



Equationally defined classes of semigroups

Peter M. Higgins¹ · Marcel Jackson²

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Abstract

We apply, in the context of semigroups, the main theorem from the authors' paper "Algebras defined by equations" (Higgins and Jackson in *J Algebra* 555:131–156, 2020) that an elementary class \mathcal{C} of algebras which is closed under the taking of direct products and homomorphic images is defined by systems of equations. We prove a dual to the Birkhoff theorem in that if the class is also closed under the taking of containing semigroups, some basis of equations of \mathcal{C} is free of the \forall quantifier. We also observe the decidability of the class of equation systems satisfied by semigroups, via a link to systems of rationally constrained equations on free semigroups. Examples are given of EHP-classes for which neither $(\forall \dots)(\exists \dots)$ equation systems nor $(\exists \dots)(\forall \dots)$ systems suffice.

Keywords Equational classes · Regular semigroups · Makanin's algorithm

1 Introduction

A theorem of universal algebra and model theory is proved in [8] that is particularly pertinent to the study of semigroups. The theorem clarifies why it is so often the case that classes of algebras which are closed under the taking of arbitrary direct products (P) and homomorphic images (H), but not necessarily subalgebras (S), may be defined by a set of equation-like sentences. The members of these bases are referred to in [8] as *equation systems* and generally involve simultaneous equations, meaning that the equations are not assumed to be independent in that a symbol may occur in

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✉ Peter M. Higgins
peth@essex.ac.uk

Marcel Jackson
m.g.jackson@latrobe.edu.au

¹ Department of Mathematical Sciences, University of Essex, Colchester, UK

² Department of Mathematical and Physical Sciences, La Trobe University, Melbourne, VIC 3086, Australia

more than one equation. More formally, a semigroup equation system is a quantified conjunction of equalities between semigroup words (elements of a free semigroup). Thus in comparison with the more familiar theory of varieties and equational logic, we allow both \forall and \exists as quantifiers at the front of the sentence and we allow the logical connective of conjunction \wedge (“and”).

For instance, Example 2.1(ii) of [8] shows that Completely regular semigroups (\mathcal{CR}) is the class consisting of all semigroups in which the following equation pair may be solved:

$$(\forall a)(\exists x): (a = axa) \wedge (ax = xa). \quad (1)$$

These equations are simultaneous and indeed both the parameter a and the variable x feature in each. Together they capture the property that each element has an inverse with which it commutes, which is one definition of completely regular semigroups. However, we shall show that this class may be defined by a single quantified equation (without conjunction), and in more than one way.

Classes defined by equation systems in this fashion are referred to as $\{E, H, P\}$ -classes or simply *EHP*-classes, with the E symbol standing for the taking of *elementary substructures*, H for *homomorphic images* and P for *direct products*. An *elementary class* (or *first order class*) is one defined by a collection of first order formulae: the quantifiers refer to elements of algebras, hence “elementary”, in contrast with second order logic where we may quantify over relations. The inclusion of the class operator E is necessary to capture equivalence with definability by equation systems, but the reader will not need familiarity with the definition of elementary substructures in this article, because we always proceed by finding equation systems that capture the classes we explore. It is not currently known if closure under the operator *EHP* is enough to ensure closure under the set of operators $\{E, H, P\}$.

For relevant background and notation on universal algebra see [1, 2] (Chapter V there also serves as a useful first introduction to model theoretic notions for those wanting to delve further), while [3, 7, 9] are texts covering semigroup facts and terminology.

In the following theorem, closure under taking elementary substructures is subsumed by the assumption that \mathcal{C} is an elementary class (all elementary classes are closed under the taking of elementary substructures).

Theorem 1.1 [8, Theorem 3.1] *An elementary class \mathcal{C} equals the class of models of some family of equation systems if and only if \mathcal{C} is closed under taking homomorphic images of direct products. If the elementary class is the model class of a single sentence, then it is a class of models of a single equation system.*

The theorem was inspired by the observation that so many of the fundamental classes of algebraic semigroups are EHP-classes, but are not varieties, which is to say the class is *not* closed under the taking of subsemigroups, and so cannot be defined by semigroup identities. Indeed since equation systems are examples of first order sentences, a class defined by the satisfaction of a set of equations is automatically an elementary class and the initial condition of the theorem is, in the (easy) forward direction, redundant.

In Sect. 2 we highlight examples showing just how rich is EHP theory in the context of semigroups. Many extensively studied semigroup classes are captured through our approach, in many cases by a single equation. We verify this in cases where the equational bases differ from those given in [8].

An EHP-class \mathcal{C} that is closed under the taking of all embeddings (and not only elementary embeddings) is a variety in which case, by Birkhoff's theorem, the equations defining \mathcal{C} may be taken to be identities, which is to say the equations do not involve the quantifier \exists . It is natural then to consider a kind of dual to the Birkhoff theorem for EHP-classes \mathcal{C} defined by equations that are free from the quantifier \forall . Clearly such a class is closed under the taking of containing algebras. In Sect. 3 we prove the converse does indeed hold for the class of semigroups.

In Theorem 1.1, the number of alternations between the \forall symbol, which qualifies *parameters*, (denoted by lower case letters from the beginning of the alphabet a, b, c, \dots), and the \exists symbol, which qualifies *variables*, (denoted typically by x, y, z) is finite but unbounded in length. In Sect. 4 we give examples of EHP-classes which have no basis comprised of equations of the form $(\forall \dots)(\exists \dots)$ nor the form $(\exists \dots)(\forall \dots)$.

2 Classical semigroup collections as EHP-classes

2.1 Classes of regular semigroups

We begin with six examples that highlight how easily important classes of regular semigroups may be characterised by a single equation. Moreover the proofs of this are simple but elegant exercises in semigroup theory. In particular we see in Proposition 2.1 how the classes in question may be characterised by small adjustments to the equation that defines regularity.

Proposition 2.1 *The classes of Regular semigroups ($\mathcal{R}eg$), Left groups ($\mathcal{L}\mathcal{G}$), Right groups ($\mathcal{R}\mathcal{G}$), Groups (\mathcal{G}), Completely regular semigroups ($\mathcal{C}\mathcal{R}$), and Completely simple semigroups ($\mathcal{C}\mathcal{S}$) are the EHP-classes of semigroups defined by the following equations.*

$$\mathcal{R}eg : (\forall a)(\exists x) : a = axa. \quad (2)$$

$$\mathcal{L}\mathcal{G} : (\forall a, b)(\exists x) : a = axb, \quad \mathcal{R}\mathcal{G} : (\forall a, b)(\exists x) : a = bxa. \quad (3)$$

$$\mathcal{G} : (\forall a, b)(\exists x) : a = bxb. \quad (4)$$

$$\mathcal{C}\mathcal{R} : (\forall a)(\exists x) : a = a^2xa^2. \quad (5)$$

$$\mathcal{C}\mathcal{S} : (\forall a, b)(\exists x) : a = abxba. \quad (6)$$

Proof The Eq. (2) may be satisfied in any regular semigroup S by taking $x \in V(a)$. Conversely, if x satisfies (2), then $xax \in V(a)$.

For Eq. (3), let S be a left group and let $a, b \in S$. Since S is left simple, there exists $y \in S$ such that $a = yb$. By regularity there exists $z \in S$ such that $a = aza$, whence $a = azyb$. Putting $x = zy$ gives $a = axb$, as required.

Conversely suppose that $a = axb$ is solvable in a semigroup S . Putting $b = a$ gives $a = axa$ is solvable so that S is regular. Moreover $a = axb$ implies that $a \leq_{\mathcal{L}} b$, and since a, b are arbitrary it follows that $a \mathcal{L} b$. Therefore S is left simple and regular and so S is a left group. The left-right dual argument shows that \mathcal{RH} is defined by the equation $a = bxa$ in like manner.

For Eq. (4), if G is a group then for given $a, b \in G$ there is a unique solution to (4), that being $x = b^{-1}ab^{-1}$.

Conversely, let S be a semigroup in which (4) is solvable. The equation implies that $H_a \leq H_b$. By interchanging a and b we obtain the reverse inequality, whence S consists of a single \mathcal{H} -class, and is therefore a group.

For Eq. (5): from the equation, it follows that $a \mathcal{H} a^2$, from which we infer that every \mathcal{H} -class is a group, whence S is a union of groups, which is to say that S is completely regular.

Conversely, let S be a completely regular semigroup. Let $a \in S$ and put $x = b^3$, where b is the inverse of a in the group H_a . Then $ab = ba$ is the identity element of H_a and so

$$a^2xa^2 = a^2b^3a^2 = a(ab)b(ba)a = aba = a,$$

in accord with (5).

Finally, let S be a semigroup that satisfies (6). By putting $b = a$ we see that (6) implies (5), so that S is completely regular. For any $a, b \in S$, (6) implies that $J_a \leq_{\mathcal{J}} J_b$, and by role reversal of a and b , the reverse inequality follows so that $J_a = J_b$. Therefore S is a simple completely regular semigroup, which is to say that S is completely simple.

Conversely let S be a completely simple semigroup and take $a, b \in S$. Then we have $a \mathcal{R} ab \mathcal{L} b \mathcal{R} ba \mathcal{L} a$. By Green’s Lemma, the mapping $\rho_{ba} : H_a \rightarrow H_a$ whereby $x\rho_{ba} = xba$ is a bijection, as is the mapping $\lambda_{ab} : H_a \rightarrow H_a$ whereby $x\lambda_{ab} = abx$. It follows that $\phi = \lambda_{ab}\rho_{ba} = \rho_{ba}\lambda_{ab} : H_a \rightarrow H_a$ is also a bijection, whereupon there exists a unique $x \in H_a$ such that $x\lambda_{ab}\rho_{ba} = abxba = a$, thereby proving that S satisfies Eq. (6). □

Other classes of regular semigroups may be defined by an equation system consisting of (2) together with one more equation. For instance, Example 2.1(iii) of [8] shows that *Semilattices of groups* (\mathcal{SLG}) is the EHP-class of regular semigroups defined by the additional equation:

$$(\forall a, b)(\exists y) : ab = bya. \tag{7}$$

Some standard properties used in the description of classes of semigroups may be expressed by equations, and that allows for abbreviation. For example, that a certain product u of some parameters and variables is idempotent we write as $u \in E$, or if v is an inverse of u we write $v \in V(u)$. Properties defined by Green’s relations are generally not intrinsically equational within the class of semigroups but may become so in the presence of the regularity Eq. (2). However the respective properties of being \mathcal{G} -simple for any of the five Green’s relations \mathcal{G} defines an EHP-class, except for the

case of \mathcal{D} . In general, bisimple semigroups are not closed under the taking of direct products. Within $\mathcal{R}eg$, the condition $u \mathcal{H} v$ is expressible through equations in S , with similar comments applying to \mathcal{L} , \mathcal{R} , \mathcal{J} , and indeed \mathcal{D} . In particular, the class of regular bisimple semigroups is defined by the regularity equation together with the equational relationships $(\forall a, b)(\exists x): a \mathcal{R} x \mathcal{L} b$. This class is however not a variety as, by a theorem of Preston, any semigroup may be embedded in a regular bisimple monoid [7, Corollary 1.2.15].

The property of u belonging to a subgroup, which we write as $u \in G$, is also equational:

$$(\exists x): (x \in V(u)) \wedge (ux = xu).$$

The ascending chain of the three important classes of \mathcal{I} (*Inverse semigroups*), \mathcal{O} (*Orthodox semigroups*), and $\mathcal{E}\mathcal{S}$ (*Idempotent-solid semigroups*), which are those regular semigroups whose idempotent generated subsemigroup is a union of groups, may be defined in a uniform fashion that is conveniently displayed if we adjoin two redundant equation types to the definition of regularity:

$$\mathbf{reg} : (\forall a, b)(\exists x, u, v): (x \in V(a)) \wedge (u \in V(a^2)) \wedge (v \in V(b^2)). \quad (8)$$

We include two further classes within this sequence. For the first, we have the *Right Inverse semigroups* introduced by Venkatesan [14] as regular semigroups in which each \mathcal{L} -class contains a unique idempotent (for that reason, they are also known as *\mathcal{L} -unipotent semigroups*). The class $\mathcal{R}\mathcal{I}$ of Right inverse semigroups is given six further characterisations in [14, Theorem 2.1], one of which is the class of all regular semigroups S for which $efe = fe$ for any idempotents e, f in S . It follows that $\mathcal{I} \subseteq \mathcal{R}\mathcal{I} \subseteq \mathcal{O}$. It also is the case that $\mathcal{R}\mathcal{I}$ is an EHP class, and the argument for closure under homomorphisms is given in Theorem 3 of [14].

The second inclusion in the chain is the class $\mathcal{C}\mathcal{N}$ of *Conventional semigroups* of Masat [13]: a regular semigroup S is *conventional* if aea' is idempotent for all $(a, a') \in V(S)$ and $e \in E(S)$. Equivalently, by [13, Lemma 2.2], if $eEe \subseteq E$, which compares as a natural weakening of the $efe = fe$ condition of Right inverse semigroups. The Conventional semigroup definition is also a weakening of the Orthodox semigroup definition, and so $\mathcal{O} \subseteq \mathcal{C}\mathcal{N}$. A consequence of Masat's Lemma 2.2 and Lallement's Lemma is that $\mathcal{C}\mathcal{N}$ is closed under the taking of homomorphisms [13, Lemma 3.1]. Since $\mathcal{C}\mathcal{N}$ is clearly closed under direct products and is an elementary class, it follows that $\mathcal{C}\mathcal{N}$ is an EHP-class. With the symbols a, b, u, v satisfying the equations of **reg**, we may define each these five classes by means of one additional equation (to be included within the scope of the quantification of **reg**). The following proposition is [8, Theorem 5.2], extended by the inclusion of $\mathcal{R}\mathcal{I}$ and $\mathcal{C}\mathcal{N}$.

Proposition 2.2 *The classes of Inverse semigroups, Right inverse semigroups, Conventional semigroups, Orthodox semigroups, and Idempotent solid semigroups, are EHP-classes defined by the following equational bases.*

$$\mathcal{I} : \mathbf{reg} \wedge aua \cdot bvb = bvb \cdot aua; \quad (9)$$

$$\mathcal{RS} : \mathbf{reg} \wedge aua \cdot bvb \cdot aua = bvb \cdot aua; \tag{10}$$

$$\mathcal{O} : \mathbf{reg} \wedge aua \cdot bvb \in E; \tag{11}$$

$$\mathcal{CN} : \mathbf{reg} \wedge aua \cdot bvb \cdot aua \in E; \tag{12}$$

$$\mathcal{ES} : \mathbf{reg} \wedge aua \cdot bvb \in G. \tag{13}$$

Proof The cases of \mathcal{I} , \mathcal{O} and \mathcal{ES} are given in [8, Theorem 5.2]. The proof for \mathcal{CN} is indicative of the approach, and we omit the very similar argument for \mathcal{RS} . Recall that \mathbf{reg} includes the conditions $u \in V(a^2)$ and $b \in V(b^2)$ so that aua and bvb are idempotents. Thus any conventional semigroup satisfies the given equation system because of the $eEe \subseteq E$ condition of [13, Lemma 2.2]. Conversely if S satisfies the equations then S is regular and for any two idempotents a and b we have:

$$aba = a^2b^2a^2 = a^2ua^2 \cdot b^2vb^2 \cdot a^2ua^2 = aua \cdot bvb \cdot aub,$$

which is idempotent. It follows by [13, Lemma 2.2] that we have an equational basis for Conventional semigroups. □

The equational bases in Propositions 2.1 and 2.2 are not unique. The bases given by Eqs. (9), (11), and (13) of Proposition 2.2 correspond to bases for these classes when considered as e-varieties in the sense of Hall [6]. These are classes of regular semigroups that are *HP*-closed, and also closed under the taking of regular subsemigroups. Indeed any e-variety (of regular semigroups) is an EHP-class of regular semigroups, and so e-varieties may be defined without the need to introduce a unary operation that selects arbitrary inverses. The only semigroup operation involved is the natural operation of semigroup multiplication. However, not all EHP-classes consisting of regular semigroups are e-varieties (see [8, Theorem 5.1]). In common with e-varieties however is the property that if \mathcal{C} is an EHP-class of regular semigroups then the class \mathcal{C}^{loc} of all semigroups S whose local subsemigroups eSe lie in \mathcal{C} ($e \in E(S)$) is also an EHP-class (see [8, Theorem 5.5]).

The abstraction of the idea of e-varieties involves taking an EHP-class of algebras \mathcal{N} , which are labelled *nice*, and then considering classes \mathcal{C} of nice algebras that are *HP*-closed and closed under the taking of nice subalgebras. Since the nice algebras are defined by first order formulae (equation systems), it follows that \mathcal{C} will be closed under the taking of elementary subalgebras, and so \mathcal{C} will automatically be another EHP-class. There could however, as in the case of regular semigroups, be EHP-classes of nice algebras that were not closed under the taking of nice subalgebras. If we declare the class of all algebras to be nice, then the corresponding class of e-varieties coincides with varieties in the usual sense of algebras defined by identities (\exists -free equation systems).

2.2 Classes defined by $(\exists \dots)(\forall \dots)$

A simple example of a fundamentally different type is the class \mathcal{M} of *monoids*.

$$\mathcal{M} : (\exists x)(\forall a): ax = xa = a. \tag{14}$$

We sometimes write these equations as $x = 1$, and similarly we write $x = 0$ to abbreviate the equations that ensure the existence of a zero element in a semigroup. The point to note here however is that the order of the existential quantifiers in (14) is $(\exists \dots)(\forall \dots)$, which is the reverse of all our previous examples. Indeed \mathcal{M} cannot be represented by equations of the type $(\forall \dots)(\exists \dots)$ because any class that has such a basis is closed under the taking of the union of an ascending chain of algebras from the class, and the class \mathcal{M} lacks this property. We will investigate this facet of the theory further in our final section.

A natural exercise then is to exchange the order of the quantifiers of the examples of Sect. 2.1. This will necessarily result in a more restricted class to that defined by the original equation system. Exchanging the order of the quantifiers in Eq. (2) defines the class \mathcal{B} of all semigroups that possess a universal pre-inverse element:

$$\mathcal{B} : (\exists x)(\forall a) : a = axa. \quad (15)$$

We note that for any $S \in \mathcal{B}$, the \mathcal{J} -class J_x is maximal. Conversely any band B with a maximal \mathcal{J} -class $J = J_x$ belongs to \mathcal{B} . To see this observe that since B is a semilattice of rectangular bands, it follows that for any $a \in \mathcal{B}$, $ax \in D_a$. Since D_a is a rectangular band, we have $ax = a^2x = a \cdot ax\mathcal{R}a$, whence $ax \cdot a = a$, showing that $B \in \mathcal{B}$.

The following is a simple reformulation of the condition defined by (15).

Lemma 2.3 *A semigroup $S \in \mathcal{B}$ if and only if $\exists x \in S$ such that $ax \in R_a \cap E(S)$ for all $a \in S$, which in turn is equivalent to the condition that $xa \in L_a \cap E(S)$ for all $a \in S$.*

Proposition 2.4 *Suppose that $S \in \mathcal{B}$, let E denote $E(S)$, and let x denote a fixed choice for satisfying (15). Then*

- (i) $x \in E$, S satisfies the identity $a^2 = a^3$, and $S = E^2$.
- (ii) In S , $\mathcal{D} = \mathcal{J}$, and \mathcal{H} is the equality relation.
- (iii) Let $J = J_x$. Then J is the maximum \mathcal{J} -class of S , the principal factor $J \cup \{0\} = S/(S - J) \in \mathcal{B}$, and

$$R_x \cup L_x \subseteq E. \quad (16)$$

Proof (i) Taking $a = x$ in (15) we get $x = x^3$. For any a we have $a^2 = a^2xa^2 = a(axa)a = a^3$. In particular, $x^2 = x^3 = x$, so that $x \in E$. Furthermore, since $(ax)^2 = ax$ and $(xa)^2 = xa$ we have $a = axa = (ax)(xa)$, and so $S = E^2$.

(ii) That $\mathcal{D} = \mathcal{J}$ follows from the satisfaction of $a^2 = a^3$, as this equality of Green's relations is true in any periodic semigroup; indeed it is true of any group-bound semigroup (see [4, Theorem 1.2.20]).

Let D be any (regular) \mathcal{D} -class of S . In any subgroup G of S , the equation $a^2 = a^3$ implies that $a = e$, the identity element of G , and so S has trivial subgroups. It follows that every group \mathcal{H} -class, and hence every \mathcal{H} -class of S is trivial, which is to say that S is a combinatorial (i.e. \mathcal{H} -trivial) semigroup.

(iii) Since $J_a \leq J_x$ it follows that $J = J_x$ is the maximum \mathcal{J} -class of S (where x represents any solution to (15)). Since EHP-classes are closed under the taking of homomorphisms, the principal factor $J \cup \{0\}$ also belongs to \mathcal{B} , and any solution to (15) in S is also a solution to (15) in $J \cup \{0\}$. Suppose that $x \mathcal{R} a$ in S , and hence in $S/(S - J)$ also. Then since $x \in E$, it follows that $a = xa$. But then $a = axa = a^2$, so that $a \in E$. Dually if $x \mathcal{L} a$ then a is also idempotent. We conclude that Condition (16) holds. \square

Remark 2.5 It is possible for a semigroup S to satisfy all three conditions of Proposition 2.4 yet for S not to belong to \mathcal{B} . For example take the six-element semigroup A_2^1 , which is the Rees matrix 0-simple semigroup with adjoined identity element 1, given by $\mathcal{M}^0[\{e\}, 2, 2; P]^1$, where $\{e\}$ is a one-element group and

$$P = \begin{bmatrix} e & e \\ e & 0 \end{bmatrix}.$$

Taking $x = 1$, we see that each of (i), (ii), and (iii) is satisfied. However $A_2^1 \notin \mathcal{B}$: taking $a = 1$ we see that we must take $x = 1$ in order to satisfy the condition of Proposition 2.4. However with $x = 1$, for the single non-idempotent element $a = (e; 2, 2)$ of A_2^1 , we have $ax = xa = a \notin E(S)$, contrary to Lemma 2.3.

We next consider the more restricted equational system $\mathcal{V} \subseteq \mathcal{B}$, consisting of all semigroups that possess a universal inverse element:

$$\mathcal{V} : (\exists x)(\forall a) : x \in V(a). \tag{17}$$

Let $S \in \mathcal{V}$, and so Proposition 2.4 applies. In particular S is a regular periodic combinatorial semigroup. Moreover, (17) implies that S is bisimple, and since any periodic bisimple semigroup is completely simple (Corollary 2.56 of [3]), we conclude that S is completely simple. However, a completely simple combinatorial semigroup is none other than a rectangular band, which certainly satisfies (17). Indeed this gives the following curious formulation of the property of the existence of a universal inverse.

Proposition 2.6 *For any semigroup S either*

$$\cap_{a \in S} V(a) = \emptyset \text{ or } \cap_{a \in S} V(a) = S,$$
the latter occurring if and only if S is a rectangular band.

2.3 Applying the EHP theorem: the Croisot theory

The previous section also serves to introduce a strategy for applying the EHP theorem. The general approach is to systematically list semigroup equations and identify the corresponding semigroup classes. The classical decomposition theory of R. Croisot [5] deals in semigroup classes defined by a collection of simple related equations, which we summarise here. The following two results, based on [5], may be found in Sect. 4.1 of [3].

In this section the existential quantifiers are suppressed with a, b standing for parameters and x, y variables. Some left-right dual statements are not explicitly stated.

Theorem 2.7 [3] *For the EHP-class defined by $a = a^m x a^n$ ($m + n \geq 2$):*

- (i) *The class defined by $(m, 0)$ is the class of all semigroups S in which each \mathcal{R} -class is a subsemigroup (which is then necessarily a right simple subsemigroup) of S . Equivalently $a\mathcal{R}a^2$ for all $a \in S$.*
- (ii) *The class defined by (m, n) with $m, n \geq 1$ and $m+n \geq 3$ is the class of completely regular semigroups.*

Theorem 2.8 [3] *The EHP-class defined by $a = x a^2 y$ is the class of all semigroups S that are unions of simple subsemigroups; S is then necessarily a semilattice of simple semigroups, and these simple components are the \mathcal{J} -classes of S .*

The proof of Theorem 2.7 involves a mix of syntactic and semantic argument, with equation manipulation and use of properties of Green's relations. Another semantic proof of Theorem 2.7(ii) comes through observing that $a = a^2 x a$ implies that S is regular and satisfies $a\mathcal{R}a^2$ so that each \mathcal{R} -class is a regular subsemigroup of S . (The general (m, n) case of Theorem 2.7 follows easily from this basic equation.) From this we may deduce that the principal factors of S are completely simple, for if not S would contain a copy of the bicyclic semigroup, B . However, the property of \mathcal{R} -classes being subsemigroups would be inherited by any inverse subsemigroup of S , which would then be a semilattice of groups, which is contradicted by the presence of B .

An advantage of the EHP theorem is that it characterises classes in the first-order language, offering a vehicle for automated theorem proving. The software package *Prover 9* [11] found a proof that $a = a^2 x a$ implies $a\mathcal{L}a^2$, whence $a\mathcal{H}a^2$, and so S is a union of groups. The proof below is the result of equation manipulation, suggesting that automated theorem provers may play a role in results of this kind.

Remark 2.9 We now present a demonstration of the statement that the equation $a = a^2 x a$ characterises completely regular semigroups.

Proof As observed above, it is enough to show that $a\mathcal{L}a^2$ for an arbitrary $a \in S$.

Fix two members $a', a^R \in S$ such that $a = a a' a$ and $a = a^2 a^R$ and put

$$r = a^R a' a. \quad (18)$$

Then r has three properties relevant to our purpose,

$$a^2 r = (a^2 a^R) a' a = a a' a = a. \quad (19)$$

$$\begin{aligned} (ar)^2 &= (a a^R a' a) (a a^R a' a) = (a a^R a') (a^2 a^R) a' a \\ &= a a^R a' (a a' a) = a (a^R a' a) = ar. \end{aligned} \quad (20)$$

Moreover $a^2 r = a$ implies $a^2 r^2 = ar$ and since (20) shows that ar is idempotent we infer:

$$a^2 r^2 = a r a^2 r^2. \quad (21)$$

We next deduce that $a = a r a$ as follows:

$$\begin{aligned}
 a &= a^2r \text{ (by (19))} \\
 &= a^2r^2r^R \text{ (by definition of } r^R\text{)} \\
 &= ara^2r^2r^R \text{ (by (21))} \\
 &= ara^2r \text{ (by definition of } r^R\text{)} \\
 &= ara \text{ (by (19)).}
 \end{aligned}$$

Now $ra \mathcal{L} a^2$ as $ra = a^R a' a^2$ (by (18)), and from (19), $a^2ra = a^2$. Since $a = ara$ we then have $a \mathcal{L} ra \mathcal{L} a^2$, which completes the proof. \square

Proposition 2.10 *The following are equivalent for a semigroup S .*

- (i) S satisfies $a = abxba$.
- (ii) S satisfies $a = abxa$.
- (iii) S satisfies $a = axba$.
- (iv) S is completely simple.

Proof We have by Proposition 2.1 that (i) and (iv) are equivalent. Clearly (i) implies (ii) as given $a = abxba$ we have $a = aby a$, where $y = xb$. Similarly (i) implies (iii). By symmetry it is enough now to prove that (ii) implies (iv). By taking $b = a$ in (ii) we see that S satisfies $a = a^2xa$, whence by Theorem 2.7(ii), S is completely regular. It also follows from (ii) that S has only one \mathcal{J} -class, whence S is completely simple. \square

3 The EHP theorem

3.1 Outline of proof of Theorem 1.1

The proof of Theorem 1.1 in the forward direction is simple for it is clear that if a class of algebras \mathcal{C} is defined by an equation system then this property is preserved under the taking of homomorphic images and arbitrary direct products. Moreover, such a class \mathcal{C} is automatically an elementary class as \mathcal{C} is defined in the first order language.

The converse direction however is a consequence of Lyndon’s positivity theorem, which states that for an elementary class closed under taking surjective homomorphic images, a sentence is equivalent to a positive sentence (one free of negations). Thus we may assume that our class of algebras \mathcal{C} such that $\mathcal{C} \subseteq HP(\mathcal{C})$ is the class of models of a set Σ of positive sentences. There is no loss of generality to assume that all quantifiers are at the front of the positive sentences. The remaining task then is to show that disjunctions in each $\rho \in \Sigma$ may be removed.

Consider a sentence $\rho \in \Sigma$. We may express the equation systems of ρ as a finite conjunction of disjunctions, $\bigwedge_{1 \leq i \leq m} \gamma_i$, where each γ_i is a finite disjunction: $\gamma_i = \alpha_{i,1} \vee \dots \vee \alpha_{i,r_i}$, and each $\alpha_{i,j}$ is an atomic formula involving some subset of the full set of parameters and variables of ρ . Suppose that for some i , $r_i \geq 2$ ($1 \leq i \leq m$). We show that the conjunct γ_i may be replaced by some $\alpha_{i,j}$ and the resulting reduced sentence is equivalent to ρ for any elementary class \mathcal{C} that is closed

under *HP*. Repeating this for each conjunct γ_i will see us arrive at the desired \bigvee -free sentence. The quantifiers remain unchanged throughout.

In view of this, we suppress the symbol i and use the corresponding symbols $r = r_i$, $\gamma = \gamma_i$, and $\alpha_j = \alpha_{i,j}$. Let ρ_j be the result of replacing γ in ρ by α_j . Note that $\rho_j \vdash \rho$ so that the class of models satisfied by $(\Sigma \cup \rho_j) \setminus \rho$ is a subclass of \mathcal{C} . We wish to show that for some j the reverse containment holds. Assume by way of contradiction that this is not the case. Then for each j ($1 \leq j \leq r$) there exists a model $M_j \in \mathcal{C}$ such that ρ_j fails in M_j . Put $M = \prod_{j=1}^r M_j \in \mathcal{C}$ and so $M \models \gamma$. (This stage of the argument only requires that \mathcal{C} is closed under the taking of finitary direct products. However since \mathcal{C} is an elementary class, \mathcal{C} is closed under the taking of ultraproducts; \mathcal{C} being closed under finitary direct products and ultraproducts then implies that \mathcal{C} is in fact closed under the taking of arbitrary direct products.)

The nature of the argument may be illustrated in the simplest case where ρ has only one pair of existential quantifiers, for instance, let us say that each α_j has a single equation:

$$\alpha_j : (\forall \mathbf{a}_j)(\exists \mathbf{x}_j) : u_j(\mathbf{a}_j, \mathbf{x}_j) = v_j(\mathbf{a}_j, \mathbf{x}_j),$$

where \mathbf{a}_j and \mathbf{x}_j are the respective vectors of parameters and variables of the equation $u_j = v_j$. Since $M_j \not\models \alpha_j$ it follows that for M_j , $\exists \mathbf{a}_j$ such that $\forall \mathbf{x}_j$ $u_j(\mathbf{a}_j, \mathbf{x}_j) \neq v_j(\mathbf{a}_j, \mathbf{x}_j)$. Since $M \models \gamma$ we may select $\tilde{\mathbf{a}} = (\mathbf{a}_1, \dots, \mathbf{a}_j, \dots, \mathbf{a}_r)$ as our parameter choices for M and there exists a corresponding choice of variables $\tilde{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_r)$ such that for some j , $u_j(\tilde{\mathbf{a}}, \tilde{\mathbf{x}}) = v_j(\tilde{\mathbf{a}}, \tilde{\mathbf{x}})$. However, taking the projection of this last equation onto the j th component then yields the contradiction that $u_j(\mathbf{a}_j, \mathbf{x}_j) = v_j(\mathbf{a}_j, \mathbf{x}_j)$.

In general however, an equation system $\rho \in \Sigma$ may have any finite number of alternations of existential quantifiers. Since satisfaction for such a sentence is defined recursively on the string of quantifiers, the above argument needs to be taken by induction on the number of quantifiers through the stages outlined in the previous discussion. This technical argument however does not require any additional facet to the proof strategy presented in the previous paragraph. The complete argument is given in [8, Theorem 3.1].

3.2 The dual variety theorem for semigroups

An EHP-class \mathcal{C} defined without the use of the \exists quantifier is a variety, and in particular the class is closed under the taking of subalgebras. (Birkhoff's theorem says that a class of algebras \mathcal{C} is defined by a countable list of identities if and only if \mathcal{C} is closed under the operator *HSP*.) On the other hand, if the class is defined without the use of the \forall symbol then the class is closed under the taking of superalgebras, meaning that if $A \in \mathcal{C}$ and $A \leq B$, where B is an algebra in the defining signature of the algebra class under consideration, then $B \in \mathcal{C}$ also.

Here we prove the converse for the class of Semigroups: if an EHP-class \mathcal{C} is closed under the taking of superalgebras it follows that \mathcal{C} may be defined by equations of the type $(\exists x_1, \dots, x_n) : (\bigwedge_{1 \leq i \leq m} u_i(x_1, \dots, x_n) = v_i(x_1, \dots, x_n))$.

Definition 3.1 An equation system is called *existential* if it has no instances of the \forall quantifier. An EHP-class is called *existential* if it has a basis of existential equation systems.

In the model theory literature what we call here an existential equation system is known as a *primitive positive* sentence. First let us suppose that \mathcal{C} is an EHP-class that is closed under the taking of containing algebras. Suppose that A is an algebra containing a trivial (one-element) subalgebra, T . Then as $T \in \mathcal{C}$ we have that $A \in \mathcal{C}$ by the containment property. It follows that if our algebra is of a type where every algebra contains a one-element algebra, such as Monoids or Groups, then \mathcal{C} is the EHP class of all algebras of the type under consideration. Moreover, in this context all algebras satisfy all existential equation systems. It follows that the converse is trivially true as a class closed under the taking of containing algebras and a class defined by a basis of existential equation systems are both necessarily equal to the class of all algebras.

However, within the class of Semigroups, there are (infinite) semigroups that are idempotent-free, and so the previous observation does not apply. For example the equation $(\exists x) : x = x^2$ defines the EHP-class of all semigroups that contain an idempotent. This is a proper class of semigroups that is contained in every existentially defined EHP-class of semigroups.

Theorem 3.2 *An EHP-class of semigroups \mathcal{C} is existential if and only if \mathcal{C} is closed under the taking of containing semigroups. Equivalently, \mathcal{C} is closed under the taking of codomains of homomorphisms.*

Before we embark on the main proof we observe the equivalence with the second sentence. Suppose that the EHP-class \mathcal{C} is closed under the taking of containing semigroups and let $S \in \mathcal{C}$ with $\alpha : S \rightarrow T$ a homomorphism. Since \mathcal{C} is closed under the taking of homomorphisms, $S\alpha \in \mathcal{C}$, and since $S\alpha \leq T$ it then follows that $T \in \mathcal{C}$. Conversely suppose that \mathcal{C} is closed under the taking of codomains of homomorphisms and suppose that $S \leq T$. We take $\alpha : S \rightarrow T$ to be the identity mapping on S with codomain T , whence by the given condition $T \in \mathcal{C}$.

Most of the remainder of the section is devoted to the proof of Theorem 3.2, which is completed after some preliminary lemmas and discussion. The main challenge of the proof is facilitating a kind of quantifier elimination, achieved using the free product construction. For any semigroup S , we consider the free product $F * S$ of S with the free semigroup $F = F_A$ on a countably infinite alphabet $A = \{A_1, A_2, \dots\}$.

Consider an arbitrary equation system ε satisfied by $F * S$: a quantified system $p_1 = q_1 \wedge \dots \wedge p_\ell = q_\ell$. As ε is satisfied, for every evaluation of the parameters (universally quantified variables in ε) in $F * S$, we may find witnesses to the existentially quantified variables, with the choice of each witness being made on the basis of prior quantified variables. As the parameters can be chosen without restriction, we are going to adopt the strategy that each parameter is chosen to be a free generator from the set A that has not appeared within the evaluation of any variable quantified before it: we refer to this as the *free dependency condition*. Thus if we have $\forall a_1 \exists x_1 \forall a_2$, we choose a_1 to be A_1 and if we have chosen the witness x_1 to evaluate as $A_2 A_1 s A_5$ for some $s \in S$, then we will choose a_2 to take the value A_3 (or any other free generator

except for those already in use: A_1, A_2, A_5 in the example). Without loss of generality however, it is clear that we may rename the free generators so that each parameter a_i is assigned the free generator A_j : in the example just given we could rename A_2 and A_3 and choose the witness $A_3A_1sA_5$ for x_1 and then choose $a_2 \mapsto A_2$. We refer to this as an instance of a *canonical evaluation of parameters*, and we say that $F * S$ satisfies ε under the canonical evaluation of parameters to mean that witnesses to the existential variables can be made to achieve equality $p_1 = q_1, \dots, p_\ell = q_\ell$ under the evaluation (and satisfying the free dependency condition). The following lemma holds in any variety within any signature of algebras, replacing “free semigroup” with a relatively free algebra in the variety. We use U to denote a semigroup instead of S to match later usage of the lemma.

Lemma 3.3 *Let ε be an equation system with parameters a_1, \dots, a_p and U be a semigroup. Let $F = F_A$ be the denumerably generated free semigroup with free generators $A = \{A_1, A_2, \dots\}$. If $F * U$ satisfies ε under the canonical parameter evaluation, then U satisfies ε .*

Proof Every function $\phi : A \rightarrow U$ extends to a retraction from $F * U$ onto the subsemigroup U . We have witnesses $X_1, \dots, X_q \in F * U$ for the canonical evaluation of parameters a_1, \dots, a_p as A_1, \dots, A_p satisfying the free dependency condition. Let γ be any evaluation of the parameters of ε in U , and let $\phi : A \rightarrow U$ be $\phi(A_i) := \gamma(a_i)$. We may extend ϕ to a retraction onto U , and use witnesses $\phi(X_1), \dots, \phi(X_q)$ in U to verify satisfaction of ε . \square

Note that the free dependency condition is used only at the final step of this proof: if x_i is (existentially) quantified prior to (the universally quantified) a_j , we should have that the choice of $\phi(X_i)$ can yield satisfaction of ε for all subsequent evaluations of a_j . But if X_i contained an occurrence of A_j , then the value of $\phi(X_i)$ would in general depend on the evaluation $\gamma(a_j) = \phi(A_j)$ of a_j .

Every element of $F * S$ may be written uniquely in the form $f_1s_1f_2 \dots s_{k-1}f_k$, where each f_i is an element of F_A , each s_i is an element of S and f_1 and possibly f_k could be empty (though note that if $k = 1$, then this expression is simply f_1 and then f_1 cannot be empty). We refer to this as the *normal form* for an element of $F * S$. If $z_1 \dots z_p$ is an arbitrary semigroup word (possibly with repeats in the sequence of letters z_1, \dots, z_p), then under any evaluation of the letters $\{z_1, \dots, z_p\}$ into $F * S$, the word $z_1 \dots z_p$ gives rise to a product of normal forms

$$\prod_{1 \leq i \leq p} (f_{i,1}s_{i,1}f_{i,2} \dots s_{i,k_i-1}f_{i,k_i}). \quad (22)$$

Depending on the value of k_i , and on whether $f_{i,1}$ or f_{i,k_i} are empty, the product in (22) may give rise to sequences of consecutive instances of elements of S ; a maximal block of consecutive elements of S that arises from such a product will be called an *S-run*. So for example, if letters z_1, z_2, z_3, z_4 are evaluated in $F * S$ as $A_1s_1, s_2A_2s_1A_3s_3, s_2$ and s_1A_1 respectively, then the word $z_1z_2z_3z_4$ evaluates as $(A_1s_1)(s_2A_2s_1A_3s_3)(s_2)(s_1A_1)$, and we have S -runs s_1s_2, s_1 and $s_3s_2s_1$. These of course collapse to individual elements of S in the reduction of (22) to normal form,

but we are interested in the uncollapsed form. For each S -run we may also create an *abstract run*, which is a matching semigroup word in the alphabet $\{x_s \mid s \in S\}$ of variables indexed by elements of S ; so the S -run $s_3s_2s_1$ becomes $x_{s_3}x_{s_2}x_{s_1}$.

If $F * S$ satisfies an equation system ε under the canonical evaluation of parameters, then the chosen witnesses in $F * S$ will provide a collection of equalities between S -runs. More precisely, each equality in ε produces an equality in $F * S$ of the form

$$f_1S_1f_2 \dots S_{k_i-1}f_{k_i} = g_1T_1g_2 \dots T_{\ell_i-1}g_{\ell_i} \tag{23}$$

where S_1, \dots, S_{k_i-1} and T_1, \dots, T_{ℓ_i-1} are S -runs. As this is an equality holding in the free product, it follows that $k_i = \ell_i$ and that $S_1 = T_1, \dots, S_{k_i-1} = T_{k_i-1}$ in S (and that $f_i = g_i$ are identical as words in A^+ , or empty). For such a choice c of witnesses to the canonical evaluation, let ε_c denote the existential equation systems consisting of the conjunction of the equalities arising from the resulting abstract S -runs, across the witnessing evaluations of all the equalities $p_1 = q_1 \wedge \dots \wedge p_\ell = q_\ell$. As an example, consider the equation system

$$\varepsilon : (\forall a_1)(\exists x_1)(\forall a_2)(\exists x_2\exists x_3) : (a_1a_2x_1x_2x_3a_1 = a_1x_2a_1a_2x_3x_1 \wedge a_1x_3a_1 = a_1x_3x_3a_1),$$

and consider a semigroup S in which the canonical evaluation of a_1, a_2 into $F * S$ has choices for x_1, x_2, x_3 satisfying the free dependency condition that lead to satisfaction of the equalities: an example might be $X_1 := s_1A_1, X_2 := A_2s_1$ and $X_3 = s_2$. (Note that as x_1 is quantified prior to a_2 the free dependency condition requires that the word X_1 should not involve A_2 , because the value of x_1 should not in general depend on the choice of a_2 .) In this hypothetical scenario, we have

$$(A_1)(A_2)(s_1A_1)(A_2s_1)(s_2)(A_1) = (A_1)(A_2s_1)(A_1)(A_2)(s_2)(s_1A_1)$$

and

$$(A_1)(s_2)(A_1) = (A_1)(s_2)(s_2)(A_1).$$

From the first equality we find that $s_1 = s_1$ and $s_1s_2 = s_2s_1$. From the second equality we find that $s_2 = s_2s_2$. Then for this choice c we obtain ε_c as

$$(\exists x_{s_1}\exists x_{s_2}) : (x_{s_1} = x_{s_1} \wedge x_{s_1}x_{s_2} = x_{s_2}x_{s_1} \wedge x_{s_2} = x_{s_2}x_{s_2}).$$

Obviously the equality $x_{s_1} = x_{s_1}$ here is redundant and could be removed. One can further see that s_1 could be replaced by s_2 in this sentence, so that ε_c is logically equivalent to $(\exists x) x = x^2$.

Lemma 3.4 *Let ε be an equation system satisfied by $F * S$, under the canonical parameter evaluation for some semigroup S . Then for any choice c of witnesses in $F * S$ to the satisfaction of the equalities in ε we have $S \models \varepsilon_c$ and $\varepsilon_c \vdash \varepsilon$ within the class of semigroups.*

Proof Fix a choice c of witnesses, $x_i \mapsto X_i \in F * S$. The construction of ε_c from this choice trivially ensures that $S \models \varepsilon_c$, which proves the first claim. Now assume that U is any semigroup satisfying ε_c ; we show that $U \models \varepsilon$. We may find witnesses to the canonical evaluation of parameters in $F * U$, as the witnesses to satisfaction of ε_c in U provide the required U -runs to match those from S in the satisfaction of ε in $F * S$ under the canonical parameter evaluation: note that these will not violate the free dependency condition for existential witnesses, as they only provide the required U -runs, with any free parameters chosen in the evaluation of a variable x_i following whatever was done in S for the choice c (which satisfied the free dependency condition). Then Lemma 3.3 shows that U satisfies ε . \square

Referring to the example given prior to Lemma 3.4, where ε_c was logically equivalent to $(\exists x) x = x^2$, the statement $\varepsilon_c \vdash \varepsilon$ in Lemma 3.4 is asserting that any semigroup containing an idempotent will satisfy

$$(\forall a_1)(\exists x_1)(\forall a_2)(\exists x_2 \exists x_3): (a_1 a_2 x_1 x_2 x_3 a_1 = a_1 x_2 a_1 a_2 x_3 x_1 \wedge a_1 x_3 a_1 = a_1 x_3 x_3 a_1).$$

For a given equation system ε and any S for which $F * S$ satisfies ε under the canonical parameter evaluation, the maximal length of any S -run is bounded by the maximal number of consecutively adjacent existential variables within a word occurring in ε . Thus there is an upper bound ℓ on the number of S -runs of length more than 1 that appear in ε_c (independent of S and c). Because equalities between S -runs of length 1 are trivial (they yield equalities $x_s = x_s$ for some $s \in S$), the number of nontrivial conjuncts within ε_c is at most ℓ also, so that up to logical equivalence, there are only finitely many different existential equation systems of the form ε_c , with the number determined by the structure of the sentence ε only. We let $H(\varepsilon)$ denote this (finite) set of existential sentences. The case of $H(\varepsilon) = \emptyset$ is possible, and corresponds to the situation where there are no nontrivial S -runs in any canonical evaluation, for any S . In this situation, note that ε is trivially equivalent to the equation system ε' obtained by including in ε the conjunct $x = x$, where x is a new (existentially quantified) variable. Because any element of $F * S$ will satisfy $x = x$, we have the same witnesses as previously for ε (which by assumption all avoided any S -runs), along with an arbitrary witness for x . Thus $H(\varepsilon') = \{(\exists x): x = x\}$, so that we may let ε_c denote the equation system $(\exists x): x = x$.

Proof of Theorem 3.2 Clearly an existential class is closed under the taking of containing semigroups. Conversely let us suppose that \mathcal{C} is an EHP-class of semigroups that is closed under the operation of taking containing semigroups.

Let \mathcal{B} be the existential class with EHP basis E' , which is the set of all existential equation systems that are satisfied by all members of \mathcal{C} . Clearly $\mathcal{C} \subseteq \mathcal{B}$, and indeed \mathcal{B} is the smallest existential class that contains \mathcal{C} . Our task is to prove the reverse containment.

Let E be a set of equation systems characterising \mathcal{C} and consider any $\varepsilon \in E$. We will show that there is an equation of the form ε_c that can replace ε . Repeated application, across all members of E leads to a subset of E' , which will complete the proof.

As a first step we need to show that there is an equation of the form ε_c that holds on all members of \mathcal{C} . We may assume that $H(\varepsilon)$ is not empty. Assume for contradiction that for each $\varepsilon_c \in H(\varepsilon)$ there is $S_c \in \mathcal{C}$ that fails ε_c . As \mathcal{C} is closed under taking direct products we have that $T := \prod_{\varepsilon_c \in H(\varepsilon)} S_c \in \mathcal{C}$ and then as \mathcal{C} is closed under taking containing semigroups we have that $F * T \models \varepsilon$. But then $T \models \varepsilon_c$ for some choice of witnesses to the canonical parameter evaluation. But then all quotients of T satisfy ε_c , contradicting the assumption that S_c fails ε_c .

Thus for every $\varepsilon \in E$ there is an existential equation system of the form ε_c such that $\varepsilon_c \in E'$. By Lemma 3.4 we have that $\varepsilon_c \vdash \varepsilon$, so that ε_c can replace ε in E . \square

The existential equations that are satisfied by every semigroup are explicitly identified in [8, Corollary 6.7]. To conclude this section we note that this may be extended, at least at the level of an algorithmic solution to arbitrary equation systems. Satisfying equations in $F * S$ by canonical parameter evaluation is somewhat reminiscent of solving equations in free semigroups, a problem originally solved by Makanin [12], and of continued interest and development. The connection turns out to be genuine, and a strong form of Makanin's algorithm can be used to show that it is decidable to determine if an equation system holds in the variety of all semigroups.

Theorem 3.5 *The class of equation systems satisfied in the class of all semigroups is decidable.*

Proof As we now explain, Theorem 3.5 is a direct corollary of Makanin's celebrated solution [12] to solvability of equations over free semigroups, as extended to allow for rational constraints (see Chapter 12 of [10]). In the context of Makanin's algorithm, a system of equations on the free semigroup consists of a finite set of equalities ε between words in alphabet $\{A_1, A_2, \dots\} \cup \{X_1, X_2, \dots\}$ and we are asked whether there is a satisfying evaluation of the variables X_i in the free semigroup A^+ (where $A = \{A_1, A_2, \dots\}$ as before). In the absence of any constraint on the choice of the X_i , this coincides with what we called the canonical parameter evaluation of the generators A_1, A_2, \dots (as themselves) in the equation system $(\forall A_1, \dots, A_n)(\exists X_1, \dots, X_m)\varepsilon$ (for suitable n, m determined by the variables that appear in ε). Lothaire [10, §12.1.8] details an extension of Makanin's algorithm to allow for the variables X_1, X_2, \dots to be constrained by rational languages $\lambda_1, \lambda_2, \dots$ over the alphabet A . This enables us to additionally enforce the free dependency condition for an equation system ε : constrain each variable X_i to lie in the rational language λ_i excluding letters in A that are quantified in ε to the right of the existential quantification of X_i . Satisfaction of this constrained instance of the equation problem coincides in definition precisely with satisfaction of ε under the canonical parameter evaluation with free dependency condition holding. But this latter property is equivalent to unconditional satisfaction of the equation system ε : for the nontrivial direction, use the universal mapping property of A^+ with respect to itself (or alternatively use Lemma 3.4, using the fact that $A^+ * A^+ \cong A^+$). Thus the extension of Makanin's algorithm in [10, §12.1.8] can be used to decide satisfaction of arbitrary equation systems on A^+ .

Finally, the equation systems true on A^+ (with A a denumerable alphabet) are precisely those in the class of all semigroups. One direction of this claim follows trivially from the fact that A^+ is a semigroup. For the other direction, use the universal

mapping property for A^+ with respect to any other semigroup S to find that if ε is satisfied by A^+ (under canonical parameter evaluation, with free dependency), then it is satisfied by S . \square

4 EHP-classes requiring both types of quantifier alternation

An open question in this new theory of EHP-classes is: For any n , does there exist an EHP-class \mathcal{C} such that in any basis for \mathcal{C} , n alternations of the existential quantifiers \forall and \exists are *necessary* in at least one equation of the basis?

Example 2.3(iii) of [8] is the EHP-class \mathcal{C} defined by:

$$(\exists y)(\forall a)(\exists x, z): a = xyz,$$

which is the class of semigroups S that have a maximum \mathcal{J} -class J such that $S/(S-J)$ is not a null semigroup. It is proved there that \mathcal{C} cannot be defined by equations systems exclusively of the type $(\forall \dots)(\exists \dots)$ nor by systems with the reverse order of quantifiers $(\exists \dots)(\forall \dots)$.

The strategy for proving this type of result is two-fold. To show that $(\forall \dots)(\exists \dots)$ quantification is not possible we find a chain of semigroups $S_1 \subseteq S_2 \subseteq \dots$, with each $S_n \in \mathcal{C}$, such that the semigroup union $S = \bigcup_{n=1}^{\infty} S_n \notin \mathcal{C}$. It follows from this that \mathcal{C} cannot be captured by an equation system based on a $(\forall \dots)(\exists \dots)$ quantification as that would imply that $S \in \mathcal{C}$ as well. (This phenomenon we have already observed in Sect. 2 in the context of the EHP-class of Monoids.)

Next we wish to show that definition by a $(\exists \dots)(\forall \dots)$ quantification for \mathcal{C} is also impossible. This is done by identifying a semigroup chain $S_1 \subseteq S_2 \subseteq \dots$, where no member of the chain lies in \mathcal{C} , yet their union $S = \bigcup_{n=1}^{\infty} S_n$ is a semigroup in \mathcal{C} . Given that $S \in \mathcal{C}$, if \mathcal{C} possessed a quantification of the form $(\exists \dots)(\forall \dots)$, it would follow that $S_n \in \mathcal{C}$ for all sufficiently large n .

Our Example 4.1 is complementary to the previous one in that it is defined by quantification of the type $(\forall \dots)(\exists \dots)(\forall \dots)$.

We are working with the class of Semigroups throughout and all quantifications are assumed to take place in some arbitrary semigroup S . Letters at the front (resp. end) of the alphabet a, b, c , (resp. x, y, z) denote parameters (resp. variables) in a given equation, typically written $e : p = q$.

Example 4.1 Let \mathcal{C} be the EHP-class defined by

$$\mathcal{C} : (\forall a)(\exists x)(\forall b): axb = abx. \quad (24)$$

Then \mathcal{C} cannot be defined by equation systems of the type $(\forall \dots)(\exists \dots)$ nor by systems with the reverse order of quantifiers $(\exists \dots)(\forall \dots)$.

Proof We prove the claim by applying the strategy we have just outlined.

Let F_A be the free semigroup on the infinite set of generators

$$A = \{a_1, a_2, \dots, x_1, x_2, \dots\}.$$

Denote the finite subset $\{a_1, \dots, a_n, x_1, \dots, x_n\}$ of A by A_n . Let ρ be the congruence on F_A generated by

$$\rho^0 = \{(b_i x_j, x_j b_i) : \text{where } b_i \in \{a_i, x_i\}, i \leq j\},$$

and put $S = F_A/\rho$. Let $S_n \leq S$, where $S_n = \langle a_1\rho, \dots, a_n\rho, x_1\rho, \dots, x_n\rho \rangle$. In the following argument we suppress the symbol ρ and write $a \in S_n, b \in S$ and such like to stand for $a\rho \in S_n, b\rho \in S$.

Thus we have a semigroup chain $S_1 \leq S_2 \leq \dots \leq S_n \leq \dots \leq S$. Observe that $S_n \in \mathcal{C}$ as for any $a, b \in S_n$ we put $x = x_n$ and note that $axb = abx$ as x_n commutes with each member of A_n . However we now show that $S \notin \mathcal{C}$. Put $a = a_1$ and let $x \in S$. Then $x \in S_n$ for some n . Put $b = a_{n+1}$ and consider $axb = a_1 x a_{n+1}$. Since a_{n+1} commutes only with x_j where $j \geq n + 1$, we infer that any word $w \in F_A$ such that $a_1 x a_{n+1} = w$ in S has the form $w = w_1 a_{n+1}$ where w_1 is the result of a permutation of the letters of $a_1 x$. In particular $a_1 x a_{n+1} \neq a_1 a_{n+1} x$ in S . Hence S does not satisfy the equation system (24). Therefore \mathcal{C} cannot be defined by equations using only $(\forall \dots)(\exists \dots)$ quantification.

Next we wish to show that definition by a $(\exists \dots)(\forall \dots)$ quantification for \mathcal{C} is also impossible. This is done by identifying a semigroup chain $S_1 \leq S_2 \leq \dots$, where no member of the chain lies in \mathcal{C} , yet their union $S = \cup_{n=1}^\infty S_n$ is a semigroup in \mathcal{C} . The existence of such a chain together with the chain of the previous paragraph establishes our claim.

Let F_n be the free semigroup on $A_n = \{a_1, \dots, a_n, x_1, \dots, x_n\}$ ($n \geq 1$) and let F_A denote the free semigroup on $A = \cup_{n=1}^\infty A_n$. Let ρ be the congruence on F_A generated by ρ^0 where

$$\rho^0 = \{(ax_{n+1}b, abx_{n+1}), n \geq 1, a \in F_n, b \in F_A\}$$

and put $S = F_A/\rho$. Then $S \in \mathcal{C}$ for let us take any $a\rho \in S$. Then $a \in F_n$ for some $n \geq 1$. Take $x = x_{n+1}$ and let $b\rho \in S$. Then $(ax_{n+1}b, abx_{n+1}) \in \rho^0$, so that in S we have $axb = abx$, and therefore $S \in \mathcal{C}$.

Now define $S_n = \{a\rho : a \in F_n\}$. Then $S_n \leq S$ and we have the semigroup chain $S_1 \leq S_2 \leq \dots \leq S_n \leq S_{n+1} \leq \dots$, with S being the union of this chain. Now take $a = a_n$ and consider $a\rho, x\rho, b\rho \in S_n$. Then $a\rho x\rho b\rho = (a_n x b)\rho$. Since $a_n \notin F_m$ for any $m < n$, the only ρ^0 pairs involving a word with initial letter a_n contain some letter $x_{n+m} \notin A_n$ ($m \geq 1$). Hence $a_n x b \neq a_n b x$ in S_n , for any b that is not a power of x , from which we infer that S_n does not satisfy the equation system (24). We conclude that $S_n \notin \mathcal{C}$ for all $n \geq 1$ but, as we have witnessed, $S \in \mathcal{C}$.

We therefore conclude that \mathcal{C} cannot be defined by a basis of equation systems of the type $(\forall \dots)(\exists \dots)$ nor of the type $(\exists \dots)(\forall \dots)$. □

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