n, at least a positive payoff obtainable from constant consumption and balances.

Suppose for a contradiction that there is a point of discontinuity in $\bar{V}$, call it $\check{b}>0$. Then there is $\varepsilon>0$ and a sequence $(\check{b}_n)_{n=1}^{\infty}$ convergent to $\check{b}$ with $|\bar{V}(\check{b}_n)-\bar{V}(\check{b})|\geq\varepsilon$ for all n. We will suppose first there is a subsequence $(\check{b}_{n_k})_{k=1}^{\infty}$ along which $\bar{V}(\check{b}_{n_k})\leq\bar{V}(\check{b})-\varepsilon$ for all k.

Denote $\check{c}$ and $\check{b}'$ consumption and next-period balance that achieve within $\varepsilon/2$ of the supremum in the program (42) to (44) when the initial balance is $b_t=\check{b}$ and $W=\bar{V}$. We may assume that $\hat{e}(\check{c},\check{b},\check{b}')$ is strictly positive, as otherwise $\delta\bar{V}(\check{b}')-(\delta\check{b}'-\check{b}+\check{c})>0$ and $\check{c}$ can be increased, so the payment $\delta\check{b}'-\check{b}+\check{c}$ increases, and the implied effort $\hat{e}(\check{c},\check{b},\check{b}')$ increases, yielding an increase in profit. To obtain a contradiction, we then note that, for k sufficiently large, it is possible in the program given $b_t=\check{b}_{n_k}$ and $W=\bar{V}$ to choose subsequent balance $b_{t+1}=\check{b}'$ and consumption c_t equal to $\check{c}+\check{b}_{n_k}-\check{b}$. Note that the payment $\delta b_{t+1}-b_t+c_t$ is the same as when the initial balance is $\check{b}$ (hence equal to $\delta\check{b}'-\check{b}+\check{c}$) while, as k becomes large, effort $\hat{e}(c_t,b_t,b_{t+1})$ is arbitrarily close to $\hat{e}(\check{c},\check{b},\check{b}')$. This shows that $\bar{V}(\check{b}_{n_k})=T\bar{V}(\check{b}_{n_k})>\bar{V}(\check{b})-\varepsilon$, a contradiction.

The remaining case is where there is a subsequence $(\check{b}_{n_k})_{k=1}^{\infty}$ for which $\bar{V}(\check{b}) \le \bar{V}(\check{b}_{n_k}) - \varepsilon$. For each k, consider the program (42) to (44) when $b_t = \check{b}_{n_k}$ and $W = \bar{V}$, and pick $c_t = \check{c}_{n_k}$ and $b_{t+1} = \check{b}'_{n_k}$ satisfying the constraints and such that

$$\hat{e}\big(\check{c}_{n_k}, \check{b}_{n_k}, \check{b}'_{n_k}\big) - \big(\delta \check{b}'_{n_k} - \check{b}_{n_k} + \check{c}_{n_k}\big) + \delta \bar{V}\big(\check{b}'_{n_k}\big) > \bar{V}(\check{b}_{n_k}) - |\check{b} - \check{b}_{n_k}|. \tag{48}$$

Using equation (46), there exists $u^b > 0$ such that, necessarily, $\check{b}'_{n_k} \le u^b$ for all k. Using (47) and the arguments from Observation 2, there exists $l^b > 0$ such that $\check{b}'_{n_k} \ge l^b$ for all k. From this, and using constraints (43) and (44), we conclude also that there are $l^c, u^c > 0$ with $\check{c}_{n_k} \in [l^c, u^c]$ for all k.

Note we may assume that $\hat{e}(\check{c}_{n_k}, \check{b}_{n_k}, \check{b}'_{n_k})$ remains bounded below by some $\bar{e} > 0$ for all k sufficiently large. This follows from examining the program (42) to (44), and by the existence of a strictly positive function $\kappa(b)$ such that $\bar{V}(b) \ge \kappa(b)$ for all $b > 0$, as mentioned above. In particular, if there is no such lower bound, we can find an $\bar{e} > 0$ small enough that the following is true. For all large enough k, if $\hat{e}(\check{c}_{n_k}, \check{b}_{n_k}, \check{b}'_{n_k}) < \bar{e}$, $\check{c}_{n_k}$ can be increased to yield effort $\hat{e}(\check{c}_{n_k}, \check{b}_{n_k}, \check{b}'_{n_k}) \ge \bar{e}$ (preserving, in particular, the constraint (43), as this must initially have sufficient slack). These changes can be made so as to increase the payoffs in the program (42) to (44), so the inequality (48) continues to hold for all k.

Finally, for any large enough k, in the problem (42) to (44) with $b_t = \check{b}$ and $W = \bar{V}$, we may specify $c_t = \check{c}_{n_k} + \check{b} - \check{b}_{n_k}$ and $b_{t+1} = \check{b}'_{n_k}$ (thus specifying the same payment $\delta \check{b}'_{n_k} - \check{b}_{n_k} + \check{c}_{n_k}$ and next-period balance $\check{b}'_{n_k}$ as when the initial balance is $\check{b}_{n_k}$). As $k \to \infty$, we have

$$\hat{e}\big(\check{c}_{n_k} + \check{b} - \check{b}_{n_k}, \check{b}, \check{b}'_{n_k}\big) - \big(\delta \check{b}'_{n_k} - \check{b}_{n_k} + \check{c}_{n_k}\big) + \delta \bar{V}\big(\check{b}'_{n_k}\big) - \bar{V}(\check{b}_{n_k}) \to 0.$$

But, for all large enough k,

$$\bar{V}(\check{b}) \ge \hat{e}\big(\check{c}_{n_k} + \check{b} - \check{b}_{n_k}, \check{b}, \check{b}'_{n_k}\big) - \big(\delta \check{b}'_{n_k} - \check{b}_{n_k} + \check{c}_{n_k}\big) + \delta \bar{V}\big(\check{b}'_{n_k}\big)$$

and it cannot be that $\bar{V}(\check{b}) \le \bar{V}(\check{b}_{n_k}) - \varepsilon$. This establishes the contradiction.

Observation 4. The function $\bar{V}$ is strictly decreasing. To see that $\bar{V}$ is decreasing, we show that if W is a strictly positive and nonincreasing function with $W \le V^{\text{FB}}$, then TW is strictly decreasing. That $\bar{V}$ is nonincreasing then follows because $\bar{V}$ is the pointwise limit of $T^n V^{\text{FB}}$ as $n \to \infty$.

That it is strictly decreasing follows because $\bar{V} = T\bar{V}$. To see this, consider any strictly positive and nonincreasing function W, with $W \le V^{\text{FB}}$. Note that TW is continuous (the argument is the same as for the continuity of $\bar{V}$ above and so omitted). Also, $TW \le V^{\text{FB}}$. Furthermore, it is easy to see that if TW fails to be strictly decreasing, then there exists a value $b^* > 0$ such that, for every $\varepsilon > 0$, there is a $\check{b} \in (b^* - \varepsilon, b^*)$ satisfying $TW(\check{b}) \le V(b^*)$.

To establish the claim, note that given $b_t = b^*$ in the optimization of equations (42) to (44), the value $TW(b^*)$ can be derived by considering choices of c_t and b_{t+1} such that, for some fixed $\eta > 0$, either $c_t > b_t(1 - \delta) + \eta$ or $b_{t+1} > b_t + \eta$. Otherwise, the supremum would be approached by choices such that $\hat{e}(c_t, b_t, b_{t+1})$ and $\delta b_{t+1} - b_t + c_t$ approach zero, which is not the case as c_t (and hence the payment $c_t + \delta b_{t+1} - b_t$) could be increased achieving a higher payoff than the claimed supremum (that this would be possible follows by the assumption that W is strictly positive).

Now, considering the aforementioned values of (c_t, b_{t+1}) chosen when $b_t = b^*$, there is a strictly positive function $\kappa(\nu)$ such that the following is true. For all ν sufficiently small, we have that if $c_t > b_t(1-\delta) + \eta$,

$$\hat{e}(c_t - \nu, b_t - \nu, b_{t+1}) > \hat{e}(c_t, b_t, b_{t+1}) + \kappa(\nu).$$

Also, if $b_{t+1} > b_t + \eta$, then

$$\hat{e}\left(c_t, b_t - \nu, b_{t+1} - \frac{\nu}{\delta}\right) + \delta W\left(b_{t+1} - \frac{\nu}{\delta}\right) > \hat{e}(c_t, b_t, b_{t+1}) + \delta W(b_{t+1}) + \kappa(\nu).$$

In either case, this shows that given a choice (c_t, b_{t+1}) when the balance is $b_t = b^*$, a payoff that is greater by at least $\kappa(\nu)$ can be obtained when the balance is reduced by ν. This can be achieved by keeping the payment $\delta b_{t+1} - b_t + c_t$ unchanged and either decreasing consumption by ν (keeping the next-period balance the same) or decreasing the next-period balance by $\frac{\nu}{\delta}$ (keeping consumption the same). That the latter is possible given the constraints follows because W is nonincreasing. This shows that indeed $TW(b^* - \nu) \geq TW(b^*) + \kappa(\nu)$ whenever ν is sufficiently small, which establishes the result.

Completion of the proof. We now show that, for any $b_1 > 0$, $\bar{V}(b_1) = V(b_1)$. Also, this payoff is attained by a feasible self-enforceable contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, with $\tilde{b}_1 = b_1$. The latter is sufficient for $\bar{V}(b_1) = V(b_1)$ as $\bar{V}(b_1) \geq V(b_1)$ follows straightforwardly from the definition of $\bar{V}$.

Given any date-1 balance $b_1 > 0$, a sequence $(c_t, b_{t+1})_{t=1}^{\infty}$ can be determined by iteratively solving the program given by equations (42) to (44) for $W = \bar{V}$. That a maximum exists, given each balance b_t, follows by Observations 2 and 3 above. We can then define a contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. For all t, we have $\tilde{c}_t = c_t$, $\tilde{b}_t = b_t$, $\tilde{w}_t = \delta \tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t$, and $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$. We now argue that this contract gives the principal a payoff $\bar{V}(b_1)$ and that it is feasible and self-enforceable.

First, we argue that $\bar{V}(\tilde{b}_t)$ is bounded along the sequence of balances $(\tilde{b}_t)_{t=1}^{\infty}$. To see this, recall Observation 4 that $\bar{V}$ is a decreasing function. Also, balances remain positive and bounded away from zero. This follows because there is a $\bar{b} > 0$ such that, if the balance $\tilde{b}_t$ is less than $\bar{b}$, $\bar{V}(\tilde{b}_t)$ equals $V^{\mathrm{FB}}(\tilde{b}_t)$, and the balance necessarily remains constant from then on (and effort and consumption from then on are equal to first-best values). (That the first-best payoff is achievable when $b_t \leq \bar{b}$, for some $\bar{b} > 0$, follows by the argument in Step 3 of the proof of Lemma A.12.) We can then conclude that, for all t,

$$\bar{V}(\tilde{b}_t) = \sum_{\tau=t}^{\infty} \delta^{\tau-t}(\tilde{e}_\tau - \tilde{w}_\tau).$$

In particular, the contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ attains the payoff $\bar{V}(\tilde{b}_1)$ and also the principal's constraints (PC_t) are satisfied (using equation (43)).

Now let us show that the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied. It will be enough to show that, for all t,

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t}\big(v(\tilde{c}_\tau) - \psi(\tilde{e}_\tau)\big) = v\big(\tilde{b}_t(1-\delta)\big). \tag{49}$$

From the perspective of date t, the agent's payoff from obediently working and consuming until date $s > t$ and then "quitting" in the subsequent period and smoothing the available balance is

$$\sum_{\tau=t}^{s} \delta^{\tau-t}\big(v(\tilde{c}_\tau) - \psi(\tilde{e}_\tau)\big) + \delta^{s+1-t} v\big(\tilde{b}_{s+1}(1-\delta)\big).$$

By construction of the proposed contract, this is constant in s for any fixed t, and equal to $v(\tilde{b}_t(1-\delta))$. Equation (49) will follow if we can show that $\delta^{s+1-t}v(\tilde{b}_{s+1}(1-\delta)) \to 0$ as $s \to \infty$.

To establish our claim note that, for each date τ, $\tilde{c}_\tau \geq (1-\delta)\tilde{b}_{\tau+1}$. This follows from the optimality of $c_\tau = \tilde{c}_\tau$ and $b_{\tau+1} = \tilde{b}_{\tau+1}$ in the maximization (42) to (44), with $W = \bar{V}$ and initial balance $b_\tau = \tilde{b}_\tau$. If instead $\tilde{c}_\tau < (1-\delta)\tilde{b}_{\tau+1}$, c_τ can be increased and $b_{\tau+1}$ decreased, holding the payment $\delta b_{\tau+1} - b_\tau + c_\tau$ unchanged, increasing $v(c_\tau) + \frac{\delta}{1-\delta} v((1-\delta)b_{\tau+1})$ (by strict concavity of v), and hence increasing effort $\hat{e}(c_\tau, b_\tau, b_{\tau+1})$. Recalling again that $\bar{V}$ is a decreasing function (Observation 4), the objective in equation (42) is increased, and both constraints (43) and (44) remain intact.

Using equation (1) and the previous observation, we have that, for any date s, $\tilde{b}_{s+1} \leq \tilde{b}_s + \tilde{w}_s$. Iterating, for any $t \leq s$, we have $\tilde{b}_{s+1} \leq \tilde{b}_t + \sum_{\tau=t}^{s} \tilde{w}_\tau$. Recalling that $\bar{V}(\tilde{b}_{\tau+1})$ remains bounded across dates τ, payments $\tilde{w}_\tau$ must remain bounded (due to (43)), say by a value $\bar{w} > 0$. Let also $\check{b} = v^{-1}(0)$. We then have, for any dates t and s, $t \leq s$,

$$\begin{aligned}
\delta^{s+1-t} v\big(\tilde{b}_{s+1}(1-\delta)\big) &\leq \delta^{s+1-t} v\left((1-\delta)\left(\tilde{b}_t + \sum_{\tau=t}^{s} \tilde{w}_\tau\right)\right) \\
&\leq \delta^{s+1-t} v\big((1-\delta)\big(\tilde{b}_t + \bar{w}(s+1-t)\big)\big) \\
&\leq \delta^{s+1-t} v'(\check{b})(1-\delta)\big(\tilde{b}_t + \bar{w}(s+1-t)\big).
\end{aligned} \tag{50}$$

The right-hand side approaches zero as $s \to \infty$ for fixed t. This, together with the fact that balances are positive and bounded away from zero as noted above, establishes $\delta^{s+1-t}v(\tilde{b}_{s+1}(1-\delta)) \to 0$. This proves the equality in equation (49).

Finally, we check feasibility of $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Similar to the observations from equation (50), we have for fixed t that $\delta^{s+1-t}\tilde{b}_{s+1} \to 0$ as $s \to \infty$. This implies that the agent's intertemporal budget constraint (2) is satisfied as an equality. Equation (1) is satisfied by choice of each $\tilde{w}_t$ given the sequence of balances and consumption. We have already argued that these payments remain bounded. Because balances $\tilde{b}_t$ are bounded away from zero, bounded payments in turn imply bounded efforts $\tilde{e}_t$.

It remains to check that consumption $\tilde{c}_t$ is bounded. Note that all of the arguments in the proof of this lemma are unaffected if the feasibility requirement of bounded consumption is dropped, so we have established that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is optimal with the relaxed feasibility condition. In addition, because the agent's intertemporal budget constraint (2) is satisfied as an equality, equation (36) in the proof of Lemma A.11 holds for all t. Given this, the argument in Lemma A.11 continues to apply, implying that consumption $\tilde{c}_t$ is weakly decreasing. Hence, consumption is in fact bounded, and all the original feasibility conditions are satisfied. *Q.E.D.*

From the proof of Lemma A.13 we have that, in any equilibrium, V is continuous and strictly decreasing. As a result, we have the following dynamics for the principal's continuation payoff.

LEMMA A.14: *Assume* $V(b_1) < V^{\text{FB}}(b_1)$. *Then* $(V(\tilde{b}_t))_{t\geq 1}$ *is a strictly increasing sequence.*

PROOF: Recall from Lemma A.12 we have that, if $V(b_1) < V^{\text{FB}}(b_1)$, then $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing. The result then follows from the fact that $V(\cdot)$ is strictly decreasing (see Observation 4 in the proof of Lemma A.13). *Q.E.D.*

We then show that, if the first-best outcome is not attainable in a self-enforceable relational contract, effort is always downward distorted.

LEMMA A.15: *Assume* $V(b_1) < V^{\text{FB}}(b_1)$. *Then, in any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, $v'(\tilde{c}_t) > \psi'(\tilde{e}_t)$ *for all* t.

PROOF: We first show that, if there is a date $\check{t}$ with $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$, then $v'(\tilde{c}_{\check{t}+1}) = \psi'(\tilde{e}_{\check{t}+1})$. To do so, assume for a contradiction that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ and $v'(\tilde{c}_{\check{t}+1}) \neq \psi'(\tilde{e}_{\check{t}+1})$ for some date $\check{t}$. Then, by Lemma A.10, we have $v'(\tilde{c}_{\check{t}+1}) > \psi'(\tilde{e}_{\check{t}+1})$ and, therefore, $\tilde{w}_{\check{t}+1} = \delta V(\tilde{b}_{\check{t}+2})$. In turn, this implies

$$\tilde{e}_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \tilde{w}_{\check{t}+1} + \delta V(\tilde{b}_{\check{t}+2}) = V(\tilde{b}_{\check{t}+1}) > V(\tilde{b}_{\check{t}}) = \tilde{e}_{\check{t}} - \tilde{w}_{\check{t}} + \delta V(\tilde{b}_{\check{t}+1}) \geq \tilde{e}_{\check{t}},$$

where the strict inequality follows by the previous lemma, and the weak inequality follows because the principal's constraint is satisfied in an optimal contract at $\check{t}$.

There are two cases: either $\tilde{w}_{\check{t}} < \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}}(\tilde{e}_s - \tilde{w}_s)$ or $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}}(\tilde{e}_s - \tilde{w}_s)$. Consider the first. Define a new contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$, which is identical to the original, except that $\tilde{e}'_{\check{t}} = \tilde{e}_{\check{t}} + \varepsilon$ and $\tilde{e}'_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \nu(\varepsilon)$, with $\nu(\varepsilon)$ defined by

$$\psi(\tilde{e}_{\check{t}} + \varepsilon) + \delta\psi\big(\tilde{e}_{\check{t}+1} - \nu(\varepsilon)\big) = \psi(\tilde{e}_{\check{t}}) + \delta\psi(\tilde{e}_{\check{t}+1}).$$

Thus, $\nu'(0) = \frac{\psi'(\tilde{e}_{\check{t}})}{\delta\psi'(\tilde{e}_{\check{t}+1})}$, and so the change in the NPV of effort is

$$\varepsilon - \delta\nu(\varepsilon) = \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon),$$

which is strictly positive for ε sufficiently small. It is easy to see that the agent's constraint (AC_t^{ob}) is unchanged at all dates except $\check{t}+1$, when the constraint is relaxed. The principal's constraint (PC_t) is unchanged from date $\check{t}+1$ onwards, relaxed at date $\check{t}-1$ and earlier (because the NPV of effort increases), but is tightened at date $\check{t}$. Provided ε is small enough, the date-$\check{t}$ constraint remains intact. Profits increase, contradicting the optimality of the original contract.

Now suppose that $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}}(\tilde{e}_s - \tilde{w}_s)$, and note the above adjustment now leads to a violation of the principal's constraint (PC_t) at date $\check{t}$. In this case, we reduce slightly the payment, effort, and consumption at date $\check{t}$, keeping the agent's payoff unchanged, but ensuring the principal's constraint (PC_t) is satisfied. This has a negligible effect on profits since $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$. Hence, we again contradict the optimality of the original contract.

Now let us demonstrate precisely an adjustment that yields a self-enforceable contract. We further adjust the modified contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$ by reducing the date-$\check{t}$ payment and consumption by an amount $\gamma(\varepsilon)$, and reducing date-$\check{t}$ effort by an amount $\eta(\varepsilon)$ to leave agent payoffs unchanged. The date-$\check{t}$ principal constraint (PC_t) will then hold as equality by setting $\gamma(\varepsilon) = \delta\nu(\varepsilon)$. The requirement that the agent's payoff is unaffected by the adjustment is

$$v\big(\tilde{c}_{\check{t}} - \gamma(\varepsilon)\big) - \psi\big(\tilde{e}_{\check{t}} + \varepsilon - \eta(\varepsilon)\big) = v(\tilde{c}_{\check{t}}) - \psi(\tilde{e}_{\check{t}} + \varepsilon).$$

We then have $v'(\tilde{c}_{\check{t}})\gamma(0) = \psi'(\tilde{e}_{\check{t}})\eta'(0)$. Therefore, the overall increase in date-$\check{t}$ profits from all changes to $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is

$$\begin{aligned}\varepsilon - \delta\nu(\varepsilon) - \big(\eta(\varepsilon) - \gamma(\varepsilon)\big) &= \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon - \delta\nu'(0)\left(\frac{v'(\tilde{c}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}})} - 1\right)\varepsilon + o(\varepsilon) \\ &= \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon),\end{aligned}$$

which is strictly positive for ε sufficiently small.

Hence, for small enough ε, the overall effect on profits of all changes to the original contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is positive, with the continuation profits from date $\check{t}$ increasing. The principal's constraint (PC_t) is relaxed at dates $\check{t} - 1$ and earlier, it is satisfied by construction at $\check{t}$, and it is unchanged from date $\check{t} + 1$ onwards. Again, the fact profits strictly increase contradicts the optimality of $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$.

From the above, and by induction, we have that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at some $\check{t}$ implies $v'(\tilde{c}_t) = \psi'(\tilde{e}_t)$ for all $t \geq \check{t}$. By the Euler equation (14), consumption and effort remain constant from $\check{t}$ onwards. Moreover, the agent's indifference condition in equation (8) is presumed to hold at all dates. This shows that the agent's balances $\tilde{b}_t$ remain constant from date $\check{t}$ onwards, which given the assumption $V(b_1) < V^{\text{FB}}(b_1)$, contradicts the finding of Lemma A.12 that balances strictly decrease. Hence, we cannot have $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at any $\check{t}$. *Q.E.D.*

We use the above results to shed light on the dynamics of effort, pay, and consumption.

LEMMA A.16: *If $V(b_1) < V^{\text{FB}}(b_1)$, then in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, effort $\tilde{e}_t$ and payments $\tilde{w}_t$ strictly increase over time, while consumption $\tilde{c}_t$ strictly declines over time.*

PROOF: Suppose $V(b_1) < V^{\text{FB}}(b_1)$. By the previous lemma and Lemma A.10, the principal's constraint must bind at each date, which can be stated as $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ for all t. Hence, payments are strictly increasing in t by Lemma A.14. We also have $V(\tilde{b}_t) = \tilde{e}_t$ for all t, so effort is strictly increasing as well. Now consider consumption. By Lemma A.11, we know that $\tilde{c}_{t-1} \geq \tilde{c}_t$ for all $t \geq 2$. Hence, if consumption fails to be strictly decreasing, we must have $\tilde{c}_{t-1} = \tilde{c}_t$ for some t. We then have, by equation (14) (and noting that $\tilde{c}_t > (1-\delta)\tilde{b}_t$, also by Lemma A.11), that $\psi'(\tilde{e}_t) = v'(\tilde{c}_t)$. However, this contradicts Lemma A.15. *Q.E.D.*

We now prove convergence to efficiency.

LEMMA A.17: $\lim_{t\to\infty} V(\tilde{b}_t) = V(\tilde{b}_\infty)$, *with $V(\tilde{b}_\infty) = V^{\text{FB}}(\tilde{b}_\infty)$, and where $\tilde{b}_\infty$ is the limit of balances defined in Lemma A.12.*

PROOF: Suppose that $V(b_1) < V^{\text{FB}}(b_1)$ and consider an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. By the continuity of V established in Observation 3 in the proof of Lemma A.13, we have that $\lim_{t\to\infty} V(\tilde{b}_t) = V(\tilde{b}_\infty)$. Lemma A.15 states that $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ for all t, implying by Lemma A.10 that the principal's constraint (PC_t) binds for all t. Therefore, $V(\tilde{b}_t) = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$ for all t. By continuity of $\hat{e}(\cdot,\cdot,\cdot)$, we have

$$\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t = \lim_{t\to\infty} \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty),$$

where $\tilde{c}_\infty \equiv \lim_{t\to\infty} \tilde{c}_t$, which exists because $\tilde{c}_t$ is decreasing and remains above $(1-\delta)\tilde{b}_\infty$ by Lemma A.11. Therefore,

$$V(\tilde{b}_\infty) = \lim_{t\to\infty} V(\tilde{b}_t) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty) = \psi^{-1}\big(v(\tilde{c}_\infty) - v\big((1-\delta)\tilde{b}_\infty\big)\big).$$

Since $V(\tilde{b}_\infty) > 0$ (recall Lemma A.10), $\tilde{c}_\infty > (1-\delta)\tilde{b}_\infty$. Therefore, the Euler equation (14) implies that, necessarily, $\lim_{t\to\infty} \frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} = 1$ and, therefore, $\frac{v'(\tilde{c}_\infty)}{\psi'(\tilde{e}_\infty)} = 1$. It is then clear that both Conditions 1 and 2 of Proposition 3.1 hold for $\tilde{e}_\infty$, $\tilde{c}_\infty$, and $\tilde{b}_\infty$ (instead of $e^{\text{FB}}(b_1)$, $c^{\text{FB}}(b_1)$, and b_1). Also, using that the principal's constraint (PC_t) binds for all t, we can conclude that the limiting payments to the agent are $\tilde{c}_\infty - \tilde{b}_\infty(1-\delta) = \delta\tilde{e}_\infty$, which confirms $V(\tilde{b}_\infty) = \tilde{e}_\infty$ is the principal's first-best payoff (recall Proposition 3.1's expression for V^{FB}). This establishes the result. *Q.E.D.*

(End of the proof of Proposition 5.3.) *Q.E.D.*

PROOF OF PROPOSITION 5.4: If the result does not hold, then there is a date t such that

$$\frac{1}{1-\delta} v\big(\tilde{b}_t(1-\delta)\big) < \sum_{s\geq t} \delta^{s-t}\big(v(\tilde{c}_s) - \psi(\tilde{e}_s)\big).$$

If this date is $t = 1$, then date-t effort can be increased to obtain another self-enforceable contract that is more profitable for the principal, so we may assume $t > 1$. We can then increase the payment to the agent at date $t-1$ by $\varepsilon\delta$ for $\varepsilon > 0$, and reduce the date-t payment by ε. All other variables are unchanged. Provided ε is small enough, all constraints are preserved. Because the date-t payment is reduced, the principal's constraint (PC_t) is then slack at date t.

Because the contract is optimal, but not first best, we have that effort strictly increases over time. We can then change the date-t effort to a value $\tilde{e}'_t$, and the date-$t+1$ effort to $\tilde{e}'_{t+1}$, with $\tilde{e}_t < \tilde{e}'_t < \tilde{e}'_{t+1} < \tilde{e}_{t+1}$, and with

$$\psi(\tilde{e}'_t) + \delta\psi(\tilde{e}'_{t+1}) = \psi(\tilde{e}_t) + \delta\psi(\tilde{e}_{t+1}).$$

All other variables remain unchanged. This affects the agent constraints (AC_t^{ob}) by increasing the profitability of remaining in the contract from date $t+1$ onwards (i.e., the date-$t+1$ constraint is slackened). It relaxes the principal's constraint at date $t-1$ and earlier, because the NPV of effort increases (by convexity of ψ). It tightens the principal's constraint at date t, but provided the changes are small, it remains slack. The principal's constraints are unaffected from date $t+1$ onwards. Because the NPV of effort increases, profits strictly increase. This contradicts the optimality of the original contract, which establishes the result. *Q.E.D.*

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