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Limiting behaviour of pairs with equal images

for polynomial functions

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Abstract:	For any non-constant polynomial p(x) with real coefficients, for all sufficiently large x there is a unique y, distinct from x, such that p(x) and p(y) are equal in absolute value. We prove that, in the limit, the mean of x and y is equal to the mean of the roots of p(x).

Limiting behaviour of pairs with equal images for polynomial functions

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Abstract

For any polynomial $p(x)$ with real coefficients and of degree $n \geq 1$, for sufficiently large positive x there is a unique y, distinct from x, such that $p(x)$ and $p(y)$ are equal in absolute value. We show that, in the limit, the mean of x and y is equal to the mean of the roots of $p(x)$.

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ $(a_n \neq 0, n \geq 1)$ be a polynomial with real coefficients.

Theorem For all sufficiently large positive x there exists a unique y, distinct from x, such $p(x) = p(y)$ if n is even, and $p(x) = -p(y)$ if n is odd. Then

$$\lim_{x\to\infty}(x+y) = -\frac{2a_{n-1}}{na_n}. \tag{1}$$

Remarks It is simple to check that for $n = 1$ and $n = 2$, (1) is true for all x without the need to pass to the limit. We also observe that our theorem may be formulated as saying that the limit of the mean of $x+y$ is equal to the mean of the roots of $p(x)$, allowing of course for pairs of complex conjugate roots, and roots having multiplicities greater than 1.

The first statement of the theorem follows easily from the nature of polynomial functions. Since $p(x)$ and the corresponding monic polynomial $p(x)/a_n$, share the same roots, and the same linked pairs (x, y), we see that the general result follows from the case where $a_n = 1$, which we henceforth assume.

Proof First take the even degree case: $n = 2k$. Write $p(x) = x^{2k} + q(x)$, where $q(x) = a_{2k-1}x^{2k-1} + \cdots + a_0$. For all sufficiently large x we have $y < 0$. Since $p(x) = p(y)$ we get $x^{2k} + q(x) = y^{2k} + q(y)$, whence

$$x^{2k}(1 + \frac{a_{2k-1}}{x} + \cdots + \frac{a_0}{x^{2k}}) = y^{2k}(1 + \frac{a_{2k-1}}{y} + \cdots + \frac{a_0}{y^{2k}})$$

$$\Rightarrow (\frac{x}{y})^{2k} = \frac{1 + \frac{a_{2k-1}}{y} + \cdots + \frac{a_0}{y^{2k}}}{1 + \frac{a_{2k-1}}{x} + \cdots + \frac{a_0}{x^{2k}}}$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{x}{-y}\right) = 1, \text{ whence } \lim_{x\to\infty}\frac{x}{y} = -1 = \lim_{x\to\infty}\frac{y}{x}. \tag{2}$$

Now for any real numbers x and y we have

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}). \tag{3}$$

Applying (3) to $(x^2)^k - (y^2)^k$ gives

$$x^{2k} - y^{2k} = (x+y)(x-y)(x^{2k-2} + x^{2k-4}y^2 + \cdots + x^2y^{2k-4} + y^{2k-2}). \tag{4}$$

Now $p(x) = p(y) \Leftrightarrow x^{2k} - y^{2k} = q(y) - q(x)$, which is to say

$$x^{2k} - y^{2k} = a_1(y-x) + a_2(y^2-x^2) + \cdots + a_i(y^i - x^i) + \cdots + a_{2k-1}(y^{2k-1} - x^{2k-1}). \tag{5}$$

We next equate the right hand sides of (4) and (5) and make $x+y$ the subject. Applying (3) to the ith term in (5) yields:

$$\begin{aligned} \frac{a_i(y^i - x^i)}{x-y} &= -a_i\frac{(x-y)(x^{i-1} + x^{i-2}y + \cdots + xy^{i-2} + y^{i-1})}{x-y} \\ &= -a_i(x^{i-1} + x^{i-2}y + \cdots + xy^{i-2} + y^{i-1}). \end{aligned} \tag{6}$$

Therefore the overall result is:

$$x + y = \frac{-\sum_{i=1}^{2k-1} a_i(x^{i-1} + x^{i-2}y + \cdots + xy^{i-2} + y^{i-1})}{x^{2k-2} + x^{2k-4}y^2 + \cdots + x^2y^{2k-4} + y^{2k-2}}. \tag{7}$$

Divide top and bottom of (7) by x^{2k-2}. By (2), $\lim_{x\to\infty}\frac{y}{x} = -1$. Since (6) is a homogeneous polynomial of degree $i-1$, it follows that the contribution to the limit of all terms in the numerator of (7) will be 0, except when $i = 2k-1$. These observations yield:

$$\lim_{x\to\infty}(x+y) = -a_{2k-1}\lim_{x\to\infty}\frac{1 + (y/x) + \cdots + (y/x)^{2k-3} + (y/x)^{2k-2}}{1 + (y/x)^2 + (y/x)^4 + \cdots + (y/x)^{2k-2}}. \tag{8}$$

By (2), each term in (8) approaches ± 1 according as the power involved is even or odd. Therefore we reach the required conclusion in the case of even degree:

$$\lim_{x\to\infty}(x+y) = -\frac{a_{2k-1}}{k} = -\frac{2a_{n-1}}{n}.$$

Next we turn to the case of odd degree, so that $n = 2k+1$ say. We may write $p(x) = x^{2k+1} + q(x)$, where $q(x) = a_{2k}x^{2k} + \cdots + a_0$. Then $p(x) = -p(y)$ and a similar argument to that which yielded (2) gives likewise that

$$\lim_{x\to\infty}\frac{x}{y} = -1 = \lim_{x\to\infty}\frac{y}{x}. \tag{9}$$

Now we employ the factorization of the sum of two odd powers:

$$x^{2k+1} + y^{2k+1} = (x+y)(x^{2k} - x^{2k-1}y + x^{2k-2}y^2 - \cdots - xy^{2k-1} + y^{2k}). \tag{10}$$

Since $p(x) = -p(y)$ we have $x^{2k+1} + y^{2k+1} = -q(x) - q(y)$, whence from this together with (10) we infer that

$$x + y = -\frac{a_{2k}(x^{2k} + y^{2k}) + a_{2k-1}(x^{2k-1} + y^{2k-1}) + \cdots + a_1(x + y) + 2a_0}{x^{2k} - x^{2k-1}y + x^{2k-2}y^2 - \cdots - xy^{2k-1} + y^{2k}}. \tag{11}$$

Divide top and bottom of (11) by x^{2k}. Since by (9) $\lim_{x\to\infty} \frac{y}{x} = -1$, the contribution to the limit of all terms in the numerator of (11) will be 0, except for the first bracketed pair. Those of the denominator approach a series of $2k + 1$ instances of 1. Therefore

$$\lim_{x\to\infty} (x + y) = -\frac{2a_{2k}}{2k + 1} = -\frac{2a_{n-1}}{n}. \quad \square$$