SOJOURNS OF FRACTIONAL BROWNIAN MOTION QUEUES: TRANSIENT ASYMPTOTICS

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Abstract: We study the asymptotics of sojourn time of the stationary queueing process $Q(t), t \ge 0$ fed by a fractional Brownian motion with Hurst parameter $H \in (0,1)$ above a high threshold u. For the Brownian motion case $H = 1/2$, we derive the exact asymptotics of

$$\mathbb{P}\left\{ \int_{T_1}^{T_2} \mathbb{I}(Q(t) > u + h(u))dt > x \Big| Q(0) > u \right\}$$

as $u \to \infty$, where $T_1, T_2, x \ge 0$ and $T_2 - T_1 > x$, whereas for all $H \in (0,1)$, we obtain sharp asymptotic approximations of

$$\mathbb{P}\left\{ \frac{1}{v(u)} \int_{[T_2(u), T_3(u)]} \mathbb{I}(Q(t) > u + h(u))dt > y \Big| \frac{1}{v(u)} \int_{[0, T_1(u)]} \mathbb{I}(Q(t) > u)dt > x \right\}, \quad x, y > 0$$

as $u \to \infty$, for appropriately chosen T_i's and v. Two regimes of the ratio between u and $h(u)$, that lead to qualitatively different approximations, are considered.

Key Words: sojourn time; fractional Brownian motion; stationary queueing process; exact asymptotics; generalized Berman-type constants.

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1. Introduction

Fluid queueing systems with Gaussian-driven structure attained a substantial research interest over the last years; see, e.g., the monograph [1] and references therein. Following the seminal contributions [2–4] the class of fractional Brownian motions (fBm's) is a well legitimated model for the traffic stream in modern communication networks.

Let $B_H(t), t \in \mathbb{R}$ be a standard fBm with Hurst index $H \in (0,1)$, that is a Gaussian process with continuous sample paths, zero mean and covariance function satisfying

$$2Cov(B_H(t), B_H(s)) = |s|^{2H} + |t|^{2H} - |t-s|^{2H}, \quad s, t \in \mathbb{R}.$$

Consider the fluid queue fed by B_H and emptied with a constant rate $c > 0$. Using the interpretation that for $s < t$, the increment $B_H(t) - B_H(s)$ models the amount of traffic that entered the system in

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the time interval $[s,t)$, we define the workload process $Q(t), t \geq 0$ by

$$Q(t) = B_H(t) - ct + \max\left(Q(0), -\inf_{s\in[0,t]}(B_H(s) - cs)\right). \tag{1}$$

The unique stationary solution to the above equation, that is the object of the analysis in this contribution, takes the following form (see. e.g., [1])

$$\{Q(t), t \geq 0\} \stackrel{d}{=} \left\{\sup_{s\geq t}(B_H(s) - B_H(t) - c(s-t)), t \geq 0\right\}. \tag{2}$$

The vast majority of the analysis of queueing models with Gaussian inputs deals with the asymptotic results, with particular focus on the asymptotics of the probability

$$\mathbb{P}\{Q(t) > u\}$$

as $u \to \infty$, see e.g., [1, 2, 5–8]. Much less is known about transient characteristics of Q, such as

$$\mathbb{P}\left\{Q(T) > \omega \middle| Q(0) > u\right\},$$

with a notable exception for the Brownian motion ($H = 1/2$). In particular, in view of [9], see also related works [10–12], it is known that for $H = 1/2$ and $u, \omega, T > 0$

$$\mathbb{P}\left\{Q(T) > \omega \middle| Q(0) = u\right\} = \Phi\left(\frac{u-\omega-cT}{\sqrt{T}}\right) + e^{-2c\omega}\Phi\left(\frac{\omega+u-cT}{\sqrt{T}}\right) \tag{3}$$

and

$$\begin{aligned}\mathbb{P}\left\{Q(T) > \omega \middle| Q(0) > u\right\} &= -e^{2uc}\Phi\left(\frac{-\omega-u-cT}{\sqrt{T}}\right) + e^{-2c(\omega-u)}\Phi\left(\frac{\omega-u-cT}{\sqrt{T}}\right)\\ &\quad +\Phi\left(\frac{u-\omega-cT}{\sqrt{T}}\right) + e^{-2c\omega}\Phi\left(\frac{\omega+u-cT}{\sqrt{T}}\right),\end{aligned} \tag{4}$$

where $\Phi(\cdot)$ denotes the distribution function of a standard Gaussian random variable. Since $Q(0)$ is exponentially distributed for $H = 1/2$, (3)-(4) lead to explicit formula for $\mathbb{P}\{Q(0) > u, Q(T) > \omega\}$, which compared with $\mathbb{P}\{Q(0) > u\}\mathbb{P}\{Q(T) > \omega\}$ gives some insight to the dependence structure of the workload process $Q(t), t \geq 0$. Since the general case $H \in (0,1)$ is very complicated, the findings available in the literature concern mainly large deviation-type results; see e.g., [13] where the asymptotics of

$$\ln(\mathbb{P}\{Q(0) > pu, Q(Tu) > qu\}), \quad u \to \infty$$

was derived for $H \in (0,1)$. See also [14] for corresponding results the *many-source* model.
In addition to the conditional probability (4), it is also interesting to know how much time the queue spends above a given threshold during a given time period. This motivates us to consider the following quantity

$$\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t) > \omega)dt > x \middle| Q(0) > u\right\}, \quad x \in [0, T_2 - T_1) \tag{5}$$

for given non-negative $T_1 < T_2$.

In Section 2, for $H = 1/2$ we derive exact asymptotics of the above conditional sojourn time by letting $u, \omega = \omega(u) \to \infty$ in an appropriate way. Specifically, we shall distinguish between two regimes that lead to qualitatively different results:

(i) small fluctuation regime: $\omega = u + w + o(1), w \in \mathbb{R}$, for which the asymptotics of (5) tends to a positive constant as $u \to \infty$;

(ii) large fluctuation regime: $\omega = (1 + a)u + o(u), a \in (-1, \infty)$, for which (5) tends to 0 as $u \to \infty$ with the speed controlled by a

see Propositions 2.1, 2.2 respectively.

Then, in Section 3 for all $H \in (0, 1)$ and x, y non-negative we shall investigate approximations, as $u \to \infty$, of the following conditional sojourn times probabilities

$$\mathscr{P}^{x,y}_{T_1,T_2,T_3}(\omega, u) := \mathbb{P}\left\{ \frac{1}{v(u)} \int_{[T_2(u),T_3(u)]} \mathbb{I}(Q(t) > \omega)dt > y \Big| \frac{1}{v(u)} \int_{[0,T_1(u)]} \mathbb{I}(Q(t) > u)dt > x \right\}, \tag{6}$$

where $T_i(u), i = 1, 2, 3$ and $\omega = u + h(u), v(u)$ are suitably chosen functions, see assumption (**T**). In Theorem 3.1, complementing the findings of Proposition 2.1, we shall determine

$$\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}(u + au^{2H-1}, u) \tag{7}$$

under some asymptotic restrictions on $T_i(u)$'s and $a \in \mathbb{R}$, which yield a positive and finite limit. The idea of its proof is based on a modification of recently developed extension of the uniform double-sum technique for functionals of Gaussian processes [15]. Then, in Theorem 3.3 we shall obtain approximations of $\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1 + a)u, u)$ as $u \to \infty$, which correspond to the results derived in Proposition 2.2. The main findings of this section go in line with recently derived asymptotics for

$$\mathbb{P}\left\{ \frac{1}{v(u)} \int_{[0,T_1(u)]} \mathbb{I}(Q(t) > u)dt > x \right\}$$

as $u \to \infty$, see [15].

Structure of the paper: Section 2 is devoted to the analysis of the exact asymptotics of (5) for the classical model of the Brownian-driven queue, while in Section 3 we shall investigate asymptotic properties of $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(\omega, u)$ for $H \in (0, 1)$. Proofs of all the results are deferred to Section 4 and Appendix.

2. Preliminary results

In this section we shall focus on the exact asymptotics of (5) for the queueing process (2) driven by the Brownian motion.

Let in the following for $T_2 - T_1 > x \geq 0$ and $w \in \mathbb{R}$

$$\mathcal{C}(T_1, T_2, x; w) = 2c \int_{-\infty}^{w} e^{2cy} \mathbb{P}\left\{ \int_{T_1}^{T_2} \mathbb{I}\left(B_{1/2}(t) - ct > y\right) dt > x \right\} dy \in (0, \infty). \tag{8}$$

We begin with a *small fluctuation* result concerning the case when the distance between u and $\omega = \omega(u)$ in (5) is asymptotically constant. Below the term $o(1)$ is means for $u \to \infty$.

Proposition 2.1. *If $H = 1/2$ and $T_2 - T_1 > x \geq 0$, then for $\omega(u) = u + w + o(1), w \in \mathbb{R}$*

$$\mathbb{P}\left\{ \int_{T_1}^{T_2} \mathbb{I}(Q(t) > \omega(u))dt > x \Big| Q(0) > u \right\} \sim e^{-2cw} \mathcal{C}(T_1, T_2, x; w)$$

as $u \to \infty$.

Next, we consider the *large fluctuation* scenario, i.e., $\omega = \omega(u)$ in (5) is asymptotically proportional to u.

Proposition 2.2. *Suppose that $H = 1/2, T_2 - T_1 > x \geq 0$ and $\omega(u) = (1 + a)u + o(u)$.*

(i) If $a \in (-1, 0)$, then as $u \to \infty$

$$\mathbb{P}\left\{ \int_{T_1}^{T_2} \mathbb{I}(Q(t) > \omega(u))dt > x \Big| Q(0) > u \right\} \sim 1.$$

(ii) If $a > 0$, then as $u \to \infty$

$$\mathbb{P}\left\{ \int_{T_1}^{T_2} \mathbb{I}(Q(t) > \omega(u))dt > x \Big| Q(0) > u \right\} \sim e^{-2c(\omega(u) - u)} \mathcal{C}(T_1, T_2, x; \infty).$$

Both (i) and (ii) in Proposition 2.2 also hold if T_1, T_2 depend on u in such a way that as $u \to \infty$, these converge to positive constants $T_1 < T_2$ with $T_2 - T_1 > x \geq 0$.

3. Main results

This section is devoted to the asymptotic analysis of (6) for the queueing process Q defined in (2) with fBm input B_H, $H \in (0, 1)$. Before proceeding to the main results of this contribution, we introduce some notation and assumptions. Let $W_H(t) = \sqrt{2}B_H(t) - |t|^{2H}, t \in \mathbb{R}$ and define for

$$x \geq 0, y \geq 0$$

and $\lambda \in \mathbb{R}, \mathscr{T}_1 > 0$, $0 < \mathscr{T}_2 < \mathscr{T}_3 < \infty$

$$\overline{\mathcal{B}}_H^{x,y}(\mathscr{T}_1; \lambda, \mathscr{T}_2, \mathscr{T}_3) = \int_{\mathbb{R}} e^z \mathbb{P}\left\{ \int_{[0,\mathscr{T}_1]} \mathbb{I}(W_H(t) > z)dt > x, \int_{[\mathscr{T}_2,\mathscr{T}_3]} \mathbb{I}(W_H(t) > z + \lambda)dt > y \right\} dz$$

and set

$$\overline{\mathcal{B}}_H^{x}(\mathscr{T}_1) = \int_{\mathbb{R}} e^z \mathbb{P}\left\{ \int_{[0,\mathscr{T}_1]} \mathbb{I}(W_H(t) > z)dt > x \right\} dz.$$

Further, given $H \in (0, 1)$, $c > 0, u > 0$ let

$$A = \left(\frac{H}{c(1 - H)} \right)^{-H} \frac{1}{1 - H}, \quad t^* = \frac{H}{c(1 - H)}, \quad \Delta(u) = 2^{\frac{1}{2H}} t^* A^{-\frac{1}{H}} u^{-\frac{1-H}{H}} \tag{9}$$

and set

$$v(u) = u\Delta(u).$$

In the rest of this section, for a given function h, we analyse the asymptotics of $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(\omega(u), u)$ defined in (6) with $\omega(u) = u + h(u)$ as $u \to \infty$, where $T_i(u)$'s depend on u in such a way that

(T) $\lim_{u\to\infty} \frac{T_i(u)}{v(u)} = \mathscr{T}_i \in (0, \infty)$, for $i = 1, 2, 3$ with $\mathscr{T}_1 > x$ and $\mathscr{T}_3 - \mathscr{T}_2 > y$

is satisfied.

We note in passing that for $H = 1/2$, $v(u) = u\Delta(u) = 2^{\frac{1}{2H}} t^* A^{-\frac{1}{H}}$ is a constant. Hence, under **(T)**, we have $T_i(u) \to \mathscr{C}_i \in (0,\infty)$. Thus **(T)** included the model considered in Propositions 2.1 and 2.2.

We shall consider two scenarios that depend on the relative size of $h(u)$ with respect to u:

- ⋄ *small fluctuation* case: $|h(u)|$ is relatively *small* with respect to u, i.e., $h(u) = \lambda u^{2H-1}$ with $\lambda \in \mathbb{R}$ and $H \in (0,1)$, which leads to $\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}(u+h(u),u) > 0$;
- ⋄ *large fluctuation* case: $h(u) = au$ is proportional to u, which leads to $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(u+h(u),u) \to 0$ if $h(u) > 0$ and $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(u+h(u),u) \to 1$ if $h(u) < 0$ as $u \to \infty$.

Small fluctuation regime. We begin with the case when $h(u)$ is relatively small with comparison to u and thus the conditional probability $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(u+h(u),u)$ is cut away from 0, as $u \to \infty$.

Theorem 3.1. *If* **(T)** *holds, then with Q defined in (2) and $\lambda \in \mathbb{R}$*

$$\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}\left(u + \frac{\lambda}{A^2(1-H)}u^{2H-1}, u\right) = \frac{\overline{\mathcal{B}}^{x,y}_H(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)}{\overline{\mathcal{B}}^{x}_H(\mathscr{T}_1)} \in (0,\infty). \tag{10}$$

Remark 3.2. *(i) In the case of Brownian motion with $H = 1/2$, function $v(u) = 1/(2c^2)$ does not depend on u and the above reads*

$$\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}\left(u + \frac{\lambda}{2c}, u\right) = \frac{\overline{\mathcal{B}}^{x,y}_H(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)}{\overline{\mathcal{B}}^{x}_H(\mathscr{T}_1)} \in (0,\infty). \tag{11}$$

Since $v(u)$ is constant in this case, we can take $T_i = 2c^2\mathscr{T}_i > 0, i \le 3$ in (11). In the particular case that $x = 0, H = 1/2$ we have

$$\lim_{u\to\infty} \mathbb{P}\left\{\int_{[T_2,T_3]} \mathbb{I}\left(Q(t) > u + \frac{\lambda}{2c}\right) dt > 2c^2 y \Big| \sup_{t\in[0,T_1]} Q(t) > u\right\} = \frac{\overline{\mathcal{B}}^{0,y}_H(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)}{\overline{\mathcal{B}}^{0}_H(\mathscr{T}_1)} \in (0,\infty)$$

and taking $y = 0$ yields

$$\lim_{u\to\infty} \mathbb{P}\left\{\sup_{t\in[T_2,T_3]} Q(t) > u + \frac{\lambda}{2c} \Big| \sup_{t\in[0,T_1]} Q(t) > u\right\} = \frac{\overline{\mathcal{B}}^{0,0}_H(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)}{\overline{\mathcal{B}}^{0}_H(\mathscr{T}_1)} \in (0,\infty).$$

(ii) It follows from Theorem 3.1 that for $h(u) = o(u^{2H-1})$

$$\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}(u+h(u),u) = \frac{\overline{\mathcal{B}}^{x,y}_H(\mathscr{T}_1;0,\mathscr{T}_2,\mathscr{T}_3)}{\overline{\mathcal{B}}^{x}_H(\mathscr{T}_1)} \in (0,\infty). \tag{12}$$

Notably, if $H \in (1/2,1)$, then $T_i(u) \sim \mathscr{T}_i u^{(2H-1)/H}$ as $u \to \infty$ for $i = 1,2,3$. Hence $\lim_{u\to\infty}(T_2(u) - T_1(u)) = \infty$ and one can take $h(u) \to \infty$, as $u \to \infty$. Thus, the insensitivity of limit (12) on $h(u)$ is yet another manifestation of the long range dependence property of Q inherited from the input process B_H. This observation goes in line with the Piterbarg property

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,T(u)]} Q(t) > u\right\}}{\mathbb{P}\{Q(0) > u\}} = 1$$

derived in [16] and the strong Piterbarg property see [17], namely

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{\inf_{t\in[0,T(u)]} Q(t) > u\right\}}{\mathbb{P}\left\{Q(0) > u\right\}} = 1,$$

where $T(u) = o(u^{(2H-1)/H})$ *as* $u \to \infty$.

Large fluctuation regime. Suppose next that $h(u) = au$, $a \neq 0$. It appears that in this case the fluctuation $h(u)$ substantially influences the asymptotics of $\mathscr{P}^{x,y}_{T_1,T_2,T_3}(u + h(u), u)$ as $u \to \infty$. We point out the lack of symmetry with respect to the sign of a in the results given in the following theorem, which is due to the non-reversibility in time of the queueing process Q, i.e., the fact that

$$\mathbb{P}\left\{Q(s) > u, Q(t) > v\right\} \neq \mathbb{P}\left\{Q(t) > u, Q(s) > v\right\}$$

for $u \neq v$.

Theorem 3.3. *Let Q be defined in (2) and set $\tilde{a} = (1 + a)^{(1-2H)/H}$. Suppose that* **(T)** *holds.*

(i) If $a \in (-1, 0)$, then

$$\lim_{u\to\infty} \mathscr{P}^{x,y}_{T_1,T_2,T_3}((1 + a)u, u) = 1.$$

(ii) If $a > 0$, then

$$\limsup_{u\to\infty} \frac{\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1 + a)u, u)}{\exp\left(-\frac{A^2((1+a)^{2-2H}-1)}{2} u^{2-2H}\right)} \leq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}^{\tilde{a}y}_H(\tilde{a}(\mathscr{T}_3 - \mathscr{T}_2))}{\overline{\mathcal{B}}^{x}_H(\mathscr{T}_1)}$$

and

$$\liminf_{u\to\infty} \frac{\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1 + a)u, u)}{\exp\left(-\frac{A^2((1+a)^{2-2H}-1)}{2} u^{2-2H}\right)} \geq \tilde{a}^{1-H} \frac{\overline{\mathcal{B}}^{\tilde{a}x,\tilde{a}y}_H(\tilde{a}\mathscr{T}_1; 0, \tilde{a}\mathscr{T}_2, \tilde{a}\mathscr{T}_3)}{\overline{\mathcal{B}}^{x}_H(\mathscr{T}_1)}.$$

Remark 3.4. *Theorem 3.3 straightforwardly implies that*

$$\lim_{u\to\infty} \frac{\ln\left(\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1 + a)u, u)\right)}{u^{2-2H}} = -\frac{1}{2}A^2\left((1 + a)^{2-2H} - 1\right), \quad \forall a > 0.$$

4. Proofs

In this section we present detailed proofs of Proposition 2.1, 2.2 and Theorem 3.1, 3.3.

4.1. **Proof of Proposition 2.1.** Recall that by (1)

$$Q(t) = B_{1/2}(t) - ct + \max\left(Q(0), -\inf_{s\in[0,t]}(B_{1/2}(s) - cs)\right),$$

where $Q(0)$ is independent of $B_{1/2}(t) - ct$ and $\inf_{s\in[0,t]}(B_{1/2}(s) - cs)$ for $t > 0$. By [18, Eq. (5)] we have

$$\mathbb{P}\left\{Q(0) > u\right\} = \mathbb{P}\left\{\sup_{t\geq 0}(B_{1/2}(t) - ct) > u\right\} = e^{-2cu}, \quad u \geq 0. \tag{13}$$

Hence it suffices to analyse

$$\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u\right\}.$$

We note first that

$$\begin{aligned}
&\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u\right\}\\
&\geq \mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(0)+B_{1/2}(t)-ct>\omega(u))dt>x, Q(0)>u\right\}. \qquad (14)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u\right\}\\
&= \mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u, \sup_{s\in[0,T_2]}(cs-B_{1/2}(s))\leq u\right\}\\
&\quad +\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u, \sup_{s\in[0,T_2]}(cs-B_{1/2}(s))> u\right\}\\
&= P_1(u)+P_2(u). \qquad (15)
\end{aligned}$$

For $P_1(u)$ we have the following upper bound

$$P_1(u)\leq \mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(0)+B_{1/2}(t)-ct>\omega(u))dt>x, Q(0)>u\right\}$$

and for $P_2(u)$ by Borell-TIS inequality (see, e.g., [19])

$$P_2(u) \quad \leq \quad \mathbb{P}\left\{\sup_{s\in[0,T_2]}(cs-B_{1/2}(s))>u\right\}\leq e^{-Cu^2} \qquad (16)$$

for some $C>0$ and sufficiently large u.

Next, we note that

$$\begin{aligned}
&\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(0)+B_{1/2}(t)-ct>\omega(u))dt>x, Q(0)>u\right\}\\
&=2c\int_u^\infty e^{-2cy}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(y+B_{1/2}(t)-ct>\omega(u)\right)dt>x\right\}dy\\
&=2ce^{-2c\omega(u)}\int_{u-\omega(u)}^\infty e^{-2cy}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>-y\right)dt>x\right\}dy\\
&=2ce^{-2c\omega(u)}\int_{-\infty}^{\omega(u)-u} e^{2cy}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>y\right)dt>x\right\}dy\\
&=e^{-2c\omega(u)}\mathcal{C}(T_1,T_2,x;\omega(u)-u),
\end{aligned}$$

where $\mathcal{C}(T_1,T_2,x;z)$ is defined in (8). Hence, by (16) applied to (14) and (15), we arrive at

$$\mathbb{P}\left\{\int_{T_1}^{T_2} \mathbb{I}(Q(t)>\omega(u))dt>x, Q(0)>u\right\}\sim e^{-2cu}e^{-2cw}\mathcal{C}(T_1,T_2,x;w) \qquad (17)$$

as $u\to\infty$. Finally, by (13) we get

$$\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}(Q(t)>\omega(u))dt>x\Big|Q(0)>u\right\}\sim e^{-2cw}\mathcal{C}(T_1,T_2,x;w)$$

as $u\to\infty$. This completes the proof. □

4.2. **Proof of Proposition 2.2.** The idea of the proof is the same as the proof of Proposition 2.1. Since by Borell-TIS inequality

$$\begin{aligned}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>y\right)dt>x\right\} &\leq \mathbb{P}\left\{\sup_{t\in[T_1,T_2]}\left(B_{1/2}(t)-ct\right)>y\right\}\\ &\leq C_1\exp(-C_2y^2)\end{aligned}$$

for some positive constants C_1,C_2, we conclude that

$$\mathcal{C}(T_1,T_2,x;\infty)=\int_{-\infty}^{\infty}e^{2cy}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>y\right)dt>x\right\}dy<\infty.$$

Thus if $a\in(-1,0)$, then as $u\to\infty$

$$\begin{aligned}&\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}(Q(t)>\omega(u))dt>x,Q(0)>u\right\}\\ &\sim 2ce^{-2c\omega(u)}\int_{-\infty}^{\omega(u)-u}e^{2cy}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>y\right)dt>x\right\}dy\\ &\sim e^{-2cu},\end{aligned}$$

where we used that uniformly for $y\in(-\infty,\omega(u)-u]$

$$\lim_{u\to\infty}\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}\left(B_{1/2}(t)-ct>y\right)dt>x\right\}=1.$$

Similarly for $a>0$, we have that

$$\mathbb{P}\left\{\int_{T_1}^{T_2}\mathbb{I}(Q(t)>\omega(u))dt>x,Q(0)>u\right\}\sim e^{-2c\omega(u)}\mathcal{C}(T_1,T_2,x;\infty)$$

as $u\to\infty$. Thus, combining the above with (13), we complete the proof. □

4.3. **Proof of Theorem 3.1.** We begin with a result which is crucial for the proof and of some interests on its own right. Recall that Q is defined in (2). For B_H,B'_H two independent fBm's with Hurst indexes H, we set

$$W_H(t)=\sqrt{2}B_H(t)-|t|^{2H},\quad W'_H(t)=\sqrt{2}B'_H(t)-|t|^{2H} \tag{18}$$

and

$$V_H(S)=\sup_{s\in[0,S]}W_H(s),\quad t\in\mathbb{R},S\geq 0.$$

Define further for all x, y non-negative and $\lambda \in \mathbb{R}$, the generalized Berman-type constants by

$$\begin{aligned}&\mathcal{B}_H^{x,y}(T_1;\lambda,T_2,T_3)([0,S])\\&=\int_{\mathbb{R}} e^w \mathbb{P}\left\{\int_{[0,T_1]}\mathbb{I}(W'_H(t)+V_H(S)>w)dt>x,\int_{[T_2,T_3]}\mathbb{I}(W'_H(t)+V_H(S)>w+\lambda)dt>y\right\}dw.\end{aligned}$$

Denote further by $\mathcal{H}_{2H}$ the Pickands constant corresponding to B_H, i.e.,

$$\mathcal{H}_{2H}=\lim_{S\to\infty}S^{-1}\mathbb{E}\left\{e^{V_H(S)}\right\}=\mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}}e^{W_H(t)}}{\int_{t\in\mathbb{R}}e^{W_H(t)}dt}\right\}\in(0,\infty).$$

Lemma 4.1. *For all T_1, T_2, T_3 positive, $\lambda \in \mathbb{R}$, and all x, y non-negative we have*

$$\mathcal{B}_H^{x,y}(T_1;\lambda,T_2,T_3):=\lim_{S\to\infty}S^{-1}\mathcal{B}_H^{x,y}(T_1;\lambda,T_2,T_3)([0,S]) \quad = \quad \mathcal{H}_{2H}\overline{\mathcal{B}}_H^{x,y}(T_1;\lambda,T_2,T_3)\in(0,\infty).$$

It is worth mentioning that both sides of equation in the above lemma is equal to zero if $x \ge T_1$ or $y \ge T_3 - T_2$. Hence it is valid for all nonnegative x and y.

Proof of Lemma 4.1 First note that for any $S > 0$ we have using the Fubini-Tonelli theorem and the independence of V_H and W'_H

$$\begin{aligned}&\mathcal{B}_H^{x,y}(T_1;\lambda,T_2,T_3)([0,S])\\
&= \mathbb{E}\left\{\int_{\mathbb{R}}e^w\mathbb{I}(\int_{[0,T_1]}\mathbb{I}(W'_H(t)+V_H(S)>w)dt>x,\int_{[T_2,T_3]}\mathbb{I}(W'_H(t)+V_H(S)>w+\lambda)dt>y)dw\right\}\\
&= \mathbb{E}\left\{e^{V_H(S)}\int_{\mathbb{R}}e^w\mathbb{I}(\int_{[0,T_1]}\mathbb{I}(W'_H(t)>w)dt>x,\int_{[T_2,T_3]}\mathbb{I}(W'_H(t)>w+\lambda)dt>y)dw\right\}\\
&= \mathbb{E}\left\{e^{V_H(S)}\right\}\int_{\mathbb{R}}e^w\mathbb{P}\left\{\int_{[0,T_1]}\mathbb{I}(W_H(t)>w)dt>x,\int_{[T_2,T_3]}\mathbb{I}(W_H(t)>w+\lambda)dt>y\right\}dw\\
&\le \mathbb{E}\left\{e^{V_H(S)}\right\}\int_{\mathbb{R}}e^w\mathbb{P}\left\{\int_{[0,T_1]}\mathbb{I}(W_H(t)>w)dt>0\right\}dw\\
&= \mathbb{E}\left\{e^{V_H(S)}\right\}\int_0^\infty e^w\mathbb{P}\left\{V_H(T_1)>w\right\}dw\\
&= \mathbb{E}\left\{e^{V_H(S)}\right\}\mathbb{E}\left\{e^{V_H(T_1)}\right\}.\end{aligned}$$

Hence the claim follows by the definition of the Pickands constant and the sample continuity of V_H.
□

Let in the following

$$B=\left(\frac{H}{c(1-H)}\right)^{-H-2}H$$

and recall that

$$\Delta(u)=2^{\frac{1}{2H}}t^*A^{-\frac{1}{H}}u^{-\frac{1-H}{H}},\quad v(u)=u\Delta(u). \tag{19}$$

Applying [15, Lem 4.1] we obtain the following result.

Proposition 4.2. *If* **(T)** *holds, then*

$$\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\sim\mathcal{H}_{2H}\overline{\mathcal{B}}_H^x(\mathscr{T}_1)\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(Au^{1-H}),\quad u\to\infty. \tag{20}$$

The next proposition plays a key role in the proof of Theorem 3.1.

Proposition 4.3. *If* **(T)** *holds, then for all* $\lambda\in\mathbb{R},\tau=\lambda/(A^2(1-H))$

$$\begin{aligned}&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>u+\tau u^{2H-1})dt>y,\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\\&\sim\mathcal{H}_{2H}\overline{\mathcal{B}}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(Au^{1-H}),\quad u\to\infty.\end{aligned} \tag{21}$$

Hereafter, for any non-constant random variable Z, we denote $\overline{Z}=Z/\sqrt{Var(Z)}$.

Proof of Proposition 4.3 Using the self-similarity of B_H, i.e.,

$$\{B_H(ut),t\in\mathbb{R}\}\overset{d}{=}\{u^HB_H(t),t\in\mathbb{R}\},\quad u>0$$

we have with $\Delta(u)$ given in (9) and $\widetilde{u}=u+\tau u^{2H-1}$

$$\begin{aligned}&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x,\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>\widetilde{u})dt>y\right\}\\
&=\mathbb{P}\left\{\frac{1}{\Delta(u)}\int_{[0,T_1(u)/u]}\mathbb{I}(\sup_{s\ge t}(u^H(B_H(s)-B_H(t))-cu(s-t))>u)dt>x,\right.\\
&\qquad\left.\frac{1}{\Delta(u)}\int_{[T_2(u)/\widetilde{u},T_3(u)/\widetilde{u}]}\mathbb{I}(\sup_{s\ge t}(\widetilde{u}^H(B_H(s)-B_H(t))-c\widetilde{u}(s-t))>\widetilde{u})dt>y\right\}\\
&=\mathbb{P}\left\{\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{s\ge t}Z(s,t)>u_\star)dt>x,\right.\\
&\qquad\left.\frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}(\sup_{s\ge t}Z(s,t)>\widetilde{u}_\star)dt>y\right\},\end{aligned}$$

where

$$Z(s,t)=A\frac{B_H(s)-B_H(t)}{1+c(s-t)}$$

and

$$u_\star=Au^{1-H},\quad\widetilde{u}_\star=A\widetilde{u}^{1-H},\quad\overline{T}_1(u)=T_1(u)/u,\overline{T}_i(u)=T_i(u)/\widetilde{u},i=2,3.$$

Note that as $u\to\infty$

$$\widetilde{u}_\star=u_\star+\frac{\lambda}{u_\star},\quad\widetilde{u}_\star^2\sim{u_\star}^2+2\lambda+o(1). \tag{22}$$

Direct calculation shows that

$$\max_{s\ge t}\sqrt{Var(Z(s,t))}=\max_{s\ge t}\frac{A(s-t)^H}{1+c(s-t)}=1$$

and the maximum is attained for all s,t such that

$$s-t=t^*=\frac{H}{c(1-H)}$$

and

$$1-A\frac{t^H}{1+ct}\sim\frac{B}{2A}(t-t^*)^2,\quad t\to t^*. \tag{23}$$

Moreover, we have

$$\lim_{\delta\to 0}\sup_{|s-t-t^*|,|s'-t'-t^*|<\delta,|s-s'|<\delta}\left|\frac{1-Cor(Z(s,t),Z(s',t'))}{|s-s'|^{2H}+|t-t'|^{2H}}-2^{-1}(t^*)^{-2H}\right|=0. \tag{24}$$

In the following we tacitly assume that

$$S>\max(x,y).$$

Observe that

$$\begin{aligned}\pi_1(u) &\le \mathbb{P}\left\{\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{s\ge t}Z(s,t)>u_\star)dt>x,\frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}(\sup_{s\ge t}Z(s,t)>\widetilde{u}_\star)dt>y\right\}\\ &\le \pi_1(u)+\pi_2(u),\end{aligned}$$

where

$$\begin{aligned}\pi_1(u) &= \mathbb{P}\Bigg\{\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{|s-t^*|\le(\ln u)/u^{1-H}}Z(s,t)>u_\star)dt>x,\\ &\qquad\frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}(\sup_{|s-t^*|\le(\ln u)/u^{1-H}}Z(s,t)>\widetilde{u}_\star)dt>y\Bigg\},\end{aligned}$$

$$\pi_2(u)=\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{|s-t^*|\ge(\ln u)/(2u^{1-H}),s\ge t}Z(s,t)>\hat{u}\right\},$$

with $\overline{T^*}(u)=\max(\overline{T}_1(u),\overline{T}_2(u),\overline{T}_3(u))$ and $\hat{u}=\min(u_\star,\widetilde{u}_\star)$.

$\diamond$ *Upper bound of $\pi_2(u)$*. Next, for some $T>0$ we have

$$\pi_2(u)\le\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{|s-t^*|\ge(\ln u)/(2u^{1-H}),t\le s\le T}Z(s,t)>\hat{u}\right\}+\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{s\ge T}Z(s,t)>\hat{u}\right\}.$$

In view of (23) for u sufficiently large

$$\sup_{t\in[0,\overline{T^*}(u)]}\sup_{|s-t^*|\ge(\ln u)/(2u^{1-H}),t\le s\le T}Var(Z(s,t))\le 1-\mathbb{Q}\left(\frac{\ln u}{u^{1-H}}\right)^2$$

and by (24)

$$\mathbb{E}\left\{\left(Z(s,t)-Z(s',t')\right)^2\right\}\le\mathbb{Q}_1(|s-s'|^H+|t-t'|^H),\quad t\in[0,\overline{T^*}(u)],|s-t^*|\ge(\ln u)/(2u^{1-H}),t\le s\le T.$$

Hence, in light of [20, Thm 8.1] for all u large enough

$$\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{|s-t^*|\ge(\ln u)/(2u^{1-H}),t\le s\le T}Z(s,t)>\hat{u}\right\}\le\mathbb{Q}_2u^{\frac{4(1-H)}{H}}\Psi\left(\frac{\hat{u}}{\sqrt{1-\mathbb{Q}\left(\frac{\ln u}{u^{1-H}}\right)^2}}\right).$$

Moreover, for T sufficiently large

$$\sqrt{Var(Z(s,t))} = \frac{A(s-t)^H}{1+c(s-t)} \le \frac{2A}{c}(T+k)^{-(1-H)}, \quad s \in [T+k, T+k+1], t \in [0, \overline{T^*}(u)].$$

Hence for some $\varepsilon \in (0,1)$ (set $c_\varepsilon = (1+\varepsilon)c$)

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{s\ge T} Z(s,t) > \hat{u}\right\} &\le \sum_{k=0}^{\infty}\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{s\in[T+k,T+k+1]} Z(s,t) > \hat{u}\right\}\\
&\le \sum_{k=0}^{\infty}\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{s\in[T+k,T+k+1]} \overline{Z}(s,t) > \frac{1}{2}c_\varepsilon (T+k)^{(1-H)}u^{1-H}\right\}.
\end{aligned}$$

Additionally, for T sufficiently large and $k \ge 0$, we have

$$\mathbb{E}\left\{(\overline{Z}(s,t) - \overline{Z}(s',t'))^2\right\} \le \mathbb{Q}_3(|s-s'|^H + |t-t'|^H), \quad s,s' \in [T+k, T+k+1], t,t' \in [0,1].$$

Thus by [20, Thm 8.1] for all T and u sufficiently large we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,\overline{T^*}(u)]}\sup_{s\ge T} Z(s,t) > \hat{u}\right\} &\le \sum_{k=0}^{\infty}\mathbb{P}\left\{\sup_{t\in[0,1]}\sup_{s\in[T+k,T+k+1]} \overline{Z}(s,t) > \frac{1}{2}c_\varepsilon (T+k)^{(1-H)}u^{1-H}\right\}\\
&\le \sum_{k=0}^{\infty}\mathbb{Q}_4 u^{\frac{4(1-H)}{H}}\Psi\left(\frac{1}{2}c_\varepsilon (T+k)^{(1-H)}u^{1-H}\right)\\
&\le \sum_{k=0}^{\infty}\mathbb{Q}_4 u^{\frac{4(1-H)}{H}} e^{-\frac{1}{2}\left(\frac{1}{2}c_\varepsilon (T+k)^{(1-H)}u^{1-H}\right)^2}\\
&\le \sum_{k=0}^{\infty}\mathbb{Q}_4 u^{\frac{4(1-H)}{H}} \int_{T-1}^{\infty} e^{-\frac{1}{2}\left(\frac{1}{2}c_\varepsilon z^{(1-H)}u^{1-H}\right)^2}dz\\
&\le \mathbb{Q}_4 u^{\frac{4(1-H)}{H}}\Psi\left(\mathbb{Q}_5 (Tu)^{1-H}\right).
\end{aligned}$$

Therefore we conclude that for all u, T sufficiently large

$$\pi_2(u) \le \mathbb{Q}_2 u^{\frac{4(1-H)}{H}}\Psi\left(\frac{\hat{u}}{\sqrt{1-\mathbb{Q}\left(\frac{\ln u}{u^{1-H}}\right)^2}}\right) + \mathbb{Q}_4 u^{\frac{4(1-H)}{H}}\Psi\left(2Au^{1-H}\right). \tag{25}$$

◇ *Upper bound of $\pi_1(u)$*. Given a positive integer k and $u > 0$ define

$$I_k(u) = [k\Delta(u)S, (k+1)\Delta(u)S], \quad N(u) = \left[\frac{\ln u}{u^{1-H}\Delta(u)S}\right] + 1.$$

It follows that

$$\begin{aligned}
\pi_1(u) &= \mathbb{P}\Bigg\{\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}\Big(\sup_{|s|\le(\ln u)/u^{1-H}} Z(s+t^*,t) > u_\star\Big)dt > x,\\
&\qquad \frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}\Big(\sup_{|s|\le(\ln u)/u^{1-H}} Z(s+t^*,t) > \widetilde{u}_\star\Big)dt > y\Bigg\}\\
&\le \Sigma_1^+(u) + 2\Sigma\Sigma_1(u) + 2\Sigma\Sigma_2(u),
\end{aligned}$$

where

$$
\begin{aligned}
\Sigma_1^+(u) &= \sum_{k=-N(u)-1}^{N(u)+1} \mathbb{P}\left\{ \frac{1}{\Delta(u)} \int_{[0,\overline{T}_1(u)]} \mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t) > u_\star) dt > x, \right. \\
&\qquad \left. \frac{1}{\Delta(u)} \int_{[\overline{T}_2(u),\overline{T}_3(u)]} \mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t) > \widetilde{u}_\star) dt > y \right\} \\
&\le \sum_{k=-N(u)-1}^{N(u)+1} \mathbb{P}\left\{ \int_{[0,\mathscr{T}_1+\epsilon]} \mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t) > u_k^-) dt > x, \right. \\
&\qquad \left. \int_{[\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon]} \mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t) > \widetilde{u}_k^-) dt > y \right\}, \\
\Sigma\Sigma_1(u) &= \sum_{|k|,|l|\le N(u)+1, l=k+1} \mathbb{P}\left\{ \sup_{t\in[0,T^*], s\in[kS,(k+1)S]} Z(\Delta(u)s+t^*,\Delta(u)t) > \widetilde{u}_\star, \right. \\
&\qquad \left. \sup_{t\in[0,T^*], s\in[lS,(l+1)S]} Z(\Delta(u)s+t^*,\Delta(u)t) > u_\star \right\}, \\
\Sigma\Sigma_2(u) &= \sum_{|k|,|l|\le N(u)+1, l\ge k+2} \mathbb{P}\left\{ \sup_{t\in[0,T^*], s\in[kS,(k+1)S]} Z(\Delta(u)s+t^*,\Delta(u)t) > \widetilde{u}_\star, \right. \\
&\qquad \left. \sup_{t\in[0,T^*], s\in[lS,(l+1)S]} Z(\Delta(u)s+t^*,\Delta(u)t) > u_\star \right\},
\end{aligned}
$$

with

$$
T^* = \max(\mathscr{T}_1+\epsilon, \mathscr{T}_2-\epsilon, \mathscr{T}_3+\epsilon), \epsilon < \mathscr{T}_2, \quad \Delta(u) = Cu^{-\frac{1-H}{H}}, \quad C = 2^{\frac{1}{2H}} t^* A^{-\frac{1}{H}},
$$

$$
Z_{u,k}(s,t) = \overline{Z}(t^* + \Delta(u)(kS+s), \Delta(u)t),
$$

$$
u_k^- = u_\star \left(1 + \frac{(1-\epsilon)B}{2A} \Delta^2(u) \eta_{k,S}\right), \quad \eta_{k,S} = \inf_{s\in[kS,(k+1)S], t\in[0,T_*]} (s-t)^2,
$$

$$
\widetilde{u_k}^- = \widetilde{u}_\star \left(1 + \frac{(1-\epsilon)B}{2A} \Delta^2(u) \eta_{k,S}\right).
$$

Since the maximal value of k is $N(u) = \left[\frac{\ln u}{u^{1-H}\Delta(u)S}\right] + 1$ and $\eta_{k,S}$ is non-negative and bounded up to some constant by k^2S^2 using further (22) we have

$$
u_k^- = u_\star(1+o(u^{H-1}\ln u)), \quad \widetilde{u_k}^- = (u_\star + \lambda/u_\star)(1+o(u^{H-1}\ln u)) = u_k^- + \lambda_{u,k}/u_k^-, \tag{26}
$$

where $o(u^{H-1}\ln u)$ does not depend on k, S and further

$$
\lim_{u\to\infty} \sup_{|k|\le N(u)} |\lambda - \lambda_{u,k}| = 0.
$$

We analyse next the uniform asymptotics of

$$
p_k(u) := \mathbb{P}\left\{ \int_{[0,\mathscr{T}_1+\epsilon]} \mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t) > u_k^-) dt > x, \int_{[\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon]} \mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t) > u_k^- + \lambda_{u,k}/u_k^-) dt > y \right\}
$$

as $u\to\infty$ with respect to $|k|\le N(u)+1$. In order to apply Lemma 5.1 in Appendix, we need to check conditions **C1-C3** therein. The first condition **C1** follows immediately from (26). The second condition **C2** is a consequence of (24), while **C3** follows from (26). Consequently, using further (26), the application of the aforementioned lemma is justified and we obtain

$$\lim_{u\to\infty}\sup_{|k|\le N(u)+1}\left|\frac{p_k(u)}{\Psi(u_k^-)}-\mathcal{B}_H^{x,y}(\mathscr{T}_1+\epsilon;\lambda,\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon)([0,S])\right|=0. \tag{27}$$

Hence

$$\begin{aligned}
\Sigma_1^+(u) &\le \sum_{|k|\le N(u)+1}\mathcal{B}_H^{x,y}(\mathscr{T}_1+\epsilon;\lambda,\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon)([0,S])\Psi(u_k^-)\\
&\le \mathcal{B}_H^{x,y}(\mathscr{T}_1+\epsilon;\lambda,\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon)([0,S])\Psi(u_\star)\sum_{|k|\le N(u)+1}e^{-A^2u^{2(1-H)}\times\frac{(1-\epsilon)B}{2A}\Delta^2(u)\times(kS)^2}\\
&\sim \frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1+\epsilon;\lambda,\mathscr{T}_2-\epsilon,\mathscr{T}_3+\epsilon)([0,S])}{S}\frac{\sqrt{2}(AB)^{-1/2}(1-\epsilon)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star)\int_{\mathbb{R}}e^{-t^2}dt\\
&\sim \frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)([0,S])}{S}\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star),\quad u\to\infty,\epsilon\to 0. \qquad (28)
\end{aligned}$$

<u>*Upper bound of* $\Sigma\Sigma_1(u)$</u>. Suppose for notational simplicity that $\lambda=0$. Then $\widetilde{u}_\star=u_\star$ and

$$\Sigma\Sigma_1(u)\le\sum_{|k|\le N(u)+1}(q_{k,1}(u)+q_{k,2}(u)),$$

where

$$\begin{aligned}
q_{k,1}(u) &= \mathbb{P}\left\{\sup_{t\in[0,T_3^*],s\in[kS,(k+1)S]}Z_u(s,t)>u_\star\ \sup_{t\in[0,T_3^*],s\in[(k+1)S,(k+1)S+\sqrt{S}]}Z_u(s,t)>u_\star\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,T_3^*],s\in[(k+1)S,(k+1)S+\sqrt{S}]}\overline{Z}_u(s,t)>u_{k+1}^-\right\},\\
q_{k,2}(u) &= \mathbb{P}\left\{\sup_{t\in[0,T_3^*],s\in[kS,(k+1)S]}Z_u(s,t)>u_\star,\ \sup_{t\in[0,T_3^*],s\in[(k+1)S+\sqrt{S},(k+2)S]}Z_u(s,t)>u_\star\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,T_3^*],s\in[kS,(k+1)S]}\overline{Z}_u(s,t)>u_k^-,\ \sup_{t\in[0,T_3^*],s\in[(k+1)S+\sqrt{S},(k+2)S]}\overline{Z}_u(s,t)>u_{k+1}^-\right\},
\end{aligned}$$

with

$$Z_u(s,t)=Z(t^*+\Delta(u)s,\Delta(u)t).$$

Analogously as in (27), we have that

$$\lim_{u\to\infty}\sup_{|k|\le N(u)+1}\left|\frac{\mathbb{P}\left\{\sup_{t\in[0,T_3^*],s\in[(k+1)S,(k+1)S+\sqrt{S}]}\overline{Z}_u(s,t)>u_{k+1}^-\right\}}{\Psi(u_{k+1}^-)}-\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})\right|=0.$$

Thus in view of (28)

$$\begin{aligned}\sum_{|k|\le N(u)+1} q_{k,1}(u) &\le \sum_{|k|\le N(u)+1} \overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})\Psi(u_{k+1}^-)\\ &\le \frac{\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})}{S}\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty.\end{aligned}$$

Additionally, in light of (24) for u sufficiently large

$$|s-s'|^{2H}+|t-t'|^{2H} \le 2(u_\star)^2\left(1-Cor(\overline{Z}_u(s,t),\overline{Z}_u(s',t'))\right) \le 4(|s-s'|^{2H}+|t-t'|^{2H}) \tag{29}$$

for all $|s|,|s'|\le \frac{2\ln u}{u^{1-H}\Delta(u)}, t,t'\in[0,T_*]$. Thus by [21, Cor 3.1] there exist two positive constants $\mathcal{C},\mathcal{C}_1$ such that for u sufficiently large and $S>1$

$$q_{k,2}(u)\le \mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}}\Psi(u_{k,k+1}^-), \quad u_{k,l}^-=\min(u_k^-,u_l^-).$$

Hence

$$\begin{aligned}\sum_{|k|\le N(u)+1} q_{k,2}(u) &\le \sum_{|k|\le N(u)+1} \mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}}\Psi(u_{k,k+1}^-)\\ &\le \mathcal{C}S^3 e^{-\mathcal{C}_1 S^{\frac{H}{2}}}\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty.\end{aligned}$$

Therefore we conclude that

$$\Sigma\Sigma_1(u)\le\left(\frac{\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})}{S}+\mathcal{C}S^3e^{-\mathcal{C}_1S^{\frac{H}{2}}}\right)\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty. \tag{30}$$

Note that if $\lambda\neq 0$, the bound derived in (30) changes only by a multiplication by some constant, which does not affect the negligibility of $\Sigma\Sigma_1(u)$.

<u>*Upper bound of* $\Sigma\Sigma_2(u)$</u>. In light of (29) and applying [21, Cor 3.1], we have that

$$\begin{aligned}\Sigma\Sigma_2(u) &\le \sum_{|k|,|l|\le N(u)+1, l\ge k+2}\mathcal{C}S^4e^{-\mathcal{C}_1|l-k-1|^HS^H}\Psi(u_{k,l}^-)\\ &\le \sum_{|k|\le N(u)+1}\mathcal{C}S^4\Psi(u_k^-)\sum_{l=1}^{\infty}e^{-\mathcal{C}_1l^HS^H}\\ &\le \sum_{|k|\le N(u)+1}\mathcal{C}S^4e^{-\mathbb{Q}_6S^H}\Psi(u_k^-)\\ &\le \mathcal{C}S^3e^{-\mathbb{Q}_6S^H}\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty. \end{aligned} \tag{31}$$

Consequently, as $u\to\infty$

$$\begin{aligned}\pi_1(u) \le &\left(\frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)([0,S])}{S}+\frac{\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})}{S}+\mathcal{C}S^3[e^{-\mathcal{C}_1S^{\frac{H}{2}}}+e^{-\mathbb{Q}_6S^H}]\right)\\ &\times\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star).\end{aligned} \tag{32}$$

$\diamond$ <u>*Lower bound of* $\pi_1(u)$</u>. Again for notation simplicity we assume $\lambda = 0$. Observe that

$$
\begin{aligned}
&\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{|s-t-t^*|\le(\ln u)/u^{1-H}} Z(s,t)>u_\star)dt\\
&\ge \sum_{|k|\le N(u)}\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star)dt\\
&\quad-\sum_{|k|,|l|\le N(u),k<l}\frac{1}{\Delta(u)}\int_{[0,T^*(u)/u]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star,\sup_{s\in I_l(u)} Z(s+t^*,t)>u_\star)dt\\
&:= F_1(u)-F_2(u),\\
&\frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}(\sup_{|s-t-t^*|\le(\ln u)/u^{1-H}} Z(s,t)>u_\star)dt\\
&\ge \sum_{|k|\le N(u)}\frac{1}{\Delta(u)}\int_{[\overline{T}_2(u),\overline{T}_3(u)]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star)dt\\
&\quad-\sum_{|k|,|l|\le N(u),k<l}\frac{1}{\Delta(u)}\int_{[0,\overline{T^*}(u)]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star,\sup_{s\in I_l(u)} Z(s+t^*,t)>u_\star)dt\\
&:= F_3(u)-F_2(u).
\end{aligned}
$$

Hence, for $0<\epsilon<1$ (write $s_\epsilon=(1+\epsilon)s)$)

$$
\begin{aligned}
\pi_1(u) &\ge \mathbb{P}\left\{F_1(u)-F_2(u)>x, F_3(u)-F_2(u)>y\right\}\\
&\ge \mathbb{P}\left\{F_1(u)>x_\epsilon, F_3(u)>x_\epsilon, F_2(u)<\epsilon\min(x,y)\right\}\\
&\ge \mathbb{P}\left\{F_1(u)>x_\epsilon, F_3(u)>y_\epsilon\right\}-\mathbb{P}\left\{F_2(u)\ge\epsilon\min(x,y)\right\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{P}\left\{F_1(u)>x_\epsilon, F_3(u)>y_\epsilon\right\}\\
&\ge \mathbb{P}\left\{\exists|k|\le N(u): \frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star)dt>x_\epsilon, F_3(u)>y_\epsilon\right\}\\
&\ge \sum_{|k|\le N(u)}\mathbb{P}\left\{\frac{1}{\Delta(u)}\int_{[0,\overline{T}_1(u)]}\mathbb{I}(\sup_{s\in I_k(u)} Z(s+t^*,t)>u_\star)dt>x_\epsilon, F_3(u)>y_\epsilon\right\}\\
&\quad-\Sigma\Sigma_1(u)-\Sigma\Sigma_2(u)\\
&\ge \Sigma_1^-(u)-\Sigma\Sigma_1(u)-\Sigma\Sigma_2(u),
\end{aligned}
$$

and

$$
\mathbb{P}\left\{F_2(u)\ge\epsilon\min(x,y)\right\}\le\mathbb{P}\left\{F_2(u)>0\right\}\le\Sigma\Sigma_1(u)+\Sigma\Sigma_2(u),
$$

where

$$
\Sigma_1^-(u)=\sum_{k=-N(u)}^{N(u)}\mathbb{P}\left\{\int_{[0,\mathscr{T}_1]}\mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t)>u_k^-)dt>x_\epsilon, \int_{[\mathscr{T}_2,\mathscr{T}_3]}\mathbb{I}(\sup_{s\in[0,S]} Z_{u,k}(s,t)>u_k^-)dt>y_\epsilon\right\}.
$$

Hence

$$\pi_1(u) \geq \Sigma_1^-(u) - 2\Sigma\Sigma_1(u) - 2\Sigma\Sigma_2(u).$$

Analogously as in (28), it follows that

$$\Sigma_1^-(u) \sim \frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)([0,S])}{S}\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty, \epsilon\to 0,$$

which together with the upper bound of $\Sigma\Sigma_i, i=1,2$ leads to

$$\begin{aligned}\pi_1(u) \quad &\geq \quad \left(\frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)([0,S])}{S} - \frac{2\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})}{S} - 2\mathcal{C}S^3[e^{-\mathcal{C}_1 S^{\frac{H}{2}}} + e^{-\mathbb{Q}_6 S^H}]\right)\\ &\qquad \times\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty. \qquad (33)\end{aligned}$$

Next by Lemma 4.1, we have

$$\lim_{S\to\infty}\frac{\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)([0,S])}{S} = \mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)\in(0,\infty), \quad \lim_{S\to\infty}\frac{\overline{\mathcal{B}}_H^0(T_3^*)\overline{\mathcal{B}}_H^0(\sqrt{S})}{S}=0.$$

Thus letting $S\to\infty$ in (32) and (33) yields

$$\pi_1(u)\sim\mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty,$$

which combined with (25) leads to

$$\begin{aligned}&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>u+\tau u^{2H-1})dt>y, \frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\\ &\quad\sim\quad \mathcal{B}_H^{x,y}(\mathscr{T}_1;\lambda,\mathscr{T}_2,\mathscr{T}_3)\frac{\sqrt{2\pi}(AB)^{-1/2}}{u^{1-H}\Delta(u)}\Psi(u_\star), \quad u\to\infty\end{aligned}$$

establishing the proof. □

Proof of Theorem 3.1 Clearly, for all x,y non-negative with $\widetilde{u}=u+\tau u^{2H-1}$

$$\mathscr{P}_{T_1,T_2,T_3}^{x,y}(\widetilde{u},u)=\frac{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x, \frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>\widetilde{u})dt>y\right\}}{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}}.$$

The asymptotics of the denominator and the nominator are derived in Proposition 4.2 and Proposition 4.3, respectively. Hence, using further (22) establishes the claim. □

4.4. **Proof of Theorem 3.3.** Case $a\in(-1,0)$. Observe that

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>y,\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\\
&\quad=\quad\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\\
&\qquad\quad-\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)\le(1+a)u)dt>T_3(u)-T_2(u)-y,\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\\
&\quad=:\quad P_1(u)-P_2(u).
\end{aligned}$$

Next, recalling that $T^*(u)=\max(T_1(u),T_2(u),T_3(u))$ and using that $T^*(u)\sim Cu^{(2H-1)/H}$ as $u\to\infty$ for some $C>0$, we obtain

$$\begin{aligned}
P_2(u) &\le \mathbb{P}\left\{\inf_{t\in[T_2(u),T_3(u)]}Q(t)\le(1+a)u,\sup_{t\in[0,T_1(u)]}Q(t)>u\right\}\\
&\le \mathbb{P}\left\{\text{there exist } t,s\in[0,T^*(u)],Q(t)-Q(s)\ge -au\right\}\\
&\le \mathbb{P}\left\{\sup_{0\le t\le s\le T^*(u)}(B_H(t)-B_H(s)-c(t-s))>-au\right\}\\
&\le \mathbb{P}\left\{\sup_{0\le t\le s\le 1}{T^*}^H(u)(B_H(t)-B_H(s))>-au\right\}\\
&\le \mathrm{C}_1e^{-\mathrm{C}_2u^{4-4H}}
\end{aligned}$$

for some $C_1,C_2>0$, where the third inequality is because of (1) and the last inequality above is due to Borell-TIS inequality. Hence, in view of Proposition 4.2, $P_2(u)=o(P_1(u))$ as $u\to\infty$, which leads to

$$\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>y\Big|\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}\sim 1$$

as $u\to\infty$.

Case $a>0$. First, we consider the asymptotic upper bound. We note that

$$\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1+a)u,u)\le\frac{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>y\right\}}{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}}$$

and

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>y\right\}\\
&\quad=\quad\mathbb{P}\left\{\frac{1}{v((1+a)u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>(1+a)^{(1-2H)/H}y\right\},
\end{aligned}$$

where, by **(T)** we have

$$\lim_{u\to\infty}\frac{T_i(u)}{v((1+a)u)}=\mathscr{T}_i(1+a)^{(1-2H)/H},\quad i=1,2.$$

Consequently, by the stationarity of $Q(t), t \ge 0$ and Proposition 4.2, with $\tilde{a} = (1+a)^{(1-2H)/H}$ we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt > y\right\}\\
&= \mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_3(u)-T_2(u)]}\mathbb{I}(Q(t)>(1+a)u)dt > y\right\}\\
&\sim \mathcal{H}_{2H}\overline{\mathcal{B}}_H^{\tilde{a}y}((\mathscr{T}_3-\mathscr{T}_2)\tilde{a})\frac{\sqrt{2\pi}(AB)^{-1/2}}{(1+a)^{1-H}u^{1-H}\Delta((1+a)u)}\Psi(A((1+a)u)^{1-H})
\end{aligned}$$

as $u \to \infty$. Hence

$$\limsup_{u\to\infty}\frac{\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1+a)u,u)}{\exp\left(-\frac{A^2((1+a)^{2-2H}-1)}{2}u^{2-2H}\right)} \le \tilde{a}^{1-H}\frac{\overline{\mathcal{B}}_H^{\tilde{a}y}(\tilde{a}(\mathscr{T}_3-\mathscr{T}_2))}{\overline{\mathcal{B}}_H^{x}(\mathscr{T}_1)}.$$

For the proof of the asymptotic lower bound we have

$$\begin{aligned}
&\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1+a)u,u)\\
&\ge \frac{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[T_2(u),T_3(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>y, \frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>(1+a)u)dt>x\right\}}{\mathbb{P}\left\{\frac{1}{v(u)}\int_{[0,T_1(u)]}\mathbb{I}(Q(t)>u)dt>x\right\}}.
\end{aligned}$$

Then, following the same line of arguments as for the asymptotic upper bound, by Proposition 4.3 we obtain

$$\liminf_{u\to\infty}\frac{\mathscr{P}^{x,y}_{T_1,T_2,T_3}((1+a)u,u)}{\exp\left(-\frac{A^2((1+a)^{2-2H}-1)}{2}u^{2-2H}\right)} \ge \tilde{a}^{1-H}\frac{\overline{\mathcal{B}}_H^{\tilde{a}x,\tilde{a}y}(\tilde{a}\mathscr{T}_1;0,\tilde{a}\mathscr{T}_2,\tilde{a}\mathscr{T}_3)}{\overline{\mathcal{B}}_H^{x}(\mathscr{T}_1)}.$$

□

5. Appendix

In this Section we present a lemma that plays a crucial lemma for proof of Proposition 4.3. Consider next

$$\xi_{u,j}(s,t), \quad (s,t) \in E = [0,S]\times[0,T], \quad j \in S_u$$

a family of centered Gaussian random fields with continuous sample paths and unit variance, where S_u is a countable index set. For $S>0$, $0<b_1,b_2,b_3\le T$, $b_1>x\ge 0$ and $b_3-b_2>y\ge 0$, we are interested in the uniform asymptotics of

$$p_{u,j}(S;\lambda_{u,j}) = \mathbb{P}\left\{\int_{[0,b_1]}\mathbb{I}\left(\sup_{s\in[0,S]}\xi_{u,j}(s,t)>g_{u,j}\right)dt>x, \int_{[b_2,b_3]}\mathbb{I}\left(\sup_{s\in[0,S]}\xi_{u,j}(s,t)>g_{u,j}+\lambda_{u,j}/g_{u,j}\right)dt>y\right\}$$

with respect to $j \in S_u$, as $u\to\infty$, where $g_{u,j}$'s and $\lambda_{u,j}$'s are given constants depending on u and j. Suppose next that S_u's are finite index. The following assumptions will be imposed in the lemma below:

C1: $g_{u,j}, j \in S_u, u >$ are constants satisfying

$$\lim_{u\to\infty} \inf_{j\in S_u} g_{u,j} = \infty.$$

C2: There exists $\alpha \in (0,2]$ such that

$$\lim_{u\to\infty} \sup_{j\in S_u} \sup_{(s,t)\neq(s',t'),(s,t),(s',t')\in E} \left| g_{u,j}^2 \frac{1 - Corr(\xi_{u,j}(s,t), \xi_{u,j}(s',t'))}{|s-s'|^\alpha + |t-t'|^\alpha} - 1 \right| = 0.$$

C3: The sequence $\lambda_{u,j}$ is such that

$$\lim_{u\to\infty} \sup_{j\in S_u} |\lambda_{u,j} - \lambda| = 0$$

for some $\lambda \in \mathbb{R}$.

We state next a modification of [15, Lem 4.1].

Lemma 5.1. *Let $\{\xi_{u,j}(s,t), (s,t) \in E, j \in S_u\}$ be a family of centered Gaussian random fields defined as above. If* **C1-C3** *holds, then for all $S > 0$, $0 < b_1, b_2, b_3 \le b$, $b_1 > x \ge 0$ and $b_3 - b_2 > y \ge 0$ we have*

$$\lim_{u\to\infty} \sup_{j\in S_u} \left| \frac{p_{u,j}(S;\lambda_{u,j})}{\Psi(g_{u,j})} - \mathcal{B}^{x,y}_{\alpha/2}(b_1;\lambda,b_2,b_3)([0,S]) \right| = 0. \tag{34}$$

Proof of Lemma 5.1 The proof of Lemma 5.1 follows by similar argumentation as given in the proof of [15, Lemma 4.1]. For completeness, we present details of the main steps of the argumentation. Let

$$\chi_{u,j}(s,t) := g_{u,j}(\xi_{u,j}(s,t) - \rho_{u,j}(s,t)\xi_{u,j}(0,0)), \quad (s,t) \in E,$$

and

$$f_{u,j}(s,t,w) := w\rho_{u,j}(s,t) - g_{u,j}^2 \left(1 - \rho_{u,j}(s,t)\right), \quad (s,t) \in E, w \in \mathbb{R},$$

where $\rho_{u,j}(s,t) = Cov(\xi_{u,j}(s,t), \xi_{u,j}(0,0))$. Conditioning on $\xi_{u,j}(0,0)$ and using the fact that $\xi_{u,j}(0,0)$ and $\xi_{u,j}(s,t) - \rho_{u,j}(s,t)\xi_{u,j}(0,0)$ are mutually independent, we obtain

$$\begin{aligned}
&p_{u,j}(S;\lambda_{u,j}) \\
&= \frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi} g_{u,j}} \int_{\mathbb{R}} \exp\left(-w - \frac{w^2}{2g_{u,j}^2}\right) \mathbb{P}\left\{ \int_0^{b_1} \mathbb{I}\left(\sup_{s\in[0,S]} (g_{u,j}(\xi_{u,j}(s,t) - g_{u,j})) > 0 \right) dt > x, \right. \\
&\quad \left. \int_{b_2}^{b_3} \mathbb{I}\left(\sup_{s\in[0,S]} (g_{u,j}(\xi_{u,j}(s,t) - g_{u,j}) - \lambda_{u,j}) > 0 \right) dt > y \middle| \xi_{u,j}(0,0) = g_{u,j} + w g_{u,j}^{-1} \right\} dw \\
&= \frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi} g_{u,j}} \int_{\mathbb{R}} \exp\left(-w - \frac{w^2}{2g_{u,j}^2}\right) \mathbb{P}\left\{ \int_0^{b_1} \mathbb{I}\left(\sup_{s\in[0,S]} (\chi_{u,j}(s,t) + f_{u,j}(s,t,w)) > 0 \right) dt > x, \right. \\
&\quad \left. \int_{b_2}^{b_3} \mathbb{I}\left(\sup_{s\in[0,S]} (\chi_{u,j}(s,t) + f_{u,j}(s,t,w) - \lambda_{u,j}) > 0 \right) dt > y \right\} dw \\
&:= \frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi} g_{u,j}} \int_{\mathbb{R}} \exp\left(-w - \frac{w^2}{2g_{u,j}^2}\right) \mathcal{I}_{u,j}(w;x,y) dw,
\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{u,j}(w;x,y) &= \mathbb{P}\left\{\int_0^{b_1} \mathbb{I}\left(\sup_{s\in[0,S]} (\chi_{u,j}(s,t)+f_{u,j}(s,t,w))>0\right)dt>x,\right.\\ &\qquad \left.\int_{b_2}^{b_3} \mathbb{I}\left(\sup_{s\in[0,S]} (\chi_{u,j}(s,t)+f_{u,j}(s,t,w)-\lambda_{u,j})>0\right)dt>y\right\}.\end{aligned}$$

Noting that

$$\lim_{u\to\infty}\sup_{j\in S_u}\left|\frac{\frac{e^{-g_{u,j}^2/2}}{\sqrt{2\pi}g_{u,j}}}{\Psi(g_{u,j})}-1\right|=0$$

and for any $M>0$

$$\lim_{u\to\infty}\inf_{|w|\le M} e^{-\frac{w^2}{2g_{u,j}^2}}=1$$

we can establish the claim if we show that

$$\lim_{u\to\infty}\sup_{j\in S_u}\left|\int_{\mathbb{R}}\exp(-w)\,\mathcal{I}_{u,j}(w,x,y)dw-\mathcal{B}^{x,y}_{\alpha/2}(b_1;\lambda,b_2,b_3)([0,S])\right|=0. \tag{35}$$

Weak convergence. We next show the weak convergence of $\{\chi_{u,j}(s,t)+f_{u,j}(s,t,w),(s,t)\in E\}$ as $u\to\infty$. By **C1** and **C2** we have, for $(s,t),(s',t')\in E$, as $u\to\infty$, uniformly with respect to $j\in S_u$

$$\begin{aligned}Var(\chi_{u,j}(s,t)-\chi_{u,j}(s',t')) &= g_{u,j}^2\left(\mathbb{E}\left\{\xi_{u,j}(s,t)-\xi_{u,j}(s',t')\right\}^2-\left(\rho_{\xi_{u,j}}(s,t)-\rho_{\xi_{u,j}}(s',t')\right)^2\right)\\ &\to 2Var(\zeta(s,t)-\zeta(s',t')),\end{aligned}$$

where $\zeta(s,t)=B_{\alpha/2}(s)+B'_{\alpha/2}(t),\ (s,t)\in E$ with B and B' being independent fBm's. This implies that the finite-dimensional distributions of $\{\chi_{u,j}(s,t),(s,t)\in E\}$ weakly converge to that of $\{\sqrt{2}\zeta(s,t),(s,t)\in E\}$ as $u\to\infty$ uniformly with respect to $j\in S_u$. Moreover, it follows from **C2** that, for u sufficiently large

$$Var(\chi_{u,j}(s,t)-\chi_{u,j}(s',t'))\le g_{u,j}^2\mathbb{E}\left\{\xi_{u,j}(s,t)-\xi_{u,j}(s',t')\right\}^2\le 4(|s-s_1|^\alpha+|t-t_1|^\alpha),\ (s,t),(s_1,t_1)\in E.$$

This implies that uniform tightness of $\{\chi_{u,j}(s,t),(s,t)\in E\}$ for large u with respect to $j\in S_u$. Hence $\{\chi_{u,j}(s,t),(s,t)\in E\}$ weakly converges to $\{\sqrt{2}\zeta(s,t),(s,t)\in E\}$ as $u\to\infty$ uniformly with respect to $j\in S_u$. Additionally, by **C1-C2**, $\{f_{u,j}(s,t,w),(s,t)\in E\}$ converges to $\{w-|s|^\alpha-|t|^\alpha,(s,t)\in E\}$ uniformly with respect to $j\in S_u$. Therefore, we conclude that as $u\to\infty$, $\{\chi_{u,j}(s,t)+f_{u,j}(s,t,w),(s,t)\in E\}$ weakly converges to $\{\sqrt{2}\zeta(s,t)+w-|s|^\alpha-|t|^\alpha,(s,t)\in E\}$ uniformly with respect to $j\in S_u$. Then continuous mapping theorem implies that

$$\{z_{u,j}(t,w)=\sup_{s\in[0,S]}\left(\chi_{u,j}(s,t)+f_{u,j}(s,t,w)\right),t\in[0,b]\}$$

weakly converges to

$$\{z(t)+w=\sup_{s\in[0,S]}\left(\sqrt{2}\zeta(s,t)+w-|s|^\alpha-|t|^\alpha\right),t\in[0,b]\}$$

uniformly with respect to $j \in S_u$ for each $w \in \mathbb{R}$.

Repeating the arguments, in view of **C3** the same convergence holds for $\chi_{u,j}(s,t) + f_{u,j}(s,t,w) + \lambda_{u,j}$.

In order to show the weak convergence of

$$\left(\int_0^{b_1} \mathbb{I}(z_{u,j}(t,w) > 0)dt, \int_{b_2}^{b_3} \mathbb{I}(z_{u,j}(t,w) - \lambda_{u,j} > 0)dt\right), \quad u \to \infty$$

we have to prove that

$$\left(\int_0^{b_1} \mathbb{I}(f(t) > 0)dt, \int_{b_2}^{b_3} \mathbb{I}(f(t) > \lambda)dt\right)$$

is a continuous functional from $C([0,b_1] \cup [b_2,b_3])$ to $\mathbb{R}^2$ except a zero probability subset of $C([0,b_1] \cup [b_2,b_3])$ under the probability induced by $\{z(t)+w, t \in [0,b_1] \cup [b_2,b_3]\}$. The idea of the proof follows from Lemma 4.2 of [22]. Observe that the discontinuity set is

$$E^* = \left\{ f \in C([0,b_1] \cup [b_2,b_3]) : \int_{[0,b_1]} \mathbb{I}(f(t) = 0)dt > 0 \text{ or } \int_{[b_1,b_2]} \mathbb{I}(f(t) = \lambda)dt > 0 \right\}.$$

Note that for any $c \in \mathbb{R}$

$$\int_{\mathbb{R}} \mathbb{E}\left(\int_{[0,b_1]\cup[b_2,b_3]} \mathbb{I}(z(t)+w=c)dt\right) dw = \int_{[0,b_1]\cup[b_2,b_3]} \int_{\mathbb{R}} \mathbb{P}\left\{z(t)+w=c\right\} dwdt = 0.$$

Hence E^* has probability zero under the probability induced by $\{z(t)+w, t \in [0,b_1] \cup [b_2,b_3]\}$ for a.e. $w \in \mathbb{R}$. Application of the continuous mapping theorem yields that

$$\left(\int_0^{b_1} \mathbb{I}(z_{u,j}(t,w) > 0)dt, \int_{b_2}^{b_3} \mathbb{I}(z_{u,j}(t,w) > \lambda)dt\right)$$

weakly converges to

$$\left(\int_0^{b_1} \mathbb{I}(z(t)+w > 0)dt, \int_{b_2}^{b_3} \mathbb{I}(z(t)+w > \lambda)dt\right)$$

as $u \to \infty$, uniformly with respect to $j \in S_u$ for a.e. $w \in \mathbb{R}$.

Convergence on continuous points. Let

$$\mathcal{I}(w; x, y) := \mathbb{P}\left\{\int_0^{b_1} \mathbb{I}(z(t)+w > 0)dt > x, \int_{b_2}^{b_3} \mathbb{I}(z(t)+w > \lambda)dt > y\right\}.$$

Using similar arguments as in the proof of [23, Thm 1.3.1], we show that (35) holds for continuity points (x,y) with $x, y > 0$, i.e.,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(\mathcal{I}(w; x+\varepsilon, y+\varepsilon) - \mathcal{I}(w; x-\varepsilon, y-\varepsilon)\right) e^{-w} dw = 0.$$

Note that for all $x, y > 0$

$$\begin{aligned}
\mathcal{I}(w; x, y) &\le \mathbb{P}\left\{\sup_{(s,t)\in E} \sqrt{2}\zeta(s,t) - |s|^\alpha - |t|^\alpha > -w\right\} \\
&\le \mathbb{P}\left\{\sup_{(s,t)\in E} \sqrt{2}\zeta(s,t) > -w + C\right\} \\
&\le C_1 e^{-Cw^2}, \ w < -M \qquad (36)
\end{aligned}$$

for M sufficiently large, where C, C_1 are positive constants and in the last inequality, we used the Piterbarg inequality [20, Thm 8.1]. Hence the dominated convergence theorem gives

$$\int_{\mathbb{R}} \left(\mathcal{I}(w; x+, y+) - \mathcal{I}(w; x-, y-)\right) e^{-w} dw = 0.$$

This implies that if (x, y) is a continuity point, then $\mathcal{I}(w;)$ is continuous at (x, y) for a.e. $w \in \mathbb{R}$. Hence if (x, y) is a continuity point, then

$$\lim_{u\to\infty} \sup_{j\in S_u} |\mathcal{I}_{u,j}(w; x, y) - \mathcal{I}(w; x, y)| = 0, \text{ for a.e. } w \in \mathbb{R}. \tag{37}$$

Applying again the Piterbarg inequality, analogously as in (36), we obtain

$$\sup_{j\in S_u} \mathcal{I}_{u,j}(w; x, y) \le C_1 e^{-Cw^2}, \ w < -M \tag{38}$$

for M and u sufficiently large. Consequently, in view of (36), (37) and (38), the dominated convergence theorem establishes (35).

Continuity of $\mathcal{B}^{x,y}_{\alpha/2}(b_1; \lambda, b_2, b_3)([0, S])$. Clearly, $\mathcal{B}^{x,y}_{\alpha/2}(b_1; \lambda, b_2, b_3)([0, S])$ is right-continuous at $(x, y) = (0, 0)$. We next focus on its continuity over $([0, b_1) \times [0, b_3 - b_2))\backslash\{(0, 0)\}$. To show $\mathcal{B}^{x,y}_{\alpha/2}(b_1; \lambda, b_2, b_3)([0, S])$ is continuous at $(x, y) \in (0, b_1) \times (0, b_3 - b_2)$, it suffices to prove that

$$\int_{\mathbb{R}} e^{-w} \left(\mathbb{P}\left\{\int_0^{b_1} \mathbb{I}(z(t) + w > 0)dt = x\right\} + \mathbb{P}\left\{\int_{b_2}^{b_3} \mathbb{I}(z(t) + w > \lambda)dt = y\right\}\right) dw = 0.$$

Denote $A_w = \{z_\kappa(t) : \int_0^{b_1} \mathbb{I}(z_\kappa(t) + w > 0)dt = x\}$, where $z_\kappa(t) = z(t)(\kappa)$ with $\kappa \in \Omega$ the sample space. In light of the continuity of $z_\kappa(t)$, if $\int_0^{b_1} \mathbb{I}(z_\kappa(t) + w > 0)dt = x$ for $x \in (0, b_1)$ and $w' > w$, then

$$\int_0^{b_1} \mathbb{I}(z_\kappa(t) + w' > 0)dt > x.$$

Hence $A_w \cap A_{w'} = \emptyset$ if $w \neq w'$. Noting that the continuity of $z(s)$ guarantees the measurability of A_w, and

$$\sup_{\Lambda\subset\mathbb{R}, \#\Lambda<\infty} \sum_{w\in\Lambda} \mathbb{P}\{A_w\} \le 1,$$

where $\#\Lambda$ stands for the cardinality of the set Λ.

Note in passing the important fact that $\mathbb{P}$-measurability of A_w is a consequence of the Fubini-Tonelli theorem. Next, it follows that

$$\{w : w \in \mathbb{R} \quad \text{such that} \quad \mathbb{P}\{A_w\} > 0\}$$

is a countable set, which implies that for $x \in (0, b_1)$

$$\int_{\mathbb{R}} \mathbb{P}\{A_w\} e^{-w} dw = 0.$$

Using similar argument, we can show

$$\int_{\mathbb{R}} e^{-w} \mathbb{P}\left\{\int_{b_2}^{b_3} \mathbb{I}(z(t) + w > \lambda)dt = y\right\} dw = 0.$$

Therefore, we conclude that $\mathcal{B}^{x,y}_{\alpha/2}(b_1; \lambda, b_2, b_3)([0, S])$ is continuous at $(x, y) \in (0, b_1) \times (0, b_3 - b_2)$. Analogously, we can show the continuity on $\{0\} \times (0, b_3 - b_2)$ and $(0, b_1) \times \{0\}$. This completes the proof. □

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