

Precise large deviations for dependent subexponential variables

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In this paper, we study precise large deviations for the partial sums of a stationary sequence with a subexponential marginal distribution. Our main focus is on distributions which either have a regularly varying or a lognormal-type tail. We apply the results to prove limit theory for the maxima of the entries large sample covariance matrices.

Keywords: Large deviation probability; subexponential distribution; maximum domain of attraction; Gumbel distribution; Fréchet distribution; regular variation; stationary sequence

1. Introduction

We consider a (strictly) stationary real-valued sequence (X_t) with generic element X and distribution function F with finite first moment. The corresponding centered partial sums are given by

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n - n\mathbb{E}[X], \quad n \geq 1. \quad (1.1)$$

To ease notation we will always assume that X is centered. We also assume that F is *subexponential*.

1.1. Subexponential distributions

For the moment assume (X_i) are i.i.d. Following the classical definition of Čistyakov [11] (cf. Embrechts et al. [17], page 39), F is subexponential if X is non-negative and has the tail-equivalence property for convolutions, that is,

$$\mathbb{P}(S_n > x) \sim n(1 - F(x)) = n\bar{F}(x), \quad n \geq 2, x \rightarrow \infty; \quad (1.2)$$

we write $F \in \mathcal{S}_+$. Here $f(x) \sim g(x)$ for positive functions f, g means that $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$. In this paper, we will consider two-sided subexponential distribution functions, i.e., $X^+ = X \vee 0$ has a subexponential distribution and a *tail balance condition* holds

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} = p_+, \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(-X > x)}{\mathbb{P}(|X| > x)} = p_- \quad (1.3)$$

for some $p_+ > 0$, $p_- \geq 0$, and we write \mathcal{S} for this enlarged class of distributions. The property (1.2) has the interpretation that S_n and $M_n = \max(X_1, \dots, X_n)$ are tail-equivalent for every n . Therefore, it is considered a very natural class of *heavy-tailed distributions* which has multiple applications in insurance mathematics, telecommunications, queuing and branching theory. Textbook treatments can be found in Embrechts et al. [17], Rolski et al. [38], and Asmussen [2].

The class \mathcal{S}_+ covers a wide range of tail behaviors from power laws with certain moments infinite to semi-exponential tails such that X has all moments finite but no moment generating function. We will mainly be interested in two sub-classes of distribution functions $F \in \mathcal{S}$:

- **RV**(α). We say that X and its distribution F are regularly varying with index $\alpha > 0$ ($F \in \mathbf{RV}(\alpha)$) if $\bar{F}(x) = 1 - F(x) = L(x)x^{-\alpha}$ for some slowly varying function L .
- **LN**. This class consists of subexponential distributions F such that $\bar{F}(x) = \exp(-S(x))$ where S is a slowly varying function such that

$$S(x)/\log x \rightarrow \infty, \quad x \rightarrow \infty.$$

Well-known representatives $F \in \mathbf{RV}(\alpha)$ with positive tail index α are the Pareto, Burr, student distributions. A representative of **LN** is the (standard) lognormal distribution with tail

$$\mathbb{P}(X > x) \sim \frac{e^{-(\log x)^2/2}}{\sqrt{2\pi} \log x} = e^{-(\log x)^2/2 + \log(\sqrt{2\pi} \log x)}. \tag{1.4}$$

An interesting third subclass of \mathcal{S}_+ are the Weibull-type distributions with tail $\bar{F}(x) = \exp(-x^\alpha L(x))$ for a slowly varying function L and $\alpha \in (0, 1)$. Unfortunately, the techniques developed in this paper fail for these distributions, see Remark 2.3 below.

1.2. Precise large deviations of subexponential type in the i.i.d. case

Early on, it was discovered that the defining property of a subexponential distribution (1.2) extends to situations when $n \rightarrow \infty$ and $x = x_n \rightarrow \infty$. To be more precise, a relation of the type

$$\sup_{x > t_n} \left| \frac{\mathbb{P}(\mathcal{S}_n > x)}{n\bar{F}(x)} - 1 \right| \rightarrow 0, \quad n \rightarrow \infty, \tag{1.5}$$

holds for a suitable sequence (t_n) ; we call it a *separating sequence*, and (1.5) a (*precise*) *large deviation of subexponential type*. As a matter of fact, Cline and Hsing [12] discovered that $F \in \mathcal{S}$ is an ‘‘almost’’ necessary and sufficient condition for (1.5) to hold. Pioneering work on large deviations of type (1.5) is due to A.V. Nagaev [26–28], S.V. Nagaev [29,30], Rozovskii [39]; see also Cline and Hsing [12], Denisov et al. [15]. Large deviations for the sample paths of a Lévy process and random walks with regularly varying increments were considered by Hult et al. [21], Rhee et al. [37].

The perhaps best known result in this context is due to S.V. Nagaev [30]. For $F \in \mathbf{RV}(\alpha)$ and $\alpha > 2$, assuming $\mathbb{E}[X] = 0$ and $\text{var}(X) = 1$, he proved that (1.5) holds for $x > t_n = \sqrt{(\alpha - 2)n \log n}$, while for $x < t_n$ one has

$$\sup_{x < t_n} \left| \frac{\mathbb{P}(\mathcal{S}_n > x)}{\Phi(x/\sqrt{n})} - 1 \right| \rightarrow 0, \quad n \rightarrow \infty, \tag{1.6}$$

where Φ is the standard normal distribution function.

Results of the types of (1.5) and (1.6) are also valid for various other distributions in \mathcal{S} . In particular, the lognormal distribution with tail (1.4) satisfies (1.5) for $x \gg t_n$ and (1.6) for $x \ll t_n$ where $t_n = \sqrt{n} \log n$ and $x \gg t_n$ means that $x \geq t_n h_n$ for any sequence $h_n \rightarrow \infty$, and $x \ll t_n$ is defined correspondingly. Rozovskii [39] found that the separating sequences (t_n) in (1.5) and (1.6) have to be distinct if F is lighter than the tail of a lognormal distribution.

Extensions of large deviations of subexponential type to stationary sequences only exist in a few cases. Mikosch and Samorodnitsky [23] proved large deviations of subexponential type for *regularly*

varying linear processes driven by i.i.d. regularly varying noise. The main difference to the i.i.d. case is that the limit of $\mathbb{P}(S_n > x)/(n\overline{F}(x))$ converges uniformly for $x \gg t_n$ to a constant depending on the coefficients of the linear process and the tail index of the noise. This fact shows that *extremal clustering* in the X -sequence causes that exceedances of S_n above high thresholds x appear in clumps and not separated from each other, and the limiting constant is a measure of the size of these clumps. Solutions to affine stochastic recurrence equations $X_t = A_t X_{t-1} + B_t$, $t \in \mathbb{Z}$, for an i.i.d. sequence (A_t, B_t) , $t \in \mathbb{Z}$, may have power-law tails $\mathbb{P}(\pm X > x) \sim c_{\pm} x^{-\alpha}$ for some $\alpha > 0$ either due to regular variation of B_1 with index α and $\mathbb{E}[|A_1|^\alpha] < 1$ (the so-called Grincevičius–Grey case) or due to the condition $\mathbb{E}[|A_1|^\alpha] = 1$ (the so-called Kesten–Goldie case); see Section 3.4.2 in Buraczewski et al. [9] for an overview. Buraczewski et al. [10] proved large deviation results of subexponential type in the Kesten–Goldie case, and Konstantinides and Mikosch [22] in the Grincevičius–Grey case. Mikosch and Wintenberger [24,25] derived large deviation results for regularly varying Markov chains and m -dependent processes and applied these results to get bounds for ruin probabilities.

1.3. Goals of this paper

In this paper, we aim at proving analogs of the subexponential large deviation results for a stationary dependent sequence (X_t) . In most cases, we have to restrict ourselves to an m -dependent sequence, i.e., the dependence ranges only over m lags. We work under the heavy-tail assumption $F \in \mathcal{S}$ which is a natural condition, as we explained in Section 1.2. We also have to impose an *asymptotic tail independence condition* on the distributions of the pairs (X_0, X_h) for $1 \leq h \leq m$. Under the aforementioned conditions and for $F \in \mathbf{RV}(\alpha)$ and $F \in \mathbf{LN}$, we prove results of the type (1.5). The strong asymptotic tail independence conditions ensure that (1.5) is valid for suitable sequences (t_n) . Based on the m -dependence of (X_t) we make heavy use of the known large deviation results in the i.i.d. case. This is the topic of Section 3.

In Section 4, we study subexponential large deviations for a linear process driven by an i.i.d. noise sequence with a common subexponential distribution F in the class \mathbf{LN} . In this case, a result of type (1.5) does in general not hold but the denominator $n\overline{F}(x)$ has to be replaced by $n\overline{F}(x/|m_0|)$ for some number m_0 which depends on the coefficients of the linear process. The proof makes heavy use of the linear structure and exploits the known large deviation results for an i.i.d. sequence. We also mention that the distribution of X is tail-equivalent to the subexponential noise distribution.

In Section 5, we show how large deviations of subexponential type can be applied to determine the limits of the maxima of the diagonal or off-diagonal entries of a large sample covariance matrix with row-wise dependent entries.

2. Preliminaries

2.1. Maximum domains of attraction

Assume that (X_i) is i.i.d. with common distribution F .

The condition $F \in \mathbf{RV}(\alpha)$ for $\alpha > 0$ is equivalent to membership of F in the *maximum domain of attraction* of the Fréchet distribution Φ_α ($F \in \text{MDA}(\Phi_\alpha)$). This means that there exist constants $a_n > 0$ such that

$$\mathbb{P}(a_n^{-1}M_n \leq x) \rightarrow \Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x \geq 0, n \rightarrow \infty.$$

For $F \in \mathbf{LN} \cap \mathcal{S}$ we also require that it is a member of the maximum domain of attraction of the Gumbel distribution Λ ($F \in \text{MDA}(\Lambda)$), that is, there exist constants $c_n > 0, d_n \in \mathbb{R}$ such that

$$\mathbb{P}(c_n^{-1}(M_n - d_n) \leq x) \rightarrow \Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}, n \rightarrow \infty.$$

According to the Pickands–Balkema–de Haan theorem (Pickands [32], Balkema and de Haan [4], cf. Theorem 3.4.5 in Embrechts et al. [17]) a distribution with infinite right endpoint $F \in \text{MDA}(\Lambda)$ if and only if there exists a positive function a with Lebesgue density a' such that $a'(x) \rightarrow 0$ as $x \rightarrow \infty$ and

$$\frac{\overline{F}(x + ya(x))}{\overline{F}(x)} \rightarrow e^{-y}, \quad x \rightarrow \infty, y \in \mathbb{R}. \tag{2.1}$$

The auxiliary function a can be chosen as the *mean-excess function* of F

$$a(x) = \int_x^\infty \frac{\overline{F}(y)}{\overline{F}(x)} dy, \quad x > 0;$$

cf. Resnick [35], Proposition 1.9. We have $F \in \text{MDA}(\Lambda)$ if and only if

$$\overline{F}(x) = c(x) \exp\left(-\int_z^x \frac{1}{a(t)} dt\right), \quad x > z, \tag{2.2}$$

for some z and $c(x) \rightarrow c > 0$ as $x \rightarrow \infty$.

2.2. Long-tailed distributions

A distribution function F is said to be long-tailed if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1, \quad \text{for any } y > 0.$$

For the properties of long-tailed distributions we refer to Foss et al. [18]. In particular, $F \in \mathcal{S}$ implies long-tailedness of F ; see Lemma 3.4 in Foss et al. [18]. Moreover, for each long-tailed distribution F there exists a non-decreasing function h with $h(x) \uparrow \infty$ as $x \rightarrow \infty$ such that

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + h(x))}{\overline{F}(x)} = 1,$$

and F is called *h-insensitive*. In particular, $F \in \text{MDA}(\Lambda)$ satisfies (2.1) for some auxiliary function $a(x) \rightarrow \infty$. Hence, we can choose $h(x) = o(a(x))$, and if $F \in \text{MDA}(\Phi_\alpha)$ we can take any function h with $h(x) = o(x)$ as $x \rightarrow \infty$.

2.3. Condition (C)

We consider a stationary sequence (X_i) with mean zero and partial sum process (S_n) given in (1.1). In this section, we assume that $F \in \text{MDA}(\Lambda) \cap \mathbf{LN}$. Hence, in particular, $F \in \mathcal{S}$, F has infinite right endpoint and $S(x) = -\log \overline{F}(x)$ is slowly varying such that $S(x)/\log x \rightarrow \infty$ as $x \rightarrow \infty$. Characterizations of $\text{MDA}(\Lambda)$ are given in Section 2.1.

In what follows, we introduce and discuss a set of conditions which will be assumed in our main result, Theorem 3.1. A crucial object in this context is a positive function g which describes the region (t_n, ∞) where the large deviation results hold.

Condition (C)

C₁ $g(x) \uparrow \infty$ as $x \rightarrow \infty$ and there is $C > 0$ such that for large x ,

$$g(x) \leq Cx/S(x). \tag{2.3}$$

C₂ There is a sequence $t_n \rightarrow \infty$ such that for any $\delta > 0$,

$$\sup_{x > t_n \delta} \left| \frac{S(x)}{S(g(x))} - 1 \right| \rightarrow 0 \quad \text{and} \quad \frac{g(t_n)}{\sqrt{n}} \rightarrow \infty, \quad n \rightarrow \infty, \tag{2.4}$$

and for an i.i.d. sequence (X'_i) with common distribution F and partial sums $S'_n = X'_1 + \dots + X'_n$ we have the large deviation result

$$\lim_{n \rightarrow \infty} \sup_{x > t_n \delta} \left| \frac{\mathbb{P}(S'_n > x)}{n\bar{F}(x)} - 1 \right| = 0, \quad \text{for any } \delta > 0. \tag{2.5}$$

C₃ (X_i) is m -dependent for some $m \geq 1$, and for any $\varepsilon > 0$,

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(|X_0| > \varepsilon g(x), |X_h| > \varepsilon x)}{\bar{F}(x)} = 0, \quad h = 1, \dots, m. \tag{2.6}$$

The size of $g(x)$. It follows from the monotone density theorem (cf. Theorem 1.7. in Bingham et al. [6]) and (2.2) that

$$\frac{a(x)S(x)}{x} \rightarrow \infty, \quad x \rightarrow \infty.$$

Therefore, $g(x) = o(a(x))$ in agreement with condition (2.3) which also implies that $g(x)/x \rightarrow 0$ since $S(x) \rightarrow \infty$ for $F \in \text{MDA}(\Lambda)$ with infinite right endpoint. Moreover, we conclude from (2.1) that for any $c \in \mathbb{R}$,

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X > x - cg(x))}{\mathbb{P}(X > x)} = 1, \tag{2.7}$$

that is, F is (cg) -insensitive for any $c \in \mathbb{R}$. The latter condition will be frequently used in the remainder of this paper. On the other hand, the first condition in (2.4) ensures that $g(x)$ increases not too slowly.

Lemma 2.1. *If (2.4) holds then for any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{x > t_n} \frac{n\bar{F}(\varepsilon x)\bar{F}(\varepsilon g(x))}{\bar{F}(x)} = 0. \tag{2.8}$$

Proof. Since $F \in \text{LN} \cap \text{MDA}(\Lambda)$ we have $\lim_{x \rightarrow \infty} S(x)/\log x = \infty$. It follows from (2.4) uniformly for $x > t_n$,

$$S(x) \sim S(g(x)) \geq S(g(t_n)) \geq S(\sqrt{n}) \gg \log n.$$

Hence by slow variation of $S(x)$ and (2.4), uniformly for $x > t_n$,

$$\frac{n\overline{F}(\varepsilon x)\overline{F}(\varepsilon g(x))}{\overline{F}(x)} = \exp(\log n - S(x)(1 + o(1))) \rightarrow 0. \quad \square$$

Example 2.2. If we choose $g(x) = x/S(x)$ and $S(x) = f(\log x)$ for a differentiable regularly varying function f with index $\alpha > 1$, then (2.3) and (2.4) are satisfied. Indeed, if $T(x)$ is slowly varying then, according to Bojanic and Seneta [7], the condition

$$\lim_{x \rightarrow \infty} \frac{xT'(x)}{T(x)} \log T(x) = 0 \tag{2.9}$$

implies that for any $\rho \in \mathbb{R}$

$$\lim_{x \rightarrow \infty} \frac{T(xT^\rho(x))}{T(x)} = 1 \tag{2.10}$$

holds and $T(x) = S(x)$ satisfies (2.9). In particular, one can choose $S_1(x) = c(\log x)^\alpha(1 + o(1))$, or $S_2(x) = \exp(c(\log \log x)^\alpha)(1 + o(1))$ for $c > 0$, and Lemma 2.1 applies.

The lognormal distribution is of type S_1 with $\alpha = 2$; see (1.4). For this distribution we can choose $t_n \gg \sqrt{n}(\log n)^2$. Therefore, the conditions $g(t_n)/\sqrt{n} \rightarrow \infty$ and (2.5) hold as well; see the discussion in Section 1.2.

Remark 2.3. Note that Weibull-type distributions do not satisfy conditions \mathbf{C}_1 – \mathbf{C}_2 . Indeed, if a distribution F has a tail $\overline{F}(x) = \exp(-x^\alpha L(x))$ for a slowly varying function L and $\alpha \in (0, 1)$, then \mathbf{C}_1 implies that $g(x) \leq Cx^{1-\alpha}/L(x)$ as $x \rightarrow \infty$. Thus, $S(x)/S(g(x)) \rightarrow \infty$ as $x \rightarrow \infty$ and (2.4) is not satisfied.

2.4. Time series models satisfying (C)

In the previous section, we verified conditions \mathbf{C}_1 – \mathbf{C}_2 on some examples. These conditions depended only on the marginal distribution F of (X_i) . In this section, we provide some examples of time series for which we can verify condition \mathbf{C}_3 which depends on the pairwise dependence structure of $(X_0, X_1), h = 1, \dots, m$. Here and in what follows, c denotes any positive constant whose value is not of interest.

Example 2.4. Let $\mathbf{Y} = (Y_i)$ be a Gaussian m -dependent stationary sequence with mean μ , variance $\sigma^2 > 0$ and correlation function $\rho(h) < 1$ for $h \neq 0$. Consider a stationary sequence $\mathbf{X} = e^{b(\mathbf{Y})} = (e^{b(Y_i)})$, where $b(x) = \text{sign}(x)|x|^\alpha$, $\alpha \in (0, 2)$. We observe that for large x

$$S(x) = \frac{1}{2\sigma^2}(\log x)^{2/\alpha}(1 + o(1)),$$

thus $S(x)$ satisfies (2.9) by Example 2.2 and then $g(x) = x/S(x)$ satisfies \mathbf{C}_1 . The conditions $g(t_n)/\sqrt{n} \rightarrow \infty$ and (2.5) hold with $t_n \gg \sqrt{n}(\log n)^{2/\alpha}$. Indeed, according to Rozovskii [39], the large deviation result (2.5) holds with $t_n \gg \sqrt{n}(\log n)^{2/\alpha-1}$ for $\alpha \in (0, 1]$ and with $t_n \gg \sqrt{n}(\log n)^{1/\alpha}$ for $\alpha \in (1, 2)$. Note also that for $\alpha = 1$ the random vector $(X_1, \dots, X_d), d \in \mathbb{N}$, has a multivariate lognormal distribution in the sense of Asmussen and Rojas-Nandayapa [3].

Next, we verify \mathbf{C}_3 . We assume $\mu = 0$ and observe that $\rho(h) = 0$ for $h > m$. An adapted version of Shibuya’s classical estimate, Shibuya [40], and the tail-balance condition (1.3) yield for $\varepsilon > 0$ and

large x ,

$$\begin{aligned} \mathbb{P}(|X_0| > \varepsilon x, |X_h| > \varepsilon g(x)) &\leq c\mathbb{P}(X_0 > \varepsilon g(x), X_h > \varepsilon g(x)) \\ &= c\mathbb{P}(\min(Y_0, Y_h) > (\log(\varepsilon g(x)))^{1/\alpha}) \\ &\leq c\mathbb{P}(Y_0 + Y_h > 2\log(\varepsilon g(x)))^{1/\alpha} \\ &= c\bar{\Phi}\left(\frac{2(\log(\varepsilon g(x)))^{1/\alpha}}{\sigma\sqrt{1+\rho(h)}}\right) = o\left(\bar{\Phi}\left(\frac{(\log x)^{1/\alpha}}{\sigma}\right)\right), \end{aligned}$$

where Φ is the standard normal distribution function. In the last step, we used the facts that $\rho(h) < 1$ and

$$\frac{2(\log(\varepsilon g(x)))^{1/\alpha}}{\sqrt{1+\rho(h)}} = (\log x)^{1/\alpha} \frac{2}{\sqrt{1+\rho(h)}} (1 + o(1)).$$

We conclude that for $h \geq 1$,

$$\mathbb{P}(|X_0| > \varepsilon x, |X_h| > \varepsilon g(x)) = o(\bar{F}(x)), \quad x \rightarrow \infty,$$

and thus \mathbf{C}_3 is satisfied.

Example 2.5. Let (Y_i) be an i.i.d. sequence with common distribution given by

$$\mathbb{P}(Y > x) = \exp(-(\log x)^\alpha), \quad x > 1, \tag{2.11}$$

for some $\alpha > 1$. The sequence

$$X_i = \min(a_0 Y_i, a_1 Y_{i+1}, \dots, a_m Y_{i+m})$$

for some positive a_0, \dots, a_m is m -dependent, stationary and has tail

$$\mathbb{P}(X > x) = \exp\left(-\sum_{i=0}^m S(x/a_i)\right) = \exp(-mS(x)(1 + o(1))).$$

Thus the distribution of X is also subexponential. This follows by checking Pitman’s condition, Pitman [33]: integrability of the function $\exp(xF'(x)/\bar{F}(x))F'(x)$ on $(0, \infty)$. We verify that (\mathbf{C}) holds with $g(x) = x/(\log x)^\alpha$. \mathbf{C}_1 is immediate. \mathbf{C}_2 follows by virtue of Example 2.2. It remains to verify \mathbf{C}_3 . Direct calculation yields for $\varepsilon > 0$ and $h = 1, \dots, m$,

$$\begin{aligned} &\frac{\mathbb{P}(X_0 > \varepsilon g(x), X_h > \varepsilon x)}{\bar{F}(x)} \\ &\leq \frac{\mathbb{P}(\min(a_0 Y_0, \dots, a_{h-1} Y_{h-1}) > \varepsilon g(x), \min(a_0 Y_h, \dots, a_m Y_{m+h}) > \varepsilon x)}{\mathbb{P}(\min(a_0 Y_0, \dots, a_m Y_m) > x)} \\ &= \exp\left(\sum_{i=0}^m S(x/a_i) - \sum_{i=0}^{h-1} S(\varepsilon g(x)/a_i) - \sum_{i=0}^m S(\varepsilon x/a_i)\right) \\ &= \exp((1 + o(1))S(x)((m + 1) - (m + 1 + h))) \rightarrow 0, \quad x \rightarrow \infty. \end{aligned}$$

Example 2.6. Consider the stochastic volatility model

$$X_i = \sigma_i Y_i,$$

where (σ_i) is a stationary sequence with $\mathbb{P}(a \leq \sigma_1 \leq b) = 1$, $0 < a < 1 < b$, and (Y_i) is an i.i.d. sequence with common distribution function $F_Y(x) = 1 - e^{-S_Y(x)}$, such that $F_Y \in \text{MDA}(\Lambda) \cap \mathcal{S}$, it satisfies the tail-balance condition (1.3), and (2.9) holds for S_Y . We also assume that the distribution F of X is subexponential. This is not automatic even though it is easily verified that $S(x) = S_Y(x)(1 + o(1))$, hence $S(x)$ is slowly varying, but this fact does not necessarily imply subexponentiality of F ; see comments on page 52 in Embrechts et al. [17]. Subexponentiality of F can be verified in simple situations, see, for example, if σ has a binomial distribution on (a, b) , by using Pitman’s aforementioned condition. We choose as before $g(x) = x/S(x)$ and assume that it increases. Hence, \mathbf{C}_1 – \mathbf{C}_2 are satisfied. It remains to show \mathbf{C}_3 . Applying the slow variation of $S(x)$, the tail-balance condition (1.3) and \mathbf{C}_2 , we have for $h = 1, \dots, m$ and $\varepsilon > 0$,

$$\begin{aligned} \frac{\mathbb{P}(|X_0| > \varepsilon g(x), |X_h| > \varepsilon x)}{\overline{F}(x)} &\leq \frac{\mathbb{P}(|Y_0| > \varepsilon g(x)/b, |Y_h| > \varepsilon x/b)}{\overline{F}_Y(x/a)} \\ &\leq c \frac{\overline{F}_Y(\varepsilon g(x)/b) \overline{F}_Y(\varepsilon x/b)}{\overline{F}_Y(x/a)} \\ &= \exp(-S_Y(x)(1 + o(1))) \rightarrow 0, \quad x \rightarrow \infty. \end{aligned}$$

2.5. Regularly varying stationary sequences

A random vector \mathbf{X} with values in \mathbb{R}^d and its distribution are *regularly varying with index $\alpha > 0$* if

$$\mathbb{P}\left(\left(\frac{\mathbf{X}}{x}, \frac{\mathbf{X}}{|\mathbf{X}|}\right) \in \cdot \mid |\mathbf{X}| > x\right) \xrightarrow{w} \mathbb{P}((Y, \Theta) \in \cdot), \quad x \rightarrow \infty,$$

where Y is Pareto distributed, $\mathbb{P}(Y > x) = x^{-\alpha}$, $x > 1$, independent of Θ ; see Resnick [35,36] for some reading on multivariate regular variation. Davis and Hsing [14] introduced regularly varying stationary sequences (X_t) by assuming that each lagged vector (X_0, \dots, X_h) , $h \geq 0$, is regularly varying with index α . Basrak and Segers [5] characterized such sequences by showing that regular variation of (X_t) is equivalent to the existence of a *spectral tail process* (Θ_t) defined via the limit relations

$$\mathbb{P}(x^{-1}(X_0, \dots, X_h) \in \cdot \mid |X_0| > x) \xrightarrow{w} \mathbb{P}(Y(\Theta_0, \dots, \Theta_h) \in \cdot), \quad h \geq 0, x \rightarrow \infty, \tag{2.12}$$

where Y is Pareto distributed and independent of (Θ_t) . Obviously, $|\Theta_0| = 1$. If $\Theta_t = 0$ a.s. for $t \neq 0$ then (X_t) is called asymptotically independent. In Section 3.2, we will work under this assumption.

We will work under the following set of conditions.

Condition (RV)

RV₁ The separating sequence (t_n) satisfies

$$\lim_{n \rightarrow \infty} \frac{t_n}{\sqrt{n \log n}} = \infty. \tag{2.13}$$

RV₂ $F \in \mathbf{RV}(\alpha)$ for some $\alpha > 2$ and satisfies the tail-balance condition (1.3).

RV₃ (X_t) is m -dependent for some $m \geq 1$ and satisfies

$$\lim_{x \rightarrow \infty} \mathbb{P}(|X_h| > x \mid |X_0| > x) = 0, \quad h = 1, \dots, m. \tag{2.14}$$

Condition **RV₂** implies in particular that $\mathbb{E}[|X|^{2+\delta}] < \infty$ for $0 < \delta < \alpha - 2$. Moreover, $S(x) = -\log F(x) = \alpha \log x - \log L(x)$ for some slowly varying function L . We conclude that any function g satisfying $g(x)/x \rightarrow 0$ as $x \rightarrow \infty$ has the property

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + g(x))}{\overline{F}(x)} = 1.$$

Condition **RV₃** implies the asymptotic independence of the sequence (X_t) , that is, $\Theta_t = 0$ a.s., $t \neq 0$, in (2.12). In particular, (X_t) is regularly varying with index α . By regular variation we can rewrite (2.14) in the form

$$\lim_{x \rightarrow \infty} \mathbb{P}(|X_h| > \varepsilon x \mid |X_0| > \varepsilon x) = 0, \quad h = 1, \dots, m, \varepsilon > 0.$$

Condition **RV₃** is slightly stronger than the corresponding one in Mikosch and Wintenberger [25] who proved their large deviation result under the assumption that all Θ_t , $t = 1, \dots, m$, have an atom at zero. However, the proof in this paper is direct in contrast to Mikosch and Wintenberger [25] who use techniques from the theory of regularly varying processes. In Section 1.2, we mentioned that the best separating sequence in the i.i.d. regularly varying case is $t_n = \sqrt{(\alpha - 2)n \log n}$. Thus, **RV₁** is not too far away from the latter growth condition.

Example 2.7. We consider the stochastic volatility model $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, where (σ_t) is a positive stationary sequence independent of the i.i.d. regularly varying sequence (Z_t) with index $\alpha > 0$. If $\mathbb{E}[\sigma^{\alpha+\delta}] < \infty$ for some $\delta > 0$, then it is not difficult to see that (X_t) is regularly varying with index α . Moreover, it is asymptotic ally independent. Condition (2.14) can be verified as follows: for $h \geq 1$,

$$\begin{aligned} \mathbb{P}(|X_h| > x, |X_0| > x) &= \mathbb{P}(\min(\sigma_h |Z_h|, \sigma_0 |Z_0|) > x) \\ &\leq \mathbb{P}((\sigma_h \vee \sigma_0)(|Z_0| \wedge |Z_h|) > x) =: I(x). \end{aligned}$$

We observe that $|Z_0| \wedge |Z_h|$ has regularly varying tail with index -2α . By a result of Breiman [8], we have

$$I(x) \sim \mathbb{E}[(\sigma_h \vee \sigma_0)^{2\alpha}] \mathbb{P}(|Z_0| \wedge |Z_h| > x) = \mathbb{E}[(\sigma_h \vee \sigma_0)^{2\alpha}] [\mathbb{P}(|Z| > x)]^2,$$

provided $\mathbb{E}[\sigma^{2\alpha+\delta}] < \infty$ for some δ . The latter condition is satisfied e.g. if σ has a lognormal distribution. This is a standard assumption in financial time series analysis; see Andersen et al. [1]. Since we also have $\mathbb{P}(|X| > x) \sim \mathbb{E}[\sigma^\alpha] \mathbb{P}(|Z| > x)$ relation (2.14) is immediate.

3. Main results

3.1. X has semi-exponential tails

The following result is our main precise large deviation result for a stationary sequence with semi-exponential tails. The proof is given in Section 6.

Theorem 3.1. Consider an m -dependent stationary process (X_i) with marginal distribution $F \in \text{MDA}(\Lambda) \cap \text{LN}$ for some $m \geq 1$. Assume condition **(C)**. Then we have

$$\lim_{n \rightarrow \infty} \sup_{x > t_n \delta} \left| \frac{\mathbb{P}(S_n > x)}{n \overline{F}(x)} - 1 \right| = 0, \quad \text{for any } \delta > 0. \tag{3.1}$$

An inspection of the proof of Theorem 3.1 shows that it can be generalized in various directions. Instead of $F \in \text{MDA}(\Lambda) \cap \text{LN}$ we may require $\mathbb{E}[|X|^{2+\delta}] < \infty$ for some $\delta > 0$, that $S(x)$ is slowly varying, g satisfies **(C)** and F is (εg) -insensitive for any $\varepsilon > 0$. Hence, we may also take into consideration distributions with infinite moments, in particular the class **RV**(α) for some $\alpha > 2$. However, the method of proof does not allow one to get an ‘‘almost’’ optimal separating sequence $t_n \gg \sqrt{n \log n}$ under **RV**(α). For this reason, we provide Theorem 3.2 under the latter condition which proves (3.1) for a best possible separating sequence.

3.2. X has regularly varying tails

The following theorem complements the large deviation result for m -dependent stationary regularly varying sequences by Mikosch and Wintenberger [25]. The methods of proof are distinct and do not make direct use of techniques for regularly varying sequences.

Theorem 3.2. Assume (X_t) is an m -dependent stationary sequence which is regularly varying with index $\alpha > 2$ and condition **(RV)** is satisfied. Then the large deviation result (3.1) holds.

The proof of this result is given in Section 7.

4. Linear process with subexponential noise

Assume that Z has a subexponential distribution F_Z ($F_Z \in \mathcal{S}$) in the sense that Z_+ has a subexponential distribution and a tail-balance condition holds:

$$\frac{\mathbb{P}(Z > x)}{\mathbb{P}(|Z| > x)} \rightarrow p_+, \quad \frac{\mathbb{P}(-Z > x)}{\mathbb{P}(|Z| > x)} \rightarrow p_-, \quad x \rightarrow \infty, \tag{4.1}$$

for some $p_+ > 0, p_- \geq 0$ such that $p_+ + p_- = 1$. Throughout this section, we assume $F_Z \in \text{MDA}(\Lambda) \cap \mathcal{S}$. Consider real coefficients (ψ_j) such that $\psi_j = 0$ for $j < 0, \max_j |\psi_j| = 1$ and

$$\sum_{j=0}^{\infty} |\psi_j|^\delta < \infty \quad \text{for some } \delta \in (0, 1). \tag{4.2}$$

Let $k_\pm = \#\{j : \psi_j = \pm 1\}, m_0 = \sum_{j=0}^{\infty} \psi_j$ and $m_1 = \sum_{j=1}^{\infty} |\psi_j|$ which are finite in view of (4.2). Then the infinite series

$$X = \sum_{j=0}^{\infty} \psi_j Z_j$$

converges a.s. provided (Z_t) is an i.i.d. sequence with generic element Z . Indeed, $F_Z \in \text{MDA}(\Lambda) \cap \mathcal{S}$ implies that Z has finite first moment and therefore $\mathbb{E}[\sum_{j=0}^{\infty} |\psi_j Z_j|] = (m_1 + |\psi_0|)\mathbb{E}[|Z|] < \infty$.

4.1. Tail behavior of X

The following result was proved by Davis and Resnick [13].

Lemma 4.1. *If (4.2) and $F \in \text{MDA}(\Lambda) \cap \mathcal{S}$ hold, then*

$$\mathbb{P}(X > x) \sim k_+ \mathbb{P}(Z > x) + k_- \mathbb{P}(Z < -x) \sim (k_+ p_+ + k_- p_-) \mathbb{P}(|Z| > x). \tag{4.3}$$

We may conclude that the distribution of X is tail-equivalent to F_Z . Hence, it inherits subexponentiality.

4.2. Large deviations of linear processes

We consider the causal linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

with generic element X for an i.i.d. sequence (Z_t) with generic element Z , $\mathbb{E}[Z] = 0$ and

$$\sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} |\psi_j| \right)^{\delta} < \infty \quad \text{for some } \delta \in (0, 1). \tag{4.4}$$

This condition implies (4.2), and it is satisfied if $\sum_{j=1}^{\infty} j |\psi_j|^{\delta} < \infty$. Thus, by virtue of (4.3), X_t has a subexponential distribution with tail balance condition. The next result shows that a large deviation result for the i.i.d. subexponential (Z_t) with separating sequence (t_n) implies a corresponding result with separating sequence $(|m_0|t_n)$.

In what follows, we write $S_{n,Z} = Z_1 + \dots + Z_n$.

Proposition 4.2. *Consider a causal linear process (X_t) with i.i.d. mean-zero noise (Z_t) with distribution $F_Z \in \mathcal{S} \cap \text{MDA}(\Lambda)$, $m_0 \neq 0$ and real weights (ψ_j) satisfying (4.4). Choose a function $g(x)$ such that $g(x) = o(a(x))$ where $a(x)$ is an auxiliary function for F_Z in the sense of (2.1).*

1. *Assume that for a separating sequence (t_n) and a set $\Lambda_n \subset (|m_0|t_n(1 + \delta), \infty)$ for any small $\delta > 0$, we have*

$$\limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_1 |Z| > g(x))}{n \mathbb{P}(|m_0| |Z| > x)} = 0. \tag{4.5}$$

If $m_0 > 0$, we assume that for any small $\delta > 0$,

$$\sup_{x > m_0(1+\delta)t_n} \left| \frac{\mathbb{P}(S_{n,Z} > x)}{n \mathbb{P}(Z > x)} - 1 \right| \rightarrow 0, \tag{4.6}$$

and if $m_0 < 0$ and $0 < p_+ < 1$,

$$\sup_{x > |m_0|t_n(1+\delta)} \left| \frac{\mathbb{P}(-S_{n,Z} > x)}{n \mathbb{P}(Z \leq -x)} - 1 \right| \rightarrow 0. \tag{4.7}$$

Then

$$\limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \left| \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(|Z| > x/|m_0|)} - p_{\pm} \mathbb{1}_{(0, \infty)}(\pm m_0) \right| = 0. \tag{4.8}$$

2. Assume $\psi_j = 0, j > m$, for some $m \geq 1$, and (4.6) or (4.7) hold according as $m_0 > 0$ or $m_0 < 0$. Moreover, assume that there is a set Λ_n such that $\Lambda_n \subset (|m_0|t_n(1 + \delta), \infty)$ for any $\delta > 0$ and, for $m'_0 = \sum_{i=0}^m |\psi_i|$,

$$\limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \left[\frac{\mathbb{P}(m'_0|Z| > x)}{n\mathbb{P}(|m_0||Z| > x)} + \frac{[\mathbb{P}(m'_0|Z| > g(x))]^2}{\mathbb{P}(|m_0||Z| > x)} \right] = 0. \tag{4.9}$$

Then (4.8) holds.

Remark 4.3. We observe that $m'_0 = |m_0|$ if either $\psi_j \geq 0$ for all j or $\psi_j \leq 0$ for all j . In both situations, the first ratio in (4.9) vanishes for $\Lambda_n = (|m_0|t_n(1 + \delta), \infty)$. The second ratio vanishes if $-2S(g(x)/|m_0|) + S(x/|m_0|) \rightarrow -\infty$ as $x \rightarrow \infty$. This condition holds if we can ensure that $\sup_{x > t_n} |S(g(x))/S(x) - 1| \rightarrow 0$. The latter condition is satisfied for lognormal Z if we choose $g(x) = x/S(x)$ and $t_n \gg \sqrt{n} \log n$.

Example 4.4. Condition (4.5) is quite restrictive. We illustrate this for an i.i.d. sequence (Z_i) with distribution given by

$$\bar{F}_Z(x) = \mathbb{P}(Z > x) = \exp(-(\log x)^\alpha), \quad x > 1, \tag{4.10}$$

for $\alpha > 1$ and $g(x) = \varepsilon a(x)$, where

$$\varepsilon = \varepsilon(x) \rightarrow 0 \quad \text{and} \quad \varepsilon(x) \log \log x \rightarrow \infty, \quad x \rightarrow \infty.$$

Calculation yields $a(x) \sim cx/(\log x)^{\alpha-1}$ for some $c > 0$. For convenience, we assume $m_0 > 0$. We have

$$\begin{aligned} \frac{\mathbb{P}(m_1 Z > g(x))}{n\mathbb{P}(m_0|Z| > x)} &= \exp(-(\log(\varepsilon a(x)/m_1))^\alpha + (\log(x/m_0))^\alpha - \log n) \\ &= \exp((\alpha - 1)(\log x)^{\alpha-1} \log \log x(1 + o(1)) - \log n). \end{aligned}$$

For $\alpha \geq 2$ one can choose $t_n \gg \sqrt{n}(\log n)^{\alpha-1}$ in (4.7); see the discussion in Example 2.4. In this case Λ_n is empty. For $\alpha \in (1, 2)$, we can choose

$$\Lambda_n = (c_n, b_n), \quad c_n \gg \sqrt{n(\log n)^\alpha}, \quad b_n = \exp\left(\left(\frac{(1 - \delta) \log n}{(\alpha - 1) \log \log n}\right)^{1/(\alpha-1)}\right)$$

for arbitrarily small $\delta > 0$. In particular, $cn \in \Lambda_n$ for any $c > 0$.

Example 4.5. We assume $m'_0 > m_0 > 0$ in (4.9). In this case (4.9) is as restrictive as (4.5). To illustrate this, choose F_Z as in (4.10). As mentioned in Remark 4.3, the second summand in (4.9) vanishes for $x > t_n$ if $\sup_{x > t_n} |S(g(x))/S(x) - 1| \rightarrow 0$. We investigate the first summand. We have

$$\begin{aligned} \frac{\mathbb{P}(m'_0 Z > x)}{n\mathbb{P}(m_0|Z| > x)} &= \exp(-(\log x - \log m'_0)^\alpha + (\log x - \log m_0)^\alpha - \log n) \\ &= \exp(\alpha \log(m'_0/m_0)(\log x)^{\alpha-1}(1 + o(1)) - \log n). \end{aligned}$$

Thus we get similar restrictions as in Example 4.4. The set Λ_n is empty for $\alpha > 2$. For $1 < \alpha < 2$ we can choose

$$\Lambda_n = (c_n, b_n), \quad c_n \gg \sqrt{n(\log n)^\alpha}, \quad b_n = \exp\left(\left(\frac{(1 - \delta) \log n}{\alpha \log(m'_0/m_0)}\right)^{1/(\alpha-1)}\right)$$

for arbitrarily small $\delta > 0$, and we observe that $cn \in \Lambda_n$ for any $c > 0$. If $\alpha = 2$, Λ_n is not empty if $m'_0/m_0 < e$ and contains the sequence cn for any $c > 0$ if $m'_0/m_0 < \sqrt{e}$.

Proof of Proposition 4.2. 1. We follow the ideas of the proof of Lemma A.5 in Mikosch and Samorodnitsky [23]. It will be convenient to write $\psi_j = 0$ for $j \leq 0$. We prove the result for $m_0 > 0$; the case $m_0 < 0$ is analogous. We start with the decomposition

$$S_n = \sum_{j=-\infty}^0 Z_j \beta_{n,j} + \sum_{j=1}^n Z_j \beta_{n,j} =: S_{n,1} + S_{n,2}, \quad \text{where } \beta_{n,j} = \sum_{i=1-j}^{n-j} \psi_i.$$

We have

$$\begin{aligned} \mathbb{P}(S_{n,1} > x) &\leq \mathbb{P}\left(\sum_{j=0}^{\infty} |Z_j| \sum_{i=1+j}^{n+j} |\psi_i| > x\right) \\ &\leq \mathbb{P}\left(\sum_{j=0}^{\infty} |Z_j| \sum_{i=1+j}^{\infty} |\psi_i| > x\right) \leq c\mathbb{P}(|Z|m_1 > x), \end{aligned}$$

where we used Lemma 4.1 in the last step. Indeed, the conditions of this lemma are satisfied by virtue of (4.4). We have

$$S_{n,2} = \sum_{j=1}^n Z_j \sum_{i=0}^{n-j} \psi_i \stackrel{d}{=} \sum_{j=1}^n Z_j \sum_{i=0}^{j-1} \psi_j = m_0 S_{n,Z} - \sum_{j=1}^n Z_j \sum_{i=j}^{\infty} \psi_j = m_0 S_{n,Z} - S_{n,21}.$$

Applying Lemma 4.1, we obtain

$$\mathbb{P}(|S_{n,21}| > x) \leq \mathbb{P}\left(\sum_{j=1}^{\infty} |Z_j| \sum_{i=j}^{\infty} |\psi_i| > x\right) \leq c\mathbb{P}(|Z|m_1 > x).$$

By independence of $S_{n,1}$ and $S_{n,2}$ we observe that

$$\mathbb{P}(S_n > x) \leq \mathbb{P}(S_{n,1} > x - g(x)) + \mathbb{P}(S_{n,2} > x - g(x)) + \mathbb{P}(S_{n,1} > g(x))\mathbb{P}(S_{n,2} > g(x)).$$

Hence for $m_0 > 0$, $x > t_n m_0(1 + \delta)$ and sufficiently large n ,

$$\begin{aligned} \mathbb{P}(S_n > x) &\leq c\mathbb{P}(m_1|Z| > x - g(x)) + \mathbb{P}(S_{n,2} > x - g(x)) \\ &\quad + c\mathbb{P}(S_{n,2} > g(x))\mathbb{P}(m_1|Z| > g(x)) \\ &\leq c\mathbb{P}(m_1|Z| > x - g(x)) + \mathbb{P}(m_0 S_{n,Z} > x - 2g(x)) + \mathbb{P}(-S_{n,21} > g(x)) \\ &\quad + c(\mathbb{P}(m_0 S_{n,Z} > g(x)/2) + \mathbb{P}(-S_{n,21} > g(x)/2))\mathbb{P}(m_1|Z| > g(x)). \end{aligned}$$

We conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(m_0|Z| > x)} \\ & \leq \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \left[\frac{\mathbb{P}(m_0 Z > x)}{\mathbb{P}(m_0|Z| > x)} + c \frac{\mathbb{P}(m_1|Z| > g(x))}{n\mathbb{P}(m_0|Z| > x)} \right] \\ & = p_+ + c \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_1|Z| > g(x))}{n\mathbb{P}(m_0|Z| > x)} = p_+. \end{aligned}$$

In the last steps we used (4.5), the large deviation result (4.6) and the tail balance condition (4.1).

We also have

$$\begin{aligned} \mathbb{P}(S_n > x) & \geq \mathbb{P}(S_n > x, S_{n,1} \leq g(x)) \geq \mathbb{P}(S_{n,2} > x + g(x), S_{n,1} \leq g(x)) \\ & = \mathbb{P}(S_{n,2} > x + g(x))(1 - \mathbb{P}(S_{n,1} > g(x))) = \mathbb{P}(S_{n,2} > x + g(x))(1 + o(1)). \end{aligned}$$

Thus it suffices to find a lower bound for

$$\begin{aligned} \mathbb{P}(S_{n,2} > x + g(x)) & \geq \mathbb{P}(m_0 S_{n,Z} - S_{n,21} > x + g(x), |S_{n,21}| \leq g(x)) \\ & \geq \mathbb{P}(m_0 S_{n,Z} > x, |S_{n,21}| \leq g(x)) \\ & \geq \mathbb{P}(m_0 S_{n,Z} > x) - \mathbb{P}(|S_{n,21}| > g(x)) \\ & \geq n\mathbb{P}(m_0 Z > x)(1 + o(1)) - c\mathbb{P}(m_1|Z| > g(x)). \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(m_0|Z| > x)} & \geq \lim_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_0 Z > x)}{\mathbb{P}(m_0|Z| > x)} \\ & \quad - c \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(m_1|Z| > g(x))}{n\mathbb{P}(m_0|Z| > x)} = p_-. \end{aligned}$$

2. We again assume $m_0 > 0$. In this case, we have

$$S_n = \sum_{j=-m+1}^0 Z_j \sum_{i=1-j}^m \psi_i + \sum_{j=n-m}^n Z_j \sum_{i=0}^{n-j} \psi_i + \sum_{j=1}^{n-m-1} Z_j m_0 =: T_{n,1} + T_{n,2} + T_{n,3}.$$

Hence,

$$\begin{aligned} \mathbb{P}(S_n > x) & \leq \mathbb{P}(T_{n,1} + T_{n,2} > x - g(x)) + \mathbb{P}(T_{n,3} > x - g(x)) \\ & \quad + \mathbb{P}(T_{n,1} + T_{n,2} > g(x))\mathbb{P}(T_{n,3} > g(x)). \end{aligned}$$

We have by Lemma 4.1 for sufficiently large x ,

$$\begin{aligned} \mathbb{P}(T_{n,1} + T_{n,2} > x) & \leq \mathbb{P}\left(\sum_{j=-m+1}^0 |Z_j| \sum_{i=1-j}^m |\psi_i| + \sum_{j=1}^{m+1} |Z_j| \sum_{i=0}^{j-1} |\psi_i| > x \right) \\ & \leq c\mathbb{P}(m'_0|Z| > x). \end{aligned}$$

Under (4.9),

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(m_0|Z| > x)} \\ & \leq \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \left[\frac{\mathbb{P}(T_{n,1} + T_{n,2} > x - g(x))}{n\mathbb{P}(m_0|Z| > x)} + \frac{\mathbb{P}(T_{n,3} > x - g(x))}{n\mathbb{P}(m_0|Z| > x)} \right. \\ & \quad \left. + \frac{\mathbb{P}(T_{n,1} + T_{n,2} > g(x))\mathbb{P}(T_{n,3} > g(x))}{n\mathbb{P}(|m_0||Z| > x)} \right] \\ & \leq \limsup_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \left[\frac{\mathbb{P}(m'_0|Z| > x)}{n\mathbb{P}(m_0|Z| > x)} + \frac{\mathbb{P}(m_0Z > x)}{\mathbb{P}(m_0|Z| > x)} + \frac{[\mathbb{P}(m'_0|Z| > g(x))]^2}{\mathbb{P}(m_0|Z| > x)} \right] = p_+. \end{aligned}$$

As regards the lower bound, we have uniformly for $x > m_0t_n(1 + \delta)$,

$$\begin{aligned} \mathbb{P}(S_n > x) & \geq \mathbb{P}(T_{n,3} > x + g(x))\mathbb{P}(T_{n,1} + T_{n,2} > -g(x)) \\ & = \mathbb{P}(T_{n,3} > x + g(x))(1 - o(1)) \\ & \sim n\mathbb{P}(m_0Z > x + g(x)) \sim p_+n\mathbb{P}(m_0|Z| > x). \end{aligned} \quad \square$$

5. Application to a large sample covariance matrix

Consider a real-valued field (X_{it}) . We assume that the rows $(X_{it})_{t \in \mathbb{Z}}, i = 1, 2, \dots$, constitute i.i.d. stationary m -dependent sequences. We observe the matrix $\mathbf{X} = (X_{it})_{i=1, \dots, p; t=1, \dots, n}$. The corresponding sample covariance matrix is given by

$$\mathbf{X}\mathbf{X}^\top = \left(\sum_{t=1}^n X_{it}X_{jt} \right)_{i,j=1, \dots, p} =: (S_{ij}^{(n)})_{i,j=1, \dots, p}.$$

We assume that $p = p_n \rightarrow \infty$. In what follows, X stands for a generic element of the field with distribution F , and we also write (X_i) for an i.i.d. sequence with common distribution F .

5.1. The case $F \in \mathbf{RV}(\alpha)$

Lemma 5.1. *Assume the following conditions:*

- $X \in \mathbf{RV}(\alpha)$ for some $\alpha > 4$, in particular there is (c_n) such that $n\mathbb{P}(X^2 > c_n) \rightarrow 1$ and $c_n^{-1} \max_{i=1, \dots, n} X_i^2 \xrightarrow{d} Y \sim \Phi_{\alpha/2}$.
- The asymptotic tail relations are valid:

$$\mathbb{P}(S_{11}^{(n)} - n\mathbb{E}[X^2] > c_{np}x) \sim n\mathbb{P}(X^2 > c_{np}x), \quad x > 0, \tag{5.1}$$

$$\mathbb{P}(|S_{12}^{(n)} - n(\mathbb{E}[X])^2| > c_{np}x) \leq cn\mathbb{P}(|X_1X_2| > c_{np}x) = o(p^{-2}), \quad x > 0. \tag{5.2}$$

Then the following limit relations hold:

$$c_{np}^{-1} \max_{1 \leq i < j \leq p} |S_{ij}^{(n)} - n(\mathbb{E}[X])^2| \xrightarrow{\mathbb{P}} 0, \tag{5.3}$$

$$c_{np}^{-1} \max_{i=1, \dots, p} (S_{ii}^{(n)} - n\mathbb{E}[X^2]) \xrightarrow{d} Y. \tag{5.4}$$

Proof of Lemma 5.1. By assumption (5.1) we have for any $x > 0$,

$$p\mathbb{P}(S_{11}^{(n)} - n\mathbb{E}[X^2] > c_{np}x) \sim (np)\mathbb{P}(X^2 > c_{np}x) \rightarrow x^{-\alpha/2}.$$

The random variables $(S_{ii}^{(n)})$ are i.i.d. and therefore (5.4) holds if and only if the latter relation does.

Next, we show that (5.3) holds. We have for any positive x ,

$$\begin{aligned} & \mathbb{P}\left(c_{np}^{-1} \max_{1 \leq i < j \leq p} |(S_{ij}^{(n)} - n(\mathbb{E}[X])^2)| > x\right) \\ & \leq \mathbb{P}\left(\max_{1 \leq i < j \leq p} (S_{ij}^{(n)} - n(\mathbb{E}[X])^2) > c_{np}x\right) \\ & \quad + \mathbb{P}\left(\max_{1 \leq i < j \leq p} (-S_{ij}^{(n)} + n(\mathbb{E}[X])^2) > c_{np}x\right) =: I_1 + I_2. \end{aligned}$$

We restrict ourselves to prove $I_1 \rightarrow 0$. We have by assumption

$$I_1 \leq p^2\mathbb{P}(S_{12}^{(n)} - n(\mathbb{E}[X])^2 > c_{np}x) \leq cp^2n\mathbb{P}(|X_1X_2| > c_{np}x) \rightarrow 0. \quad \square$$

Example 5.2. Assume that $X \in \text{RV}(\alpha)$ for some $\alpha > 4$ and the conditions of Theorem 3.2 are satisfied for the sequences (X_{1t}^2) and $(X_{1t}X_{2t})$ with the same separating sequence (t_n) satisfying $t_n \gg \sqrt{n \log n}$. Thus, X^2 is regularly varying with index $\alpha/2 > 2$ and $\mathbb{P}(X_1X_2 > x) \sim x^{-\alpha}l(x)$ for some slowly varying l ; see Embrechts et al. [16]. We choose (c_n) such that $n\mathbb{P}(X^2 > c_n) \rightarrow 1$, that is, $c_n = n^{2/\alpha}\ell(n)$ for some slowly varying ℓ . We take (p_n) such that $p = n^\beta$ with $\beta > \alpha/4 - 1$, then we have $c_{np} \gg t_n$. An application of Theorem 3.2 yields for $x > 0$,

$$\mathbb{P}(S_{11}^{(n)} - n\mathbb{E}[X^2] > c_{np}x) \sim n\mathbb{P}(X^2 > c_{np}x).$$

This is the desired relation (5.1). Next, we consider

$$q_n = \mathbb{P}(|S_{12}^{(n)} - n(\mathbb{E}[X])^2| > c_{np}x).$$

By Theorem 3.2, we have for some slowly varying \tilde{l} ,

$$q_n \sim cn\mathbb{P}(|X_1X_2| > c_{np}x) = cn(np)^{-2}\tilde{l}(np) = n^{-1}p^{-2}\tilde{l}(np) = o(p^{-2})$$

provided $\tilde{l}(np)/n \rightarrow 0$. This condition is satisfied since we chose $p = n^\beta$. Thus, we have the desired relation (5.2). We conclude that the limit relations (5.3) and (5.4) for the maxima of the diagonal and off-diagonal terms $S_{ii}^{(n)}$ and $S_{ij}^{(n)}$, $i \neq j$, hold.

5.2. The case $F \in \text{MDA}(\Lambda) \cap \mathcal{S}$

Lemma 5.3. Assume the following conditions:

- The distribution of X^2 is in $\text{MDA}(\Lambda)$, that is, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that $c_n^{-1}(\max_{i=1, \dots, n} X_i^2 - d_n) \xrightarrow{d} Y$ with standard Gumbel limit.

- The asymptotic tail relations are valid:

$$\mathbb{P}(S_{11}^{(n)} - n\mathbb{E}[X^2] > c_{np}x + d_{np}) \sim n\mathbb{P}(X^2 > c_{np}x + d_{np}), \quad x \in \mathbb{R}, \tag{5.5}$$

$$\mathbb{P}(S_{12}^{(n)} - n(\mathbb{E}[X])^2 > c_{np}x + d_{np}) \leq cn\mathbb{P}(X_1X_2 > c_{np}x + d_{np}) = o(p^{-2}), \quad x \in \mathbb{R}. \tag{5.6}$$

Then the following limit relations hold:

$$c_{np}^{-1} \max_{1 \leq i < j \leq p} ((S_{ij}^{(n)} - n(\mathbb{E}[X])^2) - d_{np}) \xrightarrow{\mathbb{P}} -\infty, \quad i \neq j, \tag{5.7}$$

$$c_{np}^{-1} \max_{i=1, \dots, p} ((S_{ii}^{(n)} - n\mathbb{E}[X^2]) - d_{np}) \xrightarrow{d} Y. \tag{5.8}$$

Proof of Lemma 5.3. By assumption (5.5) we have for any x ,

$$p\mathbb{P}(S_{11}^{(n)} - n\mathbb{E}[X^2] > c_{np}x + d_{np}) \sim (np)\mathbb{P}(X^2 > c_{np}x + d_{np}) \rightarrow e^{-x}.$$

The random variables $(S_{ii}^{(n)})$ are i.i.d. and therefore (5.8) holds if and only if the latter relation does.

By assumption (5.6), we have for any $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i < j \leq p} (S_{ij}^{(n)} - n(\mathbb{E}[X])^2) > c_{np}x + d_{np}\right) &\leq p^2\mathbb{P}(S_{12}^{(n)} - n(\mathbb{E}[X])^2 > c_{np}x + d_{np}) \\ &\leq cp^2n\mathbb{P}(X_1X_2 > c_{np}x + d_{np}) \rightarrow 0. \end{aligned}$$

Relation (5.7) follows. □

Example 5.4. Assume that (X_{1t}) is m -dependent stationary with a lognormal generic element X and the conditions of Example 2.4 are met for $\alpha = 1$. We standardize the marginal distribution such that $X \stackrel{d}{=} e^N$ for a standard normal random variable N . Thus, $X^2 \stackrel{d}{=} e^{2N}$ and $X_1X_2 \stackrel{d}{=} e^{\sqrt{2}N}$ for independent copies (X_i) of X . According to Example 2.4 we can apply Theorem 3.1 to both (X_{it}^2) and $(X_{it}X_{jt})$ for $i \neq j$ and in both cases we can choose any separating sequence $t_n \gg \sqrt{n}(\log n)^2$.

We set

$$c_n = 2(2 \log n)^{-1/2}d_n, \quad d_n = \exp(2(\sqrt{2 \log n} - (\log(4\pi) + \log \log n)/(2\sqrt{2 \log n}))).$$

It is well known that $n\mathbb{P}(X^2 > c_nx + d_n) \rightarrow e^{-x}$ for any $x \in \mathbb{R}$; see Embrechts et al. [17], Example 3.3.31. We take (p_n) such that $p \gg n^{-1} \exp(C(\log n)^2)$ for some $C > 1/16$. Since $c_n = o(d_n)$ we have

$$c_{np}x + d_{np} \gg t_n \quad \text{for any negative } x.$$

Therefore, (5.5) follows from Theorem 3.1.

Next, we verify (5.6). To get the first bound in this relation we apply Theorem 3.1. Again observing that $c_n = o(d_n)$, we have for $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(S_{12}^{(n)} - n(\mathbb{E}[X])^2 > c_{np}x + d_{np}) &\sim n\mathbb{P}(X_1X_2 > c_{np}x + d_{np}) \\ &= p^2n\mathbb{P}(N > \log(c_{np}x + d_{np})/\sqrt{2}) \end{aligned}$$

$$\begin{aligned}
 &= p^2 n \mathbb{P}(N > 2\sqrt{\log(np)}(1 + o(1))) \\
 &\sim p^2 n \frac{e^{-2\log(np)(1+o(1))}}{\sqrt{\pi} \log d_{np}} \rightarrow 0.
 \end{aligned}$$

Therefore (5.6) holds. We conclude that the statements of Lemma 5.3 are valid.

6. Proof of Theorem 3.1

For later use, we recall two classical inequalities. Consider a sequence (X_i) of independent mean-zero random variables, set $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \text{var}(S_n)$.

- *Prokhorov’s inequality* (Prokhorov [34]; see Petrov [31], page 77) If $|X_i| \leq c$ a.s. for $i = 1, \dots, n$ and some constant c then

$$\mathbb{P}(S_n > x) \leq \exp\left(-\frac{x}{2c} \operatorname{arsinh}\left(\frac{cx}{2\sigma_n^2}\right)\right), \quad x > 0,$$

where $\operatorname{arsinh}(y) = \log(y + \sqrt{y^2 + 1})$.

- *Fuk–Nagaev’s inequality* (Fuk and Nagaev S.V. [19,20]; see Petrov [31], page 78) If $\mathbb{E}[|X_i|^p] < \infty$ for some $p \geq 2, i = 1, \dots, n, m_{p,n} = \sum_{i=1}^n \mathbb{E}[|X_i|^p]$, then for constants $c_p, d_p > 0$ only depending on p ,

$$\mathbb{P}(S_n > x) \leq c_p m_{p,n} x^{-p} + e^{-d_p(x/\sigma_n)^2}, \quad x > 0.$$

The lower bound

We have

$$\{S_n > x\} \supset \bigcup_{i=1}^n \{|S_n - X_i| \leq g(x), X_i > x + g(x), |X_j| \leq g(x), 1 \leq j \neq i \leq n\}.$$

The events on the right-hand side are disjoint. Therefore,

$$\begin{aligned}
 \mathbb{P}(S_n > x) &\geq \sum_{i=1}^n \mathbb{P}\left(|S_n - X_i| \leq g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x)\right) \\
 &= \sum_{i=1}^n \mathbb{P}\left(X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x)\right) \\
 &\quad - \sum_{i=1}^n \mathbb{P}\left(|S_n - X_i| > g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x)\right) \\
 &= \sum_{i=1}^n \mathbb{P}(X_i > x + g(x)) - \sum_{i=1}^n \mathbb{P}\left(X_i > x + g(x), \max_{j \neq i} |X_j| > g(x)\right) \tag{6.1}
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=1}^n \mathbb{P}(|S_n - X_i| > g(x), X_i > x + g(x), \max_{j \neq i} |X_j| \leq g(x)) \\
 & = J_1(x) - J_2(x) - J_3(x).
 \end{aligned} \tag{6.2}$$

We have $\sup_{x > t_n} |J_1(x)/(n\bar{F}(x)) - 1| \rightarrow 0$ as $n \rightarrow \infty$ and

$$\begin{aligned}
 \sup_{x > t_n} \frac{J_2(x)}{n\bar{F}(x)} & \leq \sup_{x > t_n} \sum_{i=1}^n \frac{\mathbb{P}(X_i > x + g(x), \max_{j \neq i: |j-i| \leq m} |X_j| > g(x))}{n\bar{F}(x)} \\
 & \quad + \sup_{x > t_n} \sum_{i=1}^n \frac{\mathbb{P}(X > x + g(x))}{n\bar{F}(x)} \mathbb{P}\left(\max_{j \neq i: |j-i| > m} |X_j| > g(x)\right) \\
 & \leq 2 \sup_{x > t_n} \sum_{h=1}^m \frac{\mathbb{P}(X_0 > x + g(x), |X_h| > g(x))}{\bar{F}(x)} \\
 & \quad + \sup_{x > t_n} \frac{\mathbb{P}(X > x + g(x))}{\bar{F}(x)} \mathbb{P}\left(\max_{i=1, \dots, n} |X_i| > g(x)\right) \\
 & \leq o(1) + \sup_{x > t_n} n\mathbb{P}(|X| > g(x)) = o(1),
 \end{aligned}$$

where the latter relation follows by **C₃** and since $g(t_n)/\sqrt{n} \rightarrow \infty$ holds. Thus, it is enough to show that $J_3(x) \rightarrow 0$ as $n \rightarrow \infty$ to derive the required lower bound for $\mathbb{P}(S_n > x)$.

In the sequel, we will use the notation,

$$\widehat{X}_j = X_j \mathbb{1}(|X_j| \leq g(x)), \quad \widehat{S}_n^{(i)} = \sum_{j=1, j \neq i}^n \widehat{X}_j. \tag{6.3}$$

Hence,

$$\begin{aligned}
 J_3(x) & = \sum_{i=1}^n \mathbb{P}(|\widehat{S}_n^{(i)}| > g(x), X_i > x + g(x)) \\
 & \leq \sum_{i=1}^n \mathbb{P}\left(\left| \sum_{1 \leq t \neq i \leq n, |t-i| \leq m} \widehat{X}_t \right| > g(x)/2, X_i > x + g(x)\right) \\
 & \quad + \sum_{i=1}^n \mathbb{P}\left(\left| \sum_{1 \leq t \leq n, |t-i| > m} \widehat{X}_t \right| > g(x)/2\right) \mathbb{P}(X > x + g(x)) \\
 & = J_{31}(x) + J_{32}(x).
 \end{aligned}$$

We have by **C₃**,

$$\frac{J_{31}(x)}{n\bar{F}(x)} \leq 2 \sum_{h=1}^m \frac{\mathbb{P}(|X_h| > g(x)/(4m), X_0 > x + g(x))}{\bar{F}(x)} = o(1).$$

Finally, we deal with $J_{32}(x)$. Since X has mean zero, we derive for arbitrary $\delta > 0$

$$n|\mathbb{E}[\widehat{X}]| = n|\mathbb{E}[X\mathbb{1}(|X| < g(x))]| = n|\mathbb{E}[X\mathbb{1}(|X| > g(x))]| \leq \frac{n\mathbb{E}[|X|^{2+\delta}]}{g^{1+\delta}(x)}.$$

We deduce from (2.4) and the fact that $\mathbb{E}[|X|^{2+\delta}] < \infty$,

$$n|\mathbb{E}[\widehat{X}]| \leq n \frac{\mathbb{E}[|X|^{2+\delta}]}{g^{1+\delta}(t_n)} = o(g^{1-\delta}(t_n)), \quad n \rightarrow \infty. \tag{6.4}$$

Write $\tilde{g}_r(x) = g(x)/(2m) - \#N_r\mathbb{E}[\widehat{X}]$ where

$$N_r = \{1 \leq t \leq n : t \equiv r \pmod{m}, |t - i| > m\},$$

and observe that $n|\mathbb{E}[\widehat{X}]| = o(g^{1-\delta}(x))$ and $|\widehat{X} - \mathbb{E}[\widehat{X}]| \leq 2g(x)$. Using the m -dependence and Prokhorov's inequality, we have for i.i.d. copies (X'_i) of X and large n ,

$$\begin{aligned} \frac{J_{32}(x)}{n\overline{F}(x)} &\leq \sum_{i=1}^n \sum_{r=1}^m \mathbb{P}\left(\left|\sum_{t \in N_r} \widehat{X}'_t\right| > g(x)/(2m)\right) \frac{\mathbb{P}(X > x + g(x))}{n\overline{F}(x)} \\ &\leq \sum_{i=1}^n \sum_{r=1}^m \mathbb{P}\left(\left|\sum_{t \in N_r} (\widehat{X}'_t - \mathbb{E}[\widehat{X}])\right| > \tilde{g}_r(x)\right) \frac{\mathbb{P}(X > x + g(x))}{n\overline{F}(x)} \\ &\leq c \sum_{i=1}^n \sum_{r=1}^m \exp\left(-\frac{\tilde{g}_r(x)}{4g(x)} \operatorname{arsinh}\left(\frac{2g(x)\tilde{g}_r(x)}{2\#N_r \operatorname{var}(\widehat{X})}\right)\right) \frac{\mathbb{P}(X > x + g(x))}{n\overline{F}(x)} \\ &\leq cm \exp\left(-\left(\frac{1}{8m} + o(1)\right) \log\left((1 + o(1)) \frac{m(g(x))^2}{2n \operatorname{var}(X)}\right)\right) \rightarrow 0. \end{aligned}$$

In the last step, we used that $(g(x))^2/n \geq (g(t_n))^2/n \rightarrow \infty$; see (2.4).

The upper bound

Consider the following disjoint partition of Ω :

$$\begin{aligned} B_1 &= \bigcup_{1 \leq i < j \leq n} \{|X_i| > g(x), |X_j| > g(x)\}, \\ B_2 &= \bigcup_{i=1}^n \left\{ |X_i| > g(x), \max_{j=1, \dots, n, i \neq j} |X_j| \leq g(x) \right\}, \\ B_3 &= \left\{ \max_{j=1, \dots, n} |X_j| \leq g(x) \right\}. \end{aligned}$$

The bound on B_1

We observe that for any $\xi \in (0, 1)$,

$$\begin{aligned} \mathbb{P}(\{S_n > x\} \cap B_1) &\leq \sum_{1 \leq i < j \leq n} \mathbb{P}(S_n > x, |X_i| > g(x), |X_j| > g(x)) \\ &\leq \sum_{1 \leq i < j \leq n} \mathbb{P}(S_{ij}^{(1)} > \xi x, |X_i| > g(x), |X_j| > g(x)) \\ &\quad + \sum_{1 \leq i < j \leq n} \mathbb{P}(S_{ij}^{(2)} > (1 - \xi)x, |X_i| > g(x), |X_j| > g(x)) \\ &=: R_1(x) + R_2(x), \end{aligned}$$

where

$$\begin{aligned} S_{ij}^{(1)} &= \sum_{h \leq n: |i-h| \wedge |j-h| > m} X_h = \sum_{r=1}^m (S_{ij}^{(r)})', & S_{ij}^{(2)} &= \sum_{h \leq n: |i-h| \wedge |j-h| \leq m} X_h, \\ (S_{ij}^{(r)})' &= \sum_{h \in Q_{ij}^{(r)}} X_h, & Q_{ij}^{(r)} &= \{h \leq n : |i-h| \wedge |j-h| > m, h \equiv r \pmod{m}\}. \end{aligned}$$

For a given r , the summands in $(S_{ij}^{(r)})'$ are independent due to m -dependence and also independent of X_i, X_j . We have $\#Q_{ij}^{(r)} \leq n/m$ while the number of summands in $S_{ij}^{(2)}$ does not exceed $4m + 2$. Thus, by the large deviation result (2.5), m -dependence and stationarity,

$$\begin{aligned} \frac{R_1(x)}{n\bar{F}(x)} &\leq \sum_{1 \leq i < j \leq n} \sum_{r=1}^m \frac{\mathbb{P}((S_{ij}^{(r)})' > \xi x/m) \mathbb{P}(|X_i| > g(x), |X_j| > g(x))}{n\bar{F}(x)} \\ &\sim \frac{\bar{F}(\xi x/m)}{\bar{F}(x)} \sum_{1 \leq i < j \leq n} \mathbb{P}(|X_i| > g(x), |X_j| > g(x)) \\ &= \frac{\bar{F}(\xi x/m)}{\bar{F}(x)} \sum_{h=1}^{n-1} (n-h) \mathbb{P}(|X_0| > g(x), |X_h| > g(x)) \\ &\leq n \frac{\bar{F}(\xi x/m)}{\bar{F}(x)} \sum_{h=1}^m \mathbb{P}(|X_0| > g(x), |X_h| > g(x)) \\ &\quad + \frac{\bar{F}(\xi x/m)}{\bar{F}(x)} [n\mathbb{P}(|X| > g(x))]^2 \\ &=: R_{11}(x) + R_{12}(x). \end{aligned}$$

Applying the tail balance condition, C_3 and (2.8), we have

$$\sup_{x > t_n} R_{11}(x) \leq cm \sup_{x > t_n} \frac{n\bar{F}(\xi x/m)\bar{F}(g(x))}{\bar{F}(x)} \rightarrow 0.$$

Since $g(t_n)/\sqrt{n} \rightarrow \infty$ and $\mathbb{E}[X^2] < \infty$ we also have

$$\sup_{x>t_n} n\mathbb{P}(|X| > g(x)) \leq \sup_{x>t_n} n\mathbb{P}(|X| > \sqrt{n}) \rightarrow 0. \tag{6.5}$$

Hence, the tail balance condition, Lemma 2.1 and (6.5) immediately imply that $R_{12} \rightarrow 0$.

We have

$$\begin{aligned} R_2(x) &\leq \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{(i+2m)\wedge n} + \sum_{j-i>2m, j\leq n} \right) \mathbb{P}(S_{ij}^{(2)} > (1-\xi)x, |X_i| > g(x), |X_j| > g(x)) \\ &= R_{21}(x) + R_{22}(x). \end{aligned}$$

We restrict ourselves to the study of $R_{21}(x)$; $R_{22}(x)$ can be treated by similar methods. We note that $S_{ij}^{(2)}$ has representation

$$S_{ij}^{(2)} = \sum_{h=(i-m)\vee 1}^{(j+m)\wedge n} X_h.$$

Observe that the number of summands in $S_{ij}^{(2)}$ does not exceed $4m + 2$. Therefore and by stationarity, taking care of the cases $h = i$ and $h = j$,

$$R_{21}(x) \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{(i+2m)\wedge n} \sum_{h=(i-m)\vee 1}^{(j+m)\wedge n} \mathbb{P}\left(|X_h| > \frac{(1-\xi)x}{4m+2}, |X_i| > g(x), |X_j| > g(x)\right) \tag{6.6}$$

$$\leq cn \sum_{h=1}^m \mathbb{P}\left(|X_0| > \frac{(1-\xi)x}{4m+2}, |X_h| > g(x)\right) + cn \bar{F}\left(\frac{(1-\xi)x}{4m+2}\right) \bar{F}(g(x)). \tag{6.7}$$

By C_3 and (2.8) we conclude that

$$\limsup_{n \rightarrow \infty} \sup_{x>t_n} \frac{R_{21}(x)}{n\bar{F}(x)} = 0.$$

Combining the previous bounds, we conclude that

$$\lim_{n \rightarrow \infty} \sup_{x>t_n} \frac{\mathbb{P}(\{S_n > x\} \cap B_1)}{n\bar{F}(x)} = 0.$$

The bound on B_2

Next, we bound $\mathbb{P}(\{S_n > x\} \cap B_2)$. Recall the notation \widehat{X}_j and $\widehat{S}_n^{(i)}$ from (6.3). Fix $b \in (0, 1)$. Since $g(x)/x \rightarrow 0$ as $x \rightarrow \infty$ we have

$$\begin{aligned} \mathbb{P}(\{S_n > x\} \cap B_2) &\leq \sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, |X_i| > g(x)) \\ &= \sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, |X_i| \in (g(x), x - bx]) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, |X_i| \in (x - bx, x - g(x))) \\
 & + \sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, |X_i| > x - g(x)) \\
 & =: I_1(x) + I_2(x) + I_3(x).
 \end{aligned}$$

Bounding $I_1(x)$

We show that $I_1(x) = o(n\overline{F}(x))$. Similarly to the bound for $R_1(x)$ estimation, using the m -dependence, we split $\widehat{S}_n^{(i)}$ into m sums of i.i.d. summands:

$$\widehat{S}_n^{(i)} = \sum_{r=1}^m \widehat{S}_{i,r}, \quad \widehat{S}_{i,r} = \sum_{h \in Q_{i,r}^*} \widehat{X}_h, \text{ where } Q_{i,r}^* = \{h \leq n : h \equiv r \pmod{m}, h \neq i\}.$$

In view of (6.4), we have $n|\mathbb{E}[\widehat{X}]| \leq g(x)$ for large x . Moreover, $|\widehat{X} - \mathbb{E}[\widehat{X}]| \leq 2g(x)$ and $\#Q_{i,r} \leq n/m$. An application of Prokhorov’s inequality for large n yields

$$\begin{aligned}
 I_1(x) & \leq \sum_{i=1}^n \mathbb{P}(\widehat{S}_n^{(i)} > bx) \leq \sum_{i=1}^n \sum_{r=1}^m \mathbb{P}(\widehat{S}_{i,r} > bx/m) \\
 & \leq \sum_{i=1}^n \sum_{r=1}^m \mathbb{P}(\widehat{S}_{i,r} - \#Q_{i,r}\mathbb{E}[\widehat{X}] > bx/m - g(x)) \\
 & \leq \sum_{i=1}^n \sum_{r=1}^m \exp\left(-\frac{bx/m - g(x)}{4g(x)} \operatorname{arsinh}\left(\frac{2g(x)(bx/m - g(x))}{2\#Q_{i,r} \operatorname{var}(\widehat{X})}\right)\right) \\
 & \leq nm \exp\left(-c \frac{x}{g(x)} \log\left(\frac{xg(x)}{n}\right)\right) = o(n\overline{F}(x))
 \end{aligned}$$

uniformly for $x > t_n$. In the last step, we used the bounds on $g(x)$ in \mathbf{C}_1 and \mathbf{C}_2 .

Bounding $I_2(x)$

Write

$$\widetilde{S}_n^{(i)} = \sum_{t \notin [i-m, i+m]} \widehat{X}_t.$$

We observe that $|\widehat{X}| \leq g(x)$ and conclude by independence between X_i and $\widetilde{S}_n^{(i)}$, the tail-balance condition and integration by parts that

$$\begin{aligned}
 I_2(x) & \leq \sum_{i=1}^n \mathbb{P}(X_i + \widetilde{S}_n^{(i)} > x - 2mg(x), |X_i| \in (x - bx, x - g(x))) \\
 & \leq c \sum_{i=1}^n \int_{x-bx}^{x-g(x)} \mathbb{P}(\widetilde{S}_n^{(i)} > x - 2mg(x) - y) dF(y)
 \end{aligned}$$

$$\begin{aligned} &\leq c\bar{F}(x - bx) \sum_{i=1}^n \mathbb{P}(\tilde{S}_n^{(i)} > bx - 2mg(x)) + c \int_{g(x)}^{bx} \sum_{i=1}^n \bar{F}(x - y) \mathbb{P}(\tilde{S}_n^{(i)} \in dy) \\ &=: (I_{21}(x) + I_{22}(x)). \end{aligned}$$

In view of (2.7), for every $\delta > 0$ there is u_δ such that

$$\sup_{y \geq u_\delta / (1-b)} \frac{\bar{F}(y)}{\bar{F}(y + g(y))} \leq e^\delta.$$

By a telescoping argument for sufficiently large x ,

$$\frac{\bar{F}(x - bx)}{\bar{F}(x)} \leq \prod_{h=1}^{\lfloor bx/g(x) \rfloor} \frac{\bar{F}(x - hg(x))}{\bar{F}(x - (h - 1)g(x))} \frac{\bar{F}(x - bx)}{\bar{F}(x - g(x)\lfloor bx/g(x) \rfloor)} \leq e^{\delta(\lfloor bx/g(x) \rfloor + 1)}.$$

Now the same argument as for $I_1(x)$ combined with C_1 , (2.4) and Prokhorov’s inequality yields uniformly for $x > t_n$,

$$\begin{aligned} \frac{I_{21}(x)}{n\bar{F}(x)} &\leq e^{\delta(\lfloor bx/g(x) \rfloor + 1)} m \exp\left(-\frac{bx/m - 3g(x)}{4g(x)} \log\left(\frac{mg(x)(bx/m - 3g(x))}{n \text{var}(X)}\right)\right) \\ &\leq \exp\left(-c \frac{x}{g(x)} \log\left(\frac{xg(x)}{n}\right)\right) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

A similar argument yields

$$\begin{aligned} \frac{I_{22}(x)}{n\bar{F}(x)} &\leq \frac{1}{n\bar{F}(x)} \sum_{k=1}^{\lfloor bx/g(x) \rfloor} \int_{g(x)k}^{g(x)(k+1)} \sum_{i=1}^n \bar{F}(x - y) \mathbb{P}(\tilde{S}_n^{(i)} \in dy) \\ &\leq \sum_{k=1}^{\lfloor bx/g(x) \rfloor} \frac{\bar{F}(x - (k + 1)g(x))}{n\bar{F}(x)} \sum_{i=1}^n \mathbb{P}(\tilde{S}_n^{(i)} \in g(x)(k, k + 1]) \\ &\leq \frac{1}{n} \sum_{k=1}^{\infty} e^{(k+1)\delta} \sum_{i=1}^n \mathbb{P}(\tilde{S}_n^{(i)} > kg(x)) \\ &\leq \sum_{k=1}^{\infty} \exp\left(-ck \log\left(\frac{kg^2(x)}{n}\right) + (k + 1)\delta\right) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Bounding $I_3(x)$

We have

$$\begin{aligned} \limsup_{x > t_n} \frac{I_3(x)}{n\bar{F}(x)} &\leq \limsup_{n \rightarrow \infty} \sup_{x > t_n} \frac{1}{n\bar{F}(x)} \left(\sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, X_i > x - g(x)) \right. \\ &\quad \left. + \sum_{i=1}^n \mathbb{P}(X_i + S_n^{(i)} > x, X_i < -x + g(x)) \right) \end{aligned}$$

$$\begin{aligned} &\leq \limsup_{n \rightarrow \infty} \sup_{x > t_n} \frac{\overline{F}(x - g(x))}{\overline{F}(x)} + \limsup_{n \rightarrow \infty} \sup_{x > t_n} \frac{1}{n\overline{F}(x)} \sum_{i=1}^n \mathbb{P}(\widehat{S}_n^{(i)} > 2x - g(x)) \\ &\leq 1 + \lim_{n \rightarrow \infty} \sup_{x > t_n} \exp\left(-c \frac{x}{g(x)} \log\left(\frac{xg(x)}{n}\right)\right) = 1. \end{aligned}$$

The second term is bounded in the same way as I_1 , by exploiting the m -dependence, \mathbf{C}_1 , (2.4) and Prokhorov’s inequality.

Collecting the bounds for all $I_i(x)$, we obtain the desired relation

$$\lim_{n \rightarrow \infty} \sup_{x > t_n} \frac{\mathbb{P}(\{S_n > x\} \cap B_2)}{n\overline{F}(x)} \leq 1.$$

The bound on B_3

It remains to show that $\mathbb{P}(\{S_n > x\} \cap B_3) = o(n\overline{F}(x))$. We observe that $\{S_n > x\} \cap B_3 = \{\widehat{S}_n > x\}$ where $\widehat{S}_n = \sum_{i=1}^n \widehat{X}_i$ and $|\widehat{X}_i| \leq g(x)$. Now the same techniques as for bounding $I_1(x)$ apply. We omit further details. This finishes the proof of the upper bound. \square

7. Proof of Theorem 3.2

The proof is similar to the one of Theorem 3.1. We follow the lines of this proof and also use the same notation. We set $g(x) = g_\varepsilon(x) = \varepsilon x$ for any $\varepsilon > 0$.

The lower bound

We start with the bound (6.2): $\mathbb{P}(S_n > x) \geq J_1(x) - J_2(x) - J_3(x)$. For the first term, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{x > t_n} \frac{J_1(x)}{n\overline{F}(x)} &= \lim_{n \rightarrow \infty} \sup_{x > t_n} \frac{n\overline{F}(x(1 + \varepsilon))}{n\overline{F}(x)} \\ &= (1 + \varepsilon)^{-\alpha}, \end{aligned}$$

and the right-hand side converges to 1 as $\varepsilon \downarrow 0$. By regular variation the bound on $J_2(x)$ turns into

$$\begin{aligned} \sup_{x > t_n} \frac{J_2(x)}{n\overline{F}(x)} &\leq 2 \sup_{x > t_n} \sum_{h=1}^m \frac{\mathbb{P}(X_0 > (1 + \varepsilon)x, |X_h| > \varepsilon x)}{\overline{F}(x)} \\ &\quad + \sup_{x > t_n} \frac{\mathbb{P}(X > (1 + \varepsilon)x)}{\overline{F}(x)} \mathbb{P}\left(\max_{i=1, \dots, n} |X_i| > \varepsilon x\right) \\ &\leq c \sup_{x > t_n} \sum_{h=1}^m \mathbb{P}(|X_h| > \varepsilon x \mid |X_0| > \varepsilon x) + cn\mathbb{P}(|X| > t_n) \rightarrow 0. \end{aligned}$$

Here we used condition (2.14) for the first term and the facts that $t_n \gg \sqrt{n}$ and $\text{var}(X) < \infty$ for the second term. Next, we consider the bound $J_3(x) \leq J_{31}(x) + J_{32}(x)$. The relation

$\lim_{n \rightarrow \infty} \sup_{x > t_n} J_{31}(x)/(n\bar{F}(x)) = 0$ follows in the same way as for the first term in the last display. The negligibility of $J_{32}(x)/(n\bar{F}(x))$ is again proved by Prokhorov’s inequality for i.i.d. regularly varying random variables, also observing that uniformly for $x > t_n$ and fixed $\varepsilon > 0$, by Karamata’s theorem and the choice of $t_n \gg \sqrt{n \log n}$,

$$n|\mathbb{E}[\widehat{X}]| \leq n\mathbb{E}[|X|\mathbb{1}(|X| > \varepsilon x)] \sim cn(\varepsilon x)\mathbb{P}(|X| > \varepsilon x) = o(x).$$

The upper bound

We start with the bound $\mathbb{P}(\{S_n > x\} \cap B_1) \leq R_1(x) + R_2(x)$. Since $x > t_n \gg \sqrt{n \log n}$ the classical Nagaev large deviation result, S.V. Nagaev [30], applies to each of the $\mathbb{P}((S_{ij}^{(r)})' > \xi x/m)$ uniformly for x . Hence, uniformly for $x > t_n$,

$$\frac{R_1(x)}{n\bar{F}(x)} \leq c \sum_{h=1}^m \mathbb{P}(|X_h| > \varepsilon x \mid |X_0| > \varepsilon x) + c[n\mathbb{P}(|X| > \varepsilon x)] = o(1).$$

Next, having $R_2(x) = R_{21}(x) + R_{22}(x)$, we restrict ourselves to the investigation of $R_{21}(x)$ as in the proof of Theorem 3.1. The relation (6.6) remains true, thus we have by **RV**₃

$$\frac{R_{21}(x)}{n\bar{F}(x)} \leq c \sum_{h=1}^m \mathbb{P}\left(|X_h| > \varepsilon x \mid |X_0| > \frac{(1 - \xi)x}{4m + 2}\right) + c[n\bar{F}(x)] = o(1)$$

uniformly for $x > t_n$.

Next, we comment on $\mathbb{P}(\{S_n > x\} \cap B_2)$. For any small $\delta > 0$ write $A_\delta = \cup_{i=1}^n \{X_i > (1 - \delta)x\}$. Thus, we derive

$$\begin{aligned} \mathbb{P}(\{S_n > x\} \cap B_2) &= \mathbb{P}(\{S_n > x\} \cap B_2 \cap A_\delta) + \mathbb{P}(\{S_n > x\} \cap B_2 \cap A_\delta^c) \\ &\leq n\bar{F}((1 - \delta)x) + \sum_{i=1}^n \mathbb{P}(X_i + \widehat{S}_n^{(i)} > x, |X_i| > \varepsilon x, X_i \leq (1 - \delta)x) \\ &\leq n\bar{F}((1 - \delta)x) + \sum_{i=1}^n \mathbb{P}(\widehat{S}_n^{(i)} > \delta x, |X_i| > \varepsilon x) =: C_1(x) + C_2(x). \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \sup_{x > t_n} \frac{C_1(x)}{n\bar{F}(x)} \leq (1 - \delta)^{-\alpha},$$

and the right-hand side converges to 1 as $\delta \downarrow 0$. We have by m -dependence

$$\begin{aligned} C_2(x) &\leq \sum_{i=1}^n \mathbb{P}\left(\sum_{j \in [i-m, i+m], j \neq i} \widehat{X}_j > \delta x/2\right) \\ &\quad + \sum_{i=1}^n \mathbb{P}\left(\sum_{j \leq n, j \notin [i-m, i+m]} \widehat{X}_j > \delta x/2\right) \mathbb{P}(|X| > \varepsilon x). \end{aligned}$$

The sums in the first right-hand probabilities can be bounded by $\widehat{2mg}(x) = 2m\epsilon x$. Choosing $2m\epsilon < \delta/2$ the first term vanishes. Writing $P_{n,i}(x)$ for the summands in the second term, we have

$$\frac{\sum_{i=1}^n P_{n,i}(x)}{n\overline{F}(x)} \leq c\epsilon^{-\alpha} \frac{1}{n} \sum_{i=1}^n \mathbb{P}\left(\sum_{j \leq n, j \notin [i-m, i+m]} \widehat{X}_j > \delta x/2\right).$$

The probabilities in the sum can be bounded uniformly for x and i by splitting the sum into m sums of i.i.d. summands and then applying Prokhorov’s inequality. Moreover, this bound converges to zero since we can choose $\epsilon > 0$ arbitrarily small. Thus, $\sup_{x > t_n} C_2(x)/(n\overline{F}(x))$ is negligible as $n \rightarrow \infty$.

Finally, we bound $\mathbb{P}(\{S_n > x\} \cap B_3) = \mathbb{P}(\widehat{S}_n > x)$. We split \widehat{S}_n into m independent sums and apply Fuk–Nagaev’s inequality for $p > \alpha$. Thus, we obtain for universal constants $c, d > 0$ only depending on p ,

$$\mathbb{P}(\widehat{S}_n > x) \leq cn\mathbb{E}[|\widehat{X}|^p]x^{-p} + \exp(-dx^2/n).$$

We have by Karamata’s theorem uniformly for $x > t_n$,

$$\frac{n\mathbb{E}[|\widehat{X}|^p/x^p]}{n\overline{F}(x)} = \frac{\mathbb{E}[|\widehat{X}|^p]}{(\epsilon x)^p \mathbb{P}(|X| > \epsilon x)} \epsilon^p \frac{\mathbb{P}(|X| > \epsilon x)}{\overline{F}(x)} \rightarrow c\epsilon^{p-\alpha}, \quad n \rightarrow \infty,$$

and the right-hand side vanishes as $\epsilon \downarrow 0$. Moreover, for large x

$$\frac{\exp(-dx^2/n)}{n\overline{F}(x)} = \exp(-dx^2/n - \log n + S(x)) \leq \exp(-0.5dx^2/n - \log n + \alpha \log x)$$

and the right-hand side converges to zero as $n \rightarrow \infty$ uniformly for $x > t_n \gg \sqrt{n \log n}$. □

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