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K-moduli of log Fano Complete Intersections

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Abstract

Classification problems are an important subclass of problems in algebraic geometry that are mainly considered for projective varieties. One modern approach for such classification problems is the minimal model programme (MMP), which (conjecturally) classifies varieties into certain building blocks. One of those fundamental building blocks are Fano varieties, which although aren't minimal in the sense of the MMP, appear as fibres of Fano fibrations, which are relatively minimal. A natural next step in such classifications is to add all building blocks in a moduli space; moduli spaces are quite sophisticated mathematical objects, which parametrise all objects of a given kind (usually projective varieties with well-defined algebrogeometric conditions). The advantage they bring into classification problems, is that they capture information on the degenerations of the objects that are added to them. As such, much of the recent focus of classification problems has shifted to finding and describing explicitly these moduli spaces.

The construction of such moduli spaces, although highly beneficial, can be quite challenging. Thankfully, at least in the case of smooth Fano varieties, the notion of K-stability has demonstrated such a construction. K-stability serves as a bridge between algebraic and differential geometry; it was initially developed to answer which smooth Fano varieties admit a Kähler-Einstein metric. The answer to this question was achieved as a joint effort by many mathematicians, including the key development by Chen-Donaldson-Sun, and showed that a Fano variety admits such a metric if and only if it is K-polystable. In recent developments, it was demonstrated that K-polystable Fano varieties form a moduli space named the K-moduli space. The added benefit of this construction is due to the advantageous properties the moduli space has, which have been generalised through several iterations, including key developments by Odaka and Xu and his collaborators. However, since this construction is not explicit, the explicit description of this moduli space for specific Fano families can prove quite challenging, with insofar only a handful of known examples due

to Odaka–Spotti–Sun and Liu–Xu. In this thesis, we provide an explicit example for one of the 105 families of Fano threefolds in the Iskovskikh–Mori–Mukai classification, in particular we describe the K-moduli space of the family 2.25, which can be described as the blow up of $\mathbb{P}^3$ along an elliptic curve. In achieving this we make use of the 'reverse moduli continuity method', which is a modification of the original 'moduli continuity method' introduced by Odaka–Spotti–Sun.

The added advantage of K-stability is that the above theories also expand to log pairs, formed by varieties and divisors. In this particular case, the construction is not rigid, which allows for certain flexibility by introducing 'stability conditions'. These stability conditions depend on a continuous parameter, which determines what objects should and shouldn't be added in the moduli space as it varies. The end result is an array of moduli spaces depending on this parameter, which give rise to wall-crossing phenomena. Wall-crossing refers to the fact that there are finite intervals of stability where moduli spaces remain the same. While the theory of K-stability has been developed for log Fano pairs, including a recent proof of the existence of K-moduli for log Fano pairs by Ascher–DeVleming–Liu, higher dimensional examples of such wall crossings are lacking in the literature. The only explicit examples of such wall-crossings exist in lower dimensions, which are also due to Ascher–DeVleming–Liu. In this thesis, we present the first such higher dimensional examples of wall-crossing for the K-moduli of del Pezzo surfaces of degree 4 and their anticanonical divisors. In achieving the above, we make use of the 'moduli continuity method', introduced by Odaka–Spotti–Sun and later generalised to log pairs by Gallardo–Martinez-Garcia–Spotti.

To complete the two main aims and examples presented above, we use Geometric Invariant Theory (GIT). GIT was developed by Mumford in the 1960s and studies quotients of projective varieties by algebraic groups. It is one of the first stability theories, which has inspired the construction of K-stability, and can construct geometric quotients which are similar to moduli spaces. In this thesis, we particularly make use of computational GIT relying on the Hilbert-Mumford numerical criterion. In the process of doing so, we have developed computational material, including the development of theory, algorithms and code to study GIT quotients parametrising the moduli of log pairs formed by complete intersections and hyperplane sections. This computational approach to the GIT of log pairs of complete intersections generalises the work of Gallardo–Martinez-Garcia for hypersurfaces. Our expectation is that the computational techniques developed in this thesis will have applications beyond those explored in this thesis.

Contents

Declaration of Originality

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

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CHAPTER 1

Introduction

1.1 Overview

Classification problems form a large part of the study of modern mathematics, and especially so in algebraic geometry. Out of theories developed recently, and most particularly the Minimal Model Program (MMP), Fano varieties have emerged as a particular type of projective variety which are essential in the study of modern algebraic geometry, as they are conjectured to be fundamental building blocks for other varieties, in some sense.

A smooth Fano variety X has an ample anti-canonical line bundle $-K_X$ [IP99], and equivalently, it has positive Ricci curvature. Fano varieties are bounded in any given dimension if their singularities are somehow bounded (e.g. if they are epsilon-log canonical) by a renowned theorem of Birkar [Bir21] and that as a result, once epsilon is fixed, they can theoretically be classified into a finite number of families. The simplest case is the smooth one. The only Fano curve is the sphere, while smooth Fano surfaces were classified in the 19th century by Pasquale del Pezzo [Pez85; Pez87], and are known as del Pezzo surfaces. There are 10 deformation families overall, eight given as the blow up of $\mathbb{P}^2$ along $1 \leq n \leq 8$ points (with $(-K_X)^2 = 9 - n$) along with $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

For Fano threefolds, Iskovskikh [Isk80], Mori and Mukai classified smooth Fano threefolds [MM03] into 105 distinct families. The subject of classification of Fano varieties in higher dimensions remains an active area of research. The overall number of families of Fano n-folds is unknown, but a result of Kollár–Miyaoka–Mori [KMM92] shows that this number is bounded in the smooth case.

An important aim of many algebraic geometers is to compactify these families into compact moduli spaces. A moduli space is a modern mathematical tool, an algebraic object whose points represent all algebro-geometric objects with some fixed topological invariants, or isomorphism classes of such objects. For us, the moduli spaces of interest will parametrise Fano varieties in the same family, modulo some algebro-geometric conditions. The attempts of the mathematical community at such classifications are ongoing. In particular, the classification of del Pezzo surfaces into a compact moduli space was recently completed by Odaka–Spotti–Sun in the 2000s [OSS16]. For Fano threefolds, only a few number of families have been compactified into a moduli space. Liu–Xu [LX19] compactified families of cubic threefolds, while Spotti–Sun [SS17] compactified families of complete intersections of two quadrics in $\mathbb{P}^n$ where $n \geq 5$. Meanwhile, Liu [Liu22] compactified families of cubic fourfolds. In this thesis, we will compactify the moduli space of the family 2.25 in the Mori- Mukai classification, which can be presented as blow-ups of $\mathbb{P}^3$ along a complete intersection of two quadrics.

Recent attempts at classification have used the theories K-stability and Geometric Invariant Theory (GIT). Geometric invariant theory (GIT), pioneered by Mumford [MFK94], based on Hilbert's classical invariant theory, is an effective method to study the construction of quotients by group actions in algebraic geometry, which can often be used to construct moduli spaces. K-stability, on the other hand, is an algebro-geometric theory which was initially developed to answer which smooth Fano manifolds admit a Kähler-Einstein metric. By Chen–Donaldson-Sun these are K-polystable manifolds [CDS13]. Recent results (e.g. [CP21; LWX19; BX19; AHH19; XZ20]) show that the space parametrising these K-polystable objects gives a compact moduli space which satisfies the criteria of classification we mentioned above. This space is often called a K-moduli space. However, explicit descriptions of these moduli spaces are hard to achieve, as the construction varies with each particular family one wishes to compactify. An interesting observation [ADL19; GMS21], is that a similar K-moduli construction holds for log Fano pairs $(X, (1-\beta)D)$, and that in fact these moduli constructions depend on the parameter β. The different moduli spaces one obtains are given by a finite number of *walls* and *chambers* related by a series of explicit birational transformations. The variety and the divisor are deformed before and after each wall in each moduli space.

The aim of this thesis is to compactify the K-moduli space for family 2.25 of Fano threefolds, in the Mori-Mukai classification, and to ultimately study the K-moduli compactifications of log pairs formed by a Fano complete intersection of hypersurfaces of fixed degree and a hyperplane. Specifically, we aim to provide the first higher dimensional example of wall crossing for the K-moduli of log pairs, where both the variety and divisor admit deformations in the K-moduli both before and after the wall crossing. We aim to introduce a compactification for the moduli space of log Fano pairs of complete intersections and hyperplane sections. We achieve the above by providing a computational setting to characterise all polarizations which give rise to different GIT quotients, in the particular cases of tuples of complete intersection of hypersurfaces of fixed degree and hyperplanes.

Computational approaches to GIT quotients are not a novelty, but what makes this approach novel is that we create the first methods to study GIT quotients computationally where the ambient scheme is the Grassmanian times the projective space. In this setting, the GIT construction depends on the choice of the linearisation, as analysed in great generality by the theory of variations of GIT quotients of Thaddeus [Tha96] and Dolgachev-Hu [DH98]. Essentially, one obtains a division of the space of linearisations into a finite number of *chambers* and *walls*, giving rise to only a finite number of different GIT quotients, related by a series of explicit birational transformations. We specialise these general results to our particular situation, tuples $(X, H_1, \ldots, H_m)$ where X is a complete intersection of the same degree hypersurfaces and the H_i are distinct hyperplanes. Since these methods are very general and depend abstractly only on the dimension, number and degree of polynomials in the complete intersections, they provide a concrete, comprehensive setting to study such GIT problems computationally. To achieve the computational GIT results, we expand the setting of Gallardo–Martinez-Garcia–Zhang [GM18; GMZ18], which in turn comes as a continuation from ideas in Laza [Laz09a]. With emphasis to the case of pairs (S, H) our first main result is the following:

Theorem 1.1 (see Theorem 3.14)**.** *Every point in the above GIT quotient parametrises a closed orbit associated to a pair (S, D) with $D := S \cap H$ in the cases where S is a Calabi-Yau or a Fano complete intersection of k hypersurfaces of degree $d > 1$. Furthermore, if S is Fano, and (S, D) is semistable, then S does not contain a hyperplane in the support of at least one of the hypersurfaces in the complete intersection.*

We also demonstrate an algorithmical method to find all unstable and non-stable tuples.

Theorem 1.2 (see Theorem 3.17)**.** *A tuple $(S, H_1, \ldots, H_m)$ is not $\vec{t}$-stable ($\vec{t}$-unstable, respectively), if and only if there exists $g \in G$ and one-parameter subgroup λ in a finite set $P_{n,d,k,m}$, such that the set of monomials associated to $(g \cdot S, g \cdot H_1, \ldots, g \cdot H_m)$ is contained in a pair of sets $N^{\ominus}_{\vec{t}}(\lambda)$ ($N^{-}_{\vec{t}}(\lambda)$, respectively).*

In the computational GIT setting, we also extend the results of [Zan22] for general complete intersections. In this setting, we show the following:

Corollary 1.2.1 (see Corollary 3.28.1)**.** *If a tuple $\mathcal{T} = \{\sum_{i=1}^{k} \alpha_i f_i = 0 | (\alpha_1 : \cdots : \alpha_k) \in \mathbb{P}^{k-1}\}$ is such that $\operatorname{lct}(\mathbb{P}^n, F) \geq \frac{n+1}{d}$ (respectively, $> \frac{n+1}{d}$) for any hypersurface $F = \{f = 0\}$ in $\mathcal{T}$, then $\mathcal{T}$ is GIT semistable (respectively, stable).*

In addition, we will demonstrate this algorithmic approach by classifying in detail the GIT compactification for all elements of a complete intersection of two quadrics in dimension 1, i.e. an intersection of two quadrics in $\mathbb{P}^3$, and dimension 2, i.e. a del Pezzo surface of degree 4, with a hyperplane section. For the specific GIT classification results, we make use of classifications of intersections of quadrics found in Sommerville [Som59, §XIII] and Dolgachev [Dol12, §8.6]. We expand these results along with ideas presented in [MM90] in order to obtain a complete classification of such intersections and their hyperplane sections based on singularities.

We will use these specific classifications, alongside with the *moduli continuity method* which first appeared in [OSS16], and was expanded for log pairs in [GMS21], along with a generalisation of this method for our particular study which we will term *reverse moduli continuity method*. While the original continuity method constructed a map from the K-moduli to the GIT quotient and then proved it was injective and surjective, the reverse moduli continuity method will construct a map from the GIT quotient to the K-moduli and prove it is an isomorphism. The tools required in each case are very different. In addition to the above methods, a detailed computation of the CM line bundle for the K-moduli allows us to explicitly compactify the K-moduli of the family 2.25 and provide the first explicit example of a K-moduli wall-crossing.

The main conclusion of this thesis is that the computational techniques that are developed to describe GIT problems explicitly can assist in describing K-moduli spaces both in cases of varieties and log Fano pairs. We will have to mention that the connection of GIT and K-stability is not new, and has been known since Odaka–Spotti–Sun used it to compactify the

K-moduli of del Pezzo surfaces [OSS16]. This was further demonstrated when Li–Wang–Xu [LWX19] constructed K-moduli spaces by using GIT charts. This connection was further expanded by Liu–Xu [LX19] and Liu [Liu22] to describe the K-moduli of cubic threefolds and fourfolds respectively. Our first result regarding K-moduli is the following connection between the K-moduli of Fano threefold family 2.25 and the GIT quotient compactifying complete intersections of two quadrics in $\mathbb{P}^3$. We have:

Theorem 1.3 (see Theorem 6.3)**.** *The K-moduli component of family 2.25 is isomorphic as a scheme to the GIT compactification of complete intersections of two quadrics in $\mathbb{P}^3$.*

The connection between K-moduli and GIT also found application in [GMS21], where the K-moduli of log pairs $(S, (1-\beta)D)$ for S a del Pezzo surface of degree 3 and D an anticanonical divisor are studied, and then in Ascher–DeVlemming–Liu [ADL19], where wall crossings in the K-moduli of log pairs $(\mathbb{P}^n, cD)$ were studied, with a particular emphasis on the two-dimensional case. In this thesis, we will study log pairs $(S, (1-\beta)D)$ for S a del Pezzo surface of degree 4, which are smooth complete intersections of 2 quadrics in $\mathbb{P}^4$ and D an anticanonical divisor, which is a hyperplane section, and we will establish a similar link between log K-stability and VGIT. For this particular example, we prove a direct link between log K-stability and VGIT:

Theorem 1.4 (see Theorem 8.6)**.** *Suppose $(S, (1-\beta)D)$ is log K-(semi/poly)stable. Then, (S, D) is $\mathrm{GIT}_{t(\beta)}$-(semi/poly)stable, with slope $t(\beta) = \frac{6(1-\beta)}{6-\beta}$.*

This in turn allows us to produce the first example of higher dimensional wall crossing for K-moduli:

Theorem 1.5 (see Theorem 9.2)**.** *Let $\beta > \frac{3}{4}$. Then the K-moduli component of log pairs $(S, (1-\beta)D)$ is isomorphic as a scheme to the VGIT compactification of pairs (S, D) complete intersections of two quadrics in $\mathbb{P}^4$, and an anticanonical divisor.*

This Theorem in particular allows us to show where the first wall crossing occurs. Our description of the VGIT quotient then gives an explicit description of this wall crossing:

Corollary 1.5.1 (see Corollary 9.2.1)**.** *The first wall crossing occurs at $t(\beta) = \frac{1}{6}$, $\beta = \frac{6}{7}$. In particular, a log Fano pair $(S, \frac{1}{7}D)$ is log strictly K-polystable if S is a complete intersection of two quadrics with at worse $\mathbf{A}_2$ singularities, where D is two lines and a double line tangent at two points, or if S and D have 4 or 2 $\mathbf{A}_1$ singularities.*

One has to note that the GIT descriptions for the examples in [ADL19] were already known, and in most cases relied on computational GIT results; in particular, in one of the cases, [GMS21], the GIT description [GM19] was constructed for K-stability purposes. In addition, one must note, that as of completion of this thesis, the above examples constitute the only known examples of K-moduli compactifications either for Fano families or log Fano pairs, which have been expanded by Theorems 1.3 and 1.5. As such, although the link between GIT and K-stability is well-known, it is still quite hard to establish concretely in specific cases, which is the main focus of this thesis.

In conclusion, this thesis demonstrates how such computational results from the GIT side can give rise to explicit descriptions of K-moduli, especially in the cases of explicit wall-crossings where both variety and divisor are deformed.

1.2 Background

As we mentioned before, constructing moduli spaces for varieties is a persistent aim of modern algebraic geometry. In addition, we mentioned how GIT and K-stability play important roles in such descriptions. In this segment, we provide an introduction of these concepts, for the convenience of the reader. This is an introduction to Chapter 2 which will go into much greater detail in explaining these concepts. We direct the reader to Chapter 2 for a more detailed treatment of these concepts, and, in particular, to [MFK94; Muk03; Xu21; Ols16].

1.2.1 Geometric Invariant Theory

GIT studies how we can construct quotients of actions of algebraic groups on algebraic varieties.

Consider a projective variety X over a field k and let G be a reductive group acting on X. Considering an ample line bundle L on X, i.e. a line bundle inducing projective embedding $X \subseteq \mathbb{P}^n$; the action of G on X extends to an action on $\mathbb{P}^n$ given by a representation $\rho : G \to \mathrm{GL}(n+1)$. Then there exists an induced action of G on the homogeneous coordinate ring:

$$k[X] \colon = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m}).$$

We define

$$k[X]^G\colon=\bigoplus_{m\geq 0} H^0(X,L^{\otimes m})^G$$

to be all the invariant elements of $k[X]$ under the action of G, which for reductive groups is finitely generated by a result of Hilbert [Hil90]. Furthermore, since $k[X]^G \subseteq k[X]$ we obtain an induced rational map

$$X=\mathrm{Proj}(k[X]) \overset{\phi}{\dashrightarrow} X /\!\!/ G := \mathrm{Proj}(k[X]^G)$$

which is constant in orbits, but which is not a morphism as there are some points of X such that for all $f \in k[X]^G$, $f(x)=0$.

To overcome this, we define the set of semi-stable points, X^{ss}, which consists of all points $x \in X$ such that there exists some m and invariant section $s \in H^0(X,L^{\otimes m})$ such that $s(x)\neq 0$. Under this definition we have the following.

Theorem 1.6 (Mumford, [MFK94, Theorem 1.10]). *X^{ss} is an open subset of X and the map ϕ restricts to a well-defined categorical quotient*

$$\phi\colon X^{ss} \longrightarrow X /\!\!/ G.$$

Extending this type of thinking, we define a set of stable points X^s as the set of $x \in X^{ss}$, with the orbit $G\cdot x$ closed, and finite stabiliser $\mathrm{Stab}(x)$. Under this definition, we have the following

Theorem 1.7 (Mumford, [MFK94, Converse 1.12]). *X^s is an open subset of X with $X^s \subseteq X^{ss} \subseteq X$ and the map ϕ restricts to a well-defined geometric quotient*

$$\phi\colon X^s \to X/G.$$

One of the main tasks when considering GIT quotients is to describe the good loci X^s, X^{ss}, X^{ps} that allow us to describe quotients of group actions that are varieties, where:

1. $X^s \equiv$ closed orbits of the action with finite stabilisers;

2. $X^{ss} \equiv$ all semistable orbits;

3. $X^{ps} \equiv$ all semistable closed orbits.

The above description is a difficult task without the introduction of a numerical criterion (the Hilbert-Mumford numerical criterion) [MFK94]. The Hilbert-Mumford numerical criterion can reduce the question to checking stability for one-parameter subgroups of G which in turn permits to turn it into a discrete, in fact finite, computation.

This method is the best tool for recognising stable (Case 1), semi-stable (Case 2) and polystable (Case 3) points. We do have to note that although extremely useful, the Hilbert-Mumford numerical criterion can be hard to verify in a number of cases. As such, it is beneficial to develop computational techniques in order to describe these three loci explicitly. This algorithmical technique was studied in [Laz09a] and expanded upon in [GM18; GMZ18].

The study of semi-stable and polystable points is especially significant for a number of reasons. For us, the most important is that the categorical quotient $X /\!\!/ G$ defines a well-defined moduli-space $\overline{M}_X^{GIT}$ of closed semistable orbits, as well as a moduli stack $\mathcal{M}_X^{GIT}$. When we are referring to a moduli problem, we are essentially seeking a classification of geometric, algebraic or topological objects based on a specific property they hold, up to some equivalence. One of the important consequences of GIT is that such a classification is achievable through the GIT quotient construction.

An important consequence of the GIT construction is that the choice of linearisation affects the GIT quotient. If $\mathrm{Pic}^G(X) = \mathbb{Z}$ the linearisation is independent of the quotient, but as Dolgachev-Hu [DH98] and Thaddeus [Tha96] noticed independently, if $\dim(\mathrm{Pic}^G(X)) = 2$ then the dependence on L can be replaced by a single rational parameter $t \in \mathbb{Q}_{\geq 0}$, which we call the slope of L. Thus, we obtain different quotients $X /\!\!/_t G$ and a number of moduli spaces $M_X^{GIT}(t)$, depending on each choice of t. Moreover, it is known, by the general results of Thaddeus [Tha96] and Dolgachev-Hu [DH98], that there exists only a finite number of non-isomorphic quotients. These are a finite number of critical slopes called walls, such that $X /\!\!/_t G \cong X /\!\!/_{t'} G$ for all $t, t' \in (t_i, t_{i+1})$, but $X /\!\!/_t G \ncong X /\!\!/_{t_i} G \ncong X /\!\!/_{t_{i+1}} G$. The open intervals (t_i, t_{i+1}) are called chambers.

One also observes that if $\dim(\mathrm{Pic}^G(X)) = m > 2$ then each quotient depends on a vector $(t_1, \ldots, t_{m-1})$ where similar properties hold. This dependence on linearisations on GIT quotient is termed Variational GIT or VGIT for short. The computational study of (V)GIT has had plenty of advances as of late, particularly aided by the emergence of mathematical computer software. Gallardo– Martinez-Garcia [GM18] have studied a pair (Z, H) of Z a degree d hypersurface, i.e. a hypersurface given as the solution of a degree d homogeneous polynomial in $n+2$ variables $Z = \{f_d = 0\}$ with H a hyperplane in $\mathbb{P}^{n+1}$, acted upon by an algebraic group G (where in their case $G = SL(n+2)$). In their case, they wanted to study the map

$$X = Y \times \mathbb{P}^{n+1} \to \mathbb{P}^N,$$

where here Y is the parametrising space of degree d hypersurfaces, and $\mathbb{P}^{n+1}$ the parametrising space of hyperplanes in $\mathbb{P}^{n+1}$. In more detail, $Y = \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))$. Notice that as the Picard rank of X is 2, the relevant discussion on Section 1.2.1 applies here, giving a number of distinct compactifications $\bar{M}_{\tau_i}$. Gallardo and Martinez-Garcia managed to produce an algorithm to find the τ_i, the number of 1-PS λ and, as such, they produced strong results regarding the (semi)-stability of specific points and the corresponding GIT quotients. This idea was further expanded in [GMZ18], where the case of tuples $(Z, H_1, H_2, \ldots, H_m)$ is analysed.

1.2.2 K-stability

$K-$stability is a modern theory developed initially by Yau–Tian–Donaldson [Tia97], using methods from analytical geometry to describe which varieties admit a Kähler - Einstein metric. Their work came as a solution to the Yau-Tian-Donaldson conjecture, and via the introduction of notions such as the Donaldson-Futaki invariant and test configurations (which generalise one-parameter subgroups) they established which varieties admit such a metric.

In more detail, for a pair (X, L) where X is a projective variety and L is an ample line bundle, a test configuration is a pair $(\mathcal{X}, \mathcal{L})$ such that $\mathcal{X}$ is a $\mathbb{C}^*$-scheme with a flat $\mathbb{C}^*$-equivariant morphism $\pi\colon \mathcal{X} \to \mathbb{C}$ and $\mathcal{L}$ is a relatively ample line bundle on $\mathcal{X}$ such that for all $t \neq 0$ there exists an isomorphism $(\mathcal{X}_t, \mathcal{L}_t) \simeq (X, L)$. The simplest test configuration to define is the *trivial test configuration*, where $\mathcal{X} = X \times \mathbb{C}^*$. Test configurations that are non isomorphic to the trivial one are called *non-trivial test configurations*. The Donaldson-Futaki invariant for the test configuration is a numerical invariant

$$\mathrm{DF}(\mathcal{X}, \mathcal{L})\colon = \frac{b_0 a_1 - a_0 b_1}{a_0^2},$$

where the a_i are the first coefficients of the Hilbert polynomial for the central fiber of the test configuration, and the b_i are the coefficients of the weight function $w(k)$ of the $\mathbb{C}^*$-action on $\mathcal{L}^*|_{\mathcal{X}_0}$, which is a polynomial for $k \gg 0$. Yau [Yau96], Tian [Tia97] and Donaldson [Don02] defined:

1. X is K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all non-trivial test configurations;

2. X is K-polystable if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ only for trivial test configurations.

One of the main results of K-stability is the following theorem.

Theorem 1.8 (Chen–Donaldson–Sun [CDS13])**.** *Let X be a smooth Fano manifold. Then X admits a Kähler-Einstein metric if and only if it is K-polystable.*

This is particularly important since the emergence of the study of K-moduli spaces. Similarly to the GIT constructions and results, the space $\overline{M}^K$ which parametrises K-polystable varieties is a good moduli space in the sense of Alper [Alp+20]. In addition, recent research shows that this K-moduli space satisfies a number of desirable properties, which make it an ideal candidate for the compactification of the moduli spaces of Fano varieties. In [Alp+20], it is shown that this moduli space must be separated. Another property is that the moduli space is projective. Projectivity arises from [CP21; XZ20] which observed that a natural line bundle, called the CM line bundle, on the K-moduli, must be ample.

In [LXZ22], the authors solved the Higher Rank Finite Generation Conjecture, which also showed that the moduli space is proper.

These results rely on tremendous expansions of the theories of moduli spaces, which unfortunately is impossible to detail in this section. They, however, make the explicit descriptions of K-moduli spaces even more desirable. When we compactify the moduli spaces, the difficulty arises from 'adding' limit elements (obtained either as limits of degenerations or as Gromov-Hausdorff limits) that can be singular. Although we do know their singularities have to be at worse klt, since they are limits of K-polystable elements, knowing exactly what their singularities are might be difficult. As such, we need to study the singularities of the limit points that could potentially compactify the moduli spaces. In addition, we can use GIT to provide explicit compactifications. GIT quotients are much better understood than K-stability, so there are plenty of benefits of this approach.

Trying to relate K-stability with GIT is not something new. This idea was explored even before most of these results on K-moduli were known, when Mabuchi–Mukai [MM90] studied the moduli space M^{KE} of complete intersections of two quadrics in $\mathbb{P}^4$ that admit a Kähler-Einstein metric, and they showed that it had to be isomorphic to the GIT quotient of the Grassmanian parametrising the intersection of these quadrics under the action of $\mathrm{PGL}(5)$. In short, they showed:

Theorem 1.9 (Mabuchi–Mukai)**.** *There exists a homeomorphism between* $\overline{M}^{KE}$ *and* $\overline{M}^{GIT}$*.*

Here, $\overline{M}^{GIT} = \mathrm{Gr}(2, 15) /\!\!/ \mathrm{PGL}(5)$. After the Yau–Tian–Donaldson conjecture was solved, the space was shown to be equivalent to the K-moduli space.

Odaka–Spotti–Sun [OSS16] expanded upon this idea and used the *moduli continuity method* to show that similar homeomorphisms exist for other families of del Pezzo surfaces that admit K-polystable members. As an example

Theorem 1.10 (Odaka–Spotti–Sun)**.** *There exists a homeomorphism between* $\overline{M}^{K}$ *and* $\overline{M}^{GIT}$*, where* $\overline{M}^{K}$ *parametrises K-polystable cubic surfaces and* $\overline{M}^{GIT} = \mathbb{P}\big(H^0(\mathbb{P}^3), \mathcal{O}(3)\big) /\!\!/ \mathrm{PGL}(4)$.

The moduli continuity method is a powerful technique that uses properties of the moduli spaces (such as properness) to show that if one can define a map from the K-moduli to the GIT quotient, it is going to be injective due to the uniqueness of Kähler-Einstein metrics, and that it is surjective upon dependence on the GIT compactification, due to the properties of the moduli spaces in the analytic and Euclidean topologies. In order to show this, they use the Gromov-Hausdorff compactification which is known to be canonically homeomorphic to the K-moduli space.

Spotti–Sun [SS17] expanded Mabuchi–Mukai's result to general dimension. Furthermore, Liu–Xu [LX19] showed how one can get such explicit isomorphisms on level of stacks in the case of cubic threefolds. In addition, they provided details of how such isomorphisms may be constructed. After these results, Gallardo–Martinez-Garcia–Spotti [GMS21] extended the moduli continuity method to log Fano pairs and showed that similar homeomorphisms exist in the case of pairs (S, D) where S is a cubic surface and D is a hyperplane section, but they failed to provide such a homeomorphism on the wall crossings. In a further step, Ascher–DeVlemming–Liu [ADL19] extended these isomorphisms to the case of pairs, of a fixed variety and a varying divisor, by introducing a condition on the walls.

1.3 Organization and results

This thesis is organised in three main parts, in addition to a preliminaries section where we survey some of the tools used throughout the thesis (Chapter 2). The first part (Chapter 3) deals with constructing a computational framework to describe GIT quotients of the moduli of polarisations of tuples $(S, H_1, \dots, H_m)$ of a complete intersection of k hypersurfaces of fixed degree d in projective space $\mathbb{P}^n$, S, and hyperplanes H_i, with an emphasis on pairs (S, H). In particular, in this Chapter we prove Theorems 1.1 and 1.2.

The second part (Chapters 4, 5, 7) deals with the explicit description of GIT quotients of intersections of two quadrics in $\mathbb{P}^2$, $\mathbb{P}^3$ and $\mathbb{P}^4$, with a hyperplane section. This includes a detailed classification of pairs of such complete intersections and their hyperplane sections given their singularities. The third part (Chapters 6, 8, 9) focuses on the relation of K-moduli with GIT quotients. In this part, we prove Theorem 1.3. Using similar methods, we obtain our wall crossing example, Theorem 1.5 and Corollary 1.5.1, for log pairs (S, D) of a complete intersection of two quadrics in $\mathbb{P}^4$, S, and the hyperplane section D.

In more detail, Chapter 3 focuses on GIT quotients of tuples $(X, H_1, \dots, H_m)$ of a complete intersection of k hypersurfaces of fixed degree d in projective space $\mathbb{P}^n$, X, and hyperplanes H_i, with an exceptional emphasis on pairs (X, H).

For the case of pairs, the keen reader will notice that $\dim \operatorname{Pic}^G(\mathcal{R}) = 2$ and as such, we are in the VGIT situation we described in Section 1.2.1, where we have a finite number of walls $\{t_0 = 0, t_1, \dots, t_{\max}\}$ that determine the GIT quotient. We formulate a computationally explicit form of the Hilbert-Mumford numerical criterion for this particular case of interest. Using this, we show that there is a finite set of one-parameter subgroups, denoted by $P_{n,k,d}$, that depends only on n, k, d, such that if a pair is unstable or non-stable with respect to some wall/chamber t, there exists a one-parameter subgroup $\lambda \in P_{n,k,d}$ such that the Hilbert-Mumford numerical criterion fails for that λ. We also provide a computational way to find this set. We denote the GIT moduli space (GIT quotient) by $M^{GIT}_{n,k,d}$.

To expand further on the computational setting, we introduce a criterion (*Centroid Criterion*) which will allow us to determine whether a pair is stable or strictly semistable. This criterion depends on the combinatorial nature of the theory, and is polyhedral in nature. Here, we prove Theorem 1.1. We characterise all unstable pairs by providing a family to which they must belong, in terms of equations, and how to obtain it computationally (Theorem 1.2).

We finish the Chapter, by extending some results found in [Zan22], relating log canonical thresholds and the VGIT quotients we have studied. The results of this thesis have appeared in a number of talks and in two separate papers by the author, (c.f. [Gar+21; Pap22b; Pap22a]).

In Chapter 4, we go over some results for the singularities of complete intersections of two quadrics in a general $\mathbb{P}^n$, and we provide the GIT classification of the complete intersection of two conics in $\mathbb{P}^2$ as a toy example for the next chapters to follow. In particular, we show that in this GIT problem, the stable and semistable locus coincide, and are comprised by smooth complete intersections (i.e. pencils whose base locus is 4 distinct points).

In Chapter 5, we completely solve the VGIT problem of the complete intersection of two quadrics in $\mathbb{P}^3$ with a hyperplane section, which describes compactifications of the moduli of log pairs (C, D) of an elliptic curve C with an ample divisor D, by classifying the GIT stable and polystable elements for each wall. This also includes the wall $t = 0$ which corresponds to the GIT problem without the hyperplane section. We achieve this by first obtaining a full classification based on singularities of pairs $(S, D = S \cap H)$. We then apply our VGIT algorithm, which is detailed in Chapter 3 and has been implemented in the computer software SageMath [Pap22c]. This gives us all walls and their corresponding non-stable and strictly polystable elements. We then proceed to use our classification results to solve the VGIT problem. In the case of $t = 0$, we see that a complete intersection will be GIT stable if and only if it is smooth, and will be strictly polystable if and only if it is the unique curve $\tilde{S} = \{x_0x_1 = x_2x_3 = 0\}$, which has 4 ordinary double points ($\mathbf{A}_1$ singularities). In our classification, we notice that each wall corresponds to a particular singularity type of S. As t increases, the singularities of S become worse, from at worse $\mathbf{A}_1$ at $t = 0$, to at worse $\mathbf{D}_4$ at the final chamber. Conversely, the singularities of D in a polystable pair get better as t increases. At the first chamber, D can be a quadruple point at worse, while at the final chamber D can only be smooth (4 distinct points).

Chapter 6 is dedicated to proving Theorem 1.3. The smooth elements of family 2.25 are known to be K-stable (see e.g. [Ara+21, Corollary 4.3.16]). We show, that a singular element of this family, obtained by blowing up along the curve $\tilde{C} = \{x_0x_1 = x_2x_3 = 0\}$ is also K-polystable. Using results from [Ara+21, §4.3] and adapting the moduli continuity method of Odaka–Spotti–Sun [OSS16] to the new *reverse continuity method*, we prove the desired isomorphism of Theorem 1.3.

In Chapter 7 we completely solve the VGIT problem of the complete intersection of two quadrics in $\mathbb{P}^4$ with a hyperplane section, by classifying the GIT stable and polystable elements for each wall. This also includes the wall $t = 0$ which corresponds to the GIT problem without the hyperplane section, and verifies the results found in [MM90] and [AL00]. For $t = 0$, we see that a complete intersection will be GIT stable if and only if it is smooth, and will be strictly polystable if and only if it has 2 or 4 $\mathbf{A}_1$ singularities. We achieve this by first obtaining a full classification based on singularities of pairs $(S, D = S \cap H)$. We then apply our VGIT algorithm, which gives us all walls and their corresponding non-stable and strictly polystable elements. We then proceed to use our classification results to solve the VGIT problem. Here, each wall corresponds to a particular singularity type of S, i.e. polystable elements of each wall have at worse a particular type of singularity. As t increases, the singularities of S become worse, from at worse $\mathbf{A}_1$ at $t = 0$ to $\mathbf{D}_5$ at the final wall. Conversely, the singularities of D in a polystable pair get 'better' as t increases.

In Chapter 8 we introduce a compactification for the moduli space of log Fano pairs of complete intersections and hyperplane sections. We achieve this by studying the CM line bundle for the K-moduli space of log Fano pairs $(X, (1 - \beta D))$, where X is a complete intersection of hypersurfaces of fixed degree and $D = X \cap H$ is a hyperplane section. The CM line bundle, originally introduced by Paul and Tian in the absolute case [PT09] and extended for log Fano pairs by Gallardo–Martinez-Garcia–Spotti [GMS21], is an invariant which plays an important role to the link between K-stability and GIT-stability. We make such an explicit link by proving Theorem 1.4. This allows us to get an explicit relation between t and β.

In Chapter 9 we prove the main Theorem of this thesis, Theorem 1.5. We restrict ourselves to consider log Fano pairs $(S, (1 - \beta)D)$ where S is a complete intersection of two quadrics in $\mathbb{P}^4$ and $D = S \cap H$ is a hyperplane section. The existence of the desired isomorphism is not obvious at first, but it comes from the combination of the previous results in Chapters 7, 8. We use our VGIT classification from Chapter 7 and results by Liu [Liu18] and Kollár [KS88] to show that the singular limits of degenerations of log pairs $(S, (1 - \beta)D)$ up to a specific wall are complete intersections of quadrics and hyperplane sections. This result, along with Theorem 1.4 and the moduli continuity method for log Fano pairs [GMS21] allows us to prove Theorem 1.5. Theorem 1.5 along with our results in Chapter 7 allows us to obtain a full description of the wall crossing, in Corollary 1.5.1.

CHAPTER 2

Preliminaries

2.1 Geometric Invariant Theory

Geometric invariant theory (GIT), pioneered by Mumford [MFK94], based on Hilbert's classical invariant theory, is an effective method to study the construction of quotients by group actions in algebraic geometry, which can often be used to construct moduli spaces. More explicitly, GIT studies how we can construct quotients of actions of algebraic groups on algebraic varieties. The study of GIT has played an integral part in the recent effort of the classification of Fano manifolds, in particular in the compactification of moduli spaces of families of Fano varieties. Odaka–Spotti–Sun used GIT to compactify moduli spaces of Fano surfaces (del Pezzo surfaces) [OSS16], while Spotti–Sun [SS17], Liu–Xu [LX19] and Liu [Liu22] compactified families of complete intersections of two quadrics, cubic threefolds and fourfolds respectively, using explicit descriptions of GIT quotients.

In this section, we provide the preliminaries of algebraic group actions on varieties and how these group actions can be described using representation theory. We then provide a detailed account on how GIT quotients are constructed. In particular, we define geometric and categorical quotients, and we construct projective GIT quotients on actions of reductive algebraic groups. We show that these quotients are geometric and categorical in each case, and we give a description of Variational GIT (VGIT). We then move to describe the Hilbert-Mumford numerical criterion, which is essential in obtaining explicit examples of such quotients. We end this Section with a detailed exposition to the theory of moduli spaces. We introduce moduli problems and why these are studied, and then we proceed to introduce stacks and their relationship to moduli problems. We end the Section by showing that the GIT quotients we have constructed provide good moduli spaces and moduli stacks.

2.1.1 Algebraic Group Actions

2.1.1.1 Definitions and Constructions

To begin our discussion on GIT we first need to give a short introduction to algebraic groups, their representations and how they act on varieties. We will follow the treaty and notation in Mukai [Muk03]. In this section, we let k to be an algebraically closed field.

Definition 2.1. Let A be a finitely generated k-algebra. $G = \operatorname{Spec} A$ is an *affine algebraic group* if there exist k-algebra homomorphisms

$$\begin{aligned} \mu\colon A \to A \otimes_k A \quad &\text{(coproduct)}, \\ \epsilon\colon A \to k \quad &\text{(coidentity)}, \\ \iota\colon A \to A \quad &\text{(coinverse)}, \end{aligned}$$

which satisfy the following three conditions:

1. The diagram

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\mu"] & A \otimes_k A \arrow[d, "1_A \otimes_k \mu"] \\ A \otimes_k A \arrow[r, "\mu \otimes_k 1_A"] & A \otimes_k A \otimes_k A \end{tikzcd}$$

commutes;

2. both of the compositions

$$\begin{tikzcd} & & k \otimes_k A \arrow[dr, "p"] & \\ A \arrow[r, "\mu"] & A \otimes_k A \arrow[ur, "\epsilon \otimes_k 1_A"] \arrow[dr, "1_A \otimes_k \epsilon"] & & A \\ & & A \otimes_k k \arrow[ur, "p"] & \end{tikzcd}$$

where p is the projection, are equal to the identity;

3. the composition, with m the algebra multiplication

$$A \xrightarrow{\mu} A \otimes_k A \xrightarrow{1_A \otimes_k \iota} A \otimes_k A. \xrightarrow{m} A$$

coincide with ϵ.

Remark 2.1.1. One should realise that after taking Spec, the maps μ, ϵ, ι become by abuse of notation $\mu\colon G \times G \to G$, $\epsilon\colon \operatorname{Spec} k \to G$ and $\iota\colon G \to G$ representing the group product, identity element and group inversion respectively. In fact, the three conditions in Definition 2.1 guarantee that G is a group with multiplication μ, identity $\epsilon(\operatorname{Spec} k)$ and inverse ι.

A categorical way of thinking of algebraic groups is the following. An algebraic group G is an algebraic scheme where, along with the definitions of μ, ϵ, ι the functor

$$G\colon \{\text{algebras over } k\} \to \{\text{sets}\},$$

takes values in the category of groups.

Example 2.1.1. Consider the Laurent polynomials $A = k[t, t^{-1}]$. The algebraic scheme $G = \operatorname{Spec} A$ becomes an affine algebraic group by defining the coproduct, coidentity, and coinverse as follows. Let $\mu(t) = t \otimes t$, $\epsilon(t) = 1$, $\iota(t) = t^{-1}$. Then

$$\begin{aligned}
(\mu \otimes_k 1_A)(\mu(t)) &= t \otimes_k t \otimes_k t \\
&= (1_A \otimes_k \mu)(\mu(t)), \\
p\big((\epsilon \otimes_k 1_A)(\mu(t))\big) &= p\big((\epsilon \otimes_k 1_A)(t \otimes t)\big) \\
&= p(1 \otimes_k t) \\
&= t, \\
p\big((1_A \otimes_k \epsilon)(\mu(t))\big) &= p\big((1_A \otimes_k \epsilon)(t \otimes t)\big) \\
&= p(t \otimes_k 1) \\
&= t.
\end{aligned}$$

Hence, the compositions are equivalent to the identity. Also,

$$\begin{aligned}
\big(1_A \otimes_k \iota(\mu(t))\big) &= m\big((1_A \otimes_k \iota)(t \otimes t)\big) \\
&= m\big(t \otimes t^{-1}\big) \\
&= 1,
\end{aligned}$$

and hence the composition is equivalent to ϵ. This group $G = \operatorname{Spec} A$ is denoted by $\mathbb{G}_m$ and is called the *multiplicative group*. It also admits a natural embedding to $\mathbb{A}^1$ via taking Spec to the map

$$k[t] \hookrightarrow k[t, t^{-1}].$$

Any algebraic torus is isomorphic to $\mathbb{G}_m^N$ and as such is an affine algebraic group.

Example 2.1.2. The matrix groups $\mathrm{GL}_n, \mathrm{SL}_n$ and PGL_n are algebraic groups. To see this, first consider the ring $A = k[x_{i,j}, (\det(x))^{-1}]$, where here $x = [x_{i,j}]$, i.e. the polynomial ring with n^2 variables $x_{i,j}$ where $1 \leq i, j \leq n$, and the inverse of the determinant added as a generator modulo the obvious relations. Then $G = \operatorname{Spec} A \hookrightarrow \mathbb{A}^{n^2}$ becomes an algebraic group if we define

$$\mu(x_{i,j}) = \sum_{l=1}^{n} x_{i,l} \otimes x_{l,j} \quad \epsilon(x_{i,j}) = \delta_{i,j} \quad \iota(x_{i,j}) = (\det x)^{-1} (\operatorname{adj} x)_{i,j}.$$

Checking the conditions of Definition 2.1 is a bit more challenging than in Example 2.1.1, but not hard to see.

Here,

$$\begin{aligned}
(\mu \otimes_k 1_A)(\mu(x_{i,j})) &= (\mu \otimes_k 1_A)\big(\sum_{l=1}^{n} x_{i,l} \otimes x_{l,j}\big) \\
&= \sum_{l=1}^{n} \big(\sum_{m=1}^{n} x_{i,m} \otimes x_{m,l} \otimes x_{l,j}\big) \\
&= (1_A \otimes_k \mu)(\mu(x_{i,j})), \\
p\big((\epsilon \otimes_k 1_A)(\mu(x_{i,j}))\big) &= p\big((\epsilon \otimes_k 1_A)(\sum_{l=1}^{n} x_{i,l} \otimes x_{l,j})\big) \\
&= p(1 \otimes x_{i,j}) \\
&= 1, \\
p\big((1_A \otimes_k \epsilon)(\mu(x_{i,j}))\big) &= p\big((1_A \otimes_k \epsilon)(\sum_{l=1}^{n} x_{i,l} \otimes x_{l,j})\big) \\
&= p(x_{i,j} \otimes 1) \\
&= 1.
\end{aligned}$$

Hence, the compositions are equivalent to the identity. Also,

$$\begin{aligned} m\big((1_A \otimes_k \iota)(\mu(x_{i,j}))\big) &= m\big((1_A \otimes_k \iota)(\sum_{l=1}^{n} x_{i,l} \otimes x_{l,j})\big) \\ &= m\big(\sum_{l=1}^{n} x_{i,l} \otimes (\det x)^{-1} (\operatorname{adj} x)_{l,j}\big) \\ &= (\det x)^{-1} \det x \\ &= 1. \end{aligned}$$

Thus, $G = \operatorname{Spec} A$ is an algebraic group. In fact $G = \mathrm{GL}(n)$. One way to see this is that one can interpret A as the localization of $k[x_{i,j}]$ at the polynomial $p = (\det(x))^{-1}$. Then, $G = \operatorname{Spec} A$, is the localization of $\operatorname{Spec} k[x_{i,j}] = \mathbb{A}^{n^2}$ away from $p = 0$. In particular, G is an open subscheme of $\mathbb{A}^{n^2}$ which contains all $n \times n$ matrices where $(\det(x))^{-1} \neq 0$.

We also define $B = k[x_{i,j}]/(\det x - 1)$, which is a quotient of A with the ideal $I = (\det x - 1) \subset A$. In particular, $I \subseteq \ker \epsilon$, and hence we can define maps μ, ϵ, ι on B by restriction (and some abuse of notation) which make $S = \operatorname{Spec} B$ an affine algebraic group. As one would expect, $S = \mathrm{SL}(n)$.

For $\mathrm{PGL}(n)$, notice that $\mathrm{PGL}(n) = \mathrm{GL}(n)/Z$ where $Z := \{aI_n | a \in k^\times\}$ is the centre. In particular Z is a subgroup of $\mathrm{GL}(n)$ and hence $\mathrm{PGL}(n)$ is an affine algebraic group, since it is the quotient of two affine algebraic groups.

Definition 2.2. A *homomorphism of algebraic groups* G and H is a morphism of varieties $f\colon G \to H$ such that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f \times f"] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

commutes.

2.1.1.2 Representation Theory and Algebraic Group Actions

An integral part needed to define algebraic group actions on varieties is the representation theory of algebraic groups. See [FH91; CR06] for more information. Below are two different definitions of representations of algebraic groups.

Definition 2.3. A *representation* of an (algebraic) group G on a finite dimensional vector space V over a field k is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ such that $\rho(xy) = \rho(x)\rho(y)$. A representation is called *faithful*, if the group homomorphism ρ is injective.

Two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\pi\colon G \to \mathrm{GL}(W)$ are *equivalent* or *isomorphic* if there exists a vector space isomorphism $\zeta\colon V \to W$ such that for all $x \in G$, $\zeta \circ \rho(x) \circ \zeta^{-1} = \pi(x)$.

For algebraic groups specifically, one can use the following definition:

Definition 2.4. An *algebraic representation* of the affine algebraic group $\mathrm{Spec}\, A$ is a pair consisting of a vector space V over a field k and a linear map $\mu_V\colon V \to V \otimes_k A$ such that:

1. The composition
$$V \xrightarrow{\mu_V} V \otimes_k A \xrightarrow{1\otimes\epsilon} V \otimes k \cong V$$
is the identity;

2. the diagram
$$\begin{tikzcd} V \arrow[r, "\mu_V"] \arrow[d, "\mu_V"] & V \otimes_k A \arrow[d, "\mu_V \otimes_k 1_A"] \\ V \otimes_k A \arrow[r, "1_V \otimes_k \mu_A"] & V \otimes_k A \otimes_k A \end{tikzcd}$$
commutes.

Lemma 2.5. *Definitions 2.3 and 2.4 are equivalent for affine algebraic groups.*

Proof. Let $G = \mathrm{Spec}\, A$ be an affine algebraic group and $\rho\colon G \to \mathrm{GL}(V) \cong \mathrm{GL}(n)$ be a representation, where $n = \dim V$. Then as in Example 2.1.2, we have $B = k[x_{i,j}, (\det(x))^{-1}]$, $GL(n) = \mathrm{Spec}\, B$ and we can define a number of pullbacks of the $x_{i,j}$ along ρ (at the ring level), i.e. $f_{i,j} := \rho^* x_{i,j}$ which are functions on G. By taking a basis $\{e_k\}$ of V, we can define $\mu_V\colon e_i \to \sum_k e_k \otimes f_{k,i}$. Under this definition,

$$(1 \otimes \epsilon) \circ \mu_V(e_i) = (1 \otimes \epsilon)(\sum_k e_k \otimes f_{k,i}) = e_i \otimes 1 = e_i$$

since $\epsilon(f_{k,i}) = \rho^*(\epsilon(x_{i,j})) = \rho^*(\delta_{i,j})$.

Also,

$$(1_V \otimes \mu_A) \circ \mu_V(e_i) = (1_V \otimes \mu_A)(\sum_k e_k \otimes f_{k,i}) = \sum_k \big(\sum_j e_k \otimes f_{k,j} \otimes f_{j,i}\big) = (\mu_V \otimes 1_A) \circ \mu_V(e_i),$$

since $\mu_A(f_{k,i}) = \rho^*(\mu_B(x_{k,i})) = \rho^*(\sum_j x_{k,j} \otimes x_{j,i}) = \sum_j f_{k,j} \otimes f_{j,i}$.

For the converse, see [Muk03, Remark 4.3]. □

Example 2.5.1. For $\mathbb{G}_m$, where $m \in \mathbb{Z}$ one can find particularly easy representations. For a vector space V over k, one can define

$$\mu_V \colon V \to V \otimes k[t, t^{-1}], \quad v \to v \otimes t^m.$$

Verifying that this definition satisfies the conditions of Definition 2.4 is an easy exercise. An important consequence is that every such representation $\rho \colon \mathbb{G}_m \to \mathrm{GL}(V)$ has a weight decomposition with respect to the above representations $V \cong \bigoplus V_m$ where here $V_m := \{v \in V | \mu(v) = v \otimes t^m\}$.

We are now in a position to define algebraic group actions on (affine) varieties.

Definition 2.6. Let X be a scheme and $G = \operatorname{Spec} A$ an affine algebraic group. An *action* of G on X is a morphism of schemes $\sigma \colon G \times X \to X$ such that the following diagrams

$$\begin{tikzcd} \operatorname{Spec} k \times X \arrow[r, "\epsilon \times \mathrm{id}_X"] \arrow[dr, "\cong"] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

and

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "\mu_G \times \mathrm{id}_X"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

commute. Here, μ_G and ϵ are the coproduct and coidentity of G after taking Spec.

One easily notices the similarity between Definitions 2.4 and 2.6. This is not by chance; in fact, our knowledge of representation theory for algebraic groups is integral for investigating how to study such actions. In fact, in the case of affine schemes, the representation induces the group action.

Remark 2.6.1. Notice that this definition is compatible with the usual classical definition of group actions. To see this, note that the first commutative diagram shows that the composition $X \to X \times G \to X$ is given by $x \to (e, x) \to e \cdot x$ is the identity (i.e. that $e \cdot x = x$).

Similarly, the second commutative diagram guarantees that $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.

These are the axioms of the classical (left) group action. In most examples we will use this notation to give describe specific instances of group actions.

Definition 2.7. Let G be an algebraic group and X a scheme. Given an algebraic group action $\sigma\colon G \times X \to X$ and x a point of X

1. The *orbit* $G \cdot x$ of the point x is defined to be the set theoretic image of the morphism $\sigma_x := \sigma(-, x)\colon G(k) \to X(k)$ which is given by $g \to g \cdot x$;

2. The *stabiliser* $\mathrm{Stab}(x)$ or G_x of x is defined to be the fibre product of the morphism $\sigma_x\colon G \to X$ and $x\colon \operatorname{Spec} k \to X$. Furthermore, it is a closed subscheme of G, which is closed under multiplication, and hence is a subgroup of G.

3. If all orbits of the action are Zariski-closed, then the action is called *closed*.

Example 2.7.1. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ via scalar multiplication, i.e. $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. This is a group action easily verifiable from Definition 2.6 and Remark 2.6.1, and it has only two possible orbits: the origin, and lines through the origin minus the origin.

Now consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x_1, x_2) = (tx_1, t^{-1}x_2)$. As before, the origin is one of the orbits of this action, as well as each of the two axes minus the origin. The other orbit occurs at points where $x_1x_2 = c$, where c is non-zero, i.e. at conics $\big\{(x_1, x_2) | x_1x_2 = c, x \in \mathbb{A}^1 \setminus \{0\}\big\}$. The origin and conics are both closed orbits, while the axes minus the origin are not closed, as they contain the origin in their closure.

Notice, that the theoretical framework we have set up does not allow us to separate the orbit that corresponds to the origin and the orbits of the axes minus the origin topologically, which would be necessary in constructing a topological quotient [Muk03, page 159, Example 5.1]. Hence, the quotient space of $\mathbb{A}^2/\mathbb{G}_m$ cannot be an algebraic scheme, as it cannot be separated. Thus, in order to construct a topological quotient, we have to remove the orbit corresponding to the origin $\{0, 0\}$, as this is the orbit that breaks separation in the quotient. This will allow us to obtain a quotient scheme, $(\mathbb{A}^2 \setminus \{0, 0\})/\mathbb{G}_m$. In this example, the bad locus (the origin) needed to be removed in order to obtain a quotient scheme is easy to identify, but in many cases it is much more complex to identify what the bad locus is. In fact, a big part of GIT focuses on identifying limits of orbits and removing them in order to obtain well-defined algebraic quotients.

Theorem 2.8. *Let G be an affine algebraic group acting on a scheme X. Then, the dimension of the stabiliser subgroup (respectively, orbit) viewed as a function $X \to \mathbb{N}$ is upper semi-continuous (respectively, lower-semi-continuous); that is, for every n, the sets $\{x \in X | \dim G_x \geq n\}$ and $\{x \in X | \dim G_x \leq n\}$ are closed in X.*

Proof. Consider the graph of the action

$$\Gamma = (\mathrm{pr}_X, \sigma)\colon G \times X \to X \times X$$

and the fibre product P

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "\Delta"] \\ G \times X \arrow[r, "\Gamma"] & X \times X \end{tikzcd}$$

where $\Delta\colon X \to X \times X$ is the diagonal. The k-points of the fibre product P consist of pairs (g, x) such that $g \in G_x$. The function on P which sends $p = (g, x) \in P$ to the dimension of $P_{\phi(p)} := \phi^{-1}(\phi(p))$ is upper semi-continuous by [Gro66, p. 13.1.3] or by [Har77, p. III 12.8]. By restricting to the closed subscheme $X \cong \{(e, x)|x \in X\} \subset P$ the result follows. $\square$

An important definition we will require when talking about quotients, is the invariant elements of the action.

Definition 2.9. Given a representation $\mu\colon V \to V \otimes A$ of the affine algebraic group $G = \operatorname{Spec} A$, a vector $v \in V$ is *G-invariant* if $\mu(v) = v \otimes 1$. The subspace of invariant vectors is denoted by

$$V^G := \{v \in V|\mu(v) = v \otimes 1\}.$$

Definition 2.10. Let $G = \operatorname{Spec} A$ be an affine algebraic group. A 1-*dimensional character* χ is a function $\chi \in A$ such that $\mu(\chi) = \chi \otimes \chi$ and $\iota(\chi)\chi = 1$.

2.1.2 GIT Construction

2.1.2.1 Geometric and Categorical Quotients

Before we begin to discuss group quotients, it is beneficial to discuss what constitutes a good quotient in our framework. This important distinction is necessary, because the typical orbit space $X/G = \{G \cdot x|x \in X\}$ may not always admit the structure of a scheme. This was seen in Example 2.7.1.

Definition 2.11. Let X be a scheme. A *categorical quotient* of X by the action of G is a G-equivariant morphism $\psi\colon X \to Y$, where Y is a G-scheme which is G-invariant with respect to the action (i.e. ψ is constant on G-orbits, and every point $y \in Y$ is G-invariant), and is universal, i.e. any G-invariant morphism $\delta\colon X \to A$ factors as $\delta = \chi \circ \psi$ for unique $\chi\colon Y \to A$.

Definition 2.12. Let X be a scheme. A *geometric quotient* of X by an algebraic G group is a categorical quotient $\phi\colon X \to Y$ which is affine, such that the fibres $\phi^{-1}(y)$ are preorbits for each $y \in Y$ (i.e. they are G-orbits for each $y \in Y$) with additional properties [DK07]:

1. if $U \subseteq Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ is an isomorphism onto the G-invariant functions;
2. for W_1, W_2 disjoint, closed G-invariant subschemes of X, $\phi(W_1), \phi(W_2)$ are disjoint, closed subschemes of Y.

Definition 2.13. A morphism of schemes X and Y, $\phi : X \to Y$ is a *good quotient* for the action of G on X if

1. ϕ is affine, G-invariant (i.e. ϕ is constant on orbits) and surjective;
2. if $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\phi^{-1}(U))^G$;
3. if $W \subset X$ is a G-invariant closed subset of X, its image $\phi(W)$ is closed in Y;
4. if W_1 and W_2 are disjoint G-invariant closed subsets, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint.

In fact, every good quotient is categorical.

Theorem 2.14. *Let G be an algebraic group acting on a scheme X and let $\phi\colon X \to Y$ be a good quotient. Then ϕ is a categorical quotient.*

Proof. See [MFK94, Chapter 0.2, Remark 6]. □

The reverse statement is not true. However, counterexamples are hard to produce; we refer the reader to [AH99; AH00] for counterexamples. The below example demonstrates that a good quotient need not be geometric.

Example 2.14.1. Let $G = \mathbb{G}_m$ act on $\mathbb{C}^n$ by $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. The map $\mathbb{C}^n \to \mathrm{pt} = \mathrm{Spec}(\mathbb{C}^n)$ is a good quotient for this action, but not a geometric quotient, as the preimage of pt consists of multiple orbits.

Remark 2.14.1. The definitions of good and geometric quotients are local in the target; this implies that if $\phi\colon X \to Y$ is G-invariant and there exists a cover of Y by open subsets U_i such that the maps $\phi|_{U_i}\colon \phi^{-1}(U_i) \to U_i$ are all good (respectively, geometric) quotients, then so is $\phi\colon X \to Y$. This follows directly from the properties in Definitions 2.12 and 2.13.

After our discussion in Example 2.7.1 it becomes more apparent why we seek such quotients. Namely, we aim to find geometric and categorical quotients because these allow us to distinguish between the orbits of the action and carry a universal property. In particular, for geometric and good quotients we also obtain 'nice' properties between the rings of invariants of the action, and we can guarantee that disjoint sets are disjoint in the action.

In attempting to find a quotient scheme by an algebraic group action, the below method was first developed by Hilbert [Hil90]. Consider a (quasi-)projective algebraic scheme X embedded into some projective space, $\mathbb{P}^n$, i.e. $X \subset \mathbb{P}^n$, acted upon by a group G. Hilbert's initial approach was to find enough invariant homogeneous polynomials $f_0, \ldots, f_N$, where $deg(f_i) = k_i$ that would generate the invariant ring of sections of X by the G-action. Here, invariance implies that for any $g \in G$, we have $f_i(gx) = f_i(x)$. Having obtained the $N+1$ invariant polynomials, we can define a rational map

$$\begin{aligned} X & \overset{\pi}{\dashrightarrow} \mathbb{P}^N \\ p & \longmapsto (f_0(p) : \ \ldots : f_N(p)). \end{aligned}$$

Since the f_i are invariant under the action, the preimage of a point in the image of π, $Im(\pi)$, is an orbit of the action, hence π is a categorical quotient. Although this approach has its merits, we quickly run into a few problems:

1. If the ring of invariant sections is not finitely generated, N will not be finite, and as such the rational map will not be defined. This is why, as we will show later, one is required to consider reductive groups, which have a finitely generated ring of global G-invariant sections [Hil90];

2. another issue that arises, is that we require the image of π to be a closed algebraic scheme, i.e. for the closure of the image of π to be equal to the image of π. This approach certainly does not guarantee such a condition, and even for the reductive case it is a hard condition to check [BJK17];

3. we still have to remove the bad locus in order to construct the quotient, as the map π is not everywhere defined (specifically for points where $f_0 = \cdots = f_N = 0$);

4. invariant polynomials such as the f_i presented above are extremely difficult to find in most, if not every, case.

It is apparent that one has to take different routes in order to describe good or categorical quotients of such actions.

2.1.2.2 Reductive Groups

Mumford [MFK94] achieved the construction of a quotient scheme by considering reductive groups. Before we introduce these, we have to introduce some notation.

Definition 2.15. Let $G = \operatorname{Spec} A$ be an affine algebraic group. An element $g \in G$ is called *unipotent* if there exists a faithful linear representation $\rho\colon G \to \mathrm{GL}(n)$ such that $\rho(g)$ is unipotent. Furthermore, G is *unipotent* if every non-trivial linear representation $\rho\colon V \to V \otimes A$ has a non-zero G-invariant vector v.

Alternatively, one can think of unipotent groups as groups whose elements are unipotent.

Definition 2.16. An affine algebraic subgroup $H = \operatorname{Spec} A_0$ of $G = \operatorname{Spec} A$ is called *normal* if the conjugation action $H \times G \to G$ of H on G given by $(h, g) \to ghg^{-1}$ factors through the inclusion $H \hookrightarrow G$.

Definition 2.17. An affine algebraic group $G = \operatorname{Spec} A$ is *reductive* if it is non-singular (smooth) and every smooth unipotent normal algebraic subgroup of G is trivial.

Definition 2.18 ([Muk03], Definition 4.36)**.** An affine algebraic group $G = \operatorname{Spec} A$ is *linearly reductive* if for any epimorphism $\phi\colon V \to W$ of G-representations, the induced map on G-invariant vectors $\phi^G\colon V^G \to W^G$ is surjective.

Theorem 2.19. *The affine algebraic group G is linearly reductive if and only if every finite dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ is completely reducible, i.e. V factors as a direct sum of irreducible subrepresentations.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite dimensional representation and $V' \subset V$ a G-invariant subspace. The vector spaces $\operatorname{Hom}(V, V')$ and $\operatorname{Hom}(V', V')$ are both representations, and the natural map

$$\phi\colon \operatorname{Hom}(V, V') \to \operatorname{Hom}(V', V')$$

is an epimorphism of G-representations. Since G is linearly reductive, the induced map ϕ^G is surjective. Hence, the identity map $1_{V'}$ lifts to a G-equivariant morphism $f\colon V \to V'$, with a complement $V'' = \ker f$ and a decomposition $V = V' \oplus V''$. One can repeat this procedure as long as there is a non-trivial G-invariant subspace. Since every time $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)) < \infty$ eventually this process stops.

For the reverse statement, let $\phi\colon V \to W$ be an epimorphism of G-representations and define $V' := \ker \phi$. By the assumption, there exists V'' such that $V = V' \oplus V''$. In particular, since ϕ is an epimorphism, we have $V'' \cong W$. Notice that both V' and V'' are G-invariant, hence $V^G = (V')^G \oplus (V'')^G$, which implies that the induced map $\phi^G\colon V^G = (V')^G \oplus (V'')^G \to W^G \cong (V'')^G$ is surjective. □

Corollary 2.19.1. *Every finite algebraic group G is linearly reductive.*

Proof. If one applies the techniques in the proof of [FH91, Proposition 1.5] to Theorem 2.19, one obtains the required statement. □

Example 2.19.1. We have seen already in Example 2.5.1 that the representations of the group $\mathbb{G}_m$ are completely reducible, hence $\mathbb{G}_m$ is linearly reductive.

Definition 2.20. An affine algebraic group $G = \operatorname{Spec} A$ is *geometrically reductive* if for every finite dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ and every non-zero G-invariant vector $v \in V$ there exists a G-invariant non-constant homogeneous polynomial $f \in \mathcal{O}(V)$ such that $f(v) \neq 0$.

Theorem 2.21. *If an affine algebraic group $G = \operatorname{Spec} A$ is linearly reductive, then it is geometrically reductive.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite dimensional linear G-representation and $v \in V^G$ be a non-zero G-invariant vector. Then v determines a G-invariant linear form $f\colon V^* \to k$, where f is the dual of v. By letting G act trivially on the field k, k becomes a representation of G and f can be seen as an epimorphism of G-representations. Since G is linearly reductive, the induced map f^G is surjective, and hence for the fixed point $1 \in k = k^G$, there exists a non-zero vector $v \in V^G$ such that $f(v) = 1 \neq 0$. □

The following is an important result by Weyl, Nagata, Mumford, Haboush:

Theorem 2.22 ([MFK94; Nag64; Hab75])**.** *An affine algebraic group G is a reductive group if and only if it is geometrically reductive. A linearly reductive affine algebraic group G is reductive (but the converse is not always true). In particular, in characteristic zero, an affine algebraic group G is a reductive group if and only if it is linearly reductive.*

The above result is particularly useful for us, as in the rest of the thesis we will always work in characteristic zero, and in particular over $\mathbb{C}$. In fact, from now one we will only refer to reductive groups, instead of linearly reductive groups. The following result is arguably the most important component for formulating Mumford's GIT.

Definition 2.23. A G-action on a k-algebra A is *rational* if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Remark 2.23.1. Let G be an affine algebraic group acting on X. The induced action of G on $A = \mathcal{O}(X)$ is rational.

Theorem 2.24 ([Hil90])**.** *Let G be a (linearly) reductive group acting rationally on a finitely generated k-algebra A. Then A^G is finitely generated.*

Example 2.24.1. Although we will not prove so, many of the matrix groups are reductive (in characteristic 0, see [Nag62]). These include the groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$ and $\mathrm{PGL}(n)$.

2.1.2.3 Construction of Projective GIT Quotients

The above discussion allows us to progress to the details on how to construct projective quotients via algebraic group actions following Mumford's approach. Since we will be working with projective varieties/schemes, we will only define the projective GIT quotient; the construction for the affine GIT quotient follows a similar, and somewhat simpler rationale.

Consider a projective scheme X over a field k and let G be a reductive group acting on X. It is useful to recall that there is a projective embedding $X \hookrightarrow \mathbb{P}^n$ induced by an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ of X. We will first study how if there exists a linearisation, we can extend the G-action to these line bundles. Considering an ample line bundle $\mathcal{L}$ on X, if the action of G on X extends to an action on $\mathbb{P}^n$, this is induced by a representation $\rho : G \to \mathrm{GL}(n+1)$.

Definition 2.25. Let X be a projective scheme with an action of an affine algebraic group G. A *linear G-equivariant projective embedding $f\colon X \hookrightarrow \mathbb{P}^n$* is an embedding which a choice of group homomorphism $G \to \mathrm{GL}(n+1)$ making f G-equivariant. We say that the G-action on $X \hookrightarrow \mathbb{P}^n$ is *linear* when we have a linear G-equivariant projective embedding of X as above. The line bundle $\mathcal{L}$ inducing the above embedding is called a G-linearised line bundle.

More formally, in the above setting the action of G on the projective space $\mathbb{P}^n$ lifts to an action on the affine space $\mathbb{A}^{n+1}$ and there is an induced G-action on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$

over X. Here:

$$k[\mathbb{A}^{n+1}] = k[x_0, \dots, x_n] = \bigoplus_{m \geq 0} H^0(\mathbb{P}^n, \mathcal{O}(m)).$$

Similarly, we define

$$k[X] := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{\otimes m})$$

the homogeneous coordinate ring of X, which inherits an induced action of G. Notice that in both cases, the G-invariant elements of the coordinate rings

$$k[\mathbb{A}^{n+1}]^G = k[x_0, \dots, x_n] = \bigoplus_{m \geq 0} H^0(\mathbb{P}^n, \mathcal{O}(m))^G$$

and

$$k[X]^G := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{\otimes m})^G$$

are well-defined as all the invariant elements of $k[A]$ and $k[X]$ under the action of G respectively. By Theorem 2.24 both of these rings are finitely generated as G is reductive. We also define $k[X]^G_+$ to be the positive degree part of the invariant ring.

Furthermore, since $k[X]^G \subseteq k[X]$ we obtain an induced rational map

$$X = \operatorname{Proj}(k[X]) \overset{\phi}{\dashrightarrow} X /\!\!/ G := \operatorname{Proj}(k[X]^G).$$

Definition 2.26. For a linear action of G on a projective scheme $X \subset \mathbb{P}^n$ we define the *set of semi-stable points* X^{ss} as follows:

$$X^{ss} := \{x \in X | \exists m \in \mathbb{Z}_{>0}, s \in H^0(X, \mathcal{L}^{\otimes m})^G \text{ such that } s(x) \neq 0\}.$$

The restriction of the map ϕ

$$\phi \colon X^{ss} \longrightarrow X /\!\!/ G$$

is called the *GIT quotient* of this action.

From the above discussion and since $k[X]^G$ is finitely generated, by [Har77, II, Corollary 5.16] it follows that $X /\!\!/ G$ is in fact a projective scheme. Set-theoretically, $X /\!\!/ G$ is the quotient of X^{ss} by the equivalence relation for which $x, y \in X^{ss}$ are equivalent if and only if the closures $\overline{G \cdot x}$ and $\overline{G \cdot y}$ of the G-orbits of x and y meet in X^{ss}. As such, it becomes apparent that this definition will allow us to overcome the problems detailed in Example 2.7.1. In fact, the map ϕ is a categorical quotient, which follows directly from its definition. Thus, we have:

Theorem 2.27. *For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, the GIT quotient $\phi\colon X^{ss} \to X /\!\!/ G$ is a categorical quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X /\!\!/ G$ is a projective scheme.*

The question that remains is whether we can define a geometric quotient of the action. This is achieved by the following construction:

Definition 2.28. For a linear action of G on a projective scheme $X \subset \mathbb{P}^n$ we define the *set of stable points* X^s as follows:

$$X^s := \{x \in X^{ss} | G \cdot x \text{ is closed in } X^{ss} \text{ and } \dim G_x = 0\}.$$

We also define the set of *polystable points* X^{ps} as the set:

$$X^{ps} := \{x \in X^{ss} | G \cdot x \text{ is closed}\}.$$

The set $X \setminus X^{ss}$ is the set of *unstable points*.

By the orbit-stabiliser theorem, the condition $\dim G_x = 0$ is equivalent to $\dim G_x = \dim G$, i.e. stable points are polystable points with maximal orbits. Notice also that, by the above Definition, we have

$$X^s \subseteq X^{ps} \subseteq X^{ss}.$$

Lemma 2.29. *The sets X^{ss} and X^s are open in X.*

Proof. By construction, the set of semistable points is defined as the complement $X \setminus \mathrm{Null}(X)$ where

$$\mathrm{Null}(X) := \{x \in X | \exists m \in \mathbb{Z}, \forall s \in H^0(X, \mathcal{L}^{\otimes m})^G \text{ such that } s(x) = 0\}$$

is by definition a closed set as G is reductive.

For the set of stable points, consider the set $X' = \cup X_f$ where here $X_f := \{x \in X | f(x) \neq 0\}$ is affine, $f \in k[X]^G_+$ and the union is taken over $k[X]^G_+$ such that the action of G on X_f is closed in X^{ss}, and hence each X_f is open in X, and X is also open in X.

In particular, by Theorem 2.8, the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X, since it is the dimension function of the stabiliser subgroup of an affine group action, and so the subset of X' consisting of points with zero dimensional stabiliser is open in X. Hence, we have open inclusions $X^s \subseteq X' \subseteq X$ □

Theorem 2.30. *For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, the restriction of the map ϕ, $\phi\colon X^s \to X^s/G$ is a geometric quotient of the G-action on the open subset X^s of stable points in X. Note that, by abuse of notation, we will denote X^s/G by X/G.*

Proof. We will first prove the result locally. Let $f \in k[X]_+^G$ be an invariant section, and let $X_f = \{x \in X | f(x) \neq 0\}$. As in the proof of Theorem 2.29 each X_f is affine, and let $X' = \cup_f X_f$ be the union over all the X_f; the G-action is closed in X^{ss}. For X/G we define $(X/G)_f = \phi(X_f)$ and $Y' = \phi(X')$. For each f we have maps $\phi_f\colon X_f \to (X/G)_f$ which glue to give a map $\phi\colon X' \to Y'$. Notice that by definition each ϕ_f is a good quotient, and since the action of G on each X_f is closed it is also a geometric quotient. As discussed in Remark 2.14.1, the ϕ_f glue to give a geometric quotient $\phi\colon X' \to Y'$.

Since $X^s \subset X'$ and X/GY' are open subsets of X' and Y' respectively, the restriction on ϕ is the map $\phi\colon X^s \to X/G$ which is also a geometric quotient, due to the local structure of geometric quotients. □

We will later study how the map

$$\phi\colon X^s \to X/G$$

defines a coarse moduli space of stable orbits. One of the main tasks when considering GIT quotients is to describe the good loci X^s, X^{ss}, X^{ps} that allow us to describe quotients of group actions that are projective varieties/schemes.

2.1.3 The Hilbert-Mumford Numerical Criterion

The explicit description of GIT quotients is a difficult task without the introduction of a numerical criterion (the Hilbert-Mumford numerical criterion) [MFK94] which we will now briefly describe.

Definition 2.31. Let G be a linear algebraic group. A 1*-parameter subgroup* (1-PS) is a group homomorphism $\lambda\colon \mathbb{G}_m \to G$. The action of G on X induces an action of $\mathbb{G}_m$ on X via the 1-PS, i.e. $x \mapsto \lambda(t) \cdot x$, $t \in \mathbb{G}_m$.

Definition 2.32. Let X be a projective k-scheme acted upon by an affine algebraic group G induced by a G-linearisation $\mathcal{L}$, and let $x \in X$ and $\lambda\colon \mathbb{G}_m \to G$ be a 1-PS. We define the

Hilbert-Mumford weight $\mu^{\mathcal{L}}(x, \lambda)$ to be the negative of the weight of the induced action of $\mathbb{G}_m$ on the fibre $\mathcal{L}_{x_0}$ where $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$. The *Hilbert-Mumford function*, is defined as:

$$\mu^{\mathcal{L}}(-, \lambda) \colon X \to \mathbb{Z}.$$

Theorem 2.33 ([MFK94, Theorem 2.1])**.** *Let $\mathcal{L}$ be an ample G-linearised line bundle on a projective k-scheme X. Then $x \in X$ is stable (respectively, semistable) with respect to $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x, \lambda) > 0$ (respectively, $\mu^{\mathcal{L}}(x, \lambda) \geq 0$) for all non-trivial 1-PS λ of G.*

Example 2.33.1 ([Muk03, Example 7.12])**.** We will study the (semi-)stability of plane cubics, i.e. cubic curves in $\mathbb{P}^2$ under the action of $\mathrm{SL}(3)$. In this case, the numerical criterion takes the following form: we can pick a maximal torus T in $\mathrm{SL}(3)$, such that all 1-PS belong to that torus. Since all maximal tori in $\mathrm{SL}(3)$ are conjugate, given a 1-PS λ, we can choose a maximal torus T such that λ is diagonal. We can use the Hilbert-Mumford numerical criterion on diagonal 1-PS to prove that a specific hypersurface is unstable or strictly semistable, but unfortunately not stable.

As such, we can think of the 1-PS as diagonal elements $\lambda(t) = \mathrm{Diag}(s^{a_0}, s^{a_1}, s^{a_2}) \in \mathrm{SL}(3)$, so it is customary to think of them as vectors $\boldsymbol{r} = (a_0, a_1, a_2)$ where $\sum a_i = 0$. Each cubic curve consists of a linear span of monomials of the form $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ which can be represented as points on the set

$$\Xi_3 := \{(i_1, i_2, i_3) \in \mathbb{Z}^3 | i_1 + i_2 + i_3 = 3\},$$

and non-zero coefficients c_i. For a cubic curve f we define its *support* as $\mathrm{Supp}(f) := \{I \in \Xi_3 | f = \sum c_I x^I,\ c_I \neq 0\}$. The 1-PS are points in the dual space of monomials, and as such we have a pairing $\langle I, \lambda \rangle = \sum i_i a_i$ where the $I \in \Xi_3$ represents a monomial. The action of the one parameter subgroups can be described via this pairing.

Notice that for all diagonal 1-PS, we have $\langle (1,1,1), \lambda \rangle = \sum_{i=1}^{3} a_i = 0$. Hence, if we arrange the 10 possible monomials in a triangle, the centroid is the monomial $x_0 x_1 x_2$ and every λ has to pass through this monomial. If we pick 1-PS $\boldsymbol{r} = (2, -1, -1)$, the monomials with negative or zero pairing are x_1^3, $x_0 x_1^2$, $x_0 x_2^2$, $x_0 x_1 x_2$, $x_1 x_2^2$, $x_2 x_1^2$, x_2^3, which is the condition that the cubic curve has a singular point at $(1 : 0 : 0)$.

Choosing $\boldsymbol{r} = (1, -2, 1)$ the monomials with negative or zero pairing are x_1^3, $x_0 x_1^2$, $x_0^2 x_1$, $x_0 x_1 x_2$, $x_1 x_2^2$, $x_2 x_1^2$. This is equivalent to the cubic curve having a line $x_1 = 0$ (see Figure 2.1). By a projective change of coordinates, we can assume that if f contains a line, this is the line $x_1 = 0$. In particular, if we consider the symmetry of the triangle the above two choices are

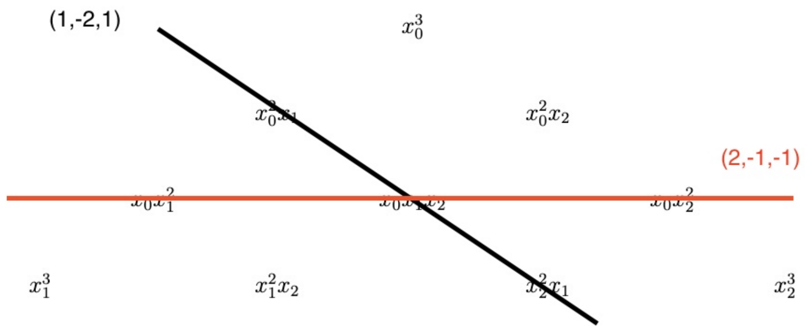

Figure 2.1: Monomials of cubics and action of one-parameter subgroups

the only possibilities as the line r rotates about the centroid. As such, the strictly semistable cubic curves are cubics with only ordinary double points as singularities. Similarly, one can show that a cubic curve is stable if and only if it is smooth.

Although the numerical criterion is the only concrete tool for checking (semi-)stability criteria, the above example is one of the few simple cases. In reality, applying the Hilbert-Mumford numerical criterion can be a challenge, especially for problems in higher codimension.

2.1.3.1 Variations of GIT Quotients

An interesting phenomenon occurs when the Picard rank of the projective G-scheme is bigger than one. In this case, the choice of very ample linearisation $\mathcal{L}$ affects how the GIT quotients are constructed. In this Section we will cover a general enough case which will be relevant to this thesis, but we will not cover the most general one. If the reader is interested, we prompt them to [DH98] and [Tha96], where the authors study the existence of variations of GIT quotients for arbitrary reductive G-action on an arbitrary scheme X. In our case we will deal with schemes $X = X_1 \times \dots X_m$ with $\dim(\mathrm{Pic}(X_i)) = 1$. In our setting, we have:

Lemma 2.34. *Let G be an algebraic group such that $\mathrm{Pic}(G) = \{1\}$ and let X be a normal projective k-scheme such that $X = X_1 \times \dots X_m$ with $\dim(\mathrm{Pic})(X_i) = 1$, such that the action of G on X extends to an action of G on each X_i. Let also every line bundle of X have at most one linearisation class. Then the set of G-linearisable line bundles $\mathrm{Pic}^G(X)$ is isomorphic to $\mathbb{Z}^m$. A line bundle $\mathcal{L} \in \mathrm{Pic}^G(X)$ is ample if and only if*

$$\mathcal{L} = \mathcal{O}(a_1, \dots, a_m) := \pi_1^*\mathcal{O}_{X_1}(a_1) \otimes \pi_m^*\mathcal{O}_{X_m}(a_m),$$

where the p_i are the natural projections on X_i, and $a_i > 0$, and we denote by $\mathcal{O}_{X_i}(a_i)$ an ample generator (over $\mathbb{Z}$) of the Picard group.

Proof. The proof follows [GMZ18, Section 2.1] which comes as a generalisation of [GM18, Lemma 2.1].

Let p_i be the projections. Since the action of G on X extends to an action of G on each X_i, the p_i are morphisms of G-varieties. Recall there is an exact sequence (see [Dol03, Theorem 7.2])

$$0 \longrightarrow \mathcal{X}(G) \longrightarrow \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G)$$

where $\mathcal{X}(G)$ is the kernel of the forgetful morphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$. By assumption $\mathcal{X}(G) = \{1\}$, hence we have an isomorphism $\mathrm{Pic}^G(X) \cong \mathrm{Pic}(X)$. Moreover, given that $\mathrm{Pic}^G(X) \subseteq \mathrm{Pic}(X)^G \subseteq \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)^G$ is the group of G-invariant line bundles, there result follows from

$$\mathrm{Pic}^G(X) \cong \mathrm{Pic}(X)^G \cong p_1^*(\mathrm{Pic}(X_1)) \otimes \cdots \otimes p_m^*(\mathrm{Pic}(X_m)) \cong \mathbb{Z}^m.$$

□

Remark 2.34.1. The condition that the algebraic group G has a trivial Picard group, i.e. $\mathrm{Pic}(G) = \{1\}$, in the above Lemma may seem restrictive, but for this thesis, where $G = \mathrm{SL}(n+1)$, this condition will always be satisfied (see [Dol03, Chapter 7.2]).

If $\dim \mathrm{Pic}(X) = 1$ the choice of linearisation does not affect the construction of $X /\!\!/ G$. This is because in our construction picking $\mathcal{L}$ or $\mathcal{L}^k$ for some $k \in \mathbb{Z}$ does not change the stability conditions as

$$\bigoplus_{m\geq 0} H^0(X, (\mathcal{L}^k)^{\otimes m})^G \cong \bigoplus_{m'\geq 0} H^0(X, \mathcal{L}^{\otimes m'})^G = k[X]^G.$$

Consider now a projective scheme X satisfying the hypotheses of Lemma 2.34, where $\dim \mathrm{Pic}(X) = m > 1$. In this case, $\mathcal{L} \cong \mathcal{O}(a_1, \dots, a_m)$ and the construction of the projective scheme $X /\!\!/ G$ is affected by the choice. The quotient is denoted by $X /\!\!/_{\mathcal{L}} G$ to specify the choice of G-linearisation. We refer to this situation, i.e. the variation of the line bundles and the understanding of how the quotient changes, as variations of GIT quotients (VGIT). In more detail, given the choice of linearisation $\mathcal{L} \cong \mathcal{O}(a_1, \dots, a_m)$, the Hilbert-Mumford function decomposes as (see [Laz09a; GM18; GMZ18]) follows:

Lemma 2.35. *Let $\mathcal{L} \cong \mathcal{O}(a_1, \dots, a_m) \in \mathrm{Pic}^G(X)$ be ample. Then*

$$\mu^{\mathcal{L}}(X, \lambda) = \sum_{i=1}^{m} a_i \mu(X_i, \lambda).$$

Proof. By the properties of the Hilbert-Mumford function [MFK94, Definition 2.2], we have that $\mu^{\mathcal{L}}(-,\lambda) \to \mathbb{Z}$ is a group homomorphism, and that given any G-equivariant morphism of G-varieties $f\colon X \to Y$ and ample $\mathcal{M} \in \operatorname{Pic}^G(Y)$, we have that $\mu^{f^*\mathcal{M}}(X,\lambda) = \mu^{\mathcal{M}}(f(X),\lambda)$. Hence, we have:

$$\begin{aligned}\mu^{\mathcal{O}(a_1,\dots,a_m)}(X,\lambda) &= \mu^{\pi_1^*\mathcal{O}_{X_1}(a_1)\otimes\pi_m^*\mathcal{O}_{X_m}(a_m)}(X,\lambda)\\ &= \sum_{i=1}^m \mu^{\pi_i^*\mathcal{O}_{X_i}(a_i)}(X_i,\lambda)\\ &= \sum_{i=1}^m a_i\mu(X_i,\lambda).\end{aligned}$$

□

Since $a_i > 0$ for all i, we may pick one a_j (usually and without loss of generality this will be chosen as a_1) and divide through. Then, each quotient depends on a vector $\vec{t} = (t_1,\dots,t_m)$ where $t_i = \frac{a_i}{a_j}$ (and $\hat{t}_j$), and is denoted by $X /\!\!/_{\vec{t}}\, G$. By a result of Dolgachev–Hu [DH98, 0.2.3 Theorem] and independently Thaddeus [Tha96, (2.4) Theorem] the number of non-isomorphic compactifications is finite.

2.1.4 Moduli Spaces

The study of semistable and polystable points, in GIT, is especially significant for a number of reasons. The categorical quotient $X /\!\!/ G$ defines a well-defined moduli-stack $\mathcal{M}_X^{GIT}$ of closed orbits, which is beneficial for the study of the classification of varieties.

2.1.4.1 Moduli Problems

When we are referring to a moduli problem, essentially we are seeking a classification of geometric, algebraic or topological objects based on a specific property they hold, up to some equivalence. More formally (see [New78, Chapter 1], [Hos16]):

Definition 2.36 ([Hos16, Definition 2.8])**.** A *naive moduli problem* (in algebraic geometry) is a collection $\mathcal{A}$ of objects in a category (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 2.36.1. The primary example to illustrate the above is $\mathbb{P}^1$. If we think of $\mathbb{P}^1 = k^2 \setminus \{0\}/\sim$ where $(x_0,x_2) \sim (y_0,y_1)$ if and only if $(x_0,x_1) = c(y_0,y_1)$ for some $c \in k^*$ we see that $\mathbb{P}^1$ is the collection of all 1-dimensional linear subspaces of k^2 (i.e. lines) under the equivalence $\sim$, i.e. the moduli problem of linear 1-dimensional subspaces of k^2 under the equivalence $\sim$.

We can extend this rationale for the Grassmanian $\mathrm{Gr}(m, n)$ which is the moduli problem for linear m-dimensional subspaces in some n-dimensional space under the same equivalence.

In order to be able to derive more properties out of more complex moduli problems, we need to use category theory to make the above more formal. In particular, we want to find a scheme $\mathcal{M}$ which encodes how the parametrised objects deform in families. Essentially, we aim to find moduli functors, which will be representable by a scheme whose k-points will be in bijection with the set of equivalence classes of $\mathcal{A}/\sim$. Hence, we are looking for a functor from the category of schemes Sch to the category of sets Set.

Definition 2.37 ([Ols16, Definition 2.2.1])**.** Contravariant functors from the category of schemes Sch to the category of sets Set are called *presheaves* on Sch and form a category, denoted by $\mathrm{Psh}(\mathrm{Sch})$. The morphisms of this category are given by natural transformations.

More generally, contravariant functors from a category S to the category of sets Set are called *presheaves* on S.

The categorical language allows us to use a vast number of useful pre-existing results (e.g. Yoneda's lemma) and tap into the very rich theory of stacks, which will be discussed later on. Re-formulating our moduli problem in categorical language, we obtain the following definition.

Definition 2.38 ([Hos16, Definition 2.10])**.** An *extended moduli problem* or for simplicity a *moduli problem*, based on a moduli problem $(\mathcal{A}, \sim)$, is given by the following data:

(a) For any scheme S, sets $\mathcal{A}_S$ of families of objects in $\mathcal{A}$ and an equivalence relation $\sim_S$ for objects of each $\mathcal{A}_S$;

(b) for each morphism $f\colon T \to S$ of schemes, a pullback map $f^*\colon \mathcal{A}_S \to \mathcal{A}_T$,

satisfying the following conditions:

1. $(\mathcal{A}_{\operatorname{Spec} k}, \sim_{\operatorname{Spec} k}) = (\mathcal{A}, \sim)$;

2. for the identity $\mathrm{Id}\colon S \to S$ and any family $\mathcal{F}/S$, we have $\mathrm{Id}^* \mathcal{F} = \mathcal{F}$;

3. for a morphism $f\colon T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$ over S, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$;

4. for morphisms $f\colon T \to S$ and $g\colon S \to R$, and a family $\mathcal{F}/R$, we have an equivalence $(g \circ f)^*\mathcal{F} \sim_T f^*g^*\mathcal{F}$.

The extra conditions we have imposed here serve as a guarantee that the moduli functor will take into account the deformations of families over a scheme S. The second condition guarantees that identity maps preserve families, while the third condition ensures that equivalent families over one scheme give rise to equivalent families in another scheme under some map f. The fourth condition, ensures that composition maps preserve families. The above definition allows us to construct the moduli functor $\mathcal{M}$ as seen below:

Lemma 2.39 ([Hos16, Lemma 2.11]). *A moduli problem $(\mathcal{A}, \sim)$ defines a presheaf $\mathcal{M} \in \mathrm{Psh}(\mathrm{Sch})$ which is given by*

$$\mathcal{M}(S) := \{\textit{families over } S\}/\sim_S \quad \mathcal{M}(f\colon T \to S) := f^*\colon \mathcal{M}(S) \to \mathcal{M}(T).$$

It is customary when discussing moduli problems to only consider the moduli functor $\mathcal{M}$. If the moduli functor $\mathcal{M}$ is representable by a scheme M, we call the latter its *fine moduli space*. The family $\mathcal{U}$ in $\mathcal{M}(M)$ which corresponds to the identity morphism of M is called the universal family.

2.1.4.2 An Introduction to Stacks

Definition 2.40 (Sites). A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e. collections of morphisms $\{X_i \to X\}$ in $\mathcal{S}$. We require that:

1. (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

2. (restriction) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism, then the fibre products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\} \in \mathrm{Cov}(Y)$.

3. (composition) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category S with a Grothendieck topology.

Example 2.40.1. The *big étale site* Sch_{Et} on the category Sch is a site where a covering of a scheme X is a collection of étale morphisms $\{X_i \to X\}$ such that $\coprod_i X_i \to X$ is surjective.

Definition 2.41 ([Ols16, Definition 2.2.2])**.** A *sheaf* on a site $\mathcal{S}$ on the category Sch is a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ such that for every object S of $\mathcal{S}$ and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is exact, where the two maps $F(S_i) \to F(S_i \times_S S_j)$ are induced by the two projections $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_j$.

Let $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories, visualised as:

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] & a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathcal{S} & S \arrow[r, "f"] & T \end{tikzcd}$$

where a, b are objects in $\mathcal{X}$, and S, T are objects in $\mathcal{S}$ respectively. We say that the morphism $\alpha\colon a \to b$ is *over* $f\colon S \to T$ if in addition $p(\alpha) = f$, and that a is *over* S.

Definition 2.42 ([Alp22, Definition 2.3.1])**.** A functor $p\colon \mathcal{X} \to \mathcal{S}$ of categories is a *prestack* over a category $\mathcal{S}$ if

1. For each diagram of solid arrows

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, dashed, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

there exists a morphism $\alpha\colon a \to b$ over $f\colon S \to T$;

2. for each diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

there exists a unique arrow $a \to b$ over $R \to S$ which fills the diagram making it commute, where commutativity means that the composition $a \dashrightarrow b \to c$ equals the morphism $a \to c$ given by the top arrow;

3. for all coverings $\{S_i \to S\}$, objects a and b in $\mathcal{X}$ over $\mathcal{S}$, objects $a|_{S_i}$ in $\mathcal{X}$ over S_i and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} := \phi_i|_{a_{|S_{ij}}} = \phi_j|_{a_{|S_{ij}}} =: \phi_j|_{S_{ij}}$ as presented in the diagram

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S \\ & S_j \arrow[ur] & \end{tikzcd}$$

there exists a unique morphism $\phi \colon a \to b$ such that $\phi_{a|_{S_i}} = \phi_i$.

The first condition in the previous definition ensures that a pullback from $\mathcal{S}$ to $\mathcal{X}$ exists, the second one provides a universal property, and the third one guarantees that morphisms glue.

Definition 2.43 ([Alp22, Definition 2.4.1]). A prestack $p \colon \mathcal{X} \to \mathcal{S}$ is a *stack* if the following *stack axiom* holds: for covering $\{S_i \to S\}$ and objects a_i over S_i and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ with cocycle condition $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ over S_{ijk}, as presented in the following diagram

$$\begin{tikzcd} & & a_i \arrow[dr, dashed] & \\ a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \\ & & a_j \arrow[ur, dashed] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S \\ & S_j \arrow[ur] & \end{tikzcd}$$

then there exists an object a over S and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

This condition guarantees that the objects inside the stack glue. A reader might notice similarities between the above definitions and the classical definitions of presheaves and sheaves. In fact, for the readers who are more comfortable in that language there is an equivalent definition [Ols16, Definition 4.6.1], where the stack axiom (i.e. the condition in Definition 2.43) corresponds to the exactness of the following diagram (similar to Definition 2.41)

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k),$$

where here, for a prestack $\mathcal{X}$ over $\mathcal{S}$, $\mathcal{X}(S)$ is called the *fibre category* of $S \in \mathcal{S}$. This is the category of objects in $\mathcal{X}$ over S with morphisms over id_S. The two and three maps to the right are induced by the two and three corresponding projections. Here, the exactness of the sequence implies that the map $\mathcal{X}(S) \to \prod_i \mathcal{X}(S_i)$ identifies $\mathcal{X}(S)$ with the equalizer of the two maps $\prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j)$, and so on. We will often refer to a stack just by $\mathcal{X}$ and omitting the target category $\mathcal{S}$, which will be explicit in most cases.

Remark 2.43.1. We have to emphasise that our terminology is not standard. Prestacks are usually referred to as categories fibred in groupoids. In the literature (c.f. [Fan+05, Part 1], [Ols16]) a prestack is sometimes defined as a category fibred in groupoids together with the gluing axiom of morphisms of stacks. In this section, we follow the terminology presented in [Alp22].

Definition 2.44 ([Alp22, Definition C.3.1])**.** Let $G \to S$ be a flat group scheme locally of finite presentation. A *principal G-bundle* or *G-torsor* over an S-scheme X is a flat morphism $P \to X$ locally of finite presentation with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is G-invariant and

$$(\sigma, p_2)\colon G \times_S P \to P \times_X P, \quad (g, p) \to (g \cdot p, p)$$

is an isomorphism.

The following is the main example of quotient stacks that will also demonstrate that the GIT quotient induces a moduli stack. We will explore this more in Theorem 2.46.

Example 2.44.1. Let $G \to S$ be an algebraic group scheme acting on a k-scheme $X \to S$ as defined in Section 2.1.1.1. The *quotient stack* $[X/G]$ is defined as a category over Sch/S (i.e. the category of schemes over S). The objects of $[X/G]$ over an S-scheme T are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ T & \end{tikzcd}$$

where $P \to T$ is a G-torsor (i.e. a principal G-bundle) and f is G-equivariant. A morphism of objects $a := (P' \to T', P' \xrightarrow{f'} X)$, $b := (P \to T, P \xrightarrow{f} X)$, $a \to b$, consists of map $g\colon T' \to T$ and G-equivariant map $\phi\colon P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r, "\phi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is commutative, and the left square is a fibre product. We will show that the quotient stack is indeed a stack over Sch_{Et}/S. The first step, is to show that $[X/G]$ is a prestack. Notice that this will be a functor p that sends object $a = (P' \to T', P' \xrightarrow{f'} X)$ to T', i.e. $p(a) = T'$, and morphism $a \to b$ to $g\colon T' \to T$, i.e. $p(a \to b) = g\colon T' \to T$. The first axiom of prestacks is verified as follows:

Let $T, T' \in \mathrm{Obj}(\mathrm{Sch}/S)$ and $b = (P \to T, P \xrightarrow{f} X)$ be an object of the quotient stack. Let $g\colon T' \to T$. Then we can define $P' := P \times_T T'$ to be the fiber product such that we have a Cartesian diagram

$$\begin{tikzcd} P' \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ T' \arrow[r, "g"] & T. \end{tikzcd}$$

Defining $a := (P' \to T', P' \xrightarrow{f'} X)$, we see that there exists a morphism $a \to b$ constructed as above. This shows axiom 1 is satisfied.

For axiom 2, further suppose that $c = (P'' \to T'', P'' \xrightarrow{f''} X)$ and that we have a diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ T' \arrow[r, "g"] & T \arrow[r, "g'"] & T'' \end{tikzcd}$$

where $g\colon T' \to T$ and $g'\colon T \to T''$. From the map $a \to c$, we have a Cartesian diagram

$$\begin{tikzcd} P'' \times_{T''} T' \arrow[r, "p_1"] \arrow[d, "p_2"] & P'' \arrow[d] \\ T' \arrow[r, "g' \circ g"] & T'' \end{tikzcd}$$

and maps $P' \to T'$ and $P' \to P''$. Hence, by the uniqueness of the fibre product, we have a unique map $P' \to P'' \times_{T''} T'$. Since these are G-torsors over the same base, they must be isomorphic, i.e. $P' \cong P'' \times_{T''} T'$. Using a similar argument, one can show that $P \cong P'' \times_{T''} T$. Notice that we also have a map $P' \to T$ given by $g \circ p_2$. Hence, since P is a fibre product, there exists a unique map $P' \xrightarrow{\phi} P$. Hence, we obtain the following diagram

$$
\begin{tikzcd}
P' \arrow[r, "\phi"] \arrow[d] \arrow[rrr, bend left=50, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "\phi'"] \arrow[d] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "\square"] & P'' \arrow[r, "f''"] \arrow[d] & X \\
T' \arrow[r, "g"] & T \arrow[r, "g'"] & T''
\end{tikzcd}
$$

and since both diagrams are fibre products the morphism $a \to b$ is unique.

For axiom 3 for the prestacks and the stack axiom, we will use the theorems of descent for morphism of schemes (see [Gro63, SGA I.8] or [Vis05] or [Alp22, Proposition B.2.1]) and G-torsors (see [Alp22, Proposition C.3.11]) respectively. In fact, axiom 3 follows directly from the descent for morphisms of schemes. For the stack axiom, consider $\{T_i \to T\}$ an étale covering and objects $(P_i \to T_i, P_i \to X)$ over T_i with isomorphisms on the restrictions satisfying the cocycle condition. The existence of a G-torsor $P \to T$ follows from the descent for G-torsors and the existence of a map $P \to X$ follows again from the descent for morphisms of schemes (see [Alp22, Proposition B.2.1]).

2.1.4.3 Algebraic Stacks and Moduli Spaces

The two most important type of stacks are the Deligne-Mumford stacks and the algebraic (Artin) stacks defined in [DM69]. Here, we follow the notation of [Alp22].

Definition 2.45 ([Alp22, Definitions 3.1.2, 3.1.4, 3.1.6],)**.** An *algebraic space* is a sheaf X on Sch_{Et} such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes. The map $U \to X$ is called an *étale presentation*.

An *algebraic stack* is a stack $\mathcal{X}$ over Sch_{Et} such that there exist a scheme X and a surjective, smooth and representable morphism $X \to \mathcal{X}$. The morphism $X \to \mathcal{X}$ is called a *smooth presentation*.

An algebraic stack $\mathcal{X}/S$ is called *Deligne-Mumford* (DM) if there exists a scheme X and a surjective, étale and representable morphism $X \to \mathcal{X}$.

Based on this definition, the reader may be interested to know how the above notions are connected. The following is the 'hierarchy' of the above.

$$\text{schemes} \subset \text{algebraic spaces} \subset \text{Deligne-Mumford stacks} \subset \text{algebraic stacks}$$

Theorem 2.46 ([Alp22]). *If G/S is a smooth, affine group scheme acting on a scheme X/S, the quotient stack $[X/G]$ is an algebraic stack over S such that $X \to [X/G]$, is a G-torsor and in particular surjective, smooth and affine.*

Remark 2.46.1. There exists an object of $[X/G]$ over X given by

$$\begin{tikzcd} X \times G \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X & \end{tikzcd}$$

where σ is the map defining the action of G on X. This object is the one that defines the map $X \to [X/G]$, by the 2-Yoneda Lemma (see [Vis05, p. 3.6.2]).

Properties of morphisms of schemes also extend to morphism of stacks.

Definition 2.47 ([Alp22, Definition 2.2.1]). Let $\mathcal{P}$ be a property of morphisms of schemes.

1. If $\mathcal{P}$ is stable under composition and base change and is étale-local (resp. smooth-local) on the source and target, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks (resp. algebraic stacks) *has property* $\mathcal{P}$ if for all étale (resp. smooth) presentations (equivalently there exists presentations) $V \to \mathcal{Y}$ and $U \to V \times_{\mathcal{Y}} \mathcal{X}$,in the diagram

$$\begin{tikzcd} U \arrow[r] & V \times_{\mathcal{Y}} \mathcal{X} \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{Y}, \end{tikzcd}$$

   the composition $U \to V$ has the property $\mathcal{P}$.

2. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to Y$ from a scheme, the base change $T \times_{\mathcal{Y}} \mathcal{X}$ has $\mathcal{P}$.

3. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *open immersion, closed immersion, locally closed immersion, affine, or quasi-affine* if it is representable by schemes and has the corresponding property in the sense of 2.

Definition 2.48 ([Alp22, Definition 2.4.15(2)]). A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is separated if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes, is proper.

We are now in a position to define good and coarse moduli spaces.

Definition 2.49 ([Alp22, Definition 4.3.1])**.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

1. for any algebraically closed field k, the induced map $\mathcal{X}(k)/\tilde{\to}X(k)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over k, is bijective,

2. π is universal for maps to algebraic spaces, i.e. any other map from $\mathcal{X} \to Y$ factors uniquely as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \arrow[dr] & \\ X \arrow[r] & Y. \end{tikzcd}$$

If in addition $\mathcal{X} = [U/G]$ is a quotient stack, we often write the coarse moduli space as U/G and call it the geometric quotient of U by G.

Definition 2.50 ([Alp13, Definition 4.1])**.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if

1. π is cohomollogically affine,

2. the natural map $\mathcal{O}_{\mathcal{X}} \xrightarrow{\cong} \pi_*\mathcal{O}_X$ is an isomorphism.

As one notices, the above Definitions closely resemble Definitions 2.11 and 2.12. This is not a coincidence; the above theory was developed after GIT, and the authors attempted to emulate the GIT methods that ensured one obtains good/coarse moduli spaces. As such, and based on our discussion in Section 2.1.2.1, we have the following:

Theorem 2.51 ([Alp13, Theorem 13.6], [MFK94, Theorem 1.10])**.** *Following our notation, the map $[X^{ss}/G] \to X /\!\!/ G$ is a coarse moduli space. The map $[X^s/G] \to X/G$ is a good moduli space.*

Remark 2.51.1. Throughout this thesis, we will use a somewhat different notation. The main moduli space we will deal with will be $\overline{M}^{GIT}$ which is thought as the closure of the quotient of stable orbits, i.e. roughly

$$\overline{M}^{GIT} := \{\text{stable and polystable orbits}\}.$$

In a similar notation we will denote the stack $\mathcal{M}^{GIT} = [X^{ss}/G]$.

Definition 2.52. Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space. We say that X is *proper* if π induces a bijection between the isomorphism classes of k-points of $\mathcal{X}$ and the k-points of X.

Remark 2.52.1. One of the most important results of Mumford, is that the moduli stack of smooth curves $\mathcal{M}_g$ is a proper DM-stack, which admits a coarse moduli space $\overline{M}_g$. Although we haven't covered this, the reader is prompted to [DM69] for more information.

Remark 2.52.2. This is but a short introduction to a very vast, fascinating and wealthy part of modern algebraic geometry. If the reader is interested more on GIT, we prompt them to [MFK94; Muk03; BJK17], and for stacks [Ols16; Knu71; LM00; Alp22].

2.2 K-stability

K-stability is a modern theory developed initially by Yau [Yau96] and Tian [Tia97] in a differential geometric setting and extended later by Donaldson [Don02], using methods from analytical geometry to describe which toric surfaces admit a constant scalar curvature Kähler (cscK metric). The definitions for K-stability and K-polystability were modified by Li–Xu [LX14] and were placed in a more algebro-geometric setting. A renowned achievement in K-stability has been the work of Chen–Donaldson–Sun [CDS13] which came as a solution to the Yau-Tian-Donaldson conjecture and has been one of the most important recent contributions to algebraic geometry. In particular, this remarkable result shows that Fano manifolds admit a Kähler-Einstein (KE) metric if they are K-polystable.

K-stability has continued to evolve, and recent developments have placed it within a stronger algebro-geometric setting. Furthermore, valuative criteria have been developed in order to determine when a Fano variety is K-(poly/semi)stable [Fuj21].

2.2.1 K-stability Definitions

Throughout this thesis, a variety is a separated integral scheme of finite type over the field $k = \mathbb{C}$. Throughout the rest of this thesis, we will work with Fano varieties [IP99] over the complex numbers $\mathbb{C}$. Unless stated otherwise, these are going to be $\mathbb{Q}$-Gorenstein Fano varieties, i.e. varieties X where there will exist some $r \in \mathbb{Q}_{>0}$ such that $-rK_X$ is an ample Cartier divisor.

2.2.1.1 Fano Varieties and Test Configurations

Definition 2.53. Let X be a projective variety. X is a *Fano variety* if the anticanonical line bundle $-K_X$ is ample.

An equivalent way to think of smooth Fano varieties, is as varieties with positive Ricci curvature. The only smooth Fano curve is the sphere, while smooth Fano surfaces were classified in the 19th century by Pasquale del Pezzo [Pez85; Pez87] and their classification into a compact moduli space was completed by Odaka–Spotti–Sun in 2016 [OSS16] building on the work of Mabuchi–Mukai [MM90]. The Del Pezzo surfaces are classified into 10 deformation families [Bea03]. The subject of classification of Fano varieties in higher dimensions, such as

Fano threefolds, remains an active area of research, following the works of Iskovskih [Isk80], Mori and Mukai, who classified smooth Fano threefolds [MM03] into 105 deformation families. In higher dimensions, such classifications are not known, although we know of explicit examples of such Fano varieties.

Before introducing the main notions of K-stability, we have to first introduce polarised pairs and test configurations (which generalise one-parameter subgroups).

Definition 2.54. A pair (X, L) where X is a projective variety and L is an ample line bundle is called a *polarised pair*.

The motivation for the definition of test configurations arises from the fact that stability is usually defined by a numerical criterion on degenerations of the objects in question. In our situation, we consider polarised varieties (X, L) where a multiple of L induces the embedding $X \hookrightarrow \mathbb{P}^n$ to some projective space via its sections [Har77]. The test configurations are essentially the data encoding these degenerations [Don02]. Our notation follows [RT07] and [Oda13b].

Definition 2.55. Let (X, L) be a polarised variety. A *test configuration* of this polarised variety is a pair $(\mathcal{X}, \mathcal{L})$ such that $\mathcal{X}$ is a scheme and $\mathcal{L}$ is an invertible sheaf of $\mathcal{X}$ with

1. a $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$,

2. a proper flat morphism $\alpha\colon \mathcal{X} \to \mathbb{A}^1$

such that α is $\mathbb{G}_m$-equivariant under the usual $\mathbb{G}_m$-action, $\mathcal{L}$ is relatively ample, and the restriction $(\mathcal{X}, \mathcal{L}|_{\alpha^{-1}(\mathbb{A}^1\setminus\{0\})})$ is $\mathbb{G}_m$-equivariantly isomorphic via the map $\phi\colon (\mathcal{X}, \mathcal{L}|_{\alpha^{-1}(\mathbb{A}^1\setminus\{0\})}) \to (X, L^{\otimes r}) \times (\mathbb{A}^1 \setminus \{0\})$ for some $r \in \mathbb{N}$ called the exponent.

Since we are talking about $\mathbb{G}_m$-actions it is reasonable to ask ourselves whether test configurations arise naturally from one-parameter subgroups. In fact, we have:

Theorem 2.56. *A one-parameter subgroup of* $\mathrm{GL}(H^0(X, L^{\otimes r}))$ *is equivalent to the data of a test configuration* $(\mathcal{X}, \mathcal{L})$ *of* (X, L) *with the polarization* $\mathcal{M}$ *very ample (over* $\mathbb{A}^1$*) and of exponent* r *for* $r \gg 0$.

Proof. We will construct a test-configuration from a 1-PS. Let $G = \mathrm{GL}(H^0(X, L^{\otimes r}))$ and let $\lambda\colon \mathbb{G}_m \to G$ be a 1-PS. Let ρ be the natural multiplication $\mathbb{G}_m$-action on $\mathbb{A}^1$. Then we have a natural action $\lambda \times \rho$ of $\mathbb{G}_m$ on $G \times \mathbb{A}^1$. The closure of the orbit $\mathcal{X} := \mathbb{G}_m \cdot (X \times \{1\})$ defines a test configuration with the natural polarisation $\mathcal{O}(1)|_{\mathcal{X}}$ and the restriction of the natural action on $(\mathbb{P}(H^0(X, L^{\otimes r})) \times \mathbb{A}^1, \mathcal{O}(1))$. $\square$

Example 2.56.1. Let (X, L) be a polarised variety. Then the pair $(\mathcal{X} = X \times \mathbb{C}, \mathcal{L} = p_1^*(-K_X))$, where p_1 is the projection on the first factor, is a test configuration with $\mathbb{G}_m$-action given by $t \cdot (x, a) \to (t(x), t \cdot a)$. This is called a *product test configuration*.

The test configuration $(\mathcal{X} = X \times \mathbb{C}, \mathcal{L} = p_1^*(-K_X))$, where $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$ with the trivial action, is called a *trivial test configuration*.

Remark 2.56.1. Notice that in Definition 2.55 we can consider the exponent r to be 1. The reason for this is, that if $r \neq 1$ we can consider the polarised pair $(X, L^{\otimes r})$ instead of (X, L).

Let G be a reductive subgroup of $\mathrm{Aut}(X)$. A given test configuration $(\mathcal{X}, \mathcal{L})$ is *G-equivariant* if the product $G \times \mathbb{G}_m$ acts on $(\mathcal{X}, \mathcal{L})$ such that

1. $\{1\} \times \mathbb{G}_m$ acting on $(\mathcal{X}, \mathcal{L})$ is the original $\mathbb{G}_m$-action,

2. the $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X}, \mathcal{L}|_{\alpha^{-1}(\mathbb{A}^1 \setminus \{0\})}) \cong (X, L^{\otimes r}) \times (\mathbb{A}^1 \setminus \{0\})$$

   is $G \times \mathbb{G}_m$-equivariant.

The first invariant for K-stability is the Donaldson-Futaki invariant [Oda12], which is based on the original Futaki invariant [Don02]. Let X be a projective variety, with $\dim X = n$ and L an ample line bundle, and let $(\mathcal{X}, \mathcal{L})$ be a test configuration for (X, L). For each $k \in \mathbb{Z}_+$ we have vector spaces $H^0(X, L^k)$; denoting by $d(k) := \dim H^0(X, L^k)$ the dimension of each vector space, we notice by the Riemann-Roch theorem that, since L is ample, for large k, $d(k)$ is given by a Hilbert polynomial of degree n. Since the $\mathbb{G}_m$- action on $(\mathcal{X}, \mathcal{L})$ fixes the central fibre $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$, $\mathbb{G}_m$ acts also on the vector space $H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$. We define $w(k)$ to be the weight of this action on the highest exterior power of $H^0(\mathcal{X}_0, \mathcal{L}^k|_{\mathcal{X}_0})$. By the Riemann-Roch theorem, Mumford's droll Lemma [Mum77, Lemma 2.14] and by [Oda13a, Lemma 3.3], $w(k)$ is a polynomial of degree $n + 1$. Here, the total weight of an action of $\mathbb{G}_m$ on some finite-dimensional vector space is defined as the sum of all weights, where the weights mean the exponents of eigenvalues which should be powers of $t \in \mathbb{A}^1$. Hence, we have that

$$d(k) = a_0 + a_1 k + \dots,$$

$$w(k) = b_0 + b_1 k + \dots$$

and

$$\frac{d(k)}{kw(k)} = F_0 + k^{-1} F_1 + \dots$$

The *Futaki invariant* is defined to be the coefficient F_0. The *Donaldson-Futaki invariant* is defined as

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) := \frac{b_0 a_1 - a_0 b_1}{a_0^2}.$$

Having defined the above, we are in a position to define K-stability.

Definition 2.57. We say that the polarised pair (X, L) is *K-stable* (resp. *K-semistable*) if and only if $\mathrm{DF}(\mathcal{X}; \mathcal{L}) > 0$ (respectively, $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$) for any non-trivial test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L). We say that the polarised pair (X, L) is *K-polystable* if $\mathrm{DF}(\mathcal{X}; \mathcal{L}) \geq 0$ for any non-trivial test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L), and $\mathrm{DF}(\mathcal{X}; \mathcal{L}) = 0$ only if a test configuration $(\mathcal{X}, \mathcal{L})$ is a product test configuration.

As one can notice from the above definitions, K-stability implies K-polystability and K-polystability implies K-semistability.

Remark 2.57.1. When we are studying anti-canonically polarised pairs $(X, -K_X)$ we will omit the anticanonical line bundle $-K_X$ and say that X is K-(semi/poly)stable.

Similarly, we can define G-equivariant K-stability.

Definition 2.58. The Fano variety X is said to be *G-equivariantly K-polystable* if for every G-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ one has $DF(\mathcal{X}; \mathcal{L}) \geq 0$, and $DF(\mathcal{X}; \mathcal{L}) = 0$ if only if $(\mathcal{X}, \mathcal{L})$ is of the product type. The Fano variety X is said to be *G-equivariantly K-semistable* (respectively *G-equivariantly K-stable*) if for every (non-trivial) G-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ one has $DF(\mathcal{X}; \mathcal{L}) \geq 0$ (respectively > 0).

From the definition we see that if X is K-polystable it is G-equivariantly K-polystable. In fact, there is a remarkable implication that the opposite holds true.

Theorem 2.59 ([DS16; LWX19; LZ22; Zhu21]). *Let X be a smooth Fano variety, and $L = -K_X$. Suppose that X is G-equivariantly K-polystable. Then X is K-polystable.*

Donaldson's [Don02] and Tian's [Tia97] initial definition was based on the Futaki invariant. It is more beneficial, when in an algebro-geometric setting, to refer to Definition 2.57. This is because there exist algebro-geometric formulae for computing the Donaldson-Futaki invariant [Wan12].

Theorem 2.60 ([Wan12], [Oda13a, Theorem 3.2]). *For a normal test configuration $(\mathcal{X}, \mathcal{L})$ of the polarised pair (X, L) of dimension n, and $r > 0$, we can glue the test configuration with $(X \times (\mathbb{P}^1 \setminus \{0\}, pr_1^*(-rL))$ to get a proper family $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ over $\mathbb{P}^1$. Furthermore, we have*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L})) = \frac{1}{2(n+1)(-K_{\mathcal{X}}^n)} \left(n \left(\frac{1}{r} \overline{\mathcal{L}} \right)^{n+1} + (n+1) K_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \left(\frac{1}{r} \overline{\mathcal{L}} \right)^n \right).$$

Here, pr_1 is the projection to the first factor, and we are gluing $(\mathcal{X}, \mathcal{L})$ to $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ by the $\mathbb{G}_m$-equivariant isomorphism ϕ from Definition 2.55 along $\mathbb{A}^1 \setminus \{0\}$.

Although Theorem 2.60 provides us with a formula for computing the Donaldson-Futaki invariant, in reality detecting K-stability explicitly using just this is a complicated and hard process, as finding all possible test configurations is a hard task. For this, new methods have been introduced that allow us to detect K-stability. These will be covered later on in this chapter.

The following two theorems provide striking results on the nature of the automorphism groups of K-(poly)stable Fano varieties.

Theorem 2.61 ([Alp+20, Theorem 1.3],[Mat57]). *If X is Fano and K-polystable, then $\mathrm{Aut}(X)$ is reductive.*

Theorem 2.62 ([BX19, Corollary 1.3]). *If X is Fano and K-stable, then $\mathrm{Aut}(X)$ is finite.*

In particular, the above two theorems can help us to rule out if a Fano variety is K-(poly)/stable. If we know that $\mathrm{Aut}(X)$ is not reductive, then we can surmise that it is not K-polystable without having to check the Donaldson-Futaki invariant. Similarly, if we know that $\mathrm{Aut}(X)$ is not finite, then we can deduce that it is not K-stable.

2.2.1.2 The Case of Pairs (X, D)

A natural extension of the above notions is to the setting of log Fano pairs (X, D), where X is a variety and D is a divisor on this variety. More specifically we let $(X, D = \sum_{i=1}^{k} a_i D_i)$ be a projective log pair and L an ample line bundle on X. If $L = -K_X$ we have the following definition.

Definition 2.63. Let X be a normal variety and D a $\mathbb{Q}$-effective divisor. A pair (X, D) is called a *log Fano pair* if $-K_X - D$ is ample and $\mathbb{Q}$-Cartier.

Below is the natural extension to test configurations for log Fano pairs.

Definition 2.64. Let $(X, D = \sum_{i=1}^{k} a_i D_i)$ be a projective log pair and L an ample line bundle on X, where $a_i \geq 0$ and $a_i \in \mathbb{Q}$. A *test configuration* $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of this log pair is a tuple such that

1. $(\mathcal{X}; \mathcal{L})$ is a test configuration of $(X; L)$;
2. the formal sum $\mathcal{D} = \sum_{i=1}^{k} a_i \mathcal{D}_i$ of codimension one closed integral subschemes $\mathcal{D}_i$ of $\mathcal{X}$ is such that $\mathcal{D}_i$ is the Zariski closure of $D_i \times (\mathbb{A}^1 \setminus \{0\})$ under the isomorphism between $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{A}^1 \setminus \{0\})$.

Notice, that, under the above conditions, $(\mathcal{D}_i, \mathcal{L}|_{\mathcal{D}_i})$ is a test configuration of $(D_i, L|_{D_i})$. Similar to our discussion in the previous section, we can define a Donaldson-Futaki invariant, and we can generalise Definition 2.57. In particular, for the log pairs $(\mathcal{D}, \mathcal{L}|_{\mathcal{D}})$ of $(D, L|_D)$ we can define Hilbert and weight polynomials as we did before, denoted respectively by

$$\tilde{d}(k) = \tilde{a}_0 + \tilde{a}_1 k + \dots$$
$$\tilde{w}(k) = \tilde{b}_0 + \tilde{b}_1 k + \dots.$$

Definition 2.65. We define the *log Donaldson-Futaki invariant* of test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of $(X, D; L)$ to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) := \frac{b_0 a_1 - a_0 b_1}{a_0^2} + \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{2a_0}.$$

We define the *log Donaldson-Futaki invariant with angle* β, with $0 \leq \beta \leq 1$, of test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of $(X, D; L)$ to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) := \frac{b_0 a_1 - a_0 b_1}{a_0^2} + (1 - \beta) \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{2a_0}.$$

Example 2.65.1. Let (X, D) be a projective log pair and L be an ample line bundle. Then the tuple $(\mathcal{X} = X \times \mathbb{C}, \mathcal{D} = D \times \mathbb{C}; \mathcal{L} = p_1^*(-K_X - D))$, where p_1 is the projection on the first factor, is a test configuration with $\mathbb{G}_m$-action given by $t \cdot (x, a) \to (t(x), t \cdot a)$. This is called a *product test configuration*.

The test configuration $(\mathcal{X} = X \times \mathbb{C}, \mathcal{D} = D \times \mathbb{C}; \mathcal{L} = p_1^*(-K_X - D))$, where $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$, and $\mathcal{D}$ is isomorphic to $D \times \mathbb{C}$ with the trivial action, is called a *trivial test configuration*.

Definition 2.66. We say that the log Fano pair (X, D) is *K-stable* (resp. *K-semistable*) if and only if $\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) > 0$ (respectively, $\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) \geq 0$) for any non-trivial test configuration. We say that the log Fano pair (X, D) is K-polystable if $\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) \geq 0$ for any non-trivial test configuration, and $\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = 0$ only if a test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ is a product test configuration.

As before, K-stability implies K-polystability and K-polystability implies K-semistability. The formula in Theorem 2.60 can also be extended for the case of pairs.

Theorem 2.67 ([Oda13a])**.** *For a normal test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of the log Fano pair $(X, D; L)$, we can glue the test configuration to get a proper family $(\overline{\mathcal{X}}, \overline{\mathcal{D}}; \overline{\mathcal{L}})$. Furthermore, we have*

$$\mathrm{DF}(\mathcal{X}, \mathcal{D}; \mathcal{L})) = \frac{1}{2(n+1)(-K_{\mathcal{X}} - D)^n} \left(n \left(\frac{1}{r} \overline{\mathcal{L}} \right)^{n+1} + (n+1)(K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\mathcal{D}}) \cdot \left(\frac{1}{r} \overline{\mathcal{L}} \right)^{n} \right).$$

Here, pr_1 is the projection to the first factor, and we are gluing by the $\mathbb{G}_m$-equivariant isomorphism ϕ along $\mathbb{A}^1 \setminus \{0\}$.

An interesting point of study is the case of log Fano pairs $(X, (1-\beta)D)$ with $\beta \in (0, 1) \cap \mathbb{Q}$. A historical reason to study the K-stability of log Fano pairs of this form can be found in [CDS13], where such pairs were studied in order to answer the Yau-Tian-Donaldson Conjecture. A great part of this thesis will be devoted to studying specific examples of such log Fano pairs, that will arise as complete intersections of the same degree polynomials and their hyperplane sections.

2.2.2 K-moduli

In Section 2.1.4 we discussed that obtaining a moduli stack with a compact good moduli space is an important step in the classification of algebraic varieties. It turns out that K-stability is in many cases the 'right' theory for the construction of this good moduli space, especially for the case of Fano varieties. This is both quite remarkable and somewhat expected by experts. K-stability was initially developed to answer questions that arise from differential geometry, in particular, which manifolds admit a Kähler-Einstein metric. The fact that this theory is algebro-geometric in nature displays an interesting interplay between mathematical fields, which is on its own a particularly fascinating part of modern mathematics. Further, the fact that this theory can be used in classification problems in algebraic geometry is a remarkable consequence, and a tool which is sure to be used extensively in the future.

On the other hand, K-stability is a stability theory. We have already seen in Section 2.2.1 that the definitions of K-stability follow GIT-like constructions. As such, the expectation that such a stability theory can give rise to moduli spaces may be natural. In this section, we will give an overview of the construction of moduli stacks and spaces using K-stability, which we will refer to as K-moduli from now on. We will demonstrate that the K-moduli space is a good moduli space, and we will give specific examples of its compactification.

2.2.2.1 Brief Review of Minimal Model Programme techniques

Throughout this subsection, we will go over some tools we require from the Minimal Model Programme (MMP) which will allow us to show specific properties of K-moduli. Let X, Y be normal projective varieties. The following are some basic notions of the MMP [KM98], [LLX20].

Definition 2.68. Let X be a reduced, irreducible variety defined over $\mathbb{C}$. A *real valuation* of its function field $K(X)$ is a non-constant map $v\colon K(X)^\times \to \mathbb{R}$, satisfying:

1. $v(fg) = v(f) + v(g)$;
2. $v(f+g) \geq \min\{v(f), v(g)\}$;
3. $v(\mathbb{C}^*) = \{0\}$.

We set $v(0) = +\infty$. A valuation v gives rise to a local valuation ring $\mathcal{O}_v := \{f \in K(X) |\, v(f) \geq 0\}$. We say a real valuation v is *centred* at a scheme-theoretic point $\xi := c_X(v) \in X$ if we have a local inclusion $\mathcal{O}_{\xi,X} \to \mathcal{O}_v$ of local rings. The point ξ is called a *centre* of the valuation. Notice that the centre of a valuation, if it exists, is unique since X is separated, by the valuative criterion of separatedness. We denote by Val_X the set of real valuations of $K(X)$ that admit a centre. For a closed point $x \in X$, we denote by $\mathrm{Val}_{X,x}$ the set of real valuations of $K(X)$ centred at x.

Definition 2.69. Let X be a normal projective variety. Assume that mK_X is Cartier for some $m > 0$. Let $\mu\colon Y \to X$ be a birational morphism, and $E \subset Y$ be an irreducible exceptional Cartier divisor, $D \subset Y$ and irreducible Cartier divisor. In particular, D is called a *divisor over* X, which we will usually denote by D/X. The closure of $f(D) \subset X$ is called the *centre* of D on X, denoted by $\mathrm{centre}_X(D)$ or $c_X(D)$.

If we further assume that K_Y is $\mathbb{Q}$-Cartier (e.g. if Y is smooth) then we have

$$K_Y \sim_{\mathbb{Q}} \mu^*(K_X) + \sum_i a(E_i, X)E_i$$

where E_i are all the exceptional divisors of the morphism f and the number $a(E_i, X)$ is the *discrepancy* of divisor E_i with respect to X. The quantity $A_X(E) := a(E, X) + 1$ is the *log discrepancy* of divisor E with respect to X.

Example 2.69.1. Take a proper birational morphism $\mu\colon Y \to X$, with Y a normal variety, and a Cartier divisor E over X. We will define a valuation $\mathrm{ord}_E \in \mathrm{Val}_X$ as follows:

For each $f \in K(X)^\times = K(Y)^\times$, we define $\mathrm{ord}_E(f)$ to be the order of vanishing of f along E. Then, the centre $c_X(\mathrm{ord}_E)$ is the generic point of $\mu(E)$. We say that $v \in \mathrm{Val}_X$ is a *divisorial valuation* if there exists E as above and $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda \cdot \mathrm{ord}_E$.

Example 2.69.2. Let $\mu\colon Y \to X$ be a proper birational morphism between two normal varieties Y, X and let $\eta \in Y$ be a point such that Y is regular at η. Given a local system of coordinates $y_1, \ldots, y_r \in \mathcal{O}_{Y,\eta}$ at η and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$, we define a valuation v_α as follows. For $f \in \mathcal{O}_{Y,\eta}$ we can write f as $f = \sum_{\beta \in \mathbb{Z}^r_{\geq 0}} c_\beta y_\beta$ with $c_\beta \in \hat{\mathcal{O}}_{Y,\eta}$ either zero or a unit. We set

$$v_\alpha(f) := \min\{\langle \alpha, \beta \rangle | c_\beta \neq 0\}.$$

Every valuation that can be written in this form is called a *quasi-monomial valuation*.

Notice that the discrepancy does not depend on μ but depends on each E_i. The reason for this is the following. Take another birational morphism $\mu'\colon Y' \to X$ with irreducible divisor E'/X, such that $\mathrm{ord}_E = \mathrm{ord}_{E'}$. Then, $a(E, X) = \mathrm{ord}_E(K_Y - \mu^*(K_X)) = \mathrm{ord}_{E'}(K_Y - (\mu')^*(K_X)) = a(E', X)$, i.e. we have that $a(E, X) = a(E', X)$ and hence, $a(E, X)$ depends only on the valuation ord_E.

The above notions also extend to the case of pairs (X, Δ).

Definition 2.70. Let (X, Δ) be a log pair where X is a normal variety and $\Delta = \sum_i a_i D_i$ is a sum of distinct prime divisors, with $a_i \geq 0$ for all i. Assume that $m(K_X + \Delta)$ is Cartier for some $m > 0$. Let $\mu\colon Y \to X$ be a birational morphism from a normal variety Y, with exceptional locus $E \subset Y$ and irreducible exceptional Cartier divisors $E_i \subset E$. Let also $\mu_*^{-1}\Delta := \sum_i a_i \mu_*^{-1} D_i$ be the proper transform of Δ. If we further assume that K_Y is $\mathbb{Q}$-Cartier (e.g. if Y is smooth) then we have

$$K_Y + \mu_*^{-1}\Delta \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta) E_i.$$

By definition $a(D_i, X, \Delta) = -a_i$ and $a(D, X, \Delta) = 0$ for any divisor $D \subset X$ which is different from the D_i. $a(E, X, \Delta)$ is called the *discrepancy of* E with respect to (X, Δ). We frequently write $a(E)$ if no confusion is likely. Similarly, we define $A_{(X,\Delta)}(E) := a(E, X, \Delta) + 1$ to be the *log discrepancy* of divisor E with respect to (X, Δ).

The *discrepancy* of (X, Δ) is given by

$$\operatorname{discrep}(X, \Delta) := \inf_{E/X} \{a(E, X, \Delta) |\ E \text{ exceptional}\}.$$

Usually, we build such birational morphisms by blowing up along singularities or subvarieties of X. An important distinction to the types of singularities comes from the classification below.

Definition 2.71 ([KM98, Definition 2.34]). We say that (X, Δ) is

1. *terminal* if $\operatorname{discrep}(X, \Delta) > 0$. If $\Delta = 0$, this is the smallest class that is necessary to run the minimal model program for smooth varieties;

2. *canonical* if $\operatorname{discrep}(X, \Delta) \geq 0$. If $\Delta = 0$ these are precisely the singularities that appear on the canonical models of varieties of general type;

3. *klt (Kawamata log terminal)* if $\operatorname{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor \geq 0$, where $\lfloor \Delta \rfloor = \sum \lfloor a_i \rfloor D_i$. The proofs of the vanishing theorems seem to run naturally in this class;

4. *plt (purely log terminal)* if $\operatorname{discrep}(X, \Delta) > -1$. If $\Delta = 0$ then klt =plt;

5. *lc (log canonical)* if $\operatorname{discrep}(X, \Delta) \geq -1$. This is where the discrepancy of a pair is bounded.

From here on out we will focus our discussion on the case of log pairs (X, Δ) as most of the results can be generalised from X in the case where $\Delta = 0$. One more invariant we need to introduce is the log canonical threshold of a log pair. This is a numerical invariant depending on our log pair and is an indicator of how 'bad' the singularities are.

Definition 2.72. Let X be a normal variety.

1. The *log canonical threshold* of the log pair (X, Δ) is

$$\mathrm{lct}(X, \Delta) = \max\{\lambda |\ (X, \lambda\Delta) \text{ is log canonical}\};$$

2. the *local log canonical threshold* of the log pair (X, Δ) at $p \in X$ is

$$\mathrm{lct}_p(X, \Delta) = \max\{\lambda |\ (X, \lambda\Delta) \text{ is log canonical at } p\}$$

Here:

$$\mathrm{lct}(X, \Delta) = \min_{p \in X} \mathrm{lct}_p(X, \Delta).$$

Example 2.72.1. We will demonstrate the above notions by studying the discrepancies and log canonical thresholds of the pair (S, C), where S is the smooth affine plane and C is the affine curve $C = \{y^2 - x^3 = 0\} \subset S = \mathbb{A}^2$. Take $p = (0, 0)$ and consider the minimal log resolution of (S, C), $\mu \colon Y \to S$, where μ is the composition of three blow-ups. This morphism has 3 exceptional divisors E_1, E_2, E_3. Then by the pullback formula, we have

$$\mu^*(K_X + C) + a(E_1, S, C)E_1 + a(E_2, S, C)E_2 + a(E_3, S, C)E_3 \sim K_Y + C_Y,$$

where C_Y is the strict transform of C. Notice that after the first blow-up we obtain a smooth curve $\tilde{C}$ and the exceptional divisor E_1 tangent at $\tilde{C}$ at p. Similarly, the second blow up gives a line $\bar{C}$, and the exceptional divisor E_2, which cuts E_1 and $\bar{C}$ at p. The third blow-up gives the line C_Y and the exceptional divisor E_3 which intersects C_Y, E_1 and E_2. From this geometric image we see that $a(E_1, S, C) = 1 - 2 = -1$, $a(E_2, S, C) = 2 - 3 = -1$ and $a(E_3, S, C) = 4 - 6 = -2$. Since $a(E_3, S, C) = -2$, we see that (S, C) is not log canonical.

If we study the same resolution of singularities of $(S, \lambda C)$ we obtain

$$\mu^*(K_X + \lambda C) + a(E_1, S, \lambda C)E_1 + a(E_2, S, \lambda C)E_2 + a(E_3, S, \lambda C)E_3 \sim K_Y + \lambda C_Y$$

and $a(E_1, S, \lambda C) = 1 - 2\lambda$, $a(E_2, S, \lambda C) = 2 - 3\lambda$ and $a(E_3, S, \lambda C) = 4 - 6\lambda$. This implies that $a(E_1, S, \lambda C) \geq -1$ and $a(E_2, S, \lambda C) \geq -1$ for $\lambda \leq 1$ and $a(E_3, S, \lambda C) = 4 - 6\lambda \geq -1$ for $\lambda \leq \frac{5}{6}$, i.e. $\mathrm{lct}_p(S, C) = \frac{5}{6}$.

Let us briefly return to Fano varieties.

Definition 2.73. Let X be a projective variety. We say that X is a $\mathbb{Q}$-*Fano variety* if X is klt and $-K_X$ is $\mathbb{Q}$-Cartier.

The following theorem provides us with the first instance of a connection between K-stability and the MMP. It shows that K-stability imposes a bound on the singularities of varieties, which comes directly from the MMP.

Theorem 2.74 ([Oda13b])**.** *Let X be an n-dimensional normal $\mathbb{Q}$-Gorenstein Fano variety. If X is K-semistable, then X has (at worst) klt singularities, i.e., X is a $\mathbb{Q}$-Fano variety.*

Definition 2.75. Let X be a $\mathbb{Q}$-Fano variety. Let $D \sim_{\mathbb{Q}} -K_X$ be an effective $\mathbb{Q}$-divisor on X. We say that (X, D) is *K-semistable* if $(X, D; L)$ is K-semistable for some Cartier divisor $L \sim_{\mathbb{Q}} -lK_X$ and some $l \in \mathbb{Z}_{>0}$.

Remark 2.75.1. From Theorem 2.74 and [Oda13b] we see that if (X, D) is K-semistable, then it (at worst) log canonical.

For toric Fano varieties, it turns out that one can verify if they are K-polystable by studying their polytopes.

Theorem 2.76 ([Bat81], [Fuj16, Theorem 1.2], [Ber16, Corollary 1.2.])**.** *Let X be a normal toric Fano variety, and let P be its associated anticanonical polytope in $M \otimes_{\mathbb{Z}} \mathbb{R}$, where M be the character lattice of the torus. Then X is K-polystable if and only if the barycentre of P is the origin.*

2.2.2.2 Deformations, Families and K-moduli

The theory of K-moduli has been developed mainly in the last decade, with the results presented here achieved by Alper, Hacon, McKernan, Blum, Halpern-Leistner, Liu, Xu, Wang and Zhuang, in a series of papers ([LWX21; BX19; Alp+20; Xu20; BLX22; XZ20; XZ21; LXZ22]), moving away from previous analytically-inspired work in the $\mathbb{Q}$-Gorenstein smoothable case by Odaka [Oda15], Spotti-Sun-Yao [SSY16] and Li–Wang–Xu [LWX19], as well as work in projectivity of the K-moduli space by Codogni–Patakfalvi [CP21].

From our discussion of moduli problems and moduli spaces in Section 2.1.4 we understand that we need an appropriate definition of families of Fano varieties, if we are to define a moduli stack that admits a good moduli space. We need this definition over a general base to determine the scheme structure of the moduli space. From our discussion so far, since we are aiming to parametrise families of K-semistable Fano varieties, it is natural to consider families over $\mathbb{Q}$-Fano varieties. This is truly beneficial for us, since a great deal of the relevant theory has already been surveyed [Kol09].

Definition 2.77 ([Xu21, Definition 6.1])**.** A *$\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties* $f\colon \mathcal{X} \to B$ over a normal base B, is a flat proper morphism such that:

1. f has normal, connected fibres, which implies that $\mathcal{X}$ is also normal,
2. $-K_{\mathcal{X}/B}$ is an f-ample $\mathbb{Q}$-Cartier divisor,
3. the fibres $\mathcal{X}_t$ are klt for all $t \in B$.

If in addition each fibre $\mathcal{X}_t$ is K-semistable we call f a *$\mathbb{Q}$-Gorenstein family of K-semistable $\mathbb{Q}$-Fano varieties*.

We will define the K-moduli functor as follows:

Definition 2.78 ([BX19, §1])**.** The *K-moduli functor* $\mathcal{M}^K_{n,V}$ is a functor that sends a scheme $S \in$ Sch to

$$\mathcal{M}^K_{n,V}(S) := \left\{ \begin{array}{r} \text{flat proper morphisms } f\colon \mathcal{X} \to S \text{, with fibres } \mathcal{X}_t \text{ that are} \\ n\text{-dimensional } K\text{-semistable } \mathbb{Q}\text{-Fano varieties with volume } V \\ \text{satisfying Kollár's condition (see, [Kol09, p. 24])} \end{array} \right\}.$$

We will usually omit n, V from our notation, as in most cases the choice will be explicit. Showing that this functor is an algebraic stack is in fact a difficult process. In addition, demonstrating that it admits a good moduli space M^K is also an arduous process that involves some groundbreaking results in moduli theory, and many results from the MMP. We will not go over these in great detail, but we will make a summary of the results. In particular, in order to obtain the good moduli space from K-stability which also holds 'nice' properties, we need to go over the steps below (see, [Xu21, §6, 7, 8] and [Wan19, Chapter 4]):

1. Prove that $\mathbb{Q}$-Gorenstein K-semistable $\mathbb{Q}$-Fano varieties belong to a finite number of families in any given dimension (boundedness condition);
2. show that K-semistable varieties form a Zariski open set in families;

3. show that, for special kinds of pointed surfaces $0 \in S$, a family of K-semistable Fano varieties over the punctured surface $S \setminus \{0\}$ can be uniquely extended to a family over the entire surface S ([AHH19]);

4. show that any $\mathbb{Q}$-Gorenstein family of K-semistable Fano varieties over a punctured curve $C^\circ = C \setminus \{0\}$ can be can be filled in over 0 to a $\mathbb{Q}$-Gorenstein family of K-semistable Fano varieties over C;

5. show that there exists a natural ample $\mathbb{Q}$-line bundle on M^K.

Steps 1 and 2 guarantee that $\mathcal{M}^K$ is in fact an algebraic stack of finite type. The boundedness condition would imply that there exists a positive integer N such that $-NK_X$ is a very ample Cartier divisor for any n-dimensional K-semistable $\mathbb{Q}$-Fano variety X. This implies that the linear system $|-NK_X|$ defines an embedding $|-NK_X|\colon X \to \mathbb{P}^m$ for some uniform m. In turn, this implies that there exists a Hilbert scheme $\mathrm{Hilb}(X; \mathbb{P}^m)$, where any such embedding gives a point in $\mathrm{Hilb}(X; \mathbb{P}^m)$. Hence, there exists a locally closed subscheme $W \subset \mathrm{Hilb}(X; \mathbb{P}^m)$ such that a map $B \to W$ factors through W if and only if the pullback family Univ_B (the *universal family*) is a $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties and $O(-NK_{\mathrm{Univ}_B/B}) \sim_B O(1)$. The openness condition would further imply that we could find an open subscheme $U \subset W$, such that $\mathcal{M}^K = [U/\,\mathrm{PGL}(m+1)]$, which is an algebraic stack.

Step 3 is used to verify that in fact there exists a good moduli space M^K. The first step, is to show that if M^K exists it must be separated. The existence of a good moduli space, is a subtle condition and, in general, one that is hard to show, but one that contains strong implications regarding the orbit geometry with respect to $\mathrm{PGL}(m+1)$. The difficulty here arises from the fact that a family of $\mathbb{Q}$-Fano varieties $\mathcal{X}^\circ$ over a punctured curve $C^\circ = C \setminus \{0\}$ could admit many possible fillings to give a family of $\mathbb{Q}$-Fano varieties $\mathcal{X}$ over C. Hence, we need to be careful and only consider the fillings which are K-semistable up to an equivalence condition. Moreover, since a K-polystable $\mathbb{Q}$-Fano could have an infinite automorphism group in general, we can not expect that the extension family is unique. Hence, we need to rely on the separatedness of M^K and define carefully a notion of equivalence for different fillings. Step 4 is used to show that M^K is proper, and step 5 is used to show that it is projective. For the convenience of the reader, there exists a fantastic survey of the above results and discussion [Xu21], which includes a more in-depth analysis than the one we will present here.

Before we move on to discuss steps $1-5$ in more detail, we will end this part with the following picture, which can motivate the reader to understand the above construction in more depth. For more information, one is prompted to [Ser06; Man09]. For a Fano variety X recall that Def_X is the *infinitesimal deformation functor* of X. For an Artinian local $\mathbb{C}$-algebra A with residue field $\mathbb{C}$, $\mathrm{Def}_X(A)$ consists of isomorphism classes of commutative diagrams:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X}_S \arrow[d] \\ \{0\} = \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & S = \mathrm{Spec}(A) \end{tikzcd}$$

where $\{\mathcal{X}_S \to S\} \in \mathrm{Def}_X(A)$ is a *deformation family* of X over S. If we further assume that X is K-polystable, this implies that $G = \mathrm{Aut}(X)$ is reductive by Theorem 2.61. G acts on A and we have a good quotient $[S/G] \to S^G$. If we take it for granted that $\mathcal{M}^K$ is an algebraic stack which admits a good moduli space M^K, the Luna étale slice theorem for algebraic stacks [AHR20, Theorem 1.1] gives a fibre product

$$\begin{tikzcd} {[S/G]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^K \arrow[d] \\ S^G \arrow[r] & M^K \end{tikzcd}$$

which allows us to get a better understanding of the structure of $\mathcal{M}^K$.

2.2.2.3 Boundedness and Openness

Boundedness is the first result one can establish for K-stability. This was accomplished in [Jia20] using modern algebro-geometric techniques developed in [HMX14; Bir21; Bir19]. Before we discuss these, we will introduce some notation.

For a closed point $x \in X$, a valuation $v \in \mathrm{Val}_{X,x}$ and an integer m we can define the *valuation ideal* $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{x,X} | \, v(f) \geq m\}$. Verifying that this indeed is an ideal is not hard; for $f, g \in \mathfrak{a}_m(v)$ we have $v(f+g) \geq \min\{v(f), v(g)\} \geq m$, i.e. $f + g \in \mathfrak{a}_m(v)$, and for $h \in \mathcal{O}_{x,X}$ $v(fh) = v(f) + v(h) \geq m + v(h) \geq m$, i.e. $fh \in \mathfrak{a}_m(v)$. In fact, this ideal is an $\mathfrak{m}_x$-primary ideal for all $m > 0$. This discussion allows us to define the volume of a valuation v as in [ELS03].

Definition 2.79. Let X be an n-dimensional normal variety. Let $x \in X$ be a closed point. We define the *volume of a valuation* $v \in \mathrm{Val}_{X,x}$ as

$$\mathrm{vol}_{X,x}(v) := \limsup_{m\to\infty} \frac{l(\mathcal{O}_{x,X}/\mathfrak{a}_m(v))}{\frac{m^n}{n!}}$$

where l denotes the length of the Artinian module.

For a log pair (X, Δ) we can extend the definition of the log discrepancy from divisors to valuations [LLX20]. This allows us to define a *log discrepancy function* $A_{(X,\Delta)}\colon \operatorname{Val}_X \to (0, \infty]$. This is done in three successive steps:

1. for a divisor E/X we set $A_{(X,\Delta)}(\operatorname{ord}_E) := A_{(X,\Delta)}(E)$;

2. For a quasi-monomial valuation v_α (see Example 2.69.2) where $\alpha \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$, let $(Y, E = \sum_{i=1}^N E_i)$ be a log smooth model for X and η be the generic point of a connected component of $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}$ of codimension r. Let $(y_1, \ldots, y_r)$ be a system of parameters of the local ring $\mathcal{O}_{Y,\eta}$ such that $E_{i_j} = (y_j = 0)$. Then for any tuple $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \setminus \{0\}$ we define $A_{(X,\Delta)}(v_\alpha)$ as
$$A_{(X,\Delta)}(v_\alpha) := \sum_{j=1}^r \alpha_j A_{(X,\Delta)}(\operatorname{ord}_{E_{i_j}});$$

3. let (Y, E) be a log smooth model for X. Keeping the notation from step 2, we let $QM_\eta(Y, E)$ be the *set of all quasi-monomial valuations* v that can be described at the point η as above. We put $QM(Y, E) := \bigcup_\eta QM_\eta(Y, E)$ where the union runs over generic points of all irreducible components of intersections of some of the divisors E_i. By [JM12], there exists a topology on Val_X and $\mathrm{QM}(Y, E)$ and a retraction map $r_{Y,E}\colon \operatorname{Val}_X \to QM(Y, E)$ which induces a homeomorphism $\operatorname{Val}_X \to \varprojlim_{(Y,E)} QM(Y, E)$. This allows us to define $A_{(X,\Delta)}$ for all real valuations v as follows:
$$A_{(X,\Delta)}(v) := \sup_{(Y,E)} A_{(X,\Delta)}(r_{(Y,E)}(v)).$$
Here, the supremum ranges over all log smooth models (Y, E) of X.

Philosophically, steps 1-3 should make sense to the reader. Step 2 is reminiscent of a local definition of $A_{(X,\Delta)}$ chosen over a resolution of singularities for X, and step 3 uses the results in [JM12; Bou+15] to extend the local definition to all real valuations by considering all possible such resolutions of singularities.

Definition 2.80 ([Li18, Chapter 2])**.** Let (X, Δ) be an n-dimensional klt log pair, $x \in X$ be a closed point. The *normalised volume function of valuations* $\widehat{\operatorname{vol}}_{(X,\Delta),x}\colon \operatorname{Val}_{X,x} \to (0, \infty]$ is defined as:
$$\widehat{\operatorname{vol}}_{(X,\Delta),x}(v) := \begin{cases} A_{(X,\Delta)}(v)^n \operatorname{vol}_{X,x}(v), & \text{if } A_{(X,\Delta)}(v) < +\infty \\ +\infty, & \text{if } A_{(X,\Delta)}(v) = +\infty. \end{cases}$$

The *volume of the singularity* $(x \in (X, \Delta))$ is defined as

$$\widehat{\mathrm{vol}}(x, (X, \Delta)) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_{(X,\Delta),x}(v).$$

We can think of this volume as a measure of how 'bad' the singularity is. It is the local analogue of the volume of the anticanonical divisor $-K_X$. In recent years, it has been found that this volume is in fact bounded. We will use this result to show the boundedness of K-semistable varieties. This is different to the results presented in [Jia20] and more akin to the treatment in [LLX20]. Restricting to the case $\Delta = 0$ we have the following result.

Theorem 2.81. *[Fuj18; Liu18; BJ20] Let X be a $\mathbb{Q}$-Fano variety. For any $x \in X$ we have*

$$\widehat{\mathrm{vol}}(x, X)\left(\frac{n+1}{n}\right)^n \geq (-K_X)^n \cdot \delta(X)^n.$$

Here, $\delta(X, \Delta)$ is the *delta invariant*, defined as $\delta(X, \Delta) := \inf_{E/X} \frac{A_{(X,\Delta)}(E)}{S_{(X,\Delta)}(E)}$, where

$$S_{(X,\Delta)}(E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty (\mathrm{vol}(-K_X - \Delta - xE) dx.$$

This Theorem implies that if we bound $(-K_X)^n$ and $\delta(X)$ from below we obtain a lower bound for $\widehat{\mathrm{vol}}(x, X)$. Using the above, we obtain the following:

Theorem 2.82 ([Jia20], [Xu21, Theorem 6.5] (Boundedness))**.** *Fix $n \in N$ and $V > 0$. All n-dimensional K-semistable $\mathbb{Q}$-Fano varieties with volume at least V are contained in a bounded family.*

Proof. Let X be a K-semistable $\mathbb{Q}$-Fano variety. Take a finite cover $f\colon (y \in Y) \to (x \in X)$ étale in codimension 1. Then by [XZ21, Theorem 1.3] and [XZ21, Theorem 4.22 (2)]

$$\widehat{\mathrm{vol}}(y, Y) = \deg(f) \cdot \widehat{\mathrm{vol}}(x, X).$$

This implies that $-K_Y$ is $\mathbb{Q}$-Cartier, and the local Cartier index is $\deg(f)$. We apply this locally to the index-1 cover of $-K_X$, and by [LX19, Theorem 1.6] we have $\widehat{\mathrm{vol}}(y, Y) \leq n^n$. This implies that the Cartier index of any point $x \in X$ is bounded from above by $\frac{n^n}{\widehat{\mathrm{vol}}(x,X)}$. Note, that since $X$is K-semistable, x is at worse a klt singularity, and by [Li18, Theorem 3.1 and Theorem 4.1] we know that $\widehat{\mathrm{vol}}(x, X)$ is always positive. Thus, the Cartier index of X is bounded from above, and by [HMX14, Theorem 1.8] we know that it belongs in a bounded family. □

For the openness result, we will omit the proof. We will note that one can obtain stronger results, as in the following theorem.

Theorem 2.83 ([Xu21, Theorem 6.9]). *Let $\mathcal{X} \to B$ be a $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties, and let $s \in B$ be a point. We have the following:*

1. *The function*
$$(s \in B) \to \min\{\delta(\mathcal{X}_{\bar{s}}), 1\}$$
is a constructible, lower-semicontinous function,

2. *if the family is a family of klt singularities, the function*
$$(s \in B) \to \widehat{\mathrm{vol}}(s, \mathcal{X}_s)$$
is a constructible, lower-semicontinous function.

Both of these results imply the following

Corollary 2.83.1 ([Xu20; BLX22]). *Let $\mathcal{X} \to B$ be a $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties, then the locus where the fibre is K-semistable is an open set.*

Our discussion in Section 2.2.2.2 thus shows that $\mathcal{M}^K_{n,V} = [U/\operatorname{PGL}(m+1)]$ for some open scheme U. Hence, since it is a quotient stack, it is in fact a moduli functor which is represented by an algebraic stack of finite type by Theorem 2.46 and Example 2.44.1.

2.2.2.4 Existence of Good Moduli Space

As we mentioned before, we have to establish that the Artin stack $\mathcal{M}^K$ does in fact admit a good moduli space M^K. We will make a brief account of the results presented in [LWX21; BX19; Alp+20] which in turn rely on the abstract results on moduli theory presented in [AHH19]. We will not give a very deep account of the above theories, but we invite the reader to consult [Xu21] for more information.

From our previous discussion, it should be apparent that we need to study fillings to punctured families. For this, we introduce the notions below:

Definition 2.84. A test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$ is called a *special test configuration* if $\mathcal{L} \sim_{\mathbb{Q}} -rK_{\mathcal{X}}$ and the special fibre $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety. By inversion of adjunction, this is equivalent to saying $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein and $-K_{\mathcal{X}}$ is ample and $(\mathcal{X}, \mathcal{X}_0)$ is plt.

The usefulness of considering special test configurations arises from the fact that it is enough to consider only special test configurations in determining K-stability [LX14].

Definition 2.85. Two K-semistable $\mathbb{Q}$-Fano varieties X and X' are *S-equivalent* if they degenerate to a common K-semistable variety via special test configurations.

The notion of S-equivalence seems to be the 'right' notion in our attempt to go over step 3 in Section 2.2.2.2. This is bolstered by the following result.

Theorem 2.86 ([BX19, Theorem 1.1]). *Let $f\colon X \to C$ and $f'\colon X' \to C$ be $\mathbb{Q}$-Gorenstein families of Fano varieties over a smooth pointed curve $0 \in C$. Assume there exists an isomorphism*

$$\phi\colon X \times_C C^\circ \to X' \times_C C^\circ$$

over $C^\circ := C \setminus \{0\}$. If X_0 and X_0' are K-semistable, then they are S-equivalent.

Remark 2.86.1. The above Theorem also generalises to the case of log Fano pairs (see [BX19]).

Roughly, Theorem 2.86 says that if there exist two different K-semistable fillings of some families of $\mathbb{Q}$-Fano varieties, then these should be S-equivalent, and we should focus on studying S-equivalence in order to cover step 3. To formalise the above, we need to introduce some notation. Let R be a DVR with fraction field K, and let $\eta = \operatorname{Spec}(K)$ be the generic point. Let $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ with the natural multiplicative action, and set $\Theta_R = \Theta \times \operatorname{Spec}(R)$. We also set $0 \in \Theta_R$ to be the unique closed point.

Definition 2.87 ([AHH19, Definition 3.10]). We say that an algebraic stack $\mathcal{Y}$ is Θ-*reductive* if a morphism $\Theta_R \setminus 0 \to \mathcal{Y}$ can be uniquely extended to a morphism $\Theta_R \to \mathcal{Y}$.

Fix an uniformiser π of R. As in [Hei17, Chapter 2.B] define the quotient stack

$$\overline{\mathrm{ST}}(R) := [\operatorname{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m]$$

where the action is $(s,t) \to (\mu \cdot s, \mu^{-1} \cdot t)$. Let $0 = [(0,0)/\mathbb{G}_m]$, then $\overline{\mathrm{ST}}(R) \setminus 0$ is isomorphic to the curve with two origins $\operatorname{Spec}(R) \cup_{\operatorname{Spec}(K)} \operatorname{Spec}(R)$.

Definition 2.88 ([AHH19, Definition 3.37]). A stack $\mathcal{Y}$ is *S-complete* if any morphism $\pi^\circ\colon \overline{\mathrm{ST}}(R) \setminus 0 \to \mathcal{Y}$ can be uniquely extended to a morphism $\pi\colon \overline{\mathrm{ST}}(R) \to \mathcal{Y}$.

The notions of Θ-reductivity and S-completeness are particularly important due to the next theorem.

Theorem 2.89 ([AHH19, Theorem A]). *Let $\mathcal{Y}$ be an algebraic stack of finite type. $\mathcal{Y}$ admits a good moduli space if it is S-complete and Θ-reductive.*

We will omit the proof of this Theorem, but we prompt the reader to [Alp+20; Xu21] for more details. From Theorem 2.89 we can show that $\mathcal{M}^K$ admits a good moduli space if we show that $\mathcal{M}^K$ is Θ-reductive and S-complete. The following result is equivalent to establishing Θ-reductivity for $\mathcal{M}^K$.

Theorem 2.90 ([Alp+20, Thoerem 5.2])**.** *Let R be a DVR of essentially finite type and η the generic point of $\mathrm{Spec}(R)$. For any $\mathbb{Q}$-Gorenstein family of K-semistable Fano varieties X_R over R, any special K-semistable degeneration $X_\eta/\mathbb{A}^1_\eta$ of the generic fibre X_η can be extended to a family of K-semistable degenerations $X_R/\mathbb{A}^1_R$ of X_R.*

For S-equivalence, Definition 2.88 may seem complicated, but a keen reader will notice that for our case, Theorem 2.86 gives us the S-completeness of $\mathcal{M}^K$. Let $f\colon X \to \mathrm{Spec}(R)$ and $f'\colon X' \to \mathrm{Spec}(R)$ be two families of $\mathbb{Q}$-Fano varieties, such that we have an isomorphism $\phi\colon X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \cong X' \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$. Then, we can define a family $\pi^\circ\colon X^\circ \to \overline{\mathrm{ST}}(R)\backslash 0$. Our aim is to define an appropriate $\mathcal{X}$ such that this family extends to $\mathcal{X} \to \overline{\mathrm{ST}}(R)$, which is precisely the claim of S-completeness for the functor of K-semistable Fano varieties with fixed numerical invariants. Let $L := -rK_X$, $L' := -rK_{X'}$ where r is a positive integer so that L and L' are Cartier. We also define

$$\mathcal{R}_m := H^0(X, \mathcal{O}_X(mL)) \quad \mathcal{R}'_m := H^0(X', \mathcal{O}_{X'}(mL'))$$

$$R_m := H^0(X_0, \mathcal{O}_{X_0}(mL_0)) \quad R'_m := H^0(X'_0, \mathcal{O}_{X'_0}(mL'_0)).$$

We fix a common log resolution $\hat{X}$ of X and X', such that we have diagram

$$\begin{tikzcd} & \hat{X} \arrow[dl, "\psi"'] \arrow[dr, "\psi'"] & \\ X \arrow[rr, dashed, "\phi"] & & X'. \end{tikzcd}$$

We write $\hat{X}_0$ and $\hat{X}'_0$ for the birational transforms of X_0 and X'_0 respectively, where $\hat{X}_0 \ncong \hat{X}'_0$, since ϕ does not extend to an isomorphism over the 0-fibers. Then, both $\hat{X}_0$ and $\hat{X}'_0$ are divisors over X and X', and as such we can define valuations $u := \mathrm{ord}_{\hat{X}_0}$ and $u' := \mathrm{ord}_{\hat{X}'_0}$. The existence of these valuations allows us to define filtrations $\mathcal{F}$ and $\mathcal{F}'$ as follows. For each $p \in \mathbb{Z}$ and $m \in \mathbb{N}$, set $\mathcal{F}^p\mathcal{R}_m := \{s \in \mathcal{R}_m | u'(s) \geq p\}$, and $\mathcal{F}'^p\mathcal{R}'_m := \{s \in \mathcal{R}'_m | u(s) \geq p\}$. In particular, we can extend these definitions to filtrations on R_m and R'_m as follows. We set $\mathcal{F}^p R_m := \mathrm{im}(\mathcal{F}^p\mathcal{R}_m \to R_m)$, $\mathcal{F}'^p R'_m := \mathrm{im}(\mathcal{F}'^p\mathcal{R}'_m \to R'_m)$, where the maps are given by restrictions of sections. Note that a section $s \in R_m$ lies in $\mathcal{F}^p R_m$ if and only if there exists an extension $\tilde{s} \in \mathcal{R}_m$ of s such that $\tilde{s} \in \mathcal{F}^p\mathcal{R}_m$. Notice that these filtrations are related, and in fact we have an isomorphism of graded rings

$$\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} \operatorname{gr}^p_{\mathcal{F}} R_m \cong \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} \operatorname{gr}^p_{\mathcal{F}'} R'_m.$$

We denote by $\iota\colon \overline{\mathrm{ST}}(R) \setminus 0 \hookrightarrow \overline{\mathrm{ST}}(R)$ the open inclusion of stacks. Notice, that the pushforward $\pi^\circ_*(-rmK_{X^\circ})$ is a vector bundle on $\overline{\mathrm{ST}}(R)\setminus 0$. Also, the point 0 is of codimension 2 in $\overline{\mathrm{ST}}(R)$, and as such the pushforward $\iota_*(\pi^\circ_*(-rmK_{X^\circ}))$ is also a vector bundle of $\overline{\mathrm{ST}}(R)$. In particular, in [Alp+20, Proposition 3.7], it is shown that for any m

$$\iota_*(\pi^\circ_*(-rmK_{X^\circ})) \cong \bigoplus_{p\in\mathbb{Z}} \operatorname{gr}^p_{\mathcal{F}} R_m.$$

We then define $\mathcal{X}$ as

$$\mathcal{X} := \operatorname{Proj}\Big(\bigoplus_m \iota_*(\pi^\circ_*(-rmK_{X^\circ}))\Big).$$

The remaining part of the proof is to show that the graded ring $\bigoplus_m \iota_*(\pi^\circ_*(-rmK_{X^\circ}))$ is finitely generated and via restriction to R yields normal test configurations, via filtrations, for X'_0 and X_0. Then, by [LWX21] this will imply that $\mathcal{X}_0$ is a K-semistable $\mathbb{Q}$-Fano variety. Hence, the two families $X \to \operatorname{Spec}(R)$ and $X' \to \operatorname{Spec}(R)$ which define a family $\pi^\circ\colon \overline{\mathrm{ST}}(R) \setminus 0$ will extend to a family $\mathcal{X} \to \overline{\mathrm{ST}}(R)$, with K-semistable fibers, and as such the valuative criterion of S-completeness will be satisfied. These results mainly follow from [BX19] and the proof of Theorem 2.86 which gives finite generation of the graded ring and ensures that it provides normal test configurations (see, [BX19]).

The discussion in this section, in addition with [LWX21, Theorem 1.3], which shows the separatedness of the K-moduli space M^K, allows us to show the following:

Theorem 2.91 ([Alp+20, Corollary 1.2])**.** *The finite type algebraic stack $\mathcal{M}^K$ admits a separated good moduli space $\phi\colon \mathcal{M}^K \to M^K$.*

In the next section, we will focus on the properties of this good moduli space.

2.2.2.5 Projectivity and Properness of the K-moduli Space

The good moduli space M^K admits a natural $\mathbb{Q}$-line bundle, called the *Chow-Mumford* or more typically *CM line bundle*. This has been known analytically since the early days of the study of K-stability [Tia87; FS90; Tia97], and analytical results had already shown that this canonical line bundle is big and nef on the locus of smooth K-polystable Fano varieties [LWX19]. Projectivity of the K-moduli space is shown by showing this line bundle is ample. This has been established using results from [CP21] and [XZ20]. In this section, we will sketch the methods needed to establish these results. We will define the CM line bundle, but we will do so for the case of log pairs, as that will be of particular importance to us in Chapter 9.

Definition 2.92 (See [PT09; GMS21]). Let $f\colon \mathcal{X} \to \mathcal{T}$ be a proper flat morphism of varieties of relative dimension n such that the general fibre is normal, and $\mathcal{L}$ an f-ample $\mathbb{Q}$-line bundle on $\mathcal{X}$. Let $\mathcal{D} \subseteq \mathcal{X}$ be an effective Weil $\mathbb{Q}$-divisor of $\mathcal{X}$ such that $\mathcal{D}|_t$ is equidimensional of dimension $n-1$ for all $t \in \mathcal{T}$. By the Knudsen-Mumford theorem [KM76], there exist functorially defined line bundles on $\mathcal{T}$, $\lambda_j := \lambda_j(\mathcal{X}, \mathcal{T}, \mathcal{L})$, and $\tilde{\lambda}_j := \lambda_j(\mathcal{D}, \mathcal{T}, \mathcal{L}|_{\mathcal{D}})$ where $1 \leq j \leq n+1$, such that

$$\det(f!_*(\mathcal{L}^r)) = \bigotimes_{i=1}^{n+1} \lambda_i^{\binom{r}{i}}$$

$$\det\left(f!_*((\mathcal{L}|_{\mathcal{D}})^r)\right) = \bigotimes_{i=1}^{n+1} \tilde{\lambda}_i^{\binom{r}{i}}.$$

Since f is flat, the Hilbert polynomial is constant along fibres $t \in \mathcal{T}$. Let $p(k)$ and $\tilde{p}(k)$ be the Hilbert polynomials of $\mathcal{L}_t$ and $\mathcal{L}_t|_{\mathcal{D}}$ on fibres $\mathcal{X}_t$ and $\mathcal{D}_t$, respectively. For k sufficiently large, we have $p(k) = a_0 k^n + a_1 k^{n-1+} \dots$ and $\tilde{p}(k) = \tilde{a}_0 k^n + \tilde{a}_1 k^{n-1} + \dots$.

For tuple $(\mathcal{X}, \mathcal{D}, \mathcal{T}, \mathcal{L}^r)$ we define the *log CM line bundle with angle* $\beta \in \mathbb{Q}_{>0}$ *on* $\mathcal{T}$ *to be*

$$\Lambda_{CM,\beta}(\mathcal{X}, \mathcal{D}, \mathcal{L}) := \lambda_{n+1}^{n(n+1)+\frac{2a_1-(1-\beta)\tilde{a_0}}{a_0}} \otimes \lambda_n^{-2(n+1)} \otimes \tilde{\lambda}^{(1-\beta)(n+1)}$$

Notice, that if $\beta = 1$, we recover the original definition of Paul–Tian [PT09] for varieties. We also the following theorem which lets us classify the CM line bundle.

Theorem 2.93 ([GMS21, Theorem 2.7]). *Let (X, D, L) be the restriction of a family $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ where Grothendieck- Riemann- Roch applies (e.g. if the fibres have mild singularities, for instance if they are locally complete intersections) to a general $b \in \mathcal{B}$ and assume that X is $\mathbb{Q}$-factorial. Moreover, if $\mathcal{L} = -K_{\mathcal{X}/\mathcal{B}}$ and $\mathcal{D}|_{\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$ for all $b \in \mathcal{B}$, we have*

$$\deg(\Lambda_{CM,\beta}) = \pi_*\Big(c_1(-K_{\mathcal{X}/\mathbb{B}})^{n-1}\Big) \cdot \Bigg(- c_1(-K_{\mathcal{X}/\mathbb{B}}) + (1-\beta)\big(n\mathcal{D} - (n-1)c_1(-K_{\mathcal{X}/\mathbb{B}})\big)\Bigg)$$

This construction is natural in our setting, since we can set $\mathcal{T} = \mathcal{M}^K$ and take the universal family $\mathcal{X} \to \mathcal{M}^K$. In this case, λ_{CM} is a $\mathbb{Q}$-line bundle on $\mathcal{M}^K$. In fact, this descends to a line bundle (here denoted by Λ_{CM}) on the good moduli space M^K.

Theorem 2.94 ((Projectivity) [CP21; XZ20]). *The restriction of Λ_{CM} to any proper subspace of $\mathcal{M}^K$ whose points parametrise reduced uniformly K-stable Fano varieties, is ample. In particular, $\Lambda_{CM}|_{M^K}$ is ample.*

Properness is arguably the most difficult of the properties of the good moduli space to show. Properness here would imply that any family of K-semistable Fano varieties over a punctured curve $C^\circ = C \setminus \{0\}$, after a possible finite base change, can be filled in over 0 to a family of K-semistable Fano varieties over C. In [Blu+21], a specific strategy was proposed in order to show properness of the K-moduli space. This is sometimes called Langton's algorithm, after Langton [Lan75], who proved the valuative criterion of properness for the moduli space of polystable sheaves on a smooth projective variety X of arbitrary dimension. As before, let R be a DVR and K be its fraction field. For a semistable sheaf F_K on $X \times \operatorname{Spec}(K)$, Langton's approach shows that F_κ of F_K on $X \times \operatorname{Spec}(\kappa)$, where κ is the residue field of R, with a sequence of uniquely determined elementary transformations, so that the 'instability' of F_κ decreases. Moreover, he showed that after finitely many steps this process terminates with the degeneration becoming semistable. This method was abstracted in [AHH19], where it is shown that Langton's algorithm can be carried out on an Artin stack as long as it admits a Θ-*Stratification*. The introduction of the Θ-Stratification is based on three key observations:

1. for a point x in a stack $\mathcal{X}$, the stability of x is determined by considering maps $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ such that $f(1) = x$,
2. if x is unstable, then there should be a unique optimal destabilising map,
3. these optimal destabilisations should satisfy certain properties in families and can be used to stratify the unstable locus of the stack.

These observations are not in vacuum; they arise from the study of the Harder-Narasimhan stratification of the moduli of coherent sheaves on a projective scheme as well as the Kempf-Ness stratification in GIT.

We will not define the notion of Θ-Stratification here, or go into terrible detail, but we should note that in [Blu+21], the authors showed that the notion of Θ-Stratification for $\mathcal{M}^K$, is equivalent to the existence of optimal destabilisations. This stems from the observation that maps $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{M}^K$ such that $f(1) = [X]$ are equivalent to special test configurations of

X, which further to the observation that in order to attain properness one must identify a unique optimal destabilizing test configuration for each K-unstable Fano variety X. Hence, the existence of optimal destabilisations show that there exists a Θ-Stratification for $\mathcal{M}^K$, which in turn, in combination with the theory established in [AHH19], shows properness of K-moduli spaces.

The Optimal Destabilisation Conjecture, along with other results, was recently established in [LXZ22] using results from [Alp+20] and [XZ20]. We should note that the results in [LXZ22] are proven for the K-moduli of log Fano pairs (X, Δ), which is a more extensive setting. Although we have not introduced these already, their definitions occur as a natural extension to the definitions in our discussion in Section 2.2.2.2. Although results on boundedness (Theorem 2.82) and openness (Corollary 2.83.1) extend to the log Fano setting very naturally, the results on projectivity do not. This is due to the nature of deformations of the divisor Δ. The result in [LXZ22] establishing properness is the following:

Theorem 2.95 ([LXZ22], Optimal Destabilization Conjecture [Blu+21, Conjecture 1.1]). *Let (X, Δ) be a log Fano pair of dimension n and let $r > 0$ be an integer such that $r(K_X + \Delta)$ is Cartier. Assume that $\delta(X, \Delta) < \frac{n+1}{n}$. Then $\delta(X, \Delta) \in \mathbb{Q}$ and there exists a divisorial valuation $v = \mathrm{ord}_E$, for divisor E/X, such that*

$$\delta(X, \Delta) = \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}.$$

In particular, if $\delta(X, \Delta) \leq 1$, then there exists a non-trivial special test configuration $(\mathcal{X}, \Delta_{\mathcal{X}})$ with a central fiber (X_0, Δ_0) such that $\delta(X, \Delta) = \delta(X_0, \Delta_0)$ and $\delta(X_0, \Delta_0)$ is computed by the $\mathbb{G}_m$-action induced by the test configuration structure. Here, $\delta(X, \Delta)$ is the delta invariant for log pairs (see [FO18, Definition 0.2], [BJ20, §4.2]).

In the notation of this Theorem, the delta invariant has been extended to valuations. The above result, combined with [Blu+21, Theorem 1.1] shows that there exists a Θ-Stratification for $\mathcal{M}^K$, which in turn, combined with the general results on good moduli spaces in [AHH19], yields properness of the K-moduli space M^K. As a final result on the properties of the K-moduli space, we present the following, which gives a bound on the singularity type of the moduli space.

Theorem 2.96 ([Bra+21, Theorem 4]). *Let $M^K_{n,V}$ be the good moduli space of K-polystable Fano manifolds of dimension n and volume V. Then, there exists an effective divisor $B_{n,V}$ on $M^K_{n,V}$ so that the pair $(M^K_{n,V}, B_{n,V})$ has Kawamata log terminal singularities.*

Remark 2.96.1. An interesting phenomenon occurs when we consider log Fano pairs $(X,(1-\beta)D)$ where $\beta \in (0,1) \cap \mathbb{Q}$. In this case, there exists a wall-chamber decomposition of the K-moduli spaces, similar to the situation of VGIT we discussed in Section 2.1.3.1 (see e.g. [GMS21; ADL19]). Explicit results for these compactifications are very hard to come by, and are usually fixing the variety before and after the wall. Later on in this thesis will examine a specific example where both variety and divisor are deformed before and after the wall, and we will obtain explicit results on the wall-crossing.

2.2.2.6 Examples and Compactifications

So far, we have seen that the moduli stack $\mathcal{M}^K$ admits a moduli space, M^K which is separated, projective and proper, and whose closed points are in bijection with n-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume V. Hence, we can ask ourselves whether these compact K-moduli can be described explicitly. This means that we are both interested in explicitly describing which varieties are in the compactification, or describing what the compactification is as a scheme. The answer is (thankfully) yes to this question, but at the same time, results are scarce and the method for obtaining such explicit compactifications is very highly dependent on the specific example of a Fano family we wish to compactify.

To give perspective to the above statement, there are 1, 10 [Pez85; Pez87] and 105 families of (smooth) Fano curves, surfaces and threefolds respectively. Smooth Fano threefolds were classified initially in Picard rank 1 by Iskovskih [Isk77; Isk78; Isk79], and later on in the remaining cases by Mori–Mukai [MM82; MM03] using results by Shokurov [Šok79]. For Fano surfaces, or del Pezzo surfaces, 4 of these admit moduli, and their K-moduli compactification was recently achieved by Odaka–Spotti–Sun [OSS16], using known results from Mabuchi–Mukai [MM90] and Ding–Tian [DT92]. For the families of Fano threefolds, only two have been compactified into moduli spaces by [SS17] (who show a more general result in compactifying the K-moduli of del Pezzo varieties of degree 4, i.e. a complete intersection of two quadrics, in any dimension), and [LX19]. For Fano fourfolds, only cubic 4-folds have been compactified recently [Liu22]. It is important to note that all the above compactifications have been achieved by relating the K-moduli to GIT quotients in some way or another. However, we should note that these compactifications would not have been achieved if not for previous work on finding which deformations have K-polystable elements (see [Tia90; Che08; Mar14]). Later on in this thesis, we will compactify another family of Fano threefolds, using similar methods.

When we describe the K-moduli components of smooth Fano varieties, the difficulty arises because we are 'adding' limit elements (obtained as limits of degenerations) that can be singular. Although we do know their singularities have to be at worse klt, since they are limits of K-polystable elements, knowing exactly what their singularities are might be difficult. As such, we need to study the singularities of the limit points that could potentially compactify the moduli spaces. In addition, we can use GIT to provide explicit compactifications. GIT quotients are much better understood than K-stability, so there are plenty of benefits in this approach. Trying to relate K-stability to GIT is not something new.

Theorem 2.97 ([PT09], [OSS16, Theorem 3.4])**.** *Let G be a reductive algebraic group without non-trivial characters. Let $\pi\colon (\mathcal{X}, \mathcal{L}) \to S$ be a G-equivariant polarised projective flat family of equidimensional varieties over a projective scheme. Here, "polarised" means that $\mathcal{L}$ is a relatively ample line bundle on $\mathcal{X}$, and "equidimensional" means that all the irreducible components have the same dimension. Suppose that*

1. *the Picard rank $\rho(S)$ is one;*
2. *there is at least one K-polystable $(\mathcal{X}_t, \mathcal{L}_t)$ which degenerates in S to $(\mathcal{X}_0, \mathcal{L}_0) \neq (\mathcal{X}_t, \mathcal{L}_t)$ via a one parameter subgroup λ in G, i.e. the corresponding test configuration is not of product type.*

Then a point $s \in S$ is GIT (poly/ semi)stable if $(\mathcal{X}_s, \mathcal{L}_s)$ is K-(poly/ semi)stable.

In studying the singularities of the limit of a degeneration family, one effective method is to study the normalised volume. In lower dimensions, alongside with a classification of singularities of the relevant deformation family, this becomes effective. The theorems below provide us with a very powerful tool in determining the singularities of limits of degeneration families.

Theorem 2.98 ([LL19, Proposition 4.6])**.** *Let (X, Δ) be a K-semistable log Fano pair of dimension n. Then for any closed point $x \in X$, we have*

$$(-K_X - \Delta)^n \leq \left(\frac{n+1}{n}\right)^n \widehat{\mathrm{vol}}(x, X, \Delta).$$

Theorem 2.99 ([Liu18, Theorem 3])**.** *Let X be K-polystable $\mathbb{Q}$-Fano variety of dimension n. Let $p \in X$ be a closed point. Suppose (X, p) is a quotient singularity with local analytic model $\mathbb{C}^n/G$ where $G \subset \mathrm{GL}(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$\widehat{\mathrm{vol}}(p, X) = \frac{n^n}{|G|}$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ *and* $X \cong \mathbb{P}^n/G$*, where* $\mathbb{G}_m$ *is the diagonal in* $\mathrm{GL}(n, \mathbb{C})$*.*

Let us consider an example, which first appeared in [DT92] and later expanded upon in [OSS16, Chapter 4]. Consider del Pezzo surfaces of degree 3, which (in the smooth case) are smooth cubic surfaces X in $\mathbb{P}^3$, with volume $(-K_X)^2 = 3$. If we consider a K-polystable limit W of a degeneration family $\mathcal{X}$ of smooth K-polystable del Pezzo cubics of degree 3 and a singular point $x \in W$ we have by continuity of volumes that $(-K_W)^2 = (-K_{\mathcal{X}_t})^2 = 3$, and thus Theorem 2.98 (with $\Delta = 0$) implies that

$$\widehat{\mathrm{vol}}(x, W) \geq \frac{4}{3}.$$

In addition, since W is at worse klt, W must have only isolated quotient singularities locally analytically isomorphic to $\mathbb{C}^2/G$, where G is a finite subgroup of $U(2)$ acting freely on S^3. This is implied by the fact that klt surface singularities are precisely quotient singularities [CKM88, Proposition 6.11], and that normal surfaces have only isolated singularities. Moreover, by Theorem 2.99 the local normalised volume for quotient singularities is given by $\widehat{\mathrm{vol}}(x, W) = \frac{4}{|G|}$, which implies that $|G| \leq 3$.

Recall that the order of orbifold group at an A_k singularity is $k+1$, a D_k singularity is $4(k-2)$, an E_6 singularity is 24, an E_7 singularity is 48, and an E_8 singularity is 120. Hence, since $|G| = 1, 2, 3$, we see that x is an A_1 or A_2 singularity (or W is smooth). The above discussion implies that if a del Pezzo surface of degree 3 is K-polystable, then it must be either smooth or have at worse $A1$ or $A2$ singularities. In addition, by [KS88] such a sungular del Pezzo surface of degree 3 is a cubic surface with at worse $A1$ or $A2$ singularities. Thus, all the elements of the K-moduli are cubic surfaces.

We now consider the GIT quotient of cubic surfaces, acted by $\mathrm{PGL}(4)$. In this case, the parameter scheme is $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_3)) = \mathbb{P}^{19}$, and the GIT quotient we want to study is $M^{GIT} := \mathbb{P}^{19} /\!\!/ \mathrm{PGL}(4)$. Thankfully, this quotient is one of the first GIT quotients which was explicitly described, by Hilbert using his Invariant Theory, even before GIT was developed. This was studied by Hilbert in [Hil93] but has also been reformulated by [MFK94] in a more modern language.

Theorem 2.100 ([Hil93], [Muk03, §7]). *A cubic surface* $X \subset \mathbb{P}^3$ *is*

1. *GIT stable if and only if it has at worst A_1 singularities,*

2. *strictly GIT polystable if and only if X is isomorphic to $S = \{x_1x_2x_3 - x_0^3 = 0\}$, which has 3 A_2 singularities,*

3. *GIT semistable if and only if it has at worst A_1 or A_2 singularities.*

Defining the universal family of cubics $\pi \colon \mathcal{X} \to \mathbb{P}^{19}$, where

$$\mathcal{X} = \left\{ (x_0 : \cdots : x_3) \times a_I \in \mathbb{P}^3 \times \mathbb{P}^{19} \;\middle|\; \sum a_I x^I = 0 \right\}$$

we can apply Theorem 2.97 to see that a cubic X is GIT-(poly/semi)stable if it is K-(poly/semi)stable. This is because the universal family is flat and proper and $\rho(\mathbb{P}^{19}) = 1$.

We can now apply the *moduli continuity method* as it appeared in [OSS16] to show that the two moduli stacks are isomorphic.

Let $\mathcal{X}$ be the Hilbert polynomial of smooth elements of the family of cubic surfaces, pluri-anticanonically embedded by $-mK_{\mathcal{X}}$ in $\mathbb{P}^N$, and let $\mathbb{H}^{\mathcal{X};N} :- \mathrm{Hilb}_{\mathcal{X}}(\mathbb{P}^N)$. Given a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\mathcal{X}(X, \mathcal{O}_{\mathbb{P}^N}(k)|_X) = \mathcal{X}(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\mathcal{X};N}$ denote its Hilbert point. Let

$$\hat{Z}_m := \left\{ \mathrm{Hilb}(X) \in \mathbb{H}^{\mathcal{X};N} \;\middle|\; \begin{array}{l} X \text{ is a Fano manifold of the family of cubic surfaces,} \\ \mathcal{O}_{P^N}(1)|_X \sim \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)) \end{array} \right\}$$

which is a locally closed subscheme of $\mathbb{H}^{\mathcal{X};N}$. Let $\overline{Z}_m$ be its Zariski closure in $\mathbb{H}^{\mathcal{X};N}$ and Z_m be the subset of $\hat{Z}_m$ consisting of K-semistable varieties.

Since by [Tia90; TY87] smooth cubic surfaces in $\mathbb{P}^3$ are K-stable and by [Oda15], the smooth K-stable loci is a Zariski open set of M^K, in the definition of moduli stack of $\mathcal{M}^K = [Z_m/PGL(N_m + 1)]$ for appropriate $m > 0$ and in fact $\mathcal{M}^{GIT} \cong [\overline{Z}_m/PGL(N_m + 1)]$.

Thus, by the above discussion we have an open immersion of representable morphism of stacks:

$$\phi \colon \mathcal{M}^K \longrightarrow \mathcal{M}^{GIT}$$

which descends to a map on the moduli spaces:

$$\overline{\phi} \colon M^K \longrightarrow M^{GIT}.$$

Note that representability follows once we prove that the base-change of a scheme mapping to the K-moduli stack is itself a scheme. Such a scheme mapping to the K-moduli stack is the same as a PGL-torsor over $\overline{Z}_m$, which produces a PGL-torsor over Z_m after a PGL-equivariant base change. This PGL-torsor over Z_m shows the desired pullback is a scheme. By [The22, Lemma 06MY], since ϕ is an open immersion of stacks, ϕ is separated and, since it is injective, it is also quasi-finite. In particular $\overline{\phi}$ is also injective.

Now, by [Alp13, Prop 6.4], since ϕ is representable, quasi-finite and separated, $\overline{\phi}$ is finite and ϕ maps closed points to closed points, we obtain that ϕ is finite. Thus, by Zariski's Main Theorem, as $\overline{\phi}$ is a birational morphism with finite fibers to a normal variety, $\overline{\phi}$ is an isomorphism to an open subset, but it is also an open immersion, thus it is an isomorphism. Moreover, by [Sal65] we know that $M^{GIT} \cong M^K$ is isomorphic to $\mathbb{P}(1, 2, 3, 4, 5)$.

The proofs for similar explicit descriptions of K-moduli use the moduli continuity method in one way or another. [LX19] use a variation of this method, taking into account that they need different methods to determine the singularities of limit varieties or deformations in the compact K-moduli. Later in this thesis, we will explore a variation of this method for compactifying the K-moduli of the family of Fano threefolds $2-25$.

CHAPTER 3

Variations of GIT Quotients

In this section, we will study how to generalise results in [Laz09b; GM18; GMZ18], in order to obtain a computational toolkit that will allow us to study specific GIT problems algorithmically. In more detail, we will consider GIT quotients of tuples $(X, H_1, \ldots, H_m)$ where X is the complete intersection of k hypersurfaces of degree d in $\mathbb{P}^n$, and the H_i are hyperplanes. We will explain what makes these types of quotients variational, and we will explain how one can define a Hilbert-Mumford numerical criterion in order to study these quotients.

We will then proceed to construct the computational setting necessary for our analysis. We will introduce a finite *fundamental set of one-parameter subgroups*, which, roughly, determines which of these tuples are not-stable/unstable with respect to a specific polarisation. Using this, we will demonstrate, with extra care in the case of pairs (X, H) how one can computationally obtain the walls and chambers of this VGIT problem.

We will also use a polyhedral criterion, the *Centroid Criterion*, following [GM18], which will allow us to distinguish between stable, and strictly semistable tuples. We will demonstrate how this, in addition with the extra condition that X is Fano, shows that in the case of pairs, the GIT quotient parametrises log pairs $(X, D = S \cap H)$. We also compute the dimension of the VGIT quotient.

We conclude the computational side of VGIT by introducing the concept of semi-destabilising families, following [GM18], which can roughly be thought as sets of weights which in turn will define the polynomials in the support of the pair of complete intersection and hyperplane such that these are unstable/non-stable for a specific parameter t. We will show that these families are maximal, in the sense that a pair (X, H) which is unstable/non-stable for some t must have their weights in these families.

3.1 Preliminaries

Throughout this Section, we will work over an algebraically closed field k. Let $G \coloneqq \mathrm{SL}(n+1)$. Consider a variety S which is the complete intersection of k hypersurfaces of degree d in $\mathbb{P}^n$, i.e, $S = \{f_1 = f_2 = \cdots = f_k = 0\}$, where each $f_i(x) = \sum f_{I_i} x^{I_i}$ with $I_i = \{d_{i,0}, \ldots, d_{i,n}\}$, $\sum_{j=0}^{n} d_{i,j} = d$ for all i. Here, $x^{I_i} = x_0^{d_{i,0}} x_1^{d_{i,1}} \ldots x_n^{d_{i,n}}$. Also consider $H_1, \ldots, H_m$, m distinct hyperplanes with defining polynomials $h_i(x) = \sum h_{i,j} x_j$. Let Ξ_d be the set of monomials of degree d in variables $x_0, \ldots, x_n$, written in the vector notation $I_i = (d_{i,0}, \ldots, d_{i,n})$. As in Gallardo –Martinez-Garcia [GM18; GMZ18] we define the associated set of monomials

$$\mathrm{Supp}(f_i) = \{x^{I_i} \in \Xi_d | f_{I_i} \neq 0\}, \quad \mathrm{Supp}(h_i) = \{x_j \in \Xi_1 | h_{i,j} \neq 0\}.$$

Let $V \coloneqq \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $W \coloneqq \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \simeq \mathrm{Sym}^d V$ be the vector space of degree d forms. For an embedded variety $S = (f_1, \ldots, f_k) \subseteq \mathbb{P}^n$ we associate its Hilbert point

$$[S] = [f_1 \wedge \cdots \wedge f_k] \in \mathrm{Gr}(k, W) \subset \mathbb{P} \bigwedge^k W.$$

Note that $\mathrm{Gr}(k, W)$ is embedded in $\mathbb{P} \bigwedge^k W$ via the Plücker embedding $(w_1, \ldots, w_r) \to [w_1 \wedge \cdots \wedge w_r]$, where the w_r are the basis vectors of W. We denote by $[\overline{S}] \coloneqq f_1 \wedge \cdots \wedge f_k$ some lift in $\bigwedge^k W$. We will consider the natural G action, given by $A \cdot f(x) = f(Ax)$ for $A \in G$.

For simplicity, we will denote $\mathrm{Gr}(k, W)$ by $\mathcal{R}_{n,d,k}$, and we let $\mathcal{R}_m \coloneqq \mathcal{R}_{n,d,k} \times \left(\mathcal{R}_{n,1,1}\right)^m$ be the parameter scheme of tuples $(f_1, \ldots f_k, h_1, \ldots, h_m)$, under the identification $(f_1, \ldots f_k) = c(f_1, \ldots f_k)$ and $h_i = c_i h_i$ for $c, c_i \in \mathbb{G}_m$, where:

$$\mathcal{R}_m \cong \mathrm{Gr}\left(k, \binom{n+d}{d}\right) \times \left(\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))\right)^m \hookrightarrow \mathbb{P} \bigwedge^k W \times (\mathbb{P}^n)^m.$$

In the case where $m = 1$, we will just write $\mathcal{R} = \mathcal{R}_1$. There is a natural G action on V and W given by the action of G on $\mathbb{P}^n$. This action induces an action of G on $\mathcal{R}_{n,d,k}$ via the natural maps, and by the inclusion map to the Plücker embedding $\mathbb{P} \bigwedge^k W$. By extension, we also obtain an induced action of G to $\mathcal{R}$. We aim to study the GIT quotients $\mathcal{R}_m /\!\!/ G$.

Let $\mathcal{C} := \mathbb{P}(W)$. We will begin our analysis by first studying the GIT quotients $\mathcal{C} /\!\!/ G$. $\mathcal{C}$ parametrises hypersurfaces $X = \{f = 0\}$ of degree d in $\mathbb{P}^n$, where $f = \sum f_I x^I$ is a polynomial of degree d. As we saw in Section 2.1.3 we must study the Hilbert-Mumford numerical criterion in order to study this quotient. In order to do so, we fix a maximal torus $T \cong (\mathbb{G}_m)^n \subset G$ which in turn induces lattices of characters $M = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{G}_m) \cong \mathbb{Z}^{n+1}$ and one-parameter subgroups $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G}_m, T) \cong \mathbb{Z}^{n+1}$ with natural pairing

$$\langle -, - \rangle \colon M \times N \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

given by the composition $(\chi, \lambda) \mapsto \chi \circ \lambda$. We also choose projective coordinates $(x_0 \colon \dots : x_n)$ such that the maximal torus T is diagonal in G. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to T \subset SL(n+1)$ we say λ is *normalised* if

$$\lambda(s) = \mathrm{Diag}(s^{\mu_0}, \dots, s^{\mu_n})$$

where $\mu_0 \geq \cdots \geq \mu_n$ with $\sum \mu_i = 0$ (implying $\mu_0 > 0, \mu_n < 0$ if λ is not trivial).

From Lemma 2.34 with $m = 1$, we can choose an ample G-linearisation $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(1)$. Hence, the set of characters, with respect to this G-linearisation corresponds to degree d polynomials f, as sections $s \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1))$ are polynomials of degree d. In particular, for a monomial x^I of degree d and a normalised one-parameter subgroup λ as above, the natural pairing is given by $\langle x^I, \lambda \rangle = \sum_{i=0}^n d_i \mu_i$. Note, that in many cases we will abuse the notation and write $\langle I, \lambda \rangle$ instead of $\langle x^I, \lambda \rangle$. Then, the G-action induced by λ on a monomial x^I is given by $\lambda(s) \cdot x^I = s^{\langle I, \lambda \rangle} x^I$. Naturally, the G-action induced by λ on the polynomial f is given by

$$\lambda(s) \cdot f = \sum_{I \in \mathrm{Supp}(f)} s^{\langle I, \lambda \rangle} f_I x^I.$$

In addition, notice that the action of λ on a fiber is equivalent to the action of λ on the polynomial f, and by the above discussion, we have that $\mathrm{weight}(f, \lambda) = \min_{x^I \in \mathrm{Supp}(f)} \{\langle I, \lambda \rangle\}$. Thus the Hilbert-Mumford function reads

$$\mu(f, \lambda) = - \min_{x^I \in \mathrm{Supp}(f)} \{\langle I, \lambda \rangle\},$$

and the Hilbert-Mumford numerical criterion (following Theorem 2.33) is:

Lemma 3.1. *With respect to a maximal torus T:*

1. *$X = \{f = 0\}$ is* semi-stable *if and only if $\mu(f, \lambda) \geq 0$, i.e. if $\min_{x^I \in \mathrm{Supp}(f)} \{\langle I, \lambda \rangle\} \leq 0$, for all non-trivial one-parameter subgroups λ of T.*

2. $X = \{f = 0\}$ *is* stable *if and only if* $\mu(f, \lambda) > 0$*, i.e. if* $\min_{x^I \in \mathrm{Supp}(f)}\{\langle I, \lambda\rangle\} < 0$*, for all non-trivial one-parameter subgroups* λ *of* T*.*

Remark 3.1.1. Fix a torus T, coordinates $(x_0 \colon \dots \colon x_n)$ such that the torus is diagonal, and a normalised one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s^{\mu_0}, \dots, s^{\mu_n})$. Let f be a degree d polynomial, and assume that for some $I = (d_0, \dots, d_n)$, for $x^I \in \mathrm{Supp}(f)$, $\langle I, \lambda\rangle$ is minimal, i.e.

$$\mu(f, \lambda) = -\langle I, \lambda\rangle = -\sum_{i=0}^{n} d_i \mu_i.$$

Notice that $\overline{\lambda(s)} = \mathrm{Diag}(s^{-\mu_n}, \dots, s^{-\mu_0})$ defines another normalised one-parameter subgroup, since $\sum_{j=0}^{n}(-\mu_{n-j}) = -\sum_{i=0}^{n} \mu_i = 0$, and $-\mu_{n-i} \geq -\mu_{n-i-1}$ since $\mu_{n-i-1} \geq \mu_{n-i}$. Let also $\overline{I} = (d_n, d_{n-1}, \dots, d_0)$ be another monomial vector. Then, we have $\langle \overline{I}, \overline{\lambda}\rangle = -\langle I, \lambda\rangle$. We can think of obtaining this new vector $\overline{I}$ by making the change of coordinates $x_{n-i} \leftrightarrow x_i$; this is a projective change of coordinates, such that f is projectively equivalent and isomorphic to $\overline{f}$, where $\overline{f} = \sum f_{\overline{I}} x^{\overline{I}}$. We define

$$\overline{\mu(\overline{f}, \overline{\lambda})} \coloneqq \max_{x^{\overline{J}} \in \mathrm{Supp}(\overline{f})} \{\langle \overline{J}, \overline{\lambda}\rangle\}$$

and we will show that $\overline{\mu(\overline{f}, \overline{\lambda})} = \langle \overline{I}, \overline{\lambda}\rangle$, i.e. that $\langle \overline{I}, \overline{\lambda}\rangle$ is maximal for $x^{\overline{I}} \in \mathrm{Supp}(\overline{f})$. Suppose it is not. Then, there exists $\overline{I}' = ({d_n}', {d_{n-1}}', \dots, {d_0}')$, with $x^{\overline{I}'} \in \mathrm{Supp}(\overline{f})$, such that $\langle \overline{I}, \overline{\lambda}\rangle < \langle \overline{I}', \overline{\lambda}\rangle$. But, this in turn would imply that $\langle I', \lambda\rangle < \langle I, \lambda\rangle$, where $I' = (d'_0, \dots, d'_n)$, and $x^{I'} \in \mathrm{Supp}(f)$ since $f \cong \overline{f}$. This contradicts the original assumption that $\langle I, \lambda\rangle$ is minimal. Thus, we have shown that

$$\mu(f, \lambda) = \overline{\mu(\overline{f}, \overline{\lambda})}.$$

The implication of the above is that we may reformulate Lemma 3.1 as follows:

Lemma 3.2. *With respect to a maximal torus* T*:*

1. $X = \{f = 0\}$ *is* semi-stable *if and only if* $\max_{x^I \in \mathrm{Supp}(f)}\{\langle I, \lambda\rangle\} \geq 0$*, for all non-trivial one-parameter subgroups* λ *of* T*.*

2. $X = \{f = 0\}$ *is* stable *if and only if* $\max_{x^I \in \mathrm{Supp}(f)}\{\langle I, \lambda\rangle\} > 0$*, for all non-trivial one-parameter subgroups* λ *of* T*.*

Remark 3.2.1. Throughout Sections 3.1, 3.2 and 3.3, as well as Chapters 4, 5 and 7 we will use the description of the Hilbert-Mumford numerical criterion in Lemma 3.2 (adapted for our VGIT problem of complete intersections and hyperplanes). Throughout these sections, to ease notation, we will denote the Hilbert-Mumford function by μ and not $\overline{\mu}$. In Section 3.6, in order to aide the reader, we will use the convention of Lemma 3.1 (adapted for our VGIT problem of complete intersections and hyperplanes), in keeping with the convention of the notation in [Zan22].

It is also noteworthy to point out that we would have obtained an identical Hilbert-Mumford function if we considered the conjugate action $A \cdot f(x) = f(A^{-1}x)$ for $A \in G$, as in [FS13; AL00]. This should not be a surprise, as the actions are conjugate to each other. We are choosing the natural G-action to be consistent with the moduli descriptions we are looking for.

We will proceed to analyse the GIT quotient $\mathcal{R}_{n,d,k} /\!\!/ G$. We will keep the notation as before, and we will fix a maximal torus T, with coordinates $(x_0 \colon \dots \colon x_n)$, such that T is diagonal. Recall that $I_i = (d_{i,0}, \dots, d_{i,n})$ is a monomial vector such that $x^{I_i} \in \mathrm{Supp}(f_i)$. Let λ be a normalised one-parameter subgroup. In this case, the natural pairing gives us $\langle I_i, \lambda \rangle = \sum_{j=0}^{n} d_{i,j}\mu_j$, and more specifically $\lambda(s) \cdot x^{I_i} = s^{\langle I_i, \lambda \rangle} x^{I_i}$. Similarly, the action of λ on a Plücker coordinate $\bigwedge_{i=1}^{k} x^{I_i}$ is induced as:

$$\lambda(s) \cdot \bigwedge_{i=1}^{k} x^{I_i} = s^{\sum_{i=1}^{k} \langle I_i, \lambda \rangle} \bigwedge_{i=1}^{k} x^{I_i},$$

hence the λ-action on $[\overline{S}] := \bigwedge_{i=1}^{k} f_i = \bigwedge_{i=1}^{k} \sum f_{I_i} x^{I_i}$ is:

$$\lambda(s) \cdot [\overline{S}] = \sum_{x^{I_i} \in \Xi_d} s^{\sum_{i=1}^{k} \langle I_i, \lambda \rangle} \bigwedge_{i=1}^{k} f_{I_i} x^{I_i}$$

for λ a normalised one-parameter subgroup.

From our discussion above, we have that

$$\mu(f_1 \wedge \cdots \wedge f_k, \lambda) := \max \left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle \;\middle|\; (I_1, \dots, I_k) \in (\Xi_d)^k, I_i \neq I_j \text{ if } i \neq j \text{ and } x^{I_i} \in \mathrm{Supp}(f_i) \right\}.$$

Lemma 3.3. *With respect to a maximal torus T:*

1. $[f_1 \wedge \cdots \wedge f_k] \in \mathrm{Gr}(k, W)$ *is* semi-stable *if and only if* $\mu(f_1 \wedge \cdots \wedge f_k, \lambda) \geq 0$ *for all non-trivial one-parameter subgroups* λ *of* T.

2. $[f_1 \wedge \cdots \wedge f_k] \in \mathrm{Gr}(k, W)$ *is* stable *if and only if* $\mu(f_1 \wedge \cdots \wedge f_k, \lambda) > 0$ *for all non-trivial one-parameter subgroups* λ *of* T.

Proof. From the above discussion, and Theorem 2.33, we are left to show that $\operatorname{sign}(\mu(f_1 \wedge \cdots \wedge f_k, \lambda)) = \operatorname{sign}(\mu([f_1 \wedge \cdots \wedge f_k], \lambda))$. A general element of $[f_1 \wedge \cdots \wedge f_k]$ is of the form

$$\alpha \bigwedge_{i=1}^{k} f_i,$$

and hence we have:

$$\begin{aligned} &\operatorname{sign}(\mu([f_1 \wedge \cdots \wedge f_k], \lambda)) = \\ &\operatorname{sign}\left(\max \left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle \,\middle|\, (I_1, \dots, I_k) \in (\Xi_d)^k,\ I_i \neq I_j \text{ if } i \neq j,\, x^{I_i} \in \operatorname{Supp}(f_i) \right\} \right) \\ &= \operatorname{sign}(\mu(f_1 \wedge \cdots \wedge f_k, \lambda)). \end{aligned}$$

□

Remark 3.3.1. If the choice of the f_i is clear, we will write $\mu(S, \lambda)$ instead of $\mu(f_1 \wedge \cdots \wedge f_k, \lambda)$

Remark 3.3.2. The complete intersection S induces an element of the Grassmanian $\Phi := \left\{ \sum \alpha_i f_i | (a_i : \cdots : a_k) \in \mathbb{P}^{k-1} \right\}$, which is a linear system of dimension $k-1$, whose base locus is S. As in [MM90], when Φ is (semi-)stable we will also say that S is (semi-)stable.

Extending to the VGIT case, i.e to the quotient $\mathcal{R}_m /\!\!/ G$, as in [GM18; GMZ18] we have the following Lemma.

Lemma 3.4. *The set of G-linearisable line bundles $\operatorname{Pic}^G(\mathcal{R}_m)$ is isomorphic to $\mathbb{Z}^{m+1}$. A line bundle $\mathcal{L} \in \operatorname{Pic}^G(\mathcal{R}_m)$, is ample if and only if*

$$\mathcal{L} = \mathcal{O}(a, b_1, \dots, b_m) := \mathcal{O}(a, \vec{b}) := \pi_1^* \mathcal{O}_{\mathrm{Gr}}(a) \otimes \pi_{2_1}^* \mathcal{O}_{\mathbb{P}^n}(b) \otimes \cdots \otimes \pi_{2_m}^* \mathcal{O}_{\mathbb{P}^n}(b_m)$$

where π_1 and π_{2_i} are the natural projections and $a, b_i > 0$.

Proof. The proof follows the argument the proof of [GM18, Lemma 3.2], and Lemma 2.34, noting that all the assumptions of Lemma 2.34 are satisfied. □

For ample $\mathcal{L} = \mathcal{O}(a, \vec{b})$ the GIT quotient $\mathcal{R} /\!\!/ G$ is

$$\overline{M}^{GIT}_{n,d,k,m,\vec{t}} := \overline{M}^{GIT}(\vec{t})_{n,d,k,m} := \operatorname{Proj} \bigoplus_j H^0(\mathcal{R}, \mathcal{L}^{\otimes j})^G,$$

where $\vec{t} = (t_1, \dots, t_m)$ with $t_i = \frac{b_i}{a} \in \mathbb{Q}_{>0}$. By Lemma 2.35, for $\mathcal{L}$ as above, and by the functoriality of the Hilbert-Mumford function, we have $\mu^{\mathcal{L}}((S, H_1, \dots, H_m), \lambda) = a\mu_{\vec{t}}(S, H_1, \dots, H_m, \lambda)$

where

$$\begin{aligned}\mu_{\vec{t}}(S, H_1, \ldots, H_m, \lambda) &:= \mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, \lambda) \\ &:= \mu(f_1 \wedge \cdots \wedge f_k, \lambda) + \sum_{p=1}^{m} t_p \mu(H_p, \lambda) \\ &= \max \left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle \middle| (I_1, \ldots, I_k) \in (\Xi_d)^k, I_i \neq I_j \text{ if } i \neq j, x^{I_i} \in \operatorname{Supp}(f_i) \right\} \\ &+ \sum_{p=1}^{m} t_p \max\{ r_j | h_{j,p} \neq 0, h_{j,p} \in \operatorname{Supp}(h_p) \}.\end{aligned}$$

Definition 3.5. Let $\vec{t}$ be such that all $t_i \in \mathbb{Q}_{>0}$. The tuple $(f_1, \ldots, f_k, h_1, \ldots, h_m)$ is *$\vec{t}$-stable* (resp. *$\vec{t}$-semistable*) if $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, \lambda) > 0$ (respectively, $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, \lambda) \geq 0$) for all non-trivial normalised one-parameter subgroups λ of G. A tuple $(f_1, \ldots, f_k, h_1, \ldots, h_m)$ is *$\vec{t}$-unstable* if it is not $\vec{t}$-semistable. A tuple $(f_1, \ldots, f_k, h_1, \ldots, h_m)$ is *strictly $\vec{t}$-semistable* if it is $\vec{t}$-semistable but not $\vec{t}$-stable.

Remark 3.5.1. We will often write $(S, H_1, \ldots, H_m)$ instead of the tuple $(f_1, \ldots, f_k, h_1, \ldots, h_m)$ for ease of notation.

3.2 Stability Conditions

As before, we fix a maximal torus $T \subset G$ and coordinates such that T is diagonal. We define a partial order on Ξ_d (following Mukai [Muk03, §7]): given $v, v' \in \Xi_d$ we have

$$v \leq v' \quad \text{if and only if} \quad \langle v, \lambda \rangle \leq \langle v', \lambda \rangle$$

for all normalised one-parameter subgroups $\lambda \colon \mathbb{G}_m \to T \subset G$. It is also useful to note there exists a lexicographic order $<_{lex}$, e.g. $x_0 x_1 <_{lex} x_0^2$ in Ξ_d, which is a total order [Har05, Theorem 1.8], i.e. it is antisymmetric, transitive, and equipped with a connex relation, where if $x \neq y$ we have either $x <_{lex} y$ or $y <_{lex} x$. This gives rise to the order $\leq_\lambda$ for a normalised one-parameter subgroup λ which we call the λ-order: $v \leq_\lambda v'$ if and only if:

1. $\langle v, \lambda \rangle < \langle v', \lambda \rangle$ or
2. $\langle v, \lambda \rangle = \langle v', \lambda \rangle$ and $v \leq_{lex} v'$.

Lemma 3.6. *The λ-order is a total order .*

Proof. This follows, since the lexicographic order is a total order, [Har05, Theorem 1.8]. □

The λ-order is necessary in introducing an order on Ξ_d. This is particularly beneficial as, as in [Muk03, §7], we will later define maximal unstable sets that contain elements of Ξ_d. The maximality of these sets is guaranteed by the λ-order, and these sets will allow us to computationally find the monomials in Ξ_d necessary to define unstable elements.

As in Gallardo–Martinez- Garcia [GM18, Section 5] we define a fundamental set of one-parameter subgroups $\lambda \in T$.

Definition 3.7. The *fundamental set of one-parameter subgroups* $\lambda \in T$ for k hypersurfaces of degree d and m hyperplanes, $P_{n,k,d,m}$ consists of all non-trivial elements $\lambda(s) = \mathrm{Diag}(s^{\mu_0}, \dots, s^{\mu_n})$ where

$$(\mu_0, \dots, \mu_n) = c(\gamma_0, \dots, \gamma_n) \in \mathbb{Z}^{n+1}$$

satisfying the following:

1. $\gamma_i = \frac{a_i}{b_i} \in \mathbb{Q}$ such that $\gcd(a_i, b_i) = 1$ for all $i = 0, \dots n$ and $c = \mathrm{lcm}(b_0, \dots, b_n)$,

2. $1 = \gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_n = -1 - \sum_{i=1}^{n-1} \gamma_i$,

3. $(\gamma_0, \dots, \gamma_n)$ is the unique solution of a consistent linear system given by $n-1$ equations chosen from the union of the following sets:

$$\mathrm{Eq}(n-1, k, d) := \{\gamma_i - \gamma_{i+1} = 0\} \cup$$

$$\Big\{ \sum_{j=0}^{n} \big[\sum_{i=1}^{k} d_{i,j} - \sum_{i=1}^{k} \overline{d}_{i,j} \big] \gamma_j = 0 \mid d_{i,j},\ \overline{d}_{i,j} \in \mathbb{Z}_{\geq 0}, \forall i, j, \sum_{j=0}^{n} d_{i,j} = \sum_{j=0}^{n} \overline{d}_{i,j} = d \Big\}.$$

Since there is a finite number of monomials in Ξ_d and equations in $\mathrm{Eq}(n-1, k, d)$, the set $P_{n,k,d,m}$ is finite.

Lemma 3.8. *A tuple $(S, H_1, \dots, H_m)$ is not $\vec{t}$-stable (respectively not $\vec{t}$-semistable) if and only if there is $g \in G$ such that*

$$\mu_{\vec{t}}(S, H_1, \dots, H_m) := \max_{\lambda \in P_{n,d,k,m}} \{\mu_t(g \cdot S, g \cdot H_1, \dots, g \cdot H_m, \lambda)\} \leq 0 \quad (\textit{respectively} < 0)$$

where $\mu_{\vec{t}}$ is the Hilbert-Mumford function defined above and $P_{n,d,k,m}$ is the fundamental set of Definition 3.7.

Proof. The proof follows the argument in the proof of [GM18, Lemma 3.2]. Let $R^n_{m,T}$ be the non-stable loci of $\mathcal{R}_m$ with respect to a maximal torus $T < G$; and let $\mathcal{R}^n_m$ be the non-stable loci of $\mathcal{R}_m$. By Mumford [MFK94, p.137], the following holds

$$\mathcal{R}^n_m = \bigcup_{T_i \subset G,\text{ maximal}} R^n_{m,T_i}.$$

Let $(f'_1, \ldots, f'_k, h_1, \ldots, h_m)$ be a tuple which is not $\vec{t}$-stable. Then, $\mu_{\vec{t}}(f'_1 \wedge \cdots \wedge f'_k, h'_1, \ldots, h'_m, \rho) \leq 0$ for some $\rho \in T'$ in a maximal torus T'. All the maximal tori are conjugate to each other in G, and by [MFK94, Definition 2.2] the following holds:

$$\mu_{\vec{t}}(f'_1 \wedge \cdots \wedge f'_k, h'_1, \ldots, h'_m, \rho) = \mu_{\vec{t}}(g \cdot (f'_1 \wedge \cdots \wedge f'_k, h'_1, \ldots, h'_m), g\rho g^{-1}).$$

Then, there is $g_0 \in G$ such that $\lambda := g_0 \rho {g_0}^{-1}$ is normalised with respect to a torus T whose generators define the variables for the monomials of f_i and h_j and $(f_1, \ldots, f_k, h_1, \ldots, h_m) := g_0 \cdot (f'_1, \ldots, f'_k, h'_1, \ldots, h'_m)$ has coordinates in the coordinate system which diagonalises λ such that $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, \lambda) \leq 0$. In this coordinate system normalised one-parameter subgroups $\lambda = \mathrm{Diag}(s^{\mu_0}, \ldots, s^{\mu_n})$, with fixed $\mu_0 > 0$ form a closed convex polyhedral subset Δ of dimension n in $N \otimes \mathbb{Q} \cong \mathbb{Q}^{n+1}$ (in fact Δ is a standard simplex). Indeed, this is the case since for any normalised one-parameter subgroup, $\lambda(s) = \mathrm{Diag}(s^{\mu_0}, \ldots, s^{\mu_n})$ where $\mu_0 \geq \cdots \geq \mu_n$ with $\sum \mu_i = 0$ and we may assume without loss of generality that $\mu_0 = 1$.

For any fixed $f_1, \ldots, f_k, h_1, \ldots, h_m$, the function $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, -)\colon N \otimes \mathbb{Q} \to \mathbb{Q}$ is piecewise linear, as for a fixed maximal torus T and normalised one-parameter subgroup λ the function is the maximum of a finite number of linear forms. The critical points of $\mu_{\vec{t}}$ (i.e. the points where $\mu_{\vec{t}}$ fails to be linear) correspond to those points in $N \otimes \mathbb{Q}$ where $\sum_{i=1}^k \langle I_i, \lambda \rangle + \sum_{p=1}^m t_p \langle x_{l_p}, \lambda \rangle = \sum_{i=1}^k \langle \overline{I}_i, \lambda \rangle + \sum_{p=1}^m t_p \langle \overline{x}_{l_p}, \lambda \rangle$, for $I_i = (d_{i,0}, \ldots, d_{i,n})$, $\overline{I}_i = (\overline{d}_{i,0}, \ldots, \overline{d}_{i,n})$ representing monomials of degree d, with $I_i \neq I_j$ for all $i \neq j$, where $f_i = \sum f_{I_i} x^{I_i}$, and x_{l_p} are monomials of degree 1, such that $x_{l_p} := \max \mathrm{Supp}(h_p)$. Notice that x_{l_p} is unique for each of the $H_p = \{h_p = 0\}$, hence $\sum_{p=1}^m t_p \langle x_{l_p}, \lambda \rangle = \sum_{p=1}^m t_p \langle \overline{x}_{l_p}, \lambda \rangle$ implies that $x_{l_p} = \overline{x}_{l_p}$ for each p. Hence the $\sum_{i=1}^m t_i x_{j_i}$ component is always linear, and as such the critical points of $\mu_{\vec{t}}$ correspond to those points in $N \otimes \mathbb{Q}$ where $\sum_{i=1}^k \langle I_i, \lambda \rangle = \sum_{i=1}^k \langle \overline{I}_i, \lambda \rangle$. Since $\langle -, - \rangle$ is bilinear, that is equivalent to say that $\left\langle \sum_{i=1}^k \left(I_i - \overline{I}_i \right), \lambda \right\rangle = 0$. These points define a hyperplane in $N \otimes \mathbb{Q}$ and the intersection of this hyperplane with Δ is a simplex $\Delta_{(I_i, \overline{I}_i)}$ of dimension $n-1$.

The function $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, -)$ is linear on the complement of the union of hyperplanes defined by $\left\langle \sum_{i=1}^k \left(I_i - \overline{I}_i \right), \lambda \right\rangle = 0$. Hence, its maximum is achieved on the boundary, i.e. either on $\partial\Delta$ or on $\Delta_{(I_i,\bar{I}_i)}$ which are all convex polytopes of dimension $n-1$. We can repeat this reasoning by finite inverse induction on the dimension until we conclude that the maximum of $\mu_{\vec{t}}(f_1 \wedge \cdots \wedge f_k, h_1, \ldots, h_m, -)$ is achieved at one of the vertices of Δ or $\Delta_{(I_i,\bar{I}_i)}$. Notice that these correspond precisely, up to multiplication by a constant, to the finite set of one-parameter subgroups in $P_{n,k,d,m}$. $\square$

Corollary 3.8.1. *Let $(S, H) \in \mathcal{R}$ and*

$$a = \min\{t \in \mathbb{Q}_{>0} | \mu_t(g \cdot S, g \cdot H, \lambda) \geq 0,\ \forall \lambda \in P_{n,d,k,1}, g \in G\},$$
$$b = \max\{t \in \mathbb{Q}_{>0} | \mu_t(g \cdot S, g \cdot H, \lambda) \geq 0,\ \forall \lambda \in P_{n,d,k,1}, g \in G\}.$$

If (S, H) is t-semistable for some t, then

1. *(S, H) is t-semistable if and only if $t \in [a, b] \cap \mathbb{Q}_{>0}$,*

2. *if (S, H) is t-stable for some $t \in (a, b)$, then (S, H) is t-stable for all $t \in (a, b) \cap \mathbb{Q}_{>0}$.*

The interval $[a, b]$ is called the interval of stability of the pair. If $[a, b] = \emptyset$ then the pair is t-unstable for all t.

Proof. The proof follows the argument of the proof of [GM18, Corollary 3.3].

$\square$

We can also extend the above for the case of tuples.

Definition 3.9 (see [GMZ18, Definition 2.4])**.** The *space of stability conditions* is

$$\mathrm{Stab}(n, d, k, m) := \left\{\vec{t} \in (\mathbb{Q}_{\geq 0})^m | \text{ there exists a } \vec{t}\text{-semistable tuple } (S, H_1, \ldots, H_m)\right\}.$$

Lemma 3.8 determines the $\vec{t}$-stability for all tuples $(S, H_1, \ldots, H_m)$ as the fundamental set does not depend on t. In more detail, in checking the stability conditions for such tuples, Lemma 3.8 shows that we may only consider a finite set of one-parameter subgroups $P_{n,d,k,m}$, to check which tuples are not $\vec{t}$-semistable, In particular, this Lemma allows us to determine the space of stability conditions, which is compact by [DH98]. This verifies the prediction in [KW06] that there should be a finite set of one-parameter subgroups that determines the stability of tuples.

3.3 The Centroid Criterion

We will state a *centroid criterion*, which will allow us to computationally deduce whether a point is stable or strictly semistable. This is known to be always possible by [DH98, Theorem 9.2], and is a well-known direct application of the Hilbert-Mumford numerical criterion. The contribution of this thesis in this application, is the very explicit description of the proof in the notation we have introduced for Section 3.1.

We define a map $A : (\Xi_d)^k \times (\Xi_1)^m \to \mathbb{Q}^{n+1}$ as follows: Given distinct monomials $x^{I_1}, \ldots, x^{I_k} \in \Xi_d$, and distinct monomials $x_{l_1}, \ldots, x_{l_m} \in \Xi_1$, let

$$A\Big((x^{I_1}, \ldots, x^{I_k}), x_{l_1}, \ldots, x_{l_m}\Big) = \Big(\sum_{i=0}^{k} d_{i,0}, \ldots, \sum_{i=0}^{k} d_{i,l_1} + t_1, \ldots, \sum_{i=0}^{k} d_{i,l_m} + t_m, \ldots, \sum_{i=0}^{k} d_{i,n}\Big)$$

where we add the t_p at the position of monomial x_{l_p}, i.e. at position l_p, e.g. if we have monomials $x_{l_1} = x_0$, $x_{l_2} = x_3$ and $m = 2$ we have

$$A\Big((x^{I_1}, \ldots, x^{I_k}), x_0, x_3\Big) = \left(\sum_{i=0}^{k} d_{i,0} + t_1, \sum_{i=0}^{k} d_{i,1}, \sum_{i=0}^{k} d_{i,2}, \sum_{i=0}^{k} d_{i,3} + t_2, \sum_{i=0}^{k} d_{i,4}, \ldots, \sum_{i=0}^{k} d_{i,n}\right).$$

Since:

$$\sum_{j=0}^{n} \big(\sum_{i=1}^{k} (d_{i,j})\big) + \sum_{p=1}^{m} t_p = kd + \sum_{p=1}^{m} t_p$$

the image of A is contained on the first quadrant of the hyperplane

$$H_{n,k,d,m,\vec{t}} := \left\{(z_0, \ldots, z_n) \in \mathbb{Q}^{n+1} \middle| \sum_{i=0}^{n} z_i = kd + \sum_{p=1}^{m} t_p\right\}.$$

Then, for S, a complete intersection of k hypersurfaces of degree d, $S = \{f_1 = \cdots = f_k = 0\}$, as before and $H_1, \ldots, H_m$ distinct hyperplanes embedded in $\mathbb{P}^n$, where $H_i = \{h_i = 0\}$, we define their convex hull $\mathrm{Conv}(S, H_1, \ldots, H_m)$ as the convex hull of

$$\left\{A((x^{I_1}, \ldots, x^{I_k}), x_{l_1}, \ldots, x_{l_m}) \middle| \, x^{I_i} \in \mathrm{Supp}(f_i) \text{ for all } i, \ I_i \neq I_j, x_{l_p} = \max(\mathrm{Supp}(h_p))\right\},$$

where $\max$ is given with respect to the λ order and is unique since the λ-order is a total order. Then $\overline{\mathrm{Conv}(S, H_1, \ldots, H_m)}$ is a convex polytope in $H_{n,k,d,m,\vec{t}}$. Note that in particular, for $A\big((f_1, \ldots, f_k), h_1, \ldots, h_m\big) := A\Big(\big(\sum f_{I_1} x^{I_1}, \ldots, \sum f_{I_k} x^{I_k}\big), x_{l_1}, \ldots, x_{l_m}\Big)$ we have

$$A\big((f_1, \ldots, f_k), h_1, \ldots, h_m\big) \in \overline{\mathrm{Conv}(S, H_1, \ldots, H_m)}.$$

We also define the *$\vec{t}$-centroid* of $S, H_1, \ldots, H_m$ as

$$\mathcal{C}_{n,d,k,m,\vec{t}} = \left(\frac{kd + \sum_{p=1}^m t_p}{n+1}, \ldots, \frac{kd + \sum_{p=1}^m t_p}{n+1} \right) \in H_{n,k,d,m,\vec{t}} \subset \mathbb{Q}^{n+1}.$$

Theorem 3.10. *A tuple $(S, H_1, \ldots, H_m)$ is $\vec{t}$-semistable (respectively, $\vec{t}$-stable) if and only if $\mathcal{C}_{n,d,k,m,\vec{t}} \in \overline{\mathrm{Conv}(S, H_1, \ldots, H_m)}$ (respectively, $\mathcal{C}_{n,d,k,m,\vec{t}} \in \mathrm{Int}(S, H_1, \ldots, H_m)$, where $\mathrm{Int}(S, H_1, \ldots, H_m)$ is the interior of $\overline{\mathrm{Conv}(S, H_1, \ldots, H_m)}$).*

Proof. The proof follows the argument of the proof of [GM18, Lemma 1.5]. Let $x_{l_p} = \max \mathrm{Supp}(h_p)$, for $p = 1, \ldots, m$. First, note that since

$$\begin{aligned} \mu_{\vec{t}}(S, H_1, \ldots, H_m, \lambda) &= \mu(S, \lambda) + \sum_{p=1}^m t_p \max_{x_z \in \mathrm{Supp}(H_p)} \langle x_z, \lambda \rangle \\ &= \mu(S, \lambda) + \sum_{p=1}^m t_p \langle x_{l_p}, \lambda \rangle \\ &= \mu_{\vec{t}}(S, S \cap \{x_{l_1} = 0\} \cap \cdots \cap \{x_{l_m} = 0\}, \lambda). \end{aligned}$$

We first show that if $(S, H_1, \ldots, H_m) = ((f_1, \ldots, f_k), H_1, \ldots, H_m)$ is $\vec{t}$-semistable then the condition holds. Suppose $\mathcal{C}_{n,d,k,\vec{t}} \notin \overline{\mathrm{Conv}(S, H_1, \ldots, H_m)}$, then, there exists an affine map $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\psi(\mathcal{C}_{n,d,k,\vec{t}}) = 0$ and $\psi|_{\overline{\mathrm{Conv}(S,H_1,\ldots,H_m)}} > 0$. We can write

$$\psi(z_0, \ldots, z_n) = \sum_{i=0}^n \alpha_j z_j + q$$

where α_i are integers, since the convex hull has vertices with rational coefficients. Thus, for $x^{I_i} \in \mathrm{Supp}(f_i)$:

$$\psi(A(x^{I_1}, \ldots, x^{I_k}, x_{l_1}, \ldots, x_{l_m})) = \sum_{j=0}^n \alpha_j \Big(\sum_{i=1}^k d_{i,j} \Big) + \sum_{p=1}^m t_p \alpha_{l_p} + q > 0.$$

$$0 = \psi(\mathcal{C}_{n,d,k,\vec{t}}) = \frac{kd + \sum_{p=1}^m t_p}{n+1} \sum_{i=0}^n \alpha_i + q.$$

Let $\delta := -\frac{q}{kd + \sum_{p=1}^m t_p} \in \mathbb{Q}$. Then $(n+1)\delta = \sum_{i=0}^n \alpha_i$, and we can choose some $r \in \mathbb{Z}_{\leq 0}$ such that $r\delta \in \mathbb{Z}$. Then we can define one-parameter subgroup:

$$\lambda(s) := \mathrm{Diag}(s^{r(\alpha_0 - \delta)}, \ldots, s^{r(\alpha_n - \delta)}).$$

Using this one-parameter subgroup we have:

$$\begin{aligned}
\mu_{\vec{t}}(S, H_1, \ldots, H_m, \lambda) &= \max_{x^{I_i} \in \operatorname{Supp}(f_i)} \left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle | \text{ for all } i, I_i \neq I_j \text{ for all } i, j \right\} + \sum_{p=1}^{m} t_p \langle x_{l_p}, \lambda \rangle \\
&= \max_{x^{I_i} \in \operatorname{Supp}(f_i)} \left\{ \sum_{j=0}^{n} r(\alpha_j - \delta) \sum_{i=1}^{k} d_{i,j} \right\} + \sum_{p=1}^{m} t_p r(\alpha_{l_p} - \delta) \\
&= r \left(\max_{x^{I_i} \in \operatorname{Supp}(f_i)} \left\{ \sum_{j=0}^{n} \alpha_j \sum_{i=1}^{k} d_{i,j} + \sum_{p=1}^{m} t_p \alpha_{l_p} \right\} - (kd + \sum_{p=1}^{m} t_p)\delta \right) \\
&= r \max_{x^{I_i} \in \operatorname{Supp}(f_i)} \left\{ \psi(A((x^{I_1}, \ldots, x^{I_k}), x_{l_1}, \ldots, x_{l_m}) \right\} \\
&\leq r \max_{v \in \overline{\operatorname{Conv}(S, H_1, \ldots, H_m)}} \psi(v) < 0
\end{aligned}$$

hence $(S, H_1, \ldots, H_m)$ is not $\vec{t}$-semistable. This shows that if $(S, H_1, \ldots, H_m)$ is $\vec{t}$-semistable then $\mathcal{C}_{n,d,k,m,\vec{t}} \in \overline{\operatorname{Conv}(S, H_1, \ldots, H_m)}$.

Conversely, let $(S, H_1, \ldots, H_m)$ be not $\vec{t}$-semistable. As such, there exists a normalised one-parameter subgroup $\lambda(s) := \operatorname{Diag}(s^{\alpha_0}, \ldots, s^{\alpha_n})$ with $\sum_{j=0}^{n} \alpha_j = 0$ and

$$0 > \mu_{\vec{t}}(S, H_1, \ldots, H_m, \lambda) = \max_{I_i \in \operatorname{Supp}(f_i)} \left\{ \sum_{j=0}^{n} \alpha_i \sum_{i=1}^{k} d_{i,j} \right\} + \sum_{p=1}^{m} t_p a_{l_p}.$$

We define the affine transformation $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ as $\psi(z_0, \ldots, z_n) = \sum_{i=0}^{n} \alpha_i z_i$, such that $\psi|_{\overline{\operatorname{Conv}(S, H_1, \ldots, H_m)}} < 0$ (by convexity), and

$$\psi(\mathcal{C}_{n,d,k,m,\vec{t}}) = \frac{kd + \sum_{p=1}^{m} t_p}{n+1} \sum_{j=0}^{n} \alpha_j = 0.$$

Hence $\mathcal{C}_{n,d,k,m,\vec{t}} \notin \overline{\operatorname{Conv}(S, H_1, \ldots, H_m)}$.

A similar argument shows the result for stable orbits, where the $>$ and $<$ are replaced accordingly by $\geq$ and $\leq$ respectively.

$\square$

We can also find the dimension of the moduli space.

Theorem 3.11. *Assume that the ground field is algebraically closed with characteristic* 0 *and that the locus of stable points is not empty and* $d > 2$*. Then*

$$\dim \overline{M}^{GIT}_{n,d,k,m}(\vec{t}) = k(n+1) \left(\frac{(n+2) \ldots (n+d)}{d!} - (n+1) \right) - n(n+m-2) - k^2.$$

Proof. Let $p = (S, H_1, \dots, H_m) \in \mathcal{R}_m$ be a tuple. Then, we have

$$0 \leq \dim(G_p) \leq \dim(G_S \cap G_{H_1} \cap \cdots \cap G_{H_m}) \leq \dim(G_S) \leq \dim(\mathrm{Aut}(S)) = 0,$$

where the last inequality follows from [JL17, Lemma 2.13]. We obtain the result using the following identity from [Dol03, Corollary 6.2]:

$$\begin{aligned}\dim \overline{M}^{GIT}_{n,d,k,\vec{t}} &= \dim(\mathcal{R}_m) - \dim(G) + \min_{S\in\mathcal{R}_m} G_S = \\ &= k\left(\binom{n+d}{d} - k\right) + mn - \left((n+1)^2 - 1\right) \\ &= k(n+1)\left(\frac{(n+2)\dots(n+d)}{d!} - (n+1)\right) - n(n+m-2) - k^2.\end{aligned}$$

□

For the rest of this section, we will restrict ourselves to the case $m = 1$. The following Lemma is a generalization of [GM18, Lemma 4.1].

Lemma 3.12. *Let $(S, H) \in \mathcal{R}$, with non-empty interval of stability $[a, b]$. Then*

1. *$a = 0$ if and only if S is a GIT semistable variety which is the complete intersection of k degree d hypersurfaces.*
2. *$b \leq t_{n,d,k} = \frac{kd}{n}$.*
3. *(S, H) is $t_{n,k,d}$ semistable if and only if $D = S \cap H$ is a GIT semistable complete intersection of k hypersurfaces of degree d in $H \cong \mathbb{P}^{n-1}$.*

Proof. For 1, notice that if $t = \frac{a}{b} = 0$ if and only if $a = 0$, then $\mu_t(S, H, \lambda) = \mu(S, \lambda) \geq 0$ and thus this reduces to a GIT problem for complete intersections of k degree d hypersurfaces, since the natural projection $\mathcal{R} \to \mathrm{Gr}(k, W)$ is G-invariant.

For 2, assume that $t > t_{n,d,k}$. Without loss of generality, we may assume that $S = f_1(x_0, \dots, x_n) \cap \cdots \cap f_k(x_0, \dots, x_n)$ and that $H = \{x_n = 0\}$. Then, for the normalised one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s, s \dots, s, s^{-n})$ we have

$$\mu_t(S, H, \lambda) \leq kd - tn < kd - \frac{kdn}{n} = 0,$$

hence (S, H) is unstable, and thus $b \leq t_{n,d,k}$.

For 3, first assume that $D = S \cap H$ is unstable, where we can find a coordinate system $(x_0 \colon \dots \colon x_n)$ such that $H = \{x_n = 0\}$, $D = \{f_1'(x_0, \dots, x_{n-1}) = \cdots = f_k'(x_0, \dots, x_{n-1})\}$. Notice that under the λ-order, the monomial $x_0^{d-1}x_n$ is the maximal monomial of degree d containing x_n as for any normalised one-parameter subgroup λ, where $\lambda(t) = \mathrm{Diag}(t^{\lambda_0}, \dots, t^{\lambda_n})$, we have:

$$\begin{aligned}
\langle x_0^{a_0} \dots x_n^{a_n}, \lambda \rangle &= \sum_{i=0}^{n-1} \lambda_i a_i + \lambda_n a_n \\
&\leq \lambda_0 \sum_{i=0}^{n-1} a_i + \lambda_n a_n \quad (\text{since } \lambda_i \geq \lambda_j \text{ for } i \leq j) \\
&\leq \lambda_0(d-1) + \lambda_n a_n \quad (\text{since } \sum_{i=1}^{n} a_i = d) \\
&\leq \lambda_0(d-1) + \lambda_n \\
&= \langle x_0^{d-1} x_n, \lambda \rangle.
\end{aligned}$$

Note that we do not have to consider the case $a_n = 0$, as then the monomial will not contain x_n. Hence, $\overline{S} := \{f_1'(x_0, \dots, x_{n-1}) + x_0^{d-1}x_n = \dots = f_k'(x_0, \dots, x_{n-1}) + x_0^{d-1}x_n\}$ is the complete intersection of k hypersurfaces of degree d which is maximal with respect to the Hilbert-Mumford function such that $\overline{S} \cap H = D$, i.e. $\mu(\overline{S}, \lambda) = \max\{\mu(D, \lambda) | S \cap H = D\}$. This further implies that if $\mu_{t_{n,d,k}}(\overline{S}, H, \lambda) < 0$ then for all pairs (S, H) such that $S \cap H = D$, the pair is $t_{n,k,d}$-unstable. Returning to the original assumption that D is unstable, we have by the centroid criterion (Theorem 3.10), that

$$\left\{\frac{kd}{n}, \dots, \frac{kd}{n}, 0\right\} \notin \mathrm{Conv}(f_1'(x_0, \dots, x_{n-1}) + x_0^{d-1}x_n, \dots, f_k'(x_0, \dots, x_{n-1}) + x_0^{d-1}x_n).$$

Notice that,

$$A := \{A((x^{I_1}, \dots, x^{I_k}), x_n) | x^{I_i} \in \mathrm{Supp}(f_i' + x_0^{d-1}x_n) \text{ for all } i, I_i \neq I_j\} \subset M := \{y_n = t_{n,d,k}\}$$

and $p = \left\{kd - 1, 0 \dots, 0, 1 + t_{n,d,k}\right\} \notin M$. Then, since $\overline{\mathrm{Conv}(S, H)} = \mathrm{ConvexHull}(A)$, it is a pyramid with base M and vertex p. Thus $\mathcal{C}_{n,d,k,t_{n,d,k}} \notin \overline{\mathrm{Conv}(S, H)}$ and thus (S, H) is $t_{n,k,d}$-unstable by the centroid criterion (Theorem 3.10).

Now, suppose (S, H) is $t_{n,k,d}$-unstable, i.e. $\mathcal{C}_{n,d,k,\frac{kd}{n}} \notin \mathrm{Conv}_{t_{n,d,k}}(S, H)$. Without loss of generality, we can assume that $H = \{x_i = 0\}$, and let

$$p = \left\{\frac{kd}{n}, \dots, \frac{kd}{n}, 0, \frac{kd}{n}, \dots, \frac{kd}{n}\right\}$$

with 0 in the i-th position. In particular $p \notin \overline{\mathrm{Conv}(f_1, \dots, f_k, x_i)}$. The monomials of $D = S \cap H$ are monomials of the form $I_j = (d_{0,j}, \dots, d_{i-1,j}, 0, d_{i+1,j}, \dots, d_{n,j})$, which correspond to faces

$F \in \overline{\mathrm{Conv}_{t_{n,d,k}}(S,H)}$. The projection F_M of each face F to hyperplane $M = \{y_i = 0\}$ shows $p \notin F_M$. Notice that $p = \mathcal{C}_{n,d,k}$ and $F_M \subseteq \overline{\mathrm{Conv}(f_1,\dots,f_k,h)}$, and hence D is unstable by the centroid criterion (Theorem 3.10).

□

The following theorem generalises [GM18, Theorem 1.1].

Theorem 3.13. *All GIT walls $\{t_0,\dots,t_{n,d,k}\}$ correspond to a subset of the finite set*

$$\left\{-\frac{\sum_{i=1}^k \langle I_i,\lambda\rangle}{\langle x_i,\lambda\rangle}\middle| I_i \neq I_i,\text{ for all } i\neq j,\ I_i\in\Xi_d,\ 0\le i\le n,\ \lambda\in P_{n,d,k}\right\}$$

and they are contained in the interval $[0,t_{n,d,k}]$. Every pair (S,H) has an interval of stability $[a,b]$ with $a,b\in\{t_0,\dots,t_{n,d,k}\}$. (S,H) is t-semistable if and only if $t\in[t_i,t_j]$ for some walls t_i,t_j . If (S,H) is t-stable for some t then (S,H) is t-stable if and only if $t\in(t_i,t_j)$.

Proof. Since Ξ_d is finite for each d, $\mathcal{P}(\Xi_d)^k\times\mathcal{P}(\Xi_1)$, where $\mathcal{P}$ here denotes the power set, is finite, as by the Hilbert-Mumford numerical criterion stability only depends on the support of the polynomials involved and the combination of possible supports is thus finite. In addition, by Corollary 3.8.1 there is a finite number of intervals of stability $[a_i,b_i]$. Hence, $t_j\in\cup_i\{a_i,b_i\}$ with $b_i\le t_{n,d,k}$ by Lemma 3.12. For any wall t_i there is at least a pair (S,H) such that

$$\mu_t(S,H) := \max_{\lambda\in P_{n,d,k}}\{\mu_t(S,H,\lambda)\}\ge 0$$

for all $t\le t_i$ and $\mu_t(S,H)<0$ for all $t>t_i$. By the continuity of $\mu_t(S,H)$, $\mu_{t_i}(S,H)=0$ and hence

$$t_i = -\frac{\sum_{i=1}^k\langle I_i,\lambda\rangle}{\langle x_i,\lambda\rangle}.$$

□

Remark 3.13.1. In the general case, one can find a superset of the stability walls $\vec{t}$ by solving the simultaneous equations

$$\sum_{i=1}^k\langle I_i,\lambda\rangle+\sum_{j=1}^m t_j\langle x_{i_j},\lambda\rangle = 0$$

for $I_i\in\Xi_d$, $I_i\neq I_k$ for all $i\neq k$, $x_{i_j}\in\Xi_1$ and $\lambda\in P_{n,d,k,m}$. The complexity of computations increases as m increases, and in addition the walls are not 0-dimensional, which implies that we need to treat them in a stratified way. In addition, we don't have a simple one-directional way of exploring the set of stability conditions to find all walls.

Theorem 3.14. *Every point in the GIT quotient $\overline{M}^{GIT}_{n,d,k,t} := \overline{M}^{GIT}_{n,d,k}(t)$ parametrises a closed orbit associated to a pair (S, D) with $D := S \cap H$ in the cases where S is a Calabi-Yau or a Fano complete intersection of k hypersurfaces of degree $d > 1$. Furthermore, if S is Fano, $t \leq t_{n,d,k}$ and (S, D) is t-semistable, then S does not contain a hyperplane in the support of at least one of the hypersurfaces in the complete intersection.*

Proof. Suppose that $n + 1 \geq kd$ (i.e. S is Fano or Calabi-Yau) and (S, H) is a pair such that $\mathrm{Supp}(H) \subset \mathrm{Supp}(f_i)$ for some i. Then, it suffices to show (S, H) is t-unstable for all $t \geq 0$. Without loss of generality, we take $H = \{x_n = 0\}$ and $S = \{x_n f_1(x_0, \dots, x_n) = 0\} \cap \{f_2(x_0, \dots, x_n) = 0\} \cdots \cap \{f_k(x_0, \dots, x_n) = 0\}$. Then for the λ-order, the monomial $x_0^{d-1}x_n$ is maximal in $\mathrm{Supp}(f_1) \cup \{x_0^{d-1}x_n\}$ and the monomial x_0^d is maximal in $\mathrm{Supp}(f_i) \cup \{x_0^d\}$ for all $i > 1$. For any normalised one-parameter subgroup $\lambda = \mathrm{Diag}(s^{r_0} \dots, s^{r_n})$ we have:

$$\mu_t(S, H, \lambda) \leq (k-1)dr_0 + (d-1)r_0 + r_n + tr_n.$$

Specifically, for the normalised 1-ps $\lambda = \mathrm{Diag}(s^{\boldsymbol{r}})$ with $\boldsymbol{r} = (n, -\frac{nkd}{n-1}, \dots, -\frac{nkd}{n-1}, -n(kd-1))$ we have

$$\mu_t(S, H, \lambda) \leq (k-1)dn + (d-1)n - n(kd-1) - tn(kd-1) = -tn(kd-1) < 0,$$

and hence (S, H) is unstable for all $t > 0$.

Let S be a Fano complete intersection, i.e. $kd \leq n$ and we assume that $S = \{x_n f_1^{d-1}(x_0, \dots, x_n) = 0\} \cap \dots \cap \{x_n f_k(x_0, \dots, x_n) = 0\}$ and $H \neq \{x_n = 0\}$ by the previous step. Further, assume that $\mathrm{Supp}(f_1)$ contains a hyperplane in its support.

Then for the normalised one-parameter subgroup $\lambda = \mathrm{Diag}(s, s, \dots, s, s^{-n})$ we have, noting that $t \leq t_{n,d,k} \leq 1$, since $kd \leq n$:

$$\begin{aligned}
\mu_t(S, D, \lambda) &\leq k((d-1) - n) + t \\
&= kd - k - kn + t \\
&\leq n - k - kn + 1 \\
&= (n+1)(1-k) \\
&< 0.
\end{aligned}$$

Hence, the pair (S, D) is t-unstable, so S cannot contain a a hyperplane in the support of at least one of the hypersurfaces in the complete intersection. □

3.4 Semi-destabilizing Families

Definition 3.15. We fix $\vec{t} \in \mathrm{Stab}(n, d, k, m)$ and let λ be a normalised one-parameter subgroup. A non-empty $k+m$-tuple of sets $A_1 \times \cdots \times A_k \times B_1 \times \cdots \times B_m \subseteq (\Xi_d)^k \times (\Xi_1)^m$ is *maximal $\vec{t}$-(semi-) destabilised with respect to* λ, if:

1. Each $k+m$-tuple $(v_1, \ldots, v_k, a_1, \ldots, a_m) \in A_1 \times \cdots \times A_k \times B_1 \times \cdots \times B_m$ satisfies $\sum_{i=1}^k \langle v_i, \lambda\rangle + \sum_{j=1}^m t_j \langle a_j, \lambda\rangle < 0$ (≤ 0, respectively).
2. If there is another $k+m$-tuple of sets $\overline{A}_1 \times \cdots \times \overline{A}_k \times \overline{B}_1 \times \cdots \times \overline{B}_m \subseteq (\Xi_d)^k \times (\Xi_1)^m$ such that $A_i \subseteq \overline{A}_i$, $B_i \subseteq \overline{B}_i$ for all i, and for all $(v_1, \ldots, v_k, a_1, \ldots, a_m) \in \bar{A}_1 \times \cdots \times \overline{A}_k \times \overline{B}_1 \times \cdots \times \overline{B}_m$ the inequality $\sum_{i=1}^k \langle v_i, \lambda\rangle + \sum_{j=1}^m t_j \langle a_j, \lambda\rangle < 0$ (≤ 0, respectively) holds, then $A_i = \overline{A}_i$ and $B_j = \overline{B}_j$ for all i, j.

We can characterise the semi-destabilizing sets as follows, generalizing [GM18, §5].

Lemma 3.16. *Given one-parameter subgroup λ, any maximal $\vec{t}$-destabilised Cartesian product of sets and $\vec{t}$-semi-destabilised Cartesian product of sets as in Definition 3.15 with respect to λ can be written as:*

$$N^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) :=$$
$$V^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) \times \prod_{i=1}^{k-1} B^-(\lambda, x^{J_i}) \times \prod_{p=1}^{m} B^-(\lambda, x_{j_p}),$$
$$N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) :=$$
$$V^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) \times \prod_{i=1}^{k-1} B^{\ominus}(\lambda, x^{J_i}) \times \prod_{p=1}^{m} B^{\ominus}(\lambda, x_{j_p}),$$

where $x^{J_i} \in \Xi_d$ are support monomials with $J_r \neq J_s$ for all r, s, $x_{j_p} \in \Xi_1$ are arbitrary support monomials and

$$V^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) := \{x^I \in \Xi_d | \langle I, \lambda\rangle + \sum_{i=1}^{k-1} \langle J_i, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{j_p}, \lambda\rangle < 0\},$$
$$B^-(\lambda, x^{J_i}) := \{x^J \in \Xi_d | x^J \leq_\lambda x^{J_i}\},$$
$$V^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) := \{x^I \in \Xi_d | \langle I, \lambda\rangle + \sum_{i=1}^{k-1} \langle J_i, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{j_p}, \lambda\rangle \leq 0\},$$
$$B^{\ominus}(\lambda, x^{J_i}) := \{x^J \in \Xi_d | x^J \leq_\lambda x^{J_i}\},$$
$$B^-(\lambda, x_{j_p}) := \{x_i \in \Xi_1 | x_i \leq x_{j_p}\}, \quad B^{\ominus}(\lambda, x_{j_p}) := \{x_i \in \Xi_1 | x_i \leq x_{j_p}\}.$$

Proof. Let $\Delta := (A_1, \ldots, A_k, B_1, \ldots, B_m)$ be a maximal $\vec{t}$-(semi-)destabilised $k+m$-tuple with respect to λ. Let $x^{J_{i-1}} = \max(A_i)$, for $2 \leq i \leq k$ be the maximal element of A_i with respect to the λ-order and $x_{j_p} = \max(B_p)$, for $1 \leq p \leq m$. By the λ-order we have

$$\sum_{i=1}^{k-1} \langle I_1, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{l_p}, \lambda\rangle \leq \langle I_1, \lambda\rangle + \sum_{i=1}^{k-1} \langle J_i, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{j_p}, \lambda\rangle < 0 \quad (\leq \text{ respectively})$$

for all $(x^{I_0}, \ldots, \ldots x^{I_k}, x_{l_1} \ldots, x_{l_m}) \in \Delta$. This implies that

$$\Delta \subseteq N^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m})$$

$$(\Delta \subseteq N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}), \text{ respectively})$$

and the maximality condition of Definition 3.15 implies the equality.

□

Theorem 3.17. *Let $\vec{t} \in \mathrm{Stab}(n, d, k, m)$. A tuple $(S, H_1, \ldots, H_m)$ is not $\vec{t}$-stable ($\vec{t}$-unstable, respectively), if and only if there exists $g \in G$, $\lambda \in P_{n,d,k,m}$, such that the set of monomials associated to $(g \cdot S, g \cdot H_1, \ldots, g \cdot H_m)$ is contained in a pair of sets $N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$ ($N^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$, respectively) defined in Lemma 3.8. Furthermore, the sets $N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$ and $N^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$ which are maximal with respect to the containment order of sets define families of non-$\vec{t}$-stable tuples ($\vec{t}$-unstable tuples, respectively) in $\mathcal{R}_{n,d,k,m}$. Any not $\vec{t}$-stable (respectively $\vec{t}$-unstable) tuple $(g \cdot S, g \cdot H_1, \ldots, g \cdot H_m)$ belongs to one of these families for some group element g.*

Proof. Let $(S, H_1, \ldots, H_m)$ be $\vec{t}$-unstable ($\vec{t}$-non stable respectively). Then by Lemma 3.8 there is $g \in G$ and $\lambda \in P_{n,d,k,m}$ such that

$$\mu_{\vec{t}}(g \cdot (S, H_1, \ldots, H_m), \lambda) < 0 \quad (\leq 0, \quad \text{respectively.})$$

Then, every $(x^{I_1}, \ldots, x^{I_k}, x_{j_1}, \ldots, x_{j_m}) \in g \cdot (\mathrm{Supp}(f_1), \ldots, \mathrm{Supp}(f_k), \mathrm{Supp}(h_1), \ldots, \mathrm{Supp}(h_m))$ satisfies

$$\sum_{i=1}^{k} \langle I_i, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{j_p}, \lambda\rangle < 0 \quad (\leq 0, \quad \text{respectively}).$$

By Definition 3.15 and Lemma 3.8, $g \cdot \mathrm{Supp}(f_1) \subseteq V^-_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{i-1}})$ and $g \cdot \mathrm{Supp}(f_i) \subseteq B^-(\lambda, x^{J_{i-1}})$, $g \cdot \mathrm{Supp}(H_i) \subseteq B^-(\lambda, x_{j_i})$ hold for some $\lambda \in P_{n,d,k,m}$, $x^{J_i} \in \Xi_d$, and $x_{j_i} \in \Xi_1$ ($g \cdot \mathrm{Supp}(f_1) \subseteq V^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{i-1}})$ and $g \cdot \mathrm{Supp}(f_i) \subseteq B^{\ominus}(\lambda, x^{J_{i-1}})$, $g \cdot \mathrm{Supp}(H_i) \subseteq B^{\ominus}(\lambda, x_{j_i})$, respectively). Choosing the maximal Cartesian products of sets $N^-_t(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$,

$(N_{\vec{t}}^{\ominus}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1} \ldots, x_{j_m})$ respectively) under the containment order where $\lambda \in P_{n,d,k,m}$, $x^{J_i} \in \Xi_d$, and $x_{j_p} \in \Xi_1$ we obtain families of Cartesian products of sets whose coefficients belong to maximal $\vec{t}$-(semi-)destabilised $k+m$-tuples. For the opposite direction, note that if the monomials associated to $(g \cdot S, g \cdot H_1, \ldots, g \cdot H_m)$ are contained in $N_{\vec{t}}^{-}$ ($N_{\vec{t}}^{\ominus}$, respectively), then

$$\mu_{\vec{t}}(g \cdot (S, H_1, \ldots, H_m), \lambda) < 0 \quad (\leq 0, \quad \text{respectively})$$

and $(S, H_1, \ldots, H_m)$ is $\vec{t}$-unstable ($\vec{t}$-non stable respectively).

□

We can also define the annihilator as in [GM18, §5]:

Proposition 3.18. *For $\vec{t} \in \operatorname{Int}\big(\operatorname{Stab}(n,d,k,m)\big)$ and normalised one-parameter subgroup λ the annihilator of λ with respect to $x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}$ is the set*

$$\begin{aligned}\operatorname{Ann}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) &:= \Big\{(x^{I}, x^{I_1}, \ldots x^{I_{k-1}}, x_{i_1}, \ldots, x_{i_m}) \in \\ N_{\vec{t}}^{\ominus}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) &\Big| \langle I, \lambda\rangle + \sum_{i=1}^{k-1}\langle I_i, \lambda\rangle + \sum_{p=1}^{m} t_p \langle x_{i_p}, \lambda\rangle = 0\Big\}.\end{aligned}$$

If this is not empty, it is equal to the Cartesian product

$$V_{\vec{t}}^{0}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) \times \prod_{i=1}^{k-1} B^0(\lambda, x^{J_i}) \times \prod_{p=1}^{m} B^0(\lambda, x_{j_p}),$$

where

$$\begin{aligned} &V_{\vec{t}}^{0}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) := \\ &= \{x^I \in V_{\vec{t}}^{\ominus}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) | \exists x^{J_i'} \in B^{\ominus}(x^{J_i}), x_{i_p} \in B^{\ominus}(\lambda, x_{j_p}),\end{aligned}$$

such that $\langle I, \lambda\rangle + \sum_{i=1}^{k-1}\langle J_i', \lambda\rangle + \sum_{p=1}^{m} t_p\langle x_{i_p}, \lambda\rangle = 0\}$,

$$\begin{aligned} B^0(\lambda, x^{J_i}) &:= \{x^J \in B^{\ominus}(\lambda, x^{J_i}) | x^{\overline{J}} \leq_\lambda x^J \text{ for all } x^{\overline{J}} \in B^{\ominus}(\lambda, x^{J_i})\}, \\ B^0(\lambda, x_{j_p}) &:= \{x_i \in B^{\ominus}(\lambda, x_{j_p}) | \langle x_k, \lambda\rangle \leq \langle x_i, \lambda\rangle \text{ for all } x_k \in B^{\ominus}(\lambda, x_{j_p})\}. \end{aligned}$$

Proof. For one direction, let $(x^{I_1}, \ldots, x^{I_k}, x_{i_1}, \ldots, x_{i_m}) \in \operatorname{Ann}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m})$. This implies that $x^{I_1} \in V_{\vec{t}}^{0}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m})$. Suppose that there exist $x^{\overline{I}_i}$, for $2 \leq i \leq k$ such that $x^{I_i} <_\lambda x^{\overline{I}_i}$ and x_{l_p} such that $x_{i_p} <_\lambda x_{l_p}$. Without loss of generality we can take $i = 2$ such that there exists $x^{\overline{I}_2} \in B^{\ominus}(\lambda, x^{J_2})$ such that $x^{I_2} <_\lambda x^{\overline{I}_2}$. Then either

$$0 = \sum_{i=1}^{k}\langle I_i, \lambda\rangle + \sum_{p=1}^{m} t_p\langle x_{i_p}, \lambda\rangle < \langle I_1, \lambda\rangle + \langle \overline{I}_2, \lambda\rangle + \sum_{i=2}^{k}\langle I_i, \lambda\rangle + \sum_{p=1}^{m} t_p\langle x_{i_p}, \lambda\rangle,$$

or

$$0 = \sum_{i=1}^{k} \langle I_i, \lambda \rangle + \sum_{p=1}^{m} t_p \langle x_{i_p}, \lambda \rangle < \sum_{i=1}^{k} \langle I_i, \lambda \rangle + \sum_{p=1}^{m} t_p \langle x_{l_p}, \lambda \rangle.$$

In both cases this would imply that

$$(x^{I_1}, x^{\overline{I}_2}, \ldots, x^{\overline{I}_k}, x_{i_1}, \ldots, x_{i_m}) \notin N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}),$$

which is a contradiction.

Now let

$$(x^{I_1}, \ldots, x^{I_k}, x_{i_1}, \ldots, x_{i_m}) \in V^0_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) \times \prod_{i=1}^{k-1} B^0(\lambda, x^{J_{i-1}}) \times \prod_{p=1}^{m} B^0(\lambda, x_{j_p}).$$

Then, there exist $x^{I'_i} \in B^{\ominus}(\lambda, x^{J_{i-1}})$ for $2 \leq i \leq k$ and $x_{i'_p} \in B^{\ominus}(\lambda, x_{j_p})$ such that

$$\langle I_1, \lambda \rangle + \sum_{i=2}^{k} \langle I'_i, \lambda \rangle + \sum_{p=1}^{m} t_p \langle x_{i'_p}, \lambda \rangle = 0.$$

Also, because $x^{I_i} \in B^0(\lambda, x^{J_i})$, $x^{I'_i} \leq x^{I_i}$, and $x_{i_p} \in B^0(\lambda, x_{j_p})$, $x_{i'_p} \leq x_{i_p}$ we obtain:

$$0 = \langle I_1, \lambda \rangle + \sum_{i=2}^{k} \langle I'_i, \lambda \rangle + \sum_{p=1}^{m} t_p \langle x_{i'_p}, \lambda \rangle \leq \sum_{i=1}^{k} \langle I_i, \lambda \rangle + \sum_{p=1}^{m} t_p \langle x_{i_p}, \lambda \rangle \leq 0$$

since $(x^{I_1}, \ldots, x^{I_k}, x_{i_1}, \ldots, x_{i_m}) \in N^{\ominus}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m})$. Hence

$$(x^{I_1}, \ldots, x^{I_k}, x_{i_1}, \ldots, x_{i_m}) \in \mathrm{Ann}_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m})$$

and the result follows.

□

Theorem 3.19. *If the tuple $(S, H_1, \ldots, H_m)$ belongs to a closed strictly $\vec{t}$-semistable orbit, there is $g \in SL(n+1)$, $\lambda \in P_{n,k,d,m}$ and support monomials $x^{J_1}, \ldots, x^{J_{k-1}} \in \Xi_d$, $x_{j_1}, \ldots, x_{j_m} \in \Xi_1$ such that the set of monomials associated to $g \cdot \big((\mathrm{Supp}(f_1) \times \cdots \times \mathrm{Supp}(f_k)), \mathrm{Supp}(h_1), \ldots, \mathrm{Supp}(h_m)\big)$ corresponds to those in the k-product of sets*

$$V^0_{\vec{t}}(\lambda, x^{J_1}, \ldots, x^{J_{k-1}}, x_{j_1}, \ldots, x_{j_m}) \times \prod_{i=1}^{i} B^0(\lambda, x^{J_{k-1}}) \times \prod_{p=1}^{m} B^0(\lambda, x_{j_p}).$$

Proof. Let $M = (S, H_1, \ldots, H_m)$. By [Dol03, Remark 8.1 (5)], since M is strictly $\vec{t}$-semistable and represents a closed orbit, the stabilizer subgroup $G_M \subset G$ is infinite. Hence, there exists one parameter subgroup $\lambda \in G_M$ where $\lim_{s \to 0} \lambda(s) \cdot M = M$, i.e. $\mu_{\vec{t}}(S, H_1, \ldots, H_m, \lambda) = 0$.

By choosing an appropriate coordinate system and applying Lemma 3.8, we may assume that $\lambda \in P_{n,d,k,m}$ and

$$g\cdot\big((\mathrm{Supp}(f_1),\dots,\mathrm{Supp}(f_k)),\mathrm{Supp}(h_1),\dots,\mathrm{Supp}(h_m)\big) = \mathrm{Ann}_{\vec{t}}(\lambda, x^{J_1},\dots,x^{J_{k-1}},x_{j_1},\dots,x_{j_m})$$
$$= V^0_{\vec{t}}(\lambda, x^{J_1},\dots,x^{J_{k-1}},x_{j_1},\dots,x_{j_m}) \times \prod_{i=1}^{k-1} B^0(\lambda, x^{J_i}) \times \prod_{p=1}^{m} B^0(\lambda, x_{j_p})$$

by Proposition 3.18.

□

Remark 3.19.1. Note, that the converse statement is not true in principle.

Remark 3.19.2. Both characterisations of the annihilator are necessary. The original definition given in Proposition 3.18 is mostly beneficial for proofs, while the 'product' definition is mostly beneficial for algorithms and their implementations in generating all unstable sets.

3.5 How to Study VGIT Quotients Computationally

We will describe how the above sections give us a toolkit to study VGIT quotients (in the case $m = 1$) computationally. This follows the methodology in Gallardo–Martinez-Garcia [GM18; GMZ18; Laz09b].

- By Theorem 3.13, we know that the stability conditions of (S, H) are determined by a finite set of one-parameter subgroups $P_{n,d,k}$ which can be determined using Definition 3.7 computationally in computer programs such as Python and Sage.
- Using this set one can find a superset of the GIT stability walls through solving the equations
$$\sum_{i=1}^{k} \langle I_i, \lambda\rangle + t\langle x_i, \lambda\rangle = 0$$
for $I_i \in \Xi_d$, $I_i \neq I_k$ for all $i \neq k$, $x_i \in \Xi_1$ and $\lambda \in P_{n,d,k}$.
- Knowing the above, for each t, either a potential stability interval or equal to a potential stability wall, as determined by the superset previously introduced, one can compute the $N_t^{\ominus}(\lambda)$ and $N_t^{-}(\lambda)$ for each $\lambda \in P_{n,d,k}$ from their definitions in Lemma 3.16 and find the maximal ones among them. For elements of $N_t^{\ominus}$ which parametrise non-t-stable pairs by Theorem 3.17 the centroid criterion distinguishes which families are strictly $t-$semistable.

- For each of these of the strictly t-semistable families one can compute the annihilator $N_t^0(\lambda) := \mathrm{Ann}(\lambda)$, using Proposition 3.18, which correspond to potentially strictly t-polystable orbits.

- For each wall t, we compare the different sets $N_t^{\ominus}(\lambda)$ and $N_t^-(\lambda)$ obtained between the wall and the previous chamber, and in the cases where these are identical or the maximal families of the chamber are contained in the wall, we remove the wall as a false wall. Although this does reduce the steps to follow, it does not give a complete characterisation of the VGIT quotient.

- The above have been implemented in a computational package written in Sage (and in Python) which can be used for arbitrary initial parameters n, k, d. [Pap22c]. This in turn generalises a computational package [GM17], which deals with VGIT quotients of hypersurfaces, based on [GM18].

- For a geometric characterisation of VGIT quotients, one needs to study the singularities of pairs (S, H) for each stability condition, which are dependent only on n, d, k. In the following sections, we will do so for the case of a complete intersection of two quadrics in $\mathbb{P}^2, \mathbb{P}^3$ and $\mathbb{P}^4$ and a single hyperplane section, which we obtained using our computer program.

- A precise characterisation of each VGIT quotient allows us to remove excess walls, but the algorithm alone cannot guarantee this at the moment. This is due to the fact that we don't have a method to prove that two non-isomorphic maximally destabilised families, which are not isomorphic via transformations by the specific maximal torus used which normalises the one-parameter subgroups, are not isomorphic via a different element of G.

The above computational methodology could potentially be extended to the general case of tuples $(S, H_1, \ldots, H_m)$ as well. This has, however, not yet been implemented in Sage. The reason for this, is the added algorithmical complexity needed in calculating the stability walls $\vec{t}$, and thereafter the maximal (semi-)destabilised sets.

3.6 Stability of k-tuples of Hypersurfaces and Log Canonical Thresholds

In this Section we will study how we can use the Hilbert-Mumford numerical criterion, introduced in Section 3.1, to study a connection between GIT and log canonical thresholds. We will also, study the above through the scope of VGIT. It serves as a generalization of the results presented in Zanardini [Zan22], where the link between log canonical thresholds and the GIT stability of pencils of hypersurfaces are studies, with an emphasis on the pencils of curves. Here, we generalise this setting, considering complete intersections of k hypersurfaces of degree d in $\mathbb{P}^n$, and hence, elements $\mathcal{T} \in \mathrm{Gr}\left(k, \binom{n+d}{d}\right) = \mathcal{R}_{n,d,k}$.

3.6.1 GIT and log Canonical Thresholds

As in Section 3.1, we are interested in studying the GIT quotient

$$\mathcal{R} /\!\!/ G := \mathrm{Gr}\left(k, \binom{n+d}{d}\right) /\!\!/ \mathrm{SL}(n+1).$$

Recall that we let the embedded variety S be defined as the zero locus of k degree d polynomials f_i, where:

$$f_i = \sum (f_i)_{I_i} x^{I_i}.$$

As before, we fix a maximal torus T of G, and a coordinate system such that T is diagonal. In Section 3.1, we demonstrated that the Hilbert-Mumford function is given by

$$\mu(f_1 \wedge \cdots \wedge f_k, \lambda) := -\min\left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle \,\middle|\, (I_1, \ldots, I_k) \in (\Xi_d)^k, I_i \neq I_j \text{ if } i \neq j \text{ and } x^{I_i} \in \mathrm{Supp}(f_i) \right\},$$

where λ is a normalised one-parameter subgroup.

Let $I_i = (d_{i,0}, \ldots, d_{i,n})$ with $i = 1, \ldots, k$, $\sum_{j=0}^{n} d_{i,j} = d$ be distinct monomials with $I_i \in \mathrm{Supp}(f_i)$. For any normalised one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s^{a_0}, \ldots, s^{a_n})$ and since $d_{i,n} = d - \sum_{j=0}^{n-1} d_{i,j}$ and $a_n = -\sum_{k=0}^{n-1} a_k$ we have:

$$\begin{aligned}
\mu(f_1 \wedge \cdots \wedge f_k, \lambda) &= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{i=1}^{k} \langle I_i, \lambda \rangle \middle| (1) \right\} \\
&= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{i=1}^{k} \left(\sum_{j=0}^{n} d_{i,j} a_j \right) \middle| (1) \right\} \\
&= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{j=0}^{n} \left(\sum_{i=1}^{k} d_{i,j} a_j \right) \middle| (1) \right\} \\
&= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j} a_j \right) - \left(kd - \sum_{i=1}^{k} \sum_{j=0}^{n-1} d_{i,j} \right) a_n \middle| (1) \right\} \\
&= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j} (a_j - a_n) \right) - kda_n \middle| (1) \right\} \\
&= -\min_{x^{I_i} \in \mathrm{Supp}(f_i)} \left\{ \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j} (a_j - a_n) \middle| (1) \right) \right\} + \frac{kd}{n+1} \left(\sum_{k=0}^{n-1} a_k - na_n \right)
\end{aligned}$$

where condition (1) refers to the condition $(I_1, \dots, I_k) \in (\Xi_d)^k, I_i \neq I_j$ if $i \neq j$ and $x^{I_i} \in \mathrm{Supp}(f_i)$. Throughout, we will denote by $\mathcal{T}$ an element of the Grassmanian $\mathrm{Gr}(k, W)$. Recall, that $\mathcal{R}$ parametrises the space of tuples $\{\sum_{i=1}^k z_i f_i | (z_1 \colon \dots \colon z_k) \in \mathbb{P}^{k-1}\}$ of k hypersurfaces, and as such, we can write $\mathcal{T} = \{\sum_{i=1}^k z_i f_i | (z_1 \colon \dots \colon z_k) \in \mathbb{P}^{k-1}\}$. We define the following.

Definition 3.20. Fix a maximal torus T. For any k-tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ and normalised one-parameter subgroup λ we define the *affine weight* of $\mathcal{T}$ as

$$\omega(\mathcal{T}, \lambda) := \min_{x^{I_i} \in \mathrm{Supp(f_i)}} \left\{ \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j} (a_j - a_n) \right) \right\}.$$

With this definition and the above discussion, we can reformulate the Hilbert-Mumford numerical criterion (Theorem 2.33):

Lemma 3.21. *With respect to a maximal torus T, $\mathcal{T}$ is unstable (respectively, non-stable) if for some normalised one-parameter subgroup λ*

$$\omega(\mathcal{T}, \lambda) > \frac{kd}{n+1} \left(\sum_{k=0}^{n-1} a_k - na_n \right) \ (\textit{resp. } \geq).$$

Similarly, $\mathcal{T}$ is (semi-)stable if for all normalized one-parameter subgroups λ

$$\omega(\mathcal{T}, \lambda) < \frac{kd}{n+1} \left(\sum_{k=0}^{n-1} a_k - na_n \right) \ (\textit{resp. } \leq).$$

We can also define the affine weight of a hypersurface f, by following the same discussion for the case $k = 1$.

Definition 3.22 ([Zan22, Definition 3.3])**.** Fix a maximal torus T. For a hypersurface f of degree d and normalised one-parameter subgroup λ we define its *affine weight* as

$$\omega(f, \lambda) := \min \left\{ \sum_{i=0}^{n-1} d_i(a_j - a_n) \middle| I = (d_1, \dots, d_n) \in \mathrm{Supp}(f_i) \right\}.$$

This definition allows us to rewrite Lemma 3.1 as follows.

Proposition 3.23. *With respect to a maximal torus T, f is unstable (respectively, non-stable) if for some normalised one-parameter subgroup λ*

$$\omega(f, \lambda) > \frac{d}{n+1} \left(\sum_{k=0}^{n-1} a_k - n a_n \right) \text{ (resp. } \geq \text{)}.$$

Proposition 3.24 (Analogue of [Zan22, Proposition 3.5])**.** *Given a k-tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ and k distinct hypersurfaces $g_1, \dots, g_k \in \mathcal{T}$ we have*

$$\omega(g_i, \lambda) \leq \sum_{i=1}^{k} \omega(g_i, \lambda) \leq \omega(\mathcal{T}, \lambda)$$

for all one-parameter subgroups λ.

Proof. Notice that $\omega(f, \lambda) \geq 0$ for all hypersurfaces f and all normalised one-parameter subgroups λ, since $a_i \geq a_n$ for all $i < n$. This gives the left-hand side of the inequality. For the right-hand side of the inequality notice that

$$\begin{aligned} \omega(\mathcal{T}, \lambda) &= \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j}(a_j - a_n) \right) \\ &= \sum_{j=0}^{n-1} \left(d_{1,j}(a_j - a_n) \right) + \sum_{j=0}^{n-1} \left(d_{2,j}(a_j - a_n) \right) + \dots + \sum_{j=0}^{n-1} \left(d_{k,j}(a_j - a_n) \right) \\ &\geq \omega(g_1, \lambda) + \dots + \omega(g_k, \lambda) \end{aligned}$$

since the g_i are distinct. □

Proposition 3.25. *Let $F = \{f = 0\} \in \mathcal{T}$ be a hypersurface of degree d, and λ a normalised one-parameter subgroup. Then there exist $k-1$ hypersurfaces $g_1, \dots, g_{k-1} \in \mathcal{T}$ such that*

$$\omega(\mathcal{T}, \lambda) = \omega(f, \lambda) + \sum_{i=1}^{k-1} \omega(g_i, \lambda).$$

Proof. Let $\mathcal{T}$ be generated by k hypersurfaces of degree d, f_i. Since $F = \{f = 0\} \in \mathcal{T}$, f is a linear combination of the f_i that generate $\mathcal{T}$, i.e. $f = \sum_{i=1}^{k} \xi_i f_i$ for some $(\xi_1 : \cdots : \xi_k) \in \mathbb{P}^{k-1}$. Then, for a fixed normalised one-parameter subgroup λ, with respect to the λ-order introduced before, the minimum monomial $I = \min\{\mathrm{Supp}(f)\} = \min\{\mathrm{Supp}(f_l)\}$, where without loss of generality we can assume that $f_l = f_1$ (if not we can rearrange the generators such that $f_l = f_1$). Hence, let $I = (d_{1,0}, \ldots, d_{1,n})$ Then, by taking $g_i \in \mathcal{T}$ recursively such that

$$\min\{\mathrm{Supp}(g_i)\} = \min\left\{ \mathrm{Supp}(f_{i+1}) \setminus \Big(\bigcup_{k \leq i} \mathrm{Supp}(f_k)\Big)\right\}$$

the result follows, as we obtain $k-1$ distinct monomials $I_i = (d_{i,0}, \ldots, d_{i,n})$, $2 \leq i \leq k$ such that

$$\omega(\mathcal{T}, \lambda) := \sum_{j=0}^{n-1} \left(\sum_{i=1}^{k} d_{i,j}(a_j - a_n) \right) = \omega(f, \lambda) + \sum_{i=1}^{k-1} \omega(g_i, \lambda).$$

$\square$

Corollary 3.25.1. *Let $F = \{f = 0\} \in \mathcal{T}$ be a hypersurface of degree d, and λ a one-parameter subgroup. Then there exist $k-1$ hypersurfaces $g_1, \ldots, g_{k-1} \in \mathcal{T}$ such that*

$$\omega(\mathcal{T}, \lambda) \leq k \max\left\{\omega(f, \lambda), \omega(g_1, \lambda), \ldots, \omega(g_{k-1}, \lambda)\right\}.$$

Corollary 3.25.2. *If a k-tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ has only semistable (respectively, stable) members, then $\mathcal{T}$ is semistable (respectively stable).*

Proof. Let $f, g_1, \ldots, g_{k-1} \in \mathcal{T}$ be k semistable hypersurfaces as in Corollary 3.25.1. Then by Proposition 3.23 for all λ and i

$$\frac{\omega(f, \lambda)}{\left(\sum_{k=0}^{n-1} a_k - n a_n\right)} \leq \frac{d}{n+1},$$

$$\frac{\omega(g_i, \lambda)}{\left(\sum_{k=0}^{n-1} a_k - n a_n\right)} \leq \frac{d}{n+1},$$

and by Corollary 3.25.1

$$\begin{aligned} \frac{\omega(\mathcal{T}, \lambda)}{\left(\sum_{k=0}^{n-1} a_k - n a_n\right)} &\leq k \max\left\{\omega(f, \lambda), \omega(g_1, \lambda), \ldots, \omega(g_{k-1}, \lambda)\right\} \\ &\leq \frac{kd}{n+1}. \end{aligned}$$

$\square$

The results of [Zan22] can also be extended as follows:

Theorem 3.26 (Analogue of [Zan22, Theorem 3.14]). *If $\mathcal{T} \in \mathcal{R}_{n,d,k}$ contains at worst one strictly semistable hypersurface (and all other hypersurfaces in $\mathcal{T}$ are stable), then $\mathcal{T}$ is stable.*

Proof. Let $f \in \mathcal{T}$ be strictly semistable, i.e. by Proposition 3.23 since f is strictly semi-stable, for some normalised λ, we have

$$\omega(f,\lambda) = \frac{d}{n+1}\Big(\sum_{k=0}^{n-1} a_k - na_n\Big).$$

By Proposition 3.25 there exist $g_1, \dots, g_{k-1} \in \mathcal{T}$ which are stable and

$$\begin{aligned}
\frac{\omega(\mathcal{T},\lambda)}{\Big(\sum_{k=0}^{n-1} a_k - na_n\Big)} &= \frac{\omega(f,\lambda) + \sum_{i=1}^{k-1} \omega(g_i,\lambda)}{\Big(\sum_{k=0}^{n-1} a_k - na_n\Big)} \\
&\leq \frac{d}{n+1} + \sum_{i=1}^{k-1} \frac{d}{n+1} \\
&= \frac{kd}{n+1}.
\end{aligned}$$

□

Proposition 3.27 (Analogue of [Zan22, Proposition 4.6]). *For a k-tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ and any base point p of $\mathcal{T}$, there exists a one-parameter subgroup λ such that for any hypersurface $F = (f = 0)$ in $\mathcal{T}$*

$$\frac{\sum_{i=0}^{n-1}(a_i) - na_n}{\omega(\mathcal{T},\lambda)} \leq \operatorname{lct}_p(\mathbb{P}^n, F).$$

Proof. The proof follows the idea of proof of [Zan22, Proposition 4.1]. Without loss of generality, we choose coordinates $(x_0 : \cdots : x_n)$ in $\mathbb{P}^n$ such that $p = (0 : \cdots : 0 : 1)$. Let λ be a normalized one-parameter subgroup with

$$\lambda = \operatorname{Diag}(s, s^{a_1}, \dots, s^{a_{n-1}}, s^{-1-\sum_{i=1}^{n-1} a_i}).$$

Notice then that

$$\begin{aligned}
\sum_{i=0}^{n-1}(a_i - a_n) &= 2 + \sum_{i=0}^{n-1} a_i + \sum_{j=0}^{n-1}\big(1 + \sum_{i=0}^{n-1} a_i + a_j\big) \\
&= (n+1) + (n+1)\sum_{i=0}^{n-1} a_i \\
&= (n+1)\big(1 + \sum_{i=0}^{n-1} a_i\big)
\end{aligned}$$

and

$$\begin{aligned}\omega(\mathcal{T},\lambda) &= \sum_{j=0}^{n-1}\Bigg(\sum_{i=1}^{k} d_{i,j}(a_j-a_n)\Bigg)\\ &\geq \sum_{i=1}^{2}\Bigg(\sum_{j=0}^{n-1} d_{i,j}(a_j-a_n)\Bigg)\\ &\geq \Bigg[(n+1)+(n+1)\sum_{i=0}^{n-1}a_i\Bigg]\end{aligned}$$

by [Zan22, Proposition 4.1], hence

$$\frac{\sum_{i=0}^{n-1}(a_i-a_n)}{\omega(\mathcal{T},\lambda)} = \frac{(n+1)\big(1+\sum_{i=0}^{n-1}a_i\big)}{\sum_{j=0}^{n-1}\Bigg(\sum_{i=1}^{k} d_{i,j}(1+a_j+\sum_{i=0}^{n-1}a_i)\Bigg)} \leq 1.$$

For contradiction, assume that there exists $F=\{f=0\}\in\mathcal{T}$ such that

$$\operatorname{lct}_p(\mathbb{P}^n,F) < \frac{\sum_{i=0}^{n-1}(a_i-a_n)}{\omega(\mathcal{T},\lambda)}.$$

Then, let $\tilde{F}(u_1,\ldots,u_n)=f(x_0:\cdots:x_{n-1}:1)$, where $u_i=\frac{x_{i-1}}{x_n}$ defined in a neighbourhood around p, which is enough to compute lct_p since it is a local invariant. Also, assign weights $\omega(u_1)=a_0-a_n$, $\omega(u_i)=a_{i-1}-a_n$, $\omega(u_n)=a_{n-1}-a_n$. Consider the finite morphism $\phi\colon\mathbb{C}^n\to\mathbb{C}^n$ where $(u_1,\ldots,u_n)\mapsto(u_1^{\omega(u_1)},\ldots,u_n^{\omega(u_n)})$. Then, let $H_{u_i}=\{u_i=0\}$ and

$$\Delta := \sum_{i=1}^{n}(1-\omega(u_i))H_{u_i}+c\tilde{F}(u_1^{\omega(u_1)},\ldots,u_n^{\omega(u_n)})$$

for some $c\in\mathbb{Q}\cap[0,1]$. Then,

$$\phi^*(K_{\mathbb{C}^n}+c\tilde{F}(u_1,\ldots,u_n)) = K_{\mathbb{C}^n}+\Delta.$$

We also know that the pair $(\mathbb{C}^n,\tilde{F})$ is log canonical at the origin if and only if the pair $(\mathbb{C}^n,\Delta)$ is log canonical at the origin. Let

$$c=\frac{\sum_{i=1}^{n}\omega(u_i)}{\omega(\mathcal{T},\lambda)}$$

where $c>\operatorname{lct}_p(\mathbb{P}^n,F)=\operatorname{lct}_{\vec{0}}(\mathbb{C}^n,\tilde{F})$ by the assumption. Blowing up $\mathbb{C}^n$ at the origin, we then have that for the log discrepancy of Δ with respect to the exceptional divisor E of the blow up

$$a(E;\mathbb{C}^n,\Delta) = -1-\sum_{i=1}^{n}\omega(u_i)-c\omega(f,\lambda) < -1$$

which in turn would imply that $\omega(f,\lambda) > \omega(\mathcal{T},\lambda)$ which contradicts Proposition 3.24.

□

Corollary 3.27.1 (Analogue of [Zan22, Theorem 4.8]). *If $\mathcal{T}$ is (semi-)stable, then for any hypersurface $F = \{f = 0\} \in \mathcal{T}$ and any base point p of $\mathcal{T}$*

$$\frac{n+1}{kd} < \mathrm{lct}_p(\mathbb{P}^n, F) \textit{ (respectively } \leq).$$

Proof. Since $\mathcal{T}$ is (semi-)stable, for all normalized one-parameter subgroups λ we have

$$\frac{\sum_{i=0}^{n-1}(a_i - a_n)}{\omega(\mathcal{T},\lambda)} \geq \frac{n+1}{kd}$$

by Lemma 3.21. □

Proposition 3.28. *Given a tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ we have that for any one-parameter subgroup λ there exists $F = \{f = 0\} \in \mathcal{T}$ such that*

$$\frac{\omega(\mathcal{T},\lambda)}{\left(\sum_{k=0}^{n-1} a_k - na_n\right)} \leq \frac{k}{\mathrm{lct}(\mathbb{P}^n, F)}.$$

Proof. By [Kol97, Proposition 8.13] we have that

$$\frac{\omega(f,\lambda)}{\left(\sum_{k=0}^{n-1} a_k - na_n\right)} \leq \frac{1}{\mathrm{lct}(\mathbb{P}^n, F)},$$

and we obtain the result by Corollary 3.25.1.

□

Corollary 3.28.1. *If a tuple $\mathcal{T} \in \mathcal{R}_{n,d,k}$ is such that $\mathrm{lct}(\mathbb{P}^n, F) \geq \frac{n+1}{d}$ (respectively, $> \frac{n+1}{d}$) for any hypersurface $F = \{f = 0\}$ in $\mathcal{T}$, then $\mathcal{T}$ is semistable (respectively, stable).*

Proof. Let $f \in \mathcal{T}$ a hypersurface with $\mathrm{lct}(\mathbb{P}^n, F) \geq \frac{n+1}{d}$ (respectively $> \frac{n+1}{d}$). Then, for any normalized one-parameter subgroup λ,

$$\begin{aligned}\frac{\omega(\mathcal{T},\lambda)}{\left(\sum_{k=0}^{n-1} a_k - na_n\right)} &\leq \frac{k}{\mathrm{lct}(\mathbb{P}^n, F)}\\ &\leq \frac{kd}{n+1} \quad \left(< \frac{kd}{n+1} \text{ resp.}\right).\end{aligned}$$

□

3.6.2 VGIT and log Canonical Thresholds

Consider now the GIT quotient $\mathcal{R}_m /\!\!/ G$ which parametrizes tuples $(\mathcal{T}, H_1, \dots, H_m)$ of complete intersections and m hyperplanes in $\mathbb{P}^n$.

From our discussion in Sections 3.1 and 3.6.1, along with Lemma 3.21 and Definition 3.5 we obtain the following:

Lemma 3.29. *With respect to a maximal torus T, the tuple $(\mathcal{T}, H_1, \dots, H_m)$ is $\vec{t}$–unstable (respectively, $\vec{t}$–non-stable) if for some λ*

$$\omega(\mathcal{T}, \lambda) + \sum_{i=1}^{m} t_i \omega(H_i, \lambda) > \frac{kd + \sum_{i=1}^{m} t_i}{n+1} \left(\sum_{k=0}^{n-1} a_k - n a_n \right) \textit{(resp. } \geq).$$

Proof. From Lemma 2.35, we have that $\mu_{\vec{t}}(\mathcal{T}, H_1, \dots, H_m, \lambda) = \mu(\mathcal{T}, \lambda) + \sum_{i=0}^{m} t_i \mu(H_i, \lambda)$. The result then follows from the discussion in Section 3.6.1. □

We expand Proposition 3.25 as follows:

Proposition 3.30. *Let $F = \{f = 0\} \in \mathcal{T}$ be a hypersurface of degree d, and λ a one-parameter subgroup. Then there exist $k-1$ hypersurfaces $g_1, \dots, g_{k-1} \in \mathcal{T}$ such that*

$$\omega(\mathcal{T}, \lambda) + \sum_{i=1}^{m} t_i \omega(H_i, \lambda) = \omega(f, \lambda) + \sum_{i=1}^{k-1} \omega(g_i, \lambda) + \sum_{i=1}^{m} t_i \omega(H_i, \lambda).$$

By Fujita [Fuj21], the log-canonical threshold $\operatorname{lct}(\mathbb{P}^n, H_i) = 1$ for all hyperplanes H_i, and hence

$$\frac{\omega(H_i, \lambda)}{\left(\sum_{k=0}^{n-1} a_k - n a_n \right)} \leq 1,$$

so in combination with Proposition 3.28 we obtain the following.

Proposition 3.31. *Given a tuple $(\mathcal{T}, H_1 \dots, H_m) \in \mathcal{R}_m$ we have that for any one-parameter subgroup λ there exists $F = \{f = 0\} \in \mathcal{T}$ such that*

$$\frac{\omega(\mathcal{T}, \lambda) + \sum_{i=1}^{m} t_i \omega(H_i, \lambda)}{\left(\sum_{k=0}^{n-1} a_k - n a_n \right)} \leq \frac{k}{\operatorname{lct}(\mathbb{P}^n, F)} + \sum_{i=1}^{m} t_i.$$

Corollary 3.31.1. *If a tuple $(\mathcal{T}, H_1 \dots, H_m) \in \mathcal{R}_m$ is such that $\operatorname{lct}(\mathbb{P}^n, F) \geq \frac{k(n+1)}{kd - n\sum_{i=1}^{m} t_i}$ (respectively, $> \frac{k(n+1)}{kd - n\sum_{i=1}^{m} t_i}$) for any hypersurface f in $\mathcal{T}$, then $(\mathcal{T}, H_1 \dots, H_m) \in \mathcal{R}_m$ is $\vec{t}$–semistable (respectively, $\vec{t}$–stable).*

Proof. From Proposition 3.27, let $f \in \mathcal{T}$ be such that $\operatorname{lct}(\mathbb{P}^n, F) \geq \frac{k(n+1)}{kd-n\sum_{i=1}^m t_i}$ (respectively, $> \frac{k(n+1)}{kd-n\sum_{i=1}^m t_i}$). Then for all λ,

$$\begin{aligned}\frac{\omega(\mathcal{T},\lambda)+\sum_{i=1}^m t_i\omega(H_i,\lambda)}{\left(\sum_{k=0}^{n-1} a_k - na_n\right)} &\leq \frac{k}{\operatorname{lct}(\mathbb{P}^n,F)}+\sum_{i=1}^m t_i \\ &\leq \frac{kd-n\sum_{i=1}^m t_i}{n+1}+\sum_{i=1}^m t_i \text{ (respectively, } <) \\ &\leq \frac{kd+\sum_{i=1}^m t_i}{n+1} \text{ (respectively, } <).\end{aligned}$$

□

CHAPTER 4

Complete Intersection of Conics in $\mathbb{P}^2$

In this chapter, we will work over $\mathbb{C}$ and we will study the GIT quotients of complete intersections of conics in $\mathbb{P}^2$, under an $\mathrm{SL}(3)$-action, computationally. This can be thought of as a "toy example" to demonstrate how the computational methods of Chapter 3 can be applied. In order to do so, we will first introduce some notation and theorems for general results on the singularities of complete intersections of quadrics in arbitrary n. This will be of benefit in later chapters as well.

4.1 Some Results on the Singularities of Pencils of Quadrics

For a quadric q in $\mathbb{P}^n$ we can write $q(x) = xQx^T$, for Q a $(n+1)\times(n+1)$ symmetric matrix with entries in $\mathbb{C}$. We denote by $\Phi(f,g) \in \mathrm{Gr}(2, \frac{(n+2)(n+1)}{2})$ the element of the Grassmanian naturally representing two quadrics f, g, in $\mathbb{P}^n$, i.e. $\Phi(f,g) := \{\lambda f + \mu g | (\lambda,\mu) \in \mathbb{P}^1\}$. This pencil can also be written in terms of the symmetric matrices F and G of f and g, respectively, i.e. $\Phi(f,g) = \{\lambda F + \mu G | (\lambda,\mu) \in \mathbb{P}^1\}$ (see [Rei72, §1]). The notion of the stability of pencils is defined in Section 3.1, where, following from our previous discussion, we now define $k = d = 2$. Note that for a complete intersection of quadrics, $S = \{f = 0\} \cap \{g = 0\}$, $S = \mathrm{Bs}(\Phi(f,g))$ is the base-locus of a pencil with no fixed part (see [Som59, §XIII], or [HP94, §XIII]).

Note here, that a quadric q is smooth (nondegenerate) if and only if the corresponding symmetric matrix Q is non-degenerate, i.e. if $\det(Q) \neq 0$ (see for example Reid [Rei72, §1]).

For a pencil $\Phi(f,g)$ of two quadrics in $\mathbb{P}^n$ with f smooth, (i.e. the rank of the corresponding symmetric matrix F is $n+1$) we consider the polynomial $\det(\lambda F + G)$ of degree $n+1$ with distinct roots $\alpha_1, \dots, \alpha_r$.

The Lemma below proves quite useful for detecting singular complete intersections of quadrics in $\mathbb{P}^n$.

Lemma 4.1. *Let f, q be two quadrics in $\mathbb{P}^n$. Their complete intersection $S = f \cap q$ is singular if and only if up to an action of $\mathrm{SL}(n+1)$ the quadrics can be written either as*

$$f(x_0, \dots, x_n) = q_1(x_1, \dots, x_n)$$
$$q(x_0, \dots, x_n) = x_0(b_1x_1 + \dots + b_nx_n) + q_2(x_1, \dots, x_n)$$

or

$$f(x_0, \dots, x_n) = a_0x_0x_n + q_1(x_1, \dots, x_n)$$
$$q(x_0, \dots, x_n) = b_nx_0x_n + q_2(x_1, \dots, x_n)$$

or a degeneration of the above.

Proof. Without loss of generality, we assume that the singular point is $P = (1 : 0 : \dots : 0)$. Then since $P \in f \cap q$ we have:

$$f(x_0, \dots, x_n) = x_0l_1(x_1, \dots, x_n) + q_1(x_1, \dots, x_n)$$
$$q(x_0, \dots, x_n) = x_0l_2(x_1, \dots, x_n) + q_2(x_1, \dots, x_n),$$

where the l_i are linear and the q_i are quadratic forms.

We can choose a coordinate transformation fixing P such that $x_n = l_1(x_1, \dots, x_n)$ and then

$$f(x_0, \dots, x_n) = a_0x_0x_n + q_1(x_1, \dots, x_n)$$
$$q(x_0, \dots, x_n) = x_0(b_1x_1 + \dots + b_nx_n) + q_2(x_1, \dots, x_n).$$

Recall that a point P is singular on the intersection if and only if the matrix

$$J = \begin{pmatrix} \frac{\partial f}{\partial x_i} \\ \frac{\partial q}{\partial x_i} \end{pmatrix}$$

at P has rank < 2. Then since

$$J_p = \begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 0 & b_1 & \dots & b_{n-1} & b_n \end{pmatrix}$$

we see that $\operatorname{rank} J_p < 2$ if either $a_0 = 0$ or $b_1 = b_2 = \dots = b_{n-1} = 0$, and the result follows. □

Corollary 4.1.1. *Let f, q be two quadrics in $\mathbb{P}^n$. Their complete intersection $S = f \cap q$ is smooth if and only if the determinant polynomial $\det(\lambda f + q)$ has only simple roots.*

Proof. We assume without loss of generality that $P = (1 : 0 : \cdots : 0) \in q \cap r$, hence we can write

$$q(x_0, \ldots, x_n) = a_0 x_0 x_n + q_1(x_1, \ldots, x_n)$$
$$r(x_0, \ldots, x_n) = x_0(b_1 x_1 + \cdots + b_n x_n) + q_2(x_1, \ldots, x_n)$$

and since S is smooth, $\alpha_0 \neq 0 \neq b_i$, for $i = 1, \ldots, n$. The determinant polynomial $\det(\lambda q + r)$ thus has $n + 1$ distinct roots. □

4.1.1 Segre Symbols

Let α_i be a root of the determinant polynomial of multiplicity e_i. Assume further that α_i is not only a zero of $\det(G + \lambda F)$, but also of all its subdeterminants of size $n - h_i + 2$, where $n + 1 \geq h_i \geq 2$, and h_i is the maximal number such that for each root α_i, α_i is a solution of all the non-trivial subdeterminants of size $n - h_i - 2$ of $G + \lambda F$. If $h_i = 1$, this implies that the solution is not a solution of any of the subdeterminants.

We then define l^i_j to be the minimum multiplicity of the root α_i for the set of subdeterminants of size $n + 1 - j$, for $j = 0, 1, \ldots, h_i - 1$. We have $l^i_j \geq l^i_{j+1}$, and we define $e^i_j = l^i_j - l^i_{j+1}$. Thus, we obtain a factorisation

$$\det(G + \lambda F) = \prod_{j=0}^{h_i - 1} (\lambda - \alpha_i)^{e^i_j} f_i(\lambda)$$

where $f_i(\alpha_i) \neq 0$ (see [HP94, §XIII] or [Som59, §13.86]).

Definition 4.2. The *Segre symbol* of the pencil is $[(e^0_0, \ldots e^0_{h_0 - 1}), \ldots, ((e^r_0, \ldots e^r_{h_r - 1}))]$.

Note, that if we only have a 1-tuple, we omit the brackets around the tuple.

Theorem 4.3 (Weierstrass, Segre, [Wei13])**.** *Consider two pencils Φ_1 and Φ_2 of quadric hypersurfaces in $\mathbb{P}^n$. Then their base loci are projectively equivalent if and only if they have the same Segre symbol and there exists an automorphism of $\mathbb{P}^1$ taking each root $(1 : \alpha_i)$ to the corresponding root $(1 : \beta_i)$, where α_i and β_i are the roots of the determinant polynomials of Φ_1 and Φ_2 respectively.*

We will present the following example to illustrate how one obtains Segre symbols.

Example 4.3.1. Let f and q be two quadrics in $\mathbb{P}^2$, given by the following equations.

$$f(x_0, x_1, x_2) = x_0 x_1 + x_2 l(x_0, x_1, x_2)$$
$$g(x_0, x_1, x_2) = x_2^2$$

where $l(x_0, x_1, x_2) = a_0 x_0 + a_1 x_1 + a_2 x_2$ is a linear form. Then

$$F = \frac{1}{2}\begin{bmatrix} 0 & 1 & a_0 \\ 1 & 0 & a_1 \\ a_0 & a_1 & 2a_2 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\lambda F + G = \begin{bmatrix} 0 & \lambda & \lambda a_0 \\ \lambda & 0 & \lambda a_1 \\ \lambda a_0 & \lambda a_1 & 2\lambda a_2 + 1 \end{bmatrix}$$

with $\det(\lambda F + G) = \frac{\lambda^2}{2}(-2\lambda a_2 - 1 + 2\lambda a_0 a_1)$.

The determinant polynomial has two solutions, $\lambda = 0$ with multiplicity 2, and $\lambda = \frac{1}{2a_0a_1 - 2a_2}$ with multiplicity 1. Notice that $\lambda = 0$ is a solution for all the subdeterminants of $\lambda F + G$ with size $2 + 1 - 1 = 2$ (i.e. all 2×2 minors). Thus, we have, for root $\alpha_0 = 0$: $l_0^0 = 2$, $l_1^0 = 1$, $l_2^0 = 0$, and hence $e_0^0 = 1$, $e_1^0 = 1$. On the other hand, since the multiplicity of the root $\alpha_1 = \frac{-1+2a_0a_1}{a_2}$ is 1, there is no such decomposition. Hence, the Segre symbol is $[(1, 1), 1]$.

4.1.2 Preliminaries of Singularity Theory

We will give some brief preliminaries on singularity theory, which will be used throughout the later chapters.

Definition 4.4 ([Arn76, p.88]). A class of singularities T_2 is *adjacent* to a class T_1, and one writes $T_1 \leftarrow T_2$ if every germ of $f \in T_2$ can be locally deformed into a germ in T_1 by an arbitrary small deformation. We say that the singularity T_2 is *worse* than T_1, or that T_2 is a degeneration of T_1.

$$\begin{tikzcd} \mathbf{A_1} & \mathbf{A_2} \arrow[l] & \mathbf{A_3} \arrow[l] & \mathbf{A_4} \arrow[l] & \mathbf{A_5} \arrow[l] \\ & & \mathbf{D_4} \arrow[u] & \mathbf{D_5} \arrow[l] \arrow[u] & \end{tikzcd}$$

Figure 4.1: Degeneration of germs of isolated singularities appearing in complete intersections of two quadrics in $\mathbb{P}^3$ and $\mathbb{P}^3$

The degenerations of the isolated singularities that appear in a complete intersection of two quadrics in $\mathbb{P}^3$ and $\mathbb{P}^4$ (or in their anticanonical divisors, which are complete intersection of two quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively) are described in Figure 4.1 (for details see [Arn76, p.88] and [Arn75, §13]. The above theory considers only local deformations of singularities. When we study degenerations in the GIT quotient, we are interested in global deformations. Thankfully, due to [HP10, Proposition 3.1], in the particular cases of complete intersections of two quadrics in $\mathbb{P}^4$, any local deformation of isolated singularities is induced by a global deformation.

Definition 4.5 ([BW79])**.** A polynomial F in $n+1$ variables is *semi-quasihomogeneous* (SQH) with respect to the weights $(w_1, w_2, \dots, w_n)$ if all the monomials of F have weight larger or equal than 1 and those monomials of weight 1 define a function with an isolated singularity. In particular, the weights associated to the ADE singularities $\mathbf{A}_k$ and $\mathbf{D}_k$ are:

$$(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{k+1}), \qquad (\frac{1}{2}, \dots, \frac{1}{2}, \frac{k-2}{2(k-1)} \frac{1}{k-1}).$$

4.2 GIT of Complete Intersections of Conics

From [HP94, §XIII] the following table summarizes the results for the Segre symbols of pencils of conics and their base loci.

Segre symbol of pencil	Base locus
$[3]$	A triple point and another point
$[(2,1)]$	A quadruple point
$[2,1]$	A double point and two other points
$[(1,1),1]$	Two double points
$[1,1,1]$	Four distinct points (smooth)

Table 4.1: Segre symbols of pencils of quadrics in $\mathbb{P}^2$ and classification.

In particular, due to Theorem 4.3, we see that any two pencils with the same Segre symbol must be projectively equivalent.

We will study the GIT quotient $\mathcal{R}_{2,2,2} /\!\!/ \mathrm{SL}(3)$. The following families have been generated using the computational package [Pap22c], based on the discussion on Chapter 3.

In particular, $P_{2,2,2} = [(4,1,-5),(1,0,-1),(5,-1,-4),(2,-1,-1),(1,1,-2)]$ and the computer package gives us:

λ	x^J	$V^-(\lambda, x^J)$	$B^-(\lambda, x^J)$
$(4,1,-5)$	x_2^2	$\{x_0,x_1,x_2\}^2$	x_2^2
$(1,0,-1)$	x_0x_2	$\{x_1,x_2\}^2$, x_0x_2	$\{x_1,x_2\}^2$, x_0x_2
$(2,-1,-1)$	x_0x_1	$\{x_1,x_2\}^2$, $x_0\{x_1,x_2\}$	$\{x_1,x_2\}^2$.

Table 4.2: Outputs of the computational package [Pap22c] for destabilized families of complete intersections of two quadrics in $\mathbb{P}^2$

λ	x^J	$V^{\ominus}(\lambda, x^J)$	$B^{\ominus}(\lambda, x^J)$
$(4,1,-5)$	x_2^2	$\{x_0,x_1,x_2\}^2$	x_2^2
$(1,0,-1)$	x_0x_2	$\{x_1,x_2\}^2$, x_0x_2	$\{x_1,x_2\}^2$, x_0x_2
$(2,-1,-1)$	x_0x_1	$\{x_1,x_2\}^2$, $x_0\{x_1,x_2\}$	$\{x_1,x_2\}^2$.

Table 4.3: Outputs of the computational package [Pap22c] for semi-destabilized families of complete intersections of two quadrics in $\mathbb{P}^2$

Theorem 4.6. *The following are equivalent:*

1. *A pencil of two quadrics $\Phi(f,g)$ in $\mathbb{P}^2$ is unstable;*
2. *a pencil of two quadrics $\Phi(f,g)$ in $\mathbb{P}^2$ is non-stable;*
3. *the pencil is generated by one of the following three families, or a degeneration of these families:*

   Family 1*:*

$$
\begin{aligned}
f(x_0,x_1,x_2) =& q_1(x_0,x_1,x_2)\\
g(x_0,x_1,x_2) =& x_2^2
\end{aligned}
$$

an irreducible conic f (i.e. $\det(Q_1) \neq 0$) and a double line intersecting at two separate double points, $\mathrm{Bs}(f,g) = 2P + 2Q$;

Family 2:

$$f(x_0, x_1, x_2) = x_2 l_1(x_0, x_1, x_2) + x_1^2$$
$$g(x_0, x_1, x_2) = x_2 l_2(x_0, x_1, x_2) + x_1^2$$

two irreducible conics f and g intersecting at a double point and two separate simple points, $\mathrm{Bs}(f,g) = 2P + Q + R$, where not all 3 points lie on the same line L;

Family 3:

$$f(x_0, x_1, x_2) = x_2 l_3(x_0, x_1, x_2) + x_1 l_4(x_0, x_1)$$
$$g(x_0, x_1, x_2) = q_2(x_1, x_2)$$

an irreducible conic f and a two intersecting lines g intersecting at a double point and two separate simple points, $\mathrm{Bs}(f,g) = 2P + Q + R$, where not all 3 points lie on the same line L.

Here, the l_i are lines in $\mathbb{P}^2$ and the q_i are quadratic forms, which are all maximal in their support, i.e. there are no zero coefficients.

4. *The base locus of the pencil is singular, i.e. it is not a union of 4 distinct points.*

Proof. The equivalence of 1 and 3 and 2 and 3 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15. In particular, we first obtain a finite set of one-parameter subgroups $P_{2,2,2} = [(4,1,-5), (1,0,-1), (5,-1,-4), (2,-1,-1), (1,1,-2)]$, that determine stability, as detailed in Definition 3.7 and Lemma 3.8. Then for each $\lambda \in P_{2,2,2}$ we compute the corresponding $N^{-}(\lambda, x^J)$ and $N^{\ominus}(\lambda, x^J)$ for various support monomials x^J, and we determine which of those are maximal with respect to Definition 3.15. The corresponding sets are presented in Tables 4.2 and 4.3.

We will now show that 1 and 2 are equivalent. We have that family 1 is given by $N^{-}(\lambda, x^J) = N^{\ominus}(\lambda, x^J)$ for $\lambda = (4,1,-5)$ and $x^J = x_2^2$, Family 2 is given by $N^{-}(\lambda, x^J)$ for $\lambda = (1,0,-1)$ and $x^J = x_0x_2$ and Family 3 is given by $N^{-}(\lambda, x^J)$ for $\lambda = (2,-1,-1)$ and $x^J = x_0x_1$. Since we have $N^{-}(\lambda, x^J) = N^{\ominus}(\lambda, x^J)$ for all λ and x^J, i.e. the sets of all destabilised and semi-destabilised families are the same, we deduce that there are no polystable elements, and that all unstable and non-stable maximal families are the same. In particular, this demonstrates the equivalence of 1, 2 and 3.

We will now show that 3 and 4 are equivalent. For Family 1 notice that the double line $x_2 = 0$ intersects the smooth conic $f(x_0, x_1, x_2)$ at the points $(1 : l : 0)$, $(1 : k : 0)$ where l, k are the solutions of $q_1(1, x_1, 0)$. The Segre symbol of this case is $[(1, 1), 1]$ and the base locus is the two double points. In particular, the degeneration

$$f(x_0, x_1, x_2) = x_0 x_2 + x_1^2$$
$$g(x_0, x_1, x_2) = x_2^2$$

gives us the complete intersection which is the quadruple point $(1 : 0 : 0)$.

For Family 2, we first make the standard change of coordinates $q(x_1, x_2) = x_1 x_2$ and we obtain

$$f(x_0, x_1, x_2) = x_2 l_1(x_0, x_1, x_2) + x_1^2$$
$$g(x_0, x_1, x_2) = x_2 l_2(x_0, x_1, x_2)$$

Since all the q_i and l_i are maximal in their support, we further make the change of coordinates $x'_0 = l_2(x_0, x_1, x_2)$, $x'_1 = x_1$, $x'_2 = x_2$, and we obtain $f(x'_0, x'_1, x'_2) = x'_2 l'_1(x'_0, x'_1, x'_2) + \alpha(x'_1)^2$, $g(x'_0, x'_1, x'_2) = x'_0 x'_2$. The intersection is then given as follows: For $x'_2 = 0$, $f(x'_0, x'_1, 0) = \alpha(x'_1)^2$ and the point of intersection is the double point $P = (1 : 0 : 0)$. For $x'_0 = 0$, $f(0, x'_1, x'_2) = q(x'_1, x'_2)$ and there are two points Q,R, which correspond to the intersection. The Segre symbol of this case is $[2, 1]$ and by [HP94, §XIII] the base locus is a double point and two other points. Furthermore, the degeneration

$$f(x_0, x_1, x_2) = x_2 x_0 + x_1^2$$
$$g(x_0, x_1, x_2) = x_2 x_1$$

is the intersection with triple point $P = (1 : 0 : 0)$, and point $Q = (0 : 0 : 1)$.

For Family 3, using a similar change of variables as in Family 2, we can write $g(x_0, x_1, x_2) = x_1 x_2$, $f(x'_0, x'_1, x'_2) = \alpha x_2 l'_3(x_0, x_1, x_2) + x_1 l'_4(x_0, x_1)$. For $x_1 = 0$, we obtain two points of intersection $P = (1 : 0 : 0)$, $Q = (1 : 0 : l)$ and for $x_2 = 0$, we obtain two points of intersection $P = (1 : 0 : 0)$, $R = (1 : k : 0)$, where l is the solution of $l'_3(1, 0, x_2)$ and k is the solution of $l'_4(1, x_1)$. The Segre symbol of the pencil is $[2, 1]$ and the base locus is a double point and two other points, where P is the double point.

Thus, the above 3 families and their degenerations are all possible singular complete intersections of conics (with one or two double points, a triple or a quadruple point). Hence, 3 and 4 are equivalent.

□

Corollary 4.6.1. *The GIT quotient of complete intersections of two plane conics is the one point scheme representing the unique stable orbit of two quadrics intersecting at* 4 *points.*

Proof. Let f, g such that $\mathrm{Bs}(f, g) = P_1 + P_2 + P_3 + P_4$. By Theorem 4.6 this intersection is not non-stable, hence it is a stable complete intersection.

Let f, g such that $\mathrm{Bs}(f, g)$ is stable. By Bezout's theorem, $\mathrm{Bs}(f, g)$ consists of at most 4 distinct points. Notice that a 1 point intersection (i.e. a quadruple point) corresponds to a degeneration of Family 1, a 2 point intersection corresponds to Family 1, 2 or 3, and a 3 point intersection corresponds to a degeneration of Family 2 or 3, which are all unstable by Theorem 4.6. $\square$

CHAPTER 5

VGIT of a complete intersection of Quadrics in $\mathbb{P}^3$ and a Hyperplane

In this chapter, we will study VGIT quotients of complete intersections of quadrics in $\mathbb{P}^3$ and a hyperplane, using the computational methods presented in Chapter 3. We will first provide some general results on the singularities of such complete intersections, and then we will provide a full GIT classification (in the absence of a hyperplane). We will then proceed to classify all possible singularities of pairs $(S, D = S \cap H)$ and provide a full VGIT classification using our computational method. The GIT classification will be of use when in later chapters we will compactify the K-moduli space of the Fano threefold family $2-25$.

5.1 General Results

For two quadratic polynomials $f, g \in \mathbb{P}^3$, let $\Phi(f, g)$ be their pencil, with general element $\lambda f + \mu g$, and let S be its base locus, such that $S = \{f = g = 0\}$. Throughout this section we will make use of the classification of the base loci of such pencils found in Sommerville [Som59, §XIII]. We summarise these results in the following table.

Segre symbol of pencil	Base locus
$[4]$	Twisted cubic and tangent line
$[(3,1)]$	A conic and two lines intersecting in one point
$[(2,2)]$	A double line and two lines in general position
$[(2,1,1)]$	Two tangent lines
$[3,1]$	Cuspidal curve
$[(2,1),1]$	Two tangent conics
$[(1,1,1),1]$	Double conic
$[2,2]$	Twisted cubic and bisecant
$[2,(1,1)]$	A conic and two lines in a triangle
$[(1,1),(1,1)]$	A quadrangle
$[2,1,1]$	Nodal curve
$[(1,1),1,1]$	Two conics in general position
$[1,1,1,1]$	Elliptic curve (smooth)

Table 5.1: Segre symbols of pencils of quadrics in $\mathbb{P}^3$ and classification of the base loci.

The Segre symbols can be computed for any pencil by a simple Python script, that solves the determinant equation and finds the multiplicities of the solutions. In the case where the multiplicity is greater than 1 one can compute the determinant minors to distinguish the different cases with similar multiplicity. In many cases, we will refer to a complete intersection S having a particular Segre symbol (and not the pencil whose base locus S is) to avoid confusion. Notice that by the above table and by the definition of the base locus description, if S has Segre symbol $[2,1,1]$ it has 1 $\mathbf{A}_1$ singularity, if it has Segre symbol $[2,2]$ it has 2 $\mathbf{A}_1$ singularities while if it has Segre symbol $[(1,1),(1,1)]$ it has 4 $\mathbf{A}_1$ singularities. If S has Segre symbol $[3,1]$ it has 1 $\mathbf{A}_2$ singularity, and if it has Segre symbol $[4]$ it has an $\mathbf{A}_3$ singularity. Finally, if S has Segre symbol $[(2,1,1)]$ or $[(1,1,1),1]$ it has non-isolated singularities.

The Lemmas below give us a geometric description of the singularities of a pencil with respect to its determinant polynomial.

Lemma 5.1. *Let S be complete intersection with Segre symbol $[(2,1),1]$. Then S has 1 $\mathbf{A}_3$ singularity.*

Proof. Since the Segre symbol of $S = \{f=0\} \cap \{g=0\}$ is $[(2,1),1]$, S is two conics tangent at a point P. We may assume, without loss of generality, that f is smooth and g is singular.

Notice that $f = \mathbb{P}^1 \times \mathbb{P}^1$ and each conic C_1, C_2 is a $(1,1)$ divisor of f as they are obtained via $H \cap f$, where H is a hyperplane. The C_i meet only at P, hence $C_1 \cdot C_2 = 2$. We blow up f at a point Q such that $Q \notin C_i$, and Q lies on the intersection of two lines G_1, G_2. We obtain $\pi\colon \operatorname{Bl}_Q f \to f$ and we then blow down twice at the proper transforms of G_i, G_i' to points Q_1, Q_2, to obtain $\epsilon\colon \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2 \to \mathbb{P}^2$. This allows us to obtain the usual birational map $f \xrightarrow{\phi} \mathbb{P}^2$. We have $Y := \operatorname{Bl}_Q f = \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2$. The ramification formula reads

$$K_Y = \pi^* K_f + E = \epsilon^* K_{\mathbb{P}^2} + G_1' + G_2'$$

where the E, G_i' are the corresponding exceptional divisors.

Notice that for the anticanonical divisor, $-K_f = -K_{\mathbb{P}^3} + f|_f = \mathcal{O}_f(2)$, by the adjunction formula, which in turn implies that $-K_f \cdot C_i = 4$. Notice also that for the proper transforms of π of the C_i, denoted by C_i' we have $C_i' = \pi^*(C_i)$, $G_i' = \pi^*(G_i) - E$, and for the proper transforms of the C_i', denoted by $\overline{C}_i$, of ϵ we have $\epsilon^*(\overline{C}_i) = C_i' + G_1' + G_2'$, and $\overline{C_1} \cdot \overline{C_2} = 4$. Hence,

$$\begin{aligned}
-K_{\mathbb{P}^2} \cdot \overline{C}_i =& \epsilon^*(-K_{\mathbb{P}^2}) \cdot \epsilon^*(\overline{C}_i) \\
=& - K_Y \cdot C_i' + 2 \\
=& \pi^*(K_f) \cdot \pi^*(C_i) + 2 \\
=& - K_f \cdot C_i + 2 \\
=& 6
\end{aligned}$$

This implies that the proper transforms $\overline{C}_i$ are also conics which intersect tangentially at P and normally at Q_1, Q_2. We take f, g conics in $\mathbb{P}^2$ such that $q \cap r = 2p + r + s$. Two such conics can be given by $q = x_0 l x_1 + x_1^2 + x_2^2$, $r = x_0 x_2 + x_1^2 + x_2^2$ with double point of intersection at $p = (1 : 0 : 0)$. Localising at p, $f|_{\text{loc}} = x_1 + x_1^2 + x_2^2$, $g|_{\text{loc}} = x_2 + x_1^2 + x_2^2$, and hence p is an $\mathbf{A}_3$ singularity. Since the morphism ϕ is a locally analytic homomorphism around p, the singularity is the same in P in f.

□

Lemma 5.2. *Let S be complete intersection with Segre symbol $[(3,1)]$. Then S has 1 $\mathbf{D}_4$ singularity.*

Proof. Since the Segre symbol of $S = \{f = 0\} \cap \{g = 0\}$ is $[(3,1)]$, S is a conic and two lines intersecting at one point P. We may assume, without loss of generality that f is smooth and g is singular.

Let C_1 be the conic, and $C_2 = L_1 + L_2$ the two lines; the C_i are $(1,1)$ divisors in $f = \mathbb{P}^1 \times \mathbb{P}^1$ and $C_1 \cdot C_2 = 2$. As in the proof of Lemma 5.1, we blow up f at point P', where $P' \notin C_i$, $\pi\colon \operatorname{Bl}_{P'} f \to f$ and then blow down twice at the proper transforms, to points Q_1, Q_2, $\epsilon\colon \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2 \to \mathbb{P}^2$ to obtain the usual birational map $f \xrightarrow{\phi} \mathbb{P}^2$. As in the proof of Lemma 5.1, the strict transform $\overline{C}_1$ of C_1 is a conic and the strict transform of C_2, $\overline{C}_2$, is two lines. $\overline{C}_1$ and $\overline{C}_2 = \overline{L}_1 + \overline{L}_2$ meet at Q_1, Q_2 and P normally, where $P = \overline{L}_1 \cap \overline{L}_2 \cap \overline{C}_1$. Hence, P is a $\mathbf{D}_4$ singularity. Since the morphism ϕ is a locally analytic homomorphism around P, the singularity is the same in f. $\square$

Lemma 5.3. *Let S be complete intersection with Segre symbol $[(2,2)]$. Then S has non-isolated singularities.*

Proof. Let S be a complete intersection with Segre symbol $[(2,2)]$. Then S can be given, up to projective equivalence, by $S = \{f = 0\} \cap \{g = 0\}$ where

$$\begin{aligned} f &= q_1(x_2,x_3) + x_0x_3 + x_1l_1(x_2,x_3) \\ g &= q_2(x_2,x_3). \end{aligned}$$

Then, the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f}{\partial x_i} \\ \frac{\partial q}{\partial x_i} \end{pmatrix}$$

is given by

$$J = \begin{pmatrix} x_3 & ax_2 + x_3 & 2a_0x_2 + a_1x_3 & 2a_2x_3 + a_1x_2 + x_0 + bx_1 \\ 0 & 0 & 2a_0x_2 + a_1x_3 & 2b_2x_3 + b_1x_2 \end{pmatrix}$$

and for any point $P = (p_0 : p_1 : 0 : 0) \in S$ we have

$$J_P = \begin{pmatrix} 0 & 0 & 0 & p_0 + bp_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\operatorname{rank}(J_p) = 1 < 2$. Hence, P is singular, and since every point in S can be written in the form of P, S has only non-isolated singularities. $\square$

Lemma 5.4. *Let S be complete intersection with Segre symbol $[(1,1),1,1]$. Then S has 2 $\mathbf{A}_1$ singularities.*

Proof. Since the Segre symbol of $S = f \cap g$ is $[(1,1),1,1]$, S is two conics C_1, C_2 in general position, intersecting at points P, Q, $C_1 \cdot C_2 = 2$. Without loss of generality, we assume that f is smooth.

As in the proof of Lemma 5.1, we blow up f at P', such that $P' \notin C_i$, $\pi\colon \operatorname{Bl}_{P'} f \to f$ and then blow down twice at the proper transforms, to points Q_1, Q_2, $\epsilon\colon \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2 \to \mathbb{P}^2$ to obtain the usual birational map $f \xrightarrow{\phi} \mathbb{P}^2$. We have $Y := \operatorname{Bl}_P f = \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2$. The strict transforms $\overline{C}_i$ of C_i are conics intersecting at 4 points, namely P, Q, and the points of the blow down Q_1, Q_2. The singularities at P and Q are $\mathbf{A}_1$ singularities, and since the morphism ϕ is a locally analytic homomorphism around P, the singularity is the same in f. □

Lemma 5.5. *Let S be complete intersection with Segre symbol $[2,(1,1)]$. Then S has 2 $\mathbf{A}_1$ singularities.*

Proof. Since the Segre symbol of $S = f \cap g$ is $[2,(1,1)]$, S is a conic C_1 and two lines $C_2 = L_1 + L_2$ in a triangle, intersecting at points P, Q, $C_1 \cdot C_2 = 2$. Without loss of generality, we assume that f is smooth.

As in the proof of Lemma 5.1, we blow up f at P', such that $P' \notin C_i$, $\pi\colon \operatorname{Bl}_{P'} f \to f$ and then blow down twice at the proper transforms, to points Q_1, Q_2, $\epsilon\colon \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2 \to \mathbb{P}^2$ to obtain the usual birational map $f \xrightarrow{\phi} \mathbb{P}^2$. We have $Y := \operatorname{Bl}_P f = \operatorname{Bl}_{Q_1,Q_2} \mathbb{P}^2$. The strict transforms $\overline{C}_i$ of C_i are conics intersecting at 4 points, namely P, Q, and the points of the blow down Q_1, Q_2. The singularities at P and Q are $\mathbf{A}_1$ singularities, and since the morphism ϕ is a locally analytic homomorphism around P, the singularity is the same in f. □

Remark 5.5.1. In order to check the type of (isolated) hypersurface singularities, one can employ the following MAGMA script, adjusted accordingly for each different case.

```
Q:=RationalField();
PP<x0,x1,x2,x3>:=ProjectiveSpace(Q,3);
f1:=x0*x1;
f2:=x2*x3;
X:=Scheme(PP,[f1,f2]);
IsNonsingular(X);
p := X![1,0,0,0];
_,f,_,fdat := IsHypersurfaceSingularity(p,2);
R<a,b> := Parent(f);
f;
NormalFormOfHypersurfaceSingularity(f);
boo,f0,typ := NormalFormOfHypersurfaceSingularity(f :
```

```
fData := [*fdat,3*]);
boo; f0; typ;
```

Here, $f1$ and $f2$ are the generating polynomials, and

```
IsNonsingular(X);
```

verifies that $X = f_1 \cap f_2$ is singular. The point $p = [1, 0, 0, 0]$ refers to a specific singular point, whose type of singularity we want to check, which is given by the last command. Note, that this code also works for higher dimensional complete intersections.

Lemma 5.6. *Let $\Phi(f, g)$ be a pencil of two quadrics $f, g \in \mathbb{P}^3$, where we assume that f is smooth. The base locus of the pencil has only $\mathbf{A}_1$ singularities if and only if the determinant polynomial $\det(\lambda f + g)$ has roots of multiplicity 2.*

Proof. Assume that the base locus S has only $\mathbf{A}_1$ singularities. Then, by Table 5.1, Theorem 4.3 and Lemmas 5.1, 5.2, 5.4, and 5.5 that can only occur if S is either

1. a nodal curve, with Segre symbol $[2, 1, 1]$, with one $\mathbf{A}_1$ singularity,
2. or a twisted cubic and a bisecant, with two $\mathbf{A}_1$ singularities and Segre symbol $[2, 2]$,
3. or two conics in general position with Segre symbol $[(1, 1), 1, 1]$, with two $\mathbf{A}_1$ singularities,
4. or a conic and two lines in a triangle, with Segre symbol $[2, (1, 1)]$, with two $\mathbf{A}_1$ singularities,
5. or a quadrangle, with Segre symbol $[(1, 1), (1, 1)]$ with four $\mathbf{A}_1$ singularities.

Notice that all of the above cases have determinant polynomial with a root of multiplicity 2. □

Given the analysis in this section, we present the following table, which lists the possible complete intersections of two quadrics in $\mathbb{P}^3$, up to projective equivalence, and their singularities. The list of polynomials can be also found in [Som59, §XIII] (up to projective equivalence). One can also generate the polynomials f, g via the Segre symbols.

Segre symbol	Generating polynomials	Singularities
$[4]$	$f = q_1(x_2, x_3) + x_3 l_1(x_0, x_1) + x_2 l_2(x_0, x_1)$ $g = q_2(x_2, x_3) + x_3 l_3(x_0, x_1) + x_2 l_4(x_0, x_1)$	$\mathbf{A}_3$
$[(3, 1)]$	$f = q_1(x_1, x_2, x_3) + x_0 x_3$ $g = x_3 l_1(x_1, x_2, x_3)$	$\mathbf{D}_4$
$[(2, 2)]$	$f = q_1(x_2, x_3) + x_0 x_3 + x_1 l_1(x_2, x_3)$ $g = q_2(x_2, x_3)$	non-isolated
$[3, 1]$	$f = q_1(x_1, x_2, x_3) + x_0 x_3$ $g = q_2(x_2, x_3) + x_1 x_3$	$\mathbf{A}_2$
$[(2, 1), 1]$	$f = q_1(x_1, x_2, x_3) + x_0 l_1(x_1, x_2, x_3) + x_1 l_1(x_2, x_3)$ $g = q_2(x_2, x_3) + x_0 x_3 + x_1 l_2(x_2, x_3)$	$\mathbf{A}_3$
$[(1, 1, 1), 1]$	$f = q_1(x_0, x_1, x_2, x_3)$ $g = x_3^2$	non-isolated
$[2, 2]$	$f = q_1(x_2, x_3) + x_0 l_1(x_2, x_3) + x_1 l_2(x_2, x_3)$ $g = q_2(x_2, x_3) + x_0 l_3(x_2, x_3) + x_1 l_4(x_2, x_3)$	$2\mathbf{A}_1$
$[2, (1, 1)]$	$f = q_1(x_2, x_3) + x_0 x_3$ $g = x_3 l_3(x_0, x_1, x_2, x_3)$	$2\mathbf{A}_1$
$[(1, 1), (1, 1)]$	$f = q_1(x_2, x_3)$ $g = q_2(x_0, x_1)$	$4\mathbf{A}_1$

$[2, 1, 1]$	$f = q_1(x_1, x_2, x_3) + x_0x_3$ $g = q_2(x_1, x_2, x_3)$	$\mathbf{A}_1$
$[(1, 1), 1, 1]$	$f = q_1(x_0, x_1, x_2, x_3)$ $g = q_2(x_2, x_3)$	$2\mathbf{A}_1$
$[1, 1, 1, 1]$	$f = q_1(x_0, x_1, x_2, x_3)$ $g = q_2(x_0, x_1, x_2, x_3)$	None

Table 5.2: Segre symbols, equations up to projective equivalence, and singularities of complete intersections of quadrics in $\mathbb{P}^3$.

Remark 5.6.1. Note, that for all singular complete intersections, by Theorem 4.3, and since any two sets of 3 points in $\mathbb{P}^1$ are isomorphic, the above description is unique, since any two singular complete intersections of quadrics in $\mathbb{P}^3$ with the same Segre symbol are progressively equivalent.

5.2 GIT Classification

In this Section we will study the VGIT quotient $\mathcal{R}_{3,2,2,1} /\!\!/ \mathrm{SL}(4)$. For wall $t = 0$, i.e. in the absence of a hyperplane, this is equivalent to the GIT quotient $\mathcal{R}_{3,2,2} /\!\!/ \mathrm{SL}(4)$. The following families have been generated using the computational package [Pap22c], based on the discussion on Chapter 3. In particular, they are maximal destabilising families, in the sense of Definition 3.15. The computational package gives us that $P_{3,2,2} =$ [(5, 4, -3, -6), (25, 9, -15, -19), (7, 3, -1, -9), (19, 15, -9, -25), (11, 3, -1, -13), (7, 3, -5, -5), (3, 0, -1, -2), (9, 3, -5, -7), (13, 1, -3, -11), (9, 7, -5, -11), (13, 1, -5, -9), (15, 3, -5, -13), (9, 5, -1, -13), (5, 2, -3, -4), (19, 3, -5, -17), (11, 5, -7, -9), (3, 1, 0, -4), (5, 1, -3, -3), (5, 1, -2, -4), (15, 11, -1, -25), (9, -1, -3, -5), (19, -1, -5, -13), (11, 7, -1, -17), (4, 1, -2, -3), (11, 3, -5, -9), (4, 2, -1, -5), (5, 3, 1, -9), (11, 7, -5, -13), (5, 0, -1, -4), (25, 1, -11, -15), (6, 3, -4, -5), (2, 0, -1, -1), (23, 3, -5, -21), (15, 1, -7, -9), (1, 1, -1, -1), (7, 5, 1, -13), (13, 5, -7, -11), (15, 7, 3, -25), (13, 5, 1, -19), (9, 5, 1, -15), (1, 1, 0, -2), (13, 9, -7, -15), (5, 5, -3, -7), (7, -1, -2, -4), (17, 9, -7, -19), (13, 9, 1, -23), (25, -3, -7, -15), (5, 1, -1, -5), (3, 2, 0, -5), (2, 1, 0, -3), (7, 1, -3, -5), (9, 1, -3, -7), (3, 1, -1, -3), (15, 7, -9, -13), (7, 3, 1, -11), (17, 5, -3, -19), (21, 5, -3, -23), (9, 5, -3, -11), (9, 7, -1, -15), (23, -1, -9, -13), (15, 11, -9, -17), (5, 3, -3, -5), (15, -1, -5, -9), (1, 1, 1, -3), (1, 0, 0, -1), (5, 0, -2, -3), (17, 1, -7, -11), (4, 3, -2, -5), (11, -1, -3, -7), (13, -1, -5, -7), (3, 3, -1, -5), (3, -1, -1, -1), (5, -1, -1, -3), (7, 5, -3, -9), (11, 5, -3, -13), (11, 1, -5, -7), (7, -1, -1, -5), (19, 7, -9, -17), (4, 0, -1, -3), (13, 5, -3, -15), (5, 1, 1, -7), (7, 5, -1, -11), (17, 9, -11, -15), (4, 2, 1, -7), (3, 2, -1, -4), (2, 1, -1, -2), (3, 1, 1, -5), (4, 1, 0, -5), (13, 3, -5, -11), (5, 3, -1, -7)]. In addition, we get:

λ	x^J	$V^-(\lambda, x^J)$	$B^-(\lambda, x^J)$
$(1,1,-1,-1)$	x_2^2	$\{x_0,x_1,x_2,x_3\}^2$	$\{x_2,x_3\}^2$
$(1,1,-1,-1)$	x_0x_2	$\{x_2,x_3\}^2$, $\{x_0,x_1\}\{x_2,x_3\}$	$\{x_2,x_3\}^2$, $\{x_0,x_1\}\{x_2,x_3\}$
$(1,1,1,-3)$	x_0x_3	$\{x_0,x_1,x_2,x_3\}^2$	$x_3\{x_0,x_1,x_2,x_3\}$
$(1,0,0,-1)$	x_0x_3	$\{x_1,x_2,x_3\}^2$, x_0x_3	$\{x_1,x_2,x_3\}^2$, x_0x_3
$(3,-1,-1,-1)$	x_1^2	$\{x_1,x_2,x_3\}^2$, $x_0\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}^2$

Table 5.3: Outputs of the computational package [Pap22c] for destabilized families of complete intersections of two quadrics in $\mathbb{P}^3$

λ	x^J	$V^\ominus(\lambda, x^J)$	$B^\ominus(\lambda, x^J)$
$(5,4,-3,-6)$	x_3^2	$\{x_0,x_1,x_2,x_3\}^2$	x_3^2
$(5,4,-3,-6)$	x_0x_3	$\{x_2,x_3\}^2$, $x_3\{x_0,x_1\}$	$\{x_2,x_3\}^2$, $x_3\{x_0,x_1\}$
$(25,9,-15,-19)$	x_2^2	$\{x_1,x_2,x_3\}^2$, $x_0\{x_2,x_3\}$	$\{x_2,x_3\}^2$
$(3,0,-1,-2)$	x_1x_3	$\{x_1,x_2,x_3\}^2$, x_0x_3	$\{x_2,x_3\}^2$, x_1x_3
$(9,-1,-3,-5)$	x_1^2	$\{x_1,x_2,x_3\}^2$	$\{x_1,x_2,x_3\}^2$.

Table 5.4: Outputs of the computational package [Pap22c] for semi-destabilized families of complete intersections of two quadrics in $\mathbb{P}^3$

λ	x^J	$V^0(\lambda, x^J)$	$B^0(\lambda, x^J)$
$(1,1,-1,-1)$	x_2^2	$\{x_0, x_1\}^2$	$\{x_2, x_3\}^2$
$(1,1,-1,-1)$	x_1x_3	$\{x_0, x_1\}\{x_2, x_3\}$	$\{x_0, x_1\}\{x_2, x_3\}$
$(1,1,1,-3)$	x_0x_3	$\{x_0, x_1, x_2\}^2$	$x_3\{x_0, x_1, x_2\}^2$
$(1,0,0,-1)$	x_0x_3	$\{x_1, x_2\}^2$, x_0x_3	$\{x_1, x_2\}^2$, x_0x_3
$(3,-1,-1,-1)$	x_1^2	$x_0\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}^2$.

Table 5.5: Outputs of the computational package [Pap22c] for potentially closed orbits of complete intersections of two quadrics in $\mathbb{P}^3$

Theorem 5.7. *The following are equivalent:*

1. *A pencil of two quadrics $\Phi(f, g)$ in $\mathbb{P}^3$ is unstable;*
2. *the base locus of the pencil $\mathrm{Bs}(f, g)$, has singularities worse than $\mathbf{A}_1$;*
3. *the pencil is generated by one of the following families, or a degeneration:*

   Family 1:

$$f(x_0, x_1, x_2, x_3) = q_0(x_1, x_2, x_3) + x_0 l_1(x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = q_1(x_2, x_3)$$

   an irreducible smooth quadric f and two intersecting planes g intersecting at two conic curves tangent at one point $(1:0:0:0)$, with isolated $\mathbf{A}_3$ singularity at $(1:0:0,0)$;

   Family 2:

$$f(x_0, x_1, x_2, x_3) = q_2(x_1, x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = q_3(x_1, x_2, x_3)$$

   two irreducible singular quadrics f, g such that $\mathrm{Bs}(f, g)$ is 4 intersecting lines at a point $P = (1:0:0:0)$, with non-hypersurface singularity at P;

   Family 3:

$$f(x_0, x_1, x_2, x_3) = q_4(x_0, x_1, x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = x_3^2$$

   an irreducible smooth quadric f and a double plane g intersecting at a double conic, with non isolated singularities at $(1:k:l:0)$ where k, l are the solutions of $q_4(1, x_1, x_2, 0) = 0$;

Family 4*:*

$$\begin{aligned}f(x_0,x_1,x_2,x_3) =& x_3x_0 + q_5(x_1,x_2,x_3)\\ g(x_0,x_1,x_2,x_3) =& q_6(x_2,x_3) + x_1x_3\end{aligned}$$

an irreducible smooth quadric f and a two intersecting planes g intersecting at a conic and two lines tangent at a point $P = (1:0:0:0)$, with isolated $\mathbf{A}_2$ singularity at $(1:0:0:0)$;

Family 5*:*

$$\begin{aligned}f(x_0,x_1,x_2,x_3) =& q_5(x_2,x_3) + x_3l_3(x_0,x_1)\\ g(x_0,x_1,x_2,x_3) =& q_6(x_2,x_3) + x_3l_4(x_0,x_1)\end{aligned}$$

two irreducible singular quadrics f, g intersecting at two double lines, with non-isolated singularities at $(1:\lambda:0:0)$.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, which are maximal in their support.

Proof. The equivalence of 1 and 3 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal semi-destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials, from Table 5.4.

For Family 1, the Segre symbol of the pencil is $[(2,1),1]$ so by the classification of pencils of quadrics in $\mathbb{P}^3$, the base locus is two conics tangent at a point $P = (1:0:0:0)$, which is an $\mathbf{A}_3$ singularity by Lemma 5.1.

For Family 2, f, g are two cones with common vertex $P = (1:0:0:0)$. We write the equations f, g in normal form: $f = x_1^2 + x_2^2 + x_3^2$, $g = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2$, where the λ_i are distinct. Solving for x_1, we have $x_1 = \pm\sqrt{-x_2^2 - x_3^2}$, hence $x_2^2(\lambda_2-\lambda_1) + x_3^2(\lambda_3-\lambda_1) = 0$. This in turn implies that $x_2 = \pm Ax_3$, with $A = \sqrt{\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_1}}$, and $x_1 = \pm Bx_3$, with $B = \sqrt{A^2-1}$. Hence, the base locus consists of 4 lines intersecting at $P = (1:0:0:0)$, the common vertex of the two cones. The MAGMA analysis for singularities seen in Remark 5.5.1 shows that P is not a hypersurface singularity.

For Family 3, notice that the base locus of the intersection is a conic $q_4(x_0,x_2,x_3,0)$, which is in fact a double conic. Furthermore, using Lemma 4.1 the intersection is singular for all points on the conic, hence there are no isolated singularities.

For Family 4, the Segre symbol of the pencil is $[3,1]$, hence due to the classification, the base locus is a cuspidal curve with $\mathbf{A}_2$ singularity at point $P = (1:0:0:0)$.

For Family 5, f, g are two cones with different vertices. The points $(1 : \lambda : 0 : 0)$ are singular points of the intersection, and they are non-isolated singularities.

Notice also that a particular degeneration of Family 1 with $f = q_1(x_1, x_2, x_3) + \alpha x_0 x_3$, $g = ax_2^2 + bx_3^2$, has a singular point at $P = (1 : 0 : 0 : 0)$. Furthermore, it has Segre symbol $[3, 1]$, and using the classification its base locus is a cuspidal curve, hence P is an $\mathbf{A}_2$ singularity.

To conclude the proof, note that by Lemma 5.6 the base locus of the pencil has $\mathbf{A}_1$ singularities if and only if the determinant polynomial $\det(\lambda f + g)$ has roots of multiplicity 2. The above Families and their degenerations represent all possible pencils of quadrics where the determinant polynomial has roots of multiplicity > 2, as families 1, 3 and 4 give the only possible complete intersections where the determinant polynomial has a root of multiplicity 3, and the rest of the families are either non-normal or do not have du Val singularities. Hence, 2 and 3 are equivalent.

□

Theorem 5.8. *The following are equivalent:*

1. *A pencil of two quadrics* $\Phi(f, g)$ *in* $\mathbb{P}^3$ *is non-stable;*

2. *the base locus of the pencil* $\mathrm{Bs}(f, g)$*, is singular;*

3. *the pencil is generated by one of the following families, or a degeneration:*

   Family 1*:*

$$f(x_0, x_1, x_2, x_3) = x_3 l_1(x_0, x_1, x_2, x_3) + x_2 l_2(x_0, x_1, x_2) + x_1 l_3(x_0, x_1)$$
$$g(x_0, x_1, x_2, x_3) = q_1(x_1, x_2, x_3)$$

   an irreducible smooth quadric f *and an irreducible singular quadric* g *intersecting at a nodal curve, with* $\mathbf{A}_1$ *singularity at* $(1 : 0 : 0, 0)$*;*

   Family 2*:*

$$f(x_0, x_1, x_2, x_3) = q_2(x_0, x_1, x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = q_3(x_2, x_3)$$

   an irreducible smooth quadric f *and a pair of intersecting planes* g *such that* $\mathrm{Bs}(f, g)$ *is a pair of conics in general position, with* $\mathbf{A}_1$ *singularities (up to a change of coordinates) at* $(1 : i : 0 : 0)$*,* $(1 : -i : 0 : 0)$*;*

Family 3*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3) =& x_3 l_4(x_0,x_1,x_2,x_3) + x_2 l_5(x_0,x_1,x_2)\\
g(x_0,x_1,x_2,x_3) =& x_3 l_6(x_0,x_1,x_2,x_3) + x_2 l_7(x_0,x_1,x_2)
\end{aligned}
$$

two irreducible non-singular quadrics f, g *intersecting at a twisted cubic and a bisecant, with* $\mathbf{A}_1$*-singularities at* $(1:0:0:0)$ *and* $(0:1:0:0)$*;*

Family 4*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3) =& x_3 l_8(x_0,x_1,x_2,x_3) + q_4(x_1,x_2)\\
g(x_0,x_1,x_2,x_3) =& x_3 l_9(x_0,x_1,x_2,x_3) + q_5(x_1,x_2)
\end{aligned}
$$

two irreducible smooth quadrics f, g *intersecting at a nodal curve, with an* $\mathbf{A}_1$*-singularity at* $(1:0:0:0)$*,*

or a degeneration of the above. Here, the l_i *are linear forms in* $\mathbb{P}^3$ *and the* q_i *are quadratic forms.*

In particular, the above 4 *families are strictly semistable, and families* 1 *and* 4 *are projectively equivalent.*

Proof. The equivalence of 1 and 3 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^{-}(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials, from Table 5.3. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3.

For the equivalence of 2 and 3 we have the following analysis. For the first Family, notice that the base locus is singular, with singularity $P = (1:0:0:0)$. Furthermore, the Segre symbol of the pencil is $[2,1,1]$, and by the classification, the base locus is a nodal curve, hence P is an $\mathbf{A}_1$ singularity.

For Family 2, notice that the Segre symbol of the pencil is $[(1,1),1,1]$ and hence that the base locus is two conics C_1, C_2 in general position, intersecting at points P, Q, with $\mathbf{A}_1$ singularities by Table 5.1.

For Family 3, the Segre symbol of the pencil is $[2,2]$. By the classification, the base locus of the pencil is a twisted cubic and a bisecant intersecting the cubic at $P = (1:0:0:0)$ and $Q = (0:1:0:0))$ which are $\mathbf{A}_1$ singularities.

For Family 4, the base locus is singular, with singularity $P = (1 : 0 : 0 : 0)$. Furthermore, the Segre symbol of the pencil is $[2, 1, 1]$, and by the classification, the base locus is a nodal curve, hence P is an $\mathbf{A}_1$ singularity. In particular, since families 1 and 4 have the same Segre symbol, they are projectively equivalent by Theorem 4.3.

In addition, notice that a degeneration of family 3 is

$$f(x_0, x_1, x_2, x_3) = x_0x_3$$
$$g(x_0, x_1, x_2, x_3) = x_1x_2$$

which is the quadrangle with Segre symbol $[(1, 1),(1, 1)]$ and 4 $\mathbf{A}_1$ singularities.

In addition, notice that a degeneration of family 4 is

$$f(x_0, x_1, x_2, x_3) = q(x_2, x_3) + x_0x_3$$
$$g(x_0, x_1, x_2, x_3) = x_3l(x_0, x_1, x_2, x_3)$$

which is the complete intersection with Segre symbol $[2,(1, 1)]$ and 2 $\mathbf{A}_1$ singularities.

Hence, from the above discussion, and from Lemma 4.1 we notice that if a pencil has singular base locus, then it belongs to one of the above Families (or its degenerations).

$\square$

Corollary 5.8.1. *A pencil $\Phi(f, g)$ of quadrics in $\mathbb{P}^3$ is stable if and only if it is smooth.*

Proof. Let Φ be a pencil which has smooth base locus. By Theorem 5.8 it is not non-stable, i.e. it is stable. $\square$

Theorem 5.9.

A pencil of two quadrics $\Phi(f, g)$ in $\mathbb{P}^3$ is strictly polystable if and only if it is generated by:

$$f(x_0, x_1, x_2, x_3) = q_2(x_0, x_1)$$
$$g(x_0, x_1, x_2, x_3) = q_3(x_2, x_3)$$

two reducible singular quadrics f, g intersecting at a quadrangle, with $\mathbf{A}_1$-singularities at (up to change of basis) $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$,

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, which are maximal in their support.

Proof. First notice that by Lemma 4.1 the above families are singular, and hence by Theorem 5.8 they are non-stable. The above families have all determinant polynomials with roots of multiplicity 2, hence by Lemma 5.6 they have $\mathbf{A}_1$ singularities. This, alongside with Theorem 5.7 implies that the above families are semistable.

In more detail, the above family has Segre symbol $[(1,1),(1,1)]$ and hence its base locus is a quadrangle, with 4 $\mathbf{A}_1$ singularities.

Note that $f \wedge g = x_0x_1 \wedge x_2x_3$; for any one parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(4)$, with $\lambda(s) = \mathrm{Diag}(s^{a_0}, \dots, s^{a_3})$, with $\sum a_i = 0$, we have $\lim_{s\to 0} \lambda(s) \cdot f \wedge g = s^{\sum a_i} x_0x_1 \wedge x_2x_3 = x_0x_1 \wedge x_2x_3 = f \wedge g$. Hence, $\dim \mathrm{Stab}(f \wedge g) = \dim(\mathrm{SL}(4))$ is maximal and thus the orbit of $f \wedge g$ is closed, i.e. the pencil is polystable in both cases.

By Theorem 3.19 a complete intersection S, defined by $S = \{f = g = 0\}$, that belongs to a closed strictly semistable orbit is generated by monomials in the set $N^0(\lambda, x^{J_1})$, for some (λ, x^{J_1}). The above family corresponds to the only such $N^0(\lambda, x^{J_1})$ (up to projective equivalence), presented in Table 5.5. In particular, these are obtained by verifying which $N^-(\lambda, x^J)$ give strictly semistable families, for various support monomials x^J, and then computing $N^0(\lambda, x^{J_1})$ by the description in Lemma 3.18. □

5.3 Classifying the Singularities of Pairs $(S, D = S \cap H)$

Following the discussion of Chapter 3, we take $S = C_1 \cap C_2$, and $D = S \cap H$, where C_i are quadrics in $\mathbb{P}^3$ and H is a hyperplane. The lemmas below serve as to help with the geometric classification of such pairs, based on the singularities of S and D. Notice, that D will be a complete intersection of two quadrics in $H \cong \mathbb{P}^2$, which has been analysed in Chapter 4.

Lemma 5.10. *Let S be a smooth complete intersection of two quadrics and H a general hyperplane. Then D can have at worse a quadruple point.*

Proof. Let S be the complete intersection of f and g and H a hyperplane, where

$$\begin{aligned} f(x_0,x_1,x_2,x_3) &= q_1(x_1,x_2,x_3) + x_0 l_1(x_2,x_3) \\ g(x_0,x_1,x_2,x_3) &= q_2(x_2,x_3) + x_3 l_2(x_0,x_1) \\ H(x_0,x_1,x_2,x_3) &= x_3 \end{aligned}$$

Then $S = \{f = 0\} \cap \{g = 0\}$ is smooth and D is generated by equations

$$\begin{aligned} f'(x_0,x_1,x_2) =& q_3(x_1,x_2) + x_0x_2 \\ g'(x_0,x_1,x_2) =& x_2^2. \end{aligned}$$

The only point on the intersection is the point $(1:0:0)$ which is a quadruple point.

□

Remark 5.10.1. In fact, if S is smooth, D can also have a triple point, two or one double points, and it can be smooth. To see this, let $S = \{f=0\} \cap \{g=0\}$, $D = S \cap H$ where

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_1,x_2,x_3) + x_0l_1(x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_2,x_3) + x_1l_2(x_2,x_3) + x_0x_3 \\ H(x_0,x_1,x_2,x_3) =& x_3. \end{aligned}$$

Then, S is a smooth complete intersection (it has Segre symbol $[1,1,1,1]$), but D is given by

$$\begin{aligned} f(x_0,x_1,x_2) =& q_1(x_1,x_2) + x_0x_2 \\ g(x_0,x_1,x_2) =& x_2^2 + x_1x_2 \end{aligned}$$

which has a triple point at $(1:0:0)$.

Similarly, if S and H are given by

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_0,x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_2,x_3) + x_3l_1(x_0,x_1) \\ H(x_0,x_1,x_2,x_3) =& x_3 \end{aligned}$$

then, S is a smooth complete intersection (it has Segre symbol $[1,1,1,1]$), but D is given by

$$\begin{aligned} f(x_0,x_1,x_2) =& q_1(x_0,x_1,x_2) \\ g(x_0,x_1,x_2,x_3) =& x_2^2 \end{aligned}$$

which has two double points (up to projective equivalence) at $(1:0:0)$, $(0:1:0)$. Finally, if S and H are given by

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_1,x_2,x_3) + x_0l_1(x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_1,x_2,x_3) + x_0x_3 \\ H(x_0,x_1,x_2,x_3) =& x_3, \end{aligned}$$

then, S is a smooth complete intersection (it has Segre symbol $[1, 1, 1, 1]$), but D is given by

$$\begin{aligned} f(x_0, x_1, x_2) =& q_1(x_1, x_2) + x_0 l_1(x_1, x_2) \\ g(x_0, x_1, x_2) =& q_2(x_1, x_2) \end{aligned}$$

which has one double point (up to projective equivalence) at $(1 : 0 : 0)$.

Lemma 5.11. *Let S be the complete intersection of two quadrics f, g which is given by two conics in general position. Then S has 2 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has up to an* $\mathrm{SL}(4)$*-action:*

1. *one double point at one of the $\mathbf{A}_1$ singularities if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*

2. *two double points at the $\mathbf{A}_1$ singularities if and only if $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$\begin{aligned} f &= q_1(x_0, x_1, x_2, x_3) \\ g &= q_2(x_2, x_3) \end{aligned}$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$\begin{aligned} f &= q_1(x_0, x_1, x_2, l(x_1, x_2, x_3)) \\ g &= q_2(x_2, x_3, l(x_1, x_2, x_3)) \\ H &= x_3 \end{aligned}$$

and hence D will be given by:

$$\begin{aligned} f' &= q_1(x_0, x_1, x_2) \\ g' &= q_2(x_1, x_2) \end{aligned}$$

which is a singular complete intersection of conics with a double point (up to $\mathrm{SL}(3)$-action) at $(1 : 0 : 0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$\begin{aligned} f &= q_1(x_0, x_1, x_2, x_3) \\ g &= q_2(x_2, x_3) \\ H &= x_3 \end{aligned}$$

and hence D will be given by:

$$f' = q_1(x_0, x_1, x_2)$$
$$g' = x_2^2$$

which is a singular complete intersection of conics with two double points (up to $\mathrm{SL}(3)$-action) $(1:0:0)$ and $(0:1:0)$. $\square$

Lemma 5.12. *Let S be the complete intersection of two quadrics f, g which is a nodal curve. Then S has one $\mathbf{A}_1$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has up to an $\mathrm{SL}(4)$-action:*

1. *one double point at the $\mathbf{A}_1$ singularity if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
2. *a triple point at the $\mathbf{A}_1$ singularity if and only if $H = \{l(x_2, x_3) = 0\}$;*
3. *a quadruple point at the $\mathbf{A}_1$ singularity if and only if $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_1, x_2, x_3) + x_0 x_3$$
$$g = q_2(x_1, x_2, x_3)$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l(x_1, x_2, x_3)$$
$$g = q_2(x_1, x_2, l(x_1, x_2, x_3))$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2) + x_0 l(x_1, x_2)$$
$$g' = q_2(x_1, x_2)$$

which is a singular complete intersection of conics with a double point (up to $\mathrm{SL}(3)$-action) $(1:0:0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$\begin{aligned} f &= q_1(x_1, x_2, x_3) + x_0 l(x_2, x_3) \\ g &= q_2(x_1, x_2, l(x_2, x_3)) \\ H &= x_3 \end{aligned}$$

and hence D will be given by:

$$\begin{aligned} f' &= q_1(x_1, x_2) + x_0 x_2 \\ g' &= q_2(x_1, x_2) \end{aligned}$$

which is a singular complete intersection of conics with a triple point (up to $\mathrm{SL}(3)$-action) at $(1 : 0 : 0)$.

Now, let $H = \{x_3 = 0\}$; then D will be given by:

$$\begin{aligned} f' &= q_1(x_1, x_2) \\ g' &= q_2(x_1, x_2) \end{aligned}$$

which is a singular complete intersection of conics with a quadruple point (up to $\mathrm{SL}(3)$-action) at $(1 : 0 : 0)$.

$\square$

Lemma 5.13. *Let S be the complete intersection of two quadrics f, g which is a quadrangle. Then S has 4 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has up to an* $\mathrm{SL}(4)$*-action:*

1. *one double point at one of the $\mathbf{A}_1$ singularity if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
2. *two double points at two of the $\mathbf{A}_1$ singularities if and only if $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$\begin{aligned} f &= q_1(x_2, x_3) \\ g &= q_2(x_0, x_1) \end{aligned}$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3)$$
$$g = q_2(x_0, x_1)$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2)$$
$$g' = q_2(x_0, x_1)$$

which is a singular complete intersection of conics with one double point.

Similarly, let $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$; using a similar suitable change of coordinates we have in both cases (by abuse of notation):

$$f = q_1(x_2, x_3)$$
$$g = q_2(x_0, x_1)$$
$$H = x_3$$

and hence D will be given by:

$$f' = x_2^2$$
$$g' = q_2(x_0, x_1)$$

which is a singular complete intersection of conics with a two double points (up to $\mathrm{SL}(3)$-action) at $(1:0:0)$ and $(0:1:0)$.

$\square$

Lemma 5.14. *Let S be the complete intersection of two quadrics f, g which is a conic and two lines in triangle. Then S has 2 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *one double point at one of the $\mathbf{A}_1$ singularities if and only if $H = \{l(x_1, x_2, x_3) = 0\}$ or $H = \{l(x_2, x_3) = 0\}$;*

2. *not a complete intersection if and only if $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_1, x_2, x_3) + x_0x_3$$
$$g = x_3l_1(x_0, x_1, x_2, x_3)$$

Now, let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0l(x_1, x_2, x_3)$$
$$g = q_2(x_1, x_2, x_3) + x_0l(x_1, x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2) + x_0l(x_1, x_2)$$
$$g' = q_2(x_1, x_2) + x_0l(x_1, x_2)$$

which is a singular complete intersection of two conics in $\mathbb{P}^2$, with a double point at $(1:0:0)$.

Let $H = \{l(x_2, x_3) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0l(x_2, x_3)$$
$$g = l(x_2, x_3)l_1(x_0, x_1, x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2) + x_0x_2$$
$$g' = x_2l_1(x_0, x_1, x_2)$$

which is a singular complete intersection of conics with a double point (up to $\mathrm{SL}(3)$-action) $(1:0:0)$.

Now, let $H = \{x_3 = 0\}$; then we have:

$$f = q_1(x_1, x_2, x_3) + x_0x_3$$
$$g = x_3l_1(x_1, x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2)$$
$$g' = 0$$

which is a smooth curve, hence not a complete intersection of conics.

□

Lemma 5.15. *Let S be the complete intersection of two quadrics f, g which is a twisted cubic and bisecant. Then S has 2 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *one double point at one of the $\mathbf{A}_1$ singularities if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*

2. *two double $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_2, x_3) + x_0 l_1(x_2, x_3) + x_1 l_2(x_2, x_3)$$
$$g = q_3(x_2, x_3) + x_0 l_3(x_2, x_3) + x_1 l_4(x_2, x_3)$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l(x_1, x_2, x_3)$$
$$g = q_2(x_1, x_2, x_3) + x_0 l(x_1, x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f' = q_1(x_1, x_2) + x_0 l(x_1, x_2)$$
$$g' = q_2(x_1, x_2) + x_0 l(x_1, x_2)$$

which is a singular complete intersection of two conics in $\mathbb{P}^2$, with a double point at $(1 : 0 : 0)$.

Let $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_2, x_3) + x_0 l_1(x_2, x_3) + x_1 l_2(x_2, x_3)$$
$$g = q_3(x_2, x_3) + x_0 l_3(x_2, x_3) + x_1 l_4(x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f' = x_2^2 + ax_0x_2 + bx_1x_2$$
$$g' = x_2^2 + a'x_0x_2 + b'x_1x_2$$

which is a line and two points on the line, hence not a complete intersection of conics.

□

Lemma 5.16. *Let S be the complete intersection of two quadrics f, g which is a double conic. Then S has non-isolated singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *two double points at the non-isolated singularities of S if and only if $H = \{l(x_1, x_2, x_3) = 0\}$ or $H = \{l(x_2, x_3) = 0\}$;*
2. *not a complete intersection if and only if $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_0, x_1, x_2, x_3)$$
$$g = x_3^2$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_0x_1, x_2, l)$$
$$g = l(x_1, x_2, x_3)^2$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_0, x_1, x_2)$$
$$g = l(x_1, x_2)^2$$

which is a singular complete intersection of conics, with two double points.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_0x_1, x_2, l)$$
$$g = l(x_2, x_3)^2$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_0, x_1, x_2)$$
$$g = x_2^2$$

which is a singular complete intersection of conics, with two double points.

Now, let $H = \{x_3 = 0\}$. Then D will be given by:

$$f = q_1(x_0, x_1, x_2)$$
$$g = 0$$

which is a smooth curve, hence not a complete intersection of conics.

□

Lemma 5.17. *Let S be the complete intersection of two quadrics f, g which is two tangent conics. Then S has 1 $\mathbf{A}_3$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *a double point at the $\mathbf{A}_3$ singularity if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
2. *a quadruple point at the $\mathbf{A}_3$ singularity if and only if $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$; two double points if and only if S is given by*

$$f = q(x_0, x_1)$$
$$g = x_2^2 + x_3 l(x_0, x_1)$$

and $H = \{x_3 = 0\}$.

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_1, x_2, x_3) + x_0 l_1(x_2, x_3)$$
$$g = q_2(x_2, x_3)$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, l) + x_0 l_1(x_2, l)$$
$$g = q_2(x_2, l)$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 l(x_1, x_2)$$
$$g = q_2(x_1, x_2)$$

which is a singular complete intersection of conics, with a double point, at $(1 : 0 : 0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l_1(x_2, x_3)$$
$$g = q_2(x_2, x_3)$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 x_2$$
$$g = x_2^2$$

which is a singular complete intersection of conics, with a quadruple point, at $(1 : 0 : 0)$.

Now, let $H = \{x_3 = 0\}$. Then D will be given by:

$$f = q_1(x_1, x_2) + x_0 x_2$$
$$g = x_2^2$$

which is a singular complete intersection of conics, with a quadruple point, at $(1 : 0 : 0)$.

□

Lemma 5.18. *Let S be the complete intersection of two quadrics f, g which is a cuspidal curve. Then S has 1 $\mathbf{A}_2$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *a double point at the* $\mathbf{A}_2$ *singularity if and only if* $H = \{l(x_1, x_2, x_3) = 0\}$ *;*

2. *a quadruple point at the* $\mathbf{A}_2$ *singularity if and only if* $H = \{l(x_2, x_3) = 0\}$ *or* $H = \{x_3 = 0\}$.

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_1, x_2, x_3) + x_0 x_3$$
$$g = q_2(x_2, x_3) + x_1 x_3$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, l) + x_0 l$$
$$g = q_2(x_2, l) + x_1 l$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 l(x_1, x_2)$$
$$g = q_2(x_1, x_2)$$

which is a singular complete intersection of conics, with a double point, at $(1 : 0 : 0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, l) + x_0 l$$
$$g = q_2(x_2, l) + x_1 l$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 x_2$$
$$g = x_2^2$$

which is a singular complete intersection of conics, with a quadruple point, at $(1 : 0 : 0)$.

Now, let $H = \{x_3 = 0\}$. Then D will be given by:

$$f = q_1(x_1, x_2) + x_0 x_2$$
$$g = x_2^2$$

which is a singular complete intersection of conics, with a quadruple point, at $(1:0:0)$.

□

Lemma 5.19. *Let S be the complete intersection of two quadrics f, g which is a double line and two lines in general position. Then S has non-isolated singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *a double point if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
2. *not a complete intersection if and only if $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_2, x_3) + x_0 x_3 + x_1 l_1(x_2, x_3)$$
$$g = q_2(x_2, x_3)$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_2, l) + x_0 l + x_1 l_1(x_2, l)$$
$$g = q_2(x_2, l)$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 l(x_1, x_2)$$
$$g = q_2(x_1, x_2)$$

which is a singular complete intersection of conics, with a double point, at $(1:0:0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_2, l) + x_0 l + x_1 l_1(x_2, l)$$
$$g = q_2(x_2, l)$$
$$H = x_3$$

and hence D will be given by:

$$
\begin{aligned}
f &= x_2^2 + ax_1x_2 + bx_0x_2 \\
g &= x_2^2
\end{aligned}
$$

which is the union of two lines, i.e., not a complete intersection of conics.

Now, let $H = \{x_3 = 0\}$; then, D will be given by:

$$
\begin{aligned}
f &= x_2^2 + ax_1x_2 \\
g &= x_2^2 + bx_1x_2
\end{aligned}
$$

which is a line and a point on the line, and hence not a complete intersection.

□

Lemma 5.20. *Let S be the complete intersection of two quadrics f, g which is a conic and two lines intersecting in one point. Then S has 1 $\mathbf{D}_4$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an $\mathrm{SL}(4)$-action:*

1. *a double point at the $\mathbf{D}_4$ singularity if and only if $H = \{l(x_1, x_2, x_3) = 0\}$ or if $H = \{x_3 = 0\}$;*
2. *a triple point at the $\mathbf{D}_4$ singularity if and only if $H = \{l(x_2, x_3) = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$
\begin{aligned}
f &= q_1(x_1, x_2, x_3) + x_0x_3 \\
g &= x_3l_1(x_1, x_2, x_3)
\end{aligned}
$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$
\begin{aligned}
f &= q_1(x_1, x_2, l) + x_0l \\
g &= l \cdot l_1(x_1, x_2, l) \\
H &= x_3
\end{aligned}
$$

and hence D will be given by:

$$
\begin{aligned}
f &= q_1(x_1, x_2) + x_0l(x_1, x_2) \\
g &= q_2(x_1, x_2)
\end{aligned}
$$

which is a singular complete intersection of conics, with a double point, at $(1:0:0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, l) + x_0 l$$
$$g = l \cdot l_1(x_1, x_2, l)$$
$$H = x_3$$

and hence D will be given by:

$$f = q_1(x_1, x_2) + x_0 x_2$$
$$g = x_2 l_1(x_1, x_2)$$

which is a complete intersection of conics with a triple point $(1:0:0)$.

Now, let $H = \{x_3 = 0\}$, and let f, g be given by

$$f = x_1 x_2 + a x_0 x_3$$
$$g = x_0 x_1 + b x_0 x_2$$

such that D is given by

$$f = x_1 x_2$$
$$g = x_0 x_1 + b x_0 x_2$$

which has a double point at $(1:0:0)$

□

Lemma 5.21. *Let S be the complete intersection of two quadrics f, g which is a a twisted cubic and tangent line. Then S has 1 $\mathbf{A}_3$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(4)$*-action:*

1. *a double point at the $\mathbf{A}_3$ singularity if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
2. *a triple point at the $\mathbf{A}_3$ singularity if and only if $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$.*

Proof. From Sections 4.2 and 5.1 and Table 5.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_2, x_3) + x_3 l_1(x_0, x_1) + x_2 l_2(x_0, x_1)$$
$$g = q_2(x_2, x_3) + x_3 l_3(x_0, x_1) + x_2 l_4(x_0, x_1)$$

Let $H = \{l(x_1, x_2, x_3) = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$
\begin{aligned}
f &= q_1(x_2, l) + l \cdot l_1(x_0, x_1) + x_2 l_2(x_0, x_1) \\
g &= q_2(x_2, l) + l \cdot l_3(x_0, x_1) + x_2 l_4(x_0, x_1) \\
H &= x_3
\end{aligned}
$$

and hence D will be given by:

$$
\begin{aligned}
f &= q_1(x_1, x_2) + x_0 l_1(x_1, x_2) \\
g &= q_2(x_1, x_2) + x_0 l_2(x_1, x_2)
\end{aligned}
$$

which is a singular complete intersection of conics, with a double point, at $(1 : 0 : 0)$.

Similarly, let $H = \{l(x_2, x_3) = 0\}$ or $H = \{x_3 = 0\}$; using a suitable change of coordinates we have (by abuse of notation):

$$
\begin{aligned}
f &= q_1(x_2, l) + l \cdot l_1(x_0, x_1) + x_2 l_2(x_0, x_1) \\
g &= q_2(x_2, l) + l \cdot l_3(x_0, x_1) + x_2 l_4(x_0, x_1) \\
H &= x_3
\end{aligned}
$$

and hence D will be given by:

$$
\begin{aligned}
f &= x_2^2 + x_2 l_1(x_0, x_1) \\
g &= x_2^2 + x_2 l_2(x_0, x_1)
\end{aligned}
$$

which has a triple point $(1 : 0 : 0)$.

□

As a direct result from the above Lemmas (Lemmas 5.10-5.21), we have the following Lemma:

Lemma 5.22. *Let (S, D) be a pair that is invariant under a non-trivial $\mathbb{G}_m$-action. Suppose the singularities of S and the type of D are given as in the first and second entries in one of the rows of Table 5.6, respectively. Then (S, D) is projectively equivalent to $(f = g = 0, f = g = H = 0)$ for f, g as in Table 7.2 corresponding to the Segre symbol of row 3 of Table 5.6, and H as in the fourth entries in the same row of Table 5.6, respectively. In particular, any such pair (S, D) is unique up to* *projective equivalence. Conversely, if (S, D) is given by Segre symbols and equations as in the third and fourth entries in a given row of Table 5.6, then (S, D) has singularities and type as in the first and second entries in the same row of Table and (S, D) is $\mathbb{G}_m$-invariant. Furthermore, the one-parameter subgroup $\lambda(s) \in \mathrm{SL}(5)$, given in the entry of the corresponding row of Table 7.5 is a generator of the $\mathbb{G}_m$-action.*

$\mathrm{Sing}(S)$	Type of D	Segre Symbol	H	$\lambda(s)$
$\mathbf{A}_2$ at P	quadruple point P	$[3,1]$	x_3	$\mathrm{Diag}(s, s, s^{-1}, s^{-1})$
$\mathbf{A}_3$ at P twisted cubic and tangent line	$3P + Q$	$[4]$	x_3	$\mathrm{Diag}(s, s, s^{-1}, s^{-1})$
$\mathbf{A}_3$ at P two tangent conics	$2P + 2R$	$[(2,1),1]$	x_3	$\mathrm{Diag}(s, s, s^{-1}, s^{-1})$
$\mathbf{D}_4$ at P	$2P + Q + R$	$[(3,1)]$	x_3	$\mathrm{Diag}(s, 1, 1, s^{-1})$

Table 5.6: Some pairs (S, D) invariant under a $\mathbb{G}_m$-action.

5.4 VGIT Classification

For the VGIT of a complete intersection of two quadrics in $\mathbb{P}^3$ and a hyperplane H we have the following walls and chambers, obtained via the computational package [Pap22c] based on the discussion in Chapter 3.

	t_0		t_1		t_2		t_3		t_4		t_5
Walls	0		$\frac{4}{9}$		$\frac{2}{3}$		$\frac{4}{5}$		1		$\frac{4}{3}$
Chambers		$\frac{74}{171}$		$\frac{37}{57}$		$\frac{31}{40}$		$\frac{33}{34}$		$\frac{74}{57}$	

We thus obtain 9 non-isomorphic quotients $M^{GIT}_{3,2,2}(t_i)$, which are characterised by the following Theorems.

Theorem 5.23. *Let (S, D) be a pair where S is a complete intersection of two quadrics in $\mathbb{P}^3$ and $D = S \cap H$ is a hyperplane section.*

1. *$t \in (0, \frac{4}{9})$: The pair (S, D) is t-stable if and only S has at worse 1 $\mathbf{A}_1$ singularity with D general, or 2 $\mathbf{A}_1$ singularities, with D having a double point at one of the $\mathbf{A}_1$ singularities, or if S is smooth and D has at worse a quadruple point.*

2. $t = \frac{4}{9}$: *The pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_1$ singularities and D is general, or if S is smooth and D has at worse a triple point.*

3. $t \in (\frac{4}{9}, \frac{2}{3})$: *The pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_2$ singularities and D is general, or if S is smooth and D has at worse a triple point.*

4. $t = \frac{2}{3}$: *The pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_2$ singularities and D is general, or if S is smooth and D has at worse two double points.*

5. $t \in (\frac{2}{3}, \frac{4}{5})$: *The pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_3$ singularities and D is general, or if S is smooth and D has at worse two double points.*

6. $t = \frac{4}{5}$: *The pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_3$ singularities and D is general, or if S is smooth and D has at worse one double point.*

7. $t \in (\frac{4}{5}, 1)$: *he pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_3$ singularities and D is general, or if S is smooth and D has at worse one double point.*

8. $t = 1$: *the pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{A}_3$ singularities and D is general, or if S is smooth and D is smooth.*

9. $t \in (1, \frac{4}{3})$: *the pair (S, D) is t-stable if and only S has at worse finitely many $\mathbf{D}_4$ singularities and D is general, or if S is smooth and D is smooth.*

In addition, for all $t \in (0, \frac{4}{3})$, a t-stable pair (S, D) is also t-semistable, i.e. there are no strictly t-polystable pairs.

Theorem 5.24. *Let $t \in (0, 4/3)$. If t is a chamber, then $\overline{M(t)} = M(t)$, and the stable loci $M(t)$ is compact as all stable orbits are also semistable. If $t = t_i$, for $i = 1, 2, 3, 4$, then $\overline{M(t_i)}$ is the compactification of the stable loci $M(t_i)$ by the closed $\mathrm{SL}(4)$-orbits in $\overline{M(t)} \setminus M(t)$ represented by the $\mathbb{G}_m$-invariant pair (S_i, D_i) uniquely defined as follows:*

1. *the complete intersection S_1 of two quadrics with Segre symbol $[3, 1]$ with an $\mathbf{A}_2$ singularity P, and the divisor $D_1 \in |-K_{S_1}|$, where $D_1 = 4P$;*

2. *the complete intersection S_2 of two quadrics with Segre symbol $[4]$ with an $\mathbf{A}_3$ singularity P, and the divisor $D_2 \in |-K_{S_2}|$, where $D_2 = 3P + R$;*

3. *the complete intersection S_3 of two quadrics with Segre symbol $[(2,1),1]$ with 1 $\mathbf{A}_3$ singularity P, and the divisor $D_3 \in |-K_{S_3}|$, where $D_3 = 2P + 2Q$;*

4. *the complete intersection S_4 of two quadrics with Segre symbol $[(3,1)]$ with 1 $\mathbf{D}_4$ singularity P, and the divisor $D_4 \in |-K_{S_4}|$, where $D_4 = 2P + Q + R$.*

In order to prove these theorems, we need to consider all destabilising families and classify the t-stable members for each wall using their singularities. The families presented in the following sections are maximal destabilising families, in the sense of Definition 3.15 and Lemma 3.16, and have been produced via the algorithm described in Chapter 3, via the computational package [Pap22c]. Specifically, for each chamber, they are all t-unstable with respect to the respective t via the centroid criterion (Theorem 3.10).

5.4.1 Chamber $t = \frac{74}{171}$

For chamber $t = \frac{74}{171} \in (0, \frac{4}{9})$ we have the following:

Lemma 5.25. *1. The pair (S, H) is non-t-stable if and only if the pair is generated by one of the following families or a degeneration of those families:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_1,x_2,x_3) + x_0 l_1(x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

an irreducible smooth quadric f and an reducible quadric g (the union of two hyperplanes) with Segre symbol $[(2,1),1]$, intersecting at two tangent conics with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section;

Family 2:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_3(x_0,x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& x_3^2 \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

an irreducible smooth quadric f and an reducible quadric g (double plane) intersecting at a double conic, with non-isolated singularities, with $D = S \cap H$ a general hyperplane section;

Family 3*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_4(x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_5(x_1,x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

two singular quadrics f, g intersecting at a singular curve with a non-hypersurface singularity at $(1:0:0:0)$, with $D = S \cap H$ a general hyperplane section;

Family 4*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_6(x_1,x_2,x_3) + x_0x_3 \\ g(x_0,x_1,x_2,x_3) =& q_7(x_2,x_3) + x_1x_3 \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

an irreducible smooth quadric f and an irreducible quadric g intersecting at a cuspidal curve with an $\mathbf{A}_1$ singularity at $(1:0:0:0)$, with $D = S \cap H$ a general hyperplane section;

Family 5*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_8(x_0,x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& x_3 l_1(x_0,x_1,x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& x_3 \end{aligned}$$

an irreducible smooth quadric f and a reducible quadric g (the union of two hyperplanes) intersecting at two conics in general position, with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a smooth curve, and not a complete intersection;

Family 6*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_9(x_1,x_2,x_3) + x_0 l_2(x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_{10}(x_1,x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& l(x_1,x_2,x_3) \end{aligned}$$

an irreducible smooth quadric f and an irreducible quadric g intersecting at a nodal curve, with $\mathbf{A}_1$ singularity at $(1:0:0,0)$, with $D = S \cap H$ a hyperplane section with a double point at the $\mathbf{A}_1$ singularity;

Family 7:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_{11}(x_0,x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_{12}(x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& l(x_2,x_3) \end{aligned}$$

an irreducible smooth quadric f and a pair of intersecting planes g such that $\mathrm{Bs}(f,g)$ is a pair of conics in general position, with $\mathbf{A}_1$ singularities at $(1:m:0:0)$, $(1:k:0:0)$, where m,k is the solution of $q_2(1,x_1,0,0)$, with $D=S\cap H$ a hyperplane section with two double points at the $\mathbf{A}_1$ singularities;

Family 8:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_{13}(x_2,x_3)+x_3l_3(x_0,x_1)++x_2l_4(x_0,x_1) \\ g(x_0,x_1,x_2,x_3) =& q_{14}(x_2,x_3)+x_3l_5(x_0,x_1)++x_2l_6(x_0,x_1) \\ H(x_0,x_1,x_2,x_3) =& l(x_2,x_3) \end{aligned}$$

two irreducible non-singular quadrics f,g intersecting at a twisted cubic and a bisecant, with $\mathbf{A}_1$-singularities at $(1:0:0:0)$ and $(0:1:0:0)$, with $D=S\cap H$ not a complete intersection;

Family 9:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_{15}(x_1,x_2,x_3)+x_0x_3 \\ g(x_0,x_1,x_2,x_3) =& q_{16}(x_1,x_2,x_3)+x_0x_3 \\ H(x_0,x_1,x_2,x_3) =& l(x_1,x_2,x_3) \end{aligned}$$

two irreducible smooth quadrics f,g intersecting at a nodal curve, with an $\mathbf{A}_1$-singularity at $(1:0:0:0)$, with $D=S\cap H$ a hyperplane section with a double point at the $\mathbf{A}_1$ singularity;

Family 10:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_{17}(x_2,x_3)+x_3l_8(x_0,x_1) \\ g(x_0,x_1,x_2,x_3) =& q_{18}(x_2,x_3)+x_3l_9(x_0,x_1) \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

two irreducible singular quadrics f,g intersecting at a non-normal curve, with $D=S\cap H$ a general hyperplane section.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, which are maximal in their support, i.e. they have non-zero coefficients;

2. *for $t \in (0, \frac{4}{9})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a general hyperplane section with at worse a quadruple point, or S has at worse 1 $\mathbf{A}_1$ singularity with D general, or 2 $\mathbf{A}_1$ singularities, with D having a double point at one of the $\mathbf{A}_1$ singularities.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^{-}(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3. In particular, Family 10 has non-isolated singularities, and from Serre's criterion, it is non-normal.

For 2, let S be stable. Then, by part 1 and Section 5.3, S cannot have $\mathbf{A}_2$ or worse singularities. From families 6 and 9 we see that S can have 1 $\mathbf{A}_1$ singularity and D can be general. In addition, from families 5, 7 and 8 and Lemmas 5.11 and 5.15 we see that S can have 2 $\mathbf{A}_1$ singularities and D can have a double point in one of the $\mathbf{A}_1$ singularities. In addition, we see that S can be smooth, and D can have a quadruple point from Lemma 5.10. □

5.4.2 Wall $t = \frac{4}{9}$

For wall $t = \frac{4}{9}$:

Lemma 5.26. *1. The pair (S, H) is non-t-stable if and only if it is generated by the non-stable families of chamber $t = 74/171$ and in addition by the following family, or a degeneration:*

Family 1:

$$f(x_0, x_1, x_2, x_3) = q_1(x_1, x_2, x_3) + x_0 l_1(x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = q_2(x_2, x_3) + x_3 l_2(x_0, x_1)$$
$$H(x_0, x_1, x_2, x_3) = x_3$$

an irreducible smooth quadric f and an irreducible quadric g intersecting at smooth elliptic curve, with $D = S \cap H$ a singular complete intersection of two conics which is a quadruple point.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support.

2. *for wall $t = \frac{4}{9}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section which can have at worse a triple point, or S has at worse 1 $\mathbf{A}_1$ singularity with D general, or 2 $\mathbf{A}_1$ singularities, with D having a double point at one of the $\mathbf{A}_1$ singularities*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above family is a maximal destabilising family as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above family with respect to singularities follows from Remark 5.10.1.

For part 2, let S be t-stable. From Remark 5.10.1 we see that S can be smooth if D does not have a quadruple point. Furthermore, from Theorem 5.25, we see that S can have 1 $\mathbf{A}_1$ singularity, if D is general, or 2 $\mathbf{A}_1$ singularities if D has at worse a double point at one of the singularities. Hence, the proof is completed by Lemma 5.25. □

We also obtain:

Lemma 5.27. *For $t = \frac{4}{9}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following family:*

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& x_1^2 + a x_0 x_2 \\ g(x_0, x_1, x_2, x_3) =& x_2^2 + b x_0 x_3 \\ H(x_0, x_1, x_2, x_3) =& x_3 \end{aligned}$$

with S a singular complete intersection with Segre symbol $[3, 1]$ and an $\mathbf{A}_2$ singularity at point P, and $D = S \cap H = 4P$.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N^0_t(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N^0_t(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above family represent the only maximal $N^0_t(\lambda, x^{J_1}, x_i)$ for $t = \frac{4}{9}$.

Let (S, D) be strictly t-semistable, as in family 1, then, notice that up tp projective equivalence we can write

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& x_1^2 + x_0x_2 \\
g(x_0, x_1, x_2, x_3) =& x_2^2 + x_0x_3 \\
H(x_0, x_1, x_2, x_3) =& x_3.
\end{aligned}
$$

The proof follows from Lemma 5.22.

□

5.4.3 Chamber $t = \frac{37}{57}$

For chamber $t = \frac{37}{57} \in (\frac{4}{9}, \frac{2}{3})$ we have the following:

Lemma 5.28. *1. The pair (S, H) is non-t-stable if and only if it is generated by the non-stable families of wall $t = \frac{4}{9}$ (minus family 9 from chamber $t = \frac{74}{171}$) and in addition by the following families, or a degeneration:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& q_1(x_2, x_3) + x_1 l_1(x_2, x_3) + x_0x_3 \\
g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) + x_1x_3 \\
H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3)
\end{aligned}
$$

an irreducible smooth quadric f and an irreducible quadric g intersecting at a twisted cubic and a tangent line with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section;

Family 2:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& q_1(x_1, x_2, x_3) + x_0x_3 \\
g(x_0, x_1, x_2, x_3) =& x_3 l_1(x_1, x_2, x_3) \\
H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3)
\end{aligned}
$$

an irreducible smooth quadric f and an two intersecting hyperplanes g intersecting at a conic and two lines intersecting at a point, with a $\mathbf{D}_4$ singularity, with $D = S \cap H$ a general hyperplane section.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support.

2. *For chamber $t = \frac{37}{57} \in (\frac{4}{9}, \frac{2}{3})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worse a triple point, or S has at worse $\mathbf{A}_2$ singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials.

For part 2, let S be t-stable. From part 1 and Section 5.3 we see that S cannot have $\mathbf{A}_3$ or worse singularities. From the above family, we see that the case where S and H are given by

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_1, x_2, x_3)x_0x_3 \\ g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) + x_1x_3 \\ H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3) \end{aligned}$$

is stable, and from the classification S has an $\mathbf{A}_2$ singularity and D is general. From Lemma 5.26 we also know that S can be smooth, in which case D cannot have a quadruple point. □

5.4.4 Wall $t = \frac{2}{3}$

For wall $t = \frac{2}{3}$ we have the following:

Lemma 5.29. *1. The pair (S, H) is non-t-stable if and only if it is generated by the non-stable families of chamber $t = \frac{37}{57}$ (minus the family of wall $t = \frac{4}{9}$) and in addition by the following family, or a degeneration:*

Family 1:

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_1, x_2, x_3) + x_0 l_1(x_2, x_3) \\ g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) + x_1 l_2(x_2, x_3) + x_0x_3 \\ H(x_0, x_1, x_2, x_3) =& x_3 \end{aligned}$$

an irreducible smooth quadric f and an irreducible smooth quadric g intersecting at a smooth elliptic curve, with $D = S \cap H$ a singular complete intersection of two conics with a triple point $(1 : 0 : 0)$.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support;

2. *for wall $t = \frac{2}{3}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worse one or two double points, or S has at worse $\mathbf{A}_2$ singularities with D general.*

Proof. The proof follows from Remark 5.10.1 and Theorem 5.28, and the proof of Lemma 5.28. □

We also obtain:

Lemma 5.30. *For $t = \frac{2}{3}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following family:*

$$f(x_0, x_1, x_2, x_3) = x_1^2 + a x_0 x_2$$

$$g(x_0, x_1, x_2, x_3) = x_1 x_2 + b x_0 x_3$$

$$H(x_0, x_1, x_2, x_3) = x_3$$

with S a singular complete intersection with Segre symbol $[4]$ and an $\mathbf{A}_3$ singularity at point P, and $D = S \cap H = 3P + Q$.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above family represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{2}{3}$.

Let (S, D) be strictly t-semistable, as in family 1, then notice that up to projective equivalence we can write

$$f(x_0, x_1, x_2, x_3) = x_1^2 + x_0 x_2$$

$$g(x_0, x_1, x_2, x_3) = x_1 x_2 + x_0 x_3$$

$$H(x_0, x_1, x_2, x_3) = x_3.$$

The proof follows from Lemma 5.22. □

5.4.5 Chamber $t = \frac{31}{40}$

For chamber $t = \frac{31}{40} \in (\frac{2}{3}, \frac{4}{5})$ we have the following:

Lemma 5.31. 1. *The pair (S, H) is non-t-stable if and only if it pencil is generated by the non-stable families of wall $t = \frac{2}{3}$ minus the first family of chamber $t = \frac{37}{57}$, or a degeneration of those;*

2. *For chamber $t = \frac{31}{40} \in (\frac{2}{3}, \frac{4}{5})$, the only t-stable pairs $(S, D = S \cap H)$ $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worse one double point, or S has at worse $\mathbf{A}_3$ singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3.

For part 2, let S be t-stable. From part 1 and Section 5.3, we see that S cannot have $\mathbf{D}_4$ or worse singularities. Since the maximal destabilising families of chamber $t = \frac{31}{40} \in (\frac{2}{3}, \frac{4}{5})$ do not include the family of chamber $t = \frac{37}{57}$, we know that

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& q_1(x_2, x_3) + x_1 l_1(x_2, x_3) + x_0 x_3 \\
g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) + x_1 x_3 \\
H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3)
\end{aligned}
$$

is t-stable where $S = f \cap g$ has Segre symbol $[4]$ and is a twisted cubic and a tangent line with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section, by Lemma 5.21. □

5.4.6 Wall $t = \frac{4}{5}$

For wall $t = \frac{4}{5}$ we have the following:

Lemma 5.32. 1. *The pair (S, H) is non-t-stable if and only if*

2. *the pencil is generated by the non-stable families of chamber $t = \frac{31}{40}$ and in addition by the following family, or a degeneration:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& q_1(x_0, x_1, x_2, x_3) \\
g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) + x_3 l_1(x_0, x_1) \\
H(x_0, x_1, x_2, x_3) =& x_3
\end{aligned}
$$

an irreducible smooth quadric f and an irreducible quadric g intersecting at a smooth elliptic curve, with $D = S \cap H$ a singular complete intersection of two conics with two double points.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support;

3. *for wall $t = \frac{4}{5}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worst one double point, or S has at worse $\mathbf{A}_3$ singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^{-}(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3.

For part 2, let S be t-stable. From part 1 and Section 5.2, we see that S cannot have $\mathbf{A}_3$ or worse singularities. From the above family, S is smooth and D has two double points, by Remark 5.10.1, which is t-unstable, hence D can have at worse one double point. □

We also obtain:

Lemma 5.33. *For $t = \frac{4}{5}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following family:*

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q(x_0, x_1) \\ g(x_0, x_1, x_2, x_3) =& x_2^2 + x_3 l(x_0, x_1) \\ H(x_0, x_1, x_2, x_3) =& x_3 \end{aligned}$$

with S a singular complete intersection with Segre symbol $[(2, 1), 1]$ and an $\mathbf{A}_3$ singularity at point P, and $D = S \cap H = 2P + 2Q$.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above family represents the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{4}{5}$.

Let (S, D) be strictly t-semistable, as in family 1, then notice that up to projective equivalence we can write

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& x_0x_1 \\ g(x_0, x_1, x_2, x_3) =& x_2x_2 + x_0x_3 + x_3x_0 \\ H(x_0, x_1, x_2, x_3) =& x_3. \end{aligned}$$

The proof follows from Lemma 5.22. □

5.4.7 Chamber $t = \frac{33}{34}$

For chamber $t = \frac{33}{34} \in (\frac{4}{5}, 1)$ we have the following:

Lemma 5.34. *1. The pair (S, H) is non-t-stable if and only if the pencil is generated by the non-stable families of wall $t = \frac{4}{5}$ (minus the first family of chamber $t = \frac{74}{171}$) and in addition by the following family, or a degeneration:*

Family 1:

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_2, x_3) + x_0l_1(x_2, x_3) + x_1l_2(x_2, x_3) \\ g(x_0, x_1, x_2, x_3) =& q_2(x_2, x_3) \\ H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3) \end{aligned}$$

an irreducible smooth quadric f and an irreducible singular quadric g intersecting at a double line and two lines in general position with non-isolated singularities, with $D = S \cap H$ a smooth conic.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support;

2. *For chamber $t = \frac{33}{34} \in (\frac{4}{5}, 1)$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worse one double point, or S has at worse $\mathbf{A}_3$ singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3.

For part 2, let S be t-stable. From part 1, the above family and Section 5.1 we see that S cannot have $\mathbf{D}_4$ or worse singularities. Notice, that the pair

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_1,x_2,x_3) + x_0 l_1(x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_2,x_3) \\ H(x_0,x_1,x_2,x_3) =& l(x_0,x_1,x_2,x_3) \end{aligned}$$

where $S = \{f = 0\} \cap \{g = 0\}$ has Segre symbol $[(2,1),1]$, and is two tangent conics with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ general, is t-stable. The proof then follows from Lemma 5.32.

□

5.4.8 Wall $t = 1$

For wall $t = 1$ we have the following:

Lemma 5.35. *1. The pair (S, H) is non-t-stable if and only if it is generated by the non-stable families of wall $t = \frac{33}{34}$ (minus the family of wall $t = \frac{2}{3}$) and in addition by the following family, or a degeneration:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3) =& q_1(x_1,x_2,x_3) + x_0 l_1(x_1,x_2,x_3) \\ g(x_0,x_1,x_2,x_3) =& q_2(x_1,x_2,x_3) + x_0 x_3 \\ H(x_0,x_1,x_2,x_3) =& x_3 \end{aligned}$$

two irreducible smooth quadrics f, g and an irreducible quadric g intersecting at a smooth elliptic curve, with $D = S \cap H$ a singular complete intersection of two conics with a double point and two other points.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms, maximal in their support;

2. for wall $t = 1$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section which is a smooth complete intersection of conics in $\mathbb{P}^2$, or S has at worse $\mathbf{A}_3$ singularities with D general.

Proof. The proof follows from Remark 5.10.1 and Lemma 5.31. □

We also obtain:

Lemma 5.36. *For $t = 1$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following family:*

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& q(x_1, x_2) + x_0x_3 \\
g(x_0, x_1, x_2, x_3) =& x_0 l(x_1, x_2) \\
H(x_0, x_1, x_2, x_3) =& x_3
\end{aligned}
$$

with S a singular complete intersection with Segre symbol $[(3, 1)]$ and a $\mathbf{D}_4$ singularity at point P, and $D = S \cap H = 2P + Q + R$.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above family represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = 1$.

Let (S, D) be strictly t-semistable, as in family 1, then notice that up to projective equivalence we can write

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3) =& x_1x_2 + x_0x_3 \\
g(x_0, x_1, x_2, x_3) =& x_0x_1 + x_0x_2 \\
H(x_0, x_1, x_2, x_3) =& x_3.
\end{aligned}
$$

The proof follows from Lemma 5.22. □

5.4.9 Chamber $t = \frac{74}{57}$

For chamber $t = \frac{74}{57} \in (1, \frac{4}{3})$ we have the following:

Lemma 5.37. 1. *The pair (S, H) is non t-stable if and only if it is generated by the families of wall $t = 1$ (minus the second family of $t = \frac{37}{57}$) or a degeneration of those;*

2. *For chamber $t = \frac{74}{57}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section which is a smooth complete intersection of quadrics in $\mathbb{P}^3$ or S has at worse $\mathbf{D}_4$ singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 5.1 and 5.3.

For part 2, let S be t-stable. From part 1 and Section 5.3, we see that the pair (S, H) given by

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_1, x_2, x_3) + x_0 x_3 \\ g(x_0, x_1, x_2, x_3) =& x_3 l_1(x_1, x_2, x_3) \\ H(x_0, x_1, x_2, x_3) =& l(x_0, x_1, x_2, x_3) \end{aligned}$$

is t-stable. Here, S has a Segre symbol $[(3,1)]$ and thus a $\mathbf{D}_4$ singularity, with D general. The proof then follows from Lemma 5.35 □

Proof of Theorem 5.23. The VGIT classification follows from the Theorems in Sections 5.4.1 to 5.4.9. □

Proof of Theorem 5.24. For the characterisation of potential closed orbits, notice that the centroid criterion (Theorem 3.10) shows that for each chamber none of the above destabilising families are strictly t-semistable. For each wall, the characterisation follows from the Theorems in Sections 5.4.1 to 5.4.9.

□

CHAPTER 6

K-moduli Compactification of Family 2.25

Consider a smooth intersection of two quadrics C_1 and C_2 in $\mathbb{P}^3$. The resulting complete intersection $C = C_1 \cap C_2$ is an elliptic curve; blowing up $\mathbb{P}^3$ along C gives a smooth Fano threefold $X = \mathrm{Bl}_C \mathbb{P}^3$, with $(-K_X)^3 = 32$. It is known (see, e.g. [Ara+21, Corollary 4.3.16.]), that all such smooth Fano threefolds which correspond to family 2.25 in the Mori-Mukai classification [MM03], are K-stable.

Let $C_1 = \{x_0x_1 = 0\}$, $C_2 = \{x_2x_3 = 0\}$ be two quadrics in $\mathbb{P}^3$. Then $\tilde{C} = C_1 \cap C_2$ is GIT-polystable by Theorem 5.9. Notice, that C_1 and C_2 are toric surfaces which intersect on a toric curve, which is the 4 lines $\{x_0 = 0\}$, $\{x_1 = 0\}$, $\{x_2 = 0\}$ and $\{x_3 = 0\}$, hence $\tilde{C}$ is toric. As such, as $\mathbb{P}^3$ is toric, the blow up of $\mathbb{P}^3$ along $\tilde{C}$, $\tilde{X} = \mathrm{Bl}_{\tilde{C}} \mathbb{P}^3$ is a toric blow up and hence $\tilde{X}$ is a toric variety. The polytope of $\tilde{X}$, $P_{\tilde{X}}$, is created by 'cutting' the corresponding polytope for $\mathbb{P}^3$, $P_{\mathbb{P}^3}$, along the 4 edges corresponding to each x_i. The corresponding polytope $P_{\tilde{X}}$, is a polytope generated by vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, $(-1, -1, 0)$, $(0, -1, -1)$, $(-1, -1, -1)$. This is the terminal toric Fano threefold with Reflexive ID #199 in the Graded Ring Database (GRDB) (3-fold #255743) [Kas10]. Notice that the sum of the vertices is $(0, 0, 0)$, and hence, the barycenter of P_X is $(0, 0, 0)$. Hence, by Theorem 2.76, $\tilde{X}$ is K-polystable. In particular, we have proved:

Lemma 6.1. *Let $\tilde{X} := \mathrm{Bl}_{\tilde{C}} \mathbb{P}^3$ where $\tilde{C} = C_1 \cap C_2$ for $C_1 = \{x_0x_1 = 0\}$, $C_2 = \{x_2x_3 = 0\}$. Then $\tilde{X}$ is K-polystable.*

The Magma code below checks that this threefold is toric and singular, and generates the vertices of the polyhedron P_X.

```
Q4:=Polytope([[0,-1,0],[-1,0,1],[2,-1,0],[1,0,-1],[-1,0,-1],[0,-1,2],
[0,1,0],[-1,2,-1]]);
> ViewWithJmol(Q4: point_labels:=true, open_in_background:=true);
> P:=Dual(Q4);
> P;
> IsCanonical(P);
> Volume(Q4);
> #Points(Q4);
> #Vertices(Q4);
> Faces(P);
> IsTerminal(P);
> IsSmooth(P);
```

Lemma 6.2. *Let $X := \mathrm{Bl}_C \mathbb{P}^3$, where C is a strictly GIT semistable complete intersection of two quadrics. Then X is strictly K-semistable.*

Proof. Let C be a strictly GIT semistable complete intersection of two quadrics. Since C is strictly GIT semistable, there exists a one-parameter subgroup λ such that the limit $\lim_{t\to 0}\lambda(t)\cdot C = \tilde{C}$, where $\tilde{C}$ is the unique strictly GIT polystable complete intersection i.e. the quadrangle with 4 $\mathbf{A}_1$ singularities as in Theorem 5.9. This one-parameter subgroup induces a family $f\colon \mathcal{C} \to B$, over a curve B, such that the fibers $\mathcal{C}_t$ are isomorphic to C for all $t \neq 0$, and $\mathcal{C}_0 \cong \tilde{C}$.

Let $\mathcal{P} := \mathbb{P}^3 \times B = \mathbb{P}^3 \times \mathbb{P}^1$. Then, notice that $\tilde{C} = \overline{\lambda(t)\cdot C}$ in $\mathcal{P}$. We define $\mathcal{X} = \mathrm{Bl}_{\mathcal{C}} \mathcal{P}$, and hence we have that $\mathcal{X}_0 \cong \tilde{X} := \mathrm{Bl}_{\tilde{C}} \mathcal{P}$. By taking the composition of $\mathcal{X} \to \mathcal{P}$, with the projection $\mathcal{P} \to B$, we thus have a map $\mathcal{X} \to B$ which is naturally a test configuration of $\mathcal{X}$ with central fibre $\mathcal{X}_0$. Hence, we have constructed a test configuration $g\colon \mathcal{X} \to B$ where the central fiber $\mathcal{X}_0 \cong \tilde{X}$ is a klt Fano threefold, which is K-polystable by Lemma 6.1, and the general fiber $\mathcal{X}_t \cong X$ is not isomorphic to $\mathcal{X}_0$. By [Ara+21, Corollary 1.1.14] the central fiber $\mathcal{X}_t \cong X$ is strictly K-semistable. □

Remark 6.2.1. In some cases, we can construct explicit descriptions of the above degeneration. For example, for Family 2 in Theorem 5.8 we can make a change of coordinates such that $C = \{f = 0\} \cap \{g = 0\}$ is given by

$$f(x_0, x_1, x_2, x_2) = x_0x_1 + q(x_2, x_3) + l(x_0, x_1)\tilde{l}(x_2, x_3)$$
$$g(x_0, x_1, x_2, x_2) = x_2x_3.$$

Then, defining for some parameter t, C_t as follows $C_t = \{f_t = 0\} \cap \{g_t = 0\}$, where

$$\begin{aligned} f_t(x_0, x_1, x_2, x_2) &= x_0x_1 + t\big(q(x_2, x_3) + l(x_0, x_1)\tilde{l}(x_2, x_3)\big) \\ g_t(x_0, x_1, x_2, x_2) &= x_2x_3. \end{aligned}$$

we have $C_t \cong C$ for all $t \neq 0$, and C_0 to be the strictly GIT polystable curve of Theorem 5.9.

In essence, we have shown:

Corollary 6.2.1. *Let $C = C_1 \cap C_2$ be a complete intersection of quadrics. If C is GIT (poly/semi-)stable then the threefold $X := \mathrm{Bl}_C\, \mathbb{P}^3$ is K-(poly/semi-)stable.*

Let $\mathcal{M}^K_{2.25}$ be the K-moduli stack parametrising K-semistable members in the Fano threefold family 2.25 and let $\mathcal{M}^{GIT}_{3,2,2}$ be the GIT quotient stack parametrising GIT semistable complete intersections of two quadrics in $\mathbb{P}^3$. By Theorem 2.91, these admit good moduli spaces $M^K_{2.25}$ of K-polystable members and a GIT quotient $M^{GIT}_{3,2,2}$ respectively.

Theorem 6.3. *There exists an isomorphism $\mathcal{M}^K_{2.25} \cong \mathcal{M}^{GIT}_{3,2,2}$*

Proof. Let $\mathfrak{X}$ be the Hilbert polynomial of smooth elements of the family of Fano threefolds 2.25 pluri-anticanonically embedded by $-mK_{\mathfrak{X}}$ in $\mathbb{P}^N$, and let $\mathbb{H}^{\mathfrak{X};N} := \mathrm{Hilb}_{\mathfrak{X}}(\mathbb{P}^N)$. Given a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\mathfrak{X}(X, \mathcal{O}_{\mathbb{P}^N}(k)|_X) = \mathfrak{X}(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\mathfrak{X};N}$ denote its Hilbert point. Let

$$\hat{Z}_m := \left\{ \mathrm{Hilb}(X) \in \mathbb{H}^{\mathfrak{X};N} \;\middle|\; \begin{array}{l} X \text{ is a Fano manifold of family } 2.25, \\ \mathcal{O}_{P^N}(1)|_X \sim \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)). \end{array} \right\}$$

which is a locally closed subscheme of $\mathbb{H}^{\mathfrak{X};N}$. Let $\overline{Z}_m$ be its Zariski closure in $\mathbb{H}^{\mathfrak{X};N}$ and Z_m be the subset of $\hat{Z}_m$ consisting of K-semistable varieties.

Since by Corollary 6.2.1 the blow-up of a smooth complete intersection of two quadrics in $\mathbb{P}^3$ is K-stable and by [Oda15], the smooth K-stable loci is a Zariski open set of $M^K_{2.25}$, in the definition of moduli stack of $\mathcal{M}^K_{2.25} = [Z_m/\,\mathrm{PGL}(N_m+1)]$ for appropriate $m > 0$ and in fact $\mathcal{M}^{GIT}_{3,2,2} \cong [\overline{Z}_m/\,\mathrm{PGL}(N_m+1)]$.

Thus, by Lemmas 6.1 and 6.2 and Corollary 6.2.1 we have an open immersion of representable morphism of stacks:

$$\mathcal{M}^{GIT}_{3,2,2} \xrightarrow{\quad \phi \quad} \mathcal{M}^{K}_{2.25}$$

$$[C_1 \cap C_2] \xmapsto{\phi} [\mathrm{Bl}_{C_1 \cap C_2} \mathbb{P}^3] \, .$$

Note that representability follows once we prove that the base-change of a scheme mapping to the K-moduli stack is itself a scheme. Such a scheme mapping to the K-moduli stack is the same as a PGL-torsor over Z_m, which produces a PGL-torsor over $\overline{Z}_m$ after a PGL-equivariant base change. This PGL-torsor over $\overline{Z}_m$ shows the desired pullback is a scheme. By [The22, Lemma 06MY], since ϕ is an open immersion of stacks, ϕ is separated and, since it is injective, it is also quasi-finite.

We now need to check that ϕ is an isomorphism that descends (as isomorphism of schemes) to the moduli spaces

$$M^{GIT}_{3,2,2} \xrightarrow{\quad \overline{\phi} \quad} M^{K}_{2.25}$$

$$[C_1 \cap C_2] \xmapsto{\overline{\phi}} [\mathrm{Bl}_{C_1 \cap C_2} \mathbb{P}^3]$$

since we have a morphism ϕ of stacks, both of which admit moduli spaces. Thus $\overline{\phi}$ is injective.

Now, by [Alp13, Prop 6.4], since ϕ is representable, quasi-finite and separated, $\overline{\phi}$ is finite and ϕ maps closed points to closed points, we obtain that ϕ is finite. Thus, by Zariski's Main Theorem, as $\overline{\phi}$ is a birational morphism with finite fibers to a normal variety, ϕ is an isomorphism to an open subset, but it is also an open immersion, thus it is an isomorphism.

□

Remark 6.3.1. The above method of proof emulates closely the moduli continuity method, which has appeared in different forms in [OSS16], [SS17],[GMS21]. In that method, one defines a map

$$\phi \colon \overline{M}^K \to \overline{M}^{GIT}$$

to prove the existence of a homeomorphism using the properties of the moduli spaces and the continuity of ϕ. In this instance, the definition of the map is reversed, due to the existence of Lemma 6.1 and Corollary 6.2.1. As such, we will call this method of proof the *reverse moduli continuity method*.

CHAPTER 7

VGIT of complete intersections of Quadrics in $\mathbb{P}^4$ and a Hyperplane

In this chapter, we will study VGIT quotients of complete intersection of quadrics in $\mathbb{P}^4$ and a hyperplane, using the computational methods presented in Chapter 3. We will first provide some general results on the singularities of such complete intersections based on [MM90], and then we will provide a full GIT classification. We will then proceed to classify all possible singularities of pairs $(S, D = S \cap H)$ and provide a full VGIT classification using our computational method. This in turn will be of use, in later chapters, when we will study the K-stability of such pairs.

7.1 General Results

Throughout this section, we will use the geometric classification of pencils of quadrics in $\mathbb{P}^4$ on their singularities based on their Segre symbols found in [Dol12, §8.6 and Table 8.6]. We summarise the results in the following table.

Segre symbol of pencil	Singularities of base locus
$[5]$	$\mathbf{A}_4$
$[(4,1)]$	$\mathbf{D}_5$
$[4,1]$	$\mathbf{A}_3$
$[(3,1),1]$	$\mathbf{D}_4$
$[3,2]$	$2\mathbf{A}_1+\mathbf{A}_2$
$[3,1,1]$	$\mathbf{A}_2$
$[3,(1,1)]$	$\mathbf{A}_2+\mathbf{A}_1$
$[(2,1),2]$	$\mathbf{A}_1+\mathbf{A}_3$
$[(2,1),1,1]$	$\mathbf{A}_3$
$[(2,1),(1,1)]$	$2\mathbf{A}_1+\mathbf{A}_3$
$[2,1,1,1]$	$\mathbf{A}_1$
$[2,2,1]$	$2\mathbf{A}_1$
$[(1,1),2,1]$	$3\mathbf{A}_1$
$[(1,1),1,1,1]$	$2\mathbf{A}_1$
$[(1,1),(1,1),1]$	$4\mathbf{A}_1$
$[1,1,1,1,1]$	Smooth

Table 7.1: Segre symbols of pencils of quadrics in $\mathbb{P}^3$ and singularities.

The lemma below gives a geometric classification of the pencils of quadrics in $\mathbb{P}^4$ above, attributed to Mabuchi–Mukai.

Lemma 7.1 ([MM90, Proposition 6.7]). *Let $\Phi(f,g)$ be a pencil of two quadrics $f,g \in \mathbb{P}^4$, where f is smooth. Then the base locus of the pencil has $\mathbf{A}_1$ singularities if and only if the highest multiplicity of the roots of the determinant polynomial $\det(\lambda F + G)$ is 2.*

We present the following table, which lists the possible complete intersections of two quadrics in $\mathbb{P}^4$, with isolated singularities, up to projective equivalence, and their singularities. We generate the polynomials f, g via the computational study of each Segre symbol.

Segre symbol	Generating polynomials	Singularities
$[(4,1)]$	$f = q_1(x_2,x_3,x_4) + x_1 l_1(x_3,x_4) + x_0 x_4$ $g = q_2(x_3,x_4) + x_4 l_2(x_1,x_2)$	$\mathbf{D}_5$
$[4,1]$	$f = q_1(x_2,x_3,x_4) + x_1 l_1(x_2,x_3,x_4) + x_0 x_4$ $g = q_2(x_2,x_3,x_4) + x_1 x_4$	$\mathbf{A}_3$
$[(3,1),1]$	$f = q_1(x_1,x_2,x_3,x_4) + x_0 x_4$ $g = q_2(x_3,x_4) + x_4 l_1(x_1,x_2)$	$\mathbf{D}_4$
$[3,2]$	$f = q_1(x_2,x_3,x_4) + x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1)$ $g = q_2(x_3,x_4) + x_4 l_3(x_0,x_1,x_2) + x_2 x_3$	$2\mathbf{A}_1 + \mathbf{A}_2$
$[3,1,1]$	$f = q_1(x_1,x_2,x_3,x_4) + x_0 x_4$ $g = q_2(x_2,x_3,x_4) + x_1 x_4$	$\mathbf{A}_2$
$[3,(1,1)]$	$f = q_1(x_1,x_2,x_3,x_4) + x_0 x_1$ $g = x_4^2 + x_3^2 + x_2 x_1$	$\mathbf{A}_2 + \mathbf{A}_1$
$[(2,1),2]$	$f = x_1^2 + x_0 l_1(x_2,x_3,x_4)$ $g = q_1(x_2,x_3,x_4)$	$\mathbf{A}_1 + \mathbf{A}_3$
$[(2,1),1,1]$	$f = q_1(x_1,x_2,x_3,x_4) + x_0 l_1(x_2,x_3,x_4)$ $g = q_2(x_2,x_3,x_4)$	$\mathbf{A}_3$
$[(2,1),(1,1)]$	$f = x_0 x_4 + x_1 x_3 + x_2^2$ $g = q_1(x_3,x_4) + x_3 l_1(x_1,x_2) + x_4 l_2(x_1,x_2)$	$2\mathbf{A}_1 + \mathbf{A}_3$

$[2,1,1,1]$	$f = q_1(x_1,x_2,x_3,x_4) + x_0x_4$ $g = q_2(x_1,x_2,x_3,x_4) + x_0x_4$	$\mathbf{A}_1$
$[2,2,1]$	$f = q_1(x_2,x_3,x_4) + x_4l_1(x_0,x_1) + x_3l_2(x_0,x_1)$ $g = q_2(x_2,x_3,x_4) + x_4l_3(x_0,x_1) + x_3l_4(x_0,x_1)$	$2\mathbf{A}_1$
$[(1,1),2,1]$	$f(x_0,x_1,x_2,x_3,x_4) = q_5(x_1,x_2,x_3,x_4) + x_0l_7(x_3,x_4)$ $g(x_0,x_1,x_2,x_3,x_4) = q_6(x_3,x_4) + x_4l_8(x_0,x_1,x_2)$	$3\mathbf{A}_1$
$[(1,1),1,1,1]$	$f = q_1(x_1,x_2,x_3) + x_0x_4$ $g = q_2(x_1,x_2,x_3) + x_0x_4$	$2\mathbf{A}_1$
$[(1,1),(1,1),1]$	$f = x_4l_1(x_0,x_1) + x_3l_2(x_0,x_1) + x_2^2$ $g = x_4l_3(x_0,x_1) + x_3l_4(x_0,x_1) + x_2^2$	$4\mathbf{A}_1$
$[1,1,1,1,1]$	$f = q_1(x_0,x_1,x_2,x_3,x_4)$ $g = q_2(x_0,x_1,x_2,x_3,x_4)$	None

Table 7.2: Segre symbols, equations up to projective equivalence, and singularities of complete intersections of quadrics in $\mathbb{P}^4$.

Remark 7.1.1. Notice, that due to Theorem 4.3, all singular pencils that have the same Segre symbol (except for $[2,1,1,1]$ and $[(1,1),1,1,1]$) are projectively equivalent, and the above table gives their full classification.

Remark 7.1.2. In order to check the type of (isolated) hypersurface singularities, one can employ the following MAGMA script, adjusted accordingly for each different case.

```
Q:=RationalField();
```

```
PP<x0,x1,x2,x3,x4>:=ProjectiveSpace(Q,4);
f1:=x1*x3-x1*x4+2*x2^2+3*x3^2-x4^2-7*x2*x4+3*x3*x4-6*x2*x3+x0*x4;
f2:=x4^2-5*x3^2 - 2*x3*x4+x4*(5*x1-6*x2);
X:=Scheme(PP,[f1,f2]);
IsNonsingular(X);
p := X![1,0,0,0,0];
_,f,_,fdat := IsHypersurfaceSingularity(p,3);
R<a,b,c> := Parent(f);
f;
NormalFormOfHypersurfaceSingularity(f);
boo,f0,typ :=
NormalFormOfHypersurfaceSingularity(f : fData := [*fdat,3*]);
boo; f0; typ;
```

Here, $f1$ and $f2$ are the generating polynomials, and

```
IsNonsingular(X);
```

verifies that $X = f1 \cap f2$ is singular. The point $p = [1, 0, 0, 0, 0]$ refers to a specific singular point, whose type of singularity we want to check, which is given by the last command.

If one is unsure about the exact singular points of the complete intersection, the following MAGMA code can check the type of singularity for each singular point in the complete intersection:

```
Q:=RationalField();
PP<x0,x1,x2,x3,x4>:=ProjectiveSpace(Q,4);
f1:=x1*x3-x1*x4+3*x3^2-x4^2+3*x3*x4+x0*x4-2*x0*x3;
f2:=2*x1*x3+6*x1*x4+x3^2+2*x4^2-3*x3*x4-5*x0*x4+2*x0*x3;
X:=Scheme(PP,[f1,f2]);
IsNonsingular(X);
sngs := SingularSubscheme(X);
Support(sngs);
pts := PointsOverSplittingField(sngs);
pts;
```

```
pts[1];
pt := pts[1];
k := Ring(Parent(pt));
k;
p := X(k)!Eltseq(pt);
_,f,_,fdat := IsHypersurfaceSingularity(p,3);
R<a,b,c> := Parent(f);
f;
NormalFormOfHypersurfaceSingularity(f);
boo,f0,typ :=
NormalFormOfHypersurfaceSingularity(f : fData := [*fdat,3*]);
boo; f0; typ;
```

7.2 GIT Classification

In this section we will study the VGIT quotient $\mathcal{R}_{4,2,2,1} /\!\!/ \mathrm{SL}(5)$. For wall $t = 0$, i.e. in the absence of a hyperplane, this is equivalent to the GIT quotient $\mathcal{R}_{4,2,2} /\!\!/ \mathrm{SL}(5)$. The following families have been generated using the computational package [Pap22c], based on the discussion on Chapter 3. Here, in contrast to Sections 4.2 and 5.2 we will not include the fundamental set of one-parameter subgroups $P_{4,2,2}$. This is mainly due to the fact that $P_{4,2,2}$ has 1972 elements, and as such it is more complicated and time-consuming to list. We obtain the following outputs:

λ	x^J	$V^-(\lambda, x^J)$	$B^-(\lambda, x^J)$
$(42, 37, 17, -43, -53)$	x_0^2	$\{x_0, x_1, x_2, x_3, x_4\}^2$	$\{x_3, x_4\}^2$
$(2, 0, 0, -1, -1)$	x_1x_3	$\{x_1, x_2, x_3, x_4\}^2$, $x_0\{x_3, x_4\}$	$\{x_3, x_4\}^2$, $\{x_0, x_1\}\{x_3, x_4\}$
$(1, 1, 0, -1, -1)$	x_0x_3	$\{x_2, x_3, x_4\}^2$, $\{x_0, x_1\}\{x_3, x_4\}$	$\{x_2, x_3, x_4\}^2$, $\{x_0, x_1\}\{x_3, x_4\}$
$(3, 1, -1, -1, -2)$	x_2^2	$\{x_1, x_2, x_3, x_4\}^2$, $x_0\{x_2, x_3, x_4\}$	$\{x_2, x_3, x_4\}^2$
$(1, 0, 0, 0, -1)$	x_0x_4	$\{x_1, x_2, x_3, x_4\}^2$, x_0x_4	$\{x_1, x_2, x_3, x_4\}^2$, x_0x_4
$(1, 1, 1, 1, -4)$	x_0x_4	$\{x_0, x_1, x_2, x_3, x_4\}^2$	$x_4\{x_0, x_1, x_2, x_3, x_4\}^2$

Table 7.3: Outputs of the computational package [Pap22c] for destabilized families of complete intersections of two quadrics in $\mathbb{P}^4$

λ	x^J	$V^{\ominus}(\lambda, x^J)$	$B^{\ominus}(\lambda, x^J)$
$(44, 19, -1, -11, -51)$	x_2^2	$\{x_2, x_3, x_4\}^2$, $x_4\{x_0, x_1\}$	$\{x_2, x_3, x_4\}^2$, $x_4\{x_0, x_1\}$
$(42, 37, 17, -43, -53)$	x_3^2	$\{x_0, x_1, x_2, x_3, x_4\}^2$	$\{x_3, x_4\}^2$
$(42, 37, 17, -43, -53)$	x_0x_3	$\{x_3, x_4\}^2$, $\{x_3, x_4\}\{x_0, x_1, x_2\}$	$\{x_3, x_4\}^2$, $\{x_3, x_4\}\{x_0, x_1, x_2\}$
$(9, -1, -1, -1, -6)$	x_1^2	$\{x_1, x_2, x_3, x_4\}^2$	$\{x_1, x_2, x_3, x_4\}^2$
$(16, 1, -4, -4, -9)$	x_1x_4	$\{x_1, x_2, x_3, x_4\}^2$, x_0x_4	$\{x_2, x_3, x_4\}^2$
$(6, 1, 1, -4, -4)$	x_1x_3	$\{x_1, x_2, x_3, x_4\}^2$, $x_0\{x_3, x_4\}$	$\{x_3, x_4\}^2$, $\{x_1, x_2\}\{x_3, x_4\}$
$(6, 6, 1, -4, -9)$	x_0x_4	$\{x_2, x_3, x_4\}^2$, $\{x_0, x_1\}\{x_3, x_4\}$	$\{x_3, x_4\}^2$, $x_4\{x_0, x_1, x_2\}$, x_2x_3
$(11, 1, -4, -4, -4)$	x_2^2	$\{x_1, x_2, x_3, x_4\}^2$, $x_0\{x_2, x_3, x_4\}$	$\{x_2, x_3, x_4\}^2$
$(1, 1, 1, 1, -4)$	x_0x_4	$\{x_0, x_1, x_2, x_3, x_4\}^2$	$x_4\{x_0, x_1, x_2, x_3, x_4\}$
.			

Table 7.4: Outputs of the computational package [Pap22c] for semi-destabilized families of complete intersections of two quadrics in $\mathbb{P}^4$

We obtain the following (see also [MM90, §6]).

Theorem 7.2. *The following are equivalent:*

1. *A pencil of two hyperquadrics $\Phi(f, g)$ in $\mathbb{P}^4$ is unstable;*
2. *the base locus of the pencil $\mathrm{Bs}(f, g)$ has singularities worse than $\mathbf{A}_1$;*
3. *the pencil is generated by one of the following families, or their degenerations:*

   Family 1:

$$f(x_0, x_1, x_2, x_3, x_4) = q_1(x_1, x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_2(x_1, x_2, x_3, x_4)$$

   two irreducible singular hyperquadrics f, g intersecting at a singular surface, with non-isolated singularities;

   Family 2:

$$f(x_0, x_1, x_2, x_3, x_4) = q_3(x_1, x_2, x_3, x_4) + x_0 l_1(x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_4(x_2, x_3, x_4)$$

   an irreducible smooth hyperquadric f and an irreducible singular hyperquadric g such that $\mathrm{Bs}(f, g)$ is singular, with Segre symbol $[(2, 1), 1, 1]$ and an isolated $\mathbf{A}_3$ singularity at $(1:0:0:0:0)$;

Family 3*:*

$$f(x_0, x_1, x_2, x_3, x_4) = q_4(x_0, x_1, x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_5(x_3, x_4)$$

an irreducible singular quadric f and two intersecting hyperplanes g, intersecting at a singular surface, with non isolated singularities at $(1 : k : l : 0 : 0)$ where k, l are the solutions of $q_4(x_1, x_2, 0, 0) = 0$;

Family 4*:*

$$f(x_0, x_1, x_2, x_3, x_4) = x_4 l_4(x_0, x_1, x_2, x_3, x_4) + x_3 l_5(x_0, x_1, x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4 l_6(x_0, x_1, x_2, x_3, x_4) + x_3 l_7(x_0, x_1, x_2, x_3)$$

two irreducible singular hyperquadrics f, g intersecting at a singular non-normal surface, with no isolated singularities;

Family 5*:*

$$f(x_0, x_1, x_2, x_3, x_4) = q_6(x_0, x_1, x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4 l_8(x_0, x_1, x_2, x_3, x_4)$$

an irreducible smooth hyperquadric f and two intersecting hyperplanes g at a singular surface, with non-isolated singularities;

Family 6*:*

$$f(x_0, x_1, x_2, x_3, x_4) = q_7(x_2, x_3, x_4) + x_4 l_9(x_0, x_1)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_8(x_2, x_3, x_4) + x_4 l_{10}(x_0, x_1)$$

two irreducible singular hyperquadrics f, g intersecting at a singular surface, with non-isolated singularities;

Family 7*:*

$$f(x_0, x_1, x_2, x_3, x_4) = q_9(x_1, x_2, x_3, x_4) + x_0 l_{11}(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_{10}(x_3, x_4) + x_3 l_{12}(x_1, x_2) + x_4 l_{13}(x_1, x_2)$$

an irreducible smooth hyperquadric f and singular hyperquadric g intersecting at a singular surface, with $\mathbf{A}_2$ singularity;

Family 8:

$$f(x_0, x_1, x_2, x_3, x_4) = q_{11}(x_2, x_3, x_4) + x_0 l_{14}(x_3, x_4) + x_1 l_{15}(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_{12}(x_3, x_4) + x_4 l_{16}(x_0, x_1, x_2) + x_2 x_3$$

an irreducible smooth hyperquadric f and singular hyperquadric g intersecting at a singular surface, with an $\mathbf{A}_2$ and two $\mathbf{A}_1$ singularities.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms. which are maximal in their support, i.e. they have non-zero coefficients.

The families presented here are maximal semi-destabilizing families in the sense of Definition 3.15.

Proof. The equivalence of 1 and 3 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal semi-destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^{-}(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials, from Table 7.4.

For Family 1, the intersection corresponds to 4 intersecting hyperplanes, which is a cone over a quadric in $\mathbb{P}^3$, with non-du Val singularity at $(1:0:0:0:0)$. For Family 2, notice that, from Table 7.2, the Segre symbol of the pencil is $[(2,1),1,1]$. Hence, the base locus of the pencil is singular, and the singular point $P = (1:0:0:0:0)$ is an $\mathbf{A}_3$ singularity.

For Family 3, notice that the intersection is a quadric $q_4'(x_1, x_2, x_3)$, where the points $(1:k:l:0:0)$, with k, l solutions of $q_4(x_1, x_2, 0, 0)$ are all singular, non-isolated points. For Family 4, notice that the intersection is singular and non-reduced. For Family 5, the analysis is identical as to Family 3. Family 6, has non-isolated singularities at points $(1:k:0:0:0)$, lying on a line.

Family 7 has Segre symbol $[3,1,1]$ and as such has an $\mathbf{A}_2$ singularity at point $P = (1:0:0:0:0)$. Family 8 has Segre symbol $[3,2]$ (see Table 7.2) and as such has an $\mathbf{A}_2$ and two $\mathbf{A}_1$ singularities.

To conclude the proof, note that by Lemma 7.1 the base locus of the pencil has $\mathbf{A}_1$ singularities if and only if the determinant polynomial $\det(\lambda f + g)$ has roots of maximal multiplicity 2. Notice that the degeneration

$$f(x_0, x_1, x_2.x_3, x_4) = x_1^2 + x_0 l_1(x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2.x_3, x_4) = q_1(x_2, x_3, x_4)$$

of Family 2 has Segre symbol $[(2,1),2]$.

The degeneration

$$f = x_0x_4 + x_1x_3 + x_2^2$$
$$g = q_1(x_3, x_4) + x_3l_1(x_1, x_2) + x_4l_2(x_1, x_2)$$

of family 8 has Segre symbol $[(2, 1), (1, 1)]$, while the degeneration

$$f = q_1(x_1, x_2, x_3, x_4) + x_0x_1$$
$$g = x_4^2 + x_3^2 + x_2x_1$$

of family 2 (after the coordinate change $l_1(x_2, x_3, x_4) \to x_1$), has Segre symbol $[3, (1, 1)]$.

Hence, the above families and their possible degenerations represent all possible pencils of quadrics where the determinant polynomial has roots of multiplicity > 2. Hence, 2 and 3 are also equivalent.

□

Theorem 7.3. *The following are equivalent:*

1. *A pencil of two hyperquadrics $\Phi(f, g)$ in $\mathbb{P}^4$ is non-stable;*

2. *the base locus of the pencil, $\mathrm{Bs}(f, g)$, is singular;*

3. *the pencil is generated by one of the following families, or their degenerations:*

   Family 1:

$$f(x_0, x_1, x_2, x_3, x_4) = x_4l_1(x_0, x_1, x_2, x_3, x_4) + x_3l_2(x_0, x_1, x_2, x_3) + x_2l_3(x_0, x_1, x_2) + \alpha x_1^2$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_1(x_2, x_3, x_4)$$

   an irreducible smooth hyperquadric f and an irreducible singular hyperquadric g intersecting at a singular surface, with isolated $\mathbf{A}_3$ singularity at $(1 : 0 : 0 : 0 : 0)$;

   Family 2:

$$f(x_0, x_1, x_2, x_3, x_4) = q_2(x_0, x_1, x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_3(x_3, x_4)$$

   an irreducible hyperquadric f, and a pair of intersecting hyperplanes g, such that $\mathrm{Bs}(f, g)$ is a singular surface, with an isolated $\mathbf{A}_3$ singularity at (up to a change of basis) $(1 : 0 : 0 : 0 : 0)$;

   Family 3:

$$f(x_0, x_1, x_2, x_3, x_4) = q_4(x_0, x_1, x_2, x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4l_4(x_0, x_1, x_2, x_4)$$

an irreducible smooth hyperquadric f, and a pair of intersecting hyperplanes g, intersecting at a singular surface, with an isolated $\mathbf{A}_3$ singularity at (up to a change of basis) $(1:0:0:0:0)$;

Family 4:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_4 l_5(x_0,x_1,x_2,x_3,x_4) + q_5(x_1,x_2,x_3)\\
g(x_0,x_1,x_2,x_3,x_4) =& x_4 l_6(x_0,x_1,x_2,x_3,x_4) + q_6(x_1,x_2,x_3)
\end{aligned}
$$

two irreducible smooth hyperquadrics f, g intersecting at singular surface, with an $\mathbf{A}_1$ singularity at $(1:0:0:0:0)$;

Family 5:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_4 l_7(x_0,x_1,x_2,x_3,x_4) + x_3 l_8(x_0,x_1,x_2,x_3) + \alpha x_2^2\\
g(x_0,x_1,x_2,x_3,x_4) =& x_4 l_9(x_0,x_1,x_2,x_3,x_4) + x_3 l_{10}(x_0,x_1,x_2,x_3) + \beta x_2^2
\end{aligned}
$$

two irreducible hyperquadrics f, g intersecting at singular surface, with $\mathbf{A}_1$-singularities at (up to a change of basis) $(1:0:0:0:0)$ and $(0:1:0:0:0)$;

Family 6:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_4 l_{11}(x_0,x_1,x_2,x_3,x_4) + x_3 l_{12}(x_0,x_1,x_2,x_3) + q_7(x_1,x_2)\\
g(x_0,x_1,x_2,x_3,x_4) =& x_4 l_{13}(x_1,x_2,x_3,x_4) + x_3 l_{14}(x_1,x_2,x_3)
\end{aligned}
$$

an irreducible smooth hyperquadric f and an irreducible singular hyperquadric g intersecting at a singular surface, with isolated $\mathbf{A}_3$-singularity at $(1:0:0:0:0)$.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms.

The families presented here are maximal semi-destabilizing families as in the terminology of Definition 3.15. In particular, the pencil is strictly semistable if it is generated by either Families 4 or 5.

Proof. The equivalence of 1 and 3 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15.

Families 1 and 3 are identical to families 2 and 5 respectively of Theorem 7.2, so we direct the reader to the proof of that Theorem for more details.

For Family 2, the Segre symbol is $[(2,1),1,1]$ so by the Dolgachev classification, the base locus contains an $\mathbf{A}_3$ singular point $P = (1:0:0:0:0)$ up to a change of basis. For Family 4 the Segre symbol is $[2,1,1,1]$ and hence the base locus contains the singular point

$P = (1 : 0 : 0 : 0 : 0)$ which is an A_1 singularity. For Family 5 the Segre symbol is $[2, 2, 1]$, and hence the base locus contains singular points $P = (1 : 0 : 0 : 0 : 0)$, $Q = (0 : 1 : 0 : 0 : 0)$ which are A_1 singularities. For Family 6, the Segre symbol is $[(2, 1), 1, 1]$, and hence base locus contains the singular point $P = (1 : 0 : 0 : 0 : 0)$ which is an $\mathbf{A}_3$ singularity.

Notice, that a degeneration

$$f = q_1(x_1, x_2, x_3) + x_0x_4$$
$$g = q_2(x_1, x_2, x_3) + x_0x_4$$

of family 4 has Segre symbol $[(1, 1), 1, 1]$, while the degeneration

$$f = q_1(x_1, x_2, x_3) + x_0x_4$$
$$g = q_2(x_3, x_4) + l(x_3, x_4)(x_0, x_1, x_2)$$

of family 4 has Segre symbol $[(1, 1), 2, 1]$. In addition, the degeneration

$$f = x_4l_1(x_0, x_1) + x_3l_2(x_0, x_1) + x_2^2$$
$$g = x_4l_3(x_0, x_1) + x_3l_4(x_0, x_1) + x_2^2$$

of family 5 has Segre symbol $[(1, 1), (1, 1), 1]$. From Lemma 4.1 and Table 7.2 we notice that the 6 Families above constitute all the possible pairs of quadrics in $\mathbb{P}^4$ (along with their degenerations) such that their complete intersection is singular.

□

Theorem 7.4.

A pencil of two quadrics $\Phi(f, g)$ in $\mathbb{P}^4$ is polystable if and only if it is generated by one of the following families:

Family 1:

$$f(x_0, x_1, x_2, x_3) = x_0x_4 + q_4(x_1, x_2, x_3)$$
$$g(x_0, x_1, x_2, x_3) = x_0x_4 + q_5(x_1, x_2, x_3)$$

two irreducible smooth quadrics f, g intersecting at a singular surface, with 2 $\mathbf{A}_1$-singularities at $(1 : 0 : 0 : 0 : 0)$ and $(0 : 0 : 0 : 0 : 1)$;

Family 2:

$$f(x_0, x_1, x_2, x_3) = x_4l_3(x_0, x_1) + x_3l_4(x_0, x_1) + x_2^2$$
$$g(x_0, x_1, x_2, x_3) = x_4l_5(x_0, x_1) + x_3l_6(x_0, x_1) + x_2^2$$

two irreducible smooth quadrics f, g intersecting at a singular surface, with $\mathbf{A}_1$-singularities at (up to a change of basis) $(1:0:0:0:0)$,$(0:1:0:0:0)$, $(0:0:0:1:0)$ *and* $(0:0:0:0:1)$,

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms.

Proof. First notice that by Lemma 4.1 the above families are singular, and hence by Theorem 7.3 they are non-stable. The above families have all determinant polynomials with roots of multiplicity 2, hence by Table 7.2, [Dol12, Table 8.6] and Lemma 7.1 they have $\mathbf{A}_1$ singularities. This, alongside with Theorem 7.2 implies that the above families are strictly semistable.

In more detail, the Segre symbol for the pencil of Family 4 is $[(1,1),1,1,1]$, and by the classification their base locus contains 2 $\mathbf{A}_1$ singularities. Similarly, the Segre symbol of Family 5 is $[(1,1),(1,1),1]$, and its base locus contains four $\mathbf{A}_1$ singularities.

For Family 1 we can choose a one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s,1,1,1,s^{-1})$ such that $\lim_{s\to 0} f\wedge g = f\wedge g$, hence the pencil is polystable.

For Family 2 we can make a suitable change of basis such that $f = ax_0x_4 + x_2^2$, $g = bx_1x_3 + x_2^2$, and choosing one parameter subgroup $\lambda(s) = \mathrm{Diag}(s,s,1,s^{-1},s^{-1})$ we have $\lim_{s\to 0} f\wedge g = f\wedge g$, which shows that the pencil is polystable.

By Theorem 3.19 a complete intersection S, defined by $S = \{f = g = 0\}$, that belongs to a closed strictly semistable orbit is generated by monomials in the set $N^0(\lambda, x^{J_1})$, for some (λ, x^{J_1}). The above families correspond to the only such $N^0(\lambda, x^{J_1})$ (up to projective equivalence). In particular, these are obtained by verifying which $N^-(\lambda, x^J)$ give strictly semistable families, for various support monomials x^J, and then computing $N^0(\lambda, x^{J_1})$ by the description in Lemma 3.18. Notice that the dimension of the moduli space is $2\Big(\binom{4+2}{2} - 2\Big) - \big((4+1)^2 - 1\big) = 2(15-2) - 24 = 2$, so these are the only two polystable families.

□

Remark 7.4.1. Notice that up to projective equivalence, from Theorem 4.3, we can write Family 1 as

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& x_0^2 + x_4^2 + x_1^2 + x_2^2 + x_3^2 \\ g(x_0,x_1,x_2,x_3,x_4) =& x_0^2 + x_4^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \end{aligned}$$

and Family 2 as

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ g(x_0,x_1,x_2,x_3,x_4) =& \lambda_0 x_0^2 + \lambda_1 x_1^2 + x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 \end{aligned}$$

where the λ_i are distinct and $\lambda_i \neq 1$, which correspond, up to projective equivalence to the polystable families given in [MM90, Theorem A, Theorem B, Remark 6.9] and [OSS16, Theorem 4.1].

7.3 Classifying the Singularities of Pairs $(S, D = S \cap H)$

Following the discussion of Section 3, we take $S = C_1 \cap C_2$, and $D = S \cap H$, where the C_i are hyperquadrics in $\mathbb{P}^4$ and H is a hyperplane. The lemmas below serve as to help with the geometric classification of such pairs.

Lemma 7.5. *Let S be a smooth complete intersection of two quadrics and H a general hyperplane. Then D has at worse $\mathbf{D}_4$ singularities.*

Proof. Let S be given by f, g and H be a hyperplane where

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_1, x_2, x_3, x_4) + x_0 l_1(x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_2(x_0, x_1, x_2) + x_3 l_3(x_1, x_2) \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4. \end{aligned}$$

Then S has Segre symbol $[1, 1, 1, 1, 1]$ and is smooth and D is given by

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_1, x_2, x_3) + x_0 l x_3 \\ g(x_0, x_1, x_2, x_3) =& x_3^2 + x_3 l_1(x_1, x_2) \end{aligned}$$

which is an intersection of two quadrics in $\mathbb{P}^3$ with Segre symbol $[(3, 1)]$ and by Sommerville [Som59, §XIII] and the proof of Theorem 5.7 it has $\mathbf{D}_4$ singularities.

Similarly let S be given by f, g and H be a hyperplane where

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_2, x_3, x_4) + x_1 l_1(x_2, x_3, x_4) + x_0 l_2(x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_2, x_3, x_4) + x_1 l_3(x_3, x_4) + x_0 x_4 \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4. \end{aligned}$$

Then S has Segre symbol $[1, 1, 1, 1, 1]$ and is smooth and D is given by

$$\begin{aligned} f(x_0, x_1, x_2, x_3) =& q_1(x_2, x_3) + x_1 l_1(x_2, x_3) + x_0 x_3 \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_2, x_3) + x_1 x_3 \end{aligned}$$

which is an intersection of two quadrics in $\mathbb{P}^3$ with Segre symbol $[4]$ and by Table 5.2 it is a twisted cubic with a tangent line, and it has $\mathbf{A}_3$ singularities.

Now let S be given by f, g and H be a hyperplane where

$$f(x_0,x_1,x_2,x_3,x_4) = q_1(x_1,x_2,x_3,x_4) + x_0 l_1(x_2,x_3,x_4)$$
$$g(x_0,x_1,x_2,x_3,x_4) = q_2(x_2,x_3,x_4) + x_4 l_1(x_0,x_1)$$
$$H(x_0,x_1,x_2,x_3,x_4) = x_4.$$

Then S has Segre symbol $[1,1,1,1,1]$ and is smooth and D is given by

$$f(x_0,x_1,x_2,x_3) = q_1(x_1,x_2,x_3) + x_0 l_1(x_2,x_3)$$
$$g(x_0,x_1,x_2,x_3,x_4) = q_2(x_2,x_3)$$

which is an intersection of two quadrics in $\mathbb{P}^3$ with Segre symbol $[3,1]$ and by Table 5.2 it is a cuspidal curve with $\mathbf{A}_2$ singularities.

To conclude, let S be given by f, g and H be a hyperplane where

$$f(x_0,x_1,x_2,x_3,x_4) = q_1(x_1,x_2,x_3,x_4) + x_0 l_1(x_3,x_4)$$
$$g(x_0,x_1,x_2,x_3,x_4) = q_2(x_2,x_3,x_4) + x_1 l_2(x_3,x_4) + x_0 x_4$$
$$H(x_0,x_1,x_2,x_3,x_4) = x_4.$$

Then S has Segre symbol $[1,1,1,1,1]$ and is smooth and D is given by

$$f(x_0,x_1,x_2,x_3) = q_1(x_1,x_2,x_3) + x_0 x_3$$
$$g(x_0,x_1,x_2,x_3,x_4) = q_2(x_2,x_3) + x_1 x_3$$

which is an intersection of two quadrics in $\mathbb{P}^3$ with Segre symbol $[2,1,1]$ and by Table 5.2 it is a nodal curve with $\mathbf{A}_1$ singularities. $\square$

Lemma 7.6. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(1,1),(1,1),1]$. Then S has 4 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has up to an $\mathrm{SL}(5)$-action:*

1. *no singularities if and only if $H = \{l(x_1,x_2,x_3,x_4) = 0\}$;*
2. *an $\mathbf{A}_1$ singularity at one of the singularities of S if and only if $H = \{l(x_2,x_3,x_4) = 0\}$;*
3. *4 $\mathbf{A}_1$ singularities at the singularities of S if and only if $H = \{x_2 = 0\}$.*

Proof. From Table 7.2 we know that S is given, up to projective equivalence, by

$$f = x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1) + x_2^2$$
$$g = x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1) + x_2^2$$

Let $H = \{(x_1, x_2, x_3, x_4) = 0\}$; using a suitable change of coordinates $\tilde{x}_4 = l(x_1, x_2, x_3, x_4)$, $\tilde{x}_i = x_i$ for $i \neq 4$ we have (by abuse of notation):

$$\begin{aligned} f &= ll_1(x_0, x_1) + x_3 l_2(x_0, x_1) + x_2^2 \\ g &= ll_3(x_0, x_1) + x_3 l_4(x_0, x_1) + x_2^2 \\ H &= x_4 \end{aligned}$$

and hence D will be given by:

$$\begin{aligned} f' &= q_1(x_1, x_2, x_3) + x_0 l_1(x_1, x_2, x_3) \\ g' &= q_2(x_1, x_2, x_3) + x_0 l_2(x_1, x_2, x_3) \end{aligned}$$

which is a smooth complete intersection of two quadrics in $\mathbb{P}^3$.

Let $H = \{l(x_2, x_3, x_4) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$\begin{aligned} f &= ll_1(x_0, x_1) + x_3 l_2(x_0, x_1) + x_2^2 \\ g &= ll_3(x_0, x_1) + x_3 l_4(x_0, x_1) + x_2^2 \\ H &= x_4 \end{aligned}$$

and hence D will be given by:

$$\begin{aligned} f' &= q_1(x_2, x_3) + x_0 l_1(x_2, x_3) + x_1 l_2(x_2, x_3) \\ g' &= q_2(x_2, x_3) + x_0 l_3(x_2, x_3) + x_1 l_4(x_2, x_3) \end{aligned}$$

which is a singular complete intersection of quadrics in $\mathbb{P}^3$ with Segre symbol $[2, 2]$ by Table 5.2 and an $\mathbf{A}_1$ singularity at $(1 : 0 : 0 : 0)$.

Similarly, let $H = \{x_2 = 0\}$; here, D will be given by:

$$\begin{aligned} f' &= x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1) \\ g' &= x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1) \end{aligned}$$

which is a singular complete intersection of quadrics in $\mathbb{P}^3$ with Segre symbol $[(1, 1), (1, 1)]$ by Table 5.2 and 4 $\mathbf{A}_1$ singularities at $(1 : 0 : 0 : 0)$.

□

Lemma 7.7. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(1,1),1,1,1]$. Then S has 2 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. 1 $\mathbf{A}_1$ *singularity at one of the singularities of S if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$ or $H = \{l(x_2, x_3, x_4) = 0\}$ or $H = \{l(x_3, x_4) = 0\}$;*
2. 2 $\mathbf{A}_1$ *singularity at the singularities of S if and only if $H = \{l(x_1, x_2, x_3) = 0\}$;*
3. *non-isolated singularities if and only if $H = \{x_4 = 0\}$.*

Proof. From Table 7.2 we know that S is given, up to projective equivalence, by

$$f = q_1(x_1, x_2, x_3) + x_0x_4$$
$$g = q_2(x_1, x_2, x_3) + x_0x_4$$

Let $H = \{l(x_1, x_2, x_3, x_4) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l$$
$$g = q_2(x_1, x_2, x_3) + x_0 l$$
$$H = x_4$$

and hence D will be given by:

$$f' = q_1(x_1, x_2, x_3) + x_0 l(x_1, x_2, x_3)$$
$$g' = q_2(x_1, x_2, x_3) + x_0 l(x_1, x_2, x_3)$$

which is a singular complete intersection of quadrics with an $\mathbf{A}_1$ singularity at point (up to $\mathrm{SL}(3)$-action) $(1:0:0:0)$.

Similarly, let $H = \{l(x_2, x_3, x_4) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l$$
$$g = q_2(x_1, x_2, x_3) + x_0 l$$
$$H = x_4$$

and hence D will be given by:

$$f' = q_1(x_1, x_2, x_3) + x_0 l(x_2, x_3)$$
$$g' = q_2(x_1, x_2, x_3) + x_0 l(x_2, x_3)$$

which is a singular complete intersection of quadrics with an $\mathbf{A}_1$ singularity at point (up to $\mathrm{SL}(3)$-action) $(1:0:0:0)$.

Similarly, let $H = \{l(x_3, x_4) = 0\}$; using a similar suitable change of coordinates we have (by abuse of notation):

$$f = q_1(x_1, x_2, x_3) + x_0 l$$
$$g = q_2(x_1, x_2, x_3) + x_0 l$$
$$H = x_4$$

and hence D will be given by:

$$f' = q_1(x_1, x_2, x_3) + x_0 x_3$$
$$g' = q_2(x_1, x_2, x_3) + x_0 x_3$$

which is a singular complete intersection of quadrics with an $\mathbf{A}_1$ singularity at point (up to $\mathrm{SL}(3)$-action) $(1:0:0:0)$.

Now, let $H = \{x_4 = 0\}$; then D will be given by:

$$f' = q_1(x_1, x_2, x_3)$$
$$g' = q_2(x_1, x_2, x_3)$$

which is a singular non-reduced complete intersection of quadrics in $\mathbb{P}^3$.

To conclude, let $H = \{l(x_1, x_2, x_3) = 0\}$. By making the coordinate change $x_3 = l(x_1, x_2, x_3)$ we get D:

$$f' = q_1(x_1, x_2) + x_0 x_4$$
$$g' = q_2(x_1, x_2) + x_0 x_4$$

which has Segre symbol $[(1,1),1,1]$, and two $A1$ singularities at $(1:0:0:0:0)$ and $(0:0:0:0:1)$, which are the singular points of S. $\square$

The rest of the proofs of this section are identical in method to the proofs of Lemmas 7.7 and 7.6, which in turn use the same method of proof for the lemmas of Section 5.3, and as such will be omitted.

Lemma 7.8. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(1,1),2,1]]$. Then S has 3 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *no singularities if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$or $H = \{l(x_2, x_3, x_4) = 0\}$;*
2. *2 $\mathbf{A}_1$ singularities at two of the singularities of S if and only if $H = \{l(x_3, x_4) = 0\}$;*
3. *non-isolated singularities if and only if $H = \{x_4 = 0\}$.*

Lemma 7.9. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[2,2,1]$. Then S has 2 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *no singularities if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$;*
2. *1 $\mathbf{A}_1$ singularity at one of the singularities of S if and only if $H = \{l(x_2, x_3, x_4) = 0\}$ or $H = \{l(x_3, x_4) = 0\}$ or $H = \{x_4 = 0\}$.*

Lemma 7.10. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[2,1,1,1]$. Then S has 1 $\mathbf{A}_1$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *1 $\mathbf{A}_1$ singularity at the singularity of S if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$ or $H = \{l(x_2, x_3, x_4) = 0\}$ or $H = \{l(x_3, x_4) = 0\}$;*
2. *non-isolated singularities if and only if $H = \{x_4 = 0\}$.*

Lemma 7.11. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(2,1),(1,1)]$. Then S has 1 $\mathbf{A}_1$ and 2 $\mathbf{A}_3$ singularities. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *1 $\mathbf{A}_1$ singularity at the $\mathbf{A}_3$ singularity if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$ or $H = \{l(x_2, x_3, x_4) = 0\}$;*
2. *1 $\mathbf{A}_3$ singularity at the $\mathbf{A}_3$ singularity if and only if $H = \{x_2 = 0\}$;*
3. *non-isolated singularities if and only if $H = \{l(x_3, x_4) = 0\}$ or $H = \{x_4 = 0\}$.*

Lemma 7.12. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(2,1),1,1]$. Then S has 1 $\mathbf{A}_3$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. 1 $\mathbf{A}_1$ *singularity at the* $\mathbf{A}_3$ *singularity if and only if* $H = \{l(x_1, x_2, x_3, x_4) = 0\}$*;*
2. 1 $\mathbf{A}_3$ *singularity at the* $\mathbf{A}_3$ *singularity if and only if* $H = \{l(x_2, x_3, x_4) = 0\}$ *or* $H = \{l(x_3, x_4) = 0\}$ *or* $H = \{x_4 = 0\}$*.*

Lemma 7.13. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(2,1),2]$. Then S has 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_3$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. 1 $\mathbf{A}_1$ *singularity at the* $\mathbf{A}_3$ *singularity if and only if* $H = \{l(x_1, x_2, x_3, x_4) = 0\}$*;*
2. 1 $\mathbf{A}_3$ *singularity at the* $\mathbf{A}_3$ *singularity if and only if* $H = \{l(x_2, x_3, x_4) = 0\}$ *or* $H = \{l(x_3, x_4) = 0\}$*;*
3. 1 $\mathbf{D}_4$ *singularity at the* $\mathbf{A}_3$ *singularity if and only if* $H = \{x_4 = 0\}$*.*

Lemma 7.14. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[3,(1,1)]$. Then S has 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_2$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. 2 $\mathbf{A}_1$ *singularities at the singularities of S if and only if* $H = \{l(x_1, x_2, x_3, x_4) = 0\}$*;*
2. 1 $\mathbf{A}_1$ *singularity at the* $\mathbf{A}_2$ *singularity if and only if* $H = \{l(x_2, x_3, x_4) = 0\}$*;*
3. 1 $\mathbf{A}_2$ *singularity at the* $\mathbf{A}_2$ *singularity if and only if or* $H = \{l(x_3, x_4) = 0\}$ *or* $H = \{x_4 = 0\}$*.*

Lemma 7.15. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[3,1,1]$. Then S has 1 $\mathbf{A}_2$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. 1 $\mathbf{A}_1$ *singularity at the* $\mathbf{A}_2$ *singularity if and only if* $H = \{l(x_1, x_2, x_3, x_4) = 0\}$*;*
2. 1 $\mathbf{A}_2$ *singularity at the* $\mathbf{A}_2$ *singularity if and only if* $H = \{l(x_2, x_3, x_4) = 0\}$*;*
3. 1 $\mathbf{A}_3$ *singularity at the* $\mathbf{A}_2$ *singularity if and only if or* $H = \{l(x_3, x_4) = 0\}$*;*
4. *non-isolated singularities if* $H = \{x_4 = 0\}$*.*

Lemma 7.16. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[3, 2]$. Then S has 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_2$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *no singularities if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$;*
2. *2 $\mathbf{A}_1$ singularities at the singularities of S if and only if $H = \{l(x_2, x_3, x_4) = 0\}$;*
3. *non-isolated singularities if $H = \{l(x_3, x_4) = 0\}$ or $H = \{x_4 = 0\}$.*

Lemma 7.17. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(3, 1), 1]$. Then S has 1 $\mathbf{D}_4$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *no singularities if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$;*
2. *2 $\mathbf{A}_1$ singularities, where one singularity is the $\mathbf{D}_4$ singularity if and only if $H = \{l(x_2, x_3, x_4) = 0\}$;*
3. *a double conic if and only if $H = \{l(x_3, x_4) = 0\}$;*
4. *1 $\mathbf{A}_3$ singularity away from the $\mathbf{D}_4$ singularity if and only if $H = \{x_4 = 0\}$.*

Lemma 7.18. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[4, 1]$. Then S has 1 $\mathbf{A}_3$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *1 $\mathbf{A}_1$ singularity if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$;*
2. *2 $\mathbf{A}_1$ singularities if and only if $H = \{l(x_2, x_3, x_4) = 0\}$;*
3. *1 $\mathbf{A}_3$ singularity if and only if $H = \{l(x_3, x_4) = 0\}$;*
4. *non-isolated singularities if and only if $H = \{x_4 = 0\}$.*

Lemma 7.19. *Let S be the complete intersection of two quadrics f, g with Segre symbol $[(4, 1)]$. Then S has 1 $\mathbf{D}_5$ singularity. Let H be a hyperplane. Then, the hyperplane section $D = S \cap H$ has/is up to an* $\mathrm{SL}(5)$*-action:*

1. *1 $\mathbf{A}_1$ singularity at the $\mathbf{D}_5$ singularity if and only if $H = \{l(x_1, x_2, x_3, x_4) = 0\}$;*

2. $2\ \mathbf{A}_1$ *singularities where one lies is the* $\mathbf{D}_5$ *singularity at* S *if and only if* $H = \{l(x_2, x_3, x_4) = 0\}$*;*

3. *non-isolated singularities if and only if* $H = \{l(x_3, x_4) = 0\}$ *or* $H = \{x_4 = 0\}$*;*

4. $1\ \mathbf{A}_2$ *singularity away from the singular point* P *if and only if* S *is given by*

$$f(x_0, x_1, x_2, x_3, x_4) = x_2^2 + ax_1x_3 + bx_0x_4$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_3^2 + cx_1x_4$$

and $H = \{x_4 = 0\}$.

As a direct result from the above theorems, we have the following Lemma:

Lemma 7.20. *Let* (S, D) *be a pair that is invariant under a non-trivial* $\mathbb{G}_m$*-action. Suppose the singularities of* S *and* D *are given as in the first and second entries in one of the rows of Table 7.5, respectively. Then* (S, D) *is projectively equivalent to* $(f = g = 0, f = g = H = 0)$ *for* f*,* g *as in Table 7.2 corresponding to the Segre symbol of row* 3 *of Table 7.5, and* H *as in the fourth entries in the same row of Table 7.5, respectively. In particular, any such pair* (S, D) *is unique up to projective equivalence. Conversely, if* (S, D) *is given by equations as in the third and fourth entries in a given row of Table 7.5, then* (S, D) *has singularities as in the first and second entries in the same row of Table and* (S, D) *is* $\mathbb{G}_m$*-invariant. Furthermore the one-parameter subgroup* $\lambda(s) \in \mathrm{SL}(5)$*, given in the entry of the corresponding row of Table 7.5 is a generator of the* $\mathbb{G}_m$*-action.*

Sing(S)	Sing(D)	Segre Symbol	H	$\lambda(s)$
$4\mathbf{A}_1$	$4\mathbf{A}_1$ at points	$[(1,1),(1,1),1]$	x_2	$\mathrm{Diag}(s,s,1,s^{-1},s^{-1})$
$2\mathbf{A}_1$ at P, Q	$2\mathbf{A}_1$ at points	$[(1,1),1,1,1]$	$l(x_1,x_2,x_3)$	$\mathrm{Diag}(s,1,1,1,s^{-1})$
$2\mathbf{A}_1+\mathbf{A}_2$ at P, Q, R	non-isolated, $D=2L+L_1+L_2$	$[3,2]$	x_4	$\mathrm{Diag}(s,1,1,1,s^{-1})$
$\mathbf{A}_1+\mathbf{A}_2$ at $P, Q,$	non-isolated, $D=2L+L_1+L_2$	$[3,(1,1)]$	x_4	$\mathrm{Diag}(s,1,1,1,s^{-1})$
$2\mathbf{A}_1+\mathbf{A}_3$ at P, Q, R	double conic	$[(2,1),(1,1)]$	x_4	$\mathrm{Diag}(s^7,s^2,s^{-3},s^{-3},s^{-3})$
$\mathbf{A}_1+\mathbf{A}_3$ at P, Q	$\mathbf{A}_3$ at P	$[(2,1),(1,1)]$	x_4	$\mathrm{Diag}(s^7,s^2,s^{-3},s^{-3},s^{-3})$
$\mathbf{A}_3$ at P	$\mathbf{D}_4$ at P	$[4,1]$	x_4	$\mathrm{Diag}(s^7,s^2,s^2,s^{-3},s^{-8})$
$\mathbf{A}_3+\mathbf{A}_1$ at P, R	$\mathbf{A}_1$ at R	$[(2,1),2]$	x_4	$\mathrm{Diag}(s^7,s^2,s^2,s^{-3},s^{-8})$
$\mathbf{D}_4$ at P	$\mathbf{A}_3$ not P	$[(3,1),1]$	x_4	$\mathrm{Diag}(s^9,s^4,s^{-1},s^{-1},s^{-11})$
$\mathbf{D}_5$ at P	$\mathbf{A}_2$ not P	$[(4,1)]$	x_0	$\mathrm{Diag}(s^9,s^4,s^{-1},s^{-1},s^{-11})$

Table 7.5: Some pairs (S,D) invariant under a $\mathbb{G}_m$-action.

7.4 VGIT Classification

From the algorithm described in Section 3 and the computational package [Pap22c] we obtain the following walls and chambers:

	t_0		t_1		t_2		t_3		t_4		t_5		t_6
walls	0		$\frac{1}{6}$		$\frac{2}{7}$		$\frac{3}{8}$		$\frac{6}{11}$		$\frac{2}{3}$		1
chambers		$\frac{37}{228}$		$\frac{327}{1162}$		$\frac{113}{304}$		$\frac{1039}{1914}$		$\frac{355}{534}$		$\frac{37}{38}$	

We thus obtain 11 non-isomorphic quotients $M^{GIT}_{3,2,2}(t_i)$, which are characterised by the following two Theorems.

Theorem 7.21. *Let (S,D) be a pair where S is a complete intersection of two quadrics in $\mathbb{P}^4$ and $D=S\cap H$ is a hyperplane section.*

1. $t \in (0, \frac{1}{6})$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_1$ *singularities, where* D *may be non-reduced, or if* S *is smooth and* D *has at worse a* $\mathbf{D}_4$ *singularity.*

2. $t = \frac{1}{6}$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_1$ *singularities and* D *is reduced and has at worst* $\mathbf{A}_1$ *singularities, or if* S *is smooth and* D *has at worse a* $\mathbf{D}_4$ *singularity.*

3. $t \in (\frac{1}{6}, \frac{2}{7})$: *The pair* (S, D) *is* t*-stable if and only if* S *has at worse finitely many* $\mathbf{A}_2$ *singularities and* D *is reduced and smooth, or if* S *is smooth and* D *has at worse a* $\mathbf{D}_4$ *singularity.*

4. $t = \frac{2}{7}$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *can have at worse singularities of type* $\mathbf{A}_3$*, or if* S *is smooth and* D *has at* D *has at worse a* $\mathbf{D}_4$ *singularity.*

5. $t \in (\frac{2}{7}, \frac{3}{8})$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *has at worse a* $\mathbf{D}_4$ *singularity.*

6. $t = \frac{3}{8}$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *has at worse an* $\mathbf{A}_3$ *singularity.*

7. $t \in (\frac{3}{8}, \frac{6}{11})$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *has at worse an* $\mathbf{A}_2$ *singularity.*

8. $t = \frac{6}{11}$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *has at worse a* $\mathbf{A}_1$ *singularities.*

9. $t \in (\frac{6}{11}, \frac{2}{3})$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *has at worse a* $\mathbf{A}_1$ *singularities.*

10. $t = \frac{2}{3}$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{A}_3$ *singularities and* D *is smooth, or if* S *is smooth and* D *is smooth.*

11. $t \in (\frac{2}{3}, 1)$: *The pair* (S, D) *is* t*-stable if and only* S *has at worse finitely many* $\mathbf{D}_5$ *singularities and* D *is smooth, or if* S *is smooth and* D *is smooth.*

Theorem 7.22. *Let* $t \in (0, 1)$*. If* t *is a chamber, or* $t = t_5$*, then* $\overline{M(t)}$ *is the compactification of the stable loci* $M(t)$ *by the closed* $\mathrm{SL}(5)$*-orbit in* $\overline{M(t)} \setminus M(t)$ *represented by the pair* $(\tilde{S}, \tilde{D})$*, where* $\tilde{S}$ *is the unique* $\mathbb{G}_m$*-invariant complete intersection of two quadrics with Segre symbol* $[(1, 1), (1, 1), 1]$

and 4 $\mathbf{A}_1$ singularities, and $\tilde{D}$ is the union of the unique four lines in $\tilde{S}$, each of them passing through two of those singularities, or the pair (S', D'), where S' is the complete intersection with Segre symbol $[(1,1),1,1,1]$ and 2 $\mathbf{A}_1$ singularities, and D' is two conics in general position with 2 $\mathbf{A}_1$ singularities at the singular points of S'. If $t = t_i$, for $i = 1,2,3,4,5$, then $\overline{M(t_i)}$ is the compactification of the stable loci $M(t_i)$ by the three closed $\mathrm{SL}(5)$-orbits in $\overline{M(t)} \setminus M(t)$ represented by the uniquely defined pairs $(\tilde{S}, \tilde{D})$, (S', D') described above, and the $\mathbb{G}_m$-invariant pairs (S_i, D_i), (S'_i, D'_i) uniquely defined as follows:

1. *the complete intersection S_1 of two quadrics with Segre symbol $[3,2]$ with 2 $\mathbf{A}_1$ and 1 $\mathbf{A}_2$ singularities, and the divisor $D_1 \in |-K_{S_1}|$, where $D_1 = 2L + L_1 + L_2$ (a double line and two lines meeting at two points), with non-isolated singularities, and the complete intersection S'_1 of two quadrics with Segre symbol $[3,(1,1)]$ with 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_2$ singularities, and the divisor $D'_1 \in |-K_{S'_1}|$, where $D'_1 = 2L + L_1 + L_2$ (a double line and two lines meeting at two points), with non-isolated singularities;*

2. *the complete intersection S_2 of two quadrics with Segre symbol $[(2,1),(1,1)]$ with 2 $\mathbf{A}_1$ and 1 $\mathbf{A}_3$ singularities, and the divisor $D_2 \in |-K_{S_2}|$, where D_2 is a double conic, and the complete intersection S'_2 of two quadrics with Segre symbol $[(2,1),2]$ with 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_3$ singularities, and the divisor $D'_2 \in |-K_{S'_2}|$, where D'_2 has an $\mathbf{A}_3$ singularity at the $\mathbf{A}_1$ singularity of S'_2;*

3. *the complete intersection S_3 of two quadrics with Segre symbol $[4,1]$ with 1 $\mathbf{A}_3$ singularity, and the divisor $D_3 \in |-K_{S_3}|$, where D_3 is a conic and two lines intersecting in one point, with a $\mathbf{D}_4$ singularity at the $\mathbf{A}_3$ singularity of S_2;*

4. *the complete intersection S_4 of two quadrics with Segre symbol $[(3,1),1]$ with 1 $\mathbf{D}_4$ singularity, and the divisor $D_4 \in |-K_{S_4}|$, where D_4 is two tangent conics with a $\mathbf{A}_3$ singularity, and D_4 does not contain the singular point of S_4;*

5. *the complete intersection S_4 of two quadrics with Segre symbol $[(4,1)]$ with 1 $\mathbf{D}_5$ singularity, and the divisor $D_5 \in |-K_{S_5}|$, where D_5 is a cuspidal curve with a $\mathbf{A}_2$ singularity, and D_5 does not contain the singular point of S_5.*

The lemma below, will aide us with the classification.

Lemma 7.23. *Let (S, D) be a pair.*

1. *If S is reducible or not normal, then (S, D) is t-unstable for all $t \in [0, 1)$.*

2. *If D is not reduced, then, (S, D) is t-unstable for all $t \in (1/6, 1]$.*

Proof. For the first part, the reducible case follows from Theorem 3.14. By Serre's criterion for normality, the complete intersection S is not normal if it has non-isolated singularities. Assume, without loss of generality that the non-isolated singularities are at the points $(a : b : 0 : 0 : 0)$, $(a : b : c : 0 : 0)$, $(a : b : c : d : 0)$. Then, the singular complete intersection will be given by either

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_3(x_2, x_3, x_4) + x_4 l_1(x_0, x_1) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_4(x_2, x_3, x_4) + x_4 l_2(x_0, x_1) \end{aligned}$$

or

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_3(x_3, x_4) + x_4 l_1(x_0, x_1, x_2) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_4(x_3, x_4) + x_4 l_2(x_0, x_1, x_2) \end{aligned}$$

or

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) &= q_1(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= x_3 x_4 \end{aligned}$$

or

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) &= q_1(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= x_4^2. \end{aligned}$$

Let $H(x_0, x_1, x_2, x_3, x_4) = l(x_0, x_1, x_2, x_3, x_4)$ be a general hyperplane.

For the first possibility, let $\lambda(s) = \mathrm{Diag}(s^4, s^4, s^{-1}, s^{-1}, s^{-6})$; then $\mu_t(S, H, \lambda) = -4 - 4t < 0$ for all $t \in [0, 1)$. Similarly, for the second case let $\lambda(s) = \mathrm{Diag}(s^4, s^4, s^{-1}, s^{-1}, s^{-6})$; then $\mu_t(S, H, \lambda) = -4 - 4t < 0$ for all $t \in [0, 1)$. For the third case let $\lambda(s) = \mathrm{Diag}(s^1, s^1, s^1, s^1, s^{-4})$; then $\mu_t(S, H, \lambda) = 2 - 3 + t = -1 + t < 0$ for all $t \in [0, 1)$. For the last case let $\lambda(s) = \mathrm{Diag}(s^2, s, 1, 0, s^{-3})$; then $\mu_t(S, H, \lambda) = 4 - 6 + 2t = -2 + 2t < 0$ for all $t \in [0, 1)$. Hence, in all cases, the pair is unstable.

For the second part, without loss of generality we may assume that the pair is given by

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_1, x_2, x_3, x_4) + x_0 l_1(x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_2(x_0, x_1, x_2) \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4. \end{aligned}$$

Then, let $\lambda(s) = \mathrm{Diag}(s^4, s^4, s^{-1}, s^{-1}, s^{-6})$ be a one-parameter subgroup. Then, $\mu_t(S, H, \lambda) = 1 - 6t < 0$ if $t > \frac{1}{6}$. □

The families presented below have been produced via the algorithm described in Chapter 3, via the computational package [Pap22c] and are all t-unstable with respect to the respective t via the centroid criterion (Theorem 3.10). In addition, they are maximal t-destabilising families with respect to each wall/chamber t, in the sense of Definition 3.15.

7.4.1 Chamber $t = \frac{37}{228}$

For the first chamber with $t = \frac{37}{228} \in (0, \frac{1}{6})$ we have:

Lemma 7.24. *1. the pair (S, H) is non-t-stable if and only if the pair is generated by one of the following families, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_1,x_2,x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_2(x_1,x_2,x_3,x_4) \\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

A singular complete intersection Swhich is a cone over a singular complete intersection in $\mathbb{P}^4$, with $D = S \cap H$ a general hyperplane section.

Family 2:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_2,x_3,x_4) + x_4 l_1(x_0,x_1) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_2,x_3,x_4) + x_4 l_2(x_0,x_1) \\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

A singular complete intersection S whose singular locus contains a line, with $D = S \cap H$ a general hyperplane section.

Family 3:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_5(x_1,x_2,x_3,x_4) + x_0x_4 \\ g(x_0,x_1,x_2,x_3,x_4) =& q_6(x_2,x_3,x_4) + x_1x_4 \\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_2$ singularity, with $D = S \cap H$ a general hyperplane section.

Family 4*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_7(x_2,x_3,x_4) + x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_8(x_2,x_3,x_4) + x_4 l_5(x_0,x_1) + x_3 l_6(x_0,x_1)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with four $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with an $\mathbf{A}_1$ singularity at one of the singular points of S.

Family 5*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_9(x_1,x_2,x_3,x_4) + x_0 l_7(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_{10}(x_3,x_4) + x_3 l_8(x_1,x_2) + x_4 l_9(x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with an $\mathbf{A}_2$ singularity, with $D = S\cap H$ a general hyperplane section.

Family 6*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_{11}(x_2,x_3,x_4) + x_4 l_{10}(x_0,x_1) + x_3 l_{11}(x_0,x_1)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_{12}(x_3,x_4) + x_4 l_{12}(x_0,x_1,x_2) + x_3 x_2\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with an $\mathbf{A}_2$ and an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a general hyperplane section.

Family 7*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_{13}(x_0,x_1,x_2,x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_{14}(x_3,x_4)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with non-isolated singularities, with $D = S \cap H$ a general hyperplane section.

Family 8*:*

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_{15}(x_3,x_4) + x_4 l_{12}(x_0,x_1,x_2) + x_3 l_{13}(x_0,x_1,x_2)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_{16}(x_3,x_4) + x_4 l_{13}(x_0,x_1,x_2) + +x_3 l_{14}(x_0,x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with non-isolated singularities, with $D = S \cap H$ a general hyperplane section.

Family 9*:*

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_{17}(x_1, x_2, x_3, x_4) + x_4 x_0 \\ g(x_0, x_1, x_2, x_3, x_4) =& q_{18}(x_1, x_2, x_3, x_4) + x_4 x_0 \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_1, x_2, x_3, x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a singular hyperplane section with an $\mathbf{A}_1$ singularity at the singular point of S.

Family 10*:*

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_{19}(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& x_4 l_{15}(x_0, x_1, x_2, x_3, x_4) \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

A singular complete intersection S with non-isolated singularities, with $D = S \cap H$ a general hyperplane section.

Family 11*:*

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_{20}(x_1, x_2, x_3, x_4) + x_0 l_{16}(x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_{21}(x_2, x_3, x_4) \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section.

Here, the l_i are linear forms in $\mathbb{P}^3$ and the q_i are quadratic forms. In particular, the pair is strictly t-semistable if it is generated by Family 4 *or* 9*;*

2. *for $t \in (0, \frac{1}{6})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a general hyperplane section with at worst $\mathbf{D}_4$ singularities, or S has at worse* 2 *or* 3 $\mathbf{A}_1$ *singularities with D non-reduced, or* 1 *or* 4 $\mathbf{A}_1$ *singularities with D general.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^{-}(\lambda, x^J, x_p)$ for some

$\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3. In particular, the families with non-isolated singularities are non-normal from Serre's criterion

For part 2, suppose S is stable. From the classification Lemmas of Section 7.3 and part 1 we see that S cannot have $\mathbf{A}_2$ or worse singularities. From the above families, we see that

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_1,x_2,x_3,x_4) + x_0 l_1(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_2(x_3,x_4) + x_4 l_2(x_0,x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& x_4
\end{aligned}$$

is a stable pair where S has Segre symbol $[(1,1),1,1,1]$ and 2 $\mathbf{A}_1$ singularities from Table 7.2, and D is given by

$$\begin{aligned}
f'(x_0,x_1,x_2,x_3) &= q_1(x_1,x_2,x_3) + x_0x_3\\
g'(x_0,x_1,x_2,x_3) &= x_3^2
\end{aligned}$$

which is a double line and two lines in a triangle and is non-reduced.

In addition, the pair

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_5(x_1,x_2,x_3,x_4) + x_0 l_7(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_6(x_3,x_4) + x_4 l_8(x_0,x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& x_4
\end{aligned}$$

where S has 3 $\mathbf{A}_1$ singularities, and $D = S \cap H$ is hyperplane section with non-isolated singularities is also stable.

To conclude, we also see that the pair (S,H) where S is smooth and H is any hyperplane section is smooth, and by Lemma 7.5, we see that D can have up to $\mathbf{D}_4$ singularities. □

We also obtain:

Lemma 7.25. *For $t \in (0, \frac{1}{6})$ the pair (S,H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1)\\
g(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1)\\
H(x_0,x_1,x_2,x_3,x_4) =& x_2
\end{aligned}$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities;

Family 2:

$$f(x_0, x_1, x_2, x_3, x_4) = x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1) + x_2^2$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1) + x_2^2$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_1, x_2, x_3)$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above families represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t \in (0, \frac{1}{6})$.

Let (S, D) be strictly t-semistable, as in family 1, and take one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s, s, 1, s^{-1}, s^{-1})$. Then $\lim_{s \to 0} \lambda(s) \cdot (S, D) = (S, D)$ hence the pair is t-polystable by [Dol03, Remark 8.1 (5)] and Lemma 7.20. Similarly, the pair of family 2 is t-polystable via Lemma 7.20.

□

Remark 7.25.1. For each chamber $t \in (0, 1)$, we have that the only t-polystable families are given by the above families in Theorem 7.25. In addition, the above pairs are t-polystable for all $t \in (0, 1)$. We will usually denote the pair of family 1 by $(\tilde{S}, \tilde{D})$ and of family 2 by (S', D').

7.4.2 Wall $t = \frac{1}{6}$

For wall $t = 1/6$ we have the following.

Lemma 7.26. *1. The pair (S, H) is non-t-stable if and only the pair is generated by the families of chamber $\frac{37}{228}$ in addition to one of the following families, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_2,x_3,x_4) + x_0 l_1(x_2,x_3,x_4) + x_1 l_2(x_2,x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_2(x_2,x_3,x_4) + x_4 l_3(x_0,x_1) \\ H(x_0,x_1,x_2,x_3,x_4) =& x_4 \end{aligned}$$

A singular complete intersection S with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a general hyperplane section with non-isolated singularities.

Family 2:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_2,x_3,x_4) + x_1 l_4(x_2,x_3,x_4) + x_0 l_5(x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_2,x_3,x_4) + x_1 l_6(x_3,x_4) \\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_3,x_4) \end{aligned}$$

A singular complete intersection S with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a general hyperplane section with an $\mathbf{A}_3$ singularity at one of the $\mathbf{A}_1$ singularities of S.

Family 3:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_5(x_1,x_2,x_3,x_4) + x_0 l_7(x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_6(x_3,x_4) + x_4 l_8(x_0,x_1,x_2) \\ H(x_0,x_1,x_2,x_3,x_4) =& x_4 \end{aligned}$$

A singular complete intersection S with 3 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a hyperplane section with non-isolated singularities.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Families 3, 4 or 5 of chamber $\frac{37}{228}$;

2. *for $t = \frac{1}{6}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a general hyperplane section that can have at worst $\mathbf{D}_4$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_1$, with D having at worst $\mathbf{A}_1$ singularities. In particular, S can have 2 $\mathbf{A}_1$ singularities, provided that D has at worst 1 $\mathbf{A}_1$ singularity at one of the singularities of S, or 3 $\mathbf{A}_1$ singularities if D has at worst 2 $\mathbf{A}_1$ singularities at two of the singularities of S.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3. In particular, families with non-isolated singularities are non-normal from Serre's criterion.

For part 2, let S be t-stable. From the classification Lemmas of Section 7.3 and part 1, we see that S cannot have $\mathbf{A}_2$ or worse singularities. In addition, we see that S can have 2 $\mathbf{A}_1$ singularities, provided that D has at worst 1 $\mathbf{A}_1$ singularity at one of the singularities of S, or 3 $\mathbf{A}_1$ singularities if D has at worst 2 $\mathbf{A}_1$ singularities at two of the singularities of S. The proof is then completed via Lemma 7.24. □

We also obtain:

Lemma 7.27. *For $t = \frac{1}{6}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1) \\ g(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1) \\ H(x_0,x_1,x_2,x_3,x_4) =& x_2 \end{aligned}$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, and $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities;

Family 2:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_2(x_2,x_3) + x_1 x_4 \\ g(x_0,x_1,x_2,x_3,x_4) =& x_1^2 + x_0 x_4 \\ H(x_0,x_1,x_2,x_3,x_4) =& x_0 \end{aligned}$$

with S a singular complete intersection with Segre symbol $[3,2]$ with an $\mathbf{A}_1$ and an $\mathbf{A}_2$ singularity, with $D = S \cap H$ a singular hyperplane section with Segre symbol $[(2,2)]$ which is a double line and two lines in general position, with non-isolated singularities;

Family 3:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) &= x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1) + x_2^2 \\ g(x_0,x_1,x_2,x_3,x_4) &= x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1) + x_2^2 \\ H(x_0,x_1,x_2,x_3,x_4) &= l(x_1,x_2,x_3) \end{aligned}$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S;

Family 4:

$$f(x_0, x_1, x_2, x_3, x_4) = q(x_1, x_2) + x_0 l_5(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_1 l_6(x_3, x_4) + x_2 l_7(x_3, x_4)$$
$$H(x_0, x_1, x_2, x_3, x_4) = x_0$$

with S a singular complete intersection with 1 $\mathbf{A}_1$ and singularities 1 $\mathbf{A}_2$, with $D = S \cap H = 2L + L_1 + L_2$ a singular hyperplane section with Segre symbol $[(2, 2)]$ which is a double line and two lines in general position, with non-isolated singularities.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above 2 families represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{1}{6}$.

Families 1 and 3 are identical to the ones in Theorem 7.25.

$$f(x_0, x_1, x_2, x_3, x_4) = q_1(x_1, x_2) + l_7(x_3, x_4) l_5(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_3 l_6(x_1, x_2) + x_4 x_0$$
$$H(x_0, x_1, x_2, x_3, x_4) = l_7(x_3, x_4)$$

and for one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s, 1, 1, 1, s^{-1})$ we have

$$\lim_{s \to 0} \lambda(s) \cdot (f \wedge g) = q_1(x_1, x_2) \wedge x_0 x_4 \qquad \lim_{s \to 0} \lambda(s) \cdot H = H$$

i.e. we obtain a triple

$$f(x_0, x_1, x_2, x_3, x_4) = q_1(x_1, x_2)$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4 x_0$$
$$H(x_0, x_1, x_2, x_3, x_4) = l_7(x_3, x_4)$$

which is strictly t-semistable by the Centroid criterion 3.10. For Family 2 we write after a suitable change of basis

$$f(x_0, x_1, x_2, x_3, x_4) = x_0 x_4 + x_1 x_3$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_1^2 + x_2 x_1$$
$$H(x_0, x_1, x_2, x_3, x_4) = x_2$$

and for one-parameter subgroup $\lambda(s) = \mathrm{Diag}(s, 1, 1, 1, s^{-1})$ we have

$$\lim_{s \to 0} \lambda(s) \cdot (S, D) = (S, D).$$

For family 4 we can write up to projective equivalence

$$f(x_0, x_1, x_2, x_3, x_4) = x_1x_2 + x_0x_3 + x_0x_4$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_1x_3 + x_2lx_4$$
$$H(x_0, x_1, x_2, x_3, x_4) = x_0$$

Since (S, D) is strictly t-semistable for all cases, the stabiliser subgroup of (S, D), namely $G_{(S,D)} \subset \mathrm{SL}(5)$ is infinite (see, [Dol03, Remark 8.1(5)]). In particular, there is a $\mathbb{G}_m$-action on (S, D). The proof then follows from Lemma 7.20.

□

7.4.3 Chamber $t = \frac{327}{1162}$

For chamber $t = \frac{327}{1162} \in (\frac{1}{6}, \frac{2}{7})$ we have the following:

Lemma 7.28. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of wall $\frac{1}{6}$ (minus Families 3 and 6 from the chamber $t = \frac{37}{228}$) in addition to one of the following families, or their degenerations:*

Family 1:

$$f(x_0, x_1, x_2, x_3, x_4) = q_3(x_1, x_2, x_3, x_4) + x_0l_3(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_4(x_3, x_4) + x_4l_3(x_1, x_2) + x_4l_5(x_1, x_2)$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_1, x_2, x_3, x_4)$$

A singular complete intersection S with $\mathbf{A}_1$ singularities, with $D = S \cap H$ a hyperplane section with $\mathbf{A}_1$ singularities;

Family 2:

$$f(x_0, x_1, x_2, x_3, x_4) = q_5(x_2, x_3, x_4) + x_1l_4(x_2, x_3, x_4) + x_0x_4$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_6(x_2, x_3, x_4) + x_1x_4$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_0, x_1, x_2, x_3, x_4)$$

A singular complete intersection S with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section.

Family 3:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_2,x_3,x_4)+x_1l_6(x_3,x_4)+x_0l_7(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_3,x_4)+x_4l_8(x_0,x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

A singular complete intersection S with $\mathbf{A}_2$ singularities, with $D=S\cap H$ a general hyperplane section,

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Family 4 of chamber $\frac{37}{228}$;

2. *for chamber $t=\frac{327}{1162}\in(\frac{1}{6},\frac{2}{7})$, the only t-stable pairs $(S,D=S\cap H)$ occur when S is smooth, in which case D is a general hyperplane section with worst $\mathbf{D}_4$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_2$, with D a general hyperplane section.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda,x^J,x_p)$ for some $\lambda\in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3.

For part 2, let S be t-stable. From Table 7.2, the Lemmas of Section 7.3 and part 1, we see that S cannot have $\mathbf{A}_3$ or worse singularities. From the above families, we know that

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_1,x_2,x_3,x_4)+x_0l_7(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_3,x_4)+x_4l_8(x_0,x_1,x_2)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

and

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_2,x_3,x_4)+x_1l_6(x_3,x_4)+x_0l_7(x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_3,x_4)+x_4l_8(x_0,x_1,x_2)+x_2x_3\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4)
\end{aligned}
$$

are stable pairs where S has $\mathbf{A}_2$ singularities, and D is a general hyperplane section. □

The only polystable pairs correspond to pairs $(\tilde{S},\tilde{D})$ and (S',D'), as in Theorem 7.25.

7.4.4 Wall $t = \frac{2}{7}$

For wall $t = \frac{2}{7}$ we have the following.

Lemma 7.29. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of chamber $\frac{327}{1162}$ (minus Families 2 and 3 from wall $t = \frac{1}{6}$) in addition to one of the following families, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_1(x_0, x_1, x_2) \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4 \end{aligned}$$

A singular complete intersection S with $\mathbf{A}_1$ singularities, with $D = S \cap H$ a general hyperplane section with an $\mathbf{A}_2$ singularity at the $\mathbf{A}_2$ singularity of S.

Family 2:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_3(x_1, x_2, x_3, x_4) + x_0 l_2(x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_4(x_2, x_3, x_4) \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4 \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_3$ singularity, with $D = S \cap H$ a general hyperplane section with $\mathbf{A}_2$ singularity at the $\mathbf{A}_3$ singularity of S;

Family 3:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_5(x_1, x_2, x_3, x_4) + x_0 l_3(x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) =& q_6(x_2, x_3, x_4) + x_1 l_4(x_3, x_4) \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_3, x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a hyperplane section with $\mathbf{A}_1$ singularities.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Family 1 or by Family 4 of chamber $\frac{37}{228}$;

2. *for wall $t = \frac{2}{7}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a general hyperplane section with at worst $\mathbf{D}_4$ singularities, or S has at worse finitely*

many singularities at worst of type $\mathbf{A}_3$, with D having at worst an $\mathbf{A}_3$ singularity at the $\mathbf{A}_3$ singularity of S.

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3. In particular, families with non-isolated singularities are non-normal from Serre's criterion.

For part 2, let S be t-stable. From Table 7.2 and the Lemmas of Section 7.3, we see that S cannot have $\mathbf{A}_4$ or worse singularities. From the above families, we know that

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_3(x_1,x_2,x_3,x_4) + x_0 l_2(x_2,x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& q_4(x_2,x_3,x_4)\\
H(x_0,x_1,x_2,x_3,x_4) =& l(x_3,x_4)
\end{aligned}$$

is a stable pair where S has one $\mathbf{A}_3$ singularity, and D has an $\mathbf{A}_3$ singularity at the $\mathbf{A}_3$ singularity of S, by Lemma 7.12. □

We also obtain the following.

Lemma 7.30. *For $t = \frac{2}{7}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1)\\
g(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1)\\
H(x_0,x_1,x_2,x_3,x_4) =& x_2
\end{aligned}$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, and $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities.

Family 2:

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_2,x_3,x_4)\\
g(x_0,x_1,x_2,x_3,x_4) =& x_1^2 + x_0 l_5(x_2,x_3,x_4)\\
H(x_0,x_1,x_2,x_3,x_4) =& x_0
\end{aligned}$$

with S a singular complete intersection with Segre symbol $[(2,1),(1,1)]$ with 2 $\mathbf{A}_1$ (points P and Q) and an $\mathbf{A}_3$ singularity (point R), with $D = S \cap H$ a singular hyperplane section with Segre symbol $[(1,1,1),1]$ which is a double conic.

Family 3:

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) &= x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1) + x_2^2 \\
g(x_0,x_1,x_2,x_3,x_4) &= x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1) + x_2^2 \\
H(x_0,x_1,x_2,x_3,x_4) &= l(x_1,x_2,x_3)
\end{aligned}$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S;

Family 4:

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) &= q_2(x_3,x_4) + x_1 x_4 \\
g(x_0,x_1,x_2,x_3,x_4) &= x_1^2 + x_0 l_5(x_2,x_3) \\
H(x_0,x_1,x_2,x_3,x_4) &= x_4
\end{aligned}$$

with S a singular complete intersection with 1 $\mathbf{A}_1$ and 1 $\mathbf{A}_3$ singularities, with $D = S \cap H$ a singular hyperplane section with 1 $\mathbf{A}_3$ singularities, at the $\mathbf{A}_1$ singular point of S.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above families represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{2}{7}$.

The fact that Families 1 and 3 are t-polystable follows from Theorem 7.25. For Family 2, we can make a change of variables such that

$$\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_2 x_3 + x_4 x_3 \\
g(x_0,x_1,x_2,x_3,x_4) =& x_1^2 + x_0 x_2 + c x_0 x_3 \\
H(x_0,x_1,x_2,x_3,x_4) =& x_0
\end{aligned}$$

where $c \neq 0$, as otherwise S would not be normal, which is impossible from Lemma 7.23.

For family 4 we can make the change of coordinates

$$f(x_0, x_1, x_2, x_3, x_4) = x_3x_4 + x_1x_4$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_1^2 + x_0x_2 + cx_0x_3$$
$$H(x_0, x_1, x_2, x_3, x_4) = x_4$$

where $c \neq 0$ otherwise S would be non-normal.

Since (S, D) is strictly t-semistable the stabiliser subgroup of (S, D), namely $G_{(S,D)} \subset \mathrm{SL}(5)$ is infinite (see, [Dol03, Remark 8.1(5)]). In particular, there is a $\mathbb{G}_m$-action on (S, D); since all the possible pairs with $\mathbb{G}_m$-action have been classified in Lemma 7.20, the proof is finished.

□

7.4.5 Chamber $t = \frac{113}{304}$

For chamber $t = \frac{113}{304} \in (\frac{2}{7}, \frac{3}{8})$ we have the following.

Lemma 7.31. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of wall $\frac{2}{7}$ (minus families 5 and 11 from chamber $t = \frac{37}{228}$, family 3 from wall $t = \frac{1}{6}$, family 2 from wall $t = \frac{2}{7}$) in addition to one of the following families, or their degenerations:*

Family 1:

$$f(x_0, x_1, x_2, x_3, x_4) = q_1(x_1, x_2, x_3, x_4) + x_0x_4$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_2(x_3, x_4) + x_4l_1(x_1, x_2)$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_0, x_1, x_2, x_3, x_4)$$

A singular complete intersection S with $\mathbf{A}_1$ singularities, with $D = S \cap H$ a general smooth hyperplane section.

Family 2:

$$f(x_0, x_1, x_2, x_3, x_4) = q_3(x_1, x_2, x_3, x_4) + x_0l_2(x_3, x_4)$$
$$g(x_0, x_1, x_2, x_3, x_4) = q_4(x_3, x_4) + x_4l_3(x_1, x_2) + x_3l_4(x_1, x_2)$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_1, x_2, x_3, x_4)$$

A singular complete intersection S with an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a singular hyperplane section with an $\mathbf{A}_1$ singularity.

Family 3*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_5(x_1,x_2,x_3,x_4) + x_0 l_5(x_2,x_3,x_4)\\ g(x_0,x_1,x_2,x_3,x_4) =& q_6(x_2,x_3,x_4) + x_1 x_4\\ H(x_0,x_1,x_2,x_3,x_4) =& x_4 \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a hyperplane section with $\mathbf{A}_2$ singularities.

Family 4*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_7(x_2,x_3,x_4) + x_0 l_6(x_2,x_3,x_4) + x_1 l_7(x_2,x_3,x_4)\\ g(x_0,x_1,x_2,x_3,x_4) =& q_8(x_2,x_3,x_4)\\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_3$ singularity with $D = S \cap H$ a general hyperplane section.

Family 5*:*

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_9(x_1,x_2,x_3,x_4) + x_0 l_8(x_2,x_3,x_4)\\ g(x_0,x_1,x_2,x_3,x_4) =& q_{10}(x_2,x_3,x_4)\\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

A singular complete intersection S with an $\mathbf{A}_1$ singularity, with $D = S \cap H$ a general hyperplane section.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Family 4 *from chamber $t = \frac{37}{228}$;*

2. *for chamber $t = \frac{113}{304} \in (\frac{2}{7}, \frac{3}{8})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a general hyperplane section with at worst $\mathbf{D}_4$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_3$, with D a general hyperplane section.*

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3.

For part 2, let S be t-stable. From Table 7.2 and the Lemmas of Section 7.3, we see that S cannot have $\mathbf{A}_4$ or worse singularities. From the above family, we know that the pair

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_7(x_1,x_2,x_3,x_4)+x_0l_6(x_2,x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_8(x_2,x_3,x_4) \\ H(x_0,x_1,x_2,x_3,x_4) =& l(x_0,x_1,x_2,x_3,x_4) \end{aligned}$$

is stable, where $S=\{f=0\}\cap\{g=0\}$ has Segre symbol $[(2,1),1,1]$ and has an $\mathbf{A}_3$ singularity, and D is a general hyperplane section.

□

The only polystable pairs correspond to pairs $(\tilde{S},\tilde{D})$ and (S',D'), as in Theorem 7.25.

7.4.6 Wall $t=\frac{3}{8}$

For wall $t=\frac{3}{8}$ we have the following.

Lemma 7.32. *1. The pair (S,H) is non-t-stable if and only if the pair (S,H) is generated by the families of chamber $\frac{113}{334}$ in addition to the following family, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_1,x_2,x_3,x_4)+x_0l_1(x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) =& q_2(x_3,x_4)+x_4l_1(x_0,x_1,x_2)+x_3l_2(x_1,x_2,x_3) \\ H(x_0,x_1,x_2,x_3,x_4) =& x_4 \end{aligned}$$

A smooth complete intersection S, with $D=S\cap H$ a hyperplane section with $\mathbf{D}_4$ singularities.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Family 1 from chamber $t=\frac{37}{228}$ and 2 from chamber $t=\frac{327}{1162}$;

2. for wall $t=\frac{3}{8}$, the only t-stable pairs $(S,D=S\cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worst $\mathbf{A}_3$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_3$, with D general.

Proof. The proof follows from the proof of Lemma 7.31 and Lemma 7.5.

□

We also obtain the following.

Lemma 7.33. *For $t = \frac{3}{8}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1) \\ g(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1) \\ H(x_0, x_1, x_2, x_3, x_4) =& x_2 \end{aligned}$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, and $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities.

Family 2:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_2, x_3) + x_1 x_0 \\ g(x_0, x_1, x_2, x_3, x_4) =& x_1 l_5(x_2, x_3) + x_0 x_4 \\ H(x_0, x_1, x_2, x_3, x_4) =& x_4 \end{aligned}$$

with S a singular complete intersection with Segre symbol $[4, 1]$ with an $\mathbf{A}_3$ singularity (at point R), with $D = S \cap H$ a singular hyperplane section with Segre symbol $[(3, 1)]$ which is a conic and two lines intersecting in one point, with a $\mathbf{D}_4$ singularity (at R);

Family 3:

$$f(x_0, x_1, x_2, x_3, x_4) = x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1) + x_2^2$$
$$g(x_0, x_1, x_2, x_3, x_4) = x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1) + x_2^2$$
$$H(x_0, x_1, x_2, x_3, x_4) = l(x_1, x_2, x_3)$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above families represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{3}{8}$).

The fact that Families 1 and 3 are t-polystable follows from Theorem 7.25. For Family 2, we can make a change of variables such that

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) &= x_2x_3 + x_1x_4 \\ g(x_0,x_1,x_2,x_3,x_4) &= x_4x_0 + x_1x_2 + cx_1x_3 \\ H(x_0,x_1,x_2,x_3,x_4) &= x_4 \end{aligned}$$

where $c \neq 0$, as otherwise S would not be normal, which is impossible from Lemma 7.23. Since (S, D) is strictly t-semistable the stabiliser subgroup of (S, D), namely $G_{(S,D)} \subset \mathrm{SL}(5)$ is infinite (see, [Dol03, Remark 8.1(5)]). In particular, there is a $\mathbb{G}_m$-action on (S, D); since all the possible pairs with $\mathbb{G}_m$-action have been classified in Lemma 7.20, the proof is finished. □

7.4.7 Chamber $t = \frac{1039}{1914}$

For chamber $t = \frac{1039}{1914} \in (\frac{3}{8}, \frac{6}{11})$ we have the following.

Lemma 7.34. 1. *The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of wall $\frac{3}{8}$ (minus family 2 from chamber $t = \frac{327}{1162}$) in addition to the following family, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) &= q_1(x_2,x_3,x_4) + x_1l_1(x_2,x_3,x_4) + x_0l_2(x_3,x_4) \\ g(x_0,x_1,x_2,x_3,x_4) &= q_2(x_2,x_3,x_4) + x_1l_3(x_3,x_4) + x_0x_4 \\ H(x_0,x_1,x_2,x_3,x_4) &= x_4 \end{aligned}$$

A smooth complete intersection S , with $D = S \cap H$ a singular hyperplane section with $\mathbf{A}_3$ singularities.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by Family 4 of chamber $t = \frac{37}{228}$;

2. *for chamber $t = \frac{1039}{1914} \in (\frac{3}{8}, \frac{6}{11})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worst $\mathbf{A}_2$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_3$, with D a general hyperplane section.*

Proof. The proof follows from the proof of Lemma 7.32 and Lemma 7.5.

□

The only polystable pairs correspond to pairs $(\tilde{S}, \tilde{D})$ and (S', D'), as in Theorem 7.25.

7.4.8 Wall $t = \frac{6}{11}$

For wall $t = \frac{6}{11}$ we have the following

Lemma 7.35. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of chamber $\frac{1039}{1914}$ (minus family 1 from wall $t = \frac{1}{6}$, and 3 from chamber $t = \frac{113}{304}$) in addition to the following family, or their degenerations:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_1, x_2, x_3, x_4) + x_0 l_1(x_2, x_3, x_4)\\
g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_2, x_3, x_4) + x_4 l_1(x_0, x_1)\\
H(x_0, x_1, x_2, x_3, x_4) =& x_4
\end{aligned}
$$

A smooth complete intersection S, with $D = S \cap H$ a singular hyperplane section with $\mathbf{A}_2$ singularities.

Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by this family and family 4 from chamber $t = \frac{37}{228}$;

2. *for wall $t = \frac{6}{11}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worst $\mathbf{A}_1$ singularities, or S has at worse finitely many singularities at worst of type $\mathbf{A}_3$, with D a general hyperplane section.*

Proof. The proof follows from Lemma 7.34 and Lemma 7.5. □

We also obtain the following.

Lemma 7.36. *For $t = \frac{6}{11}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1)\\
g(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1)\\
H(x_0, x_1, x_2, x_3, x_4) =& x_2
\end{aligned}
$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, and $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities.

Family 2:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& q_1(x_2,x_3)+x_0x_4\\
g(x_0,x_1,x_2,x_3,x_4) =& x_0l_5(x_2,x_3)+x_1^2\\
H(x_0,x_1,x_2,x_3,x_4) =& x_4
\end{aligned}
$$

with S a singular complete intersection with Segre symbol $[(3,1),1]$ with a $\mathbf{D}_4$ singularity (at point R), with $D=S\cap H$ a singular hyperplane section with Segre symbol $[(2,1),1]$, which is two tangent conics with an $\mathbf{A}_3$ singularity (away from R);

Family 3:

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) &= x_4l_1(x_0,x_1)+x_3l_2(x_0,x_1)+x_2^2\\
g(x_0,x_1,x_2,x_3,x_4) &= x_4l_3(x_0,x_1)+x_3l_4(x_0,x_1)+x_2^2\\
H(x_0,x_1,x_2,x_3,x_4) &= l(x_1,x_2,x_3)
\end{aligned}
$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D=S\cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S.

Proof. Suppose (S,D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D=S\cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda,x^{J_1},x_i)$ for some (λ,x^{J_1},x_i) such that $N_t^0(\lambda,x^{J_1},x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above families represent the only maximal $N_t^0(\lambda,x^{J_1},x_i)$ for $t=\frac{6}{11}$.

The fact that Families 1 and 3 are t-polystable follows from Theorem 7.25. For Family 2, we can make a change of variables such that

$$
\begin{aligned}
f(x_0,x_1,x_2,x_3,x_4) =& x_2x_3+x_0x_4\\
g(x_0,x_1,x_2,x_3,x_4) =& x_2x_0+x_1^2+cx_0x_3\\
H(x_0,x_1,x_2,x_3,x_4) =& x_4
\end{aligned}
$$

where $c\neq 0$, as otherwise S would not be normal, which is impossible from Lemma 7.23. Since (S,D) is strictly t-semistable the stabiliser subgroup of (S,D), namely $G_{(S,D)}\subset \mathrm{SL}(5)$ is infinite (see, [Dol03, Remark 8.1(5)]). In particular, there is a $\mathbb{G}_m$-action on (S,D); since all the possible pairs with $\mathbb{G}_m$-action have been classified in Lemma 7.20, the proof is finished. $\square$

7.4.9 Chamber $t = \frac{355}{534}$

For chamber $t = \frac{355}{534} \in (\frac{6}{11}, \frac{2}{3})$ we have the following.

Lemma 7.37. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of wall $\frac{6}{11}$ (minus family 3 from wall $t = \frac{1}{6}$) in addition to the following family, or their degenerations:*

Family 1:

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_2, x_3, x_4) + x_1 l_1(x_3, x_4) + x_0 x_4 \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_1(x_1, x_2) \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

A singular complete intersection S with $\mathbf{D}_5$ singularities, with $D = S \cap H$ a general hyperplane section,

or a degeneration of the above families. Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by family 4 from chamber $t = \frac{37}{228}$;

2. for wall $t = \frac{355}{534} \in (\frac{6}{11}, \frac{2}{3})$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section with at worst $\mathbf{A}_1$ singularities, or S has at worse finitely many singularities of type $\mathbf{A}_3$, with D general.

Proof. Part 1 follows from the computational program [Pap22c] we detailed in Chapter 3 and the centroid criterion (Theorem 3.10), where the above families are maximal destabilising families as in the sense of Definition 3.15, i.e. each family equals $N^-(\lambda, x^J, x_p)$ for some $\lambda \in P_{3,2,2}$ and x^J, x_p support monomials. The description of the above families with respect to singularities follows from Sections 7.1 and 7.3. In particular, families with non-isolated singularities are non-normal from Serre's criterion.

For part 2, let S be t-stable. From Theorem 7.37 and the classifying Lemmas, we see that S cannot have worse than $\mathbf{A}_4$ singularities. From the above family, we see that the pair

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_2, x_3, x_4) + x_1 l_1(x_2, x_3, x_4) + x_0 x_4 \\ g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_1(x_1, x_2) \\ H(x_0, x_1, x_2, x_3, x_4) =& l(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

is stable, with S having $\mathbf{A}_3$ singularities and D being general. $\square$

The only polystable pairs correspond to pairs $(\tilde{S}, \tilde{D})$ and (S', D'), as in Theorem 7.25.

7.4.10 Wall $t = \frac{2}{3}$

For wall $t = \frac{2}{3}$ we have the following.

Lemma 7.38. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of chamber $\frac{355}{534}$ (minus the family of chamber $t = \frac{1039}{1914}$ and 5 from chamber $t = \frac{37}{228}$) in addition to the following family, or their degenerations:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_1, x_2, x_3, x_4) + x_0 l_1(x_3, x_4) \\
g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_2, x_3, x_4) + x_1 l_2(x_3, x_4) + x_0 x_4 \\
H(x_0, x_1, x_2, x_3, x_4) =& x_4
\end{aligned}
$$

A smooth complete intersection S, with $D = S \cap H$ a singular hyperplane section with $\mathbf{A}_1$ singularities, or a degeneration of the above families. Here, the l_i are linear forms in $\mathbb{P}^4$ and the q_i are quadratic forms. In particular, the pencil is strictly t-semistable if it is generated by family 4 from chamber $t = \frac{37}{228}$;

2. for wall $= \frac{2}{3}$, the only t-stable pairs $(S, D = S \cap H)$ occur when S is smooth, in which case D is a hyperplane section which is smooth, or S has at worse one singularity of type $\mathbf{A}_3$, with D general.

Proof. From Table 7.2 and the Lemmas of Section 7.3, we see that S cannot have worse than $\mathbf{A}_4$ singularities. From the above family, we see that S is smooth by our classification, and D has an $\mathbf{A}_1$ singularity, hence if S is smooth D also has to be smooth in order to be stable. □

Lemma 7.39. *For $t = \frac{6}{11}$ the pair (S, H) is strictly t-polystable if and only if it is generated by the following families:*

Family 1:

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_1(x_0, x_1) + x_3 l_2(x_0, x_1) \\
g(x_0, x_1, x_2, x_3, x_4) =& x_2^2 + x_4 l_3(x_0, x_1) + x_3 l_4(x_0, x_1) \\
H(x_0, x_1, x_2, x_3, x_4) =& x_2
\end{aligned}
$$

with S a singular complete intersection with four $\mathbf{A}_1$ singularities, and $D = S \cap H$ a singular hyperplane section with 4 $\mathbf{A}_1$ singularities.

Family 2:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) =& x_2^2 + ax_1x_3 + bx_0x_4 \\ g(x_0,x_1,x_2,x_3,x_4) =& x_3^2 + cx_1x_4 \\ H(x_0,x_1,x_2,x_3,x_4) =& x_0 \end{aligned}$$

with S a singular complete intersection with Segre symbol $[(4,1)]$ with a $\mathbf{D}_t$ singularity (at point R), with $D = S \cap H$ a singular hyperplane section with Segre symbol $[3,1]$, which is a cuspidal curve with an $\mathbf{A}_2$ singularity (away from R);

Family 3:

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4) &= x_4 l_1(x_0,x_1) + x_3 l_2(x_0,x_1) + x_2^2 \\ g(x_0,x_1,x_2,x_3,x_4) &= x_4 l_3(x_0,x_1) + x_3 l_4(x_0,x_1) + x_2^2 \\ H(x_0,x_1,x_2,x_3,x_4) &= l(x_1,x_2,x_3) \end{aligned}$$

with S a singular complete intersection with 2 $\mathbf{A}_1$ singularities, with $D = S \cap H$ a singular hyperplane section with 2 $\mathbf{A}_1$ singularities, at the singular points of S.

Proof. Suppose (S, D) is a pair, where S is a complete intersection defined by polynomials f and g, and $D = S \cap H$ is defined by a polynomial H, which belongs to a closed strictly t-semistable orbit. By Theorem 3.19, they are generated by monomials in $N_t^0(\lambda, x^{J_1}, x_i)$ for some (λ, x^{J_1}, x_i) such that $N_t^0(\lambda, x^{J_1}, x_i)$ is maximal with respect to the containment of order of sets. As detailed in Chapter 3, these can be generated algorithmically [Pap22c], and the above families represent the only maximal $N_t^0(\lambda, x^{J_1}, x_i)$ for $t = \frac{2}{3}$.

The fact that Families 1 and 3 are t-polystable follows from Theorem 7.25. Since (S, D) is strictly t-semistable the stabiliser subgroup of (S, D), namely $G_{(S,D)} \subset \mathrm{SL}(5)$ is infinite (see, [Dol03, Remark 8.1(5)]). In particular, there is a $\mathbb{G}_m$-action on (S, D); since all the possible pairs with $\mathbb{G}_m$-action have been classified in Lemma 7.20, the proof is finished. □

7.4.11 Chamber $t = \frac{37}{38}$

For chamber $t = \frac{37}{38} \in (\frac{2}{3}, 1)$ we have the following:

Lemma 7.40. *1. The pair (S, H) is non-t-stable if and only if the pair (S, H) is generated by the families of wall $t = \frac{2}{3}$ (minus the family of chamber $t = \frac{355}{534}$) or their degenerations. In particular, the pencil is strictly t-semistable if it is generated by family 4 from chamber $t = \frac{37}{228}$;*

2. *for chamber* $t = \frac{37}{38} \in (\frac{2}{3}, 1)$*, the only* t*-stable pairs* $(S, D = S \cap H)$ *occur when* S *is smooth, in which case* D *is a smooth hyperplane section, or* S *has at worse finitely many singularities of type* $\mathbf{D}_5$*, with* D *general.*

Proof. The pair (S, H)

$$
\begin{aligned}
f(x_0, x_1, x_2, x_3, x_4) =& q_1(x_2, x_3, x_4) + x_1 l_1(x_3, x_4) + x_0 x_4 \\
g(x_0, x_1, x_2, x_3, x_4) =& q_2(x_3, x_4) + x_4 l_1(x_1, x_2) \\
H(x_0, x_1, x_2, x_3, x_4) =& l(x_0, x_1, x_2, x_3, x_4)
\end{aligned}
$$

is stable, where by Table 7.2 S has an $\mathbf{D}_5$ singularity and by Lemma 7.19, D is smooth. □

The only polystable pairs correspond to pairs $(\tilde{S}, \tilde{D})$ and (S', D'), as in Theorem 7.25. From Section 7.4.1 to 7.4.11 we obtain the following proofs:

Proof of Theorem 7.21. The classification of stable orbits for each wall/chamber follows from Lemmas 7.24, 7.26, 7.28, 7.29, 7.31, 7.32, 7.34, 7.35 7.37, 7.38 and 7.40. □

Proof of Theorem 7.22. The classification of closed orbits follows from Lemmas 7.25, 7.27, 7.30, 7.33, 7.36, Lemma 7.20 and Remark 7.25.1. □

CHAPTER 8

CM Line Bundle for Complete Intersections and Hyperplane Section

Following the discussion in Section 3 we aim to calculate the log CM line bundle for complete intersections with a hyperplane section, introduced in Section 2. Let $S = \{f_1 = \dots = f_k = 0\}$ be the complete intersection of k hypersurfaces of degree d, with $f_i = \sum f_{I_{i,j}} x^{I_{i,j}}$, and H a hyperplane in $\mathbb{P}^n$. We define the sets

$$\begin{aligned}
M_1 =& \Big\{ (\bigwedge f_{I_{i,j}} x^{I_{i,j}}, l) \in \mathcal{R} | S = \{f_1 = \dots = f_k\}, H = \{l = 0\}, \mathrm{Supp}(H) \subseteq \mathrm{Supp}(f_i) \text{ for some } i \Big\} \\
M_2 =& \Big\{ (\bigwedge f_{I_{i,j}} x^{I_{i,j}}, l) \in \mathcal{R} | S = \{f_1 = \dots = f_k\}, H = \{l = 0\}, \exists H' \neq H, S \cap H' = S \cap H \Big\}.
\end{aligned} \tag{8.1}$$

Notice that by Theorem 3.14 the set M_1 contains only $t-$unstable elements for all $0 \leq t \leq t_{n,d,k}$ for $kd \leq n$.

Lemma 8.1. *The elements of M_2 are $t-$unstable for all $0 \leq t \leq t_{n,d,k}$, for $kd \leq n$.*

Proof. Let (S, H) be a pair parametrised by $(\bigwedge f_i, l) \in M_2$, such that $S \cap H = S \cap H'$ where without loss of generality, we may assume that $H = \{x_n = 0\}$, $H' = \{x_{n-1} = 0\}$. Without loss of generality, we can choose a coordinate system such that $S = \{f_1 = \dots = f_k\}$ is given by $f_i = x_n^d + x_{n-1}^d + x_n x_{n-1} f^i_{d-2}(x_0, \dots, x_n)$, $H = \{l(x_n, x_{n-1})$. Then for one-parameter subgroup

$\lambda = \mathrm{Diag}(s^2, \ldots, s^2, 0, s^{-n}, s^{-n})$, and for $0 \leq t \leq t_{n,d,k}$ we have:

$$\begin{aligned}\mu(S, H, \lambda) <&k[(d-2)2-n]-tn\\ \leq&2kd-4k-kn-kd\\ <&0\end{aligned}$$

so the pair is t-unstable for all $0 \leq t \leq t_{n,d,k}$. □

Let $\mathcal{T} := \mathcal{R} \setminus (M_1 \cup M_2)$, with natural embedding $j\colon \mathcal{T} \to \mathcal{R}$. The above discussion shows:

Lemma 8.2. *Let $kd \leq n$, $0 \leq t \leq t_{n,d,k}$. Then $(\mathcal{T})_t^{ss} := (\mathcal{R} \setminus (M_1 \cup M_1))_t^{ss} = (\mathcal{R})_t^{ss}$.*

Thus $\mathcal{T}$ parametrises pairs $(\bigwedge a_{I_{i,j}}, l)$ which correspond to complete intersections of k homogeneous polynomials of degree d and polynomials of degree 1 respectively.

Lemma 8.3. *For $kd \leq n$, $d \geq 2$ we have:*

1. $\mathrm{codim}(M_1) = \binom{n+d-1}{d} - (k-1)^2 \geq 2$;

2. $\mathrm{codim}(M_2) = k\Big(\binom{n+d}{d} - \binom{n+d-2}{d-2}\Big) - k^2 + n \geq 2.$

Proof. 1. Since $\mathrm{Supp}(H) \subseteq \mathrm{Supp}(f_i)$ for some i, without loss of generality we may assume $i = 1$, and assuming $H = \{l(x_0, \ldots, x_n) = 0\}$ we can write $f_1 = l(x_0, \ldots, x_n) f^i_{d-1}(x_0, \ldots, x_n)$ $f_i = f^i_d(x_0, \ldots, x_n)$. We first find the dimension of M_1: there are $n+1$ coefficients in the equation of l and $\binom{n+d-1}{d-1}$ coefficients in f^1_{d-1}. Similarly, each f_i has $\binom{n+d}{d}$ coefficients. Hence $\dim(M_1) = (k-1)\binom{n+d}{d} + \binom{n+d-1}{d-1} + n + 1 - 2k$, where we subtract 2 degrees of freedom for each f_i. Then:

$$\begin{aligned}\mathrm{codim}(M_1) =&\dim(\mathcal{R}) - \dim(M_1)\\ =&k\Big(\binom{n+d}{d} - k\Big) + n - (k-1)\binom{n+d}{d} - \binom{n+d-1}{d-1} - n - 1 + 2k\\ =&\binom{n+d-1}{d} - (k-1)^2\\ \geq&\Big(\frac{n+d-1}{d}\Big)^d - \big(\frac{n}{d} - 1\big)(k-1)\end{aligned}$$

Now, since

$$
\begin{aligned}
2+\frac{n}{d}(k-1) \leq & 2+\left(\frac{n}{d}-1\right)^2 \\
= & \frac{3d^2+n^2-2nd}{d^2} \\
\leq & \frac{(n+d-1)^2}{d^2} \\
\leq & \frac{(n+d-1)^d}{d^d}
\end{aligned}
$$

we get $\operatorname{codim}(M_1) \geq 2$

2. Let $(\bigwedge a_{I_{i,j}}, l) \in H_1$, $S = \{f_1 = \cdots = f_k = 0\}$, $H = \{l = 0\}$ and $H' \neq H$ such that $S \cap H = S \cap H'$. Without loss of generality, we assume that $H = \{x_n = 0\}$, $H' = \{x_{n-1} = 0\}$. Since the Cartier divisor $S \cap H = S \cap H'$ is supported on $L = \{x_n = x_{n-1} = 0\} \simeq \mathbb{P}^{n-2}$ and is a complete intersection of k hypersurfaces of degree d in $\{x_n = 0\} \simeq \mathbb{P}^{n-1}$, we have that $S \cap H$ is kdL, hence we can write $f_i = a_i x_{n-1}^d + x_n f_{d-1}^i(x_0, \ldots, x_n)$, and similarly for H', $f_i = b_i x_n^d + x_{n-1}(f^i)'_{d-1}(x_0, \ldots, x_n)$, which implies that we can write each $f_i = a_i x_n^d + b_i x_{n-1}^d + x_n x_{n-1} f_{d-2}^i$, and l as $l(x_n, x_{n-1})$. Similar to the proof of 1, there are $\binom{n+d-2}{d-2}$ coefficients in each f_{d-2}^i, and 2 coefficients in l. Hence, $\dim(M_2) = k\left(\binom{n+d-2}{d-2} + 2\right) - 2k$. Thus:

$$
\begin{aligned}
\operatorname{codim}(M_2) = & \dim(\mathcal{R}) - \dim(M_2) \\
= & k\left(\binom{n+d}{d} - k\right) + n - k\left(\binom{n+d-2}{d-2}\right) \\
= & k\left(\binom{n+d}{d} - \binom{n+d-2}{d-2}\right) - k^2 + n \\
\geq & k\left(\binom{n+d}{d} - \binom{n+d-1}{d-1}\right) - k^2 + n \\
= & k\binom{n+d-1}{d} - k^2 + n \\
\geq & k\binom{n+d-1}{d} - k^2 + kd \\
\geq & k\binom{n+d-1}{d} - k^2 + 2k - 1 \\
\geq & \binom{n+d-1}{d} - k^2 + 2k - 1 \\
= & \operatorname{codim}(M_1) \\
\geq & 2.
\end{aligned}
$$

$\square$

A direct consequence of the above lemma, is the result below, via [Har77, Prop. II.6.5.b].

Lemma 8.4.

$$\mathrm{Pic}(\mathcal{T}) \simeq \mathrm{Pic}(\mathcal{R}) \simeq \mathbb{Z}^2,$$

and for $\mathcal{L} \in \mathrm{Pic}(\mathcal{U})$*,*

$$\mathcal{L} \simeq \mathcal{O}_{\mathcal{U}}(a,b) := j^*(\mathcal{O}_{\mathcal{R}_{n,d,k}}(a) \boxtimes \mathcal{O}_{\mathcal{R}_{n,1}}(b))$$

We consider now the universal family of the complete intersection of k hypersurfaces of degree d:

$$\pi_{n,d,k} \colon \mathcal{X}_{n,d,k} \to \mathcal{R}_{n,d,k}$$

where:

$$\mathcal{X}_{n,d,k} = \left\{ (x_0, \dots, x_n) \times \bigwedge a_{I_{i,j}} \in \mathbb{P}^n \times \mathcal{R}_{n,d,k} | \sum a_{I_{1,j}} x^{I_{1,j}} = \dots = \sum a_{I_{k,j}} x^{I_{k,j}} = 0 \right\}.$$

(Here, by abuse of notation, we denote the class $[\bigwedge a_{I_{i,j}}]$ by $\bigwedge a_{I_{i,j}}$). We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "\pi"] & \mathcal{X}_{n,d,k} \times \mathcal{R}_{n,1} \arrow[r] \arrow[d, "\pi_{n,d,k} \times \mathrm{Id}_{\mathcal{R}_{n,1}}"] & \mathcal{X}_{n,d,k} \arrow[d, "\pi_{n,d,k}"] \\
\mathcal{T} \arrow[r, "j"] & \mathcal{R} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathcal{R}_{n,d,k} \\
 & \mathcal{R}_{n,1} &
\end{tikzcd}$$

with

$$\mathcal{X} = \left\{ (x_0, \dots, x_n) \times \bigwedge a_{I_i} \times (b_0, \dots, b_n) \in \mathbb{P}^n \times \mathcal{U} | \sum a_{I_{1,j}} x^{I_{1,j}} = \dots = \sum a_{I_{k,j}} x^{I_{k,j}} = 0 \right\}$$

the fiber product in the first diagram. Here, j is the natural embedding and p_i the projections. Since $\pi_{n,d,k} \colon \mathcal{X}_{n,d,k} \to \mathcal{R}_{n,d,k}$ is a universal family, it is flat and proper, and thus, by commutativity, π is also flat and proper.

Defining

$$\mathcal{D} := \left\{ (x_0, \dots, x_n) \times \bigwedge a_{I_i} \times (b_0, \dots, b_n) \in \mathcal{X} | \sum b_i x_i = 0 \right\}$$

$\mathcal{D}$ is a Cartier divisor of $\mathcal{X}$, and the restriction $\pi_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{T}$ is also flat and proper. This implies that $\pi \colon (\mathcal{X}, \mathcal{D}) \to \mathcal{T}$ is a $\mathbb{Q}$-Gorenstein flat family.

Notice, that in the Fano case, i.e. $kd \leq n$, $-K_{\mathcal{X}/\mathcal{T}}$ is relatively ample and by Theorem 3.14 and Lemma 8.4 $\Lambda_{CM,\beta}(-K_{\mathcal{X}/\mathcal{T}}) \simeq \mathcal{O}(a,b)$. Hence, we can extend the CM-line bundle to $\mathcal{R}$: for $\beta \in (0,1) \cap \mathbb{Q}$

$$\Lambda_{CM,\beta} := \Lambda_{CM,\beta}(-K_{\mathcal{X}/\mathcal{T}}) := \Lambda_{CM,\beta}(\mathcal{X}, \mathcal{D}, -K_{\mathcal{X}/\mathcal{T}}).$$

Lemma 8.5. *Let $kd \leq n$, $\beta \in (0,1] \cap \mathbb{Q}$. Then $\Lambda_{CM,\beta} \simeq \mathcal{O}(a(\beta), b(\beta))$ where:*

$$\begin{aligned} a(\beta) =& \big(n+1-kd\big)^{n-k-1} d^{k-1} \cdot \Big((n-k+1)(1-\beta)\big((1-d)n+1\big) \\ &+ \big(1+(n-k)(1-\beta)\big)\big(n+1-kd\big)(d-1)(n+1) \Big) > 0 \\ b(\beta) =& (n+1-kd)^{n-k} d^k (n-k+1)(1-\beta) > 0, \end{aligned}$$

i.e. $\Lambda_{CM,\beta}$ is ample. In particular, if $kd = n$,

$$\begin{aligned} a(\beta) =& d^{k-1}\big(d(n-k+1) - \beta\big) \\ b(\beta) =& d^k (n-k+1)(1-\beta) > 0 \\ t(\beta) =& \frac{d(n-k+1)(1-\beta)}{d(n-k+1)-\beta} \end{aligned}$$

Proof. For a: Consider $k+1$ hypersurfaces of degree d $f_1, \ldots, f_{k+1}$. Then blowing up $\mathbb{P}^n$ along $C = f_1 \cap f_2$ we obtain a map $\mathrm{Bl}_C \mathbb{P}^n \to \mathbb{P}^1$. Then we have a pencil of hypersurfaces of degree d $\{a_1 f_1 + a_2 f_2 | (a_1, a_2) \in \mathbb{P}^1\}$ and we define $f_{1:2} := \{a_1 f_1 + a_2 f_2 = 0\}$. Notice that the proper transform of $f_{1:2}$ is isomorphic to $f_{1:2}$. Intersecting $\tilde{f}_3, \ldots, \tilde{f}_{k+1}$ with $\tilde{f}_{1:k}$ gives a well-defined family for complete intersections of k hypersurfaces of degree d over $\mathbb{P}^1$. Letting $\mathcal{S} = \tilde{f}_3 \cap \cdots \cap \tilde{f}_{k+1} \cap \tilde{f}_{1:2}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathcal{X} \arrow[r] \arrow[d, "\pi"] & \mathcal{X}_{n,d,k} \times \mathcal{R}_{n,1} \arrow[r] \arrow[d, "\pi_{n,d,k} \times \mathrm{Id}_{\mathcal{R}_{n,1}}"] & \mathcal{X}_{n,d,k} \arrow[d, "\pi_{n,d,k}"] \\ \mathbb{P}^1 \arrow[r, hook, "i"] & \mathcal{T} \arrow[r, "j"] & \mathcal{R} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathcal{R}_{n,d,k} \\ & & \mathcal{R}_{n,1} & \end{tikzcd}$$

For a general hyperplane H, let $p_H \in \mathcal{R}_{n,1}$ be the point that parametrises H. Then, we have $\mathcal{S} \times p_H \subset \mathbb{P}^1 \times \mathbb{P}^n \times \mathcal{R}_{n,1}$. Notice that $\mathcal{S}$ is the one dimensional family of complete intersection of a hypersurface of bidegrees $(1,d)$ and $k-1$ hypersurfaces of bidegrees $(0,d)$ in $\mathbb{P}^1 \times \mathbb{P}^n$.

Then, for a hyperplane H parametrised by $P_H \in \mathcal{R}_{n,1}$ we have that

$$\mathcal{D} = \{p \in \mathbb{P}^1 \times \mathbb{P}^n | p \in \mathcal{S}, p|_{\mathbb{P}^n} \in H\}$$

is a divisor obtained as the complete intersection of a hypersurface of bidegree $(1, d)$, and $k-1$ hypersurfaces of bidegrees $(0, d)$ and a hypersurface of bidegree $(0, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^n$. For the corresponding projections $p_{\mathbb{P}^n}, p'_{\mathbb{P}^1}$ let $H_{\mathbb{P}^n} = p^*_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1))$, $H_{\mathbb{P}^1} = (p')^*_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1))$. Then we have by adjunction:

$$\begin{aligned}
K_{\mathcal{S}} =&(K_{\mathbb{P}^1 \times \mathbb{P}^n}) + \mathcal{S})|_{\mathcal{S}} \\
=&(-2H_{\mathbb{P}^1} - (n+1-d)H_{\mathbb{P}^n} + H_{\mathbb{P}^1} + (k-1)dH_{\mathbb{P}^n})|_{\mathcal{S}} \\
=&\big(- H_{\mathbb{P}^1} + (kd-n-1)H_{\mathbb{P}^n}\big)|_{\mathcal{S}}
\end{aligned}$$

Hence

$$\begin{aligned}
K_{\mathcal{S}/\mathbb{P}^1} =&K_{\mathcal{S}} - \pi^* K_{\mathbb{P}^1} \\
=&\big(- H_{\mathbb{P}^1} + (kd-n-1)H_{\mathbb{P}^n}\big)|_{\mathcal{S}} + (2H_{\mathbb{P}^{k-1}})|_{\mathcal{S}} \\
=&(H_{\mathbb{P}^1} + (kd-n-1)H_{\mathbb{P}^n})|_{\mathcal{S}} \\
-K_{\mathcal{S}/\mathbb{P}^1} =&(n+1-kd)H_{\mathbb{P}^n}|_{\mathcal{S}} - H_{\mathbb{P}^1}|_{\mathcal{S}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k} &= (n+1-kd)^{n-k} H^{n-k}_{\mathbb{P}^n}|_{\mathcal{S}} - (n-k)(n+1-kd)^{n-k-1} H^{n-k-1}_{\mathbb{P}^n}|_{\mathcal{S}} \cdot H_{\mathbb{P}^1}|_{\mathcal{S}} \\
(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k+1} &= (n+1-kd)^{n-k+1} H^{n-k+1}_{\mathbb{P}^n}|_{\mathcal{S}} - (n-k+1)(n+1-kd)^{n-k} H^{n-k}_{\mathbb{P}^n}|_{\mathcal{S}} \cdot H_{\mathbb{P}^1}|_{\mathcal{S}}
\end{aligned}$$

and since $\mathcal{S} = (H_{\mathbb{P}^1} + dH_{\mathbb{P}^n}) \cdot (dH_{\mathbb{P}^n})^{k-1}$, $\mathcal{D}|_{\mathcal{S}} = H_{\mathbb{P}^n}$ we obtain

$$\begin{aligned}
c_1(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k+1} =&\Big((n+1-kd)^{n-k+1} H^{n-k+1}_{\mathbb{P}^n} - (n-k+1)(n+1-kd)^{n-k} H^{n-k}_{\mathbb{P}^n} \cdot H_{\mathbb{P}^1}\Big) \\
&\cdot \Big(H_{\mathbb{P}^1} + dH_{\mathbb{P}^n}\Big) \cdot d^{k-1} H^{k-1}_{\mathbb{P}^n} \\
=& - d^k (n-k+1)(n+1-kd)^{n-k} H^n_{\mathbb{P}^n} \cdot H_{\mathbb{P}^1} \\
&+ d^{k-1}(n+1-kd)^{n-k+1} H^n_{\mathbb{P}^n} \cdot H_{\mathbb{P}^1} \\
c_1(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k} \cdot \mathcal{D} =& - d^k (n-k)(n+1-kd)^{n-k-1} H^n_{\mathbb{P}^n} \cdot H_{\mathbb{P}^1} \\
&+ d^{k-1}(n+1-kd)^{n-k} H^n_{\mathbb{P}^n} \cdot H_{\mathbb{P}^1}
\end{aligned}$$

Using [GMS21, Theorem 2.7], since $\mathcal{L} = -K_{\mathcal{S}/\mathbb{P}^1}$ and $\mathcal{D}|_{\mathcal{S}_i} \in |-K_{\mathcal{S}_i}|$, we have

$$\begin{aligned}
\deg((j \circ i)^*(\Lambda_{CM,\beta})) = -\big(1 + (n-k)(1-\beta)\big)\pi_*\Big(c_1(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k+1}\Big) \\
+ (1-\beta)(n-k+1)\pi_*\Big(c_1(-K_{\mathcal{S}/\mathbb{P}^1})^{n-k} \cdot \mathcal{D}\Big)
\end{aligned}$$

and since $\deg((j \circ i)^*(\Lambda_{CM,\beta})) = \deg((j \circ i)^* \circ p_1^* \mathcal{O}_{\mathcal{R}_{n,d,k}}(a) = a$ the result follows.

For b: Consider a hypersurface S which is the complete intersection of k hypersurfaces of degree d, $S = \{f_1, \ldots, f_k\}$, represented by $p_S \in \mathcal{R}_{n,d,k}$, and pencil of hyperplanes $H(t)$, $t \in \mathbb{P}^1$. Then:

$$\mathcal{D}|_{p_S \times \mathbb{P}^1} = \{f_1 = \cdots = f_k = f_{H(t)}\} \subset p_S \times \mathbb{P}^n \times \mathbb{P}^1.$$

This implies that $\mathcal{D}|_{p_S \times \mathbb{P}^1}$ is the complete intersection of k hypersurfaces of bidegree $(d, 0)$ and one of bidegree $(1, 1)$. Notice that $S \times \mathbb{P}^1/\mathbb{P}^1$ is a trivial fibration, so $c_1(-K_{S \times \mathbb{P}^1/\mathbb{P}^1})^{n-k+1} = 0$. We have:

$$K_{(S \times \mathbb{P}^1)/\mathbb{P}^1} = K_{S \times \mathbb{P}^1} - \pi^* K_{\mathbb{P}^1} = K_S \otimes \mathcal{O}_{\mathbb{P}^1} = (kd - n - 1) H_{\mathbb{P}^n}|_S \otimes \mathcal{O}_{\mathbb{P}^1},$$

where $H_{\mathbb{P}^n} = \pi^*_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence, from [GMS21, Theorem 2.7]:

$$\begin{aligned}
\deg(\Lambda_{CM,\beta}) =& (1-\beta)(n-k+1) c_1(-K_{(S \times \mathbb{P}^1)/\mathbb{P}^1})^{n-k} \cdot \mathcal{D} \\
=& (1-\beta)(n-k+1)(n+1-kd)^{n-k} H_{\mathbb{P}^n}^{n-k} \cdot d^k H_{\mathbb{P}^n}^k \cdot (H_{\mathbb{P}^n} + H_{\mathbb{P}^1}) \\
=& (1-\beta)(n-k+1)(n+1-kd)^{n-k} d^k H_{\mathbb{P}^n}^n \cdot H_{\mathbb{P}^1}
\end{aligned}$$

i.e. $b = (n+1-dk)^{n-k} d^k (n-k+1)(1-\beta) > 0$.

□

Corollary 8.5.1. *If $n = 4$, $d = k = 2$, $\beta \in (0,1] \cap \mathbb{Q}$ then $\Lambda_{CM,\beta} \simeq \mathcal{O}(a(\beta), b(\beta))$ where:*

$$\begin{aligned}
a(\beta) =& 2(6-\beta) > 0 \\
b(\beta) =& 12(1-\beta) > 0,
\end{aligned}$$

and $\Lambda_{CM,\beta}$ is ample. In particular

$$t(\beta) = \frac{b(\beta)}{a(\beta)} = \frac{6(1-\beta)}{6-\beta}.$$

Remark 8.5.1. A similar theorem is shown in [GMS21, Theorem 3.8] for the case $k = 1$.

Theorem 8.6. *Let (S, D) be a log Fano pair, where S is a complete intersection of two quadrics in $\mathbb{P}^4$ and D an anticanonical section. Let $\pi \colon \mathcal{X} \to \mathcal{T}$ the family introduced before, with ample log CM line bundle $\Lambda_{CM,\beta} \simeq \mathcal{O}(a(\beta), b(\beta))$. Suppose $(S, (1-\beta)D)$ is log K- (semi/poly)stable. Then, (S, D) is GIT$_{t(\beta)}$-(semi/poly)stable, with slope $t(\beta) = \frac{b(\beta)}{a(\beta)} = \frac{6(1-\beta)}{6-\beta}$.*

Proof. We only need to verify that the conditions for [ADL19, Theorem 2.22] are satisfied. Theorem 8.5 shows that $\Lambda_{CM,\beta} \simeq \mathcal{O}(a(\beta), b(\beta))$ and hence the log CM line bundle is ample, hence, condition 3 is satisfied. For the second condition, note that for S, S' complete intersections of two quadrics in $\mathbb{P}^4$, notice that $\mathcal{O}_S(1) \simeq K_S^{-1}$, $\mathcal{O}_{S'}(1) \simeq K_{S'}^{-1}$ are anticanonical, and hence invariant under automorphisms. Thus, every isomorphism $S \cong S'$ of two such complete intersections lifts to an automorphism of $\mathbb{P}^4$, i.e. to an action of $\mathrm{PGL}(5)$, so condition 2 is satisfied. Similarly, since automorphisms of S are induced by actions of $\mathrm{PGL}(5)$, and $\mathrm{Aut}(S)$ is finite, so is the stabiliser G_S, hence condition 1 follows. □

In fact there is an alternative method to prove this Theorem:

Alternative proof of Theorem 8.6. The proof follows the idea of proof in [OSS16, Theorem 3.4] and [GMS21, Theorem 3.10]. Consider a one-parameter subgroup λ acting on $p \in \mathcal{T}$, representing a log pair (S, H) with $H \not\subset S$, $D = S \cap H$. We have a natural projection $\pi\colon \overline{\mathcal{Y}} := \overline{\lambda \cdot p} \subset \mathcal{T} \times \mathbb{P}^1 \to \mathbb{P}^1$. By abuse of notation, we extend π to $\pi\colon \overline{\mathcal{Y}} \setminus \{\pi^{-1}(\infty)\} \to \mathbb{C}$, with $q := \pi^{-1}(0) \in \mathcal{T}$ the central fiber of π. Here q is a pair $(\overline{S}, \overline{H})$, where $\overline{S}$ is a complete intersection of two quadrics in $\mathbb{P}^4$, $\overline{H} \not\subset S$ a hyperplane and $\overline{D} = \overline{S} \cap \overline{H} \in |-K_{\overline{S}}|$ a hyperplane section. Since λ induces a test configuration, we know from [GMS21, Theorem 2.6] that

$$w(\Lambda_{CM,\beta}(\mathcal{X}, \mathcal{D}, \mathcal{L}^r)) = (n+1)!\,\mathrm{DF}_\beta(\mathcal{X}, \mathcal{D}, \mathcal{L})$$

hence if $(S, (1-\beta)D)$ is K-semistable then, since the one-parameter subgroup is arbitrary we obtain the result from Corollary 8.5.1, since

$$w(\Lambda_{CM,\beta}(\mathcal{X}, \mathcal{D}, \mathcal{L}^r)) = \mu^{\Lambda_{CM,\beta}}(S, H, \lambda) = \mu_{t(\beta)}(S, H, \lambda).$$

Suppose now that $(S, (1-\beta)D)$ is K-polystable. Then in particular it is K-semistable, and the point $p \in \mathcal{T}$ is $\mathrm{GIT}_{t(\beta)}$ semistable from the above discussion. Suppose that p which parametrises (S, H) is not $\mathrm{GIT}_t(\beta)$ polystable. Then, there exists a one-parameter subgroup λ such that $\overline{p} = \lim_{t\to 0} \lambda(t) \cdot p$ is $\mathrm{GIT}_{t(\beta)}$ polystable but not $\mathrm{GIT}_{t(\beta)}$ stable. λ induces a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ with $\mathrm{DF}_\beta(\mathcal{X}, \mathcal{D}, \mathcal{L}) = 0$ by [GMS21, Theorem 2.6]. Since $(S, (1-\beta)D)$ is K-polystable, we know that $(\mathcal{X}_p, \mathcal{D}_p) \simeq (S \times \mathbb{C}, D \times \mathbb{C})$. But then for the central fiber $(\overline{S}, \overline{D})$ of the test configuration corresponding to $\overline{p}$, we have $(\overline{S}, \overline{D}) = (S, D)$, i.e. $\overline{p} = p$, and hence p is $\mathrm{GIT}_{t(\beta)}$ polystable. □

CHAPTER 9

Proof of Main Theorem and First Wall Crossing

Consider now a log Fano pair $(S,(1-\beta)D)$ where S is a complete intersection of 2 quadrics (degree 2 hypersurfaces) in $\mathbb{P}^4$, D is a hyperplane section and $\beta \in (0,1)\cap\mathbb{Q}$. We will consider $\mathbb{Q}$-Gorenstein smoothable K-semistable log Fano pairs $(S_\infty,(1-\beta)D_\infty)$ such that their smoothing is a log Fano pair $(S,(1-\beta)D)$ as above. We can think of these as K-polystable limits of a degeneration family $\mathcal{X}$ of smooth K-polystable log Fano pairs, that have to be "added" to the boundary in order to compactify the K-moduli spaces.

Lemma 9.1. *Let $(S_\infty,(1-\beta)D_\infty)$ be a $\mathbb{Q}$-Gorenstein smoothable K-semistable log Fano pair such that its smoothing is a log Fano pair $(S_i,(1-\beta)D_i)$, where S_i is a del Pezzo surface of degree 4 (i.e. a smooth intersection of two quadrics in $\mathbb{P}^4$) and D_i a smooth hyperplane section. For any $\beta>\frac{3}{4}$, S_∞ is also an intersection of two quadrics in $\mathbb{P}^4$ whose singular locus consists of $\mathbf{A}_1$ or $\mathbf{A}_2$ singularities, and D_∞ is also a hyperplane section.*

Proof. Since each D_i is a hyperplane section, $D_i\sim -K_{S_i}$ and hence, by continuity of volumes, the degree of the limit pair is

$$(-K_{S_\infty}-(1-\beta)D_\infty)^2=(-K_{S_i}-(1-\beta)D_i)^2=(\beta K_{S_i})^2=4\beta^2.$$

By [Liu18, Theorem 3], since S_∞ is at worse klt, it must have only isolated quotient singularities isomorphic to $\mathbb{C}^2/G$, where G is a finite subgroup of $U(2)$ acting freely on S^3. This is implied by the fact that klt surface singularities are precisely quotient singularities [CKM88, Proposition 6.11], and that normal surfaces have only isolated singularities.

Moreover, the normal localised volume for quotient singularities is given by $\widehat{\mathrm{vol}}_{\mathbb{C}^2/G,0} = \frac{4}{|G|}$. Then, by Theorem 2.98, and by [GMS21, Theorem 4.1]

$$\widehat{\mathrm{vol}}_{(X,(1-\beta)D),p} \leq \widehat{\mathrm{vol}}_{X,p},$$

and hence, we have:

$$\begin{aligned} 4\beta^2 = (-K_{S_\infty} - (1-\beta)D_\infty)^2 \leq & \left(1+\frac{1}{2}\right)^2 \widehat{\mathrm{vol}}_{S_\infty,(1-\beta)D_\infty,p} \\ \leq & \frac{9}{4}\widehat{\mathrm{vol}}_{\mathbb{C}^2/G,0} \\ = & \frac{9}{|G|} \end{aligned}$$

i.e. $|G| \leq \frac{9}{4\beta^2}$.

By the classification of $\mathbb{Q}$-Gorenstein smoothable surface singularities [KS88, Proposition 3.10], G must be a cyclic group acting in $\mathrm{SU}(2)$. Hence, the singularities of S_∞ are canonical and, by the classification of del Pezzo surfaces with canonical singularities, S_∞ is a complete intersection of two quadrics in $\mathbb{P}^4$ with at worse $\mathbf{A}_1$ or $\mathbf{A}_2$ singularities. In particular, D_∞ is a hyperplane section, as $D_\infty \sim -K_{S_\infty} \sim \mathcal{O}_{S_\infty}(1)$ by the adjunction formula. $\square$

The above Lemma allows us to prove the following:

Theorem 9.2. *Let $\beta > \frac{3}{4}$. Then there exists an isomorphism of moduli stacks between the K-moduli stack $\mathcal{M}^K_{4,2,2}(\beta)$ of K-semistable families of $\mathbb{Q}$-Gorenstein smoothable log Fano pairs $(S,(1-\beta)D)$, where S is a complete intersection of two quadrics in $\mathbb{P}^4$ and D is an anticanonical section, and the GIT$_t$-moduli stack $\mathcal{M}^{GIT}_{4,2,2}(t(\beta))$. In particular, for $\beta > \frac{3}{4}$ we also have an isomorphism $M_{4,2,2}(\beta) \cong M^{GIT}_{4,2,2}(t(\beta))$ on the restriction to moduli spaces.*

Proof. Let $\boldsymbol{\mathfrak{X}} = (\mathcal{X}, \overline{\mathcal{X}})$ be the Hilbert polynomials of (S,D), with S a smooth complete intersection of two quadrics in $\mathbb{P}^4$, and D an anticanonical divisor, pluri-anticanonically embedded by $-mK_{\mathcal{X}}$ in $\mathbb{P}^N$, and let $\mathbb{H}^{\boldsymbol{\mathfrak{X}};N} := \mathbb{H}^{\mathcal{X};N} \times \mathbb{H}^{\overline{\mathcal{X}};N} := \mathrm{Hilb}_{\mathcal{X}}(\mathbb{P}^N) \times \mathrm{Hilb}_{\overline{\mathcal{X}}}(\mathbb{P}^N)$.

Given a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\mathcal{X}(X, \mathcal{O}_{\mathbb{P}^N}(k)|_X) = \mathcal{X}(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\mathcal{X};N}$ denote its Hilbert point. Let, as in [ADL19],

$$\hat{Z}_m := \left\{ \mathrm{Hilb}(X,D) \in \mathbb{H}^{\boldsymbol{\mathfrak{X}};N} \;\middle|\; \begin{array}{l} X \text{ is a Fano manifold which is the complete intersection of} \\ \text{two quadrics in } \mathbb{P}^4,\, D \sim_{\mathbb{Q}} -K_X \text{ a smooth divisor,} \\ \mathcal{O}_{P^N}(1)|_X \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)). \end{array} \right\}$$

which is a locally closed subscheme of $\mathbb{H}^{\mathfrak{X};N}$. Let $\overline{Z}_m$ be its Zariski closure in $\mathbb{H}^{\mathfrak{X};N}$ and Z_m be the subset of $\hat{Z}_m$ consisting of K-semistable varieties.

By Theorem 8.6 we know that if a pair $(S,(1-\beta)D)$ is K-(poly/semi)stable then (S,D) is GIT$_{t(\beta)}$ (poly/semi)stable, where $t(\beta)=\frac{6(1-\beta)}{6-\beta}$. By [Oda15], the smooth K-stable loci is a Zariski open set of $M^K_{4,2,2}(\beta)$, in the definition of moduli stack of $\mathcal{M}^K_{4,2,2}(\beta)=[Z_m/\operatorname{PGL}(N_m+1)]$ for appropriate $m>0$ and in fact $\mathcal{M}^{GIT}_{4,2,2}(t(\beta))\cong[\overline{Z}_m/\operatorname{PGL}(N_m+1)]$. Hence, by Theorems 7.21, 7.22, Corollary 8.5.1, Lemma 9.1, for $\beta>\frac{3}{4}$ we have an open immersion of representable morphism of stacks:

$$\mathcal{M}^K_{4,2,2}(\beta)\xrightarrow{\phi}\mathcal{M}^{GIT}_{4,2,2}(t(\beta))$$

$$[(S,(1-\beta)D)]\xmapsto{\phi}[(S,D)]$$

with an injective decent $\overline{\phi}$ on the moduli spaces such that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}^K_{4,2,2}(\beta) \arrow[r, "\phi"] \arrow[d] & \mathcal{M}^{GIT}_{4,2,2}(t(\beta)) \arrow[d] \\
M^K_{4,2,2} \arrow[r, "\overline{\phi}"] & M^{GIT}_{4,2,2}(t(\beta)).
\end{tikzcd}$$

Note that representability follows once we prove that the base-change of a scheme mapping to the K-moduli stack is itself a scheme. Such a scheme mapping to the K-moduli stack is the same as a PGL-torsor over $\overline{Z}_m$, which produces a PGL-torsor over Z_m after a PGL-equivariant base change. This PGL-torsor over Z_m shows the desired pullback is a scheme. By [The22, Lemma 06MY], since ϕ is an open immersion of stacks, ϕ is separated and, since it is injective, it is also quasi-finite.

We now need to check that ϕ is an isomorphism that descends (as isomorphism of schemes) to the moduli spaces. Now, by [Alp13, Prop 6.4], since ϕ is representable, quasi-finite and separated, $\overline{\phi}$ is finite and ϕ maps closed points to closed points, we obtain that ϕ is finite. Thus, by Zariski's Main Theorem, as $\overline{\phi}$ is a birational morphism with finite fibers to a normal variety, ϕ is an isomorphism to an open subset, but it is also an open immersion, thus it is an isomorphism.

□

Corollary 9.2.1 (First Wall Crossing). *The first wall crossing occurs at $t(\beta)=\frac{1}{6}$, $\beta=\frac{6}{7}$. In particular there exists an isomorphism of moduli stacks between the K-moduli stack $\mathcal{M}^K_{4,2,2}(\frac{6}{7})$ parametrising*

K-semistable families of $\mathbb{Q}$-Gorenstein smoothable log Fano pairs $(S, \frac{1}{7}D)$, where S is a complete intersection of two quadrics in $\mathbb{P}^4$ and D is an anticanonical section, and the GIT$_t$-moduli stack $\mathcal{M}^{GIT}_{4,2,2}(\frac{1}{6})$. In particular, a log Fano pair $(S, \frac{1}{7}D)$ is log K-polystable if S is a complete intersection of two quadrics with at 1 $\mathbf{A}_2$ and 2 or 1 $\mathbf{A}_1$ singularities, and D a singular hyperplane section which is a double line and two lines meeting at two points, or S is a complete intersection of two quadrics with 2 or 4 1 $\mathbf{A}_1$ singularities, and D a singular hyperplane section with 2 or 4 1 $\mathbf{A}_1$ singularities.

Proof. The proof follows directly from Theorems 7.21, 7.22 and 9.2. □

Remark 9.2.1. In particular, there exists such an isomorphism up to the second chamber. After the second chamber, one needs to consider different methods, as potential toric GH compactifications with non-quotient singularities can occur.

Bibliography

[AH99] A. A'Campo-Neuen and J. Hausen. "Quotients of toric varieties by the action of a subtorus". In: *Tohoku Math. J. (2)* 51.1 (1999), pp. 1–12.

[AH00] A. A'Campo-Neuen and J. Hausen. "Examples and counterexamples for existence of categorical quotients". In: *J. Algebra* 231.1 (2000), pp. 67–85.

[Alp13] J. Alper. "Good moduli spaces for Artin stacks". In: *Ann. Inst. Fourier (Grenoble)* 63.6 (2013), pp. 2349–2402.

[Alp22] J. Alper. "Notes on Stacks and Moduli". preprint at webpage https://sites.math.washington.edu/~jarod/moduli.pdf accessed on 09 July 2022. 2022.

[Alp+20] J. Alper, H. Blum, D. Halpern-Leistner, and C. Xu. "Reductivity of the automorphism group of K-polystable Fano varieties". In: *Invent. Math.* 222.3 (2020), pp. 995–1032.

[AHR20] J. Alper, J. Hall, and D. Rydh. "A Luna étale slice theorem for algebraic stacks". In: *Ann. of Math. (2)* 191.3 (2020), pp. 675–738.

[AHH19] J. Alper, D. Halpern-Leistner, and J. Heinloth. *Existence of moduli spaces for algebraic stacks*. 2019. arXiv: 1812.01128 [math.AG].

[Ara+21] C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süss, and N. Viswanathan. *The Calabi problem for Fano threefolds*. https://www.maths.ed.ac.uk/cheltsov/pdf/Fanos.pdf. 2021.

[Arn75] V. I. Arnol'd. "Critical points of smooth functions and their normal forms". In: *Russian Mathematical Surveys* 30.5 (1975), pp. 1–75.

[Arn76] V. Arnold. "Local Normal Forms of Functions." In: *Inventiones mathematicae* 35 (1976), pp. 87–110.

[ADL19] K. Ascher, K. DeVleming, and Y. Liu. *Wall crossing for K-moduli spaces of plane curves*. 2019. arXiv: 1909.04576 [math.AG].

[AL00] D. Avritzer and H. Lange. "Pencils of quadrics, binary forms and hyperelliptic curves". In: vol. 28. 12. Special issue in honor of Robin Hartshorne. 2000, pp. 5541–5561.

[Bat81] V. V. Batyrev. "Toric Fano threefolds". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 45.4 (1981), pp. 704–717, 927.

[Bea03] A. Beauville. *Complex algebraic surfaces*. Cambridge University Press, 2003.

[BJK17] G. Bérczi, J. Jackson, and F. Kirwan. "Variation of non-reductive geometric invariant theory". In: *Surveys in Differential Geometry* 22.1 (2017), pp. 49–69.

[Ber16] R. J. Berman. "K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics". In: *Invent. Math.* 203.3 (2016), pp. 973–1025.

[Bir19] C. Birkar. "Anti-pluricanonical systems on Fano varieties". In: *Ann. of Math. (2)* 190.2 (2019), pp. 345–463.

[Bir21] C. Birkar. "Singularities of linear systems and boundedness of Fano varieties". In: *Ann. of Math. (2)* 193.2 (2021), pp. 347–405.

[Blu+21] H. Blum, D. Halpern-Leistner, Y. Liu, and C. Xu. "On properness of K-moduli spaces and optimal degenerations of Fano varieties". In: *Selecta Mathematica* 27.4 (2021).

[BJ20] H. Blum and M. Jonsson. "Thresholds, valuations, and K-stability". In: *Adv. Math.* 365 (2020), pp. 107062, 57.

[BLX22] H. Blum, Y. Liu, and C. Xu. "Openness of K-semistability for Fano varieties". In: *Duke Mathematical Journal* -1.-1 (2022).

[BX19] H. Blum and C. Xu. "Uniqueness of K-polystable degenerations of Fano varieties". In: *Ann. of Math. (2)* 190.2 (2019), pp. 609–656.

[Bou+15] S. Boucksom, T. de Fernex, C. Favre, and S. Urbinati. "Valuation spaces and multiplier ideals on singular varieties". In: *Recent advances in algebraic geometry*. Vol. 417. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2015, pp. 29–51.

[Bra+21] L. Braun, D. Greb, K. Langlois, and J. Moraga. *Reductive quotients of klt singularities*. 2021. arXiv: 2111.02812 [math.AG].

[BW79] J. W. Bruce and C. T. C. Wall. "On the Classification of Cubic Surfaces". In: *Journal of the London Mathematical Society* s2-19.2 (1979), pp. 245–256.

[Che08] I. Cheltsov. "Log canonical thresholds of del Pezzo surfaces". In: *Geom. Funct. Anal.* 18.4 (2008), pp. 1118–1144.

[CDS13] X. Chen, S. Donaldson, and S. Sun. "Kähler–Einstein Metrics and Stability". In: *International Mathematics Research Notices* 2014.8 (2013), pp. 2119–2125.

[CKM88] H. Clemens, J. Kollár, and S. Mori. "Higher-dimensional complex geometry". In: *Astérisque* 166 (1988), 144 pp. (1989).

[CP21] G. Codogni and Z. Patakfalvi. "Positivity of the CM line bundle for families of K-stable klt Fano varieties". In: *Invent. Math.* 223.3 (2021), pp. 811–894.

[CR06] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Reprint of the 1962 original. AMS Chelsea Publishing, Providence, RI, 2006, pp. xiv+689.

[DS16] V. Datar and G. Székelyhidi. "Kähler-Einstein metrics along the smooth continuity method". In: *Geom. Funct. Anal.* 26.4 (2016), pp. 975–1010.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 75–109.

[DT92] W. Y. Ding and G. Tian. "Kähler-Einstein metrics and the generalized Futaki invariant". In: *Invent. Math.* 110.2 (1992), pp. 315–335.

[Dol03] I. Dolgachev. *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[Dol12] I. V. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012, p. 401.

[DH98] I. V. Dolgachev and Y. Hu. "Variation of geometric invariant theory quotients". In: *Inst. Hautes Études Sci. Publ. Math.* 87 (1998). With an appendix by Nicolas Ressayre, pp. 5–56.

[Don02] S. K. Donaldson. "Scalar curvature and stability of toric varieties". In: *J. Differential Geom.* 62.2 (2002), pp. 289–349.

[DK07] B. Doran and F. Kirwan. "Towards non-reductive geometric invariant theory". In: *Pure Appl. Math. Q.* 3.1, Special Issue: In honor of Robert D. MacPherson. Part 3 (2007), pp. 61–105.

[ELS03] L. Ein, R. Lazarsfeld, and K. E. Smith. "Uniform approximation of Abhyankar valuation ideals in smooth function fields". In: *Amer. J. Math.* 125.2 (2003), pp. 409–440.

[Fan+05] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry*. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339.

[FS13] M. Fedorchuk and D. I. Smyth. "Stability of genus five canonical curves". In: *A celebration of algebraic geometry*. Vol. 18. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2013, pp. 281–310.

[FS90] A. Fujiki and G. Schumacher. "The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics". In: *Publ. Res. Inst. Math. Sci.* 26.1 (1990), pp. 101–183.

[Fuj16] K. Fujita. "On K-stability and the volume functions of $\mathbb{Q}$-Fano varieties". In: *Proc. Lond. Math. Soc. (3)* 113.5 (2016), pp. 541–582.

[Fuj18] K. Fujita. "Optimal bounds for the volumes of Kähler-Einstein Fano manifolds". In: *Amer. J. Math.* 140.2 (2018), pp. 391–414.

[Fuj21] K. Fujita. "K-stability of log Fano hyperplane arrangements". In: *J. Algebraic Geom.* 30.4 (2021), pp. 603–630.

[FO18] K. Fujita and Y. Odaka. "On the K-stability of Fano varieties and anticanonical divisors". In: *Tohoku Math. J. (2)* 70.4 (2018), pp. 511–521.

[FH91] W. Fulton and J. Harris. *Representation theory*. Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xvi+551.

[GM17] P. Gallardo and J. Martinez-Garcia. *Variations of GIT quotients package v0.6.13*. https://researchdata.bath.ac.uk/458. 2017.

[GM18] P. Gallardo and J. Martinez-Garcia. “Variations of Geometric Invariant Quotients for Pairs, a Computational Approach”. In: *Proceedings of the American Mathematical Society* 146.6 (2018), pp. 2395–2408.

[GM19] P. Gallardo and J. Martinez-Garcia. “Moduli of cubic surfaces and their anticanonical divisors”. In: *Revista Matematica Complutense* (2019).

[GMS21] P. Gallardo, J. Martinez-Garcia, and C. Spotti. “Applications of the moduli continuity method to log K-stable pairs”. In: *J. Lond. Math. Soc. (2)* 103.2 (2021), pp. 729–759.

[GMZ18] P. Gallardo, J. Martinez-Garcia, and Z. Zhang. “Compactifications of the moduli space of plane quartics and two lines”. In: *Eur. J. Math.* 4.3 (2018), pp. 1000–1034.

[Gar+21] B. Garn, L. Kampel, I. Kotsireas, D. Simos, and M. Wester. *ACA 2021 Book of Abstracts*. 2021.

[Gro66] A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III”. In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255.

[Gro63] A. Grothendieck. *Revêtements étales et groupe fondamental. Fasc. II: Exposés 6, 8 à 11*. Troisième édition, corrigée, Séminaire de Géométrie Algébrique, 1960/61. Institut des Hautes Études Scientifiques, Paris, 1963, i+163 pp. (not consecutively paged) (loose errata).

[Hab75] W. J. Haboush. “Reductive groups are geometrically reductive”. In: *Ann. of Math. (2)* 102.1 (1975), pp. 67–83.

[HP10] P. Hacking and Y. Prokhorov. “Smoothable del Pezzo surfaces with quotient singularities”. In: *Compositio Mathematica* 146.1 (2010), pp. 169–192.

[HMX14] C. D. Hacon, J. McKernan, and C. Xu. “ACC for log canonical thresholds”. In: *Ann. of Math. (2)* 180.2 (2014), pp. 523–571.

[Har77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Har05] E. Harzheim. *Ordered sets*. Vol. 7. Advances in Mathematics (Springer). Springer, New York, 2005, pp. xii+386.

[Hei17] J. Heinloth. “Hilbert-Mumford stability on algebraic stacks and applications to $\mathcal{G}$-bundles on curves”. In: *Épijournal Géom. Algébrique* 1 (2017), Art. 11, 37.

[Hil90] D. Hilbert. “Über die Theorie der algebraischen Formen”. In: *Math. Annalen* 36 (1890), pp. 473–534.

[Hil93] D. Hilbert. “Ueber die vollen Invariantensysteme”. In: *Math. Ann.* 42.3 (1893), pp. 313–373.

[HP94] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry*. Vol. 1. Cambridge Mathematical Library. Cambridge University Press, 1994.

[Hos16] V. Hoskins. “Moduli problems and Geometric Invariant Theory”. lecture notes at webpage https://userpage.fu-berlin.de/hoskins/M15_Lecture_notes.pdf, accessed on 05 October 2020. 2016.

[IP99] V. A. Iskovskich and Y. Prokhorov. *Fano varieties*. Springer, 1999.

[Isk77] V. A. Iskovskih. “Fano threefolds. I”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 41.3 (1977), pp. 516–562, 717.

[Isk78] V. A. Iskovskih. “Fano threefolds. II”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 42.3 (1978), pp. 506–549.

[Isk79] V. A. Iskovskih. “Anticanonical models of three-dimensional algebraic varieties”. In: *Current problems in mathematics, Vol. 12 (Russian)*. VINITI, Moscow, 1979, 59–157, 239 (loose errata).

[Isk80] V. A. Iskovskikh. “Anticanonical models of three-dimensional algebraic varieties”. In: *Journal of Soviet Mathematics* 13.6 (1980), pp. 745–814.

[JL17] A. Javanpeykar and D. Loughran. “Complete intersections: moduli, Torelli, and good reduction”. In: *Math. Ann.* 368.3-4 (2017), pp. 1191–1225.

[Jia20] C. Jiang. “Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 53.5 (2020), pp. 1235–1248.

[JM12] M. Jonsson and M. Mustaţă. “Valuations and asymptotic invariants for sequences of ideals”. In: *Ann. Inst. Fourier (Grenoble)* 62.6 (2012), 2145–2209 (2013).

[Kas10] A. M. Kasprzyk. “Canonical toric Fano threefolds”. In: 62.6 (2010), pp. 1293–1309.

[KM76] F. F. Knudsen and D. Mumford. "The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"". In: *Math. Scand.* 39.1 (1976), pp. 19–55.

[Knu71] D. Knutson. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971, pp. vi+261.

[Kol97] J. Kollár. "Singularities of pairs". In: *Algebraic geometry—Santa Cruz* (1995, 1997), pp. 221–287.

[KS88] J. Kollár and N. I. Shepherd-Barron. "Threefolds and deformations of surface singularities". In: *Invent. Math.* 91.2 (1988), pp. 299–338.

[Kol09] J. Kollár. *Hulls and Husks*. 2009. arXiv: 0805.0576 [math.AG].

[KMM92] J. Kollár, Y. Miyaoka, and S. Mori. "Rational connectedness and boundedness of Fano manifolds". In: *J. Differential Geom.* 36.3 (1992), pp. 765–779.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254.

[KW06] H. Kraft and N. R. Wallach. "On the nullcone of representations of reductive groups". In: *Pacific J. Math.* 224.1 (2006), pp. 119–139.

[Lan75] S. G. Langton. "Valuative criteria for families of vector bundles on algebraic varieties". In: *The Annals of Mathematics* 101.1 (1975), p. 88.

[LM00] G. Laumon and L. Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208.

[Laz09a] R. Laza. "Deformations of singularities and variation of GIT quotients". In: *Trans. Amer. Math. Soc.* 361.4 (2009), pp. 2109–2161.

[Laz09b] R. Laza. "The moduli space of cubic fourfolds". In: *J. Algebraic Geom.* 18.3 (2009), pp. 511–545.

[Li18] C. Li. "Minimizing normalized volumes of valuations". In: *Math. Z.* 289.1-2 (2018), pp. 491–513.

[LL19] C. Li and Y. Liu. “Kähler-Einstein metrics and volume minimization”. In: *Adv. Math.* 341 (2019), pp. 440–492.

[LLX20] C. Li, Y. Liu, and C. Xu. “A guided tour to normalized volume”. In: *Geometric analysis—in honor of Gang Tian's 60th birthday*. Vol. 333. Progr. Math. Birkhäuser/Springer, Cham, 2020, pp. 167–219.

[LWX19] C. Li, X. Wang, and C. Xu. “On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties”. In: *Duke Math. J.* 168.8 (2019), pp. 1387–1459.

[LWX21] C. Li, X. Wang, and C. Xu. “Algebraicity of the metric tangent cones and equivariant K-stability”. In: *J. Amer. Math. Soc.* 34.4 (2021), pp. 1175–1214.

[LX14] C. Li and C. Xu. “Special test configuration and K-stability of Fano varieties”. In: *Ann. of Math. (2)* 180.1 (2014), pp. 197–232.

[Liu18] Y. Liu. “The volume of singular Kähler-Einstein Fano varieties”. In: *Compos. Math.* 154.6 (2018), pp. 1131–1158.

[Liu22] Y. Liu. “K-stability of cubic fourfolds”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2022.786 (2022), pp. 55–77.

[LX19] Y. Liu and C. Xu. “K-stability of cubic threefolds”. In: *Duke Math. J.* 168.11 (2019), pp. 2029–2073.

[LXZ22] Y. Liu, C. Xu, and Z. Zhuang. “Finite Generation for valuations computing stability thresholds and applications to K-stability”. In: *Annals of Mathematics* 196.2 (2022).

[LZ22] Y. Liu and Z. Zhu. “Equivariant K-stability under finite group action”. In: *Internat. J. Math.* 33.1 (2022), Paper No. 2250007, 21.

[MM90] T. Mabuchi and S. Mukai. “Stability and Einstein-Kähler metric of a quartic del Pezzo surface, Einstein metrics and Yang-Mills connections (Sanda, 1990)”. In: *Lecture Notes in Pure and Appl. Math.* 145 (1990), pp. 133–169.

[Man09] M. Manetti. “Differential graded Lie algebras and formal deformation theory”. In: *Algebraic geometry—Seattle 2005. Part 2*. Vol. 80. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2009, pp. 785–810.

[Mar14] J. Martinez-Garcia. “Log canonical thresholds of del Pezzo surfaces in characteristic p”. In: *Manuscripta Math.* 145.1-2 (2014), pp. 89–110.

[Mat57] Y. Matsushima. "Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne". In: *Nagoya Math. J.* 11 (1957), pp. 145–150.

[MM03] S. Mori and S. Mukai. "Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$" [Manuscripta Math. **36** (1981/82), no. 2, 147–162; MR0641971 (83f:14032)]". In: *Manuscripta Math.* 110.3 (2003), p. 407.

[MM82] S. Mori and S. Mukai. "Classification of Fano 3-folds with $B_2 \geq 2$". In: *Manuscripta Math.* 36.2 (1981/82), pp. 147–162.

[Muk03] S. Mukai. *An Introduction to Invariants and Moduli Theory*. Cambridge University Press, 2003.

[Mum77] D. Mumford. "Stability of projective varieties". In: *Enseign. Math. (2)* 23.1-2 (1977), pp. 39–110.

[MFK94] D. Mumford, J. Fogarty, and F. C. Kirwan. *Geometric Invariant Theory*. Springer, 1994.

[Nag62] M. Nagata. "Complete reducibility of rational representations of a matric group". In: *J. Math. Kyoto Univ.* 1 (1961/62), pp. 87–99.

[Nag64] M. Nagata. "Invariants of a group in an affine ring". In: *J. Math. Kyoto Univ.* 3 (1963/64), pp. 369–377.

[New78] P. E. Newstead. *Introduction to moduli problems and orbit spaces*. Vol. 51. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Tata Institute of Fundamental Research, Bombay; Narosa Publishing House, New Delhi, 1978, pp. vi+183.

[Oda12] Y. Odaka. "The Calabi conjecture and K-stability". In: *Int. Math. Res. Not. IMRN* 10 (2012), pp. 2272–2288.

[Oda13a] Y. Odaka. "A generalization of the Ross-Thomas slope theory". In: *Osaka J. Math.* 50.1 (2013), pp. 171–185.

[Oda13b] Y. Odaka. "The GIT stability of polarized varieties via discrepancy". In: *Ann. of Math. (2)* 177.2 (2013), pp. 645–661.

[Oda15] Y. Odaka. "Compact moduli spaces of Kähler-Einstein Fano varieties". In: *Publ. Res. Inst. Math. Sci.* 51.3 (2015), pp. 549–565.

[OSS16] Y. Odaka, C. Spotti, and S. Sun. "Compact moduli spaces of Del Pezzo surfaces and Kähler–Einstein metrics". In: *Journal of Differential Geometry* 102.1 (2016), pp. 127–172.

[Ols16] M. Olsson. *Algebraic spaces and stacks*. Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298.

[Pap22a] T. S. Papazachariou. *GIT of complete intersections and log canonical thresholds*. 2022. arXiv: 2212.12328 [math.AG].

[Pap22b] T. S. Papazachariou. *K-moduli of log Fano complete intersections*. 2022. arXiv: 2212.09332 [math.AG].

[Pap22c] T. S. Papazachariou. *Variations of GIT quotients for complete intersections package*. https://doi.org/10.5526/ERDR-00000155. 2022.

[PT09] S. T. Paul and G. Tian. "CM stability and the generalized Futaki invariant II". In: *Astérisque* 328 (2009), 339–354 (2010).

[Pez85] P. del Pezzo. "Sulle superficie dell ordine n immerse negli spazi di $n+1$ dimensioni". In: *Rend. Della R. Acc. Delle Scienze Fis. E Mat. Di Napoli* (1885).

[Pez87] P. del Pezzo. "Sulle superficie dell n$^{\text{mo}}$ ordine immerse nello spazio di n dimension". In: *Rendiconti del Circolo Matematico di Palermo* 1 (1887), pp. 241–271.

[Rei72] M. Reid. "The complete intersection of two or more quadrics". PhD thesis. University of Cambridge, 1972.

[RT07] J. Ross and R. Thomas. "A study of the Hilbert-Mumford criterion for the stability of projective varieties". In: *J. Algebraic Geom.* 16.2 (2007), pp. 201–255.

[Sal65] G. Salmon. *A treatise on the analytic geometry of three dimensions. Vol. II*. Fifth. Edited by Reginald A. P. Rogers. Chelsea Publishing Co., New York, 1965, pp. xvi+334.

[Ser06] E. Sernesi. *Deformations of algebraic schemes*. Vol. 334. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. xii+339.

[Šok79] V. V. Šokurov. "The existence of a line on Fano varieties". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 43.4 (1979), pp. 922–964, 968.

[Som59] D. Sommerville. *Analytical Geometry of Three Dimensions*. Cambridge University Press, 1959, pp. 266–274.

[SS17] C. Spotti and S. Sun. "Explicit Gromov–Hausdorff compactifications of moduli spaces of Kähler–Einstein Fano manifolds". In: *Pure and Applied Mathematics Quarterly* 13.3 (2017), pp. 477–515.

[SSY16] C. Spotti, S. Sun, and C. Yao. "Existence and deformations of Kähler-Einstein metrics on smoothable $\mathbb{Q}$-Fano varieties". In: *Duke Math. J.* 165.16 (2016), pp. 3043–3083.

[Tha96] M. Thaddeus. "Geometric invariant theory and flips". In: *J. Amer. Math. Soc.* 6 (1996), pp. 691–723.

[The22] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

[Tia90] G. Tian. "On Calabi's conjecture for complex surfaces with positive first Chern class". In: *Invent. Math.* 101.1 (1990), pp. 101–172.

[Tia87] G. Tian. "Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric". In: *Mathematical aspects of string theory (San Diego, Calif., 1986)*. Vol. 1. Adv. Ser. Math. Phys. World Sci. Publishing, Singapore, 1987, pp. 629–646.

[Tia97] G. Tian. "Kähler-Einstein metrics with positive scalar curvature". In: *Invent. Math.* 130.1 (1997), pp. 1–37.

[TY87] G. Tian and S.-T. Yau. "Kähler-Einstein metrics on complex surfaces with $C_1 > 0$". In: *Comm. Math. Phys.* 112.1 (1987), pp. 175–203.

[Vis05] A. Vistoli. "Grothendieck topologies, fibered categories and descent theory". In: *Fundamental algebraic geometry*. Vol. 123. Math. Surveys Monogr. Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[Wan12] X. Wang. "Height and GIT weight". In: *Math. Res. Lett.* 19.4 (2012), pp. 909–926.

[Wan19] X. Wang. "GIT stability, K-stability and the moduli space of Fano varieties". In: *Moduli of K-stable varieties*. Vol. 31. Springer INdAM Ser. Springer, Cham, 2019, pp. 153–181.

[Wei13] K. Weierstrass. "Zur Theorie der bilinearen und quadratischen Formen". In: *Mathematische Werke: Herausgegeben unter Mitwirkung einer von der königlich preussischen Akademie der Wissenschaften eingesetzten Commission*. Vol. 2. Cambridge Library Collection - Mathematics. Cambridge University Press, 2013, pp. 19–44.

[Xu20] C. Xu. "A minimizing valuation is quasi-monomial". In: *Ann. of Math. (2)* 191.3 (2020), pp. 1003–1030.

[Xu21] C. Xu. "K-stability of Fano varieties: an algebro-geometric approach". In: *EMS Surv. Math. Sci.* 8.1-2 (2021), pp. 265–354.

[XZ20] C. Xu and Z. Zhuang. "On positivity of the CM line bundle on K-moduli spaces". In: *Ann. of Math. (2)* 192.3 (2020), pp. 1005–1068.

[XZ21] C. Xu and Z. Zhuang. "Uniqueness of the minimizer of the normalized volume function". In: *Camb. J. Math.* 9.1 (2021), pp. 149–176.

[Yau96] S.-T. Yau. "Review on Kähler-Einstein metrics in algebraic geometry". In: *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*. Vol. 9. Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, 1996, pp. 433–443.

[Zan22] A. Zanardini. "A note on the stability of pencils of plane curves". In: *Math. Z.* 300.2 (2022), pp. 1741–1751.

[Zhu21] Z. Zhuang. "Optimal destabilizing centers and equivariant K-stability". In: *Invent. Math.* 226.1 (2021), pp. 195–223.