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Finite groups defined by presentations in which each defining relator involves exactly two generators

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ABSTRACT

We consider two classes of groups, denoted J_Γ and M_Γ, defined by presentations in which each defining relator involves exactly two generators, and so are examples of simple Pride groups. (For M_Γ the relators are Baumslag-Solitar relators.) These presentations are, in turn, defined in terms of a non-trivial, simple directed graph Γ whose arcs are labelled by integers. When Γ is a directed triangle the groups J_Γ, M_Γ coincide with groups considered by Johnson and by Mennicke, respectively. When the arc labels are all equal the groups form families of so-called digraph groups. We show that if Γ is a non-trivial, strongly connected tournament then the groups J_Γ are finite, metabelian, of rank equal to the order of Γ and we show that the groups M_Γ are finite and, subject to a condition on the arc labels, are of rank equal to the order of Γ.

1. Introduction

We consider two classes of groups J_Γ, M_Γ defined as follows. Let Γ be a non-trivial, simple directed graph (i.e. a directed graph with more than one vertex and without loops or multiple arcs), with vertex set $V(\Gamma)$ and arc set $A(\Gamma)$, where each arc $[u,v] \in A(\Gamma)$ (from a vertex u to a vertex v) is labelled by an even integer $r_{[u,v]}$ (for J_Γ) or an integer $r_{[u,v]} \geq 2$ (for M_Γ), and define

$$J_\Gamma = \langle x_v \ (v \in V(\Gamma)) \mid x_v^{-1} x_u x_v = x_v^{r_{[u,v]}-2} x_u^{-1} x_v^{r_{[u,v]}+2} \ ([u,v] \in A(\Gamma)) \rangle,$$
$$M_\Gamma = \langle x_v \ (v \in V(\Gamma)) \mid x_v^{-1} x_u x_v = x_u^{r_{[u,v]}} \ ([u,v] \in A(\Gamma)) \rangle.$$

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Our interest in these groups is threefold: as simple Pride groups [24]; as generalizations of groups introduced by Johnson [13] and by Mennicke [19]; and, when the arc labels are equal, as so-called digraph groups [7].

A *simple Pride group* is defined to be a group given by a presentation in which each defining relation involves exactly two generators [24,18], and so J_Γ, M_Γ are examples of such groups. (Note that 'simple' in this context does not mean that the group has no nontrivial proper normal subgroups; the terminology arises because these groups form a special class of *Pride groups* that can be defined in terms of simple graphs [18].) A *directed triangle* $[u,v,w]$ is the directed graph with $V(\Gamma) = \{u,v,w\}$ and $A(\Gamma) = \{[u,v],[v,w],[w,u]\}$. If Γ is a directed triangle then J_Γ is the *Johnson group* introduced in [13] (following Wamsley [26] who considered the case where the arc labels are all equal to 2) and M_Γ is the *Mennicke group* introduced in [19].

If Λ is a directed graph with vertex set $V(\Lambda)$ and arc set $A(\Lambda)$, and $R(x,y)$ is an element of the free group with basis $\{x,y\}$ then the group

$$G_\Lambda(R) = \langle x_v \ (v \in V(\Lambda)) \mid R(x_u, x_v) \ ([u,v] \in A(\Lambda))\rangle \tag{1}$$

is called a *digraph group* [7]. If Λ is a directed cycle then the group $G_\Lambda(R)$ is an example of a *cyclically presented group* and the presentation (1) is a *cyclic presentation* [14, page 95]. Thus if Γ is a simple digraph in which each arc label $r_{[u,v]}$ is equal to a fixed integer r then J_Γ is the digraph group $G_\Gamma(xy^{-r-1}xy^{-r+1})$ and M_Γ is the digraph group $G_\Gamma(xyx^{-r}y^{-1})$. Noting that the relators of M_Γ are all Baumslag-Solitar relators [4], we observe the following: if Γ is a digraph consisting of an arc $[u,v]$ then M_Γ is the (solvable) Baumslag-Solitar group $BS(1, r_{[u,v]})$; if Γ is a directed triangle $[u,v,w]$ then M_Γ is the triangle of Baumslag-Solitar groups, denoted $G(1, r_{[u,v]}; 1, r_{[v,w]}; 1, r_{[w,u]})$ in [3]; and if Γ is a directed 4-cycle in which each arc is labelled 2 then M_Γ is the Higman group [11].

As we discuss below, in most cases the Johnson and Mennicke groups provide families of groups that are finite and of rank 3, a property that is rare amongst deficiency zero groups. Examples of simple Pride groups and digraph groups that are finite and of rank at least 3 are similarly rare, and the Johnson groups and Mennicke groups (of rank 3) are principal examples of such groups. The results of [23,8,9,3] provide further examples of finite simple Pride groups of rank 3 (and deficiency zero), but we are not aware of any other results concerning finite digraph groups of rank 3. In this article we generalize results concerning finiteness of the Johnson and Mennicke groups to give families of simple Pride groups and of digraph groups that are finite and of each rank at least 3.

Recall that the *order* of a digraph is the number of vertices it has, a *tournament* is a simple directed graph in which each pair of vertices is connected by exactly one arc, and a digraph is *strongly connected* if for each pair of vertices $u,v \in V(\Gamma)$ there is a directed path from u to v; in particular, the trivial digraph and the directed triangle are strongly connected tournaments and if a non-trivial tournament is strongly connected then it has order at least 3. Almost all tournaments are strongly connected and there is a strongly connected tournament of each order greater than 2 [20], [21, Chapters 2 and 3], [28]. Every vertex of a non-trivial, strongly connected tournament is a vertex of some directed triangle [21, Theorem 3]. A group is *metabelian* if its derived subgroup is abelian, and the *rank* of a group G is the minimum cardinality of a generating set for G.

To state our theorem concerning the groups J_Γ we introduce the following notation: given vertices u,v,w of a digraph Γ that form a directed triangle $[u,v,w]$ whose arcs $[u,v],[v,w],[w,u]$ are labelled by even integers $r_{[u,v]}, r_{[v,w]}, r_{[w,u]}$ define

$$\Theta(u,v,w) = |8r_{[w,u]}(r_{[u,v]}-1)(r_{[v,w]}-1)(r_{[w,u]}-1)(r_{[u,v]}r_{[v,w]}r_{[w,u]}-1)|$$

(compare [14, Equation (10), page 94] or the corresponding expression concerning the order of x in [13, page 60]); note that $\Theta(u,v,w) \neq \Theta(v,w,u)$ in general. For each vertex $u \in V(\Gamma)$ set

$$\theta(u) = \gcd\{\Theta(u, v, w) \mid [u, v, w] \text{ is a directed triangle of } \Gamma\}.$$

Note that if Γ is a strongly connected tournament then since each vertex is a vertex of some directed triangle, and each $r_{[u,v]}$ is even, we have $0 < \theta(u) < \infty$ for each $u \in V(\Gamma)$. It is straightforward to show that if J_Γ is finite then Γ is a non-trivial tournament (see Lemma 2.2). As a partial converse, we show that if Γ is a strongly connected tournament then J_Γ is finite.

Theorem A. *Let Γ be a non-trivial, strongly connected tournament where each arc $[u, v]$ is labelled by an even integer $r_{[u,v]}$. Then J_Γ is a finite, metabelian, group, with $\mathrm{rank}(J_\Gamma) = \mathrm{rank}(J_\Gamma^{\mathrm{ab}}) = |V(\Gamma)|$, whose order divides $\prod_{u\in V(\Gamma)} \theta(u)$. More precisely, if H_Γ is the subgroup of J_Γ generated by $\{x_v^2 \mid v \in V(\Gamma)\}$, then H_Γ is an abelian, normal, subgroup of J_Γ, whose order divides $\prod_{u\in V(\Gamma)}(\theta(u)/2)$, with J_Γ/H_Γ elementary abelian of order $2^{|V(\Gamma)|}$.*

The ‘strongly connected’ hypothesis cannot be directly removed from Theorem A since, without it, there are examples where J_Γ is infinite, as well as examples where J_Γ is finite. For instance, a computation in GAP [10] shows that if Λ is the tournament with arc set $A(\Lambda) = \{[2, 1], [3, 1], [4, 1], [3, 2], [4, 2], [3, 4]\}$, where each arc is labelled 2 then the second derived quotient of $J = J_\Lambda$ is the group $J'/J'' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^2$, and so J is infinite; whereas, if the arc $[3, 2]$ is replaced by the arc $[2, 3]$ then J is finite, metabelian, of order $2^{14} \cdot 7$.

To state our theorem concerning the groups M_Γ we introduce the following notation: given vertices u, v, w of a digraph Γ that form a directed triangle $[u, v, w]$ whose arcs $[u, v], [v, w], [w, u]$ are labelled by integers $r_{[u,v]}, r_{[v,w]}, r_{[w,u]}$ define

$$\Phi(u, v, w) = (r_{[u,v]} - 1)^2 (r_{[u,v]}^{r_{[v,w]}-1} - 1)$$

(compare the expression $y^{(b-1)^2(b^{c-1}-1)} = 1$ from [15, page 279]); note that $\Phi(u, v, w) \neq \Phi(v, w, u)$ in general. For each vertex $u \in V(\Gamma)$ set

$$\phi(u) = \gcd\{\Phi(u, v, w) \mid [u, v, w] \text{ is a directed triangle of } \Gamma\}. \tag{2}$$

Note that if Γ is a strongly connected tournament then since (as observed above) each vertex is a vertex of some directed triangle, if each $r_{[u,v]} \geq 2$ then $0 < \phi(u) < \infty$ for each $u \in V(\Gamma)$. It is straightforward to show that if $\gcd\{r_{[u,v]} - 1 \mid [u, v] \in A(\Gamma)\} > 1$ and M_Γ is finite then Γ is a non-trivial tournament (see Lemma 3.1). As a partial converse, Theorem B considers the case when Γ is a strongly connected tournament.

Theorem B. *Let Γ be a non-trivial, strongly connected tournament where each arc $[u, v]$ is labelled by an integer $r_{[u,v]} \geq 2$. Then M_Γ is a finite group of order at most*

$$2^{|V(\Gamma)|} \cdot \prod_{u\in V(\Gamma)} \phi(u).$$

Moreover, if $\gcd\{r_{[u,v]} - 1 \mid [u, v] \in A(\Gamma)\} > 1$ then $\mathrm{rank}(M_\Gamma) = \mathrm{rank}(M_\Gamma^{\mathrm{ab}}) = |V(\Gamma)|$.

The ‘gcd’ hypothesis cannot be directly removed from Theorem B. For example, if Γ is the directed triangle with vertex set u, v, w and $r_{[u,v]} = 3, r_{[v,w]} = 2, r_{[w,u]} = 2$, then $M_\Gamma \cong S_3$ (the symmetric group) so $1 = \mathrm{rank}(M_\Gamma^{\mathrm{ab}}) < \mathrm{rank}(M_\Gamma) < |V(\Gamma)| = 3$ (see also [1, Section (e)], which obtains this group by setting $r_{[u,v]} = 2, r_{[v,w]} = 2, r_{[w,u]} = -1$). Moreover, the ‘strongly connected’ hypothesis cannot be directly removed from Theorem B since, without it, there are examples where M_Γ is infinite, as well as examples where M_Γ is finite. For instance, a computation in GAP shows that if Λ is the tournament with arc set

$A(\Lambda) = \{[1,2],[1,3],[1,4],[2,4],[3,2],[4,3]\}$ where each arc is labelled 3, then M_Γ is finite of order 2^{14}, whereas if Γ has a sink then M_Γ is infinite by Lemma 3.1.

As immediate corollaries to Theorems A, B we have:

Corollary A1 *([13, Lemma], [14, Proposition 7.3]). Let Γ be a directed triangle $[u,v,w]$ where the arcs are labelled by even integers a,b,c. Then J_Γ is a finite, metabelian, group with $\mathrm{rank}(J_\Gamma) = \mathrm{rank}(J_\Gamma^{\mathrm{ab}}) = 3$, whose order divides $|512abc(a-1)^3(b-1)^3(c-1)^3(abc-1)^3|$. More precisely, if H_Γ is the subgroup of J_Γ generated by $\{x_u^2, x_v^2, x_w^2\}$, then H_Γ is an abelian, normal, subgroup of J_Γ, of order at most $|64abc(a-1)^3(b-1)^3(c-1)^3(abc-1)^3|$, with J_Γ/H_Γ elementary abelian of order 8.*

It was further shown in [13] (see also [14, Exercise 9.15]) that (in the setting of Corollary A1) J_Γ is nilpotent and of order $256|abc(abc-1)|$.

Corollary B1 *([19,25,15,2,12]). Let Γ be a directed triangle where the arcs are labelled by integers $a,b,c \geq 2$. Then M_Γ is a finite group; moreover, if $\gcd\{a-1,b-1,c-1\} > 1$ then $\mathrm{rank}(M_\Gamma) = \mathrm{rank}(M_\Gamma^{\mathrm{ab}}) = 3$.*

In more detail (where Γ is a directed triangle), Mennicke [19] showed that if $a = b = c \geq 2$ then M_Γ is finite; Wamsley and MacDonald [27, Theorem 8.1] showed that if $|a| \neq 1, |b| \neq 1, |c| \neq 1$ then M_Γ is finite and solvable; Schenkman [25, Theorem 1] showed that if $a,b,c \geq 2$ then M_Γ is finite; Johnson and Robertson [15, Section 3] showed that if $a,b,c \geq 2$ then M_Γ is finite and solvable of derived length at most 3, giving an upper bound for the order (which is the bound given by Theorem B), and they observe that if $\gcd\{a-1,b-1,c-1\} > 1$ then M_Γ has rank 3. Albar and Shuaibi [2] and Jabara [12] gave improved bounds and further information about the structure of M_Γ. It is reasonable to expect that, for arbitrary tournaments, more information about the groups M_Γ can be obtained using methods from these references (in particular, see Remark 3.4). However, since our goal is to provide classes of simple Pride groups and digraph groups that are finite and of arbitrary rank, with the order and structure of the groups being of secondary importance, we have not sought to do this.

As mentioned above, finite digraph groups of rank 3 considered in the literature appear to be limited to the groups in Corollaries A1, B1. Examples of finite digraph groups of rank 2 are provided by finite, non-cyclic groups defined by 2-generator cyclic presentations; a survey of such groups can be found in [15, Section 5] and they include the binary polyhedral groups $< 2,2,2 >$ [6, Section 1.7], MacDonald's groups $Mac(a,a)$ [17,22], and the Fibonacci groups $F(2s+1,2)$ $(s \geq 1)$ [16, Theorem 1(iii)]. Finite digraph groups of rank 1 (i.e. finite cyclic groups) can be found in [24, Theorem 3] (see also [5, Lemma 3.4]), [7, Theorem A]. In the immediate Corollaries A2, B2, below, we specialise Theorems A, B to the digraph group situation and show that there exist finite digraph groups of arbitrary rank at least 3 (noting that there is a strongly connected tournament of each order at least 3). We believe these to be the first examples of finite digraph groups of rank greater than 3.

Corollary A2. *Let Λ be a non-trivial, strongly connected tournament and let r be an even integer. Then the digraph group $G = G_\Lambda(xy^{-r-1}xy^{-r+1})$ is a finite, metabelian, group with $\mathrm{rank}(G) = \mathrm{rank}(G^{\mathrm{ab}}) = |V(\Gamma)|$ whose order divides $\left(8r(r-1)^3(r^3-1)\right)^{|V(\Lambda)|}$. More precisely, if H is the subgroup of G generated by $\{x_v^2 \mid v \in V(\Lambda)\}$, then H is an abelian, normal, subgroup of G, whose order divides $\left(4r(r-1)^3(r^3-1)\right)^{|V(\Lambda)|}$, with G/H elementary abelian of order $2^{|V(\Lambda)|}$.*

Corollary B2. *Let $r \geq 3$ and let Λ be a non-trivial, strongly connected tournament. Then the digraph group $G = G_\Lambda(xyx^{-r}y^{-1})$ is a finite group with $\mathrm{rank}(G) = \mathrm{rank}(G^{\mathrm{ab}}) = |V(\Gamma)|$ and of order at most*

$$2^{|V(\Lambda)|} \cdot \left((r-1)^2(r^{r-1}-1)\right)^{|V(\Lambda)|}.$$

If $r = 1$ in Corollary B2 then $G_\Lambda(xyx^{-r}y^{-1}) \cong \mathbb{Z}^{|V(\Lambda)|}$ and if $r = 2$ then the argument of [11, Section 3] (due to Hirsch), applied to each directed triangle of $G = G_\Lambda(xyx^{-r}y^{-1})$, shows that each generator is trivial, and hence G is trivial.

2. The groups J_Γ

We first record the following result.

Lemma 2.1. *For any non-trivial digraph* Γ, $\mathrm{rank}(J_\Gamma) = \mathrm{rank}(J_\Gamma^{\mathrm{ab}}) = |V(\Gamma)|$.

Proof. The group J_Γ maps onto its abelianisation J_Γ^{ab} which, by adjoining relators x_v^2 for each $v \in V(\Gamma)$, maps onto $\mathbb{Z}_2^{|V(\Gamma)|}$ of rank $|V(\Gamma)|$. Hence $\mathrm{rank}(J_\Gamma) \geq \mathrm{rank}(J_\Gamma^{\mathrm{ab}}) \geq |V(\Gamma)|$, and from the definition of J_Γ we have $\mathrm{rank}(J_\Gamma) \leq |V(\Gamma)|$, so the result follows. □

Now observe:

Lemma 2.2. *Let* Γ *be a non-trivial, simple digraph where each arc* $[u, v] \in A(\Gamma)$ *is labelled by an integer* $r_{[u,v]}$. *If* J_Γ *is finite then* Γ *is a non-trivial tournament.*

Proof. Suppose that Γ is not a tournament. Then Γ has at least two vertices and there is a pair of vertices $w_1, w_2 \in V(\Gamma)$ that are not joined by an arc. Adjoining relators x_u to the defining presentation of J_Γ for all $u \neq w_1, w_2$ and adjoining the relators $x_{w_1}^2, x_{w_2}^2$ shows that J_Γ has the infinite quotient $\langle x_{w_1}, x_{w_2} \mid x_{w_1}^2, x_{w_2}^2 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$. □

Lemma 2.3. *Let* Γ *be a non-trivial tournament where each arc* $[u, v]$ *is labelled by an even integer* $r_{[u,v]}$, *let* H_Γ *be the subgroup of* J_Γ *generated by* $\{x_v^2 \mid v \in V(\Gamma)\}$. *Then* H_Γ *is an abelian, normal, subgroup of* J_Γ *and* J_Γ/H_Γ *is elementary abelian of order* $2^{|V(\Gamma)|}$.

Proof. Let $[u, v] \in A(\Gamma)$. As shown in [13, page 59] (or [14, pages 93–94]), it follows from the defining relation that involves x_u, x_v (and no other defining relation) that $x_u x_v^2 = x_v^2 x_u$ (so $x_u^2 x_v^2 = x_v^2 x_u^2$ and $x_u^{-1} x_v^2 x_u = x_v^2 \in H_\Gamma$) and that $x_v^{-1} x_u^2 x_v = x_u^{-2} x_v^{4r_{[u,v]}} \in H_\Gamma$. Since Γ is a tournament, for each pair of vertices $u, v \in V(\Gamma)$ either $[u, v]$ or $[v, u] \in A(\Gamma)$ so H_Γ is abelian and $x_u^{-1} x_v^2 x_u \in H_\Gamma$ and $x_v^{-1} x_u^2 x_v \in H_\Gamma$, so H_Γ is a normal subgroup of J_Γ. The quotient J_Γ/H_Γ is obtained by adjoining the relators x_u^2 ($u \in V(\Gamma)$) to the defining presentation for J_Γ. Therefore, since each integer $r_{[u,v]}$ is even, we obtain

$$\begin{aligned} J_\Gamma/H_\Gamma &= \langle x_v \ (v \in V(\Gamma)) \mid x_v^2 \ (v \in V(\Gamma)),\ x_v^{-1} x_u x_v = x_u^{-1} \ ([u, v] \in A(\Gamma)) \rangle \\ &= \langle x_v \ (v \in V(\Gamma)) \mid x_v^2 \ (v \in V(\Gamma)) \rangle^{\mathrm{ab}} \\ &\cong \mathbb{Z}_2^{|V(\Gamma)|}. \quad \square \end{aligned}$$

We also need:

Lemma 2.4 *([13, pages 59–60], [14, pages 93–94]). Let* Γ *be a non-trivial, simple digraph where each arc* $[u, v]$ *is labelled by an even integer* $r_{[u,v]}$. *If* u *is a vertex of a directed triangle* $[u, v, w]$ *then* $x_u^{\Theta(u,v,w)} = 1$ *in* J_Γ.

We are now in a position to prove Theorem A; our proof is a generalization of the argument in [13].

Proof of Theorem A. Observe first that the statement concerning ranks follows from Lemma 2.1. By Lemma 2.3 we have $|J_\Gamma| = 2^{|V(\Gamma)|} \cdot |H_\Gamma|$. Moreover, by Lemma 2.3, H_Γ is an abelian, normal, subgroup of J_Γ, generated by $\{x_u^2 \mid u \in V(\Gamma)\}$. Therefore there is an epimorphism $\oplus_{u\in V(\Gamma)}\mathbb{Z}_{|x_u^2|} \to H_\Gamma$ (where $|x_u^2|$ denotes the order of x_u^2), and so the order of H_Γ divides $\prod_{u\in V(\Gamma)}|x_u^2|$. By Lemma 2.4 if $u \in V(\Gamma)$ then $x_u^{\Theta(u,v,w)} = 1$ in J_Γ for each directed triangle $[u,v,w]$ in Γ. Therefore $x_u^{\theta(u)} = 1$, so the order of x_u divides $\theta(u)$, and hence the order of x_u^2 divides $\theta(u)/2$ so $|H_\Gamma|$ divides $\prod_{u\in V(\Gamma)}\theta(u)/2$. Thus $|J_\Gamma| = 2^{|V(\Gamma)|}\cdot|H_\Gamma|$ divides $2^{|V(\Gamma)|}\cdot\prod_{u\in V(\Gamma)}\theta(u)/2 = \prod_{u\in V(\Gamma)}\theta(u)$, as required. □

3. The groups M_Γ

First observe:

Lemma 3.1. *Let Γ be a non-trivial, simple digraph where each arc $[u,v] \in A(\Gamma)$ is labelled by an integer $r_{[u,v]} \geq 1$ and suppose $\gcd\{r_{[u,v]} - 1 \mid [u,v] \in A(\Gamma)\} > 1$. If M_Γ is finite then Γ is a tournament without sinks.*

Proof. Suppose that Γ is not a tournament and let $d = \gcd\{r_{[u,v]} - 1 \mid [u,v] \in A(\Gamma)\} > 1$. Then Γ has a pair of distinct vertices $w_1, w_2 \in V(\Gamma)$ that are not joined by an arc. Adjoining the relators $x_{w_1}^d, x_{w_2}^d$ and the relators x_u for all $u \neq w_1, w_2$ to the defining presentation of M_Γ shows that M_Γ has the infinite quotient $\langle x_{w_1}, x_{w_2} \mid x_{w_1}^d, x_{w_2}^d\rangle \cong \mathbb{Z}_d * \mathbb{Z}_d$, so M_Γ is infinite. Suppose then that Γ is a tournament with a sink, t, say. Adjoining relators x_u for all $u \in V(\Gamma)$ where $u \neq t$ shows that M_Γ maps onto $\langle x_t \mid \ \rangle \cong \mathbb{Z}$, so M_Γ is infinite. □

Lemma 3.2. *The abelianisation M_Γ^{ab} is isomorphic to*

$$\oplus_{u^*\in V(\Gamma)}\mathbb{Z}_{\gcd\{r_{[u^*,v]-1} \mid [u^*,v]\in A(\Gamma)\}}.$$

Hence if $\gcd\{r_{[u,v]} - 1 \mid [u,v] \in A(\Gamma)\} > 1$ *then* $\mathrm{rank}(M_\Gamma) = \mathrm{rank}(M_\Gamma^{\mathrm{ab}}) = |V(\Gamma)|$.

Proof. The abelianisation

$$\begin{aligned} M_\Gamma^{\mathrm{ab}} &= \langle x_{u^*}\ (u^* \in V(\Gamma)) \mid x_{u^*}^{r_{[u^*,v]-1}}\ ([u^*,v] \in A(\Gamma))\rangle^{\mathrm{ab}} \\ &= \langle x_{u^*}\ (u^* \in V(\Gamma)) \mid x_{u^*}^{\gcd\{r_{[u^*,v]-1} \mid [u^*,v]\in A(\Gamma)\}}(u^* \in V(\Gamma))\rangle^{\mathrm{ab}} \\ &\cong \oplus_{u^*\in V(\Gamma)}\mathbb{Z}_{\gcd\{r_{[u^*,v]-1} \mid [u^*,v]\in A(\Gamma)\}}. \end{aligned}$$

Suppose that $d = \gcd\{r_{[u,v]} - 1 \mid [u,v] \in A(\Gamma)\} > 1$. Then since d divides $\gcd\{r_{[u^*,v]-1} \mid [u^*,v] \in A(\Gamma)\}$ for all $u^* \in V(\Gamma)$ the abelianisation M_Γ^{ab} maps onto $\oplus_{u\in V(\Gamma)}\mathbb{Z}_d = \mathbb{Z}_d^{|V(\Gamma)|}$ of rank $|V(\Gamma)|$ and, since (by the definition of M_Γ) $\mathrm{rank}(M_\Gamma) \leq |V(\Gamma)|$, the result follows. □

Note that Lemma 3.2 implies that if Γ has a sink then M_Γ is infinite.

Lemma 3.3 *([15, pages 278–279]). Let Γ be a non-trivial digraph where each arc $[u,v]$ is labelled by an integer $r_{[u,v]} \geq 2$. If u is a vertex of a directed triangle $[u,v,w]$ then $x_u^{\Phi(u,v,w)} = 1$ in M_Γ.*

Remark 3.4. An improved version of Lemma 3.3 was obtained in [2] (see also [12, Lemma 2]). Given vertices u, v, w of a digraph Γ that form a directed triangle $[u,v,w]$ whose arcs $[u,v], [v,w], [w,u]$ are labelled by even integers $r_{[u,v]}, r_{[v,w]}, r_{[w,u]} \geq 2$ define

$$\tilde{\Phi}(u,v,w) = (r_{[u,v]}^{r_{[v,w]}-1} - 1)\gcd\{\tilde{\Phi}_1(u,v,w), \tilde{\Phi}_2(u,v,w), \tilde{\Phi}_3(u,v,w), \tilde{\Phi}_4(u,v,w)\}$$

where $\tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3, \tilde{\Phi}_4$ are analogous to the quantities $K_{j1}, K_{j2}, K_{j3}, K_{j4}$ defined in [2, Theorem 1] (for example $\tilde{\Phi}_1(u,v,w) = (r_{[w,u]} - 1)(r_{[v,w]} - 1)$, $\tilde{\Phi}_2(u,v,w) = (r_{[u,v]} - 1)^2$, the formulae for $\tilde{\Phi}_3, \tilde{\Phi}_4$ being more complicated). Then [2, Corollary 1] states that $x_u^{\tilde{\Phi}(u,v,w)} = 1$ in M_Γ and it was shown in [2, Remark 4] that this gives a stronger result than Lemma 3.3. Therefore, if Φ is replaced by $\tilde{\Phi}$ in (2) we obtain an improved bound on the order in Theorem B.

In the next lemma (and in the proof of Theorem B) we will use the following notation. Given elements $a, b \in G$, if $ab = b^s a^t$ for some $s, t \in \mathbb{Z}$ we write $a \to b$ (or $b \leftarrow a$) to denote that we can "pull a through b"; if $a \to b$ and $a \leftarrow b$ we write $a \leftrightarrow b$. Therefore if $[u,v] \in A(\Gamma)$ then the relation $x_u x_v = x_v x_u^{r_{[u,v]}}$ holds in M_Γ, and so $x_u \to x_v$. Conversely, we now show that if $[u,v] \in A(\Gamma)$ and x_v has finite order in M_Γ then $x_v \to x_u$. Our argument is essentially that given in [23, page 1293].

Lemma 3.5. *Let Γ be a non-trivial digraph where each arc $[u,v]$ is labelled by an integer $r_{[u,v]}$ and suppose x_v has finite order in M_Γ. Then $x_v \to x_u$ in M_Γ.*

Proof. Suppose x_v has order $P < \infty$ in M_Γ. Repeated applications of the relation $x_v^{-1} x_u x_v = x_u^{r_{[u,v]}}$ give $x_v^{-P} x_u x_v^P = x_u^{r_{[u,v]}^P}$. Therefore $x_u^{r_{[u,v]}^P - 1} = e$ (where e is the identity of M_Γ) and so x_u has finite order, Q, say, which divides $r_{[u,v]}^P - 1$, and so is coprime to $r_{[u,v]}$. Thus there exists $\bar{r}_{[u,v]} \in \mathbb{Z}$ such that $r_{[u,v]} \bar{r}_{[u,v]} \equiv 1 \bmod Q$. Raising the defining relation of M_Γ that involves x_u, x_v to the power $\bar{r}_{[u,v]}$ gives $(x_v^{-1} x_u x_v)^{\bar{r}_{[u,v]}} = x_u^{r_{[u,v]} \bar{r}_{[u,v]}}$; that is, $x_v^{-1} x_u^{\bar{r}_{[u,v]}} x_v = x_u$ or $x_v x_u = x_u^{\bar{r}_{[u,v]}} x_v$ so $x_v \to x_u$. □

We are now in a position to prove Theorem B; our proof is a generalization of the argument in [15].

Proof of Theorem B. Observe first that the statement concerning the ranks follows from Lemma 3.2.

Since Γ is a non-trivial, strongly connected tournament, each vertex $u \in V(\Gamma)$ is in some directed triangle, so by Lemma 3.3 $x_u^{\Phi(u,v,w)} = 1$ in M_Γ for each directed triangle $[u,v,w]$ in Γ. Thus $x_u^{\phi(u)} = 1$ in M_Γ for all $u \in V$, so each generator has finite order. Therefore, by Lemma 3.5, if $[u,v] \in A(\Gamma)$ then $x_u \leftrightarrow x_v$, and since Γ is a tournament $x_u \leftrightarrow x_v$ for all $u, v \in V(\Gamma)$. Writing $V(\Gamma) = \{1, 2, \ldots, n\}$, each element of M_Γ can therefore be written in the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where $0 \le \alpha_v < \phi(v)$ $(1 \le v < n)$. Hence $|M_\Gamma| \le \prod_{v \in V} \phi(v)$, as required. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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